J. DIFFERENTIAL GEOMETRY
100 (2015) 533-545

A TOPOLOGICAL NECESSARY CONDITION FOR THE EXISTENCE OF COMPACT CLIFFORD–KLEIN FORMS

Yosuke Morita

Abstract

We provide a necessary condition for the existence of a compact Clifford–Klein form of a given homogeneous space of reductive type. The key to the proof is to combine a result of Kobayashi and Ono with the observation that a fiber bundle with contractible fiber induces an isomorphism between the cohomology rings of the total space and the base space. We give some examples: $SL(p+q,\mathbb{R})/SO(p,q)(p,q : \text{odd})$, for instance—of homogeneous spaces that do not admit compact Clifford–Klein forms.

1. Introduction

Let G be a Lie group and H its closed subgroup. If a discrete subgroup Γ of G acts properly discontinuously and freely on G/H, the double coset space $\Gamma\backslash G/H$ becomes a manifold locally modeled on G/H. The space $\Gamma\backslash G/H$ is then called a Clifford–Klein form of G/H.

In this paper, we study the following problem.

Problem 1.1 ([**8**])**.** When does G/H admit a compact Clifford–Klein form?

Using the results of [**4**, **23**], A. Borel [**3**] proved that when G is linear reductive and H is compact, G/H always admits a compact Clifford–Klein form. In contrast, if G is a linear reductive Lie group and H is a non-compact closed reductive subgroup of G, G/H does not necessarily admit a compact Clifford–Klein form. A systematic study of Problem 1.1 in this case was initiated by Kobayashi [**8**]. Since then, various methods derived from diverse fields in mathematics have been applied to this problem (for instance, [**1**, **10**, **15**, **22**, **24**, **25**]). Methods and results on this topic are surveyed in Kobayashi [**12**, **14**], Kobayashi and Yoshino [**16**], Labourie [**18**] and Constantine [**6**].

Example 1.2. If $(G,H) = (O(p,q+1), O(p,q))$ with $q \neq 1$, a compact Clifford–Klein form of G/H is nothing but a compact complete pseudo-Riemannian manifold of signature (p,q) with constant negative sectional curvature. When $q = 0$ (Riemannian case), G/H admits a

Received 10/26/2013.

compact Clifford–Klein form since H is compact. In contrast, when p and q are odd, G/H does not admit a compact Clifford–Klein form (see Kulkarni [**17**] or Kobayashi and Ono [**15**]). Corollary 1.6(4), below, combined with Fact 2.3 includes this result as a special case. For more information on this example, see Kobayashi and Yoshino [**16**].

Extending the idea of Kobayashi and Ono [**15**] that $H^\bullet(\Gamma\backslash G/H)$ is "larger than or equal to" $H^\bullet(G_U/H_U)$ if $\Gamma\backslash G/H$ is a compact Clifford–Klein form, we obtain a topological obstruction for the existence of compact Clifford–Klein forms:

Theorem 1.3. (see Convention 2.1 for notation and terminology) *Let G/H be a homogeneous space of reductive type, G_U/H_U the compact homogeneous space associated to G/H, and K_H the maximal compact subgroup of H. If the homomorphism*

$$\pi^* : H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/K_H;\mathbb{C})$$

induced by the projection $\pi : G_U/K_H \to G_U/H_U$ is not injective, then G/H does not admit a compact Clifford–Klein form.

The key to the proof of Theorem 1.3 is to combine the above idea of Kobayashi and Ono with an elementary fact that the cohomology rings of two different Clifford–Klein forms $H^\bullet(\Gamma\backslash G/H;\mathbb{C})$ and $H^\bullet(\Gamma\backslash G/K_H;\mathbb{C})$ are isomorphic to each other (see Proposition 2.5).

As an application of Theorem 1.3, we obtain some examples of symmetric spaces G/H that do not admit compact Clifford–Klein forms:

Corollary 1.4. *A symmetric space G/H does not admit a compact Clifford–Klein form if (G,H) is one of the following:*
(1) $(GL(2n,\mathbb{R}), GL(n,\mathbb{C}))$ $(n > 1)$
(2) $(SL(p+q,\mathbb{R}), SO(p,q))$ $(p,q : odd)$
(3) $(O(n,n), O(n,\mathbb{C}))$ $(n > 1)$
(4) $(O(p+r,q+s), O(p,q)\times O(r,s))$ $(p,q : odd,\ r > 0)$

Remark 1.5. We mention some related results that were previously obtained by using different methods:

- (1) is new to the best of the author's knowledge.
- Concerning (2), Kobayashi [**10**] proved that $SL(2p,\mathbb{R})/SO(p,p)$ $(p > 0)$ does not admit a compact Clifford–Klein form. Benoist [**1**] gave an alternative proof of this result, and also proved that $SL(2p+1,\mathbb{R})/SO(p,p+1)$ $(p > 0)$ does not admit a compact Clifford–Klein form.
- Concerning (3), Kobayashi [**10**] proved that $SO(n,n)/SO(n,\mathbb{C})$ (n : even) does not admit a compact Clifford–Klein form. For odd n, (3) is new to the best of the author's knowledge.
- Concerning (4), Kobayashi [**10**] proved that $O(p+r,q+s)/(O(p,q)\times O(r,s))$ does not admit a compact Clifford–Klein form unless

$\min\{p, q, r, s\} = 0$. We assume $s = 0$ without loss of generality. Then, furthermore, $O(p+r, q)/(O(p, q) \times O(r))$ does not admit a compact Clifford–Klein form if $p + r > q$ (Kobayashi [**10**]), $(p, q, r) = (2n, 2n+1, 1)$ (Benoist [**1**]), or p, q, r are all odd (Kobayashi and Ono [**15**]). On the other hand, it admits a compact Clifford–Klein form if $(p, q, r) = (1, 2n, 1)$, $(3, 4n, 1)$, $(7, 8, 1)$, $(1, 4, 3)$, $(1, 4, 2)$ (Kulkarni [**17**], Kobayashi [**10**, **12**]).

We can also apply our method to non-symmetric homogeneous spaces. For instance:

Corollary 1.6. *A homogeneous space G/H does not admit a compact Clifford–Klein form if (G, H) is one of the following:*

(1) $(SL(n_1 + \cdots + n_k, \mathbb{R}), SL(n_1, \mathbb{R}) \times \cdots \times SL(n_k, \mathbb{R}))$ $(n_1, n_2 > 2)$
(2) $(SL(n_1 + \cdots + n_k, \mathbb{C}), SL(n_1, \mathbb{C}) \times \cdots \times SL(n_k, \mathbb{C}))$ $(n_1, n_2 > 1)$
(3) $(SL(n_1 + \cdots + n_k, \mathbb{H}), SL(n_1, \mathbb{H}) \times \cdots \times SL(n_k, \mathbb{H}))$ $(n_1, n_2 > 1)$
(4) $(O(p_1 + \cdots + p_k, q_1 + \cdots + p_k), O(p_1, q_1) \times \cdots \times O(p_k, q_k))$ $(p_1, q_1:$ *odd,* $p_2 > 0)$

Remark 1.7. Corollary 1.4(4) is a special case of Corollary 1.6(4).

Remark 1.8.
- The existence problem of compact Clifford–Klein forms of $SL(n, \mathbb{F})/SL(m, \mathbb{F})$ $(n > m, \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ has been attracted considerable attention. The first result was obtained in [**9**] in the setting $n = 3, m = 2, \mathbb{F} = \mathbb{C}$. Some further results are in [**1**, **12**, **19**, **20**, **24**, **25**]. For example, expanding the method of [**25**, **19**], Labourie and Zimmer [**20**] proved that $SL(n, \mathbb{R})/SL(m, \mathbb{R})$ does not admit a compact Clifford–Klein form if $n - m > 2$. Unfortunately, Theorem 1.3 gives no information about this case.
- Benoist [**1**] proved that $SL(p+q, \mathbb{R})/(SL(p, \mathbb{R}) \times SL(q, \mathbb{R}))$ $(p, q > 0)$ does not admit a compact Clifford–Klein form if pq is even.
- By applying the method of [**10**], Kobayashi [**12**] gave many results that are similar to Corollary 1.6. See [**12**, Example 4.13.5, Example 4.13.6, Example 4.13.7].

Remark 1.9. We can also prove that $O(n_1 + \cdots + n_k, \mathbb{C})/(O(n_1, \mathbb{C}) \times \cdots \times O(n_k, \mathbb{C}))$ $(n_1, n_2 > 1$ or n_1 : even, $n_2 = 1)$ and $Sp(n_1 + \cdots + n_k, \mathbb{C})/(Sp(n_1, \mathbb{C}) \times \cdots \times Sp(n_k, \mathbb{C}))$ $(n_1, n_2 > 0)$ do not admit compact Clifford–Klein forms. However, these examples are not new. We can apply the method of [**10**] to these cases.

Kobayashi and Ono have already deduced a necessary condition for the existence of Clifford–Klein forms ([**15**, Corollary 5]). Later, Kobayashi gave a generalization of this result ([**8**, Proposition 4.10]). These methods hilighted the Euler class of tangent bundle. A feature of Theorem 1.3 is that it includes information not only on the Euler class of tangent bundles but also on other cohomology classes; to obtain the above examples, we use characteristic classes that are different from the Euler class of tangent bundles. We give a proof of [**8**, Proposition 4.10] in the spirit of [**15**] by using Theorem 1.3 (see Corollary 6.1).

Acknowledgments. The author expresses his sincere gratitude to Professor Toshiyuki Kobayashi for his advice and encouragement. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

2. Preliminaries and proof of Theorem 1.3

We work in the following setting unless otherwise specified:

Convention 2.1. G is a linear reductive Lie group and H is a closed connected subgroup of G that is reductive in G. Without loss of generality, we shall realize G and its subgroup H as closed subgroups of $GL(N,\mathbb{R})$ that are stable under transposition. $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ are connected Lie subgroups of $GL(N,\mathbb{C})$ with Lie algebras $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes \mathbb{C}$, respectively. We assume that $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ are closed in $GL(n,\mathbb{C})$. Put $G_U = G_{\mathbb{C}} \cap U(N)$ and $H_U = H_{\mathbb{C}} \cap U(N)$. They are compact connected real forms of $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$, respectively. Finally, put $K_G = G \cap O(N)$ and $K_H = H \cap O(N)$. They are maximal compact subgroups of G and H, respectively.

REMARK 2.2. (1) Since we assumed that H is connected, K_H is also connected by the Cartan decomposition, and hence a closed subgroup of H_U.

(2) The assumption that H is connected is not a serious restriction by the following result, which is essentially proved in Kobayashi [**8**]:

Fact 2.3. *Let G be a linear Lie group and H a closed subgroup of G. Suppose H has finitely many connected components. We denote by H_o the identity component of H. Then, G/H admits a compact Clifford–Klein form if and only if G/H_o admits a compact Clifford–Klein form.*

We prepare some results on the topology of Clifford–Klein forms to prove Theorem 1.3. First, we observe the following:

Lemma 2.4. *Let G_1 be a Lie group, G_2 a closed subgroup of G_1, G_3 a closed subgroup of G_2, and Γ a discrete subgroup of G_1.*

(1) If Γ acts properly discontinuously on G_1/G_2, it also acts properly discontinuously on G_1/G_3.
(2) If Γ acts freely on G_1/G_2, it also acts freely on G_1/G_3.
(3) If the assumptions of (1) and (2) are satisfied, the projection $\pi : \Gamma\backslash G_1/G_3 \to \Gamma\backslash G_1/G_2$ becomes a fiber bundle with typical fiber G_2/G_3.

Proof. (1) This is a special case of [**11**, Lemma 1.3 (1)].

(2) Γ acts freely on G_1/G_2 if and only if $x(\Gamma - \{1\})x^{-1} \cap G_2 = \emptyset$ for any $x \in G_1$. Thus, the statement follows.

(3) This follows immediately from (1) and (2). q.e.d.

Suppose $\Gamma\backslash G/H$ is a Clifford–Klein form of G/H. Then it follows from Lemma 2.4(1)–(2) that $\Gamma\backslash G/K_H$ is also a Clifford–Klein form. We do not assume that $\Gamma\backslash G/H$ is compact until the compactness is needed.

Proposition 2.5. *The projection $\pi : \Gamma\backslash G/K_H \to \Gamma\backslash G/H$ induces an isomorphism*

$$\pi^* : H^\bullet(\Gamma\backslash G/H; \mathbb{C}) \xrightarrow{\sim} H^\bullet(\Gamma\backslash G/K_H; \mathbb{C}).$$

Proof. By Lemma 2.4(3) and the Cartan decomposition, the projection $\pi : \Gamma\backslash G/K_H \to \Gamma\backslash G/H$ is a fiber bundle with contractible typical fiber H/K_H. Thus, the statement is an immediate consequence of the Leray–Serre spectral sequence. q.e.d.

Next, let us recall a homomorphism η constructed in Kobayashi and Ono [**15**]. The space $\mathcal{A}^p(G/H)^G$ of G-invariant p-forms on G/H is canonically isomorphic to $(\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^H = (\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ (H is connected). Likewise, $\mathcal{A}^p(G_U/H_U)^{G_U}$ is canonically isomorphic to $(\Lambda^p(\mathfrak{g}_U/\mathfrak{h}_U)^*)^{\mathfrak{h}_U}$. The natural isomorphism

$$(\Lambda^p(\mathfrak{g}_U/\mathfrak{h}_U)^*)^{\mathfrak{h}_U} \otimes \mathbb{C} \simeq (\Lambda^p(\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C})^*)^{\mathfrak{h}_\mathbb{C}} \simeq (\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \otimes \mathbb{C}$$

induces

$$\eta : \mathcal{A}^p(G_U/H_U)^{G_U} \otimes \mathbb{C} \xrightarrow{\sim} \mathcal{A}^p(G/H)^G \otimes \mathbb{C} \hookrightarrow \mathcal{A}^p(\Gamma\backslash G/H) \otimes \mathbb{C}.$$

Taking cohomology, we obtain

$$\eta : H^p(G_U/H_U; \mathbb{C}) \to H^p(\Gamma\backslash G/H; \mathbb{C}).$$

Fact 2.6. (see [**15**, Proposition 3.9]) *If $\Gamma\backslash G/H$ is compact, η is injective.*

Now we shall prove Theorem 1.3.

Proof of Theorem 1.3. $\eta : H^\bullet(G_U/K_H; \mathbb{C}) \to H^\bullet(\Gamma\backslash G/K_H; \mathbb{C})$ can be defined in the same way as above. By definition, the diagram

$$\begin{array}{ccc} H^\bullet(G_U/H_U; \mathbb{C}) & \xrightarrow{\eta} & H^\bullet(\Gamma\backslash G/H; \mathbb{C}) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ H^\bullet(G_U/K_H; \mathbb{C}) & \xrightarrow{\eta} & H^\bullet(\Gamma\backslash G/K_H; \mathbb{C}) \end{array}$$

is commutative. By Proposition 2.5, π^* on the right-hand side is isomorphic. If $\Gamma\backslash G/H$ is compact, then η on the above is injective by Fact 2.6 and, therefore, π^* on the left-hand side has to be injective. This completes the proof. q.e.d.

3. The Chern–Weil homomorphism and non-injectivity

To apply Theorem 1.3, we have to find examples of G/H such that $\pi^* : H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/K_H;\mathbb{C})$ is not injective. In this section, we give a sufficient condition for non-injectivity, which is easy to verify in typical cases.

$\pi : G_U \to G_U/H_U$ is a principal H_U-bundle. Thus, the Chern–Weil characteristic homomorphism

$$w : (S^p(\mathfrak{h}_U)^*)^{H_U} \to H^{2p}(G_U/H_U;\mathbb{R}) \subset H^{2p}(G_U/H_U;\mathbb{C})$$

is defined. It is straightforward to see that the diagram

$$\begin{tikzcd} (S(\mathfrak{h}_U)^*)^{H_U} \arrow[r, "w"] \arrow[d, "\mathrm{rest}"] & H^\bullet(G_U/H_U;\mathbb{C}) \arrow[d, "\pi^*"] \\ (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H} \arrow[r, "w"] & H^\bullet(G_U/K_H;\mathbb{C}) \end{tikzcd}$$

is commutative. Here, rest : $(S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}$ is the restriction map.

Fact 3.1. (see [**5**, §10])

$$\ker\left(w : (S(\mathfrak{h}_U)^*)^{H_U} \to H^\bullet(G_U/H_U;\mathbb{C})\right)$$

is equal to the ideal J_{G_U/H_U} generated by

$$\bigoplus_{p=1}^{\infty} \mathrm{im}\left(\mathrm{rest} : (S^p(\mathfrak{g}_U)^*)^{G_U} \to (S^p(\mathfrak{h}_U)^*)^{H_U}\right).$$

Proposition 3.2. *Assume that*

$$\ker\left(\mathrm{rest} : (S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}\right) \not\subset J_{G_U/H_U},$$

where J_{G_U/H_U} is as in Fact 3.1. *Then the homomorphism $\pi^* : H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/K_H;\mathbb{C})$ induced by the projection $\pi : G_U/K_H \to G_U/H_U$ is not injective, and hence G/H does not admit a compact Clifford–Klein form.*

Proof. By Fact 3.1, we can pick $P \in (S(\mathfrak{h}_U)^*)^{H_U}$ such that $w(P) \neq 0$ and $P|_{\mathfrak{k}_\mathfrak{h}} = 0$. Then $w(P) \in H(G_U/H_U;\mathbb{C})$ is a non-zero element of a kernel of $\pi^* : H(G_U/H_U;\mathbb{C}) \to H(G_U/K_H;\mathbb{C})$. q.e.d.

By Chevalley's restriction theorem, we can rewrite Proposition 3.2 in terms of Cartan subalgebras and Weyl groups as follows.

Convention 3.3. We take maximal tori T_{G_U} of G_U, T_{H_U} of H_U, T_{K_G} of K_G and T_{K_H} of K_H such that $T_{G_U} \supset T_{H_U} \supset T_{K_H}$ and $T_{K_G} \supset T_{K_H}$. Their Lie algebras and their Weyl groups are denoted by $\mathfrak{t}_{\mathfrak{g}_U}$, $\mathfrak{t}_{\mathfrak{h}_U}$, $\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}$, $\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}}$, W_{G_U}, W_{H_U}, W_{K_G}, and W_{K_H}, respectively.

Let us denote by I_{G_U/H_U} an ideal of $(S(\mathfrak{t}_{\mathfrak{h}_U})^*)^{W_{\mathfrak{h}_U}}$ generated by

$$\bigoplus_{p=1}^{\infty} \operatorname{im}\left(\text{rest} : (S^p(\mathfrak{t}_{\mathfrak{g}_U})^*)^{W_{G_U}} \to (S^p(\mathfrak{t}_{\mathfrak{h}_U})^*)^{W_{H_U}}\right).$$

In other words, I_{G_U/H_U} is the ideal of $(S(\mathfrak{t}_{\mathfrak{h}_U})^*)^{W_{H_U}}$ corresponding to J_{G_U/H_U} under the isomorphism $(S(\mathfrak{t}_{\mathfrak{h}_U})^*)^{W_{H_U}} \simeq (S(\mathfrak{h}_U)^*)^{H_U}$.

Corollary 3.4. *Assume that*

$$\ker\left(\text{rest} : (S(\mathfrak{t}_{\mathfrak{h}_U})^*)^{W_{H_U}} \to (S(\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})^*)^{W_{K_H}}\right) \not\subset I_{G_U/H_U}.$$

Then G/H does not admit a compact Clifford–Klein form.

4. Reduction to maximal tori

In the statement of Theorem 1.3, one can replace K_H by T_{K_H}:

Corollary 4.1. *If the homomorphism*

$$\pi^* : H^\bullet(G_U/H_U; \mathbb{C}) \to H^\bullet(G_U/T_{K_H}; \mathbb{C})$$

induced by the projection $\pi : G_U/T_{K_H} \to G_U/H_U$ is not injective, is not injective, then G/H does not admit a compact Clifford–Klein form.

In order to prove Corollary 4.1, we use the following fact:

Fact 4.2. (see [**21**, Théorème 2.2], [**7**, Theorem 6.8.3]) *Let K be a connected compact Lie group, T a maximal torus of K, and W its Weyl group. If M is a manifold on which K acts freely, the homomorphism*

$$\pi^* : H^\bullet(M/K; \mathbb{C}) \to H^\bullet(M/T; \mathbb{C})$$

induced by the projection $\pi : M/T \to M/K$ is injective and its image is $H^\bullet(M/T; \mathbb{C})^W$.

Proof of Corollary 4.1. Putting $M = G_U$ and $K = K_H$ in Fact 4.2, we obtain that

$$\pi^* : H^\bullet(G_U/K_H; \mathbb{C}) \to H^\bullet(G_U/T_{K_H}; \mathbb{C})$$

is injective. Thus, $\pi^* : H^\bullet(G_U/H_U; \mathbb{C}) \to H^\bullet(G_U/K_H; \mathbb{C})$ is injective if and only if $\pi^* : H^\bullet(G_U/H_U; \mathbb{C}) \to H^\bullet(G_U/T_{K_H}; \mathbb{C})$ is injective. Now the statement follows from Theorem 1.3. q.e.d.

5. Examples

In this section, we prove Corollary 1.4 and Corollary 1.6.

Proof of Corollary 1.4. (1) It is enough to confirm that the assumption of Proposition 3.2 is satisfied when $G_U = U(2n)$, $H_U = U(n) \times U(n)$ and

$$K_H = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in U(n) \right\}.$$

Recall that $(S(\mathfrak{u}(n))^*)^{U(n)}$ is the polynomial algebra generated by $\{c_1, \ldots, c_n\}$, where $c_i \in (S^i(\mathfrak{u}(n))^*)^{U(n)}$ refers the elementary symmetric polynomial of $U(n)$ of degree i. Geometrically, c_i corresponds to the ith Chern class of $U(n)$. Now, $(S(\mathfrak{g}_U)^*)^{G_U}$, $(S(\mathfrak{h}_U)^*)^{H_U}$ and $(S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}$ are the polynomial algebras generated by $\{c_1, \ldots, c_{2n}\}$, $\{c_1 \otimes 1, \ldots, c_n \otimes 1, 1 \otimes c_1, \ldots, 1 \otimes c_n\}$ and $\{c_1, \ldots, c_n\}$, respectively. The restriction maps are given by

$$\mathrm{rest} : (S(\mathfrak{g}_U)^*)^{G_U} \to (S(\mathfrak{h}_U)^*)^{H_U}, \quad c_i \mapsto c_i \otimes 1 + c_{i-1} \otimes c_1 + \cdots + 1 \otimes c_i$$

and

$$\mathrm{rest} : (S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}, \quad c_i \otimes 1 \mapsto c_i,\ 1 \otimes c_i \mapsto (-1)^i c_i.$$

Therefore,

$$c_2 \otimes 1 - 1 \otimes c_2 \in \ker\left(\mathrm{rest} : (S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}\right).$$

On the other hand,

$$c_2 \otimes 1 - 1 \otimes c_2 \notin J_{G_U/H_U},$$

namely, $c_2 \otimes 1 - 1 \otimes c_2$ is not contained in the ideal of $(S(\mathfrak{h}_U)^*)^{H_U}$ generated by the restrictions of the positive-degree parts of the elements of $(S(\mathfrak{g}_U)^*)^{G_U}$. Thus, the assumption of Proposition 3.2 is satisfied.

(2) We first remark that we may replace $SO(p,q)$ with its identity component $SO_o(p,q)$ by Fact 2.3. Thus, it suffices to confirm that the assumption of Proposition 3.2 is satisfied when $G_U = SU(p+q)$, $H_U = SO(p+q)$ and $K_H = SO(p) \times SO(q)$. Recall that $(S(\mathfrak{so}(n))^*)^{SO(n)}$ is the polynomial algebra generated by $\{p_1, \ldots, p_{\frac{n-1}{2}}\}$ when n is odd, and by $\{p_1, \ldots, p_{\frac{n-2}{2}}, e\}$ when n is even (note that $p_{\frac{n}{2}} = e^2$). Here $p_i \in (S^{2i}(\mathfrak{so}(n))^*)^{SO(n)}$ corresponds to the ith Pontrjagin class and $e \in (S^{\frac{n}{2}}(\mathfrak{so}(n))^*)^{SO(n)}$ corresponds to the Euler class. Since $p+q$ is even,

$$(S(\mathfrak{h}_U)^*)^{H_U} = (S(\mathfrak{so}(p+q))^*)^{SO(p+q)}$$

is freely generated by $\{p_1, \ldots, p_{\frac{p+q-2}{2}}, e\}$. Now, since p and q are odd,

$$e \in \ker\left(\mathrm{rest} : (S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}\right),$$

namely, the restriction of the Euler class e to $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ is equal to zero. On the other hand, since the restrictions of the elements of $(S(\mathfrak{g}_U)^*)^{G_U}$ are written as polynomials of Pontrjagin classes, $e \notin J_{G_U/H_U}$.

(3)–(4) The proofs are analogous to (1) and (2); we consider $p_1 \otimes 1 - 1 \otimes p_1$ and $e \otimes 1$, respectively. q.e.d.

Next, we shall prove Corollary 1.6. We use the following general results.

Proposition 5.1. *(1) Let $\widetilde{G}$ be a linear reductive Lie group, G a closed subgroup of $\widetilde{G}$, and H a closed connected subgroup of G. Assume that G is reductive in $\widetilde{G}$, and H is reductive in G. If (G, H) satisfies the assumption of Proposition* 3.2 *(or, equivalently, Corollary* 3.4*), so does $(\widetilde{G}, H)$.*

(2) Let G be a linear reductive Lie group. Let H, H' be two closed connected subgroups of G such that $H \cap H' = \{1\}$ and $H' \subset Z(H)$. Assume that $H \times H'$ is reductive in G. If (G, H) satisfies the assumption of Proposition 3.2*, so does $(G, H \times H')$.*

Proof. (1) Without loss of generality, we may assume that $\widetilde{G}$, G, and H are stable under transposition. Thus, we can define $\widetilde{G}_U$ and $\tilde{\mathfrak{g}}_U$ as in Convention 2.1. Since the restriction map $(S(\tilde{\mathfrak{g}}_U)^*)^{\widetilde{G}_U} \to (S(\mathfrak{h}_U)^*)^{H_U}$ factors $(S(\mathfrak{g}_U)^*)^{G_U}$, the image of

$$\mathrm{rest} : (S^p(\tilde{\mathfrak{g}}_U)^*)^{\widetilde{G}_U} \to (S^p(\mathfrak{h}_U)^*)^{H_U}$$

is contained in the image of

$$\mathrm{rest} : (S^p(\mathfrak{g}_U)^*)^{G_U} \to (S^p(\mathfrak{h}_U)^*)^{H_U}$$

for each p. Hence $J_{\widetilde{G}_U/H_U} \subset J_{G_U/H_U}$ and the statement follows.

(2) Without loss of generality, we may assume that G, H, and H' are stable under transposition. Thus we can define H'_U, $\mathfrak{h}'_U$, $K_{H'}$, and $\mathfrak{k}_{\mathfrak{h}'}$ as in Convention 2.1. We remark that

$$(S(\mathfrak{h}_U \oplus \mathfrak{h}'_U)^*)^{H_U \times H'_U} \simeq (S(\mathfrak{h}_U)^*)^{H_U} \otimes (S(\mathfrak{h}'_U)^*)^{H'_U}.$$

By the assumption of Proposition 3.2 for (G, H), there exists $P \in (S(\mathfrak{h}_U)^*)^{H_U}$ such that $P \notin J_{G_U/H_U}$ and $P|_{\mathfrak{k}_\mathfrak{h}} = 0$. Then $P \otimes 1 \in (S(\mathfrak{h}_U)^*)^{H_U} \otimes (S(\mathfrak{h}'_U)^*)^{H'_U}$ satisfies $P|_{\mathfrak{k}_\mathfrak{h} \oplus \mathfrak{k}_{\mathfrak{h}'}} = 0$. Furthermore, $P \otimes 1 \notin J_{G_U/(H_U \times H'_U)}$; it is a straightforward consequence of the fact that the restriction map $(S(\mathfrak{g}_U)^*)^{G_U} \to (S(\mathfrak{h}_U)^*)^{H_U}$ factors $(S(\mathfrak{h}_U)^*)^{H_U} \otimes (S(\mathfrak{h}'_U)^*)^{H'_U}$. Thus $(G, H \times H')$ also satisfies the assumption of Proposition 3.2. q.e.d.

Proof of Corollary 1.6. (1) Suppose $n_1, n_2 > 2$.

$$(SL(n_1 + n_2, \mathbb{R}), SL(n_1, \mathbb{R}) \times SL(n_2, \mathbb{R}))$$

satisfies the assumption of Proposition 3.2. Indeed,

$$c_3 \otimes 1 \in \ker \left(\mathrm{rest} : (S(\mathfrak{h}_U)^*)^{H_U} \to (S(\mathfrak{k}_\mathfrak{h})^*)^{K_H}\right),$$

while $c_3 \otimes 1 \notin J_{G_U/H_U}$. Here, $c_i \in (S^i(\mathfrak{su}(p))^*)^{SU(p)}$ refers to the ith Chern class of $SU(p)$. Then

$$(SL(n_1 + \cdots + n_k, \mathbb{R}), SL(n_1, \mathbb{R}) \times SL(n_2, \mathbb{R}))$$

also satisfies the assumption by Proposition 5.1(1) and thus so does

$$(SL(n_1 + \cdots + n_k, \mathbb{R}), SL(n_1, \mathbb{R}) \times \cdots \times SL(n_k, \mathbb{R}))$$

by Proposition 5.1(2). In particular, $SL(n_1+\cdots+n_k,\mathbb{R})/(SL(n_1,\mathbb{R})\times\cdots\times SL(n_k,\mathbb{R}))$ does not admit a Clifford–Klein form.

The proofs of (2)–(4) are parallel to that of (1). q.e.d.

Remark 5.2. More generally, if $n_1, n_2 > 2$, then

- $SL(n_1+n_2+n_3,\mathbb{R})/(SL(n_1,\mathbb{R})\times SL(n_2,\mathbb{R})\times H')$ and
- $SL(n_1+n_2+n_3,\mathbb{R})/(S(GL(n_1,\mathbb{R})\times GL(n_2,\mathbb{R}))\times H')$

do not admit compact Clifford–Klein forms for any closed subgroup H' of $SL(n_3,\mathbb{R})$ such that H' is reductive in $SL(n_3,\mathbb{R})$ and the complexification of H' is closed in $SL(n_3,\mathbb{C})$. The proof is the same as that of Corollary 1.6(1). Similar results also hold for (2)–(4).

6. Some remarks

We give a proof of [**8**, Proposition 4.10] in the spirit of [**15**] rather than [**8**] that uses an argument of spectral sequence:

Corollary 6.1. *If* $\operatorname{rank} G = \operatorname{rank} H$ *and* $\operatorname{rank} K_G > \operatorname{rank} K_H$, G/H *does not admit a compact Clifford–Klein form.*

Proof. It is well known that the Euler characteristic $\chi(G_U/H_U)$ of G_U/H_U is non-zero if and only if $\operatorname{rank} G_U = \operatorname{rank} H_U$, which is equivalent to $\operatorname{rank} G = \operatorname{rank} H$. By the Gauss–Bonnet–Chern theorem,

$$\chi(G_U/H_U) = \int_{G_U/H_U} e(\mathcal{T}(G_U/H_U)),$$

where $\mathcal{T}(G_U/H_U)$ denotes the tangent bundle of G_U/H_U. Hence the Euler class $e(\mathcal{T}(G_U/H_U)) \in H^n(G_U/H_U;\mathbb{C})$ $(n = \dim G - \dim H)$ is non-zero when $\operatorname{rank} G = \operatorname{rank} H$. Thus, by Corollary 4.1, it suffices to show that $\pi^* : H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/T_{K_H};\mathbb{C})$ sends $e(\mathcal{T}(G_U/H_U))$ to zero if $\operatorname{rank} K_G > \operatorname{rank} K_H$.

First, we note that

$$\mathcal{T}(G_U/H_U) = G_U \underset{H_U}{\times} (\mathfrak{g}_U/\mathfrak{h}_U)$$

and hence

$$\pi^*\mathcal{T}(G_U/H_U) = G_U \underset{T_{K_H}}{\times} (\mathfrak{g}_U/\mathfrak{h}_U).$$

Now, $\mathfrak{g}_U/\mathfrak{h}_U = (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}}) \oplus (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})^\perp$ as a real unitary representation of T_{K_H} (an inner product on $\mathfrak{g}_U/\mathfrak{h}_U$ is defined by the Killing form of $\mathfrak{g}_U$). Therefore,

$$\pi^*\mathcal{T}(G_U/H_U) = \left(G_U \underset{T_{K_H}}{\times} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})\right) \oplus \left(G_U \underset{T_{K_H}}{\times} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})^\perp\right).$$

$G_U \times_{T_{K_H}} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})$ is a trivial bundle because T_{K_H} acts trivially on $\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}}$. Its typical fiber $\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}}$ has non-zero dimension since $\operatorname{rank} K_G > \operatorname{rank} K_H$.

As a consequence, $e\left(G_U \times_{T_{K_H}} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})\right) = 0$ and

$$\begin{aligned}\pi^* e(\mathcal{T}(G_U/H_U)) &= e(\pi^*\mathcal{T}(G_U/H_U)) \\ &= e\left(G_U \underset{T_{K_H}}{\times} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})\right) e\left(G_U \underset{T_{K_H}}{\times} (\mathfrak{t}_{\mathfrak{k}_\mathfrak{g}}/\mathfrak{t}_{\mathfrak{k}_\mathfrak{h}})^{\perp}\right) \\ &= 0.\end{aligned}$$

q.e.d.

The following results exhibit limitations of our method:

Proposition 6.2. $H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/K_H;\mathbb{C})$ *is injective if either (1) or (2) is satisfied:*
(1) $\operatorname{rank} H = \operatorname{rank} K_H$*.*
(2) G *is a complexification of* H*.*

Proof. (1) By the proof of Corollary 4.1, it suffices to show that π^* : $H^\bullet(G_U/H_U;\mathbb{C}) \to H^\bullet(G_U/T_{K_H};\mathbb{C})$ is injective. By assumption, T_{K_H} coincides with T_{H_U}. Hence the injectivity follows from Fact 4.2 by putting $M = G_U$ and $K = H_U$.

(2) If $G = H_\mathbb{C}$, the projection $\pi : G_U/K_H \to G_U/H_U$ is rewritten as

$$\pi : (H_U \times H_U)/\Delta K_H \to (H_U \times H_U)/\Delta H_U.$$

$\pi^* : H^\bullet((H_U \times H_U)/\Delta H_U;\mathbb{C}) \to H^\bullet((H_U \times H_U)/\Delta K_H;\mathbb{C})$ is injective because a group manifold $(H\times H)/\Delta H$ admits a compact Clifford–Klein form (see Kobayashi [**8**, Example 4.8]). q.e.d.

Example 6.3. (1) Suppose $(G,H) = (U(p+r,q), U(p,q)\times U(r))$ with $p \geqslant q$ and $p,q,r > 0$. Then $\operatorname{rank} H = \operatorname{rank} K_H$, but only finite subgroups of G act properly discontinuously on G/H (in particular, G/H does not admit a compact Clifford–Klein form) by the Calabi–Markus phenomenon. See Kobayashi [**8**].

(2) Suppose $(G,H) = (SL(n,\mathbb{C}), SL(n,\mathbb{R}))$ with $n > 1$. Then G is a complexification of H, but only finite subgroups of G act properly discontinuously on G/H by the Calabi–Markus phenomenon.

References

[1] Y. Benoist, Actions propres sur les espaces homogènes réductifs. *Ann. of Math. (2)* **144** (1996), 315–347, MR1418901, Zbl 0868.22013.

[2] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math. (2)* **57** (1953), 115–207, MR0051508, Zbl 0052.40001; English translation in: *Topological library, Part 3: Spectral sequences in topology*, World Scientific (2012), 191–318, translated by V.P. Golubyatnikov, edited by S.P. Novikov and I.A. Taimanov, MR2976669, Zbl 1264.55002.

[3] A. Borel, Compact Clifford–Klein forms of symmetric spaces, *Topology* **2** (1963), 111–122, MR0146301, Zbl 0116.38603.

[4] A. Borel & Harish-Chandra, Arithmetic subgroups of algebraic groups, *Ann. of Math. (2)* **75** (1962), 485–535, MR0147566, Zbl 0107.14804.

[5] H. Cartan, La transgression dans un groupe de Lie et dans un espace fibré principal, *Colloque de Topologie (espaces fibrés), Bruxelles, 1950*, George Thone, Liège (1951), 57–71, MR0042427, Zbl 0045.30701; reprinted in [**7**].

[6] D. Constantine, Compact Clifford–Klein forms—geometry, topology and dynamics, arXiv:1307.2183 (2013), to appear in *Proceedings of the conference Geometry, Topology and Dynamics in Negative Curvature (Bangalore 2010)*, London Mathematical Society Lecture Notes Series.

[7] V. Guillemin & S. Sternberg, *Supersymmetry and Equivariant de Rham Theory*, Springer–Verlag, Berlin (1999), MR1689252, Zbl 0934.55007.

[8] T. Kobayashi, Proper action on a homogeneous space of reductive type, *Math. Ann.* **285** (1989), 249–263, MR1016093, Zbl 0662.22008.

[9] T. Kobayashi, Discontinuous groups acting on homogeneous spaces of reductive type, *Representation theory of Lie groups and Lie algebras (Fuji-Kawaguchiko, 1990)*, 59–75, World Sci. Publ., River Edge, NJ, (1992), MR1190750, Zbl 1193.22010.

[10] T. Kobayashi, A necessary condition for the existence of compact Clifford–Klein forms of homogeneous spaces of reductive type, *Duke Math. J.* **67** (1992), 653–664, MR1181319, Zbl 0799.53056.

[11] T. Kobayashi, On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups, *J. Geom. Phys.* **12** (1993), 133–144, MR1231232, Zbl 0815.57029.

[12] T. Kobayashi, Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds, *Algebraic and analytic methods in representation theory (Sønderborg, 1994)*, Perspectives in Mathematics, vol. 17, Academic Press, San Diego, CA, (1996), 99–165, MR1415843, Zbl 0899.43005.

[13] T. Kobayashi, Criterion for proper actions on homogeneous spaces of reductive groups, *J. Lie Theory* **6** (1996), 147–163, MR1424629, Zbl 0863.22010.

[14] T. Kobayashi, On discontinuous group actions on non-Riemannian homogeneous spaces, *Sūgaku* **57** (2005) 267–281 (in Japanese), MR2163672; English translation in: *Sugaku Expositions* **22** (2009), 1–19, translated by M. Reid, MR2503485.

[15] T. Kobayashi & K. Ono, Note on Hirzebruch's proportionality principle, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **37** (1990), 71–87, MR1049019, Zbl 0726.57019.

[16] T. Kobayashi & T. Yoshino, Compact Clifford–Klein forms of symmetric spaces—revisited, *Pure Appl. Math. Q.* **1** (2005), 591–663, MR2201328, Zbl 1145.22011.

[17] R. Kulkarni, Proper actions and pseudo-Riemannian space forms, *Adv. in Math.* **40** (1981), 10–51, MR0616159, Zbl 0462.53041.

[18] F. Labourie, Quelques résultats récents sur les espaces localement homogènes compacts, *Manifolds and geometry (Pisa, 1993), Sympos. Math., XXXVI* (1996), 267–283, Cambridge Univ. Press, Cambridge, MR1410076, Zbl 0861.53053.

[19] F. Labourie, S. Mozes & R.J. Zimmer, On manifolds locally modelled on non-Riemannian homogeneous spaces, *Geom. Funct. Anal.* **5** (1995), 955–965, MR1361517, Zbl 0852.22011.

[20] F. Labourie & R.J. Zimmer, On the non-existence of cocompact lattices for $\mathrm{SL}(n)/\mathrm{SL}(m)$, *Math. Res. Lett.* **2** (1995), 75–77, MR1312978, Zbl 0852.22009.

[21] J. Leray, Sur l'homologie des groupes de Lie, des espaces homogènes et des espace fibrés principaux, *Colloque de Topologie (espaces fibrés), Bruxelles, 1950*, George Thone, Liège (1951), 101–115, MR0041148, Zbl 0042.41801.

[22] G. Margulis, Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients, *Bull. Soc. Math. France* **125** (1997), 447–456, MR1605453, Zbl 0892.22009.

[23] G.D. Mostow & T. Tamagawa, On the compactness of arithmetically defined homogeneous spaces, *Ann. of Math. (2)* **76** (1962), 446–463, MR0141672, Zbl 0196.53201.

[24] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, *Ann. of Math. (2)* **152** (2000), 113–182, MR1792293, Zbl 0970.22011.

[25] R.J. Zimmer, Discrete groups and non-Riemannian homogeneous spaces, *J. Amer. Math. Soc.* **7** (1994), 159–168, MR1207014, Zbl 0801.22009.

Graduate School of Mathematical Science
The University of Tokyo
3-8-1 Komaba
Meguro-ku, Tokyo 153-8914, Japan
E-mail address: ymorita@ms.u-tokyo.ac.jp