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MIN-MAX MINIMAL HYPERSURFACE IN (M^{n+1}, g) WITH $Ric > 0$ AND $2 \leq n \leq 6$

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Abstract

In this paper, we study the shape of the min-max minimal hypersurface produced by Almgren-Pitts in [**AF2**, **P**] corresponding to the fundamental class of a Riemannian manifold (M^{n+1}, g) of positive Ricci curvature with $2 \leq n \leq 6$. We characterize the Morse index, area, and multiplicity of this min-max hypersurface. In particular, we show that the min-max hypersurface is either orientable and of index one, or is a double cover of a non-orientable minimal hypersurface with least area among all closed embedded minimal hypersurfaces.

1. Introduction

Almgren and Pitts developed a min-max theory for constructing embedded minimal hypersurfaces by global variational method [**AF1**, **AF2**, **P**]. They showed that any Riemannian manifold (M^{n+1}, g) with $2 \leq n \leq 5$ has a nontrivial smooth, closed, embedded minimal hypersurface. Later on, Schoen and Simon [**SS**] extended it to the case of dimension $n = 6$. (They also showed the existence of a nontrivial minimal hypersurface with a singular set of Hausdorff dimension $n - 7$ when $n \geq 7$.) In [**AF1**, **AF2**], Almgren showed that the fundamental class $[M] \in H_{n+1}(M)$ of an orientable manifold M can be realized as a nontrivial homotopy class in $\pi_1\big(\mathcal{Z}_n(M), \{0\}\big)$, where $\mathcal{Z}_n(M)$ is the space of integral n-cycles in M (see [**P**, §2.1]). Almgren and Pitts [**AF2**, **P**] showed that a min-max construction on the homotopy class in $\pi_1\big(\mathcal{Z}_n(M), \{0\}\big)$ corresponding to $[M]$ gives a nontrivial, smooth, embedded, minimal hypersurface with possible multiplicity, which will be called the min-max hypersurface corresponding to the fundamental class $[M]$. Besides existence, there is almost no geometric information known about this min-max minimal hypersurface, e.g. the Morse index, volume, and multiplicity. (See [**CM2**, Chap. 1.8] for the definition of Morse index.) Because of the nature of the min-max construction, it has been conjectured that these min-max hypersurfaces should have total Morse

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index less than or equal to one (see [**PR**]). In addition, for the purpose of geometric and topological applications, it is necessary to know the index bound [**Y**, §4]. Recently, Marques and Neves [**MN1**] gave a partial answer of this question when $n = 2$. They showed the existence of an index one Heegaard surface in certain three manifolds. Later on, in their celebrated proof of the Willmore conjecture [**MN2**], Marques and Neves showed that the min-max surface has index five for a five-parameter family of sweepouts in the standard three sphere S^3.

In this paper, we study the shape of the min-max hypersurface corresponding to the fundamental class $[M]$ in the case when (M^{n+1}, g) has positive Ricci curvature, i.e. $Ric_g > 0$. In this case, there do not exist closed, embedded, stable minimal hypersurfaces (see [**CM2**, Chap. 1.8]) in M. By exploring this special feature, we will characterize the Morse index, volume, and multiplicity of this min-max hypersurface. The study of Morse index was initiated by Marques-Neves in the case of dimension three [**MN1**].

We always assume that (M^{n+1}, g) is *connected, closed*, and *orientable with* $2 \leq n \leq 6$. Hypersurfaces $\Sigma^n \subset M^{n+1}$ are always assumed to be *connected, closed*, and *embedded*. Denote

$$\mathcal{S} = \{\Sigma^n \subset (M^{n+1}, g) : \ \Sigma^n \text{ is a minimal hypersurface in } M\}.$$

By [**P**, **SS**, **DT**], $\mathcal{S} \neq \emptyset$. Let

$$W_M = \min_{\Sigma \in \mathcal{S}} \left\{ \begin{array}{ll} V(\Sigma), & \text{if } \Sigma \text{ is orientable} \\ 2V(\Sigma), & \text{if } \Sigma \text{ is non-orientable} \end{array} \right\}, \tag{1.1}$$

where $V(\Sigma)$ denotes the volume (sometime called area) of Σ. Our main result is as follows.

Theorem 1.1. *Let (M^{n+1}, g) be any $(n+1)$-dimensional connected, closed, orientable Riemannian manifold with positive Ricci curvature and $2 \leq n \leq 6$. Then the min-max minimal hypersurface Σ corresponding to the fundamental class $[M]$ is* <u>either</u>*:*

(i) *orientable of multiplicity one, with Morse index one and $V(\Sigma) = W_M$;*
(ii) <u>or</u> *non-orientable of multiplicity two with $2V(\Sigma) = W_M$.*

Remark 1.2. In case (ii), Σ has the least area among all $\mathcal{S}$. Illustrative examples of the first case are the equators, S^n embedded in S^{n+1}, and for the second case the $\mathbb{RP}^n$'s embedded in $\mathbb{RP}^{n+1}$, when n is an even number. Our theorem says that those are the only possible situations.

Remark 1.3. In fact, the positive Ricci curvature condition is only used to rule out the existence of closed, two-sided, stable minimal hypersurfaces, and to derive the fact that any two closed immersed minimal hypersurfaces must intersect (Frankel's Theorem 3.4). Actually, those two facts can be derived by requiring that (M^{n+1}, g), $2 \leq n \leq 6$, does not admit closed, embedded minimal hypersurfaces with stable two-sided covering (see [**MN3**, Corollary 1.5] for discussion). So Theorem 1.1 is true when the positive Ricci curvature condition is replaced by assuming that (M^{n+1}, g) does not admit closed, embedded minimal hypersurfaces with stable two-sided covering.

If there are no non-orientable, embedded, minimal hypersurfaces in M, we have the following corollary.

Theorem 1.4. *Given (M^{n+1}, g) as above, if (M, g) has no non-orientable, embedded, minimal hypersurfaces, then there is an orientable, embedded, minimal hypersurface $\Sigma^n \subset M^{n+1}$ with Morse index one.*

REMARK 1.5. If M is simply connected, i.e. $\pi_1(M) = 0$, then by [**H**, Chap. 4, Theorem 4.7], there are no non-orientable, embedded hypersurfaces in M. If $\pi_1(M)$ is finite, and the cardinality $\#\big(\pi_1(M)\big)$ is an odd number, then M has no non-orientable, embedded, minimal hypersurfaces by looking at the universal cover.

As a by-product of the proof, we have a second corollary.

Theorem 1.6. *In the case of Theorem* 1.4*, the hypersurface $\Sigma^n \subset M^{n+1}$ has least area among all closed, embedded, minimal hypersurfaces in M^{n+1}.*

REMARK 1.7. In general, compactness of stable minimal hypersurfaces follows from curvature estimates [**SSY**, **SS**], which would imply the existence of a least area hypersurface among the class of stable minimal hypersurfaces, or even minimal hypersurfaces with uniform Morse index bound. However, the class of all closed, embedded, minimal hypersurfaces in M does not have an a priori Morse index bound. In fact, the existence of the least area minimal hypersurface comes from the min-max theory and the special structure of orientable minimal hypersurfaces in manifold (M^{n+1}, g) with $Ric_g > 0$.

The main idea is as follows. The difficulty in obtaining the desired geometric information is due to the fact that the min-max hypersurface is constructed by a very weak limit (varifold limit). To overcome this difficulty, we try to find an optimal minimal hypersurface, which lies in a "mountain pass" (see [**St**]) type sweepout (continuous family of hypersurfaces; see Definition 2.1) in this min-max construction. Given (M^{n+1}, g) as in Theorem 1.1, we will first embed any closed embedded minimal hypersurface Σ into a good sweepout. Then, such families are discretized as needed to apply the Almgren-Pitts theory. We will show that those families all lie in the same homotopy class corresponding to the fundamental class of M. The Almgren-Pitts theory applied to this homotopy class produces an optimal, embedded minimal hypersurface, for which we can characterize the Morse index, volume, and multiplicity.

There are two reasons that we must use the discrete Almgren-Pitts theory rather than other min-max theory in continuous setting [**CD**, **DT**]. One is due to the fact that the sweepouts generated by non-orientable minimal hypersurfaces (Proposition 3.8) do not satisfy the requirements for the continuous setting; the other reason is that only in the Almgren-Pitts setting could we show that the sweepouts all lie in the same homotopy class.

The paper is organized as follows. In Section 2, we give a min-max theory for manifolds with boundary using the continuous setting as in [**DT**]. In Section 3, we show that good sweepouts can be generalized from embedded minimal hypersurfaces, where orientable and non-orientable cases are discussed separately. In Section 4, we introduce the celebrated Almgren-Pitts theory [**AF2**, **P**, **SS**], especially the case of one parameter sweepouts. In Section 5, sweepouts which are continuous in the flat topology are made into discretized families under the mass norm topology, as those used in the Almgren-Pitts theory. In Section 6, we give a characterization of the orientation and multiplicity of the min-max hypersurface. Finally, we prove the main result in Section 7.

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2. Min-max theory I—continuous setting

Let us first introduce a continuous setting for the min-max theory for constructing minimal hypersurfaces. In fact, Almgren and Pitts [**AF2**, **P**] used a discretized setting. They can deal with a very generally discretized multi-parameter family of surfaces, but due to the discretized setting, the multi-parameter family is hard to apply to geometry directly. Later on, Smith [**Sm**] introduced a setting using continuous families in S^3. Recently, Colding, De Lellis [**CD**] ($n = 2$) and De Lellis, Tasnady [**DT**] ($n \geq 2$) gave a version of min-max theory using continuous setting based on the ideas in [**Sm**]. They mainly dealt with the family of level surfaces of a Morse function. Their setting is more suitable for geometric manipulation. Marques and Neves [**MN1**] extended [**CD**] to a setting for manifolds with fixed convex boundary when $n = 2$. They used that to construct a smooth sweepout by a Heegaard surface in certain three manifolds. In this section we will mainly use the version by De Lellis and Tasnady [**DT**]. We will extend Marques and Neves's min-max construction for manifolds with fixed convex boundary to high dimensions.

Let (M^{n+1}, g) be a Riemannian manifold with or without boundary ∂M. $\mathcal{H}^n$ denotes the n dimensional Hausdorff measure. When Σ^n is an n-dimensional submanifold, we use $V(\Sigma)$ to denote $\mathcal{H}^n(\Sigma)$.

Definition 2.1. A family of $\mathcal{H}^n$ measurable closed subsets $\{\Gamma_t\}_{t\in[0,1]^k}$ (the parameter space $[0,1]$ can be any other interval $[a,b]$ in $\mathbb{R}$) of M with finite $\mathcal{H}^n$ measure is called *a generalized smooth family of hypersurfaces* if

(s1): for each t, there is a finite subset $P_t \subset M$, such that Γ_t is a smooth hypersurface in $M \setminus P_t$;

(s2): $t \to \mathcal{H}^n(\Gamma_t)$ is continuous, and $t \to \Gamma_t$ is continuous in the Hausdorff topology;

(s3): $\Gamma_t \to \Gamma_{t_0}$ smoothly in any compact $U \subset\subset M \setminus P_{t_0}$ as $t \to t_0$.

When $\partial M = \emptyset$, a generalized smooth family $\{\Sigma_t\}_{t\in[0,1]}$ is called a *sweepout* of M if there exists a family of open sets $\{\Omega_t\}_{t\in[0,1]}$ such that

(sw1): $(\Sigma_t \setminus \partial\Omega_t) \subset P_t$, for any $t \in [0,1]$;

(sw2): $\mathrm{Volume}(\Omega_t \setminus \Omega_s) + \mathrm{Volume}(\Omega_s \setminus \Omega_t) \to 0$, as $s \to t$;

(sw3): $\Omega_0 = \emptyset$, and $\Omega_1 = M$.

When $\partial M \neq \emptyset$, a sweepout is required to satisfy all of the above, except with (sw3) changed by

(sw3′): $\Omega_1 = M$. $\Sigma_0 = \partial M$, $\Sigma_t \subset int(M)$ for $t > 0$, and $\{\Sigma_t\}_{0\leq t\leq\epsilon}$ is a smooth foliation of a neighborhood of ∂M for some small $\epsilon > 0$, i.e. there exists a smooth function $w : [0,\epsilon] \times \partial M \to \mathbb{R}$, with $w(0,x) = 0$ and $\frac{\partial}{\partial t} w(0,x) > 0$, such that

$$\Sigma_t = \{exp_x\big(w(t,x)\nu(x)\big) : \ x \in \partial M\}, \qquad \text{for } t \in [0,\epsilon],$$

where ν is the inward unit normal for $(M, \partial M)$.

Remark 2.2. The first part of the definition follows from [**DT**, Definition 0.2], while the second part borrows an idea from [**MN1**].

We will need the following two basic results.

Proposition 2.3. ([**DT**, Proposition 0.4]) *Assume $\partial M = \emptyset$. Let $f : M \to [0,1]$ be a smooth Morse function. Then the level sets $\big\{\{f = t\}\big\}_{t\in[0,1]}$ form a sweepout.*

Given a generalized family $\{\Gamma_t\}$, we set

$$\mathbf{L}(\{\Gamma_t\}) = \max_t \mathcal{H}^n(\Gamma_t).$$

As a consequence of the isoperimetric inequality, we have

Proposition 2.4. ([**CD**, Proposition 1.4] and [**DT**, Proposition 0.5]) *Assume $\partial M = \emptyset$. There exists a positive constant $C(M) > 0$ depending only on M, such that $\mathbf{L}(\{\Sigma_t\}) \geq C(M)$ for any sweepout $\{\Sigma_t\}_{t\in[0,1]}$.*

We need the following notion of homotopy equivalence.

Definition 2.5. When $\partial M = \emptyset$, two sweepouts $\{\Sigma^1_t\}_{t\in[0,1]}$ and $\{\Sigma^2_t\}_{t\in[0,1]}$ are *homotopic* if there is a generalized smooth family $\{\Gamma_{(s,t)}\}_{(s,t)\in[0,1]^2}$, such that $\Gamma_{(0,t)} = \Sigma^1_t$ and $\Gamma_{(1,t)} = \Sigma^2_t$. When $\partial M \neq \emptyset$, we further require the following condition:

(*): $\Gamma_{(s,0)} \equiv \partial M$, $\Gamma_{(s,t)} \subset int(M)$ for $t > 0$, and for some small $\epsilon > 0$, there exists a smooth function $w : [0,\epsilon]\times[0,\epsilon]\times\partial M \to \mathbb{R}$, with $w(s,0,x) = 0$ and $\frac{\partial}{\partial t}w(s,0,x) > 0$, such that

$$\Gamma_{(s,t)} = \{exp_x\big(w(s,t,x)\nu(x)\big) : \ x \in \partial M\}, \qquad \text{for } (s,t)\in[0,\epsilon]\times[0,\epsilon].$$

A family Λ of sweepouts is called *homotopically closed* if it contains the homotopy class of each of its elements.

Remark 2.6. Denote $\mathrm{Diff}_0(M)$ to be the isotopy group of diffeomorphisms of M. When $\partial M \neq \emptyset$, we require the isotopies to leave a neighborhood of ∂M fixed. Given a sweepout $\{\Sigma_t\}_{t\in[0,1]}$, and $\psi \in C^\infty([0,1]\times M, M)$ with $\psi(t) \in \mathrm{Diff}_0(M)$ for all t, then $\{\psi(t,\Sigma_t)\}_{t\in[0,1]}$ is also a sweepout, which is homotopic to $\{\Sigma_t\}$. Such homotopies will be called homotopies induced by ambient isotopies.

Given a homotopically closed family Λ of sweepouts, the *width of M associated with* Λ is defined as

$$W(M,\partial M,\Lambda) = \inf_{\{\Sigma_t\}\in\Lambda} \mathbf{L}\big(\{\Sigma_t\}\big). \tag{2.1}$$

When $\partial M = \emptyset$, we omit ∂M and write the width as $W(M,\Lambda)$. In case $\partial M = \emptyset$, as a corollary of Proposition 2.4, the width of M is always nontrivial, i.e. $W(M,\Lambda) \geq C(M) > 0$.

A sequence $\big\{\{\Sigma^n_t\}_{t\in[0,1]}\big\}_{n=1}^\infty \subset \Lambda$ of sweepouts is called a *minimizing sequence* if $\mathbf{L}\big(\{\Sigma^n_t\}\big) \searrow W(M,\partial M,\Lambda)$. A sequence of slices $\{\Sigma^n_{t_n}\}$ with $t_n \in [0,1]$ is called a *min-max sequence* if $\mathcal{H}^n(\Sigma^n_{t_n}) \to W(M,\partial M,\Lambda)$. The goal in min-max theory [**P**, **SS**, **CD**, **DT**] is to find a regular minimal hypersurface as a min-max limit corresponding to the width $W(M,\partial M,\Lambda)$.

If $\partial M \neq \emptyset$ and ν is the inward unit normal for $(M,\partial M)$, we denote the mean curvature of the boundary by $H(\partial M)$, and the mean curvature vector by $H(\partial M)\nu$. Here the sign convention for H is that $H(\partial M)(p) = -\sum_{i=1}^n \langle\nabla_{e_i}\nu, e_i\rangle$, where $\{e_1,\cdots,e_n\}$ is a local orthonormal basis at $p\in\partial M$. Based on the main results in [**DT**] and an idea in [**MN1**], we have the following main result for this section.

Theorem 2.7. *Let (M^{n+1},g) be a connected compact Riemannian manifold with or without boundary ∂M and $2 \leq n \leq 6$. When $\partial M \neq \emptyset$, we assume $H(\partial M) > 0$. For any homologically closed family Λ of sweepouts, with $W(M,\partial M,\Lambda) > V(\partial M)$ if $\partial M \neq \emptyset$, there exists a min-max sequence $\{\Sigma^n_{t_n}\}$ of Λ that converges in the varifold sense to an embedded minimal hypersurface Σ (possibly disconnected), which lies in* *the interior of M if $\partial M \neq \emptyset$. Furthermore, the width $W(M, \partial M, \Lambda)$ is equal to the volume of Σ if counted with multiplicities.*

Proof. When $\partial M = \emptyset$, this is just Theorem 0.7 in [**DT**].

Now let us assume $\partial M \neq \emptyset$. The result follows from an observation of Theorem 2.1 in [**MN1**] and minor modifications of the arguments in [**DT**]. Here we will state the main steps and point out the key points on how to modify arguments in [**DT**] to our setting.

Part 1: Since $H(\partial M) > 0$, by almost the same argument as in the proof of [**MN1**, Theorem 2.1], we can find $a > 0$, and a minimizing sequence of sweepouts $\big\{\{\Sigma_t^n\}_{t\in[0,1]}\big\}_{n=1}^{\infty}$, such that

$$\mathcal{H}^n(\Sigma_t^n) \geq W(M, \partial M, \Lambda) - \delta, \implies d(\Sigma_t^n, \partial M) \geq a/2, \tag{2.2}$$

where $\delta = \frac{1}{4}\big(W(M, \partial M, \Lambda) - V(\partial M)\big) > 0$, and $d(\cdot, \cdot)$ is the distance function of (M, g).

Let us discuss the minor difference between our situation and those in [**MN1**]. We can find a neighborhood of ∂M, such that using normal coordinates $[0, 2a] \times \partial M$ for some $a > 0$, the metric can be written as $g = dr^2 + g_r$. In [**MN1**] they deform an arbitrary minimizing sequence to satisfy (2.2) by ambient isotopies induced by a vector field $\varphi(r)\frac{\partial}{\partial r}$, where $\varphi(r)$ is a cutoff function supported in $[0, 2a]$. Although the argument in [**MN1**, Theorem 2.1] was given only in dimension 2, it works in all dimensions. The only difference is that in the proof of the claim on [**MN1**, page 5], we need to take the orthonormal basis $\{e_1, \cdots, e_n\}$ such that $\{e_1, \cdots, e_{n-1}\}$ is orthogonal to $\frac{\partial}{\partial r}$ and then projects e_n to the orthogonal complement of $\frac{\partial}{\partial r}$. Then all the argument follows exactly as in [**MN1**].

Part 2: Now let us sketch the main steps for modifying arguments of the min-max construction in [**DT**, **CD**] to our setting.

Given the minimizing sequence $\big\{\{\Sigma_t^n\}_{t\in[0,1]}\big\}_{n=1}^{\infty} \subset \Lambda$ as above, the first step is a tightening process as in [**CD**, §4], where we deform each $\{\Sigma_t^n\}_{t\in[0,1]}$ to another one $\{\tilde{\Sigma}_t^n\}_{t\in[0,1]}$ by ambient isotopy $\{F_t\}_{t\in[0,1]} \subset \mathrm{Diff}_0(M)$, i.e. $\{\tilde{\Sigma}_t^n = F(t, \Sigma_t^n)\}_{t\in[0,1]} \subset \Lambda$, such that every min-max sequence $\{\tilde{\Sigma}_{t_n}^n\}$ converges to a stationary varifold. Since those Σ_t^n with volume near $W(M, \partial M, \Lambda)$ have a distance $a/2 > 0$ away from ∂M, we can take all the deformation vector field to be zero near ∂M in [**CD**, §4]. Hence $\{\tilde{\Sigma}_t^n\}$ can be chosen to satisfy (2.2) too.

The second step is to find an almost minimizing min-max sequence (see Definition 2.3 and Proposition 2.4 in [**DT**]) $\{\tilde{\Sigma}_{t_n}^n\}$ among $\{\tilde{\Sigma}_t^n\}_{t\in[0,1]}$, where $\tilde{\Sigma}_{t_n}^n$ converge to a stationary varifold V. By (2.2), the slices $\tilde{\Sigma}_t^n$ with volume near $W(M, \partial M, \Lambda)$ always have a distance $a/2 > 0$ away from ∂M; hence they are almost minimizing in any open set supported near ∂M. Away from ∂M, all the arguments in [**DT**, §3] work; hence it implies the existence of an almost minimizing sequence in the sense of [**DT**, Proposition 2.4], which are supported away from ∂M.

The final step is to prove that the limiting stationary varifold V of the almost minimizing sequence is supported on a smooth embedded minimal hypersurface. This step was done in [**DT**, §4 and §5]. The arguments are purely local. By our construction, the corresponding varifold measure $|V|$ on M is supported away from ∂M; hence the regularity results in [**DT**] are true in our case. By the dimension restriction $2 \leq n \leq 6$, they imply the conclusion. q.e.d.

3. Min-max family from embedded minimal hypersurfaces

In this section, by exploring some special structures for embedded minimal hypersurfaces in positive Ricci curvature manifolds, we will show that every embedded closed connected orientable minimal hypersurface can be embedded into a sweepout, and a double cover of every embedded closed connected non-orientable minimal hypersurface can be embedded into a sweepout in a double cover of the manifold. The sweepouts constructed in both cases can be chosen to be level surfaces of a Morse function, which hence represent the fundamental class of the ambient manifold (see Theorem 5.8). We first collect some results from differentiable topology.

Theorem 3.1. ([**H**, Chap. 4, Lemma 4.1 and Theorem 4.2]) *Let Ω be a connected, compact, orientable manifold with boundary $\partial\Omega$. Then $\partial\Omega$ is orientable.*

Theorem 3.2. ([**H**, Chap. 4, Theorem 4.5]) *Let M be a connected, closed, orientable manifold, and $\Sigma \subset M$ a connected, closed, embedded submanifold of codimension 1. If Σ separates M, i.e. $M \setminus \Sigma$ has two connected components, then Σ is orientable.*

Lemma 3.3. *Given M and Σ as above, Σ is orientable if and only if the normal bundle of Σ inside M is trivial.*

Proof. The tangent bundle has a splitting $TM\big|_{\Sigma} = T\Sigma \oplus N$, where N is the normal bundle. Hence our result is a corollary of Lemma 4.1 and Theorem 4.3 in [**H**, Chap. 4]. q.e.d.

We also need the following result, which says that any two connected minimal surfaces must intersect in positive Ricci curvature manifolds.

Theorem 3.4. (Generalized Hadamard Theorem in [**F**]) *Let (M, g) be a connected manifold with $Ric_g > 0$; then any two connected closed immersed minimal hypersurfaces Σ and Σ' must intersect.*

Let $\Sigma^n \subset M^{n+1}$ be a minimal hypersurface. When Σ is two-sided, i.e. the normal bundle of Σ is trivial, there always exists a unit normal vector field ν. The *Jacobi operator* is

$$L\phi = \triangle_\Sigma \phi + (Ric(\nu, \nu) + |A|^2)\phi,$$

where $\phi \in C^\infty(\Sigma)$, $\triangle_\Sigma$ is the Laplacian operator on Σ with respect to the induced metric, and A is the second fundamental form of Σ. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of L if there exists a $\phi \in C^\infty(\Sigma)$, such that $L\phi = -\lambda\phi$. The *Morse index* (abbreviated as *index* in the following) of Σ, denoted by $ind(\Sigma)$, is the number of negative eigenvalues of L counted with multiplicity. Σ is called *stable* if $ind(\Sigma) \geq 0$, or in other words L is a nonpositive operator. Clearly $Ric_g > 0$ implies that there is no closed two-sided stable minimal hypersurface.

Using basic algebraic topology and geometric measure theory, together with the fact that there is no two-sided stable minimal hypersurface when $Ric_g > 0$, we can show the reverse of Theorem 3.2 when $2 \leq n \leq 6$.

Proposition 3.5. *Let (M^{n+1}, g) be a connected closed orientable Riemannian manifold with $2 \leq n \leq 6$ and $Ric_g > 0$; then every embedded connected closed orientable hypersurface $\Sigma^n \subset M^{n+1}$ must separate M.*

Proof. Since Σ^n is orientable, the fundamental class $[\Sigma^n]$ (see [**B**, p. 355]) of Σ represents a homology class in $H_n(M, \mathbb{Z})$. Using the language of geometric measure theory, Σ is an integral n-cycle; hence it also represents an integral n homology class $[\Sigma^n]$ in $H_n(M, \mathbb{Z})$ in the sense of currents (see [**FH**, §4.4]). Suppose that Σ^n does not separate. Take a coordinates chart $U \subset M$ such that $U \cap \Sigma \neq \emptyset$. Since Σ^n is embedded, Σ separates U into U_1 and U_2 after possibly shrinking U. Pick $p_1 \in U_1$ and $p_2 \in U_2$. We can connect p_1 to p_2 by a curve γ_1 inside U, such that γ_1 intersects Σ transversally only once. Since Σ does not separate, $M \setminus \Sigma$ is connected. We can connect p_1 to p_2 by a curve γ_2 inside $M \setminus \Sigma$. Now we get a closed curve $\gamma = \gamma_1 \cup \gamma_2$, which intersects Σ transversally only once. Hence Σ meets γ transversally, and $\Sigma \cap \gamma$ is a single point. Using the intersection theory (see [**B**, page 367]), the intersection of the n homology $[\Sigma]$ and the 1 homology $[\gamma]$ is

$$[\Sigma] \cdot [\gamma] = [\Sigma \cap \gamma] \neq 0.$$

Hence $[\Sigma] \neq 0$ in $H_n(M, \mathbb{Z})$. Now we can minimize the mass inside the integral homology class $[\Sigma]$ (as a collection of integral cycles). [**Si**, Lemma 34.3] tells us that there is a minimizing integral current $T_0 \in [\Sigma]$. Moreover, the codimension one regularity theory ([**Si**, Theorem 37.7]) when $2 \leq n \leq 6$ implies that T_0 is represented by a smooth n dimensional hypersurface Σ_0 (possibly with multiplicity), i.e. $T_0 = m[\Sigma_0]$, where $m \in \mathbb{Z}$, $m \neq 0$. Since $m[\Sigma_0]$ represents a nontrivial integral homology class, Σ_0 is orientable. The fact that both M and Σ_0 are orientable implies that the normal bundle of Σ_0 is trivial by Lemma 3.3; hence Σ_0 is two-sided. By the nature of the mass minimizing property of T, Σ_0 must be locally volume minimizing; hence Σ_0 is stable. This is a contradiction with $Ric_g > 0$. q.e.d.

From now on, we always assume that (M^{n+1}, g) is connected closed oriented with $2 \leq n \leq 6$, and hypersurfaces $\Sigma^n \subset M^{n+1}$ are connected closed and embedded in this section.

3.1. Orientable case. The following proposition, which asserts that every orientable minimal hypersurface lies in a good sweepout in manifold (M^{n+1}, g) of positive Ricci curvature when $2 \leq n \leq 6$, is a key observation in proving our main theorem. Denote

$$\mathcal{S}_+ = \{\Sigma^n : \ \Sigma^n \text{ is an orientable minimal hypersurface in } M\}. \tag{3.1}$$

Proposition 3.6. *For any $\Sigma \in \mathcal{S}_+$, there exists a sweepout $\{\Sigma_t\}_{t\in[-1,1]}$ of M such that*

(a) $\Sigma_0 = \Sigma$*;*
(b) $\mathcal{H}^n(\Sigma_t) \leq V(\Sigma)$*, with equality only if $t = 0$;*
(c) $\{\Sigma_t\}_{t\in[-\epsilon,\epsilon]}$ *forms a smooth foliation of a neighborhood of Σ, i.e. there exists $w(t, x) \in C^\infty([-\epsilon, \epsilon] \times \Sigma)$, $w(0, x) = 0$, $\frac{\partial}{\partial t} w(0, x) > 0$, such that*
$$\Sigma_t = \{exp_x\big(w(t, x)\nu(x)\big) : \ x \in \Sigma\}, \quad t \in [-\epsilon, \epsilon],$$
where ν is the unit normal vector field of Σ in M.

Proof. By Proposition 3.5, Σ separates M; hence $M \setminus \Sigma = M_1 \cup M_2$ is a disjoint union of two connected components M_1 and M_2, with $\partial M_1 = \partial M_2 = \Sigma$. Assume that the unit normal vector field ν points into M_1. We denote λ_1 to be the first eigenvalue of the Jacobi operator L, and u_1 the corresponding eigenfunction. The first eigenvalue has multiplicity 1, and $u_1 > 0$ everywhere on Σ. $Ric_g > 0$ means that Σ is unstable; hence $\lambda_1 < 0$, i.e. $Lu_1 = -\lambda_1 u_1 > 0$.

Consider the local foliation by the first eigenfunction via the exponential map,

$$\Sigma_s = \{exp_x\big(su_1(x)\nu(x)\big) : \ x \in \Sigma\}, \quad s \in [-\epsilon, \epsilon].$$

- For $\epsilon > 0$ small enough, since $u_1 > 0$, the map $F : [-\epsilon, \epsilon] \times \Sigma \to M$ given by $F(s, x) = exp_x\big(su_1(x)\nu(x)\big)$ is a smooth diffeomorphic one-to-one map; hence $\{\Sigma_s\}_{s\in[-\epsilon,\epsilon]}$ is a smooth foliation of a neighborhood of Σ.
- Since $u_1 > 0$, Σ_s is contained in M_1 (in M_2) for $0 < s < \epsilon$ (for $-\epsilon < s < 0$).
- By the first and second variational formulae (see [**CM2**][**Si**]),

$$\frac{d}{ds}\Big|_{s=0} V(\Sigma_s) = -\int_\Sigma H u_1 d\mu = 0, \quad \frac{d^2}{ds^2}\Big|_{s=0} V(\Sigma_s)$$

$$= -\int_{\Sigma} u_1 L u_1 d\mu < 0,$$

where $H \equiv 0$ is the mean curvature of Σ. So $V(\Sigma_s) \leq V(\Sigma)$ for $s \in [-\epsilon, \epsilon]$, and equality holds only if $s = 0$.

- Denote $\vec{H}_s$ to be the mean curvature operator of Σ_s; then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \langle \vec{H}_s, \nu \rangle = L u_1 > 0.$$

Hence $H(\Sigma_s) > 0$ for $0 < s < \epsilon$ with respect to the normal ν for ϵ small enough.

Denote $M_{1,s_0} = M_1 \setminus \{\Sigma_s\}_{0 \leq s \leq s_0}$ for $0 < s_0 \leq \epsilon$, which is the region bounded by Σ_{s_0}. Similarly we have M_{2,s_0}, such that $\partial M_{2,s_0} = \Sigma_{-s_0}$. We will extend the foliation $\{\Sigma_s\}$ to M_{1,s_0} and M_{2,s_0}. We need the following claim, which is proved in Appendix 8:

Claim 1. *For ϵ small enough, there exists a sweepout $\{\tilde{\Sigma}_s\}_{s \in [-1,1]}$, such that $\tilde{\Sigma}_s = \Sigma_s$ for $s \in [-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon]$, and $\tilde{\Sigma}_s \subset M_{1,\frac{1}{2}\epsilon}$ (or $\subset M_{2,\frac{1}{2}\epsilon}$) when $s > \frac{1}{2}\epsilon$ (or $s < -\frac{1}{2}\epsilon$).*

Now cut out part of the sweepout $\{\tilde{\Sigma}_s\}_{s \in [\frac{1}{4}\epsilon, 1]}$, which is then a sweepout of $(M_{1,\frac{1}{4}\epsilon}, \partial M_{1,\frac{1}{4}\epsilon})$ (abbreviated as $(M_1, \partial M_1)$) by Definition 2.1. Consider the smallest homotopically closed family $\tilde{\Lambda}_1$ of sweepouts containing $\{\tilde{\Sigma}_s\}_{s \in [\frac{1}{4}\epsilon, 1]}$. If the width $W(M_1, \partial M_1, \tilde{\Lambda}_1) > V(\partial M_1)$, then by Theorem 2.7 and the fact that $H(\partial M_1) = H(\Sigma_{\frac{1}{4}\epsilon}) > 0$, there is a nontrivial embedded minimal hypersurface $\tilde{\Sigma}$ lying in the interior of M_1, which is then disjoint with Σ, and hence is a contradiction to Theorem 3.4. So $W(M_1, \partial M_1, \tilde{\Lambda}_1) \leq V(\partial M_1)$, which means that there exist sweepouts $\{\tilde{\Sigma}'_s\}_{s \in [\frac{1}{4}\epsilon, 1]}$ of $(M_1, \partial M_1)$, with $\max_{s \in [\frac{1}{4}\epsilon, 1]} \mathcal{H}^n(\tilde{\Sigma}'_s)$ very close to $V(\partial M_1)$. Since $\partial M_1 = \Sigma_{\frac{1}{4}\epsilon}$, and $V(\Sigma_{\frac{1}{4}\epsilon}) < V(\Sigma)$ by our construction above, we can pick up one sweepout $\{\tilde{\Sigma}'_s\}_{s \in [\frac{1}{4}\epsilon, 1]}$ with $\max_{s \in [\frac{1}{4}\epsilon, 1]} \mathcal{H}^n(\tilde{\Sigma}'_s) < V(\Sigma)$.

We can do similar things to $M_{2,\frac{1}{4}\epsilon}$ to get another partial sweepout. Then we finish by putting them together with $\{\Sigma_s\}_{s \in [-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon]}$. q.e.d.

3.2. Non-orientable case. We have the following topological characterization of non-orientable embedded hypersurfaces in orientable manifold M.

Proposition 3.7. *For any non-orientable embedded hypersurface Σ^n in an orientable manifold M^{n+1}, there exists a connected double cover $\tilde{M}$ of M, such that the lift $\tilde{\Sigma}$ of Σ is a connected orientable embedded hypersurface. Furthermore, $\tilde{\Sigma}$ separates $\tilde{M}$, and both components of $\tilde{M} \setminus \tilde{\Sigma}$ are diffeomorphic to $M \setminus \Sigma$.*

Proof. Since Σ is non-orientable, $M\setminus\Sigma$ is connected by Theorem 3.2. Denote $\Omega = M \setminus \Sigma$. Ω has a topological boundary $\partial\Omega$. Ω is orientable since M is orientable; hence $\partial\Omega$ is orientable by Theorem 3.1.

Claim 2. *$\partial\Omega$ is a double cover of Σ.*

This is proved as follows. For all $x \in \Sigma$, there exists a neighborhood U of x, i.e., $x \in U \subset M$, with U diffeomorphic to a unit ball $B_1(0)$. Since Σ is embedded, after possibly shrinking U, $\Sigma\cap U$ is a topological n dimensional ball, and Σ separates U into two connected components U_1 and U_2, i.e. $U\setminus\Sigma = U_1\cup U_2$. Then the sets $U\cap\Sigma \simeq (\partial U_1)\cap\Sigma \simeq (\partial U_2)\cap\Sigma$ are diffeomorphic. The sets $\{U \cap \Sigma\}$ form a system of local coordinate charts for Σ. Moreover, $\{(\partial U_1) \cap \Sigma, (\partial U_2) \cap \Sigma\}$ form a system of local coordinate charts for $\partial\Omega$, and $\{(U_1, \partial U_1 \cap \Sigma), (U_2, \partial U_2 \cap \Sigma)\}$ form a system of local boundary coordinate charts for $(\Omega, \partial\Omega)$. Hence $\partial\Omega$ is a double covering of Σ, with the covering map given by $(\partial U_1)\cap\Sigma, (\partial U_2)\cap \Sigma \to U \cap \Sigma$.

Since Σ is connected, $\partial\Omega$ has no more than two connected components. If $\partial\Omega$ is not connected, then $\partial\Omega$ has two connected components, i.e. $\partial\Omega = (\partial\Omega)_1 \cup (\partial\Omega)_2$, with $\Sigma \simeq (\partial\Omega)_1 \simeq (\partial\Omega)_2$. Hence Σ is orientable since $\partial\Omega$ is orientable, which is a contradiction. So Ω must be connected. Let $\tilde{M} = \Omega \sqcup_{\{\partial\Omega: x\to x^*\}} \Omega$ be the gluing of two copies of $(\Omega, \partial\Omega)$ along $\partial\Omega$ using the deck transformation map $x \to x^*$ of the covering $\partial\Omega \to \Sigma$; then the lift of Σ is $\tilde{\Sigma} \simeq \partial\Omega$. $\tilde{M}$ is then orientable and satisfies all the requirements. q.e.d.

As a direct corollary of the results in the previous section, we can embed a double cover of a non-orientable minimal hypersurface to a sweepout in the double cover $\tilde{M}$ of a manifold (M^{n+1}, g) with positive Ricci curvature when $2 \leq n \leq 6$. Let

$$\mathcal{S}_- = \{\Sigma^n :\ \Sigma^n \text{ is a non-orientable minimal hypersurface in } M\}. \tag{3.2}$$

Proposition 3.8. *Given $\Sigma \in \mathcal{S}_-$, there exists a family $\{\Sigma_t\}_{t\in[0,1]}$ of closed sets such that*

(a) $\Sigma_0 = \emptyset$;
(b) $\{\Sigma_t\}_{t\in[0,1]}$ *satisfies (s1)(sw1)(sw2)(sw3) in Definition* 2.1;
(c) $\max_{t\in[0,1]} \mathcal{H}^n(\Sigma_t) = 2V(\Sigma)$ *and* $\mathcal{H}^n(\Sigma_t) < 2V(\Sigma)$ *for all* $t \in [0, 1]$;
(d) *(s2) in Definition* 2.1 *only fails when* $t \to 0$, *where* $\mathcal{H}^n(\Sigma_t) \to 2V(\Sigma)$;
(e) *(s3) in Definition* 2.1 *only fails when* $t \to 0$, *where* $\Sigma_t \to 2\Sigma$.

Proof. Consider the double cover $(\tilde{M}, g)$ given by Proposition 3.7. The lift $\tilde{\Sigma}$ is an orientable minimal hypersurface, and must have the double volume of Σ, i.e. $V(\tilde{\Sigma}) = 2V(\Sigma)$. $\tilde{\Sigma}$ separates $\tilde{M}$ into two isomorphic components $\tilde{M}_1$ and $\tilde{M}_2$, which are both isomorphic to $M \setminus \Sigma$. We can apply Proposition 3.6 to $(\tilde{M}, \tilde{\Sigma})$ to get a sweepout $\{\tilde{\Sigma}_t\}_{t\in[-1,1]}$

satisfying (a)(b)(c) there. By the construction, we know that $\tilde{\Sigma}_t \subset M_1$ for $t > 0$, and $\tilde{\Sigma}_t \subset M_2$ for $t < 0$. To define $\{\Sigma_t\}_{t\in[0,1]}$, we can let $\Sigma_t = \tilde{\Sigma}_t$ while identifying M_1 with $M \setminus \Sigma$, and let $\Sigma_0 = \emptyset$. Then the properties follow from those of $\{\tilde{\Sigma}_t\}_{t\in[-1,1]}$. q.e.d.

4. Min-max theory II—Almgren-Pitts discrete setting

Let us introduce the min-max theory developed by Almgren and Pitts [**AF1**, **AF2**, **P**]. We will briefly give the notations in [**P**, §4.1] in order to state the min-max theorem. Marques and Neves also gave a nice introduction in [**MN2**, §7 and §8]. For notations in geometric measure theory, we refer to [**Si**], [**P**, §2.1], and [**MN2**, §4].

Fix an oriented Riemannian manifold (M^{n+1}, g) of dimension $n + 1$, with $2 \leq n \leq 6$. Assume that (M^{n+1}, g) is embedded in some $\mathbb{R}^N$ for N large. We denote by $\mathbf{I}_k(M)$ the space of k-dimensional integral currents in $\mathbb{R}^N$ with support in M, $\mathcal{Z}_k(M)$ the space of integral currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$, and $\mathcal{V}_k(M)$ the weak closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^N$ with support in M, endowed with the weak topology. Given $T \in \mathbf{I}_k(M)$, $|T|$ and $\|T\|$ denote respectively the integral varifold and Radon measure in M associated with T. $\mathcal{F}$ and $\mathbf{M}$ denote respectively the flat norm and mass norm on $\mathbf{I}_k(M)$. $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ are in general assumed to have the flat norm topology. $\mathbf{I}_k(M, \mathbf{M})$ and $\mathcal{Z}_k(M, \mathbf{M})$ are the same space endowed with the mass norm topology. Given a smooth surface Σ or an open set Ω as in Definition 2.1, we use $[[\Sigma]]$, $[[\Omega]]$ and $[\Sigma]$, $[\Omega]$ to denote respectively the corresponding integral currents and integral varifolds.

We are mainly interested in the application of the Almgren-Pitts theory to the special case $\pi_1\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, so our notions will be restricted to this case.

Definition 4.1. (cell complex of $I = [0, 1]$)

(1) $I = [0, 1]$, $I_0 = \{[0], [1]\}$.
(2) For $j \in \mathbb{N}$, $I(1, j)$ is the cell complex of I, whose 1-cells are all intervals of form $[\frac{i}{3^j}, \frac{i+1}{3^j}]$, and 0-cells are all points $[\frac{i}{3^j}]$. Denote $I(1, j)_p$ the set of all p-cells in $I(1, j)$, with $p = 0, 1$, and $I_0(1, j) = \{[0], [1]\}$ the boundary 0-cells.
(3) Given α a 1-cell in $I(1, j)$ and $k \in \mathbb{N}$, $\alpha(k)$ denotes the 1-dimensional sub-complex of $I(1, j + k)$ formed by all cells contained in α, and $\alpha(k)_0$ are the boundary 0-cells of α.
(4) The boundary homeomorphism $\partial : I(1, j) \to I(1, j)$ is given by $\partial[a, b] = [b] - [a]$ and $\partial[a] = 0$.
(5) The distance function $d : I(1, j)_0 \times I(1, j)_0 \to \mathbb{Z}^+$ is defined as $d(x, y) = 3^j|x - y|$.

(6) The map $n(i,j) : I(1,i)_0 \to I(1,j)_0$ is defined as: $n(i,j)(x) \in I(1,j)_0$ is the unique element such that $d\big(x, n(i,j)(x)\big) = \inf\big\{d(x,y) : y \in I(1,j)_0\big\}$.

Consider a map to the space of integral cycles: $\phi : I(1,j)_0 \to \mathcal{Z}_n(M^{n+1})$. The *fineness of* ϕ is defined as:

$$\mathbf{f}(\phi) = \sup\Big\{\frac{\mathbf{M}\big(\phi(x) - \phi(y)\big)}{d(x,y)} :\ x, y \in I(1,j)_0, x \neq y\Big\}. \tag{4.1}$$

A map $\phi : I(1,j)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ means that $\phi\big(I(1,j)_0\big) \subset \mathcal{Z}_n(M^{n+1})$ and $\phi|_{I_0(1,j)_0} = 0$, i.e. $\phi([0]) = \phi([1]) = 0$.

Definition 4.2. Given $\delta > 0$ and $\phi_i : I(1,k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, $i = 1, 2$. We say ϕ_1 *is* 1*-homotopic to* ϕ_2 *in* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ *with fineness* δ, if $\exists\ k_3 \in \mathbb{N}$, $k_3 \geq \max\{k_1, k_2\}$, and

$$\psi : I(1,k_3)_0 \times I(1,k_3)_0 \to \mathcal{Z}_n(M^{n+1}),$$

such that

- $\mathbf{f}(\psi) \leq \delta$;
- $\psi([i-1], x) = \phi_i\big(n(k_3, k_i)(x)\big)$;
- $\psi\big(I(1,k_3)_0 \times I_0(1,k_3)_0\big) = 0$.

Definition 4.3. A $(1,\mathbf{M})$*-homotopy sequence of mappings into* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ is a sequence of mappings $\{\phi_i\}_{i\in\mathbb{N}}$,

$$\phi_i : I(1,k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big),$$

such that ϕ_i is 1-homotopic to ϕ_{i+1} in $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ with fineness δ_i, and

- $\lim_{i\to\infty} \delta_i = 0$;
- $\sup_i \big\{\mathbf{M}(\phi_i(x)) :\ x \in I(1,k_i)_0\big\} < +\infty$.

Definition 4.4. Assume that $S_1 = \{\phi_i^1\}_{i\in\mathbb{N}}$ and $S_2 = \{\phi_i^2\}_{i\in\mathbb{N}}$ are two $(1,\mathbf{M})$-homotopy sequence of mappings into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$; S_1 *is homotopic with* S_2 if $\exists\ \{\delta_i\}_{i\in\mathbb{N}}$, such that

- ϕ_i^1 is 1-homotopic to ϕ_i^2 in $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ with fineness δ_i;
- $\lim_{i\to\infty} \delta_i = 0$.

The relation "is homotopic with" is an equivalent relation on the space of $(1,\mathbf{M})$-homotopy sequences of mapping into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ (see [**P**, §4.1.2]). An equivalent class is a $(1,\mathbf{M})$ *homotopy class of mappings into* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$. Denote the set of all equivalent classes by $\pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$. Similarly, we can define the $(1,\mathcal{F})$-homotopy class and denote the set of all equivalent classes by $\pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big)$. In fact, Almgren-Pitts showed that they are all isomorphic to the top homology group.

Theorem 4.5. ([**AF1**, Theorem 13.4] and [**P**, Theorem 4.6]) *The following are all isomorphic:*

$$H_{n+1}(M^{n+1}),\ \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big),\ \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big).$$

Definition 4.6. (Min-max definition)
Given $\Pi \in \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$, define

$$\mathbf{L} : \Pi \to \mathbb{R}^+$$

as a function given by:

$$\mathbf{L}(S) = \mathbf{L}(\{\phi_i\}_{i\in\mathbb{N}}) = \limsup_{i\to\infty} \max\big\{\mathbf{M}\big(\phi_i(x)\big) :\ x \text{ lies in the domain of } \phi_i\big\}.$$

The *width of* Π is defined as

$$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S) :\ S \in \Pi\}. \tag{4.2}$$

We call $S \in \Pi$ a *critical sequence* if $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Let $K : \Pi \to \{\text{compact subsets of } \mathcal{V}_n(M^{n+1})\}$ be defined by

$$K(S) = \{V :\ V = \lim_{j\to\infty} |\phi_{i_j}(x_j)| :\ x_j \text{ lies in the domain of } \phi_{i_j}\}.$$

The *critical set* of S is $C(S) = K(S) \cap \{V :\ \mathbf{M}(V) = \mathbf{L}(S)\}$.

The celebrated min-max theorem of Almgren-Pitts (Theorem 4.3, 4.10, 7.12, Corollary 4.7 in [**P**]) and Schoen-Simon (for $n = 6$ [**SS**, Theorem 4]) is as follows.

Theorem 4.7. *Given a nontrivial* $\Pi \in \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$*; then* $\mathbf{L}(\Pi) > 0$*, and there exists a stationary integral varifold* Σ*, whose support is a closed, smooth, embedded, minimal hypersurface (which may be disconnected with multiplicity), such that*

$$\|\Sigma\|(M) = \mathbf{L}(\Pi).$$

In particular, Σ *lies in the critical set* $C(S)$ *of some critical sequence.*

5. Discretization

In this section, we will adapt the families constructed in Section 3 to the Almgren-Pitts setting. The families constructed in Section 3 are continuous under the flat norm topology, but Almgren-Pitts theory applies only to discrete family continuous under the mass norm topology. So we need to discretize our families and to make them continuous under the mass norm. A similar issue was considered in the celebrated proof of the Willmore conjecture [**MN2**]. In addition, we will show that the discretized families all belong to the same homotopy class. The proof is elementary but relatively long. Upon first perusla of this section, the reader might focus only on the statements of Proposition 5.4, Theorem 5.5, and Theorem 5.8.

5.1. Generating min-max family.

Proposition 5.1. *Given* $\Phi : [0,1] \to \mathcal{Z}_n(M^{n+1})$ *defined by*

$$\Phi(x) = [[\partial\Omega_x]], \quad x \in [0,1],$$

where $\{\Omega_t\}_{t\in[0,1]}$ *is a family of open sets which satisfies (sw1)(sw2)(sw3) in Definition* 2.1 *for some* $\{\Sigma_t\}_{t\in[0,1]}$ *satisfying (s1)(s2)(s3) there; then*

(1) $\Phi : [0,1] \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ *is continuous under the flat topology;*
(2) $\mathbf{m}(\Phi, r) = \sup\big\{\|\Phi(x)\|B(p,r) : \ p \in M, x \in [0,1]\big\} \to 0$ *when* $r \to 0$*, where* $B(p,r)$ *is the geodesic ball of radius* r *and centered at* p *on* M*. (The concept of* $\mathbf{m}$ *first appears in* [**MN2**, §4.2]*.)*

Proof. By (sw1) and (s1) in Definition 2.1, $\partial\Omega_x$ is smooth away from finitely many points; hence it lies in $\mathcal{Z}_n(M^{n+1})$. By (sw3), $\Omega_0 = \emptyset$, $\Omega_1 = M$, so that $\Phi(0) = \Phi(1) = 0$. So Φ is well-defined as a map to $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$.

From the definition of the flat norm (see [**Si**, §31]),

$$\mathcal{F}\big(\Phi(x), \Phi(y)\big) \leq \|\Omega_y - \Omega_x\|(M) = \text{Volume}(\Omega_y \Delta \Omega_x) \to 0,$$

as $y \to x$ by (sw2) in Definition 2.1. Here and in the following, we abuse Ω and Σ with the associated integral currents $[[\Omega]]$ and $[[\Sigma]]$.

So what is left is the last property, i.e. $\mathbf{m}(\Phi, r) \to 0$ when $r \to 0$. Now we will abuse the notation and write $\Phi(x) = \Sigma_x = \partial\Omega_x$ since they only differ by a finite set of points.

Lemma 5.2. *Fix* $x \in [0,1]$*, and let* P_x *be the finite set of singular points of* Σ_x*, and* $B_r(P_x)$ *the collection of geodesic balls centered at* P_x *on* M*; then* $\lim_{r\to 0} \|\Sigma_x\|\big(B_r(P_x)\big) = 0$*. (Here* $\|\Sigma\|$ *is the Radon measure corresponding to the integral current* $[[\Sigma]]$ *associated with* Σ *(see* [**Si**, §27]*).)*

Proof. We only need to show that $\lim_{r\to 0} \|\Sigma_x\|\big(B_r(p)\big) = 0$ for every $p \in P_x$. By the definition of Hausdorff measure (see [**Si**, §2]), $(\mathcal{H}^n \llcorner \Sigma_x)(\{p\}) = \mathcal{H}^n(\Sigma_x \cap \{p\}) = \mathcal{H}^n(\{p\}) = 0$. Since $\mathcal{H}^n(\Sigma_x) < +\infty$, by the basic convergence property for Radon measures (see [**R**, §11.1, Proposition 2.1]),

$$0 = (\mathcal{H}^n \llcorner \Sigma_x)(\{p\}) = \lim_{r\to 0} (\mathcal{H}^n \llcorner \Sigma_x)\big(B_r(p)\big) = \lim_{r\to 0} \|\Sigma_x\|\big(B_r(p)\big).$$

q.e.d.

Given $r_0 > 0$ small enough, define $f : [0, r_0] \times M \times [0,1] \to \mathbb{R}^+$ by

$$f(r, p, x) = \|\Sigma_x\|\big(B_r(p)\big).$$

Lemma 5.3. *The function* f *is continuous.*

Proof. For the continuity in the parameter "x", we can fix the ball $B_r(p)$. For any $\epsilon > 0$, we can take $0 < r_{x,\epsilon} \ll 1$, such that $\|\Sigma_x\|\big(B_{r_{x,\epsilon}}(P_x)\big) < \frac{\epsilon}{4}$ by the previous lemma, where P_x is the finite singular set of Σ_x.

Since Σ_y converges to Σ_x smooth on compact sets of $M\setminus P_x$ by (s3) of Definition 2.1, we can find $\delta_{x,\epsilon}$, such that whenever $|y-x|<\delta_{x,\epsilon}$,

$$\big|\|\Sigma_y\|\big(B_r(p)\setminus B_{r_{x,\epsilon}}(P_x)\big)-\|\Sigma_x\|\big(B_r(p)\setminus B_{r_{x,\epsilon}}(P_x)\big)\big|<\frac{\epsilon}{4}.$$

We claim that $\|\Sigma_y\|\big(B_{r_{x,\epsilon}}(P_x)\big)<\frac{\epsilon}{2}$ if $\delta_{x,\epsilon}$ is small enough. Suppose not; then for a subsequence $y_i\to x$, $\|\Sigma_{y_i}\|\big(B_{r_{x,\epsilon}}(P_x)\big)\geq\frac{\epsilon}{2}$. Notice (s2) in Definition 2.1, i.e. $\mathcal{H}^n(\Sigma_y)\to\mathcal{H}^n(\Sigma_x)$. Now

$$\mathcal{H}^n(\Sigma_y)=\mathcal{H}^n\big(\Sigma_y\setminus B_{r_{x,\epsilon}}(P_x)\big)+\mathcal{H}^n\big(\Sigma_y\cap B_{r_{x,\epsilon}}(P_x)\big),$$

$$\mathcal{H}^n(\Sigma_x)=\mathcal{H}^n\big(\Sigma_x\setminus B_{r_{x,\epsilon}}(P_x)\big)+\mathcal{H}^n\big(\Sigma_x\cap B_{r_{x,\epsilon}}(P_x)\big).$$

Since Σ_y converge smoothly to Σ_x on compact subsets of $M\setminus P_x$, $\mathcal{H}^n\big(\Sigma_y\setminus B_{r_{x,\epsilon}}(P_x)\big)\to\mathcal{H}^n\big(\Sigma_x\setminus B_{r_{x,\epsilon}}(P_x)\big)$; and hence we get a contradiction since $\mathcal{H}^n\big(\Sigma_y\cap B_{r_{x,\epsilon}}(P_x)\big)-\mathcal{H}^n\big(\Sigma_x\cap B_{r_{x,\epsilon}}(P_x)\big)>\frac{\epsilon}{2}-\frac{\epsilon}{4}=\frac{\epsilon}{4}$.

Combining all of the above, we have $\big|\|\Sigma_y\|\big(B_r(p)\big)-\|\Sigma_x\|\big(B_r(p)\big)\big|<\epsilon$ whenever $|y-x|<\delta_{x,\epsilon}$, and have hence proved the continuity of f w.r.t. "x".

For the continuity in the parameter "r", we can fix Σ_x and the point $p\in M$. For any $\epsilon>0$, take $r_{x,\epsilon}$ as above. For any $\Delta r>0$,

$$\begin{aligned}\mathcal{H}^n\big(\Sigma_x\cap B_{r+\Delta r}(p)\big)-\mathcal{H}^n\big(\Sigma_x\cap B_r(p)\big)&\leq\mathcal{H}^n\big(\Sigma_x\cap B_{r_{x,\epsilon}}(P_x)\big)\\&+\mathcal{H}^n\big(\Sigma_x\cap A(p,r,r+\Delta r)\setminus B_{r_{x,\epsilon}}(P_x)\big),\end{aligned}$$

where $A(p,r,r+\Delta r)$ is the closed annulus. Since Σ_x is smooth on $M\setminus P_x$ by (s1) in Definition 2.1, we can take $\delta_{x,\epsilon}>0$ such that whenever $\Delta r<\delta_{x,\epsilon}$, $\mathcal{H}^n\big(\Sigma_x\cap A(p,r,r+\Delta r)\setminus B_{r_{x,\epsilon}}(P_x)\big)<\frac{\epsilon}{4}$. Hence $\mathcal{H}^n\big(\Sigma_x\cap B_{r+\Delta r}(p)\big)-\mathcal{H}^n\big(\Sigma_x\cap B_r(p)\big)<\frac{\epsilon}{2}$. A similar argument holds for $\Delta r<0$.

The continuity in the parameter "p" follows exactly as that of "r", so we omit the details here. q.e.d.

Let us now return to the proof that $\lim_{r\to 0}\mathbf{m}(\Phi,r)=0$. Since $[0,r_0]\times M\times[0,1]$ is compact, f is uniformly continuous. So by standard argument in point-set topology, $\mathbf{m}(\Phi,r)=\sup_{p\in M,x\in[0,1]}f(r,p,x)\to 0$ when $r\to 0$, as $f(0,p,x)=\|\Sigma_x\|(\{p\})=0$. q.e.d.

Given $\Sigma\in\mathcal{S}$, we can define a mapping into $\big(\mathcal{Z}_n(M^{n+1}),\{0\}\big)$,

$$\Phi^{\Sigma}:[0,1]\to\big(\mathcal{Z}_n(M^{n+1}),\{0\}\big),$$

as follows:

- when $\Sigma\in\mathcal{S}_+$, $\Phi^{\Sigma}(x)=[[\partial\Omega_{2x-1}]]$ for $x\in[0,1]$, where $\{\Omega_t\}_{t\in[-1,1]}$ is the family of open sets of M in Definition 2.1 corresponding to the sweepout $\{\Sigma_t\}_{t\in[-1,1]}$ of Σ constructed in Proposition 3.6;
- when $\Sigma\in\mathcal{S}_-$, $\Phi^{\Sigma}(x)=[[\partial\Omega_x]]$ for $x\in[0,1]$, where $\{\Omega_t\}_{t\in[0,1]}$ is the family of open sets of M in Definition 2.1 corresponding to the family $\{\Sigma_t\}_{t\in[0,1]}$ of Σ constructed in Proposition 3.8.

Then, as a corollary of Proposition 3.6, Proposition 3.8, and Proposition 5.1, we have

Corollary 5.4. $\Phi^{\Sigma}: [0,1] \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ *is continuous under the flat topology, and*

(a) $\sup_{x\in[0,1]} \mathbf{M}\big(\Phi^{\Sigma}(x)\big) = V(\Sigma)$ *if* $\Sigma \in \mathcal{S}_+$;
(b) $\sup_{x\in[0,1]} \mathbf{M}\big(\Phi^{\Sigma}(x)\big) = 2V(\Sigma)$ *if* $\Sigma \in \mathcal{S}_-$;
(c) $\mathbf{m}(\Phi^{\Sigma}, r) \to 0$, *when* $r \to 0$.

Proof. In the case $\Sigma \in \mathcal{S}_+$, our conclusions are a direct consequence of Proposition 5.1, as Φ^{Σ} satisfies the conditions there.

If $\Sigma \in \mathcal{S}_-$, all the conclusions are true by Proposition 3.8 and the proof of Proposition 5.1, except that we need to check (c). Using notions in Proposition 3.8, let $\tilde{M}$ and $\tilde{\Sigma}$ be the double cover of M and Σ respectively. Let $\tilde{\Phi}^{\tilde{\Sigma}}$ be the mapping corresponding to $\tilde{\Sigma}$ in $\tilde{M}$. Then it is easy to see that $\mathbf{m}(\Phi^{\Sigma}, r) \leq 2\mathbf{m}(\tilde{\Phi}^{\tilde{\Sigma}}, r)$; hence we finish the proof by using the first case. q.e.d.

5.2. Discretize the min-max family. Now we will discretize the continuous family Φ^{Σ} to form a $(1, \mathbf{M})$-homotopy sequence as in Definition 4.3. The idea originates from Pitts in [**P**, §3.7 and §3.8]. Marques and Neves first gave a complete statement in [**MN2**, §13] on generating an $(m, \mathbf{M})$-homotopy sequence into the space $\mathcal{Z}_2(M^3)$ of integral two cycles in a three manifold from a given min-max family continuous under the flat norm topology. Their proof never used any special feature of the special dimensions, so Theorem 13.1 in [**MN2**] is still true to generate an $(n, \mathbf{M})$-homotopy sequence into $\mathcal{Z}_n(M^{n+1})$ from any continuous family under flat topology. While they used a contradiction argument, for the purpose of the proof of Theorem 5.8, we will give a modified direct discretization process based on ideas in [**P**, **MN2**]. Our main result is an adaption of Theorem 13.1 in [**MN2**].

Theorem 5.5. *Given a continuous mapping* $\Phi: [0,1] \to \big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big)$, *with*

$$\sup_{x\in[0,1]} \mathbf{M}(\Phi(x)) < \infty, \ \text{and} \ \lim_{r\to 0} \mathbf{m}(\Phi, r) = 0,$$

there exists a $(1, \mathbf{M})$ *homotopy sequence*

$$\phi_i : I(1, k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big),$$

and a sequence

$$\psi_i : I(1, k_i)_0 \times I(1, k_i)_0 \to \mathcal{Z}_n(M^{n+1}, \mathbf{M}),$$

with $k_i < k_{i+1}$, *and* $\{\delta_i\}_{i\in\mathbb{N}}$ *with* $\delta_i > 0$, $\delta_i \to 0$, *and* $\{l_i\}_{i\in\mathbb{N}}$, $l_i \in \mathbb{N}$ *with* $l_i \to \infty$, *such that* $\psi_i([0], \cdot) = \phi_i$, $\psi_i([1], \cdot) = \phi_{i+1}|_{I(1,k_i)_0}$, *and*

(1) $\mathbf{M}\big(\phi_i(x)\big) \leq \sup\big\{\mathbf{M}\big(\Phi(y)\big) : x, y \in \alpha,$ *for some 1-cell* $\alpha \in I(1, l_i)\big\} + \delta_i$*; hence*

$$\mathbf{L}(\{\phi_i\}_{i\in\mathbb{N}}) \leq \sup_{x\in[0,1]} \mathbf{M}\big(\Phi(x)\big). \tag{5.1}$$

(2) $\mathbf{f}(\psi_i) < \delta_i$.
(3) $\sup\big\{\mathcal{F}\big(\psi_i(y, x) - \Phi(x)\big) : y \in I(1, k_i)_0, x \in I(1, k_i)_0\big\} < \delta_i$.

Before giving the proof, we first give a result which is a variation of [**P**, Lemma 3.8] and [**MN2**, Proposition 13.3]. For completeness and for the purpose of application to the proof of Theorem 5.8, we will sketch a slightly modified proof. Denote $\mathcal{B}^{\mathcal{F}}_{\epsilon}(S)$ to be a ball of radius ϵ centered at S in $\mathcal{Z}_n(M^{n+1}, \mathcal{F})$.

Lemma 5.6. *Given* δ, r, L *positive real numbers, and* $T \in \mathcal{Z}_n(M^{n+1}) \cap \{S : \mathbf{M}(S) \leq 2L\}$, *there exists* $0 < \epsilon = \epsilon(T, \delta, r, L) < \delta$, *and* $k = k(T, \delta, r, L) \in \mathbb{N}$, *such that whenever* $S \in \mathcal{B}^{\mathcal{F}}_{\epsilon}(T) \cap \{S : \mathbf{M}(S) \leq 2L\}$, *and* $\mathbf{m}(S, r) < \frac{\delta}{4}$, *there exists a mapping* $\tilde{\phi} : I(1, k)_0 \to \mathcal{B}^{\mathcal{F}}_{\epsilon}(T)$, *satisfying*

(i) $\tilde{\phi}([0]) = S$, $\tilde{\phi}([1]) = T$;
(ii) $\mathbf{f}(\tilde{\phi}) \leq \delta$;
(iii) $\sup_{x\in I(1,k)_0} \tilde{\phi}([x]) \leq \mathbf{M}(S) + \delta$.

Proof. By [**AF1**, Corollary 1.14], there exists $\epsilon_M > 0$ such that if $\epsilon < \epsilon_M$, there exists $Q \in \mathbf{I}_{n+1}(M^{n+1})$ such that

$$\partial Q = S - T, \quad \mathbf{M}(Q) = \mathcal{F}(S - T) < \epsilon.$$

We claim that there exists $\epsilon = \epsilon(T, \delta, r, L) > 0$ small enough and $v = v(T, \delta, r, L) \in \mathbb{N}$ large enough such that for any $S \in \mathcal{B}^{\mathcal{F}}_{\epsilon}(T) \cap \{S : \mathbf{M}(S) \leq 2L\}$, there exists a finite collection of disjoint balls $\{B_{r_i}(p_i)\}_{i=1}^{v}$ with $r_i < r$, satisfying the following five equations:

$$\|S\|\big(B_{r_i}(p_i)\big) \leq \frac{\delta}{4}, \ \|S\|\big(M \setminus \cup_{i=1}^{v} B_{r_i}(p_i)\big) \leq \frac{\delta}{4}. \tag{5.2}$$

$$\|T\|\big(B_{r_i}(p_i)\big) \leq \frac{\delta}{3}, \ \|T\|\big(M \setminus \cup_{i=1}^{v} B_{r_i}(p_i)\big) \leq \frac{\delta}{3}. \tag{5.3}$$

$$(\|T\| - \|S\|)(B_{r_i}(p_i)) \leq \frac{\delta}{2v}, \ (\|T\| - \|S\|)(M \setminus \cup_{i=1}^{v} B_{r_i}(p_i)) \leq \frac{\delta}{2v}. \tag{5.4}$$

Denoting $d_i(x) = d(x, p_i)$, the slice $\langle Q, d_i, r_i\rangle \in \mathbf{I}_n(M^{n+1})$ (see [**Si**, §28] for definition of slices.), and

$$\begin{aligned} \langle Q, d_i, r_i\rangle &= \partial\big(Q \llcorner B_{r_i}(p_i)\big) - (\partial Q) \llcorner B_{r_i}(p_i) \\ &= \partial\big(Q \llcorner B_{r_i}(p_i)\big) - (S - T) \llcorner B_{r_i}(p_i). \end{aligned} \tag{5.5}$$

$$\sum_{i=1}^{v} \mathbf{M}\big(\langle Q, d_i, r_i\rangle\big) < \frac{\delta}{2}. \tag{5.6}$$

This claim follows from a contradiction argument. If it is not true, then there is a sequence $\epsilon_j \to 0$, and $S_j \in \mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T) \cap \{S: \mathbf{M}(S) \leq 2L\}$, such that there exists no finite collection of disjoint balls satisfying the above properties. Then $\lim_{j\to\infty} S_j = T$, and weak compactness of varifolds with bounded mass implies that $\lim_{j\to\infty} |S_j| = V \in \mathcal{V}_n(M^{n+1})$ for some subsequence. Using the arguments in the proof of [**MN2**, Lemma 13.4] and [**P**, Lemma 3.8], we can construct a finite collection of disjoint balls satisfying the above requirement for each S_j when j is large enough; hence a contradiction. Notice that the condition $\mathbf{m}(S, r) < \frac{\delta}{4}$ is essentially used to find the radius of the balls (see Lemma 13.4 in [**MN2**] for details).

Define the map $\tilde{\phi} : I(1,k)_0 \to \mathcal{Z}_n(M^{n+1})$, with $k = N$, where we write $v = 3^N - 1$ for some $N \in \mathbb{N}$, as follows:

$$\begin{aligned} &\tilde{\phi}([\frac{i}{3^N}]) = S - \sum_{a=1}^{i} \partial\big(Q \llcorner B_{r_a}(p_a)\big),\ 0 \leq i \leq 3^N - 1; \\ &\tilde{\phi}([1]) = T. \end{aligned} \tag{5.7}$$

By arguments similar to [**MN2**, Lemma 13.4], we can check that $\tilde{\phi}(I(1,k)_0) \subset \mathcal{B}^{\mathcal{F}}_{\epsilon}(T)$, and get the properties $(i), (ii), (iii)$ listed in the lemma using (5.2), (5.3), (5.4), (5.5). q.e.d.

REMARK 5.7. In the proof of [**MN2**, Lemma 13.4] and [**P**, Lemma 3.8], Marques, Neves, and Pitts used contradiction arguments to get the discretized maps, while we use contradiction arguments to get a good collection of balls.

Now let us sketch the proof of Theorem 5.5. Since the idea is the same as [**MN2**, Lemma 13.1], we will mainly point out the ingredients which we will use in the following.

Proof Theorem 5.5. Fix a small $\delta > 0$. Let $L = \sup_{x\in[0,1]} \mathbf{M}\big(\Phi(x)\big)$, and find $r > 0$, such that $\mathbf{m}(\Phi, r) < \frac{\delta}{4}$. By the compactness of $\mathcal{Z}_n(M^{n+1}) \cap \{S: \mathbf{M}(S) \leq 2L\}$ under flat norm topology, we can find a finite cover of $\mathcal{Z}_n(M^{n+1}) \cap \{S: \mathbf{M}(S) \leq 2L\}$, containing $\{\mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T_i): i = 1, \cdots, N\}$, with

$$T_i \in \mathcal{Z}_n(M^{n+1}) \cap \{S: \mathbf{M}(S) \leq 2L\},\ \epsilon_i = \frac{\epsilon(T_i, \delta, r, L)}{8},$$

where $\epsilon(T_i, \delta, r, L)$ and $k_i = k(T_i, \delta, r, L)$ are given by Lemma 5.6.

By the continuity of Φ, we can take $j_\delta \in \mathbb{N}$ large enough such that for any 1-cell $\alpha \in I(1, j_\delta)$, $\Phi(\alpha_0) \subset \mathcal{B}^{\mathcal{F}}_{\epsilon_{i(\alpha)}}(T_{i(\alpha)})$ for some $i(\alpha)$ depending on α.

Now fix a 1-cell $\alpha \in I(1, j_\delta)$, with $\alpha = [t^1_\alpha, t^2_\alpha]$. Then $\Phi(t^l_\alpha) \in \mathcal{B}^{\mathcal{F}}_{\epsilon_{i(\alpha)}}(T_{i(\alpha)})$, and $\mathbf{m}\big(\Phi(t^l_\alpha), r\big) < \frac{\delta}{4}$, for $l = 1, 2$. By Lemma 5.6, there exists $\tilde{\phi}^l_\alpha : I(1, k_i)_0 \to \mathcal{B}^{\mathcal{F}}_{\epsilon_{i(\alpha)}}(T_{i(\alpha)})$ such that: $\tilde{\phi}^l_\alpha([0]) = \Phi(t^l_\alpha)$, $\tilde{\phi}^l_\alpha([1]) = T_{i(\alpha)}$, $\mathbf{f}(\tilde{\phi}^l_\alpha) \leq \delta$, and $\sup\{\mathbf{M}\big(\tilde{\phi}^l_\alpha(x)\big) : x \in I(1, k_i)_0\} \leq \mathbf{M}\big(\Phi(t^l_\alpha)\big) + \delta$.

By identifying α with $[0, 1]$, we can define $\tilde{\phi}_\alpha : \alpha(k_i+1)_0 \to \mathcal{B}^{\mathcal{F}}_{\epsilon_{i(\alpha)}}(T_{i(\alpha)})$ as follows:

$$\tilde{\phi}_\alpha([\frac{j}{3^{k_i+1}}]) = \begin{cases} \tilde{\phi}^1_\alpha([\frac{j}{3^{k_i+1}}]), & \text{if } j = 0, \cdots, 3^{k_i}; \\ T_{i(\alpha)}, & \text{if } j = 3^{k_i}, \cdots, 2 \cdot 3^{k_i}; \\ \tilde{\phi}^2_\alpha([\frac{3^{k_i+1}-j}{3^{k_i+1}}]), & \text{if } j = 2 \cdot 3^{k_i}, \cdots, 3^{k_i+1}. \end{cases} \tag{5.8}$$

Then for $k_\delta = \max_{i=1}^N \{k_i\}$, we can define: $\phi_\delta : I(1, j_\delta + k_\delta + 1)_0 \to \mathcal{Z}_n(M^{n+1})$ as follows:

$$\phi_\delta|_{\alpha(k_\delta+1)_0} = \tilde{\phi}_\alpha \circ n(k_\delta + 1, k_i + 1), \quad \text{for any 1-cell } \alpha \in I(1, j_\delta), \tag{5.9}$$

where $n(i, j)$ is as in (6) of Definition 4.1. From Lemma 5.6, we know that: $\phi_\delta|_{I(1,j_\delta)_0} = \Phi|_{I(1,j_\delta)_0}$, $\mathbf{f}(\phi_\delta) \leq \sup_{\alpha \in I(1,j_\delta)_1} \mathbf{f}(\tilde{\phi}_\alpha) \leq \delta$, and

$$\mathbf{M}\big(\phi_\delta(x)\big) \leq \sup\{\mathbf{M}\big(\Phi(y)\big) : y, x \in \alpha, \text{ for some 1-cell } \alpha \in I(1, j_\delta)\} + \delta.$$

Now take a sequence of positive numbers $\{\delta_i\}_{i\in\mathbb{N}}$, with $\delta_i \to 0$ as $i \to \infty$. Construct $\phi_i = \phi_{\delta_i} : I(1, j_{\delta_i} + k_{\delta_i} + 1)_0 \to \mathcal{Z}_n(M^{n+1})$ as above. By taking a subsequence, we can construct the sequence of 1-homotopy $\{\psi_i\}_{i\in\mathbb{N}}$ as in the second part of [**MN2**, Theorem 13.1]. The properties (1)(2)(3) listed in the theorem follow from the arguments there. q.e.d.

In order to prove the final result, we need to show that the $(1, \mathbf{M})$-homotopy sequences of mappings into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, which are constructed above from the mapping Φ^Σ in Corollary 5.4 for any $\Sigma \in \mathcal{S}$, belong to the same homotopy class in $\pi^\#_1\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$. A similar issue was considered in the proof of [**MN2**, Theorem 8.4]. However, they only need to show that their sequence is nontrivial, while we need to identify all our sequences. First we have the following theorem.

Theorem 5.8. *Given Φ as in Theorem 5.5, and $\{\phi_i\}_{i\in\mathbb{N}}$ the corresponding $(1, \mathbf{M})$-homotopy sequence obtained by Theorem 5.5, assume that $\Phi(x) = [[\partial\Omega_x]]$, $x \in [0, 1]$ where $\{\Omega_t\}_{t\in[0,1]}$ is a family of open sets satisfying $(sw2)(sw3)$ in Definition 2.1. If $F : \pi^\#_1\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big) \to H_{n+1}(M^{n+1}, \mathbb{Z})$ is the isomorphism given by Almgren in Section 3.2 in* [**AF1**]*, then*

$$F\big([\{\phi_i\}_{i\in\mathbb{N}}]\big) = [[M]],$$

where $[[M]]$ is the fundamental class of M.

Proof. We will directly cite the notions in the proof of Theorem 5.5. First we review the definition of F given in [**AF1**, §3.2]. Fix an i large enough, with δ_i small enough, and we will omit the sub-index i in the following. Take $\phi_\delta = \phi_{\delta_i} : I(1, j_\delta + k_\delta + 1)_0 \to \mathcal{Z}_n(M^{n+1})$ constructed in Theorem 5.5. For any 1-cell $\beta \in I(j_\delta + k_\delta + 1)$, with $\beta = [t^1_\beta, t^2_\beta]$, $\mathcal{F}\big(\phi_\delta(t^1_\beta), \phi_\delta(t^2_\beta)\big) \leq \mathbf{M}\big(\phi_\delta(t^1_\beta), \phi_\delta(t^2_\beta)\big) \leq \mathbf{f}(\phi_\delta) \leq \delta$. By [**AF1**, Corollary 1.14], there exists an isoperimetric choice $Q_\beta \in \mathbf{I}_{n+1}(M^{n+1})$, with $\mathbf{M}(Q_\beta) = \mathcal{F}\big(\phi_\delta(t^1_\beta), \phi_\delta(t^2_\beta)\big)$, and

$$\partial Q_\beta = \phi_\delta(\partial\beta) = \phi_\delta(t^2_\beta) - \phi_\delta(t^1_\beta).$$

Then F is defined in [**AF1**, §3.2] as:

$$F\big([\{\phi_i\}_{i\in\mathbb{N}}]\big) = \sum_{\beta\in I(1,j_\delta+k_\delta+1)_1} [[Q_\beta]], \tag{5.10}$$

where the right hand side is an $n+1$ dimensional integral cycle as $\phi_\delta([0]) = \phi_\delta([1]) = 0$, which hence represents an $n+1$ dimensional integral homology class.

For any 1-cell $\alpha \in I(1, j_\delta)$, we denote

$$\tilde{F}(\alpha, \phi_\delta) = \sum_{\beta\in\alpha(k_\delta+1)_1} [[Q_\beta]]. \tag{5.11}$$

Now let us identify the right hand side of (5.10) with $[[M]]$ using our construction. Let $\{\Omega_t\}_{t\in[0,1]}$ be the defining open sets of Φ. From the construction of ϕ_δ, we know $\phi_\delta|_{I(1,j_\delta)_0} = \Phi|_{I(1,j_\delta)_0}$, so

$$\phi_\delta([\frac{j}{3^{j_\delta}}]) = \Phi(\frac{j}{3^{j_\delta}}) = [[\partial\Omega_{\frac{j}{3^{j_\delta}}}]],$$

by the definition of Φ.

Claim 3. *For the* 1*-cell* $\alpha_j = [\frac{j}{3^{j_\delta}}, \frac{j+1}{3^{j_\delta}}]$,

$$\tilde{F}(\alpha_j, \phi_\delta) = [[\Omega_{\frac{j+1}{3^{j_\delta}}}]] - [[\Omega_{\frac{j}{3^{j_\delta}}}]].$$

Hence

$$F\big([\{\phi_i\}_{i\in\mathbb{N}}]\big) = \sum_{\alpha\in I(1,j_\delta)_1} \tilde{F}(\alpha, \phi_\delta) = \sum_{j=0}^{3^{j_\delta}-1} [[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]] = [[\Omega_1]] = [[M]].$$

Let us go back to check the claim. Take $\alpha = \alpha_j = [\frac{j}{3^{j_\delta}}, \frac{j+1}{3^{j_\delta}}]$. Since $\phi_\delta|_{\alpha(k_\delta+1)_0} = \tilde{\phi}_\alpha \circ n(k_\delta + 1, k_{i(\alpha)} + 1)$ by (5.9), it is easy to see that

$$\tilde{F}(\alpha, \phi_\delta) = \tilde{F}(\alpha, \tilde{\phi}_\alpha) = \sum_{\beta\in\alpha(k_{i(\alpha)}+1)_1} [[Q_\beta]].$$

By identifying $\alpha = [0, 1]$, the mapping $\tilde{\phi}_\alpha : I(k_{i(\alpha)} + 1)_0 \to \mathcal{Z}_n(M^{n+1})$ is a combination of three parts by (5.8), especially $\tilde{\phi}_\alpha|_{[\frac{1}{3},\frac{2}{3}](k_{i(\alpha)})_0} \equiv T_{i(\alpha)}$; hence

$$\tilde{F}(\alpha, \tilde{\phi}_\alpha) = \tilde{F}(\tilde{\phi}^1_\alpha) + \tilde{F}(\tilde{\phi}^2_\alpha).$$

Take $\tilde{\phi}^1_\alpha : I(1, k_{i(\alpha)})_0 \to \mathcal{Z}_n(M^{n+1})$ for example. From the construction, there exists an isoperimetric choice $Q_{\alpha,1} \in \mathbf{I}_{n+1}(M^{n+1})$, such that $\partial Q_{\alpha,1} = \Phi([\frac{j}{3^{j_\delta}}]) - T_{i(\alpha)} = [[\partial\Omega_{\frac{j}{3^{j_\delta}}}]] - T_{i(\alpha)}$, and $\mathbf{M}(Q_{\alpha,1}) \leq \mathcal{F}\big(\Phi([\frac{j}{3^{j_\delta}}]), T_{i(\alpha)}\big) \leq \epsilon_\alpha < \delta$. Then from (5.7), we have

$$\tilde{\phi}^1_\alpha([\frac{h}{3^{k_{i(\alpha)}}}]) = [[\partial\Omega_{\frac{j}{3^{j_\delta}}}]] - \sum_{a=1}^{h} \partial\big(Q_{\alpha,1} \llcorner B_{r_a}(p_a)\big),\ 1 \leq h \leq 3^{k_{i(\alpha)}} - 1;$$
$$\tilde{\phi}^1_\alpha([1]) = T_{i(\alpha)}.$$

Take the isoperimetric choice $Q_{\alpha,1,h} \in \mathbf{I}_{n+1}(M^{n+1})$ such that

$$\partial Q_{\alpha,1,h} = \tilde{\phi}_\alpha([\frac{h}{3^{k_{i(\alpha)}}}]) - \tilde{\phi}_\alpha([\frac{h-1}{3^{k_{i(\alpha)}}}]) = -\partial\big(Q_{\alpha,1} \llcorner B_{r_h}(p_h)\big),\ 1 \leq h \leq 3^{k_{i(\alpha)}} - 1;$$

$$\partial Q_{\alpha,1,3^{k_{i(\alpha)}}} = T_{i(\alpha)} - \tilde{\phi}_\alpha([\frac{3^{k_{i(\alpha)}} - 1}{3^{k_{i(\alpha)}}}]) = -\partial\bigg(Q_{\alpha,1} \llcorner \big(M \backslash \cup_{h=1}^{v} B_{r_h}(p_h)\big)\bigg).$$

So

$$\sum_{h=1}^{3^{k_{i(\alpha)}}} \partial Q_{\alpha,1,h} = -\partial Q_{\alpha,1} = T_{i(\alpha)} - [[\partial\Omega_{\frac{j}{3^{j_\delta}}}]],$$

and from the definition of isoperimetric choice (see [**AF1**, Corollary 1.14]),

$$\sum_{h=1}^{3^{k_{i(\alpha)}}} \mathbf{M}(Q_{\alpha,1,h}) \leq \sum_{h=1}^{3^{k_{i(\alpha)}}-1} \mathbf{M}\big(Q_{\alpha,1} \llcorner B_{r_h}(p_h)\big)$$
$$+ \mathbf{M}\bigg(Q_{\alpha,1} \llcorner \big(M \setminus \cup_{h=1}^{v} B_{r_h}(p_h)\big)\bigg) = \mathbf{M}(Q_{\alpha,1}) < \delta.$$

Similar results hold for $\tilde{\phi}^2_\alpha$, so

$$\tilde{F}(\alpha, \tilde{\phi}_\alpha) = \tilde{F}(\tilde{\phi}^1_\alpha) + \tilde{F}(\tilde{\phi}^2_\alpha) = \sum_{h=1}^{3^{k_{i(\alpha)}}} [[Q_{\alpha,1,h}]] + \sum_{h=1}^{3^{k_{i(\alpha)}}} [[Q_{\alpha,2,h}]],$$

with $\mathbf{M}\big(\tilde{F}(\alpha, \tilde{\phi}_\alpha)\big) < 2\delta$, and

$$\partial\big(\tilde{F}(\alpha, \tilde{\phi}_\alpha)\big) = T_{i(\alpha)} - [[\partial\Omega_{\frac{j}{3^{j_\delta}}}]] + [[\partial\Omega_{\frac{j+1}{3^{j_\delta}}}]] - T_{i(\alpha)} = \partial[[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]].$$

Hence $\partial\big(\tilde{F}(\alpha, \tilde{\phi}_\alpha) - [[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]]\big) = 0$, so using the Constancy Theorem ([**Si**, Theorem 26.27]), we know that $\tilde{F}(\alpha, \tilde{\phi}_\alpha) - [[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]] = k[[M]]$ for some $k \in \mathbb{Z}$. Since $\mathbf{M}\big(\tilde{F}(\alpha, \tilde{\phi}_\alpha) - [[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]]\big) \leq 2\delta + \text{Volume}\big(\Omega_{\frac{j+1}{3^{j_\delta}}} \triangle \Omega_{\frac{j}{3^{j_\delta}}}\big)$ is small enough for large j_δ, we know that $k = 0$, and hence $\tilde{F}(\alpha, \tilde{\phi}_\alpha) = [[\Omega_{\frac{j+1}{3^{j_\delta}}} - \Omega_{\frac{j}{3^{j_\delta}}}]]$, so we have proved the claim and finished the proof. q.e.d.

Now we can combine all the results above to get discretized sequences and show that they all lie in the same homotopy class. Given $\Sigma \in \mathcal{S}$, let $\Phi^{\Sigma} : [0,1] \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ be the mapping given in Corollary 5.4. Then we can apply Theorem 5.5 to get a $(1, \mathbf{M})$-homotopy sequence $\{\phi_i^{\Sigma}\}_{i\in\mathbb{N}}$ into $\big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big)$. Clearly

$$\mathbf{L}(\{\phi_i^{\Sigma}\}_{i\in\mathbb{N}}) \leq \begin{cases} V(\Sigma), & \text{if } \Sigma \in \mathcal{S}_+; \\ 2V(\Sigma), & \text{if } \Sigma \in \mathcal{S}_-. \end{cases} \tag{5.12}$$

Then a direct corollary of Theorem 5.8 is

Corollary 5.9. *$[\{\phi_i^{\Sigma}\}_{i\in\mathbb{N}}] = F^{-1}([[M]]) \in \pi_1^{\#}\big(\mathcal{Z}(M^{n+1}), \{0\}\big)$, for any $\Sigma \in \mathcal{S}$.*

6. Orientation and multiplicity

In this section, we will discuss the orientation and multiplicity of the min-max hypersurface. In Theorem 4.7, the stationary varifold Σ is an integer multiple of some smooth minimal hypersurface (denoted still as Σ). The fact that Σ lies in the critical set $C(S)$ of some critical sequence S implies that Σ is a varifold limit of a sequence of integral cycles $\{\phi_{i_j}(x_j)\}_{j\in\mathbb{N}} \subset \mathcal{Z}_n(M^{n+1})$. The weak compactness implies that $\{\phi_{i_j}(x_j)\}_{j\in\mathbb{N}}$ converge to a limit integral current up to a subsequence, which is then supported on Σ. It has been conjectured that Σ should have some orientation structures by comparing the varifold limit and current limit. Hence we prove the following result. In fact, this result holds for all Riemannian manifolds.

Proposition 6.1. *Let Σ be the stationary varifold in Theorem 4.7, with $\Sigma = \cup_{i=1}^{l} k_i[\Sigma_i]$, where $\{\Sigma_i\}$ is a disjoint collection of smooth connected closed embedded minimal hypersurfaces with multiplicity $k_i \in \mathbb{N}$. If Σ_i is non-orientable, then the multiplicity k_i must be an even number.*

Remark 6.2. This is a characterization of the orientation structure of the min-max hypersurface. When a connected component of Σ is orientable, it naturally represents an integral cycle. While a connected component of Σ is non-orientable, an even multiple of it also represents an integral cycle—a zero integral cycle.

Let us first introduce our strategy for proving this result. Two key ingredients will be used. The first key ingredient is an important general property of the min-max varifold called the "almost minimizing" property [**P**, §3.1]. The almost minimizing property implies that the min-max varifold has a local replacement, which is a varifold limit of a sequence of integral cycles that are locally mass minimizing in the region where it replaces the original min-max varifold. In the case of co-dimension one theory, Pitts [**P**, Chap. 7] essentially showed that the local replacement, which is regular in the replacement region, coincides with the original min-max varifold locally. Hence it implies the regularity of the original min-max varifold. Here we will first show that good local replacements coincide with the original min-max varifold globally by exploring Pitts's idea. Then we need the second key ingredient, which is a convergence result by B. White [**W**]. In fact, for a sequence of integral currents where all the associated varifolds have locally bounded first variations, White showed that the varifold limit and the current limit of this sequence can differ at most by an even multiple of some integral varifold. By applying White's result to the sequence of integral currents that converges to the local replacement, we can show that the replacement, the same as the original min-max hypersurface, must have even multiplicity when it is non-orientable.

First let us introduce some concepts related to the "almost minimizing" property (see [**P**, §3.1]). Let M^{n+1} be an arbitrary Riemannian manifold, and U a bounded open subset of M. We use $B(p,r)$ and $A(p,s,r) = B(p,r) \setminus \overline{B(p,s)}$ to denote the open ball and open annulus in M. Let $k \in \mathbb{N}$ with $1 \leq k \leq n$.

Definition 6.3. Given $\epsilon > 0$ and $\delta > 0$, $\mathfrak{A}_k(U,\epsilon,\delta)$ is the set of integral cycles $T \in \mathcal{Z}_k(M)$ such that if $T = T_0, T_1, \cdots, T_m \in \mathcal{Z}_k(M)$ with

$$\mathrm{spt}(T - T_i) \subset U,\ \mathcal{F}(T_i, T_{i-1}) \leq \delta,\ \mathbf{M}(T_i) \leq \mathbf{M}(T) + \delta, \text{ for } i = 1, \cdots, m,$$

then $\mathbf{M}(T_m) \geq \mathbf{M}(T) - \epsilon$.

A rectifiable varifold $V \in \mathcal{V}_k(M)$ is called *almost minimizing* in U, if for any $\epsilon > 0$, there exists a $\delta > 0$ and $T \in \mathfrak{A}_k(U,\epsilon,\delta)$ such that $\mathbf{F}(V, |T|) < \epsilon$. (This $\mathbf{F}$ is the $\mathbf{F}$-metric for varifold defined in [**P**, page 66], which also defines the varifold weak topology.)

Remark 6.4. In the original work of Pitts (see [**P**, §3.1]), the definition of $\mathfrak{A}_k(U,\cdot,\cdot)$ uses comparison currents $T \in \mathcal{Z}_k(M, M \setminus U)$, i.e. integral currents with boundary outside U, and the almost minimizing varifold is defined to be approximated by $\mathfrak{A}_k(U,\cdot,\cdot)$ under $\mathbf{F}_U$ norm. Our definition is stronger and implies Pitts's definition in [**P**, §3.1], so we can use all the regularity results in [**P**]. Moreover, the min-max varifold appearing in [**P**, Theorem 4.10] does satisfy our definition (in small annulli). In fact, the contradiction arguments (see Part 2 in the proof on [**P**, Theorem 4.10, page 164]) are made with respect to our definition of "almost minimizing". The observation of this stronger version of "almost minimizing" will enable us to gain global properties of the min-max hypersurface.

Now we introduce the concept of local replacement. Let M and U be as above. We have the following result, which is exactly [**P**, Theorem 3.11] adapted to our definition of "almost minimizing."

Theorem 6.5. *Suppose $V \in \mathcal{V}_k(M)$ is almost minimizing in U, and K is a compact subset of U. Then there is a nonempty set $\mathcal{R}(V; U, K) \subset \mathcal{V}_k(M)$ such that any $V^* \in \mathcal{R}(V; U, K)$ satisfies:*

(1) $V^* \llcorner G_k(M \setminus K) = V \llcorner G_k(M \setminus K)$ *($G_k(\cdot)$ is the Grassmann manifold* [**P**, §2.1(12)]*.);*
(2) V^* *is almost minimizing in* U *;*
(3) $\|V^*\|(M) = \|V\|(M)$*;*
(4) $\forall\ \epsilon > 0$, $\exists\ T \in \mathcal{Z}_k(M)$, *such that* $\mathbf{F}(V^*, |T|) < \epsilon$, *and* $T \llcorner Z$ *is locally mass minimizing with respect to* $(Z, \emptyset)$ *for all compact Lipschitz neighborhood retract* $Z \subset Int(K)$.

We will call such V^* a replacement of V in K.

Remark 6.6. The construction of V^* is given in [**P**, §3.10]. The only difference here is property (4). Due to our definition of almost minimizing, the approximation current T can be chosen as an integral cycle rather than in $\mathcal{Z}_k(M, M \setminus U)$, and the approximation can be made under $\mathbf{F}$-norm rather than $\mathbf{F}_U$-norm.

Now let us cite some regularity results from [**P**, Chap. 7] for the replacements of almost minimizing varifolds in the codimension one case.

Lemma 6.7. ([**P**, Corollary 7.7]) *Suppose $2 \leq k \leq 6$, M^{k+1} is a given Riemannian manifold, and U is a bounded open subset of M. If K is a compact subset of U, $V \in \mathcal{V}_k(M)$ is almost minimizing in U, and $V^* \in \mathcal{R}(V; U, K)$, then $spt(\|V^*\|) \cap Int(K)$ is a k dimensional smooth submanifold, which is stable in $Int(K)$.*

Remark 6.8. The case $k = 6$ is due to [**SS**, equation (7.4)].

Another useful result in [**P**] is the following identification lemma.

Lemma 6.9. ([**P**, Lemma 7.10]) *Let k, M be as above. Given $p \in M$ and $r > 0$ small enough, if $V \in \mathcal{V}_k(M)$ is almost minimizing in $B(p, 2r)$ and $spt(\|V\|) \cap A(p, \frac{r}{2}, r)$ is a smooth submanifold in M, then for L^1 almost all $\frac{r}{2} < s < r$, if $V^* \in \mathcal{R}(V; B(p, r), \overline{B(p, s)})$, then $V^* \llcorner G_k\big(A(p, \frac{r}{2}, s)\big) = V \llcorner G_k\big(A(p, \frac{r}{2}, s)\big)$.*

Remark 6.10. The case $k = 6$ is again due to [**SS**, equation (7.40)].

Using the results above, we can show that good local replacement coincides with the min-max hypersurface globally.

Lemma 6.11. *Given k, M, p, r as above, suppose $V \in \mathcal{V}_k(M)$ is almost minimizing in $B(p, 2r) \subset M$, and $spt(\|V\|) \cap B(p, 2r)$ is a smooth connected embedded minimal hypersurface. Then for $s \in [\frac{r}{2}, r]$ as in Lemma 6.9 with $\|V\|(\partial B(p, s)) = 0$ (which exists due to transversality), if $V^* \in \mathcal{R}(V; B(p, r), \overline{B(p, s)})$, then $V^* = V$.*

Proof. By Lemma 6.9, $V^*\llcorner G_k\big(A(p,\frac{r}{2},s)\big) = V\llcorner G_k\big(A(p,\frac{r}{2},s)\big)$. By Lemma 6.7, $\mathrm{spt}(\|V^*\|)\cap B(p,s)$ is a smooth embedded minimal hypersurface. As $\mathrm{spt}(\|V\|)\cap B(p,2r)$ is connected, the classical unique continuation for minimal hypersurface (c.f. [**DT**, Theorem 5.3]) implies that $\mathrm{spt}(\|V\|)\cap B(p,s) \subset \mathrm{spt}(\|V^*\|)\cap B(p,s)$. By (1) and (3) in Theorem 6.5, it is easy to see that $\|V^*\|(B(p,s)) = \|V\|(B(p,s))$ and $\|V^*\|(\partial B(p,s)) = 0$. Hence $V^*\llcorner\overline{B(p,s)} = V\llcorner\overline{B(p,s)}$, so $V^* = V$.

q.e.d.

Finally we need the following convergence result by White [**W**].

Theorem 6.12. ([**W**, Theorem 1.2]) *Let $\{T_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be sequences of integral currents and integral varifolds with $V_i = [T_i]$. If V_i have locally bounded first variation, and if ∂T_i converge to a limit current, then for a subsequence, V_i converge to an integral varifold V and T_i converge to an integral current T, such that $V = [T] + 2W$ for some integral varifold W.*

Now we are ready to prove Proposition 6.1.

Proof. (of Proposition 6.1) By [**P**, Theorem 4.10] and Remark 6.4, for any $p\in M$, there exists $r_p > 0$ such that Σ is almost minimizing (in the sense of Definition 6.3) in $A(p,s,r_p)$ for all $0 < s < r_p$. Let Σ_1 be a non-orientable component of Σ. Hence we can take a point $p\in\Sigma_1$, and $r > 0$ small enough, such that Σ is almost minimizing in $B(p,2r)$ (can choose $B(p,2r)$ as a ball inside some open annulus $A(p',s,r_{p'})$), and $\mathrm{spt}(\|\Sigma\|)\cap B(p,2r) = \mathrm{spt}(\|\Sigma_1\|)\cap B(p,2r)$ is diffeomorphic to an n-ball. Take $s\in[\frac{r}{2},r]$ as in Lemma 6.11, and $V^*\in\mathcal{R}(V;B(p,r),\overline{B(p,s)})$, then $V^* = V$.

By (4) in Theorem 6.5, there exists a sequence of integral cycles $\{T_i\}_{i\in\mathbb{N}}\subset\mathcal{Z}_n(M^{n+1})$, satisfying: $\lim_{i\to\infty}[T_i] = \Sigma$ as varifolds, and $T_i\llcorner B(p,s)$ is locally mass minimizing in $B(p,s)$. The codimension one regularity theory (c.f. [**P**, Theorem 7.2][**Si**, Theorem 37.7]) implies that $\mathrm{spt}(T_i)\llcorner B(p,s)$ are smooth embedded stable minimal hypersurfaces.

Since $\lim_{i\to\infty}[T_i] = \Sigma$ as varifolds, $\mathbf{M}(T_i)$ are uniformly bounded; hence the weak compactness theorem for integral currents ([**Si**, Theorem 27.3]) implies that a subsequence, still denoted by $\{T_i\}$, converges to some integral current $T_0\in\mathcal{Z}_n(M^{n+1})$, i.e. $\lim_{i\to\infty}T_i = T_0$. Since the associated Radon measure $\|T_i\|$ converges to $\|\Sigma\|$ weakly by the varifold convergence, we know that T_0 must have its support in $\cup_{i=1}^l\Sigma_i$, i.e. $\mathrm{spt}(T_0)\subset\cup_{i=1}^l\Sigma_i$. As an elementary fact, we have (see the proof in Appendix 8).

Claim 4. *T_0 is an integral n-cycle in $\cup_{i=1}^l\Sigma_i$, i.e. $T_0\in\mathcal{Z}_n(\cup_{i=1}^l\Sigma_i)$.*

By the Constancy Theorem [**Si**, Theorem 26.27], $T_0 = \sum_{i=1}^l[[k_i'\Sigma_i]]$, for some $k_i'\in\mathbb{Z}$. (Here we can first find a finite covering of Σ_0, with each open set diffeomorphic to a Euclidean ball, and then apply the Constancy Theorem to each open set of the covering, and finally patch the results together.) As Σ_1 is non-orientable, k_1' must be zero, or $k_1'\Sigma_1$ could not represent an integral cycle. The lower semi-continuity of the mass implies that $|k_i'| \leq k_i$, for $i = 1, \cdots, l$.

Now let us focus on the ball $B(p, s)$. After possibly shrinking the radius, we can assume that $\partial(T_i \llcorner B(p, s))$ have uniformly bounded mass (by slicing theory [**Si**, Lemma 28.5]), and hence converge to a limit current up to a subsequence. Clearly $[T_i]\llcorner B(p, s)$ have bounded first variation since they are represented by smooth stable minimal hypersurfaces. Then Theorem 6.12 implies that $\Sigma \llcorner B(p, s) = k_1[\Sigma_1]\llcorner B(p, s) = [T_0 \llcorner B(p, s)] + 2W = 2W$, for some integral varifold W. So k_1 is even. q.e.d.

7. Proof of the main result

Now we are ready to prove the main result.

Proof. (of Theorem 1.1) For any $\Sigma \in \mathcal{S}$, take Φ^{Σ} as in Corollary 5.4, and let the corresponding $(1, \mathbf{M})$-homotopy sequence be $S_{\Sigma} = \{\phi_i^{\Sigma}\}_{i\in\mathbb{N}}$. From Corollary 5.9, all S_{Σ} lie in the same homotopy class $F^{-1}([[M]])$, which we denote by Π_M; then Π_M is nontrivial by Theorem 4.5. We know from (5.12) that

$$\mathbf{L}(\Pi_M) \leq W_M,$$

where W_M is defined in (1.1). Then we can apply the Almgren-Pitts min-max Theorem 4.7, so there exists a stationary integral varifold Σ, whose support is a closed smooth embedded minimal hypersurface Σ_0, such that $\mathbf{L}(\Pi_M) = \|\Sigma\|(M)$. Notice that Σ_0 must be connected by Theorem 3.4. Hence $\Sigma = k[\Sigma_0]$ for some $k \in \mathbb{N}$, $k \neq 0$. So

$$kV(\Sigma_0) = \|\Sigma\|(M) = \mathbf{L}(\Pi_M) \leq W_M, \tag{7.1}$$

and from the definition (1.1) of W_M,

- if $\Sigma_0 \in \mathcal{S}_+$, orientable, then $k \leq 1$, and hence $k = 1$;
- if $\Sigma_0 \in \mathcal{S}_-$, non-orientable, then $k \leq 2$, and hence $k = 1$ or $k = 2$.

First let us deal with the case $\Sigma_0 \in \mathcal{S}_-$. By Proposition 6.1, k must be even, hence $k = 2$. By (1.1) and (7.1) $W_M \leq 2V(\Sigma_0) \leq W_M$, which implies that $2V(\Sigma_0) = W_M$. So we have proved the case (ii).

If $\Sigma_0 \in \mathcal{S}_+$, then by (1.1) and (7.1) again $W_M \leq V(\Sigma_0) \leq W_M$, which implies that $V(\Sigma_0) = W_M$.

Claim 5. *In this case, Σ_0 has index one.*

Let us check the claim now. As in the proof of Proposition 3.6, there exists an eigenfunction u_1 of the Jacobi operator L_{Σ_0}, with $L_{\Sigma_0}u_1 > 0$ and $u_1 > 0$. Moreover, the sweepout $\{\Sigma_t\}_{t\in[-1,1]}$ constructed there is just the flow of Σ_0 along $u_1\nu$, where ν is the unit normal vector field of

Σ_0. Suppose the index of Σ_0 is greater than or equal to two; then we can find an L^2 orthonormal eigenbasis $\{v_1, v_2\} \subset C^\infty(\Sigma_0)$ of L_{Σ_0} with negative eigenvalues. A linear combination will give a $v_3 \in C^\infty(\Sigma_0)$ such that

$$\int_{\Sigma_0} v_3 L_{\Sigma_0} u_1 d\mu = 0, \quad v_3 \neq 0. \tag{7.2}$$

Let $\tilde{X} = v_3\nu$ be another normal vector field, and extend it to a tubular neighborhood of Σ_0. Denote $\{\tilde{F}_s\}_{s\in[-\epsilon,\epsilon]}$ to be the flow of $\tilde{X}$; hence $\tilde{F}_s$ are all isotopies. Now let $\Sigma_{s,t} = \tilde{F}_s(\Sigma_t)$, and consider the two parameter family of generalized smooth family $\{\Sigma_{s,t}\}_{(s,t)\in[-\epsilon,\epsilon]\times[-1,1]}$. Notice that $\Sigma_{s,t}$ is then a smooth family for $(s,t) \in [-\epsilon, \epsilon]\times[-\epsilon, \epsilon]$ for ϵ small enough by (c) in Proposition 3.6. Denote $\tilde{f}(s,t) = \mathcal{H}^n(\Sigma_{s,t})$. Then $\nabla\tilde{f}(0,0) = 0$ (by minimality of Σ_0), $\frac{\partial^2}{\partial t\partial s}\tilde{f}(0,0) = 0$ (by (7.2)), and $\frac{\partial^2}{\partial t^2} f(0,0) < 0$, $\frac{\partial^2}{\partial s^2} f(0,0) < 0$ (by negativity of eigenvalues). So there exists $\delta > 0$ small enough, $\tilde{f}(\delta,t) < \tilde{f}(0,0)$ for all t, since $\tilde{f}(0,t) < \tilde{f}(0,0)$ for all $t \neq 0$ by (b) in Proposition 3.6. By Remark 2.6, $\{\Sigma_{\delta,t}\}_{t\in[-1,1]}$ is a sweepout in the sense of Definition 2.1. By Proposition 5.1, Theorem 5.5, and Theorem 5.8, we can construct a $(1, \mathbf{M})$-homotopy sequence $\{\phi_i^\delta\}_{i\in\mathbb{N}}$, such that $\{\phi_i^\delta\}_{i\in\mathbb{N}} \in \Pi_M$, and

$$\mathbf{L}\big(\{\phi_i^\delta\}_{i\in\mathbb{N}}\big) \leq \sup_{t\in[-1,1]} \tilde{f}(\delta,t) < \tilde{f}(0,0) = V(\Sigma_0) = W_M,$$

which is hence a contradiction to the fact that $\mathbf{L}(\Pi_M) = W_M$. So we have proved Claim 5 and hence case (i). q.e.d.

Remark 7.1. We used the same idea to prove the index bound as in [**MN1**][**MN2**]. However, they a priori needed the existence of a least area embedded minimal surface among a family of embedded minimal surfaces, while in our case the existence of a least area minimal hypersurface is just a by-product of the min-max construction and the existence of good sweepouts (Proposition 3.6, Proposition 3.8).

8. Appendix

First we give the proof of Claim 1 in Proposition 3.6.

Proof. (of Claim 1 in Proposition 3.6) Denote $U_{s_0} = F([-s_0, s_0]\times \Sigma)$ for $0 < s_0 \leq \epsilon$. It is easy to see that $\{\Sigma_s\}_{s\in[-\epsilon,\epsilon]}$ is a foliation corresponding to the level set of a function f defined in a neighborhood U_ϵ of Σ, such that $f(\Sigma_s) = s$. In fact, using coordinates $(s,x) \in [-\epsilon, \epsilon]\times \Sigma$ for $U_\epsilon = F([-\epsilon, \epsilon]\times\Sigma)$,

$$f(s,x) = s = \frac{d^{\pm}(s,x)}{u_1(x)}, \quad f \in C^\infty(U_\epsilon),$$

where $d^{\pm}: U_\epsilon \to \mathbb{R}$ is the signed distance function with respect to Σ, i.e.

$$d^{\pm}(x) = \begin{cases} dist(x,\Sigma), & \text{if } x \in M_1; \\ -dist(x,\Sigma), & \text{if } x \in M_2. \end{cases}$$

Since $|\nabla d^{\pm}| = 1$, $|f| \leq \epsilon$, and $\nabla f = \frac{\nabla d^{\pm} - f\nabla u_1}{u_1}$, we can choose ϵ small enough depending only on u_1 such that $|\nabla f|$ is bounded away from 0 on U_ϵ. Hence f is a Morse function on U_ϵ.

We want to cook up a Morse function g on M, which coincides with f on $U_{\frac{1}{2}\epsilon}$. First extend f to be a smooth function on M (denoted still by f), such that $f|_{M_{1,\frac{3}{4}\epsilon}} > \frac{3}{4}\epsilon$ (and $f|_{M_{2,\frac{3}{4}\epsilon}} < -\frac{3}{4}\epsilon$). Using the fact that the set of Morse functions is dense in $C^k(M)$ for $k \geq 2$ (see [**H**, Chap. 6, Theorem 1.2]), we can find a C^∞ function $\tilde{f}$ such that $\|f - \tilde{f}\|_{C^2}$ is arbitrarily small. Choose a cutoff function $\varphi : M \to \mathbb{R}$ such that $\varphi \equiv 1$ on $U_{\frac{1}{2}\epsilon}$, and $\varphi \equiv 0$ outside $U_{\frac{3}{4}\epsilon}$. Let

$$g = \varphi f + (1-\varphi)\tilde{f} = f + (1-\varphi)(\tilde{f} - f).$$

Hence $g \equiv f$ in $U_{\frac{1}{2}\epsilon}$, and $g \equiv \tilde{f}$ outside $U_{\frac{3}{4}\epsilon}$. In order to check that g is a Morse function, we only need to check that in the middle region. Now

$$\nabla g = \nabla f + (1-\varphi)(\nabla \tilde{f} - \nabla f) - \nabla\varphi(\tilde{f} - f).$$

Since $|\nabla f|$ is bounded away from 0 on U_ϵ, we can take $\|\tilde{f} - f\|_{C^2}$ small enough to make sure that $|\nabla g|$ is bounded away from 0, and hence g is a Morse function.

Now take $\{\tilde{\Sigma}_s\}$ to be the sweepout given by the level surface of g (by Proposition 2.3). $\tilde{\Sigma}_s = \Sigma_s$ since $g \equiv f$ in $U_{\frac{1}{2}\epsilon}$. $\tilde{\Sigma}_s \subset M_{1,\frac{1}{2}\epsilon}$ (or $\subset M_{2,\frac{1}{2}\epsilon}$) when $s > \frac{1}{2}\epsilon$ (or $s < -\frac{1}{2}\epsilon$) follows from the fact that $g > \frac{1}{2}\epsilon$ on $M_{1,\frac{1}{2}\epsilon}$ (or $g < -\frac{1}{2}\epsilon$ on $M_{2,\frac{1}{2}\epsilon}$). A reparameterization gives the sweepout in the claim. q.e.d.

Now we give the proof of Claim 4 in the proof of Proposition 6. The proof is elementary, but does not appear in standard reference, so we add it here for completeness.

Proof. (of Claim 4 in Proposition 6) Denote $\Sigma_0 = \cup_{i=1}^{l}\Sigma_i$. First we show that T_0 is an integral current in Σ_0. Since T_0 is an integral current in M, it is represented as $T_0 = \underline{\underline{\tau}}(N,\theta,\xi)$ (see [**Si**, §27.1]), where N is a countably n-rectifiable set, θ is an integer-valued locally $\mathcal{H}^n$ integrable function, and ξ equals the orienting n-form of the approximated tangent plane T_xN for $\mathcal{H}^n$ a.e. $x \in N$. As N lies in the support of T_0, and hence in Σ_0, T_0 also represents an integral current in Σ_0, and we denote it as T_0'.

Now let us show that $\partial T_0' = 0$ as current in $\mathcal{Z}_n(\Sigma_0)$. We only need to show that for any compactly supported smooth $n-1$ form $\psi \in \Lambda_c^{n-1}(\Sigma_0)$, we have $\partial T_0'(\psi) = 0$. By using partition of unity, we can restrict to the case when ψ is supported in a local coordinate chart.

Assume that the support of ψ lies in $U \cap \Sigma_0$, where U is a coordinates chart for M, with coordinates $\{x_1, \cdots, x_{n-1}, y\}$, and $U \cap \Sigma_0$ is given by $y = 0$. We can easily extend ψ smoothly to a neighborhood of $U \cap \Sigma_0$, denoting by $\tilde{\psi} \in \Lambda_c^{n-1}(U)$, such that $\mathcal{L}_{\partial y}\tilde{\psi} = 0$ near $U \cap \Sigma_0$. In fact, this can be achieved by extending the coefficients of ψ to U trivially, so that those coefficients do not depend on y near $U \cap \Sigma_0$. Hence $d\tilde{\psi}|_{U\cap\Sigma_0} = d\psi$. So

$$\partial T_0'(\psi) = T_0'(d\psi) = T_0'(d\tilde{\psi}|_{U\cap\Sigma_0}) = T_0(d\tilde{\psi}) = \partial T_0(\tilde{\psi}) = 0,$$

where the third " = " follows from the integral formula ([**Si**, page 146]) for integral currents. Writing T_0' as T_0 again, we finish the proof. q.e.d.

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