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TOPOLOGY OF KÄHLER RICCI SOLITONS

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Abstract

We study the issue of connectedness at infinity of gradient Kähler Ricci solitons. For shrinking Kähler Ricci solitons, we show they must be connected at infinity. We also show the same holds true for expanding Kähler Ricci solitons with proper potential functions. As a separate issue, we obtain a sharp pointwise lower bound for the weight function of any smooth metric measure space, in terms of a lower bound of the associated Bakry–Émery curvature.

For a smooth function f on a Riemannian manifold (M, g), the associated Bakry–Émery curvature, denoted by Ric_f, is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\,(f)\,,$$

where Ric is the Ricci curvature of M and $\mathrm{Hess}\,(f)$ the hessian of f. This curvature notion has received much attention recently. Perhaps one of the reasons is that it admits a generalization to more general metric spaces. Indeed, if one normalizes the weighted measure $e^{-f}\,dv$ to be a probability measure on M, where dv is the volume form of M, then Lott and Villani [**11**] and, independently, Sturm [**19**, **20**] have interpreted the lower bound of Ric_f as a convexity measurement for the Nash entropy on the space of probability measures on M equipped with the Wasserstein metric. They further exploited this interpretation to define a notion of Ricci curvature for general metric measure spaces. From the geometry point of view, manifolds with constant Bakry–Émery curvature are of great interest in the study of the Ricci flows. Recall that a gradient Ricci soliton is a manifold (M, g) such that there exists a function $f \in C^{\infty}(M)$ satisfying

$$\mathrm{Ric}_f = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. Solitons are classified as shrinking, steady, or expanding, according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. The function f is called the potential. Customarily, the constant λ is assumed to be $\frac{1}{2}$, 0, or $-\frac{1}{2}$ by scaling. Obviously, Ricci solitons are natural generalizations of Einstein manifolds. More significantly, they are self similar

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solutions to the Ricci flows and play a crucial role in the study of singularities of the flows [**6**].

In this paper, we continue our study of the geometry and topology of manifolds with Bakry–Émery curvature bounded from below, paying particular attention to the gradient Ricci solitons. In our previous work [**13**], we have proved that a nontrivial steady gradient Ricci soliton must be connected at infinity. While this topological information is of interest itself, it also tells that one cannot perturb the connected sum of any two such solitons into a new one. In [**14**], a similar result was established for expanding Ricci solitons with scalar curvature bounded below by $S \geq -\frac{n}{2} + \frac{1}{2}$, where n is the dimension of the manifold. Note that by [**18**], $S \geq -\frac{n}{2}$ for any expanding Ricci soliton. Moreover, according to [**4**], there exists nontrivial expanding Ricci solitons with more than one end with scalar curvature asymptotic to $-\frac{n}{2}$ at infinity.

The case of shrinking solitons is apparently more challenging. We speculate that a shrinking Ricci soliton with more than one end must split as a direct product of the real line with an Einstein manifold of positive scalar curvature. This speculation is partially based on our work in [**16**]. Our first objective here is to confirm this speculation for gradient Kähler Ricci solitons.

Theorem 0.1. *Let (M, g, f) be a gradient Kähler shrinking Ricci soliton. Then (M, g) is connected at infinity, i.e., it has only one end.*

Let us point out that so far all known examples of nontrivial gradient shrinking Ricci soliton are Kähler [**2**]. Previously, in [**12**], Sesum and the first author have observed that Kähler shrinking Ricci solitons have at most one non-parabolic end. Under the restrictive assumption that the scalar curvature satisfies $\sup S < \frac{n}{2} - 1$, they managed to rule out the existence of parabolic ends.

Our proof relies on the following result, which is of interest itself. It generalizes a result due to P. Li [**8**], which corresponds to the case φ is constant.

Theorem 0.2. *Let (M, g) be a complete Kähler manifold. Let φ be a smooth real function on M such that $J(\nabla\varphi)$ is a Killing vector field. Assume that there exists a constant $C > 0$ such that*

$$|\nabla\varphi|(x) \leq C\left(d\left(x_0, x\right) + 1\right).$$

Then any solution u to $\Delta_{\varphi} u = 0$ with

$$\int_M |\nabla u|^2 e^{-\varphi} < \infty$$

must be pluriharmonic. In particular, u must be a constant if φ is proper.

Let us mention that we have studied other Liouville theorems under a similar setting in [**15**].

We point out that the preceding result is applicable to Kähler Ricci solitons. Indeed, in terms of local holomorphic coordinates, the Kähler Ricci soliton equation can be rewritten into

$$\begin{aligned} R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} &= \lambda\, g_{\alpha\bar{\beta}} \\ f_{\alpha\beta} &= 0. \end{aligned}$$

As pointed out in [**1**], $f_{\alpha\beta} = 0$ is equivalent to the vector field $J(\nabla f)$ being a Killing vector field. It is also well-known (see [**3**]) that $|\nabla f|$ is at most of linear growth for gradient Ricci solitons.

The general framework of our proof of Theorem 0.1 follows the function theoretic approach developed by Li and Tam [**10**], where they used harmonic functions to detect the number of ends. Here, we instead work with the weighted Laplacian

$$\Delta_\varphi u := \Delta u - \langle \nabla\varphi, \nabla u\rangle\,,$$

where $\varphi = -f$. A technically challenging step is to show that all the ends of a gradient shrinking soliton, not necessarily Kähler, are φ-nonparabolic, namely, there exists a nonconstant solution u to $\Delta_\varphi u = 0$ on each end E with $u = 1$ on ∂E and $0 \leq u \leq 1$ on E. Once this is done, one concludes there exists a nonconstant solution u to $\Delta_\varphi u = 0$ with

$$\int_M |\nabla u|^2\, e^{-\varphi} < \infty$$

if M is not connected at infinity. Notice that f, hence φ, is proper on a shrinking gradient soliton according to [**3**]. The proof of Theorem 0.1 is then finished by invoking Theorem 0.2.

Our technique can be applied to obtain a corresponding result for expanding Kähler Ricci solitons.

Theorem 0.3. *Let (M, g, f) be an expanding Kähler gradient Ricci soliton. Assume that the potential f is proper. Then (M, g) has only one end.*

Let us point out that it is necessary to impose the additional assumption that f is proper. This is because Kähler–Einstein manifolds, viewed as trivial expanding Ricci solitons, may have more than one end.

As a separate issue, we also obtain a result concerning the weight function f for a smooth metric measure space with its Bakry–Émery curvature bounded below.

Theorem 0.4. *Let $\left(M, g, e^{-f}dv\right)$ be a smooth metric measure space with Bakry–Émery curvature bounded by $\mathrm{Ric}_f \geq \lambda g$, where $\lambda \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$. Then for any $x_0 \in M$ fixed there exist $a > 0$, depending only on dimension n and $\sup_{B_{x_0}(1)} |f|$, such that*

$$f(x) \geq \frac{\lambda}{2} r^2(x) - a\, r(x)$$

for all $x \in M$ *with* $r(x)$ *sufficiently large, where* $r(x) := d(x_0, x)$.

Note that we make no assumption on $|\nabla f|$ here. For gradient Ricci solitons, Theorem 0.4 was previously known. Indeed, by the Bianchi identity, $\mathrm{Ric}_f = \lambda g$ implies (see [**7**]) $S + |\nabla f|^2 = 2\lambda f + C$, where C is some constant. Since $S \geq 0$ (see [**2**, **5**]), in the case $\lambda \geq 0$ and $S \geq n\lambda$ [**18**] if $\lambda < 0$, the aforementioned lower bound on f follows immediately when $\lambda \leq 0$. In the case $\lambda > 0$, Cao and Zhou [**3**] have obtained the following estimate on potential function f:

$$\frac{\lambda}{2}\left(r(x) - c\right)^2 \leq f(x) \leq \frac{\lambda}{2}\left(r(x) + c\right)^2$$

for all $x \in M$. Both arguments, however, rely on the information of $|\nabla f|$. Incidently, the results for gradient Ricci solitons imply Theorem 0.4 is sharp.

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1. A vanishing theorem for φ-harmonic functions

In this section we prove Theorem 0.2. Let us first fix some notation. We let (M, g) be a Kähler manifold and φ a smooth real function on M. Define unitary frame by

$$\begin{aligned} v_\alpha &= e_\alpha - iJe_\alpha \\ v_{\bar\alpha} &= e_\alpha + iJe_\alpha \end{aligned}$$

for $\alpha \in \{1, 2, .., m\}$, where $\{e_\alpha, Je_\alpha\}$ is an orthonormal frame for g and m the complex dimension of (M, g). With respect to such frames, we have

$$\begin{aligned} \Delta u &= u_{\alpha\bar\alpha} \\ \langle \nabla u, \nabla w\rangle &= \frac{1}{2}\left(u_\alpha w_{\bar\alpha} + u_{\bar\alpha} w_\alpha\right) \end{aligned}$$

for any two functions $u, w \in C^\infty(M)$. Recall that the φ-Laplacian on functions is the operator $\Delta_\varphi := \Delta - \langle \nabla\varphi, \nabla\rangle$.

We restate the theorem we are after.

Theorem 1.1. *For a Kähler manifold* (M, g), *assume* $\varphi : M \to \mathbb{R}$ *satisfies* $\varphi_{\alpha\beta} = 0$ *in any unitary frame. Additionally, assume there exists a constant* $C > 0$ *such that*

$$|\nabla\varphi|(x) \leq C\left(d(x_0, x) + 1\right),$$

for any $x \in M$*. Then any function* $u : M \to \mathbb{R}$ *such that*

$$\Delta_\varphi u = 0$$
$$\int_M |\nabla u|^2 e^{-\varphi} < \infty$$

must be pluriharmonic. Hence, it is constant if φ is proper on M.

Proof of Theorem 1.1. Our proof is an adaptation to manifolds with weights of a result in [**8**]. Indeed, in [**8**], P. Li proved that a harmonic function with finite Dirichlet integral is pluriharmonic.

Let us assume that there exists a function $u : M \to \mathbb{R}$ such that

$$\Delta_\varphi u = 0 \quad (1.1)$$
$$\int_M |\nabla u|^2 e^{-\varphi} < \infty.$$

For a fixed $x_0 \in M$, we consider the cut-off function defined by

$$\phi(x) = \begin{cases} 1 & \text{on } B_{x_0}(R) \\ \frac{2R - d(x_0,x)}{R} & \text{on } B_{x_0}(2R) \setminus B_{x_0}(R) \\ 0 & \text{on } M \setminus B_{x_0}(2R). \end{cases}$$

Integration by parts implies

$$\int_M \left|u_{\alpha\bar{\beta}}\right|^2 \phi^2 e^{-\varphi} = \int_M u_{\alpha\bar{\beta}} u_{\beta\bar{\alpha}} \phi^2 e^{-\varphi}$$
$$= -\int_M u_{\alpha\bar{\beta}\bar{\alpha}} u_\beta \phi^2 e^{-\varphi} + \int_M u_{\alpha\bar{\beta}} u_\beta \varphi_{\bar{\alpha}} \phi^2 e^{-\varphi} - \int_M u_{\alpha\bar{\beta}} u_\beta \left(\phi^2\right)_{\bar{\alpha}} e^{-\varphi}.$$

This can be rewritten into

$$\int_M \left|u_{\alpha\bar{\beta}}\right|^2 \phi^2 e^{-\varphi} = -\int_M \operatorname{Re}\left(u_{\alpha\bar{\beta}\bar{\alpha}} u_\beta\right) \phi^2 e^{-\varphi} \quad (1.2)$$
$$+ \int_M \operatorname{Re}\left(u_{\alpha\bar{\beta}} u_\beta \varphi_{\bar{\alpha}}\right) \phi^2 e^{-\varphi} - \int_M \operatorname{Re}\left(u_{\alpha\bar{\beta}} u_\beta \left(\phi^2\right)_{\bar{\alpha}}\right) e^{-\varphi},$$

where $\operatorname{Re}(z) := \frac{z+\bar{z}}{2}$ denotes the real part of the complex number z.

We now investigate each term in (1.2). First, by the Ricci identities on Kähler manifolds, we have

$$-\int_M \operatorname{Re}\left(u_{\alpha\bar{\beta}\bar{\alpha}} u_\beta\right) \phi^2 e^{-\varphi} = -\int_M \operatorname{Re}\left(u_{\alpha\bar{\alpha}\bar{\beta}} u_\beta\right) \phi^2 e^{-\varphi}$$
$$= -\int_M \operatorname{Re}\left((\Delta u)_{\bar{\beta}} u_\beta\right) \phi^2 e^{-\varphi} = -\int_M \langle \nabla \Delta u, \nabla u \rangle \phi^2 e^{-\varphi}$$
$$= \int_M (\Delta u)(\Delta_\varphi u) \phi^2 e^{-\varphi} + \int_M (\Delta u) \left\langle \nabla u, \nabla \phi^2 \right\rangle e^{-\varphi}$$
$$= \int_M (\Delta u) \left\langle \nabla u, \nabla \phi^2 \right\rangle e^{-\varphi}.$$

We thus conclude by the Cauchy–Schwarz inequality that

$$
\begin{aligned}
(1.3) \qquad -\int_M \operatorname{Re}\left(u_{\alpha\bar\beta\bar\alpha}u_\beta\right)\phi^2 e^{-\varphi} &\le \frac{1}{4m}\int_M (\Delta u)^2\,\phi^2 e^{-\varphi} \\
&\quad +4m\int_M |\nabla u|^2\,|\nabla\phi|^2\,e^{-\varphi} \\
&\le \frac{1}{4}\int_M \left|u_{\alpha\bar\beta}\right|^2\phi^2 e^{-\varphi} + 4m\int_M |\nabla u|^2\,|\nabla\phi|^2\,e^{-\varphi}.
\end{aligned}
$$

Secondly integration by parts gives

$$
\begin{aligned}
\int_M \operatorname{Re}\left(u_{\alpha\bar\beta}u_\beta\varphi_{\bar\alpha}\right)\phi^2 e^{-\varphi} &= -\int_M \operatorname{Re}\left(u_\alpha u_{\beta\bar\beta}\varphi_{\bar\alpha}\right)\phi^2 e^{-\varphi} \\
-\int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha\bar\beta}\right)\phi^2 e^{-\varphi} &+ \int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\varphi_{\bar\beta}\right)\phi^2 e^{-\varphi} \\
&-\int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\left(\phi^2\right)_{\bar\beta}\right)e^{-\varphi}.
\end{aligned}
$$

We now invoke the assumption

$$\varphi_{\alpha\beta}=\varphi_{\bar\alpha\bar\beta}=0.$$

Therefore, it follows that

$$
\begin{aligned}
(1.4) \qquad \int_M \operatorname{Re}\left(u_{\alpha\bar\beta}u_\beta\varphi_{\bar\alpha}\right)\phi^2 e^{-\varphi} &= -\int_M (\Delta u)\operatorname{Re}\left(u_\alpha\varphi_{\bar\alpha}\right)\phi^2 e^{-\varphi} \\
+\int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\varphi_{\bar\beta}\right)\phi^2 e^{-\varphi} &- \int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\left(\phi^2\right)_{\bar\beta}\right)e^{-\varphi} \\
= -\int_M \langle\nabla u,\nabla\varphi\rangle^2\,\phi^2 e^{-\varphi} &+ \int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\varphi_{\bar\beta}\right)\phi^2 e^{-\varphi} \\
&-\int_M \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\left(\phi^2\right)_{\bar\beta}\right)e^{-\varphi}.
\end{aligned}
$$

Notice, however, that

$$(1.5) \qquad \operatorname{Re}\left(u_\alpha u_\beta\varphi_{\bar\alpha}\varphi_{\bar\beta}\right) \le \langle\nabla u,\nabla\varphi\rangle^2.$$

Plugging into (1.4), we conclude

$$(1.6) \qquad \int_M \operatorname{Re}\left(u_{\alpha\bar\beta}u_\beta\varphi_{\bar\alpha}\right)\phi^2 e^{-\varphi} \le 2\int_M |\nabla\varphi|\,|\nabla u|^2\,\phi\,|\nabla\phi|\,e^{-\varphi}.$$

Using (1.3) and (1.6) in (1.2), we find

$$
\begin{aligned}
(1.7) \qquad \int_M \left|u_{\alpha\bar\beta}\right|^2\phi^2 e^{-\varphi} &\le \frac{1}{2}\int_M \left|u_{\alpha\bar\beta}\right|^2\phi^2 e^{-\varphi} \\
+8m\int_M |\nabla u|^2\,|\nabla\phi|^2\,e^{-\varphi} &+ 2\int_M |\nabla\varphi|\,|\nabla u|^2\,\phi\,|\nabla\phi|\,e^{-\varphi}.
\end{aligned}
$$

Since $\int_M |\nabla u|^2\,e^{-\varphi}<\infty$, it is easy to see that as $R\to\infty$,

$$\int_M |\nabla u|^2\,|\nabla\phi|^2\,e^{-\varphi}\to 0.$$

Furthermore, by the assumption that $|\nabla\varphi| \leq C\,(d\,(x_0, x) + 1)$, we have $|\nabla\varphi|\,|\nabla\phi| \leq C$ on M. Consequently,

$$\int_M |\nabla\varphi|\,|\nabla u|^2\,\phi\,|\nabla\phi|\,e^{-\varphi} \to 0.$$

We can now conclude that $u_{\alpha\bar{\beta}} = 0$, so u is pluriharmonic, by letting $R \to \infty$ in (1.7).

To show u is constant in the case φ is proper, we first note that $\langle \nabla u, \nabla\varphi \rangle = 0$ as u is both φ-harmonic and harmonic. Denote

$$D\,(t) := \{x : |\varphi|\,(x) \leq t\}\,,$$

which is compact as φ is proper. Now

$$\begin{aligned} \int_{D(t)} |\nabla u|^2 &= \frac{1}{2}\int_{D(t)} \Delta u^2 = \frac{1}{2}\int_{\partial D(t)} \frac{\partial u^2}{\partial\nu} \\ &= \int_{\partial D(t)} u\frac{\langle \nabla u, \nabla\varphi \rangle}{|\nabla\varphi|} \\ &= 0. \end{aligned}$$

Since this is true for any t, it follows that $|\nabla u| = 0$ on M or u is a constant. Theorem 1.1 is proved. q.e.d.

2. Ends of shrinking solitons

In this section, we assume (M, g, f) is a Kähler shrinking Ricci soliton. So, in terms of unitary frames, we have

$$\begin{aligned} R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} &= g_{\alpha\bar{\beta}} \\ f_{\alpha\beta} &= 0. \end{aligned} \tag{2.1}$$

For a shrinking Ricci soliton, the potential f satisfies [**3**]

$$\frac{1}{4}r^2\,(x) - cr\,(x) \leq f\,(x) \leq \frac{1}{4}r^2\,(x) + cr\,(x)$$

for all $r\,(x) := d\,(x_0, x)$ sufficiently large. From now on, we will take x_0 a minimum point for f; by the above estimates such a point always exists. Furthermore, it can be shown that in this case the constant c above depends only on the dimension n of the manifold.

Using the Bianchi identities, Hamilton [**7**] proved

$$S + |\nabla f|^2 = f,$$

where S denotes the scalar curvature of (M, g). Since according to a result by Chen [**5**] (see also [**2**]) we have $S \geq 0$ for any shrinker, it results in particular that

$$|\nabla f|^2 \leq f.$$

We gather these properties here for future reference.

$$\frac{1}{4}r^2(x) - cr(x) \leq f(x) \leq \frac{1}{4}r^2(x) + cr(x) \tag{2.2}$$
$$|\nabla f|^2 \leq f$$
$$S \geq 0.$$

We now prove the following theorem.

Theorem 2.1. *Let (M, g, f) be a gradient Kähler shrinking Ricci soliton. Then (M, g) has only one end.*

Proof of Theorem 2.1. While with respect to the weight function f, the smooth metric space $\left(M, g, e^{-f}dv\right)$ has constant positive Bakry–Émery curvature $\mathrm{Ric}_f = \frac{1}{2}g$, we will consider a different weight instead. Let us set throughout this section

$$\varphi := -af, \tag{2.3}$$

where $a > 0$ is a constant. We will be interested in the properties of $(M, g, e^{-\varphi}dv)$. Clearly, the Bakry–Émery curvature associated to this smooth metric measure space is given by

$$\begin{aligned} \mathrm{Ric}_\varphi &= \mathrm{Ric} + \mathrm{Hess}(\varphi) \\ &= \frac{1}{2}g - (a+1)\,\mathrm{Hess}(f). \end{aligned}$$

The important feature is that $(M, g, e^{-\varphi}dv)$ has only φ-nonparabolic ends, a fact we will demonstrate below. Recall that an end E of $(M, g, e^{-\varphi}dv)$ is said to be φ-nonparabolic if there exists a positive Green's function for the weighted Laplacian

$$\Delta_\varphi u := \Delta u - \langle \nabla\varphi, \nabla u\rangle$$

satisfying the Neumann boundary conditions on ∂E. Otherwise, it is called φ-parabolic. We refer to [**9**] for some important facts concerning parabolicity. Although [**9**] studies the usual Laplacian Δ, the results there extend without much effort to the weighted Laplacian case.

We now show that M does not admit any φ-parabolic ends. This fact is true without assuming (M, g) is Kähler. Suppose E is a φ-parabolic end of M. A convenient way to characterize a φ-parabolic end, due to Nakai [**17**], is by the existence of a proper φ-harmonic function h on the end. So we have a function $h \geq 1$ defined on E such that

$$\lim_{x \to E(\infty)} h(x) = \infty \text{ and } h = 1 \text{ on } \partial E, \tag{2.4}$$
$$\Delta_\varphi h = \Delta h + a\langle \nabla h, \nabla f\rangle = 0.$$

Our goal is to show that (2.4) leads to a contradiction, which implies that all ends of (M, g) are φ-nonparabolic.

Since this part of the proof does not make use of the fact of (M, g) being Kähler, we will use moving frame notations with indices from

$i = 1, \ldots, n$, where $n = 2m$ is the real dimension of (M, g). Hence, (u_{ij}) is the real hessian of the function u and $\langle \nabla u, \nabla v \rangle = u_i v_i$ in any orthonormal frame.

For $t > 1$ and $b > c > 1$, we denote

$$\begin{aligned} l(t) &:= \{x \in E : h(x) = t\} \\ L(c,b) &:= \{x \in E : c < h(x) < b\}. \end{aligned}$$

Notice that by (2.4) we know $l(t)$ and $L(c,b)$ are compact. By the Stokes formula we get

$$\begin{aligned} 0 &= \int_{L(c,b)} (\Delta_\varphi h)\, e^{-\varphi} \\ &= \int_{l(b)} \frac{\partial h}{\partial \nu} e^{-\varphi} - \int_{l(c)} \frac{\partial h}{\partial \nu} e^{-\varphi} \\ &= \int_{l(b)} |\nabla h|\, e^{-\varphi} - \int_{l(c)} |\nabla h|\, e^{-\varphi}, \end{aligned}$$

where we have used that $\frac{\partial}{\partial \nu} = \frac{\nabla h}{|\nabla h|}$ is the unit normal to the level set of h. This shows that $\int_{l(t)} |\nabla h|\, e^{-\varphi}$ is independent of $t \geq 1$. Now we apply co-area formula to get (cf. [**10**])

$$\begin{aligned} \int_E |\nabla \ln h|^2 e^{-\varphi} &= \int_{L(1,\infty)} |\nabla \ln h|^2 e^{-\varphi} \\ &= \int_1^\infty \left(\int_{l(t)} \frac{|\nabla h|}{h^2} e^{-\varphi} \right) dt \\ &= \left(\int_1^\infty \frac{1}{t^2} dt \right) \int_{l(t_0)} |\nabla h|\, e^{-\varphi} < \infty. \end{aligned}$$

In particular, for any $x \in E$ with $B_x(1) \subset E$, we have by (2.2)

$$\int_{B_x(1)} |\nabla \ln h|^2 \leq e^{-\frac{a}{4} r^2(x) + c r(x)}, \tag{2.5}$$

where $r(x) := d(x_0, x)$. We now transform (2.5) into a pointwise estimate. This is somewhat technical as $|\nabla \ln h|^2$ does not satisfy a convenient differential inequality. Let

$$v := \ln h.$$

Then

$$\frac{1}{2} \Delta |\nabla v|^2 = |\mathrm{Hess}(v)|^2 + \langle \nabla \Delta v, \nabla v \rangle + \mathrm{Ric}(\nabla v, \nabla v).$$

Since by (2.4)

$$\begin{aligned} \Delta v &= \frac{1}{h} \Delta h - \frac{1}{h^2} |\nabla h|^2 \\ &= -a \langle \nabla v, \nabla f \rangle - |\nabla v|^2, \end{aligned} \tag{2.6}$$

we conclude that

$$\begin{aligned}\frac{1}{2}\Delta\left|\nabla v\right|^2 &= \left|\mathrm{Hess}\left(v\right)\right|^2 - a\left\langle\nabla\left\langle\nabla v,\nabla f\right\rangle,\nabla v\right\rangle - \left\langle\nabla\left|\nabla v\right|^2,\nabla v\right\rangle \\ &\quad +\frac{1}{2}\left|\nabla v\right|^2 - \mathrm{Hess}\left(f\right)\left(\nabla v,\nabla v\right) \\ &\geq \frac{1}{2}\left|\mathrm{Hess}\left(v\right)\right|^2 - a^2 f\left|\nabla v\right|^2 - \left\langle\nabla\left|\nabla v\right|^2,\nabla v\right\rangle \\ &\quad -\left(a+1\right)\mathrm{Hess}\left(f\right)\left(\nabla v,\nabla v\right).\end{aligned}$$

By the Cauchy–Schwarz inequality and (2.6), we obtain

$$\begin{aligned}\left|\mathrm{Hess}\left(v\right)\right|^2 &\geq \frac{1}{n}\left(\Delta v\right)^2 \\ &= \frac{1}{n}\left(a\left\langle\nabla v,\nabla f\right\rangle + \left|\nabla v\right|^2\right)^2 \\ &\geq \frac{1}{2n}\left|\nabla v\right|^4 - a^2 f\left|\nabla v\right|^2.\end{aligned}$$

For convenience, let us denote

$$\sigma := \left|\nabla v\right|^2.$$

In conclusion, we have the following inequality:

$$\sigma^2 \leq 8na^2 f\sigma + 4n\left\langle\nabla\sigma,\nabla v\right\rangle + 4n\left(a+1\right)f_{ij}v_iv_j + 2n\Delta\sigma. \tag{2.7}$$

Multiplying (2.7) by $\phi^2\sigma^{p-1}$ and integrating over M, where ϕ is a cut-off function with support in $B_x\left(1\right)$ and $p \geq c(n) > 0$ is large enough depending only on dimension, we get

$$\begin{aligned}\int_M \sigma^{p+1}\,\phi^2 \leq 8na^2\int_M f\sigma^p\phi^2 + 4n\int_M \left\langle\nabla\sigma,\nabla v\right\rangle\sigma^{p-1}\phi^2 \\ +4n\left(a+1\right)\int_M f_{ij}v_iv_j\sigma^{p-1}\phi^2 + 2n\int_M \sigma^{p-1}\left(\Delta\sigma\right)\phi^2.\end{aligned} \tag{2.8}$$

Notice now that

$$\begin{aligned}\int_M \left\langle\nabla\sigma,\nabla v\right\rangle\sigma^{p-1}\phi^2 &= \frac{1}{p}\int_M \left\langle\nabla\sigma^p,\nabla v\right\rangle\;\phi^2 \\ &= -\frac{1}{p}\int_M \sigma^p\Delta v\;\phi^2 - \frac{1}{p}\int_M \sigma^p\left\langle\nabla v,\nabla\phi^2\right\rangle \\ &= \frac{1}{p}\int_M \sigma^p\left(a\left\langle\nabla v,\nabla f\right\rangle + \sigma\right)\;\phi^2 - \frac{1}{p}\int_M \sigma^p\left\langle\nabla v,\nabla\phi^2\right\rangle \\ &\leq \frac{2}{p}\int_M \sigma^{p+1}\phi^2 + \frac{2}{p}\int_M \left|\nabla\phi\right|^2\sigma^p + \frac{2}{p}a^2\int_M f\sigma^p\phi^2.\end{aligned} \tag{2.9}$$

Integrating by parts, we get

$$(2.10)\qquad \begin{aligned} \int_M \sigma^{p-1}(\Delta\sigma)\phi^2 &= -(p-1)\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2 \\ &- \int_M \sigma^{p-1}\langle\nabla\sigma,\nabla\phi^2\rangle \\ &\leq -(p-2)\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2 + \int_M \sigma^p|\nabla\phi|^2. \end{aligned}$$

Similarly, integration by parts gives

$$(2.11)\quad \int_M f_{ij}v_iv_j\sigma^{p-1}\phi^2 = -\int_M v_{ij}v_jf_i\sigma^{p-1}\phi^2 - \int_M(\Delta v)f_iv_i\sigma^{p-1}\phi^2 \\ -(p-1)\int_M f_iv_iv_j\sigma_j\sigma^{p-2}\phi^2 - \int_M f_iv_iv_j\sigma^{p-1}\left(\phi^2\right)_j.$$

We now study each term on the right-hand side.

$$\begin{aligned} -\int_M v_{ij}v_jf_i\sigma^{p-1}\phi^2 &= -\frac{1}{2}\int_M\left\langle\nabla|\nabla v|^2,\nabla f\right\rangle\sigma^{p-1}\phi^2 \\ &= -\frac{1}{2p}\int_M\langle\nabla\sigma^p,\nabla f\rangle\phi^2 \\ &= \frac{1}{2p}\int_M\sigma^p(\Delta f)\phi^2 + \frac{1}{2p}\int_M\sigma^p\langle\nabla f,\nabla\phi^2\rangle \\ &\leq \int_M f\sigma^p\phi^2 + \int_M\sigma^p|\nabla\phi|^2. \end{aligned}$$

Also,

$$-\int_M(\Delta v)f_iv_i\sigma^{p-1}\phi^2 = \int_M\left(a\langle\nabla v,\nabla f\rangle + |\nabla v|^2\right)\langle\nabla f,\nabla v\rangle\sigma^{p-1}\phi^2 \\ \leq np(a+1)\int_M f\sigma^p\phi^2 + \frac{1}{np(a+1)}\int_M\sigma^{p+1}\phi^2.$$

The third term in the right-hand side of (2.11) is estimated by

$$-(p-1)\int_M f_iv_iv_j\sigma_j\sigma^{p-2}\phi^2 \leq p\int_M|\nabla f||\nabla\sigma|\sigma^{p-1}\phi^2 \\ \leq \frac{p-2}{8n(a+1)}\int_M|\nabla\sigma|^2\sigma^{p-2}\phi^2 + 4np(a+1)\int_M f\sigma^p\phi^2.$$

The last term in (2.11) is

$$\begin{aligned} -\int_M f_iv_iv_j\sigma^{p-1}\left(\phi^2\right)_j &\leq 2\int_M|\nabla f|\sigma^p\phi|\nabla\phi| \\ &\leq \int_M f\sigma^p\phi^2 + \int_M\sigma^p|\nabla\phi|^2. \end{aligned}$$

Plugging all these estimates in (2.11), we conclude

$$4n\,(a+1)\int_M f_{ij}v_iv_j\sigma^{p-1}\phi^2 \leq c(n)(a+1)^2p\int_M f\sigma^p\phi^2 \tag{2.12}$$
$$+\frac{4}{p}\int_M \sigma^{p+1}\phi^2+\frac{p-2}{2}\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2+c(n)(a+1)\int_M \sigma^p\left|\nabla\phi\right|^2.$$

Using (2.9), (2.10), and (2.12), we get from (2.8) that for p large enough and depending on n,

$$(p-2)\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2 \leq c(n)(a+1)^2p\int_M f\sigma^p\phi^2 \tag{2.13}$$
$$+\,c(n)(a+1)\int_M \sigma^p\left|\nabla\phi\right|^2.$$

Note that (2.13) is true for any ϕ with support in $B_x(1)$. We now recall the Sobolev inequality established in [**13**], which says that there exist constants $\mu > 1$, c_1 and c_2, all depending only on $n = 2m$ such that

$$\left(\int_{B_x(1)}\psi^{2\mu}\right)^{\frac{1}{\mu}} \leq \frac{c_1e^{c_2A}}{\mathrm{V}_x(1)^{1-\frac{1}{\mu}}}\left(\int_{B_x(1)}|\nabla\psi|^2+\int_{B_x(1)}\psi^2\right)$$

for all $\psi \in C_0^\infty\left(B_x\left(1\right)\right)$, where

$$A := \mathrm{osc}_{B_x(3)}\left|f\right|,$$

the oscillation of f over the geodesic ball $B_x(3)$. We also note, see [**16**], that there exists a constant $c(n) > 0$ depending only on dimension n such that

$$\mathrm{V}_x(1)=\mathrm{Vol}\left(B_x\left(1\right)\right)\geq \mathrm{V}_{x_0}(1)\,e^{-C(n)\,d(x_0,x)}.$$

Therefore, using (2.2), we arrive at a Sobolev inequality of the form

$$\left(\int_{B_x(1)}\psi^{2\mu}\right)^{\frac{1}{\mu}} \leq Ce^{c(n)r(x)}\left(\int_{B_x(1)}|\nabla\psi|^2+\int_{B_x(1)}\psi^2\right) \tag{2.14}$$

for all $\psi \in C_0^\infty\left(B_x\left(1\right)\right)$, where $r\left(x\right) := d\left(x_0,x\right)$. Using (2.13) and (2.14), by the standard DeGiorgi–Nash–Moser iteration, we obtain for any $0 \leq \theta,\rho < 1$,

$$\sup_{B_x(\theta\rho)}\sigma \leq C\left(1+(1-\theta)^{-2}\rho^{-2}\right)^{\frac{\mu}{\mu-1}\frac{1}{p}}e^{c(n)r(x)}\left(\frac{1}{\mathrm{V}_x(\rho)}\int_{B_x(\rho)}\sigma^p\right)^{\frac{1}{p}}$$

for some p depending only on dimension n and given by (2.13). Here the constant C depends on a, but it is independent of $r\left(x\right)$.

Now a standard argument (see [**9**]) implies

$$\sigma\left(x\right)\leq Ce^{c(n)r(x)}\int_{B_x(1)}\sigma. \tag{2.15}$$

Combining (2.15) with (2.5), one concludes

$$|\nabla \ln h(x)| \leq \sqrt{\sigma}(x) \leq c\, e^{-\frac{a}{16} r^2(x)}. \tag{2.16}$$

Integrating (2.16) along minimizing geodesics, we immediately see that h must be bounded on the end E, which contradicts with (2.4). This proves all ends of (M, g) are φ-nonparabolic.

To finish the proof of the Theorem, we assume to the contrary that (M, g) has more than one end. Since each end must be φ-nonparabolic, the construction of Li and Tam [**10**, **9**] implies that there exists a non-constant φ-harmonic function $u : M \to \mathbb{R}$ such that

$$\begin{aligned} \Delta_\varphi u &= 0 \\ \int_M |\nabla u|^2 e^{-\varphi} &< \infty. \end{aligned}$$

But M as a Kähler shrinking Ricci soliton satisfies all the assumptions of Theorem 0.2 with $\varphi = -a\, f$. Indeed, (2.1) implies that $\varphi_{\alpha\beta} = 0$. Also, (2.2) shows φ is proper and $|\nabla \varphi|$ grows at most linearly in the distance function. Therefore, we conclude that u must be a constant. This contradiction shows that M has only one end. Theorem 2.1 is proved. q.e.d.

3. Ends of expanding solitons

In this section, we show an expanding Kähler Ricci soliton must be connected at infinity if its potential function is proper. First, let us recall some useful information about expanding Ricci solitons. An expanding gradient Ricci soliton is a manifold (M, g) for which there exists a smooth potential f with the property that

$$\mathrm{Ric} + \mathrm{Hess}(f) = -\frac{1}{2} g.$$

Since (M, g) is Kähler, in terms of unitary frames, we have

$$\begin{aligned} R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} &= -g_{\alpha\bar{\beta}} \\ f_{\alpha\beta} &= 0. \end{aligned} \tag{3.1}$$

Hamilton's identity now reads as

$$S + |\nabla f|^2 = -f.$$

It is known [**18**] that $S \geq -\frac{n}{2}$ with equality holding if and only if (M, g) is an Einstein manifold. So one has $|\nabla f|^2 \leq (-f) + \frac{n}{2}$ and

$$-f(x) \leq \frac{1}{4} r^2(x) + c r(x).$$

Let us summarize these facts as

$$
\begin{aligned}
-f(x) &\leq \frac{1}{4}r^2(x) + cr(x) \\
S &> -\frac{n}{2} \\
|\nabla f|^2 &< (-f) + \frac{n}{2},
\end{aligned}
\tag{3.2}
$$

for any nontrivial expanding gradient Ricci soliton.

We now prove Theorem 0.3.

Theorem 3.1. *Let (M, g, f) be an expanding Kähler gradient Ricci soliton. Assume that the potential f is proper. Then (M, g) has only one end.*

Proof of Theorem 3.1*.* Since $-f > -\frac{n}{2}$, the assumption that f is proper implies that

$$\lim_{x\to\infty} (-f)(x) = \infty.$$

We will use several ideas from [**14**] with some improvements. It was proved in [**14**] that the following weighted Poincaré inequality holds on M:

$$\int_M \left(S + \frac{n}{2}\right)\phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f} \tag{3.3}$$

for all $\phi \in C_0^\infty(M)$. For our purpose, however, a different inequality will be more useful. We compute

$$
\begin{aligned}
\Delta_f e^{af} &= \left(a\Delta_f(f) + a^2|\nabla f|^2\right)e^{af} \\
&= -\left(a\left(\frac{n}{2} + S\right) + \left(a - a^2\right)|\nabla f|^2\right)e^{af}.
\end{aligned}
$$

Choosing $a = 1$ yields (3.3). We take $a = \frac{1}{2}$ instead and get

$$\int_M \sigma\phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f}$$

for all $\phi \in C_0^\infty(M)$, where (see (3.2))

$$\sigma := \frac{1}{4}\left(-f + \frac{n}{2}\right) > 0.$$

Since f is proper, it means that there exists a compact set $K \subset M$ such that $-f \geq 3$ on $M\backslash K$. In particular, it follows that

$$\sigma \geq 1 \quad \text{on} \quad M\backslash K. \tag{3.4}$$

In view of (3.4), the argument in [**14**] now shows that all ends of M are necessarily f-nonparabolic. To finish, assume by contradiction that M has more than one end. Again, by the construction of Li and Tam [**10**], one finds a nontrivial f-harmonic function u on M with finite total energy $\int_M |\nabla u|^2 e^{-f} < \infty$. On the other hand, by (3.1) and (3.2), Theorem 0.2 is applicable. So we conclude u must be a constant. This contradiction proves the theorem. q.e.d.

4. Weight function estimate

In this section we prove Theorem 0.4. For convenience, we only provide the details for the case $\lambda = \frac{1}{2}$, as the other two cases $\lambda = 0$ and $\lambda = -\frac{1}{2}$ follow similarly.

Theorem 4.1. *Let $\left(M, g, e^{-f} dv\right)$ be a smooth metric measure space with Bakry–Émery curvature bounded by $\mathrm{Ric}_f \geq \frac{1}{2} g$. Then for any fixed $x_0 \in M$, there exists $a > 0$ depending only on dimension n and $\sup_{B_{x_0}(1)} |f|$ such that*

$$f(x) \geq \frac{1}{4} r^2(x) - a\, r(x) \tag{4.1}$$

for all $x \in M$ with $r(x)$ sufficiently large, where $r(x) := d(x_0, x)$.

Proof of Theorem 4.1*.* The strategy is to use an improved Laplacian comparison theorem established in [**16**]. We first discuss its derivation for completeness. Our derivation here is in fact slightly more direct than that in [**16**]. For a fixed point $p \in M$, we denote the volume form in geodesic coordinates by

$$dV|_{\exp_p(r\xi)} = J(p, r, \xi)\, dr d\xi$$

for $r > 0$ and $\xi \in S_p M$, the unit tangent sphere at p. For $x \in M$ a point outside the cut locus of p with $x = \exp_p(r\xi)$, we have

$$\Delta d(p, x) = \frac{d}{dr} \ln J(p, r, \xi).$$

We shall omit the dependency of these quantities on p and ξ from now on. Along a minimizing geodesic γ starting from p, by the Bochner formula, we have

$$m'(r) + \frac{1}{n-1} m^2(r) + \mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0, \tag{4.2}$$

where the differentiation is with respect to the r variable and

$$m(r) := \frac{d}{dr} \ln J(r).$$

Multiplying (4.2) by r and integrating from $r = \varepsilon > 0$ to $r = t > \varepsilon$, we get

$$\int_\varepsilon^t m'(r)\, r dr + \frac{1}{n-1} \int_\varepsilon^t m^2(r)\, r dr + \frac{1}{2} \int_\varepsilon^t r dr \leq \int_\varepsilon^t f''(r)\, r\, dr, \tag{4.3}$$

where we have used $\mathrm{Ric}_f \geq \frac{1}{2}$. Integrating by parts in (4.3), we arrive at

$$m(t)\,t - m(\varepsilon)\,\varepsilon + \frac{1}{n-1}\int_{\varepsilon}^{t} r\left(m(r) - \frac{n-1}{r}\right)^2 dr \leq -\frac{1}{4}t^2 + \frac{1}{4}\varepsilon^2 + \int_{\varepsilon}^{t}\left(\frac{n-1}{r} - m(r)\right) dr + \int_{\varepsilon}^{t} f''(r)\, r\, dr.$$

In particular, this yields the following

$$m(t)\,t + \ln\frac{J(t)}{t^{n-1}} \leq m(\varepsilon)\,\varepsilon + \ln\frac{J(\varepsilon)}{\varepsilon^{n-1}} - \frac{1}{4}t^2 + \frac{1}{4}\varepsilon^2 + \int_{\varepsilon}^{t} f''(r)\, r\, dr.$$

Now letting $\varepsilon \to 0$, noting that $\varepsilon m(\varepsilon) \to n-1$ and $\frac{J(\varepsilon)}{\varepsilon^{n-1}} \to 1$, we obtain

$$m(t) + \frac{1}{t}\ln\frac{J(t)}{t^{n-1}} \leq \frac{n-1}{t} - \frac{1}{4}t + \frac{1}{t}\int_{0}^{t} f''(r)\, r dr.$$

Integrating by parts on the last term, we get

$$\Delta_f d(p,x) \leq \frac{n-1}{r} - \frac{1}{4}r - \frac{1}{r}\ln\left(\frac{J(p,r,\xi)}{r^{n-1}}\right) - \frac{1}{r}\left(f(x) - f(p)\right). \tag{4.4}$$

The proof of the theorem is by contradiction. So we fix a large enough constant $a > 0$, to be determined explicitly later, and assume that for this fixed a there exists no r_0 such that (4.1) is true for $r(x) \geq r_0$. This means there exists a sequence $q_k \to \infty$ such that

$$f(q_k) < \frac{1}{4}r^2(q_k) - a\, r(q_k), \tag{4.5}$$

where $r(x) := d(x_0, x)$. We now adapt to our setting some ideas in [**16**] for proving a splitting theorem for certain smooth metric measure spaces containing a line.

More precisely, let $\gamma_k(t)$ be a minimizing geodesic from x_0 to q_k, where

$$0 \leq t \leq t_k := d(x_0, q_k).$$

For a fixed point $x \in M$, not in the cut locus of $q_k = \gamma_k(t_k)$, let $\tau_k(s)$ be the unique minimizing geodesic from $\tau_k(0) = q_k$ to $\tau_k(r_k) := x$. By (4.5), we have

$$f(q_k) < \frac{1}{4}t_k^2 - a t_k \quad \text{for all } k. \tag{4.6}$$

Let

$$r_k := d(\gamma(t_k), x) = d(q_k, x).$$

According to (4.4), we have

$$\Delta_f d(q_k, x) + \frac{1}{r_k}\ln J\left(q_k, r_k, \tau_k'(0)\right) \leq \frac{n-1}{r_k}(1 + \ln r_k) - \frac{1}{4}r_k + \frac{f(q_k)}{r_k} - \frac{f(x)}{r_k}. \tag{4.7}$$

We now claim that for any $\varepsilon > 0$ and k sufficiently large,

$$\frac{n-1}{r_k}\left(1+\ln r_k\right)-\frac{1}{4}r_k+\frac{f\left(q_k\right)}{r_k}-\frac{f\left(x\right)}{r_k}\leq\frac{1}{2}\left(t_k-r_k\right)-a+\varepsilon. \tag{4.8}$$

To verify this, first note that by the triangle inequality,

$$\left|t_k-r_k\right|\leq d\left(x_0,x\right). \tag{4.9}$$

Since x is fixed, for k sufficiently large we have

$$\frac{n-1}{r_k}\left(1+\ln r_k\right)-\frac{f\left(x\right)}{r_k}\leq\frac{\varepsilon}{2}.$$

Using (4.6), we have

$$\begin{aligned}
&-\frac{1}{4}r_k+\frac{f\left(q_k\right)}{r_k}-\frac{1}{2}\left(t_k-r_k\right)+a\\
&=\frac{1}{4r_k}\left(-r_k^2+4f\left(q_k\right)-2r_k\left(t_k-r_k\right)+4ar_k\right)\\
&\leq\frac{1}{4r_k}\left(-r_k^2+t_k^2-4at_k-2r_k\left(t_k-r_k\right)+4ar_k\right)\\
&=\frac{1}{4r_k}\left(\left(t_k-r_k\right)^2-4a\left(t_k-r_k\right)\right)\\
&\leq\frac{1}{4r_k}\left(d\left(x_0,x\right)^2+4ad\left(x_0,x\right)\right),
\end{aligned}$$

where we have used (4.9) in the last step. Hence, for k sufficiently large,

$$-\frac{1}{4}r_k+\frac{f\left(q_k\right)}{r_k}-\frac{1}{2}\left(t_k-r_k\right)+a\leq\frac{\varepsilon}{2}.$$

In conclusion, (4.8) holds true. It then follows from (4.7) that, for k sufficiently large,

$$\Delta_f d\left(q_k,x\right)+\frac{1}{r_k}\ln J\left(q_k,r_k,\xi_k\right)\leq\frac{1}{2}\left(t_k-r_k\right)-a+\varepsilon, \tag{4.10}$$

where $\xi_k := \tau_k'\left(0\right)$, $t_k := d\left(x_0,q_k\right)$, and $r_k := d\left(q_k,x\right)$. Let us emphasize that (4.10) holds for any x not in the cut-locus of q_k.

For a compact domain $\Omega\subset M$, there exists a constant $c>0$ independent of k so that

$$\Omega\subset B_{q_k}\left(t_k+c\right)\backslash B_{q_k}\left(t_k-c\right).$$

Consequently, there exists a constant $c_0>0$ so that $r_k>t_k-c_0$ whenever $x\in\Omega$. Also, note that the function $h\left(J\right):=J\ln J$ is bounded below by $-\frac{1}{e}$. For nonnegative smooth function ϕ with support in Ω, multiplying (4.10) by ϕe^{-f} and integrating over Ω, we conclude that

$$\begin{aligned}
\int_M\left(\Delta_f d\left(q_k,x\right)\right)\phi e^{-f}\leq&\frac{C}{t_k-c_0}\sup_\Omega\phi\\
&+\int_M\left(\frac{1}{2}\left(t_k-d\left(q_k,x\right)\right)-a\right)\phi e^{-f}+\varepsilon C\sup_\Omega\phi,
\end{aligned} \tag{4.11}$$

where we have used

$$\int_{\mathbb{S}^{n-1}} J(q_k, r_k, \xi_k) \ln J(q_k, r_k, \xi_k)\, \phi e^{-f} d\xi_k \geq -\frac{1}{e}\int_{\mathbb{S}^{n-1}} \phi e^{-f} d\xi_k \geq -C \sup_{\Omega} \phi.$$

The constant C depends on Ω, but it is independent of k. We now define

$$\beta_k(x) := t_k - d(q_k, x).$$

Clearly, $\beta_k(x_0) = 0$ and $|\nabla \beta_k| \leq 1$. Hence, a subsequence of β_k converges, as $k \to \infty$, uniformly on compact sets to a Lipschitz function β satisfying

$$|\beta|(x) \leq d(x_0, x) \text{ on } M.$$

Moreover, taking limit in (4.11) implies that in the weak sense

$$\Delta_f \beta \geq a - \frac{1}{2}\beta \quad \text{on } M. \tag{4.12}$$

Let ϕ be a cut-off function with $\phi = 1$ on $B_{x_0}\left(\frac{1}{2}\right)$, $\phi = 0$ on $M \backslash B_{x_0}(1)$, and $|\nabla \phi| \leq 2$. Multiplying (4.12) by ϕ and integrating, we get

$$\begin{aligned} a \mathrm{Vol}_f\left(B_{x_0}\left(\frac{1}{2}\right)\right) &\leq \frac{1}{2}\int_M \phi \beta e^{-f} + \int_M (\Delta_f \beta)\, \phi e^{-f} \\ &\leq \frac{1}{2}\int_{B_{x_0}(1)} d(x_0, \cdot)\, e^{-f} - \int_M \langle \nabla \beta, \nabla \phi \rangle e^{-f} \\ &\leq 3 \mathrm{Vol}_f(B_{x_0}(1)). \end{aligned}$$

This means that

$$a \leq 3 \frac{\mathrm{Vol}_f(B_{x_0}(1))}{\mathrm{Vol}_f\left(B_{x_0}\left(\frac{1}{2}\right)\right)}. \tag{4.13}$$

On the other hand, the Laplacian comparison theorem in [**21**] implies that for $\frac{1}{4} \leq r \leq 1$,

$$\Delta_f r \leq c(n) + 16 \sup_{B_{x_0}(1)} |f|.$$

Therefore, after integrating, we have

$$\frac{\mathrm{Vol}_f(B_{x_0}(1))}{\mathrm{Vol}_f\left(B_{x_0}\left(\frac{1}{2}\right)\right)} \leq c(n) \exp\left(16 \sup_{B_{x_0}(1)} |f|\right).$$

Now the theorem follows by combining this with (4.13). q.e.d.

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