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THE SPACE OF NONPOSITIVELY CURVED METRICS OF A NEGATIVELY CURVED MANIFOLD

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Abstract

We show that the space of nonpositively curved metrics of a closed negatively curved Riemannian n-manifold, $n \geq 10$, is highly non-connected.

0. Introduction

Let M^n be a closed smooth manifold of dimension $dim\, M = n$. We denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on M and we consider $\mathcal{MET}(M)$ with the smooth topology. Also, we denote by $\mathcal{MET}^{\,sec<0}(M)$ the subspace formed by all negatively curved Riemannian metrics on M. In [**10**] we proved that $\mathcal{MET}^{\,sec<0}(M)$ always has infinitely many path-components, provided $n \geq 10$ and it is non-empty. Moreover, we showed that all the groups $\pi_{2p-4}(\mathcal{MET}^{\,sec<0}(M))$ are non-trivial for every prime number $p > 2$, and such that $p < \frac{n+5}{6}$ (this is true in every component of $\mathcal{MET}^{\,sec<0}(M)$). In fact, these groups contain the infinite sum $(\mathbb{Z}_p)^\infty$ of $\mathbb{Z}_p = \mathbb{Z}/p\,\mathbb{Z}$'s. We also showed that $\pi_1(\mathcal{MET}^{\,sec<0}(M))$ contains the infinite sum $(\mathbb{Z}_2)^\infty$ when $n \geq 12$ (see also [**11**]). All these results follow from the Main Theorem in [**10**], which states that the orbit map $\Lambda_g : DIFF(M) \to \mathcal{MET}^{\,sec<0}(M)$ is "very non-trivial" at the π_k-level. Here $DIFF(M)$ is the group of self-diffeomorphisms on M and $\Lambda_g(\phi) = \phi_* g$ (see the introduction of [**10**] for more details).

Let $\mathcal{MET}^{\,sec\leq 0}(M)$ be the subspace of $\mathcal{MET}(M)$ formed by all nonpositively curved Riemannian metrics on M. In this paper we generalize to $\mathcal{MET}^{\,sec\leq 0}(M)$ the results mentioned above, provided $\pi_1 M$ is (word) hyperbolic:

Main Theorem. *Let M^n be a closed smooth manifold with hyperbolic fundamental group $\pi_1 M$. Assume $\mathcal{MET}^{\,sec\leq 0}(M)$ is non-empty. Then*

(i) *The space $\mathcal{MET}^{\,sec\leq 0}(M)$ has infinitely many components, provided $n \geq 10$.*
(ii) *The group $\pi_1(\mathcal{MET}^{\,sec\leq 0}(M^n))$ is not trivial when $n \geq 12$. In fact, it contains the infinite sum $(\mathbb{Z}_2)^\infty$ as a subgroup.*

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(iii) *The groups* $\pi_{2p-4}(\mathcal{MET}^{\,sec\,\leq\,0}(M^n))$ *are non-trivial for every prime number* $p > 2$*, and such that* $p < \frac{n+5}{6}$*. In fact, these groups contain the infinite sum* $(\mathbb{Z}_p)^\infty$ *as a subgroup.*

Remarks.

1. The results for $\pi_k\,\mathcal{MET}^{\,sec\,\leq\,0}\,(M)$, $k > 0$, given above are true relative to any base point, that is, for every component of $\mathcal{MET}^{\,sec\,\leq\,0}(M)$.

2. The decoration "$sec\,\leq\,0$" can be tightened to "$a \leq sec \leq\, 0$," for any $a < 0$.

3. The theorem above follows from a nonpositively curved version of the Main Theorem of [**10**] (which we do not state to save space). This nonpositively curved version is obtained from the Main Theorem in [**10**] by replacing $\mathcal{MET}^{\,sec\,<\,0}(M)$ by $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ and adding the hypothesis "$\pi_1(M)$ is hyperbolic." This is the result that we prove in this paper. And, as in [**10**], we obtain the following corollary.

Corollary. *Let* M *be a closed smooth* n*-manifold with* $\pi_1(M)$ *hyperbolic. Let* $I \subset (-\infty, 0]$ *and assume that* $\mathcal{MET}^{sec\in I}(M)$ *is not empty. Then the inclusion map* $\mathcal{MET}^{sec\in I}(M) \hookrightarrow \mathcal{MET}^{\,sec\,\leq\,0}(M)$ *is not null-homotopic, provided* $n \geq 10$.

Moreover, the induced maps of this inclusion, at the k*-homotopy level, are not constant for* $k = 0$*, and non-zero for* k *and* n *as in cases (ii), (iii) in the Main Theorem. Furthermore, the image of these maps satisfy a statement analogous to the one in the Addendum to the Main Theorem in* [**10**].

Here $\mathcal{MET}^{sec\in I}$ has the obvious meaning. In particular taking $I = (-\infty, 0)$ we get that the inclusion $\mathcal{MET}^{\,sec\,<\,0}(M) \hookrightarrow \mathcal{MET}^{\,sec\,\leq\,0}(M)$ is not null-homotopic, provided $n \geq 10$ and M admits a negatively curved metric.

In some sense it is quite surprising that we were able to extend the results in [**10**] to the nonpositively curved case because negative curvature is a "stable" condition (the space $\mathcal{MET}^{\,sec\,<\,0}(M)$ is open in $\mathcal{MET}(M)$) while $\mathcal{MET}^{\,sec\,\leq\,0}$ is not stable. Indeed it is not even known whether $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ is locally contractible or even locally connected. We state these as questions:

Questions.

1. Is the space $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ of nonpositively curved metrics on M locally contractible?

2. Is the space $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ of nonpositively curved metrics on M locally connected?

So, we prove here that $\mathcal{MET}^{\,sec\,\leq\,0}(M^n)$ is not (globally) connected when it is not empty, $n \geq 10$, and $\pi_1 M$ is hyperbolic. But on the other hand, it is not known whether $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ is locally connected.

In this paper there are two additional obstacles to pass from negative curvature to nonpositive curvature. First, since we can now have parallel geodesic rays emanating perpendicularly from a closed geodesic, the obstructions we defined in [**10**] (which lie in the pseudoisotopy space of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$) may not be homeomorphisms at infinity.

The second problem is that we may now have a whole family of closed geodesics freely homotopic to a given one. But in our previous papers we strongly used the fact that there is a *unique* such closed geodesic. Moreover, we strongly used the fact that such unique closed geodesics depend smoothly on the metric. This does not happen in nonpositive curvature. Even worse: there are examples of smooth families g_t, $t \in [0,1]$, of nonpositively curved metrics such that there is no continuous path of closed g_t-geodesics joining a closed g_1-geodesic to a closed g_0-geodesic (all closed geodesics in the same free homotopy class). See for instance the "swinging neck" in Appendix A. We deal with this by incorporating the closed geodesics into the system, but we pay a price for this: instead of dealing with discs (to prove that an element is zero in a homotopy group) we have to deal with more complicated spaces which we call "cellular discs." Because of this the use of shape theory becomes necessary.

The Main Theorem in [**10**] follows directly from Theorem 1 and Theorem 2 in [**10**]. Likewise, the Main Theorem in this paper follows directly from Theorem A below (which is a nonpositively curved version of Theorem 1 of [**10**]) and Theorem 2 of [**10**]. Before we state Theorem A we recall some notation and constructions of [**10**].

Let M be a closed nonpositively curved n-manifold. Let $\alpha : \mathbb{S}^1 \to M$ be an embedding. We assume that the normal bundle of α is trivial. Let V be an orthonormal trivialization of this bundle. Also, let $r > 0$, such that $2r$ is less than the width of the normal geodesic tubular neighborhood of α. Using V, and the exponential map of geodesics orthogonal to α, we identify the normal geodesic tubular neighborhood of width $2r$ (minus α), with $\mathbb{S}^1 \times \mathbb{S}^{n-2} \times (0, 2r]$. Define $\Phi = \Phi^M(\alpha, V, r) : DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial) \to DIFF(M)$ in the following way. For $\varphi \in DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial)$ let $\Phi(\varphi) : M \to M$ be the identity outside $\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [r, 2r] \subset M$, and $\Phi(\varphi) = \lambda^{-1}\varphi\lambda$, where $\lambda(z,u,t) = (z, u, \frac{t-r}{r})$, for $(z,u,t) \in \mathbb{S}^1 \times \mathbb{S}^{n-2} \times [r, 2r]$. (For more details see the first paragraph on p. 277 in [**10**].)

Denote by g the metric on M, and by Λ_g the map $\Lambda_g : DIFF(M) \to \mathcal{MET}^{sec \leq 0}(M)$, given by $\Lambda_g(\phi) = \phi_* g$. A key ingredient in the proof of the Main Theorem in [**10**] is the diagram on p. 277 in [**10**]. Here is the new version of this diagram:

$$
\begin{array}{ccccc}
DIFF(\,(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial\,) & \stackrel{\Phi}{\to} & DIFF(M) & \stackrel{\Lambda_g}{\to} & \mathcal{MET}^{sec \leq 0}(M) \\
\iota \downarrow & & & & \\
P(\mathbb{S}^1 \times \mathbb{S}^{n-2}) & & & & \\
\iota' \downarrow & & & & \\
CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1]) & & & &
\end{array}
$$

where ι and ι' are inclusions. Here $CELL(L)$ denotes the space of cellular maps on the manifold L; see Section 2 for more details. Also $P(L)$ denotes the space of (topological) pseudoisotopies on the manifold L (a pseudoisotopy on L is a self-homeomorphism on $L \times [0,1]$ that is the identity when restricted to $L \times \{0\}$). We can now state the new version of Theorem 1 of [**10**]:

Theorem A. *Let M be a closed n-manifold, $n \neq 4$, with hyperbolic fundamental group. Let g be a nonpositively curved metric on M. Assume that α is a simple closed geodesic. Then $Ker\,(\,\pi_k(\Lambda_g\Phi)\,) \subset Ker\,(\,\pi_k(\iota)\,)$, for $k < n-5$.*

Remark. The statement of Theorem A remains true if α is not a geodesic, but just a non-nullhomotopic embedded smooth curve with trivial normal bundle. We do not prove this here.

Theorem A follows from Theorem B below and Corollary 2.3.

Theorem B. *Let M be a closed n-manifold, $n \neq 4$, with hyperbolic fundamental group. Let g be a nonpositively curved metric on M. Assume that α is a simple closed geodesic. Then $Ker\,(\,\pi_k(\Lambda_g\Phi)\,) \subset Ker\,(\,\pi_k(\iota'\iota)\,)$, for $k < n-5$.*

The proof of Theorem B follows the same lines as the proof of Theorem 1 of [**10**], but some essential changes have to be made. We sketch an argument that, we hope, motivates our proof of Theorem B. This sketch is similar to the one given in the Introduction of [**10**], but with some fundamental and necessary modifications. To avoid complications, let's just consider the case $k = 0$. In this situation we essentially want to show the following:

Let $\theta \in DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times I, \partial) \subset CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1])$, and write $\varphi = \Phi(\theta) : M \to M$. Suppose that θ cannot be joined to the identity by a path in $CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1])$. Then g cannot be joined to $\varphi_* g$ by a path of nonpositively curved metrics.

Here is an argument that we could tentatively use to prove the statement above. Suppose that there is a smooth path g_u, $u \in [0,1]$, of nonpositively curved metrics on M, with $g_0 = g$ and $g_1 = \varphi_* g$. We will use g_u to show that θ can be joined to the identity in $CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1])$. Let Q be the cover of M corresponding to the infinite cyclic group generated by α. Each g_u lifts to a g_u on Q (we use the same letter). Then α lifts isometrically to (Q, g) and we can identify Q with $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ such that α corresponds to $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\}$ and such that each $\{z\} \times \mathbb{R}v$, $v \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, corresponds to a g-geodesic ray emanating perpendicularly from α. If each (M, g_u) was negatively curved, then for each u, there would be exactly one closed g_u-geodesic α_u freely homotopic to α (this is the case in [**10**]). But, as mentioned above, in the nonpositively curved case this is not true (see Appendix A). We now have, for each u, a set $\mathcal{A}_u$ (which we prove is homeomorphic to a closed disc) of closed g_u-geodesics in Q freely homotopic to α. This is a serious problem. We deal with it by replacing $[0,1]$ by the set $X = \{(u,\beta),\ u \in [0,1],\ \beta \in \mathcal{A}_u\}$. This set is not necessarily homeomorphic to $[0,1]$, but the projection map $X \to [0,1]$, $(u,\beta) \mapsto u$ is a shape equivalence. The set X, together with $X \to [0,1]$, is what we call a cellular 1-disc. Now, for each $(u,\beta) \in X$ we "deform" g_u to a nonpositively curved metric g_x such that α is a g_x-geodesic. Moreover, let us assume that g_x coincides with g in the normal tubular neighborhood W of length one of α. Note that $Q \setminus int\, W$ can be identified with $(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1,\infty)$. Using g_x-geodesic rays emanating perpendicularly from α, we can define a path of maps $f_x : (\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1,\infty) \to (\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1,\infty)$ by $f_x = [exp]^{-1} \circ exp^x$, where exp^x denotes the normal (to α) exponential map with respect to g_x, and $exp = exp^0$. Using "the space at infinity" $\partial_\infty Q$ of Q (see Section 3), we can extend f_x to $(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [1,\infty]$, which we identify with $(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [0,1]$. In the negatively curved case the maps f_x are homeomorphisms, but in our nonpositively curved case the maps f_x are cellular maps. This is why we have to deal with $CELL((\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times [0,1])$. Finally, it is proved that $f_{(1,\alpha)}$ can be joined to θ in $CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1])$ (see Claim 6 in Section 4). Note that $f_{(0,\alpha)}$ is the identity. So we can "join" θ to the identity by a map defined on a cellular 1-disc. We show in Section 1 that this implies that we can join θ and the identity by a map defined on a 1-disc, that is, by an honest path defined on $[0,1]$. This is a contradiction.

In Section 1 we define cellular discs and give some preliminary results. In Section 2 we deal with the space of cellular maps. In Section 3 we give some results about the space at infinity of a special case of a non-simply connected nonpositively curved manifold. In Section 4 we prove Theorem B.

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1. Preliminaries

A. Cellular Discs.

We will consider the k-disc $\mathbb{D}^k = \{\, x \in \mathbb{R}^k \ :\ |x| \le 1\}$ with base point $u_0 = (1,0,0,\dots,0)$. A *cellular k-disc* is a metrizable compact pointed topological space (X,x_0) together with a surjective continuous map $\eta : (X,x_0) \to (\mathbb{D}^k,u_0)$ such that the pre-image $\eta^{-1}(u)$, $u \in \mathbb{D}^k$, is homeomorphic to the ℓ_u-disc $\mathbb{D}^{\ell_u}$, with $0 \le \ell_u \le \ell$, for some $\ell < \infty$ and all $u \in \mathbb{D}^k$. We write $X_u = \eta^{-1}(u)$, $X_0 = \eta^{-1}(u_0)$ and $\partial X = \eta^{-1}(\partial \mathbb{D}^k) = \eta^{-1}(\mathbb{S}^{k-1})$.

A pair $((X,x_0),X')$, $x_0 \in X' \subset \partial X$, together with a map $\eta : X \to \mathbb{D}^k$ is a *cellular k-disc pair* if X (that is, $((X,x_0),\eta)$) is a cellular k-disc and ∂X is fibered homeomorphic to $X' \times X_0$, that is, there is a homeomorphism $X' \times X_0 \to \partial X$ that sends $\{x'\} \times X_0$ to X_u, $u = \eta(x') \in \mathbb{S}^{k-1}$. In particular, $\eta|_{X'} : X' \to \mathbb{S}^{k-1}$ is a homeomorphism. We identify X' with $\mathbb{S}^{k-1}$ and say that $(X,\mathbb{S}^{k-1})$ is a cellular k-disc pair.

Note that it makes sense to say that a map $h : \mathbb{S}^{k-1} \to Y$ extends to a cellular k-disc pair $(X,\mathbb{S}^{k-1})$.

In the proofs of the following two propositions we use shape theory (see for instance [**6**], [**15**]). Recall that the objects of the shape category are pointed spaces, and for two such objects A and B we denote the set of morphisms by $sh\{A,B\}$. There is a functor, the *shape functor*, from the pointed homotopy category of topological spaces to the shape category. Hence, for each pair of pointed spaces A and B we get a *shape map* between $[A,B]$, the set of pointed homotopy classes of maps, and $sh\{A,B\}$. In particular, there are shape maps from the homotopy groups of B to the homotopy pro-groups of B (these are the shape versions of the homotopy groups of B).

Recall that a metric space Z is LC^m if for every $z \in Z$ and $\epsilon > 0$ there is a $\delta > 0$ such that any continuous map $f : P \to B_\delta(z)$, P a locally finite polyhedron of dimension $\leq m$, is homotopic in $B_\epsilon(z)$ to a constant map. And Z is LC^∞ if it is LC^m for every m. Also recall that an onto map $f : X \to Y$ between metric spaces is *cell-like* if all pre-images $f^{-1}(y)$, $y \in Y$, have the shape of a point. We will use the following facts:

Fact 1. *A cell-like map between finite dimensional spaces is a shape equivalence* [**16**].

Fact 2. *Let W and Z be pointed spaces. Assume W is finite dimensional and Z is sufficiently nice (for instance, Z is LC^∞). Then*

$$[W,Z] \overset{shape}{\longrightarrow} sh\{W,Z\}$$

is a bijection.

Fact 2 follows from the proof of Lemma 3.1 in [**5**] (the given proof is for homology but the same proof works for homotopy) and the Whitehead Theorem in pro-homotopy (see [**4**]).

Proposition 1.1. *Let $(X,\mathbb{S}^{k-1})$ be a cellular k-disc pair with $X/\mathbb{S}^{k-1}$ finite dimensional. Let $f : (X,\mathbb{S}^{k-1}) \to (Z,z_0)$, $z_0 \in Z$, where Z is LC^∞. If $\pi_k(Z,z_0) = 0$, then f is null-homotopic rel $\mathbb{S}^{k-1}$.*

Proof. Let $\eta : X \to \mathbb{D}^k$ be the map that defines the cellular disc X. Write $W = X/\mathbb{S}^{k-1}$. The map η induces a map $\eta' : W \to \mathbb{D}^k/\mathbb{S}^{k-1} = \mathbb{S}^k$. The map η' is a cell-like map because $\partial X/\mathbb{S}^{k-1}$ is homeomorphic to $\mathbb{S}^{k-1} \times X_0/\mathbb{S}^{k-1} \times \{x_0\}$, hence contractible. Moreover, by hypothesis, the space W is finite dimensional. Therefore, by fact 1 above, the map η' is a shape equivalence, that is, an equivalence in the shape category. Consider the following commutative diagram:

$$\begin{array}{ccc} [\mathbb{S}^k, Z] & \overset{shape}{\longrightarrow} & sh\{\mathbb{S}^k, Z\} \\ (\eta')^* \downarrow & & \downarrow (\eta')^* \\ [W, Z] & \overset{shape}{\longrightarrow} & sh\{W, Z\} \end{array}$$

where $(\eta')^*$ is induced by composition with η'. By fact 2 above both horizontal arrows are bijections. Also, since η' is a shape equivalence the right vertical arrow is also a bijection. Hence the left vertical arrow is also a bijection. But $[\mathbb{S}^k, Z] = \pi_k(Z, z_0) = 0$; therefore $[W, Z]$ consists of a single element. This proves the proposition. q.e.d.

Proposition 1.2. *Let Z be LC^∞ and $f : \mathbb{S}^{k-1} \to Z$. If f extends to a cellular k-disc pair $(X, \mathbb{S}^{k-1})$, then f extends to $\mathbb{D}^k$.*

Proof. Let $\eta : X \to \mathbb{D}^k$ be the map that defines the cellular disc X. The map η is a cell-like map; hence it induces isomorphisms of all i-th homotopy pro-groups. Therefore all of these pro-groups are trivial for $i > 0$. It follows that the inclusion $\iota : \mathbb{S}^{k-1} \to X$ represents zero in the $(k-1)$ homotopy pro-group of X. Consequently $f\iota$ represents zero in the $(k-1)$ homotopy pro-group of Z. By fact 2 above, $f\iota$ represents zero in $\pi_{k-1}(Z, z_0)$. This proves the proposition. q.e.d.

Proposition 1.3. *Every principal (locally trivial) $\mathbb{S}^1$-bundle over a finite dimensional cellular disc is trivial.*

Proof. Such bundles are in one-to-one correspondence with $[X, \mathbb{C}P^\infty]$, where X is the cellular disc base space. Consider the following commutative diagram:

$$\begin{array}{ccc} [X, \mathbb{C}P^\infty] & \rightarrow & sh\{X, \mathbb{C}P^\infty\} \\ \uparrow & & \uparrow \\ [\mathbb{D}^k, \mathbb{C}P^\infty] & \rightarrow & sh\{\mathbb{D}^k, \mathbb{C}P^\infty\} \end{array}$$

The two horizontal maps are bijections because of Fact 2, and the right-hand vertical map is also a bijection because of Fact 1. Since $[\mathbb{D}^k, \mathbb{C}P^\infty]$ consists of a single point, so does $[X, \mathbb{C}P^\infty]$. This proves the proposition. q.e.d.

B. C^k-Convergence of g-Geodesics, with Varying g.

Consider Riemannian metrics g on a fixed manifold. We need to study how g-geodesics behave when the Riemannian metric g changes. We are interested in their C^k-convergence. In this section U denotes an open set of $\mathbb{R}^n$.

Proposition 1.4. *Let $S = \{g^a = (g^a_{ij})\}_{a\in A}$ be a collection of Riemannian metrics on U. Let $\mathbf{X} = \{\mathbf{x} \,/\, \mathbf{x}$ is a unit speed g^a-geodesic, $a \in A\}$. Assume that the set $\{det\, g^a(x) \,/\, x \in U,\, a \in A\}$ is bounded away from zero. Then if S is C^k-bounded for some finite $k \geq 0$, then the set of all derivatives $\frac{d^l x_i}{dt^l}(t)$, $1 \leq l \leq k+1$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$, $t \in$ Domain of $\mathbf{x}$, is bounded.*

Remarks.

1. The Riemannian metrics in S are not assumed to be complete.

2. The geodesics in $\mathbf{X}$ are defined on any interval.

3. Here "S is C^k-bounded" means that for $0 \leq l \leq k$, all l-partial derivatives of the g^a_{ij} are bounded uniformly in $a \in A$.

4. Analogously, we say that the set $\mathbf{X}$, introduced in Proposition 1.4, is "C^k-bounded" if the set of all derivatives $\frac{d^l x_i}{dt^l}(t)$, $0 \leq l \leq k$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$, $t \in$ Domain of $\mathbf{x}$, is bounded. In particular, $\mathbf{X}$ is "C^0-bounded" if the union of the images of $\mathbf{x}$, $\mathbf{x}\in\mathbf{X}$, is bounded. Note that the conclusion of Proposition 1.4 is weaker than "$\mathbf{X}$ is C^{k+1}-bounded" (which implies C^0-boundedness). That is, it does not claim that $\mathbf{X}$ is C^0-bounded. Indeed, if the open set U is not bounded, then $\mathbf{X}$ will never be C^0-bounded, and hence not C^k-bounded either; but Proposition 1.4 holds for U (assuming S is C^k-bounded).

Proof. We denote by (g^{ij}_a) the matrix inverse of $g^a = (g^a_{ij})$. First note that, since $\{det\, g^a(x) \,/\, x \in U,\, a \in A\}$ is bounded away from zero and S is C^k-bounded, we have that all l-partial derivatives of the g^{ij}_a, $0 \leq l \leq k$, are bounded. Moreover, the set $\{\, |v| \;:\; g^a(x)(v,v) = 1,\, v \in \mathbb{R}^n,\, x \in U,\, a \in A\}$ is bounded. (Here $|v|$ is the Euclidean length $\langle v, v\rangle^{1/2}$.) Hence, the set of Euclidean lengths of the velocity vectors of unit speed geodesics is bounded. Therefore, the set

$$\left\{ \frac{d\,x_i(t)}{dt},\, \mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \text{ is a unit speed } g^a\text{-geodesic}, \right.$$
$$\left. a \in A,\, t \in \text{Domain of } \mathbf{x} \right\}$$

is bounded. This proves the proposition for $k = 0$. q.e.d.

Assume S is C^1-bounded. Let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ be a unit speed g^a-geodesic. Then the x_i's satisfy a second order ODE of the form

$$\frac{d^2 x_i}{dt^2} = \Phi\left(\frac{dx_j}{dt},\, \Gamma^r_{st}(\mathbf{x}) \right)$$

where $\Gamma^r_{st} = (\Gamma^r_{st})^a$ are the Christoffel symbols of the metric g^a and the function Φ is a polynomial function independent of $\mathbf{x}$ and $a \in A$. But the Christoffel symbols $(\Gamma^r_{st})^a$ can be written canonically as a polynomial expression on the g^{ij}_a and the first partial derivatives of the g^a_{ij}. Since all these terms are bounded, we conclude that the set of all Christoffell symbols $(\Gamma^r_{st})^a$ is bounded. Therefore the set

$$\left\{ \frac{d^2\, x_i(t)}{dt^2},\ \mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \text{ is a unit speed } g^a\text{-geodesic}, \right. \\ \left. a \in A,\ t \in \text{Domain of } \mathbf{x} \right\}$$

is also bounded. This proves the proposition for $k = 1$.

Assume S is C^2-bounded. We differentiate the geodesic equation above to obtain the third order ODE

$$\frac{d^3 x_i}{dt^3} = \Psi\left(\frac{dx_j}{dt},\ \frac{dx_j^2}{dt^2},\ \Gamma^r_{st}(\mathbf{x}),\ \frac{\partial^k}{\partial x^k}\Gamma^r_{st}(\mathbf{x}) \right)$$

which is satisfied by any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$. Since Ψ is a universal polynomial, and Ψ is applied to a set of bounded variables, we conclude that the proposition holds for $k = 2$. Proceeding in this way, we prove the proposition for any $k \geq 0$. This proves the proposition.

In the next two propositions we use the following notation. For a Riemannian metric g_0 and sequence of Riemannian metrics $\{g_n\}$ on U we write $g_n \overset{C^k}{\to} g_0$ to express uniform C^k-convergence on compact supports. Also, for $p \in U$, $v \in \mathbb{R}^n$ we denote by $\alpha(p, v, g)$ the g-geodesic with value p at zero, and velocity v at zero. Also $\alpha(p_n, v_n, g_n) \overset{C^k}{\to} \alpha(p, v, g_0)$ means convergence on any closed interval $[a, b]$ where all paths are defined. (Note that in this case there is $\epsilon > 0$ such that $\alpha(p, v, g_0)$ and all $\alpha(p_n, v_n, g_n)$ are defined on $[-\epsilon, \epsilon]$.)

Lemma 1.5. *If $g_n \overset{C^1}{\to} g_0$, $p_n \to p$, $v_n \to v$, then $\alpha(p_n, v_n, g_n) \overset{C^1}{\to} \alpha(p, v, g_0)$.*

Proof. C^1-convergence follows from the general theory of first order ODE with parameters. This proves the lemma. q.e.d.

Proposition 1.6. *Let $g_n \overset{C^k}{\to} g_0$, $k \geq 1$, and $\alpha_n(t)$, $t \in [a, b]$, be g_n-geodesics such that $\alpha_n \overset{C^0}{\to} \alpha_0$. Then $\alpha_n \overset{C^{k+1}}{\to} \alpha_0$.*

Proof. It is enough to prove C^1-convergence because then the C^k-convergence, $k \geq 2$, follows using the same argument used in the proof of Proposition 1.4 involving the Φ, $\Psi, \ldots$ functions. But if α_n does not C^1-converge to α_0 we arrive, using Lemma 1.5, to a contradiction. This proves the proposition. q.e.d.

C. Sets of Parallel Lines in a Hadamard Manifold.

Let $H = H^n$ be a Hadamard manifold and $\mathcal{L}$ a set of parallel geodesic lines in H. We assume that $\mathcal{L}$ is *ribbon-convex*, i.e. if ℓ_0, $\ell_1 \in \mathcal{L}$ then $\ell \in \mathcal{L}$, for every ℓ contained in the flat ribbon bounded by ℓ_0 and ℓ_1

(for the existence of the flat ribbon see [**13**]). Write $L = \bigcup \mathcal{L}$. The Flat Ribbon Theorem of A. Wolf [**13**] implies that L is a convex set. We choose one of the two points at infinity determined by any $\ell \in \mathcal{L}$. This choice "orients" all lines $\ell \in \mathcal{L}$ and we can now make the real line $\mathbb{R}$ act isometrically on L by translations: for $t \in \mathbb{R}$ and $p \in \ell \in \mathcal{L}$, $t.p = q$, where $q \in \ell$, and q is obtained from p by a t-translation.

Now, fix $p \in \ell_0 \subset L$. Let $\ell \in \mathcal{L}$. Since ℓ_0 and ℓ bound a flat ribbon there is a unique point $p_\ell \in \ell$ which is the closest to ℓ_0, and the geodesic segment $[p, p_\ell]$ is perpendicular to both ℓ_0 and ℓ. Write $K = \{ p_\ell \mid \ell \in \mathcal{L} \}$. Note that $K \cap \ell = p_\ell$.

Proposition 1.7. *The set K is convex.*

This lemma is proved in [**8**].

Consider the map $K \times \mathbb{R} \to L$, $(p,t) \mapsto t.p_\ell$. Since K is convex, this map is an isometry, where we consider $K \times \mathbb{R}$ with the metric $d((p,t),(p',t')) = \sqrt{d_H(p,p')^2 + d_{\mathbb{R}}(t,t')^2}$. The inverse of this map is the map (π, T), where $\pi(x) = p_\ell$, $x \in \ell$, is the projection onto K, and $T(x)$ is the (oriented) distance between x and p_ℓ.

Corollary 1.8. *Assume L is a closed subset of H and that K is compact. Then K is homeomorphic to a closed disc with smoothly totally geodesic embedded interior.*

Proof. Proposition 1.7 and Theorem 1.6 of [**2**], p. 418, imply that K is homeomorphic to a compact, contractible k-manifold, $0 \le k \le n-1$. Moreover, the inclusion $K \hookrightarrow H$ restricted to the (manifold) interior of K is smooth and totally geodesic. Therefore, using the exponential map, we get that K is contained (and has non-empty interior) in a totally geodesic subspace of H; hence it is homeomorphic to a disc. This proves the corollary. q.e.d.

Remark. An argument similar to the one in the proof above shows that a compact (strongly) convex set in a Riemannian manifold is homeomorphic to a closed disc.

D. Sets of Homotopic Closed Geodesics.

Let $Q = \mathbb{S}^1 \times \mathbb{R}^{n-1}$, with a complete nonpositively curved Riemannian metric g. Write $\iota : \mathbb{S}^1 = \mathbb{S}^1 \times \{0\} \hookrightarrow Q$ for the inclusion and

$$\Omega = \{ \ \alpha \in C^\infty(\mathbb{S}^1, Q) \ / \ \alpha \simeq \iota \ \}$$

with the C^k topology, $0 \le k \le \infty$. Here $\simeq$ means *(freely) homotopic.* Note that $\mathbb{S}^1$ acts freely on Ω by $z.\alpha(w) = \alpha(zw)$, for $z, w \in \mathbb{S}^1 \subset \mathbb{C}$. Write $\Sigma = \Omega/\mathbb{S}^1$. It is straightforward to verify that with the C^k topology, $k > 0$, the quotient map $\Omega \to \Sigma$ is a (locally trivial) principal $\mathbb{S}^1$-bundle. [To see this let $\alpha \in \Omega$ and w_0 with $\alpha'(w_0) \ne 0$; now a section of $\Omega \to \Sigma$ near the image of α can be constructed using a normal bundle of $\alpha(I)$ in Q, where I is an interval around w_0.]

Note that $\mathbb{R}$ also acts on Ω by $z = x.\alpha(w) = \alpha(e^{2\pi i x}w)$, for $x \in \mathbb{R}$, $w \in \mathbb{S}^1$. Moreover, we also get $\Omega/\mathbb{R} = \Sigma$. Let $\mathcal{C} = \mathcal{C}_g$ be the set of all parametrized closed geodesics homotopic to the inclusion, i.e.

$$\mathcal{C} = \{ \ \alpha \in \Omega \ / \ \alpha \text{ is a } \ g\text{-geodesic} \ \}$$

It is straightforward to verify that every $\alpha \in \mathcal{C}$ is an embedding.

Let $\mathcal{A} = \mathcal{A}_g$ be the image of $\mathcal{C}$ by the bundle map $\Omega \to \Sigma$. That is, $\mathcal{A}$ is the set of all unparametrized closed geodesics homotopic to the inclusion. Assume that

(a) $\mathcal{C}$ is non-empty.

(b) $\mathcal{C}$ is C^0-bounded. Equivalently, the set $C = \bigcup \mathcal{C}$ is contained in a compact set.

By the Flat Ribbon Theorem of J. A. Wolf [**13**] any two elements in $\mathcal{C}$ either have the same image (hence lie in the same $\mathbb{S}^1$-orbit) or have disjoint images.

Proposition 1.9. *Under these assumptions $\mathcal{A}$ is homeomorphic to a closed l-disc, $l \leq n-1$.*

Remark. All topologies C^k, $0 \leq k \leq \infty$, induce the same topology on $\mathcal{C}$ and $\mathcal{A}$.

Proof. Let H be the universal cover of Q. Then H is a Hadamard manifold, the infinite cyclic group $\mathbb{Z}$ acts freely by isometries on H and $Q = H/\mathbb{Z}$. Let $\mathcal{L}$ be the set of all lines in H which cover elements in $\mathcal{A}$, that is, all lifts to H of unparametrized closed geodesics homotopic to ι. It is straightforward to check that $\mathcal{L}$ is ribbon-convex and $L = \bigcup \mathcal{L}$ is closed. Let K be constructed from $\mathcal{L}$ as in Section 1C. Using projections we can construct, in the obvious way, a one-to-one continuous map of K onto $\mathcal{A}$. This proves the proposition. q.e.d.

Proposition 1.10. *The space Ω deformation retracts to $\mathcal{C}$.*

Proof. Let $\Omega_\iota Q$ denote the space of all based loops (at $1 \in \mathbb{S}^1$) which are based homotopic to ι. Then we have a fibration $\Omega_\iota Q \to \Omega \to Q$, where the last map is the evaluation map at $1 \in \mathbb{S}^1$. Since $\Omega_\iota Q$ is contractible we get that the evaluation map $\Omega \to Q \simeq \mathbb{S}^1$ is a homotopy equivalence. Moreover, since $\Omega \to Q$ sends the orbit of ι to (the image of) ι, we get that the inclusion of the orbit of ι into Ω is a homotopy equivalence. The same is true for $\mathcal{C}$ because $\mathcal{A}$ is a disk. Therefore the inclusion $\mathcal{C} \to \Omega$ is a homotopy equivalence. This together with the fact that the pair $(\Omega, \mathcal{C})$ has the Homotopy Extension Property implies that Ω deformation retracts onto $\mathcal{C}$. This proves the proposition. q.e.d.

Since $\mathcal{A}$ is a disc, the bundle $\Omega \to \Sigma$ restricted to $\mathcal{A} \subset \Sigma$ is trivial. Hence $\mathcal{C}$ is homeomorphic to $\mathcal{A} \times \mathbb{S}^1$. Let $s : \mathcal{A} \to \mathcal{C}$ be any section of this bundle (equivalently, a lifting of the identity $1_{\mathcal{A}}$).

We will need the following lemma in the next section.

Lemma 1.11. *Assume the metric g on Q satisfies assumptions (a) and (b) above. Let $\pounds$ be the length of a (hence all) closed g-geodesic homotopic to ι. Then there is a bounded set $R \subset Q$ such that if the image of an $\alpha \in \Omega$ is not contained in R then the g-length of α is larger than $1 + \pounds$.*

Proof. Suppose not. Then there is a sequence α_n in Ω, such that $x_n = \alpha_n(1)$ goes to infinity and all α_n have length $\leq 1 + \pounds$. Fix $\alpha_0 \in \mathcal{C}$ and write $x = \alpha_0(1)$. Let s_n be a geodesic segment $[x, x_n]$ such that $d(x, x_n)$ is its length and write $s_n(t) = exp_x(tv_n)$, for some unit length vector $v_n \in T_xQ$. We can assume $v_n \to v$, where v also has unit length. Write $s(t) = exp_x(tv)$. Let H be the universal cover of Q. Fix a lift $\beta_0 : \mathbb{R} \to H$ of α_0. Write $y = \beta_0(0)$ and $z = \beta_0(1)$. Let s'_n, s' be liftings of s_n and s beginning at y and s''_n, s'' be liftings of s_n and s beginning at z, respectively. Note that the endpoints of s'_n and s''_n can be joined by a lifting of α_n; hence their distance lies in the interval $[\pounds, 1 + \pounds]$. Therefore $d_H(s'_n(t), s''_n(t)) \in [\pounds, 1 + \pounds]$. It follows that $d_H(s'(t), s''(t)) \in [\pounds, 1 + \pounds]$ for all $t \geq 0$. But the function $t \mapsto d_H(s'(t), s''(t))$ is convex with minimum value at $t = 0$; thus it cannot be a bounded function unless it is constant. But this contradicts assumption (b). This proves the lemma. q.e.d.

E. Sets of Homotopic Closed g-Geodesics, with Varying g.

Let Q, Ω, Σ be as in Section 1D. We denote by $\mathcal{MET}^{sec\leq 0}(Q)$ the space of all complete nonpositively curved Riemannian metrics on Q, with the weak smooth topology (i.e. the union of the weak C^s topologies, which are the topologies of the C^s-convergence on compact sets). Let $\sigma : \mathbb{D}^k \to \mathcal{MET}^{sec\leq 0}(Q)$ be continuous. Write $g_u = \sigma(u)$. Using the methods and the notation of Section 1D, for each g_u we obtain $\mathcal{A}_u$, $\mathcal{C}_u$. Write $C_u = \bigcup \mathcal{C}_u$. In what follows we assume that all g_u satisfy assumptions (a) and (b) of section 1D. In particular, for each u we get a positive number $\pounds(u)$ which is the length of an element in $\mathcal{C}_u$, that is, the length of a g_u-geodesic homotopic to the inclusion $\mathbb{S}^1 \to Q$.

Lemma 1.12. *The map $\pounds : \mathbb{D}^k \to (0, \infty)$ is upper semi-continuous.*

Proof. This follows from the following facts: (1) $\pounds(u)$ is the smallest possible length of a curve homotopic to the inclusion $\mathbb{S}^1 \to Q$, and (2) for any closed curve α, the g_u-length of α is close to the g_v-length of α, provided u is close to v. This proves the lemma. q.e.d.

Lemma 1.13. *Let $u_n \to u$ in $\mathbb{D}^k$. Then there is a compact set S of Q and a sequence $\alpha_n \in \mathcal{C}_{u_n}$ such that $\alpha_n \subset S$, for n sufficiently large.*

Proof. Let R be as in Lemma 1.11 for $g = g_u$ and assume that R is closed. Let S be any compact of Q with $R \subset int\, S$. We claim that there is a sequence $\alpha_n \in \mathcal{C}_n$ such that $\alpha_n \subset S$, for n sufficiently large. This would prove the lemma. Suppose not. Then we can assume, by passing to a subsequence, that for every $\alpha_n \in \mathcal{C}_n$ we have $\alpha_n \not\subset S$. Write $\pounds = \pounds(u)$, $g_n = g_{u_n}$ and let $\alpha \in \mathcal{C}_u$. Then the g-length $\ell_g(\alpha)$ of α is $\pounds$. Therefore $\ell_{g_n}(\alpha)$ is close to $\pounds$. For each n let α_n^t be a homotopy with $\alpha_n^0 = \alpha$, $\alpha_n^1 \in \mathcal{C}_{u_n}$ and $\ell_{g_n}(\alpha_n^t) \leq \ell_{g_n}(\alpha_n^s)$, for $t > s$. That is, the deformation $t \mapsto \alpha_n^t$ begins in α, ends in a g_n-geodesic, and is g_n-length non-increasing. (Such a deformation can be done in the usual way using evolution equations or using a polygonal deformation.) Note that, by hypothesis, $\alpha_n^1 \not\subset S$ and $\alpha = \alpha_n^0 \subset R \subset S$. This together with the continuity of the deformation implies that there is $s = s_n$ such that $\beta_n = \alpha_n^s \not\subset R$ and $\beta_n \subset S$. But Lemma 1.11 together with the convergence $g_n \to g$ implies that $\ell_{g_n}(\beta_n) > 1/2 + \pounds$ when n is sufficiently large (see remark below). This contradicts the fact that the deformation $t \mapsto \alpha_n^t$ is g_n-length non-increasing. This proves the lemma. q.e.d.

Remark. In the proof above we are using the following fact: if $|g_n - g|_g \leq \delta$ then for any PD path α we have $|\ell_{g_n}(\alpha) - \ell_g(\alpha)| \leq \frac{\delta}{1-\delta}\ell_{g_n}(\alpha)$. This fact follows from the definition of length (using integrals) and the triangular inequality.

Corollary 1.14. *The map $\pounds : \mathbb{D}^k \to (0, \infty)$ is continuous.*

Proof. Let $u_n \to u$ and S be as in 1.13. Hence there are $\alpha_n \in \mathcal{C}_{u_n}$ with $\alpha_n \subset S$. Since $g_{u_n} \to g_u$ uniformly on S, we get from Lemma 1.12 (and the remark above) that $\pounds(u_n) = \ell_{g_{u_n}}(\alpha_n)$ is close to $\ell_{g_u}(\alpha_n) \geq \pounds(u)$. This shows $\pounds$ is lower semi-continuous. This proves the corollary. q.e.d.

Proposition 1.15. *The set $\bigcup_{u\in\mathbb{D}^k} \mathcal{C}_u$ is C^0-bounded.*

That is, the set of all g_u-geodesics lie, at bounded distance from, say, the inclusion ι, for all $u \in \mathbb{D}^k$.

Proof. Suppose not. Then there are $u_n \to u$ and $\alpha'_n \in \mathcal{C}_n = \mathcal{C}_{u_n}$ with α'_n going to infinity, i.e. $\alpha'_n \not\subset K$, for any given compact K, provided n is large. Let S be as in Lemma 1.13. Hence there are $\alpha_n \in \mathcal{C}_{u_n}$ with $\alpha_n \subset S$. Let S' be a compact such that $S \subset int\, S'$. Since α_n and α'_n bound a flat two dimensional cylinder (in the $g_n = g_{u_n}$ metric) we can find $\beta_n \in \mathcal{C}_n$ with $\beta_n \in S'$ and $\beta_n \not\subset S$. By Corollary 1.14 we can assume $\pounds(u_n) \leq 1/2 + \pounds(u)$. On the other hand, by 1.11 and the uniform convergence $g_n \to g_u$ on S' we have $\pounds(u_n) = \ell_{g_n}(\beta_n)$ is close to $1 + \pounds(u)$. This is a contradiction. This proves the proposition. q.e.d.

Proposition 1.16. *The set $\bigcup_{u\in\mathbb{D}^k} \mathcal{C}_u$ is C^k-bounded, for any k, $0 \leq k < \infty$.*

Proof. The proposition follows from Proposition 1.4 by considering Q as an open set of $\mathbb{R}^n$. Note that, by Proposition 1.15, we can work on an open set with compact closure; hence all required quantities will be bounded. Note also that in Proposition 1.4 the geodesics are assumed to have speed one, but the geodesics in $\mathcal{C}_u$ have speed $\pounds(u)/2\pi$. This can be fixed by a rescaling of geodesics and using the fact that (by Corollary 1.14) the set $\{\pounds(u)\}_{u\in\mathbb{D}^k}$ is bounded and bounded away from zero. This proves the proposition. q.e.d.

Define $Y = \coprod_{u\in\mathbb{D}^k}\{u\}\times\mathcal{A}_u \subset \mathbb{D}^k\times\Sigma$, that is $Y = \{\ (u,a)\mid u\in\mathbb{D}^k,\ a\in\mathcal{A}_u\ \}$. Define also $Z = \coprod_{u\in\mathbb{D}^k}\{u\}\times\mathcal{C}_u \subset \mathbb{D}^k\times\Omega$. Each C^k-topology on Ω, $0\le k\le\infty$, induces a C^k-topology on Z. Notice that $Y\to Z$ is a principal $\mathbb{S}^1$ bundle.

Proposition 1.17. *All C^k-topologies on Z coincide.*

Proof. This follows from Proposition 1.6. q.e.d.

Proposition 1.18. *The space Z is compact and metrizable.*

Proof. The space Z is certainly metrizable. Let $\{(u_n,\alpha_n)\}$ be a sequence in Z. We can assume $u_n\to u$. By Proposition 1.16, the sequence $\{\alpha_n\}$ is C^k-bounded, for all $k\ge 0$. In particular it is C^1-bounded. Therefore the sequence $\{\alpha_n\}$ is equicontinuous. Moreover, we can assume all α_n to be Lipschitz with the same constant. Proposition 1.15 says that the set $\{\alpha_n\}$ is C^0-bounded. By the Arzela-Ascoli Theorem we can assume that $\{\alpha_n\}$ C^0-converges to a Lipschitz $\alpha\in\Omega$. Since α is Lipschitz its length is finite, where the length is defined as $sup\sum d_{g_u}(\alpha(z_i),\alpha(z_{i+1}))$, the sup taken over all partitions of $\mathbb{S}^1$. Using this definition of length it is straightforward to show that g_{u_n}-lengths of the α_n converge to the g_u-length of α. Corollary 1.14 implies now that α has minimal g_u-length; hence it is smooth and $\alpha\in\mathcal{C}_u$. Therefore $\{(u_n,\alpha_n)\}$ converges to $(u,\alpha)\in Z$. This proves the proposition. q.e.d.

It follows from Proposition 1.18 that Y is compact. The space Y is also Hausdorff because Σ is Hausdorff. Therefore the projection $Y\to\mathbb{D}^k$ is a cellular k-disc (choose any base point).

Proposition 1.19. *The space Y is finite dimensional.*

Proof. Since Y is Hausdorff compact and $Z\to Y$ is locally trivial, the proposition follows from the following claim. q.e.d.

Claim. *Let $U\subset Y$ be compact and such that $Z\to Y$ is trivial over U. Then U is homeomorphic to a compact subset of $\mathbb{R}^{n+k}$. Hence U is finite dimensional.*

Proof of the claim. Let $U'\subset Z$ be the image of a section of $Z\to Y$ over U. Hence the restriction $U'\to U$ is a homeomorphism. As before we are considering Q as an open set of $\mathbb{R}^n$. Now, just define $h: U'\to\mathbb{R}^{k+n}$ as $h(u,\alpha) = (u,\alpha(1))$. (Recall $1\in\mathbb{S}^1\subset\mathbb{C}$ and $\alpha:\mathbb{S}^1\to Q$.) This is a one-to-one continuous map with compact domain between metric spaces. Hence it is a homeomorphism onto its image. This proves the claim and Proposition 1.19. q.e.d.

By Propositions 1.3 and 1.19 the S^1-bundle $Z \to Y$ is trivial (thus Z is homeomorphic to $Y \times \mathbb{S}^1$).

Remark. Note that we cannot take $U = Y$ in the proof of Proposition 1.19 (hence the use of the sets U is necessary) because we need that proposition to deduce (together with Proposition 1.3) that $Z \to Y$ is trivial.

Now take a section $Y \to Z$ of the trivial bundle $Z \to Y$ and let X be the image of Y by this section. Write $\eta : X \to \mathbb{D}^k$ for the projection. Then $\eta : X \to \mathbb{D}^k$ is a cellular k-disc. Note that X is formed by honest *parametrized* g_u-geodesics, not unparametrized ones, like the ones in Y. And since X and Y are homeomorphic, Proposition 1.19 has the following corollary.

Corollary 1.20. *The space X is a finite dimensional space.*

Recall that the cellular discs Y and X were constructed from a map $\sigma : \mathbb{D}^k \to \mathcal{MET}^{sec \leq 0}(M)$. Now assume that $\sigma|_{\mathbb{S}^{k-1}}$ *factors through the orbit map*; that is, $\sigma|_{\mathbb{S}^{k-1}}$ factors through the map Λ_{g_0} in [**10**]. (This map is just the orbit map $\Lambda_{g_0} : DIFF(M) \to \mathcal{MET}^{sec \leq 0}(M)$ given by $\phi \mapsto \phi_* g_0$.) Explicitly $\sigma|_{\mathbb{S}^{k-1}}$ factors through the orbit map if for $u \in \mathbb{S}^{k-1}$ we can write $\sigma_u = (\phi_u)_* g_0$, for some continuous $u \mapsto \phi_u$. It follows that $\mathcal{A}_u = \phi_u(\mathcal{A}_{u_0})$, $u \in \mathbb{S}^{k-1}$. Therefore we can write $\partial Y = \mathbb{S}^{k-1} \times \mathcal{A}_{u_0}$ and we can consider $\mathbb{S}^{k-1} \subset Y$ by choosing any element in $\mathcal{A}_{u_0}$. Analogously for X. Hence we obtain cellular k-disc pairs $(Y, \mathbb{S}^{k-1})$, $(X, \mathbb{S}^{k-1})$.

Remark. The embedding $\mathbb{S}^{k-1} \subset Y$ depends on how $\sigma|_{\mathbb{S}^{k-1}} : \mathbb{S}^{k-1} \to \mathcal{MET}^{sec \leq 0}(M)$ factors through the orbit map, and on the choice of an element in $\mathcal{A}_{u_0}$. This dependence will not be essential for our arguments.

Corollary 1.21. *Assume $\sigma|_{\mathbb{S}^{k-1}} : \mathbb{S}^{k-1} \to \mathcal{MET}^{sec \leq 0}(M)$ factors through the orbit map. Then the space $X/\mathbb{S}^{k-1}$ is a finite dimensional space.*

Proof. Since $Z \to Y$ is trivial and Y is homeomorphic to X, we can take $U = X$ in the construction of the embedding h given in the proof of 1.19. The embedding $h : X \to \mathbb{R}^{k+n}$ induces an embedding $X/\mathbb{S}^{k-1} \to \mathbb{R}^{k+n}/h(\mathbb{S}^{k-1})$. But from the construction of h we see that the sphere $h(\mathbb{S}^{k-1})$ is "nicely embedded" in $\mathbb{R}^{k+n}$ (it maps to the canonical sphere by the projection $\mathbb{R}^{k+n} \to \mathbb{R}^k$). Hence $\mathbb{R}^{k+n}/h(\mathbb{S}^{k-1})$ is clearly finite dimensional, and it follows that $X/\mathbb{S}^{k-1}$ is finite dimensional. q.e.d.

2. The space of cellular maps

Let L be a manifold. Recall that a (continuous) proper onto map $f : L \to L$ is cellular if $f^{-1}(y)$ is cellular in L, for all $y \in L$. (A compact subset of a manifold is cellular if it has arbitrarily small neighborhoods homeomorphic to the Euclidean space of the same dimension as the manifold.) We denote by $CELL(L)$ the space of cellular maps on L. Also we denote by $TOP(L)$ the space of all self-homeomorphisms of L, and by $P(L)$ the space of topological pseudoisotopies on L. All these spaces are considered with the compact-open topology.

Remark. If the manifold L has boundary, then for a map $f : L \to L$ to be cellular we demand the restriction $f|_{\partial L} : \partial L \to \partial L$ to be cellular too. See [**17**], p. 271.

Lemma 2.1. *Let N be compact. The map $\pi_k\, P(N) \to \pi_k\, TOP(N \times [0,1])$, $k \geq 0$, is injective.*

Proof. Let $\alpha : \mathbb{S}^k \to P(N)$, $\beta : \mathbb{D}^{k+1} \to TOP(N \times [0,1])$, with $\beta|_{\mathbb{S}^k} = \alpha$. For $f \in TOP(N \times [0,1])$ write $f_0 : N \to N$ for its bottom, that is, for its restriction to $N \times \{0\}$. Define $\gamma : \mathbb{D}^{k+1} \to TOP(N \times [0,1])$ by $\gamma(u) = (\beta(u)_0)^{-1} \times 1_{[0,1]}$. Note that $\gamma(u) = 1_{N\times[0,1]}$ for $u \in \mathbb{S}^k$. Finally, define $\beta' : \mathbb{D}^{k+1} \to TOP(N \times [0,1])$ by $\beta'(u) = \gamma(u)\beta(u)$. Then $\beta'|_{\mathbb{S}^k} = \alpha$. This proves the lemma because $\beta' : \mathbb{D}^{k+1} \to P(N)$. q.e.d.

Lemma 2.2. *Let N be compact and $dim\, N \neq 3$. Then the map $\pi_k\, TOP(N \times [0,1]) \to \pi_k\, CELL(N \times [0,1])$, $k \geq 0$, is an isomorphism.*

This is a fibered version of the Siebenmann result [**17**]. The proof follows from Proposition 4.1 of B. Haver [**12**] together with the fact that the closure of $TOP(L)$ is $CELL(L)$, $dimL \neq 4$, proved by Siebenmann [**17**]. In the lemma above (and the corollary below), for the case $k = 0$ "isomorphism" means "bijection." These two lemmas imply:

Corollary 2.3. *Let N be compact and $dim\, N \neq 3$. Then the map $\pi_k\, P(N) \to \pi_k\, CELL(N \times [0,1])$, $k \geq 0$, is injective.*

Remark. We will use the fact that $CELL(L)$ is LC^∞ for a compact L, $dim\, L \neq 3$. This follows from Proposition 4.1 of [**12**] and the main results in [**17**] and [**7**].

3. The space at infinity of negatively curved manifolds

This section is a version (for the nonpositively curved case) of section 2 of [**10**]. We will use similar notation.

Let (Q, g) be a complete Riemannian manifold with nonpositive sectional curvatures, and $S \subset Q$ a closed totally geodesic submanifold of Q, such that the map $\pi_1(S) \to \pi_1(Q)$ is an isomorphism. Write

$\Gamma = \pi_1(S) = \pi_1(Q)$. We can assume that the universal cover $\tilde{S}$ of S is contained in the universal cover $\tilde{Q}$ of Q. The group Γ acts by isometries on $\tilde{Q}$ such that $\Gamma(S) = S$ and $Q = \tilde{Q}/\Gamma$, $S = \tilde{S}/\Gamma$. Let T be the normal bundle of S, that is, for $z \in S$, $T_z = \{v \in T_zQ \,:\, g(v,u) = 0$, for all $u \in T_zS\} \subset T_zQ$. Hence $T \oplus TS = TQ|_S$ as bundles over S. Write $\pi(v) = z$ if $v \in T_z$, that is, $\pi : T \to S$ is the bundle projection. The unit sphere bundle and unit disc bundle of T will be denoted by N and W, respectively. Note that the normal bundle, normal sphere bundle, and normal disc bundle of $\tilde{S}$ in $\tilde{Q}$ are the liftings $\tilde{T}$, $\tilde{N}$, and $\tilde{W}$ of T, N, and W, respectively. For $v \in T_qQ$ or $v \in T_q\tilde{Q}$, $v \neq 0$, the map $t \mapsto exp_q(tv)$, $t \geq 0$, will be denoted by c_v and its image will be denoted by the same symbol. Since $\tilde{Q}$ is simply connected, c_v is a geodesic ray, for every $v \in \tilde{N}$. We will denote by E the exponential map $E : T \to Q$, $E(v) = exp_{\pi(v)}(v)$, which is a diffeomorphism. Also, the exponential map $\tilde{E} : \tilde{T} \to \tilde{Q}$, $\tilde{E}(v) = exp_{\pi(v)}(v)$, is a diffeomorphism and $\tilde{E}$ is a lifting of E.

There are several facts stated for Q in section 2 of [**10**] that are clearly not true for a general nonpositively curved manifold. For this reason we need an extra condition. In what follows we assume that $\tilde{Q}$ is δ-hyperbolic (in the sense of Gromov). Hence the definition of $\partial_\infty\tilde{Q}$ remains valid, that is, the definition using quasi-geodesics. And this definition coincides with the definition using geodesics. We consider $\partial_\infty\tilde{Q}$ with the usual cone topology. Recall that, for any $q \in \tilde{Q}$, the map $\{v \in T_q\tilde{Q} \,:\, |v| = 1\} \to \partial_\infty\tilde{Q}$ given by $v \mapsto [c_v]$ is a homeomorphism. We also have that $\overline{(\tilde{Q})} = \tilde{Q} \cup \partial_\infty\tilde{Q}$ can be given a topology such that the map $\{v \in T_q\tilde{Q} \,:\, |v| \leq 1\} \to \partial_\infty\tilde{Q}$ given by $v \mapsto exp_q(\varsigma(|v|)\frac{v}{|v|})$, for $|v| < 1$ and $v \mapsto [c_v]$ for $v = 1$, is a homeomorphism. Here $\varsigma : [0,1) \to [0,\infty)$ is a homeomorphism that is the identity near 0.

Since $\tilde{S}$ is convex in $\tilde{Q}$, every geodesic ray in $\tilde{S}$ is a geodesic ray in $\tilde{Q}$. Therefore $\partial_\infty\tilde{S} \subset \partial_\infty\tilde{Q}$. For a quasi-geodesic ray β we have: $[\beta] \in \partial_\infty\tilde{Q} \setminus \partial_\infty\tilde{S}$ if and only if β diverges from $\tilde{S}$, that is, $d_{\tilde{Q}}\,(\,\beta(t)\,,\,\tilde{S}\,) \to \infty$, as $t \to \infty$. Define the map $\tilde{A} : \tilde{N} \to \partial_\infty\tilde{Q} \setminus \partial_\infty\tilde{S}$, given by $\tilde{A}(v) = [c_v]$.

Lemma 3.1. *The map $\tilde{A}$ is continuous onto and proper.*

Proof. The proof of continuity is straightforward. First we prove surjectivity. Let $p \in \tilde{S}$ and α be a geodesic ray emanating from p and not contained in $\tilde{S}$. We have to show that there is a c_v (emanating from some $q \in \tilde{S}$ with direction v perpendicular to $\tilde{S}$) such that α and c_v determine the same point in $\tilde{Q}$. Let c_{v_n}, at q_n, be perpendicular to $\tilde{S}$ and passing through $\alpha(n)$. Using the δ-hyperbolicity of $\tilde{Q}$ we get that $\{q_n\}$ is bounded so we can assume $q_n \to q$. Furthermore, we can assume $v_n \to v$. It is straightforward to verify that c_v and α determine the same point at infinity. This proves that $\tilde{A}$ is onto.

We identify $\partial\tilde{Q}$ with the unit sphere $\mathbb{S}$ in $T_p\tilde{Q}$, for some $p \in \tilde{S}$. Let $K \subset \mathbb{S} - T_p\tilde{S}$ be compact. We now prove $\tilde{A}^{-1}(K)$-bounded. Note that if α is a ray emanating from p with direction $\alpha'(0) \in K$, then the angle between $\alpha'(0)$ and $\tilde{S}$ is bounded away from zero; hence there is a $\kappa > 0$ such that $d_{\tilde{Q}}(\alpha(t), \tilde{S}) \geq \kappa t$, for every such α. Let $v \in \tilde{A}^{-1}(K)$, with $v \in T_q\tilde{S}$. Then $[c_v] = [\alpha]$, for some α as above. Using the δ-hyperbolicity of $\tilde{Q}$ we get that $d_{\tilde{Q}}(p, q)$ cannot be arbitrarily large. This proves that $\tilde{A}^{-1}(K)$ is bounded. This proves the lemma. q.e.d.

The most important change to be made here is a new version of fact 9 on p. 287 [**10**]. It is given in the next lemma. Recall $\varsigma : [0, 1) \to [0, \infty)$ is a homeomorphism such that it is the identity near 0.

Lemma 3.2. *The map $\tilde{A} : \tilde{N} \to \partial_\infty\tilde{Q} \setminus \partial_\infty\tilde{S}$, given by $\tilde{A}(v) = [c_v]$, is cellular. Furthermore, we can extend $\tilde{A}$ to $\tilde{W} \to \overline{(\tilde{Q})} \setminus \partial_\infty\tilde{S}$ by defining $\tilde{A}(v) = \tilde{E}(\varsigma(|v|)\frac{v}{|v|}) = exp_q(\varsigma(|v|)\frac{v}{|v|})$, for $|v| < 1$, $v \in \tilde{W}_q$. This extension is (continuous) cellular and a diffeomorphism on $\tilde{W} \setminus \tilde{N}$.*

Note that $\tilde{A}$ is just a reparametrization (in the "time" direction) of the normal (to $\tilde{S}$) exponential map $\tilde{E}$.

Proof. Standard Hadamard manifold techniques show that the map $\tilde{A}$ (defined on $\tilde{W}$) is continuous. Let $v \in \tilde{N}_p$, $p \in \tilde{S}$. We now prove that $C = \tilde{A}^{-1}(\tilde{A}(v))$ is homeomorphic to a convex set in $\tilde{S}$. First note that for $v, v' \in \tilde{N}_p$ we have $\tilde{A}(v) \neq \tilde{A}(v')$ because c_v and $c_{v'}$ are two geodesic rays emanating from the same point. Hence the continuous map $\pi|_C : C \to \pi(C)$ is injective. (Recall $\pi : \tilde{N} \to \tilde{S}$ is the bundle projection.)

Let $v' \in \tilde{N}_{p'}$, $p' \neq p$, be such that $\tilde{A}(v') = \tilde{A}(v)$. Let $[p, p']$ be the unique geodesic segment in $\tilde{S}$ joining p to p'. Since the geodesic rays c_v, $c_{v'}$ make a right angle with $\tilde{S}$, at p and p' respectively, we have that c_v, $c_{v'}$ and $[p, p']$ bound a flat geodesic ribbon. Hence $[p, p'] \subset \pi(C)$ and it follows that $\pi(C)$ is convex. For each $q \in \pi(C)$ there is a unique $v_q \in \tilde{N}_q \cap C$ and it is straightforward (using the ribbon property) to prove that $q \mapsto v_q$ is continuous. Hence C is homeomorphic to the convex set $\pi(C)$. Note that $\pi(C)$ is bounded: otherwise $\tilde{Q}$ would contain flat geodesic ribbons isometric to $[0, \ell] \times [0, \infty)$ with $\ell \to \infty$ which would contradict the δ-thinness of triangles in the δ-hyperbolic space $\tilde{Q}$. It follows that C is compact. Therefore $\pi(C)$ is compact and convex, and hence homeomorphic to a disc (see remark at the end of Section 1C). Since C is homeomorphic to $\pi(C)$, C is also homeomorphic to a disc. Therefore all pre-images of the map $\tilde{A}$ are homeomorphic to a disc; thus $\tilde{A}$ is a cell-like map. Since the domain and target spaces of the map $\tilde{A}$

are manifolds, we deduce that $\tilde{A}$ is a cellular map (see remarks below). This proves the lemma. q.e.d.

Remarks. 1. For maps $f : M \to N$ between manifolds we have the equivalence: cellular $\Leftrightarrow$ cell-like. Here are some references. For $dim\, M \geq 4$ see [**3**], Corollary 5E, p. 147. For $dim\, M = 2$ see [**3**], p. 122. For $dim\, M = 3$ the equivalence follows from the results in [**14**] and the Poincaré Conjecture in dim 3. (In our special case the use of Poincaré Conjecture can be avoided by using 1.4 in [**14**] and the fact that the pre-images of $\tilde{A}$ are discs.)

2. Here is an alternative elementary argument to show that $\tilde{A}$ is cellular without the equivalence cellular $\Leftrightarrow$ cell-like. The set $\pi(C)$ is compact and convex in the Hadamard manifold $\tilde{S}$. It is straightforward to show that $\pi(C)$ is cellular in $\tilde{S}$. Hence $\pi(C) = \cap_{i=1}^{\infty} U_i$, where $C \subset U_{i+1} \subset \bar{U}_{i+1} \subset U_i$ and U_i homeomorphic to Euclidean space. Since $\tilde{S}$ is contractible we can write $\tilde{N} = \tilde{S} \times \mathbb{S}^k$, for some k, and we can identify $\pi : \tilde{N} \to \tilde{S}$ with the projection $\tilde{S} \times \mathbb{S}^k \to \tilde{S}$. Therefore C is the graph of a function $\pi(C) \to \mathbb{S}^k$, i.e. $C = \{(x, f(x)),\, x \in \pi(C)\}$, for some continuous function $\theta : \pi(C) \to \mathbb{S}^k$. Define $\Theta : \pi(C) \to \tilde{N}$, by $\Theta(x) = (x, \theta(x))$. Then $\Theta(\pi(C)) = C$ and $(\pi|_C)^{-1} = \Theta$. Since $\pi(C)$ is an ANR we can extend θ continuously (in any way) to some U_{n_0}, n_0 sufficiently large. Hence we can extend $\Theta : \pi(C) \to \tilde{N}$ to an embedding $\Theta : U_{n_0} \to \tilde{N}$, by defining $\Theta(x) = (x, \theta(x))$, $x \in U_{n_0}$. Let $V_i = \Theta(U_i)$ be the image of U_i, $i \geq n_0$. Since Θ is injective we have $\cap V_i = \cap \Theta(U_i) = \Theta(\cap U_i) = \Theta(\pi(C)) = C$. We now "thicken" the V_i by considering $W_i = \{(x, u)\,,\, x \in U_i,\, d_{\mathbb{S}^k}(u, \theta(x)) < 1/i\}$. Then $C = \cap W_i$, $C \subset W_{i+1} \subset \bar{W}_{i+1} \subset W_i$ and W_i homeomorphic to Euclidean space.

Lemma 3.3. *The injectivity radius at $p \in Q$ tends to infinity, as p gets far from S.*

Proof. Suppose not. Let γ_n be non-contractible loops in Q with $d(\gamma_n, S) = n$ and the lengths $\ell(\gamma_n)$ bounded (say by $a > 0$). Each γ_n is homotopic to a closed geodesic β_n in S. Lifting to $\tilde{Q}$ we obtain $p_n, p'_n \in \tilde{S}$ and vectors v_n, v'_n such that $d(c_{v_n}(n), c_{v'_n}(n)) \leq a$ and $b \leq d(p_n, p'_n) \leq a$, where $b > 0$ is the injectivity radius of S. Since $\tilde{S}$ has a compact fundamental domain we can assume $p_n \to p$, $p'_n \to p'$, $v_n \to v$, and $v'_n \to v'$. It is straightforward to check that c_v and $c_{v'}$ bound an infinite flat (half) ribbon. We can repeat this process with the set $\{\gamma_n^k\}$, where $\alpha^k = \alpha * \cdots * \alpha$ (concatenation k times). In this way we get that $\tilde{Q}$ contains flat geodesic ribbons isometric to $[0, b] \times [0, \infty)$ with $b \to \infty$ which would contradict the δ-thinness of triangles in the δ-hyperbolic space $\tilde{Q}$. This proves the lemma. q.e.d.

Lemma 2.2 of [**10**] remains true. We can now descend to Q and define, as in Lemma 2.4 of [**10**], the map A, which is obtained from $\tilde{A}$ using the projection map $\tilde{Q} \to Q$. The map A is just a reparametrization (in the "time" direction) of the exponential map E.

Lemma 3.4. *The map $A : N \to \partial_\infty Q$, given by $A(v) = [c_v]$, is (continuous and) cellular. Furthermore, we can extend A to $W \to \partial_\infty Q \cup Q$ by defining $A(v) = E((\varsigma(|v|)\frac{v}{|v|}))$, for $|v| < 1$. This extension is (continuous and) cellular and a diffeomorphism on $W \setminus N$.*

Proof. Since $\tilde{A}$ covers A, it is enough to prove that $\gamma C \cap C = \emptyset$, for $\gamma \in \Gamma$ and C as in the proof of Lemma 3.2. Suppose there is $v \in \tilde{N}_p$ with $\tilde{A}(v) = \tilde{A}(\gamma_*(v))$, for some non-trivial $\gamma \in \Gamma$. Note that $\gamma_*(v) \in \tilde{N}_{\gamma(p)}$ and $\gamma(p) \neq p$. But then $\tilde{A}(v) = \tilde{A}(\gamma^n_*(v))$, $\gamma^n_*(v) \in \tilde{N}_{\gamma^n(p)}$ and the distance between $\gamma^n(p)$ and p becomes large. Therefore we again obtain large geodesic flat ribbons in $\tilde{Q}$, which cannot happen. This proves the lemma. q.e.d.

4. Proof of Theorem B

We will use the notation and the following diagram given in the Introduction:

$$\begin{array}{ccccc}
DIFF(\,(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial\,) & \overset{\Phi}{\to} & DIFF(M) & \overset{\Lambda_g}{\to} & \mathcal{MET}^{\,sec \leq 0}(M) \\
\iota \downarrow & & & & \\
P(\mathbb{S}^1 \times \mathbb{S}^{n-2}) & & & & \\
\iota' \downarrow & & & & \\
CELL(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0,1]) & & & &
\end{array}$$

where ι and ι' are inclusions.

Let (M^n, g) be a closed nonpositively curved Riemannian n-manifold, with $\pi_1 M$ hyperbolic.

Let α be a simple closed g-geodesic. Write $N = \mathbb{S}^1 \times \mathbb{S}^{n-2}$ and $\Sigma^M = \Lambda_g \circ \Phi^M$. The base point of the k-sphere $\mathbb{S}^k$ will always be the point $u_0 = (1, 0, \dots, 0)$. Let $\theta : \mathbb{S}^k \to DIFF(N \times I, \partial)$, $\theta(u_0) = 1_{N\times I}$, represent an element in $\pi_k(\,DIFF(N \times I, \partial)\,)$.

We have to prove that if $\pi_k(\Sigma^M)([\theta])$ is zero, then $\pi_k(\iota'_N \iota_N)([\theta])$ is also zero. Equivalently, if $\Sigma^M\,\theta$ extends to the $(k+1)$-disc $\mathbb{D}^{k+1}$, then $\iota'_N\iota_N\theta$ also extends to $\mathbb{D}^{k+1}$. So, suppose that $\Sigma^M\,\theta : \mathbb{S}^k \to \mathcal{MET}^{\,sec \leq 0}(M)$ extends to a map $\sigma' : \mathbb{D}^{k+1} \to \mathcal{MET}^{\,sec \leq 0}(M)$. Thus $\sigma'(u)$, $u \in \mathbb{D}^{k+1}$, is a nonpositively curved metric on M and $\sigma'(u_0) = g$. Write $\varphi_u = \Phi^M(\theta(u))$, $u \in \mathbb{S}^k$. Note that $\varphi_u : M \to M$ induces the identity at the π_1-level and hence φ_u is freely homotopic to 1_M.

By deforming σ', we can assume that it is radial near $\partial\,\mathbb{D}^{k+1}$. Since σ' is continuous and $\mathbb{D}^{k+1}$ is compact, we can find constants $a, b > 0$ such that $a^2 \leq \sigma'(u)(v,v) \leq b^2$ for every $v \in TM$ with $g(v,v) = 1$, $u \in \mathbb{D}^{k+1}$.

Let Q be the covering space of M with respect to the infinite cyclic subgroup of $\pi_1(M, \alpha(1))$ generated by α. Denote by $\sigma(u)$ the pullback on Q of the metric $\sigma'(u)$ on M. For the lifting of g to Q we use the same letter g. Note that α lifts to Q and we denote this lifting also by α. Let $\phi_u : Q \to Q$ be the diffeomorphism which is the unique lifting of φ_u to Q. (Note that this lifting is unique because the covering map $Q \to M$ has no nontrivial covering transformations. Equivalently, every lifting of the identity map is the identity map.)

4.1. We have some comments.

(i) $\sigma(u) = (\phi_u)_*\sigma(u_0) = (\phi_u)_*g$, for $u \in \mathbb{S}^k$.

(ii) The tubular neighborhood U of width $2r$ of α lifts to a countable number of components, with exactly one being diffeomorphic to U. We call this lifting also by U. All other components U_1, $U_2, \ldots$ are diffeomorphic to $\mathbb{D}^{n-1} \times \mathbb{R}$. Note that ϕ_u is the identity outside the union of $\bigcup U_i$ and U and inside the closed normal geodesic tubular neighborhood of width r of α.

(iii) Since $\varphi_u : M \to M$ induces the identity at the π_1-level, and $\mathbb{S}^k$ is compact, there is a constant C such that $d_{\sigma(u')}(\,p\,,\,\phi_u(p)\,) < C$, for any $u, u' \in \mathbb{S}^k$, where $d_{\sigma(u')}$ denotes the distance in the Riemannian manifold $(Q, \sigma(u'))$.

(iv) $(\phi_u)|_U = \left[\Phi^Q(\alpha, V', r)\theta(u)\right]|_U$, for $u \in \mathbb{S}^k$. Here V' is the lifting of V.

(v) We have that $a^2 \leq \sigma(u)(v, v) \leq b^2$ for every $v \in TQ$ with $g(v, v) = 1$, $u \in \mathbb{D}^{k+1}$. It follows that $\frac{a^2}{b^2} \leq \sigma(u)(v, v) \leq \frac{b^2}{a^2}$ for every $v \in TQ$ with $\sigma(u')(v, v) = 1$, $u, u' \in \mathbb{D}^{k+1}$.

(vi) All sectional curvatures of the Riemannian manifolds $(Q, \sigma(u))$, $u \in \mathbb{D}^{k+1}$, are less than or equal to 0.

(vii) The map $\sigma : \mathbb{D}^{k+1} \to \mathcal{MET}^{\,sec\leq 0}(Q)$ extends $\Sigma^Q\theta : \mathbb{S}^k \to \mathcal{MET}^{\,sec\leq 0}(Q)$, where $\Sigma^Q = \Lambda_g \circ \Phi^Q$.

Let X be the cellular $(k+1)$-disc constructed in Section 1E. Recall that the elements of X are pairs $x = (u, \beta)$, $u \in \mathbb{D}^{k+1}$ and β is a $\sigma(u)$-geodesic in Q. We consider $\mathbb{S}^k \subset X$ by identifying $u \in \mathbb{S}^k = \partial\mathbb{D}^{k+1}$ with (u, α). By 4.1(i) we get that σ factors through the orbit map (see paragraph before Corollary 1.21). We also get that $(X, \mathbb{S}^k)$ is a cellular $(k+1)$-disc pair. We have the map $X \to \mathcal{MET}^{\,sec\leq 0}(Q)$ given by $(u, \beta) \mapsto \sigma(u)$, and we will use the same letter $\sigma : X \to \mathcal{MET}^{\,sec\leq 0}(Q)$ to denote this map. Note that this map $\sigma : X \to \mathcal{MET}^{\,sec\leq 0}(Q)$ also extends $\Sigma^Q\theta$ in the sense of cellular discs (see Section 1A). The next claim says that we can modify σ (and we will use the same letter "σ" to denote this modified map) so that α is a $\sigma(x)$-geodesic, for all $x = (u, \beta) \in X$.

Claim 1. *There is a map $\sigma : X \to \mathcal{MET}^{\,sec\leq 0}(Q)$ such that:*

1. *The curve α is a $\sigma(x)$-geodesic, for all $x = (u, \beta) \in X$.*

2. *The map* $\sigma : X \to \mathcal{MET}^{sec \leq 0}(Q)$ *extends* $\Sigma^Q\theta$ *(in the sense of cellular discs, see Section 1A).*
3. *The liftings of all metrics* $\sigma(x)$, $x \in X$, *to the universal cover* $\tilde{Q}$ *are quasi-isometric with the same constants; i.e they are all* (λ, ϵ)*-quasi-isometric, for some fixed* (λ, ϵ).
4. *The liftings of all metrics* $\sigma(x)$, $x \in X$, *to the universal cover* $\tilde{Q}$ *are* δ*-hyperbolic, for some* δ.

Proof of Claim 1. Let $h : X \to Emb(\mathbb{S}^1, Q) \subset \Omega = C^\infty(\mathbb{S}^1, Q)$, $h(u, \beta) = \beta$. Note that $h(\mathbb{S}^k) = \{\alpha\}$. (The function h replaces the function h in [**10**], p. 292.) Since $\pi_i(C^\infty(\mathbb{S}^1, Q)) = 0$, $i > 1$, and the fiber of $Emb(\mathbb{S}^1, Q) \hookrightarrow C^\infty(\mathbb{S}^1, Q)$ is $(n-5)$-connected (this follows from Lemma 1.4 of [**10**]), we have that $\pi_i(Emb(\mathbb{S}^1, Q)) = 0$, $1 < i \leq n-5$. This together with Proposition 1.1, Corollary 1.21 (also see Remark 4.2(1) below), and the fact $Emb(\mathbb{S}^1, Q)$ is locally contractible (hence LC^∞) imply that, for $k > 0$, we have $h : (X, \mathbb{S}^k) \to (Emb(\mathbb{S}^1, Q), \alpha)$ is relative null-homotopic (i.e. the homotopy always sends $\mathbb{S}^k$ to $\{\alpha\}$). Therefore there is a homotopy h_t with $h_0 = h$ and $h_1 \equiv \alpha$. Hence for each $x = (u, \beta) \in X$ there is an isotopy $h_t(x) : \mathbb{S}^1 \to Q$ such that $h_0(x) = \beta$ and $h_1(x) = \alpha$.

Remarks 4.2.

1. In order to apply Corollary 1.21 we have to verify that each $\mathcal{C}_u$, $u \in \mathbb{D}^k$, satisfies conditions (a) and (b) in Section 1D. Recall that $\mathcal{C}_u$ is the set of parametrized $\sigma(u)$-geodesics homotopic to α. First $\mathcal{C}_u$ is non-empty because there is a $\sigma'(u)$-geodesic homotopic to α in M (M is closed). Second, if $\mathcal{C}_u$ is not bounded, it is straightforward to argue that $\tilde{Q}$ would contain flat geodesic ribbons isometric to $[0, \ell] \times [0, \infty)$, with $\ell \to \infty$, which would contradict the δ-thinness of triangles in the δ-hyperbolic space $\tilde{Q}$.

2. We have shown that for $k > 0$ the map $h : (X, \mathbb{S}^k) \to (Emb(\mathbb{S}^1, Q), \alpha)$ is relative null-homotopic. By modifying X a little bit we can also assume this to be true for $k = 0$, that is, when X is a cellular 1-disc. To show this consider the map $X \to Q$, $(u, \beta) \mapsto \beta(1)$. If this map represents $n \in \mathbb{Z} = \check{H}_1(\mathbb{S}^1) = \check{H}_1(Q)$ then just replace X by $\{(u, e^{-n\pi\eta(u)i}\beta) \mid (u, \beta) \in X\}$. With this new choice the map h is relative null-homotopic. Hence we can assume that the statement "for each $x = (u, \beta) \in X$ there is an isotopy $h_t(x) : \mathbb{S}^1 \to Q$ such that $h_0(x) = \beta$ and $h_1(x) = \alpha$" is true also when X is a cellular 1-disc.

Now we extend, in the usual way, the isotopies $h(x) : \mathbb{S}^1 \times [0, 1] \to Q$ to compactly supported isotopies $H(x) : Q \times [0, 1] \to Q$. Then the required map σ is defined, for $x = (u, \beta) \in X$, as $\sigma(x) = [H(x)_1]^* \sigma(u)$ (here $\sigma(u)$ is the "old" σ). Note that the metrics do not change outside a compact set of Q; hence, by Lemma 2.1 of [**10**] and the fact that X is compact, we get that the liftings of all metrics $\sigma(x)$, $x \in X$, to the universal cover $\tilde{Q}$ are quasi-isometric with the same constants. This proves item (3). Item (4) follows from item (3) and Theorem 1.9 in [**1**], p. 402. This proves Claim 1. q.e.d.

Note that the new metrics $\sigma(x)$, $x \in X - \mathbb{S}^k$, are not necessarily pullbacks from metrics in M. We shall identify, via the exponential map exp^g, the space Q with $\mathbb{S}^1 \times \mathbb{R}^{n-1}$. Thus the rays $\{z\} \times \mathbb{R}^+ v$, $v \in \mathbb{S}^{n-2}$, are geodesics (with respect to $g = \sigma(u_0)$) emanating from $z \in \mathbb{S}^1 \subset Q$ and normal to $\mathbb{S}^1$. Denote by $W_\delta = \mathbb{S}^1 \times \mathbb{D}^{n-1}(\delta)$ the closed normal tubular neighborhood of $\mathbb{S}^1$ in Q of width $\delta > 0$, with respect to the metric $\sigma(u_0)$. We have $\partial W_\delta = \mathbb{S}^1 \times \mathbb{S}^{n-2}(\delta)$.

For each $x \in X$ and $z \in \mathbb{S}^1$, let $T^x(z)$ be the orthogonal complement of the tangent space $T_z\mathbb{S}^1 \subset T_zQ$ with respect to the $\sigma(x)$ metric and denote by $exp^x_z : T^x(z) \to Q$ the normal exponential map, also with respect to the $\sigma(x)$ metric. Note that the map $exp^x : T^x \to Q$ is a diffeomorphism, where T^x is the bundle over $\mathbb{S}^1$ whose fibers are $T^x(z)$, $z \in \mathbb{S}^1$. We will denote by N^x and W^x the sphere and disc bundles of T^x, respectively.

Claim 2. *Consider the map $\partial W_\delta \to N_u$, $(z,v) \mapsto (z', \frac{s}{|s|})$, where $exp^x(z',s) = (z,v)$. (This map is just a restriction of the "normalization" of $(exp^x)^{-1}$.) Then this map is a diffeomorphism for all $x \in X$, provided δ is sufficiently small.*

Proof of Claim 2. This is proved in [**10**], pp. 292, 293. In fact, this is a slight variation of Claim 2 in [**10**], p. 293. We point out that the maps χ and τ were introduced in [**10**] for the sole purpose of applying Lemma 1.6 of [**10**]. We do not need these functions here. This proves Claim 2. q.e.d.

To simplify our notation we take $\delta = 1$ and write $W = W_1$. Thus $\partial W = N = \mathbb{S}^1 \times \mathbb{S}^{n-2}$ and we write $N \times [1,\infty) = Q \setminus int\, W$. Now define a diffeomorphism $f_x \in DIFF(N \times [1,\infty), N \times \{1\})$ by

$$f_x(z,v,t) = exp^x_{z'}(\, z', ts)$$

By definition we get $f_x(z,v,1) = (z,v,1)$, and that f_x is continuous in $x \in X$.

Here is an alternative interpretation of f_x. For $(x,z,v) \in X \times \mathbb{S}^1 \times T^x(z)$, denote by $c^x_{(z,v)} : [0,\infty) \to Q$ the $\sigma(x)$-geodesic ray given by $c^x_{(z,v)}(t) = exp^x_z(tv)$. Then f_x sends $c^{u_0}_{(z,v)}$ to $c^x_{(z',s)}$, where $exp^x_{z'}(s) = (z,v) \in Q$. Explicitly, we have $f_x(\, c^{u_0}_{(z,v)}(t)\,) = c^x_{(z',s)}(t)$, for $t \geq 1$. Claim 2 implies that $f_x(\, N \times [1,\infty)\,) = N \times [1,\infty)$ and that f_x is a diffeomorphism. We remark that the map f_x defined here coincides with the map f_u defined at the top of p. 294 in [**10**] (but replace "u" by "x").

We denote by $\partial_\infty Q$ the space at infinity of Q with respect to the $\sigma(u_0)$ metric. Recall that the elements of $\partial_\infty Q$ are equivalence classes $[\beta]$ of $\sigma(u_0)$ quasi-geodesic rays $\beta : [a,\infty) \to Q = \mathbb{S}^1 \times \mathbb{R}^{n-1}$ (see section 2 of [**10**] and Section 3 here). By Claim 1, $\partial_\infty Q$ is independent of the metric $\sigma(x)$ used.

We now extend each f_x to a map $f_x : N \times [1,\infty] \to N \times [1,\infty) \cup \partial_\infty Q$ in the following way. For (z,v,∞) define $f_x(\,z,v,\infty\,) = [\,f_x(\,c^{u_0}_{(z,v)}\,)\,]$. Recall that, as we mentioned before, we have $f_x(\,c^{u_0}_{(z,v)}(t)\,) = c^{x}_{(z',s)}(t)$, where $exp^x_{z'}(s) = (z,v) \in Q$, $t \geq 1$. Thus $f_x(\,c^{u_0}_{(z,v)}\,)$ is a $\sigma(x)$-geodesic ray; hence it is a $\sigma(u_0)$-quasi-geodesic ray. Therefore $[\,f_x(\,c^{u_0}_{(z,v)}\,)\,]$ is a well defined element in $\partial_\infty Q$.

Recall that Lemma 3.4 says that the map A (which is a reparametrization of the exponential map exp^x) is cellular. This (applied to the metric $\sigma(x)$) together with the definition of f_x implies:

Claim 3. *$f_x : N \times [1,\infty] \to N \times [1,\infty) \cup \partial_\infty Q$ is continuous, a diffeomorphism on $N \times [1,\infty)$, and cellular on $N \times \{\infty\}$.*

Denote by f the map on X defined by $f(x) = f_x$.

Claim 4. *The map f is continuous on X.*

Proof of Claim 4. The proof is the same (almost word by word) as the proof of Claim 4 in [**10**], p. 295. There are a few obvious changes in notation: replace "u" by "x", "$\mathbb{D}^{k+1}$" by "X", "α_0" by "α", and "c_v" by "(v,∞)". Replace the phrase "Let $K = K(2\lambda, 0, c_2)$ be as in item 6 of Section 2, and c_2 is as in (vi) above" by "Let $K = K(2\lambda, 0, \delta)$ be as in Theorem 1.7 of [**1**], p. 401, where δ is as in Claim 1." Finally, again for notational purposes, replace the first three lines in the proof of Claim 4 [**10**] by the following. "Note that we know that $x \mapsto f_x|_Q$ is continuous. Let $q_n = (v_n, t_n) \to (v,\infty)$, $v, v_n \in N$, $t_n \in [0,\infty]$. Thus $v_n \to v$ and $t_n \to \infty$." This proves Claim 4. q.e.d.

In the next claim we use the map $A : N \times \{\infty\} = N \to \partial_\infty Q$ of Lemma 3.4.

Claim 5. *For all $u \in \mathbb{S}^k$ we have $f_u|_{Q\setminus W} = \phi_u|_{Q\setminus W}$ and $f_u|_{N\times\{\infty\}} = A$.*

Proof of Claim 5. The proof is the same (almost word by word) as the proof of Claim 5 in [**10**]. Just replace "ψ" by "ϕ", "U'" by "U", and the phrase (at the very end): "Therefore $f_u(\,[c_{u_0}(z,v)]\,) = [\,c_{u_0}(z,v)\,)\,]$. Hence $(f_u)|_{\partial_\infty} = 1_{\partial_\infty}$" by " Therefore $f_u(v,\infty) = [\,c_{u_0}(z,v)\,)\,]$. Hence $f_u|_{N\times\{\infty\}} = A$." This proves Claim 5. q.e.d.

For $u \in \mathbb{S}^k$ define $\Theta_u : N \times [1,\infty] \to (Q \setminus int\, W) \cup \partial_\infty Q = N \times [1,\infty) \cup \partial_\infty Q$, by $\Theta_u|_{N\times[1,\infty)} = \Phi^Q(\alpha, V', r)\theta(u)$ and $\Theta_u|_{N\times\{\infty\}} = A$.

Denote by Θ the map defined on $\mathbb{S}^k$ given by $\Theta(u) = \Theta_u$. Note that the identification between Q and $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ sends V' to the canonical basis $e_1, \dots, e_{n-1}$.

Claim 6. *The maps $f|_{\mathbb{S}^k}$ and Θ defined on $\mathbb{S}^k$ are homotopic.*

Proof of Claim 6. Let $u \in \mathbb{S}^k$. By Claim 5 we have $f_u = \phi_u$ outside $N \times \{\infty\}$. Recall that ϕ_u is the identity outside the union of $\bigcup U_i$ and U and inside the closed normal geodesic tubular neighborhood of width r of $\alpha = \mathbb{S}^1$; see 4.1(ii). From 4.1(iv) we have $(\phi_u)|_U = \left[\Phi^Q(\alpha, V', r)\theta(u)\right]|_U$, for $u \in \mathbb{S}^k$. Note that each U_i is diffeomorphic to $\mathbb{D}^{n-1} \times \mathbb{R}$. Remark that U_i is the $2r$ normal geodesic tubular neighborhood of a lifting β_i of $\alpha \subset M$ which is diffeomorphic to $\mathbb{R}$. Also note that the closure of U_i in $N \times [1, \infty]$ is formed exactly by the two points at infinity determined by this geodesic line. Consequently, the closure $\bar{U}_i$ of each U_i is homeomorphic to $\mathbb{D}^n$ and intersects $N \times \{\infty\}$ in exactly two different points. Now, applying Alexander's trick to each $\phi|_{\bar{U}_i}$, we obtain an isotopy (rel U) that isotopes ϕ_u to a map that is the identity outside $U \setminus int\,(W)$, and coincides with ϕ_u on U, that is, coincides with $\Phi^Q(\alpha, V', r)\theta(u)$ on U. Finally, note that $f_u|_{N\times\{\infty\}} = \Theta_u|_{N\times\{\infty\}} = A$ and that the isotopies defined above do not change the values on $N \times \{\infty\}$. This proves Claim 6. q.e.d.

Let $B : N \times [1, \infty] \to N \times [1, \infty) \cup \partial_\infty Q$ be defined by $B|_{N\times\{\infty\}} = A$ and $B|_{N\times[1,\infty)}$ is the identity on $N \times [1, \infty)$. Hence we can write $\Theta_u = B \circ \Phi^Q\theta(u)$. Note that B is just the restriction of a reparametrization of the map A defined on W given in Lemma 3.4. Hence B is cellular. Let $B_t : N \times [1, \infty] \to N \times [1, \infty) \cup \partial_\infty Q$ be a 1-parameter family of cellular maps such that B_0 is a homeomorphism and $B_1 = B$. (To show that B_t exists, take $k = 0$ in Lemma 2.2, or use the Complement to Theorem A in [**17**].)

Now consider the map $u \mapsto B_0^{-1} \circ f_u$ defined on $\mathbb{S}^k$ with values in $CELL(N \times [1, \infty])$. We denote this map by $B_0^{-1} \circ f|_{\mathbb{S}^k}$. As mentioned at the end of Section 2, $CELL(L)$ is LC^∞, for L compact (and $dim\, L \neq 3$). This together with the fact that $B_0^{-1} \circ f|_{\mathbb{S}^k}$ extends to $B_0^{-1} \circ f$, defined on the whole of X, and Proposition 1.2 imply that $B_0^{-1} \circ f|_{\mathbb{S}^k}$ extends to $\mathbb{D}^{k+1}$. On the other hand, by Claim 6, we get that $B_0^{-1} \circ f|_{\mathbb{S}^k}$ is homotopic to $B_0^{-1} \circ \Theta$ (this is the map $u \mapsto B_0^{-1} \circ \Theta_u$) in $CELL(N \times [1, \infty])$. But $B_0^{-1} \circ \Theta_u = B_0^{-1} \circ B \circ \Phi^Q\theta(u)$. Since $B_0^{-1} \circ B$ is homotopic to the identity we get that $B_0^{-1} \circ \Theta$ is homotopic to $\Phi^Q\theta$. By reparametrizing in the t direction and identifying $[1, \infty]$ with $[0, 1]$, we can identify $\Phi^Q\theta$ with $\iota'_N\iota_N\theta$. Therefore $\iota'_N\iota_N\theta$ extends to $\mathbb{D}^{k+1}$. This proves Theorem A.

Appendix A: The swinging neck

For a function $h : \mathbb{R} \to (0, \infty)$ denote by M_h the surface of revolution obtained by rotating the graph $\{(x, h(x), 0) \; : \; x \in \mathbb{R}\}$ of h around the x-axis. We consider M_h with the Riemannian metric induced by $\mathbb{R}^3$.

Let $f : \mathbb{R} \to [1, \infty)$ be a smooth function such that: **(1)** $f \equiv 1$ on $[-1, 1]$, **(2)** $f''(x) > 0$, $|x| > 1$, **(3)** $f''(x) \geq \delta > 0$, for $|x| \geq 2$. Then M_f is nonpositively curved and contains the flat cylinder $[-1, 1] \times \mathbb{S}^1$.

Let $\alpha : \mathbb{R} \times [-2, 2] \to [0, 1]$ be a smooth function such that (we write α_t for the function $x \mapsto \alpha(x, t)$): **(1)** $\alpha \equiv 0$ for $|x| \geq 4$ and all t, **(2)** $\alpha_t''(x) > 0$, for $|x| \leq 3$ and all t, **(3)** α_t has a unique minimum value (equal to 0) on $[-3, 3]$ at t, for all t.

Define $F : \mathbb{R} \times [0, 1] \to [1, \infty)$ by $F(x, 0) = f(x)$, and for $t \in (0, 1]$ by

$$F(x, t) = f(x) + e^{-1/t} \alpha(\, x, \sin(1/t)\,)$$

and write $f_t(x) = F(x, t)$. Thus $f_0 = f$. Then F is smooth and for small enough $t > 0$ we have: **(1)** $f_t''(x) \geq 0$, $\forall x \in \mathbb{R}$; **(2)** f_t has a unique minimum value at $\sin(1/t)$; **(3)** $f_t \equiv f$ outside $[-4, 4]$.

Write $M_t = M_{f_t}$ and $M = M_f$. Then M_t is negatively curved and coincides with M outside a compact set. Note that all $\{x\} \times \mathbb{S}^1$, $x \in [-1, 1]$, are non-trivial closed geodesics of minimal length in M. But M_t has a unique non-trivial closed geodesic $\{\sin(1/t)\} \times \mathbb{S}^1$ of minimal length that oscillates between $\{-1\} \times \mathbb{S}^1$ and $\{1\} \times \mathbb{S}^1$ faster and faster, as t approaches 0.

Note that, with some care, we can fit these "necks"—the relevant parts of M_t and M—on a closed negatively curved surface.

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