J. DIFFERENTIAL GEOMETRY
99 (2015) 47-75

CONVEXITY ESTIMATES FOR SURFACES MOVING BY CURVATURE FUNCTIONS

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Abstract

We consider the evolution of compact surfaces by fully nonlinear, parabolic curvature flows for which the normal speed is given by a smooth, degree one homogeneous function of the principal curvatures of the evolving surface. Under no further restrictions on the speed function, we prove that initial surfaces on which the speed is positive become weakly convex at a singularity of the flow. This generalises the corresponding result [**26**] of Huisken and Sinestrari for the mean curvature flow to the largest possible class of degree one homogeneous surface flows.

1. Introduction

Given a smooth, compact surface immersion $X_0 : M^2 \to \mathbb{R}^3$, we consider smooth families $X : M^2 \times [0,T) \to \mathbb{R}^3$ of smooth immersions $X(\cdot,t)$ solving the curvature flow

$$\begin{aligned} \frac{\partial X}{\partial t}(x,t) &= -\, s(x,t)\nu(x,t) \\ X(x,0) &= X_0(x)\,, \end{aligned} \tag{1.1}$$

where $\nu(x,t)$ is a choice of unit normal at (x,t), and the speed s is given by a smooth, symmetric function f of the principal curvatures $\kappa_1(x,t)$, $\kappa_2(x,t)$ with respect to $\nu(x,t)$. That is,

$$s(x,t) = f\left(\kappa_1(x,t)\,, \kappa_2(x,t)\right). \tag{1.2}$$

We require that the speed function f satisfy the following conditions:

Conditions 1.1.

(i) *that $f \in C^\infty(\Gamma)$, where $\Gamma \subset \mathbb{R}^2$ is an open, symmetric, connected cone;*
(ii) *that f is strictly increasing in each argument: $\frac{\partial f}{\partial x_i} > 0$ in Γ, for $i = 1, 2$;*
(iii) *that f is homogeneous of degree 1: $f(kx) = kf(x)$ for any $k > 0$ and any $x \in \Gamma$; and*
(iv) *that f is positive on Γ.*

Received 10/19/2012.

Note that we lose no generality by assuming further that Γ contains $(1,1)$ and f is normalised such that $f(1,1)=1$. Furthermore, since f is symmetric, we may at each point $(x,t)\in M\times[0,T)$ assume that $\kappa_2(x,t)\geq\kappa_1(x,t)$.

We note that Condition (ii) ensures that (1.1) is, locally, a parabolic system. Short-time existence and uniqueness of solutions can be inferred using standard techniques (see [**19**, **13**, **18**]), so long as the principal curvatures of the initial immersion lie in Γ.

The following examples illustrate the class of flows and initial surfaces considered.

Examples. *The following speed functions satisfy Conditions* 1.1*:*

1) The mean curvature: $f(x_1,x_2)=x_1+x_2$ *on the half-space* $\Gamma=\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2>0\}$.

2) The power means: $f(x_1,x_2)=\left(|x_1|^\beta+|x_2|^\beta\right)^{\frac{1}{\beta}}$, $\beta\in\mathbb{R}$, *on the positive cone* $\Gamma=\Gamma_+$.

3) Positive linear combinations of functions satisfying Conditions 1.1: *If* $f_1,\dots,f_k$ *satisfy Conditions* 1.1 *on* Γ, *then, for all* $(s_1,\dots,s_k)\in\Gamma_+^k$, *the positive cone in* $\mathbb{R}^k$, *the function* $f=s_1f_1+\dots+s_kf_k$ *satisfies Conditions* 1.1 *on* Γ. *For example, the function* $f(x_1,x_2)=x_1+x_2+\sqrt{x_1^2+x_2^2}$ *is admissible on the cone* Γ_+. *(In fact, this speed is admissible on the much larger cone* $\Gamma=\{(x_1,x_2)\in\mathbb{R}^2 : \max\{x_1,x_2\}>0\}$*.)*

4) Homogeneous combinations of functions satisfying Conditions 1.1: *Let* $\phi:\Gamma_+^k\to\mathbb{R}$ *be smooth, homogeneous of degree one, monotone increasing in each argument, and strictly increasing in at least one argument. Then, if* $f_1,\dots,f_k$ *satisfy Conditions* 1.1 *on* Γ, *the function* $f(x_1,x_2):=\phi(f_1(x_1,x_2),\dots,f_k(x_1,x_2))$ *satisfies Conditions* 1.1 *on* Γ.

5) A general construction: *Write* x_1,x_2 *in polar coordinates* (r,θ) *defined by*

$$r=\sqrt{x_1^2+x_2^2}\,,\quad \cos\theta=\frac{x_1+x_2}{\sqrt{2(x_1^2+x_2^2)}}\,,\quad \sin\theta=\frac{x_2-x_1}{\sqrt{2(x_1^2+x_2^2)}}\,.$$

Then, writing $f=r\phi(\theta)$, *Conditions* 1.1 *become*

$$\phi>0$$

and

$$A(\theta)<\frac{\phi'}{\phi}<B(\theta)\,,$$

where

$$A(\theta)=\begin{cases}\frac{\cos\theta+\sin\theta}{\sin\theta-\cos\theta}, & -3\pi/4<\theta<\pi/4;\\ -\infty, & \pi/4\leq\theta\leq 3\pi/4;\end{cases}$$

and

$$B(\theta) = \begin{cases} +\infty, & -3\pi/4 \le \theta \le -\pi/4; \\ \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}, & -\pi/4 < \theta \le 3\pi/4\,. \end{cases}$$

Therefore, given any smooth, odd function $\psi : (-c, c) \to \mathbb{R}$, with $0 < c \le 3\pi/4$, satisfying $A(\theta) < \psi(\theta) < B(\theta)$, we can construct an admissible speed function $f = r\phi(\theta)$ on the cone $\{-c < \theta < c\}$ by taking $\phi = e^{\int_0^\theta \psi(\sigma)d\sigma}$.

Curvature problems of the form (1.1), for which the speed f satisfies Conditions 1.1, have been studied extensively, both for surfaces in $\mathbb{R}^3$ and for higher dimensional Euclidean hypersurfaces. In particular, when the initial (hyper)surface $X_0 : M \to \mathbb{R}^{n+1}$ ($n \ge 2$) is convex, much is known about the behaviour of solutions. Huisken [**24**] showed that convex hypersurfaces ($n \ge 2$) flowing by mean curvature remain convex and shrink to round points, 'round' meaning that a suitable rescaling converges smoothly to the sphere. These results were extended by Chow to flows by the n-th root of the Gauss curvature [**15**], and, in the presence of a curvature pinching condition, the square root of the scalar curvature [**16**]. Each of these speeds satisfies Conditions 1.1, with $\Gamma = \Gamma_+^n := \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i\}$, the positive cone. More general degree one homogeneous speeds were treated by the first author in [**3**, **5**, **6**], where it was shown that a very general class contract convex hypersurfaces to round points. In fact, when the dimension of the hypersurface is 2, it was shown in [**8**] that no additional restrictions on the speed are necessary; that is, all surface flows with speeds satisfying Conditions 1.1 (i)–(iii) on $\Gamma = \Gamma_+$ shrink convex surfaces to round points. Note that one cannot hope to extend this result to higher dimensions, since, in that case, there exist smooth, homogeneous degree one speeds that do not preserve convexity of the initial hypersurface [**12**, Theorem 3].

It is true in general (Proposition 2.6) that flows (1.1) satisfying Conditions 1.1 remain smooth until the curvature blows up (after a finite time), just as for convex surfaces. On the other hand, if the initial surface is not convex, the behaviour of solutions near a singularity is potentially more complicated than that of the shrinking sphere. For the mean curvature flow, a crucial part of the current understanding of singularities is the asymptotic convexity estimate of Huisken and Sinestrari [**26**] (see also White [**38**]), which states that any mean convex initial surface becomes weakly convex at a singularity. This estimate is an analogue for extrinsic flows of the famous Hamilton-Ivey estimate for three-dimensional Ricci flow [**23**, **28**]. In conjunction with the monotonicity formula of Huisken [**25**] and the Harnack inequality of Hamilton [**22**], the convexity estimate yields a rather complete description of singularities in the positive mean curvature case. In particular, asymptotic convexity is necessary in order to apply the Harnack inequality to show that 'fast-forming' or 'type-II' singularities are asymptotic to convex translation solutions of the flow. For other flows, the understanding of singularities is far less developed, for several reasons: First, there is no analogue available for the monotonicity formula, which shows that 'slowly forming' or 'type-I' singularities of the mean curvature flow are asymptotically self-similar. Second, there is in general no Harnack inequality available sufficient to classify type-II singularities, although the latter is known for quite a wide sub-class of flows [**4**]. And finally, until recently, there was no analogue of the Huisken-Sinestrari asymptotic convexity estimate for most other flows, with the notable exception of the result of Alessandroni and Sinestrari [**1**], which applies to a special class of flows by functions of the mean curvature having a certain asymptotic behaviour. In a companion paper [**11**], the authors prove that an asymptotic convexity estimate holds for fully non-linear flows (1.1) satisfying Conditions 1.1 if, in addition, the speed f is a convex function. The main purpose of this paper is to show that an asymptotic convexity estimate holds in surprising generality for flows of surfaces; namely, the assumption that f is convex is unnecessary:

Theorem 1.2. *Let $X : M^2 \times [0, T) \to \mathbb{R}^3$ be a solution of* (1.1) *for which $f : \Gamma \to \mathbb{R}$ satisfies Conditions* 1.1*. Then for any $\varepsilon > 0$ there is a constant C_ε such that for all $(x, t) \in M \times [0, T)$ we have*

$$\kappa_1(x,t) \geq -\varepsilon s(x,t) - C_\varepsilon \,.$$

Applications of the convexity estimate are discussed in [**11**]. In particular, the Harnack inequality [**4**] yields a description of type-II singularities analogous to that of the mean curvature flow, as long as the speed f satisfies a certain concavity condition on the positive cone (this condition is satisfied, for example, if f is convex, or inverse-concave). If the speed function is concave, then the results of [**10**] may be used to rule out singular profiles such as $\mathcal{G} \times \mathbb{R}$, where $\mathcal{G}$ is the Grim Reaper curve.

It is worth noting that Theorem 1.2 cannot be expected to hold in higher dimensions without additional conditions on the speed function (such as convexity) since, in general, quite different behaviour is possible; there are, for example, concave speed functions that permit loss of convexity, as mentioned earlier. The special feature of the two-dimensional case is that the 'difficult' terms involving first derivatives which arise in the evolution of the second fundamental form, which must normally be controlled by assuming some concavity condition on the speed function, turn out under careful inspection to be automatically favourable to preserve bounds on the ratios of principal curvatures. This observation was first made in [**9**], where it was used to show that convex surfaces contract to round points for a similarly general class of speeds, and has also been used in [**32**] to show that compact self-similar solutions of a wide variety of flows are spheres. Similar ideas are also present in [**34**], where they are used to obtain convergence to round points under the flow with speed given by $|h|^2$, the squared norm of the second fundamental form.

We remark that the proof of Theorem 1.2 utilises a Stampacchia iteration procedure analogous to those of [**24**, **26**, **27**], whereas the result of [**1**] is proved more directly, using the maximum principle.

Acknowledgments. This research was partly supported by ARC Discovery Projects grants DP0556211, DP120100097.

2. Preliminaries

The curvature function f is a smooth, symmetric function defined on a symmetric cone. Denote by $\mathcal{S}_\Gamma$ the cone of symmetric 2×2 matrices whose eigenvalue pair, $\lambda := (\lambda_1, \lambda_2)$, lies in Γ. A result of Glaeser [**20**] implies that there is a smooth, $GL(2)$ invariant function $F : \mathcal{S}_\Gamma \to \mathbb{R}$ such that $f(\lambda(A)) = F(A)$, where $\lambda(A) = (\lambda_1(A), \lambda_2(A))$ are the eigenvalues of A. The $GL(2)$ invariance of F implies that the speed $s(x,t) = f(\kappa_1(x,t), \kappa_2(x,t))$ is a well-defined smooth function of the Weingarten map, $\mathcal{W}$; that is, $s(x,t) = F(\mathcal{W}(x,t)) := F(W)$, where $W(x,t)$ is the component matrix of $\mathcal{W}(x,t)$ with respect to some basis of endomorphisms of T_xM. If we restrict attention to orthonormal bases, then $W_i{}^j = h_{ij}(x,t)$, where h_{ij} are the components of the second fundamental form h (which is the bilinear form related to the endomorphism $\mathcal{W}$ by the metric). This point of view will be more convenient.

We shall use dots to indicate derivatives with respect to the principal curvatures and the second fundamental form; for example,

$$\begin{aligned}
\dot{f}^i(\lambda)v_i &:= \left.\frac{d}{ds}\right|_{s=0} f(\lambda + sv)\\
\ddot{f}^{ij}(\lambda)v_iv_j &:= \left.\frac{d^2}{ds^2}\right|_{s=0} f(\lambda + sv)\\
\dot{F}^{ij}(A)B_{ij} &:= \left.\frac{d}{ds}\right|_{s=0} F(A + sB)\\
\ddot{F}^{pq,rs}(A)B_{pq}B_{rs} &:= \left.\frac{d^2}{ds^2}\right|_{s=0} F(A + sB)\,.
\end{aligned}$$

Note that the summation convention is used here, and throughout. The derivatives of f and F are related in the following way [**17**, **3**, **8**]: If A is a diagonal, and B a symmetric matrix, then

$$\dot{F}^{kl}(A) = \dot{f}^k(\lambda(A))\delta^{kl}\,,$$

and, if $\lambda_1(A) \neq \lambda_2(A)$,

$$\ddot{F}^{pq,rs}(A)B_{pq}B_{rs} = \ddot{f}^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{f}^p(\lambda(A)) - \dot{f}^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)} \left(B_{pq}\right)^2 .$$

In fact, the latter identity makes sense as a limit if $\lambda_1 = \lambda_2$. Therefore, in particular, in a local orthonormal frame of eigenvectors of $\mathcal{W}$, we have

$$\dot{F}^{kl}(\mathcal{W}) = \dot{f}^k(\kappa)\delta^{kl} \tag{2.1}$$

and

$$\ddot{F}^{pq,rs}(\mathcal{W})B_{pq}B_{rs} = \ddot{f}^{pq}(\kappa)B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{f}^p(\kappa) - \dot{f}^q(\kappa)}{\kappa_p - \kappa_q} \left(B_{pq}\right)^2 . \tag{2.2}$$

In what follows, we will drop the arguments when F and f, and their derivatives, are evaluated at $\mathcal{W}$ or κ. This convention makes the notation s for the speed obsolete, and we henceforth replace it by F. That is, we identify $F(x,t) \equiv F(\mathcal{W}(x,t))$. We remark that the preceding discussion depends only on the fact that f is a smooth, symmetric function defined on an open, symmetric cone, and not on any properties of the flow.

We now note the following evolution equations, which are well known (see, for example, [**24**, **3**, **12**]).

Lemma 2.1. *Under the flow* (1.1)*,*

(i) $\partial_t g_{ij} = -2F\, h_{ij}$ *;*
(ii) $(\partial_t - \mathcal{L})F = \dot{F}^{kl} h_k{}^m h_{ml} F$ *; and*
(iii) $\partial_t\, d\mu = -HF\, d\mu$ *,*

where g_{ij} *denote the components of the induced metric,* μ *denotes the induced measure, and* $\mathcal{L}$ *denotes the (elliptic) operator* $\dot{F}^{kl}\nabla_k\nabla_l$ *(where* ∇ *is the induced Levi-Civita connection).*

Moreover, given any smooth, symmetric function $g : \Gamma \to \mathbb{R}$*, the corresponding curvature function* $G := g(\kappa)$ *evolves according to*

(iv) $(\partial_t - \mathcal{L})G = \left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} + \dot{G}^{ij}h_{ij}\dot{F}^{kl}h^2_{kl}$ *,*

where $h^2_{kl} = h_k{}^m h_{ml}$*.*

Consider the evolution equation for F in statement (ii) of Lemma 2.1. The identity (2.1) implies that, in an orthonormal frame of eigenvectors for $\mathcal{W}$,

$$\dot{F}^{kl}h_k{}^m h_{ml} = \dot{f}^i\kappa_i^2 \geq 0 . \tag{2.3}$$

Therefore, since $F > 0$, the maximum principle implies that the minimum of F cannot decrease under the flow. In particular, since Euler's Theorem for Homogeneous Functions implies $f(\kappa_1, \kappa_2) = \dot{f}^1\kappa_1 + \dot{f}^2\kappa_2$, we find that the largest principal curvature of the solution remains positive. In fact, a time dependent lower bound for the speed is also possible (see Lemma 2.5).

Now consider a smooth, symmetric, degree zero homogeneous function $g : \Gamma \to \mathbb{R}$. By Euler's Theorem, we have that the corresponding curvature function $G = g(\kappa_1, \kappa_2)$ evolves under (1.1) according to

$$(\partial_t - \mathcal{L})G = (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\,. \tag{2.4}$$

The following lemma helps us to find preserved curvature cones. It is proved in [**9**, Proposition 2], but we give the argument here as the computations will be useful in what follows.

Lemma 2.2. *Let $g : \Gamma \to \mathbb{R}$ be a smooth, symmetric, homogeneous degree zero function, and denote by $G \equiv G(\mathcal{W}) = g(\kappa)$ the corresponding curvature function. Then, at any spatial stationary point of G for which $\dot{G}$ is non-degenerate, it holds that*

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})&\nabla_k h_{pq}\nabla_l h_{rs}\\ &= \frac{2F\dot{g}^1}{\kappa_2(\kappa_2-\kappa_1)}\left[\left(\nabla_1 h_{12}\right)^2 + \left(\nabla_2 h_{12}\right)^2\right]\,.\end{aligned}$$

Proof. We first show that $\kappa_1 \neq 0$ and $\kappa_2 \neq \kappa_1$ wherever $\dot{G}$ is non-degenerate. We compute in an orthonormal basis of eigenvectors of $\mathcal{W}$ at any point where $\dot{G}$ is non-degenerate. Then, by (2.1), $\dot{G}^{kl} = \dot{g}^k\delta^{kl}$, and it follows that $\dot{g}^k \neq 0$ for each k. Since g is homogeneous of degree zero, Euler's Theorem implies $\dot{g}^1\kappa_1 + \dot{g}^2\kappa_2 = 0$. First suppose that $\kappa_1 = \kappa_2$; then we must have $\dot{g}^2 = -\dot{g}^1$. But g is symmetric, which implies $\dot{g}^1 = \dot{g}^2$ whenever $\kappa_2 = \kappa_1$. It follows that $\dot{G} = 0$, a contradiction. Therefore $\kappa_2 \neq \kappa_1$ wherever $\dot{G}$ is non-degenerate. Now suppose $\kappa_1 = 0$. Then, again from Euler's Theorem, $\dot{g}^2\kappa_2 = 0$. But $\kappa_2 > 0$, so that $\dot{g}^2 = 0$, another contradiction. Hence $\kappa_1 \neq 0$ wherever $\dot{G}$ is non-degenerate.

Now, from (2.2), the non-zero components of $\ddot{F}$ (and similarly for G) are given by

$$\begin{aligned}\ddot{F}^{11,11} &= \ddot{f}^{11}\,; & \ddot{F}^{11,22} &= \ddot{F}^{22,11} = \ddot{f}^{12}\,;\\ \ddot{F}^{22,22} &= \ddot{f}^{22}\,; & \ddot{F}^{12,12} &= \ddot{F}^{21,21} = \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}\,.\end{aligned} \tag{2.5}$$

Therefore, defining $R_1 := \dot{F}^{kl}\ddot{G}^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs}$, we have

$$\begin{aligned}R_1 = {}&\dot{f}^1\ddot{g}^{11}(\nabla_1 h_{11})^2 + \dot{f}^2\ddot{g}^{22}(\nabla_1 h_{22})^2 + \dot{f}^1\ddot{g}^{22}(\nabla_1 h_{22})^2\\ &+ \dot{f}^2\ddot{g}^{11}(\nabla_2 h_{11})^2 + 2\dot{f}^1\ddot{g}^{12}\nabla_1 h_{11}\nabla_1 h_{22} + 2\dot{f}^2\ddot{g}^{12}\nabla_2 h_{11}\nabla_2 h_{22}\\ &+ 2\dot{f}^1\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{12})^2 + 2\dot{f}^2\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_2 h_{12})^2\,.\end{aligned}$$

This may be written in terms of $\nabla_k G = \dot{G}^{pq}\nabla_k h_{pq} = \dot{g}^1\nabla_k h_{11} + \dot{g}^2\nabla_k h_{22}$ as follows:

$$\begin{aligned}
R_1 = {} & \frac{\dot{f}^1}{\dot{g}^1}\frac{\ddot{g}^{11}}{\dot{g}^1}(\nabla_1 G)^2 + \frac{\dot{f}^2}{\dot{g}^2}\frac{\ddot{g}^{22}}{\dot{g}^2}(\nabla_2 G)^2 \\
& + 2\frac{\dot{f}^1}{\dot{g}^1}\nabla_1 G\,\nabla_1 h_{22}\left(\ddot{g}^{12} - \frac{\dot{g}^2}{\dot{g}^1}\ddot{g}^{11}\right) \\
& + 2\frac{\dot{f}^2}{\dot{g}^2}\nabla_2 G\,\nabla_2 h_{11}\left(\ddot{g}^{12} - \frac{\dot{g}^1}{\dot{g}^2}\ddot{g}^{22}\right) \\
& + \dot{f}^1\frac{\dot{g}^2}{\dot{g}^1}(\nabla_1 h_{22})^2\left(\frac{\dot{g}^2}{\dot{g}^1}\ddot{g}^{11} - 2\ddot{g}^{12} + \frac{\dot{g}^1}{\dot{g}^2}\ddot{g}^{22}\right) \\
& + \dot{f}^2\frac{\dot{g}^1}{\dot{g}^2}(\nabla_2 h_{11})^2\left(\frac{\dot{g}^1}{\dot{g}^2}\ddot{g}^{22} - 2\ddot{g}^{12} + \frac{\dot{g}^2}{\dot{g}^1}\ddot{g}^{11}\right) \\
& + 2\dot{f}^1\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{12})^2 + 2\dot{f}^2\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_2 h_{12})^2 .
\end{aligned}$$

But note that, due to Euler's Theorem, any smooth, homogeneous degree γ function k of two variables, y_1, y_2, satisfies the following identities:

$$\begin{aligned}
\dot{k}^1 y_1 + \dot{k}^2 y_2 &= \gamma k\,; \\
\ddot{k}^{11} y_1 + \ddot{k}^{12} y_2 &= (\gamma - 1)\dot{k}^1\,; \\
\ddot{k}^{22} y_2 + \ddot{k}^{12} y_1 &= (\gamma - 1)\dot{k}^2\,; \\
\text{and} \qquad \ddot{k}^{11}(y_1)^2 + 2\ddot{k}^{12} y_1 y_2 + \ddot{k}^{22}(y_2)^2 &= \gamma(\gamma - 1)k\,.
\end{aligned} \tag{2.6}$$

Since the first of these identities implies $\dot{g}^2/\dot{g}^1 = -\kappa_1/\kappa_2$, the following three imply

$$\begin{aligned}
R_1 = {} & \frac{\dot{f}^1}{\dot{g}^1}\frac{\ddot{g}^{11}}{\dot{g}^1}(\nabla_1 G)^2 + \frac{\dot{f}^2}{\dot{g}^2}\frac{\ddot{g}^{22}}{\dot{g}^2}(\nabla_2 G)^2 - 2\frac{\dot{f}^1}{\kappa_2}\nabla_1 G\,\nabla_1 h_{22} \\
& - 2\frac{\dot{f}^2}{\kappa_1}\nabla_2 G\,\nabla_2 h_{11} + 2\dot{f}^1\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{12})^2 + 2\dot{f}^2\frac{\dot{g}^2 - \dot{g}^1}{\kappa_2 - \kappa_1}(\nabla_2 h_{12})^2 .
\end{aligned}$$

We can play a similar game with $R_2 := \dot{G}^{kl}\ddot{F}^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs}$. We find

$$\begin{aligned}
R_2 = {} & \frac{\ddot{f}^{11}}{\dot{g}^1}(\nabla_1 G)^2 + \frac{\ddot{f}^{22}}{\dot{g}^2}(\nabla_2 G)^2 \\
& + 2\dot{g}^1\frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_1 h_{12})^2 + 2\dot{g}^2\frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}(\nabla_2 h_{12})^2 .
\end{aligned}$$

Therefore,

$$
\begin{aligned}
R &:= R_2 - R_1 \\
&= \left(\frac{\ddot f^{11}}{\dot g^1} - \frac{\dot f^1}{\dot g^1}\frac{\ddot g^{11}}{\dot g^1}\right)(\nabla_1 G)^2 + \left(\frac{\ddot f^{22}}{\dot g^2} - \frac{\dot f^2}{\dot g^2}\frac{\ddot g^{22}}{\dot g^2}\right)(\nabla_2 G)^2 \qquad (2.7)\\
&\quad + 2\frac{\dot f^1}{\kappa_2}\nabla_1 G\,\nabla_1 h_{22} + 2\frac{\dot f^2}{\kappa_1}\nabla_2 G\,\nabla_2 h_{11} \\
&\quad + 2\frac{\dot g^1\dot f^2 - \dot g^2\dot f^1}{\kappa_2-\kappa_1}\left[(\nabla_1 h_{12})^2 + (\nabla_2 h_{12})^2\right].
\end{aligned}
$$

The first four terms vanish at a spatial critical point of G and the coefficient of the final term is

$$
2\frac{\dot g^1\dot f^2 - \dot g^2\dot f^1}{\kappa_2-\kappa_1} = 2\frac{\dot g^1\dot f^2\kappa_2 - \dot g^2\kappa_2\dot f^1}{\kappa_2(\kappa_2-\kappa_1)} = 2\frac{\dot g^1 F}{\kappa_2(\kappa_2-\kappa_1)}.
$$

This completes the proof. q.e.d.

Corollary 2.3. *Define* $c_0 := \min_{M\times\{0\}} \frac{H}{|h|}$. *Then*

$$
H(x,t) \geq c_0|h(x,t)|
$$

for all $(x,t)\in M\times[0,T)$.

Proof. Define

$$
g(\kappa_1,\kappa_2) := \frac{\kappa_1+\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}} = \frac{H}{|h|}.
$$

Then, assuming $\kappa_2 \geq \kappa_1$, we have

$$
g(\kappa_1,\kappa_2) = \phi\left(\frac{\kappa_1}{\kappa_2}\right), \quad \text{where} \quad \phi(r) := \frac{1+r}{\sqrt{1+r^2}}.
$$

Therefore,

$$
\dot g^1(\kappa_1,\kappa_2) = \phi'\left(\frac{\kappa_1}{\kappa_2}\right)\frac{1}{\kappa_2} \quad \text{and} \quad \dot g^2(\kappa_1,\kappa_2) = -\phi'\left(\frac{\kappa_1}{\kappa_2}\right)\frac{\kappa_1}{\kappa_2^2}.
$$

Now, $\phi'(r) = \frac{1-r}{(1+r^2)^{3/2}}$. It follows that $\dot G$ is degenerate only if either $\kappa_1 = 0$ or $\kappa_1 = \kappa_2$. Since $\phi(r) < \phi(0)$ whenever $r < 0$, we cannot have $\kappa_1 = 0$ or $\kappa_1 = \kappa_2$ at a minimum point of g unless the surface is weakly convex. On the other hand, at a non-convex point, we have $\frac{\kappa_1}{\kappa_2} < 0$, so that $\dot g^1 < 0$. In view of Lemma 2.2, the result now follows from the maximum principle. q.e.d.

Now define the cone $\Gamma_{c_0} := \{x\in\mathbb{R}^2 : x_1+x_2 > c_0\sqrt{x_1^2+x_2^2}\}$. Then, by the definition of c_0, we have $\bar\Gamma_{c_0}\setminus\{0\}\subset\Gamma$. It follows that the slices $K_C := \bar\Gamma_{c_0}\cap\{x\in\mathbb{R}^2 : |x| = C > 0\}$ are compact. Since the speed (and hence also κ_2) remains positive under the flow, Corollary 2.3 implies that the cone $\bar\Gamma_{c_0}$ is preserved. This observation allows us to obtain useful estimates on homogeneous quantities. For example, we find that the flow is uniformly parabolic:

Corollary 2.4. *There is a constant $c_1 > 0$ for which*

$$\frac{1}{c_1} g^{kl} \leq \dot{F}^{kl} \leq c_1 g^{kl} \tag{2.8}$$

along the flow, where g^{kl} are the components of the inverse cometric.

Proof. Since $\bar{\Gamma}_{c_0} \setminus \{0\} \subset \Gamma$ is preserved by the flow (Corollary 2.3), it suffices to estimate $\dot{F}^{kl}$ on $\bar{\Gamma}_{c_0} \setminus \{0\}$. Since $\dot{f}^i > 0$ on Γ for each i, we have positive lower bounds for each $\dot{f}^i$ on the compact set $K := \bar{\Gamma}_{c_0} \cap \{x \in \Gamma : |x| = 1\} \subset \Gamma$. The degree zero homogeneity of $\dot{f}^i$ in κ implies that these bounds extend to the entire cone $\bar{\Gamma}_{c_0} \setminus \{0\}$. The claim now follows, since, by (2.1), $\dot{F}^{ij} = \dot{f}^i \delta^{ij}$ in an orthonormal frame of eigenvectors of the Weingarten map. q.e.d.

As promised, this leads to a time dependent lower bound for the speed:

Lemma 2.5. *There is a constant $\underline{c} > 0$ such that*

$$F \geq \frac{F_{\min}(0)}{\sqrt{1 - 2\underline{c}\, F_{\min}^2(0)\, t}} \,,$$

where $F_{\min}(0) = \min_{M \times \{0\}} F > 0$.

Proof. Applying the maximum principle to the evolution equation for F, we have that

$$\frac{d}{dt} F_{\min}(t) \ \geq \ \dot{F}^{kl} h_{km} h^m{}_l F_{\min}(t) \ = \ \dot{f}^i \kappa_i F_{\min}(t)$$

at almost every t in the interval of existence of the solution. In order to get the time dependent lower bound, we need to establish an estimate of the form

$$Q := \frac{\dot{f}^i \kappa_i^2}{f^2} \geq \underline{c} > 0 \,. \tag{2.9}$$

The result then follows from the maximum principle by comparing $F_{\min}$ with the solution of the ordinary differential equation

$$\frac{du}{dt} = \underline{c} u^3 \,.$$

Since $\bar{\Gamma}_{c_0} \setminus \{0\} \subset \Gamma$ is preserved by the flow (Corollary 2.3), it suffices to estimate Q on $\bar{\Gamma}_{c_0} \setminus \{0\}$. Now, for each i, $\dot{f}^i > 0$ on Γ, so we have a positive lower bound for $f^{-2} \dot{f}^i \kappa_i^2$ on the compact slice $K := \bar{\Gamma}_{c_0} \cap \{x \in \Gamma : |x| = 1\}$. But this bound extends to the whole cone $\bar{\Gamma}_{c_0} \setminus \{0\}$ since $f^{-2} \dot{f}^i \kappa_i^2$ is homogeneous of degree zero in the principal curvatures. q.e.d.

Remark. Lemma 2.5 motivates the distinction between type-I (or slow) and type-II (or fast) singularities, just as for the mean curvature flow: that is, those for which the curvature satisfies

$$\max_{M\times\{t\}} |h| \leq \frac{C}{\sqrt{2(T-t)}}$$

for some $C > 0$, and those for which it does not, respectively.

It follows from the preceding lemma that smooth solutions of the flow can only exist for a finite time. We now show that a singularity cannot occur whilst the curvature is bounded.

Proposition 2.6. *If f satisfies Conditions 1.1 and the principal curvatures of $X_0 : M \to \mathbb{R}^{n+1}$ lie in Γ, then the solution of equation (1.1) exists on a maximal time interval $[0, T)$, with $T < \infty$, and $\max_{M\times\{t\}} |h| \to \infty$ as $t \to T$.*

Proof. The proof is similar to that of the mean curvature flow [**24**]. We have already mentioned that $T < \infty$. Contrary to the statement of the proposition, suppose that $\max_{M\times\{t\}} |h|^2 \leq C$ for $t \to T$. We will show that this implies that $X(\cdot, t)$ approaches a smooth limit immersion X_T whose principal curvatures, by Corollary 2.3 and Lemma 2.5, must lie everywhere in Γ. This immersion could then be used as initial data in the short time existence result, extending the solution smoothly, contradicting the maximality of T.

From the evolution equation (1.1), we have for any $x \in M$,

$$|X(x, t_2) - X(x, t_1)| \leq \int_{t_1}^{t_2} F(x, \tau)\, d\tau,$$

where $0 \leq t_1 \leq t_2 < T$. Applying Conditions 1.1, we have

$$f(\kappa_1, \kappa_2) \leq f(\kappa_{\max}, \kappa_{\max}) = \kappa_{\max} \leq |h| \leq \sqrt{C},$$

so $X(\cdot, t)$ tends to a unique, continuous limit $X(\cdot, T)$ as $t \to T$.

We now show that the limit is an immersion. We recall the following theorem:

Theorem 2.7 (Hamilton [**21**]). *Let g_{ij} be a time dependent metric on a compact manifold M for $0 \leq t < T \leq \infty$. Suppose*

$$\int_0^T \max_M \left|\frac{\partial}{\partial t} g_{ij}\right| dt \leq C < \infty. \tag{2.10}$$

Then the metrics $g_{ij}(t)$ for all different times are equivalent and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

To apply Theorem 2.7, we use the evolution equation for the metric, Lemma 2.1 (i). Since $|h|$ is bounded and $T < \infty$, (2.10) is satisfied.

It remains to show that the resulting hypersurface M_T is smooth. To do this we can use a simplification of the argument for long time regularity in [**31**]. Writing our evolving surface locally as a graph $\varphi : U \subset \mathbb{R}^2 \times [0,T) \to \mathbb{R}^3$ given by

$$\varphi(x,t) = (x, z(x,t))$$

and incorporating a tangential diffeomorphism into the flow (1.1) such that this parametrisation is preserved, the graph height evolves according to

$$\frac{\partial z}{\partial t} = -\sqrt{1+|Dz|^2} F = \dot{F}^{ij} g^{-1}_{ik} D_k D_j z, \tag{2.11}$$

where D is the ordinary derivative on $\mathbb{R}^2$.

The matrix product $g^{-1}\dot{F}$ can be rewritten as $\tilde{g}\dot{F}\tilde{g}$ for the symmetric square root of the matrix of the inverse metric $\tilde{g}$, as in [**37**]. So, in view of (2.8), the equation (2.11) is uniformly parabolic.

The evolution equation for F in the local graph setting follows from Lemma 2.1 (ii):

$$\frac{\partial F}{\partial t} = \dot{F}^{ij} g^{-1}_{ik} D_k D_j F - g^{-1}_{ik} \dot{F}^{ij} \Gamma_{kj}{}^{l} D_l F + \dot{F}^{kl} h_k{}^m h_{ml} F,$$

and is likewise uniformly parabolic. Here $\Gamma_{ij}{}^{k}$, the connection coefficients of the evolving metric, do not depend on second derivatives of F. Moreover, the assumed curvature bound implies that the first derivatives of z are bounded locally. Indeed, writing $z_j = \frac{\partial z}{\partial x_j}$, in the local graph parametrisation, the spatial derivatives of z and the Weingarten map are related by

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(\frac{z_j}{\sqrt{1+|Dz|^2}}\right) &= \frac{1}{\sqrt{1+|Dz|}}\left(\delta_{ik} - \frac{z_i z_k}{1+|Dz|^2}\right) z_{kj} \\ &= -\, h^i{}_j. \end{aligned} \tag{2.12}$$

Now $|h|^2 \le C$ implies that we have bounds for each trace element of the Weingarten map

$$-\sqrt{C} \le h^i{}_i \le \sqrt{C}\,.$$

Integrating (2.12) with respect to x_i from the origin of the local parametrisation then yields

$$-\sqrt{C} x_i \le \frac{z_i}{\sqrt{1+|Dz|^2}} \le \sqrt{C} x_i\,.$$

Squaring, and summing over i, it follows that

$$\frac{|Dz|^2}{1+|Dz|^2} \le C\,|x|^2\,,$$

so $|Dz|$ is locally bounded (by 1, for example, on $\left\{|x| \le \frac{1}{2\sqrt{C}}\right\}$).

A well-known result of Krylov-Safonov [**29**] now implies that z and F are $C^{0,\beta}$ in spacetime. Now $C^{2,\beta}$ regularity in spacetime follows using results from [**14**] and [**7**], as in [**31**]. We note that the estimates of [**7**] do not require any concavity condition on F. Higher regularity follows by parabolic Schauder estimates (see, e.g., [**30**]), giving bounds in $C^{\ell,\beta}$ for all ℓ. These local estimates depend only on the curvature bound, and are easily extended to the whole of $M_T := X(M,T)$. This implies M_T is smooth, allowing us to apply the short-term existence theorem, contradicting the maximality of T. q.e.d.

3. The pinching function

Now consider the symmetric, homogeneous degree zero function

$$g(x_1,x_2) := \phi\left(\frac{x_{\min}}{x_{\max}}\right),$$

where $x_{\max} := \max\{x_1,x_2\}$, $x_{\min} := \min\{x_1,x_2\}$, and $\phi : [-a,\infty) \to \mathbb{R}$ is defined by

$$\phi(r) := \frac{-r}{a+r}, \quad a > \frac{1-c_0}{1+c_0}.$$

Then g is smooth on $\bar\Gamma_{c_0} \setminus \{x \in \mathbb{R}^2 : x_1 = x_2\}$, with (assuming $x_2 > x_1$)

$$\left(\dot g^1(x_1,x_2),\, \dot g^2(x_1,x_2)\right) = \frac{1}{x_2}\phi'\left(\frac{x_1}{x_2}\right)\left(1, -\frac{x_1}{x_2}\right).$$

Since $\phi'(r) = \frac{-a}{(a+r)^2}$, we have $\dot g^i < 0$ on $\bar\Gamma_{c_0} \setminus \bar\Gamma_+$ for each i. Moreover, g is positive on $\bar\Gamma_{c_0} \setminus \bar\Gamma_+$, vanishes on $\partial\Gamma_+$, and is negative on Γ_+. Now define $G(x,t) := g(\kappa_1(x,t),\kappa_2(x,t))$. Then, proceeding as in Corollary 2.3, we see that initial upper bounds on G are preserved:

Lemma 3.1. *The maximum of G is non-increasing under the flow:*

$$G \le c_2 := \max_{M\times\{0\}} G. \tag{3.1}$$

Proof. The proof is similar to that of Corollary 2.3. q.e.d.

Now observe that, wherever $x_2 > x_1$,

$$\ddot g^{11}(x_1,x_2) = \phi''\left(\frac{x_1}{x_2}\right)\frac{1}{x_2}.$$

Since $\phi''(r) = \frac{2a}{(a+r)^3}$, we see that $\ddot g^{11}$ is positive on $\Gamma \setminus \bar\Gamma_+$. It follows from the homogeneity identities (2.6) that $\ddot g^{ij}$ is positive on $\Gamma \setminus \bar\Gamma_+$ for each $i,j = 1,2$.

Following [**24**, **26**] we consider, for some small positive constants ε and σ,

$$G_{\varepsilon,\sigma} \;:=\; (G-\varepsilon)F^{\sigma}\,.$$

Observe that the upper bound on G implies

$$G_{\varepsilon,\sigma} \le c_2 F^{\sigma}\,. \tag{3.2}$$

Our goal is to show that for every $\varepsilon > 0$, there is some $\sigma > 0$ and some constant $K > 0$ for which $G_{\varepsilon,\sigma} < K$.

Lemma 3.2. *Wherever $\kappa_1 \neq \kappa_2$, we have*

$$\begin{aligned}(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} = &-F^{\sigma}(\dot F^{kl}\ddot G^{pq,rs} - \dot G^{kl}\ddot F^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\\ &-\frac{2\sigma}{F}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F + \frac{\sigma(\sigma+1)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon,\sigma}|h|_F^2\,,\end{aligned} \tag{3.3}$$

where we have defined $\langle u,v\rangle_F := \dot F^{ij}u_i u_j$, $|u|_F := \sqrt{\langle u,u\rangle_F}$, and $|h|_F^2 := \dot F^{kl}h_k{}^m h_{ml}$.

Proof. We first compute

$$\partial_t G_{\varepsilon,\sigma} = F^{\sigma}\partial_t G + \frac{\sigma}{F}G_{\varepsilon,\sigma}\partial_t F$$

and

$$\nabla G_{\varepsilon,\sigma} = F^{\sigma}\nabla G + \frac{\sigma}{F}G_{\varepsilon,\sigma}\nabla F\,.$$

It follows that

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} = F^{\sigma}\mathcal{L}G + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F + 2\frac{\sigma}{F}\left\langle\nabla G_{\varepsilon,\sigma},\nabla F\right\rangle_F\\ -\frac{\sigma(\sigma+1)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.\end{aligned} \tag{3.4}$$

Combining the first and third of these and applying the evolution equations (ii) and (iv) of Lemma 2.1 yields the result. q.e.d.

Unfortunately, the final two terms of the evolution equation (3.3) can be positive, and we cannot obtain the required estimate directly from the maximum principle, as in [**1**, **35**]. However, the Stampacchia iteration method of [**24**, **26**] is still available to us. The first step is to show that the spatial L^p norms of the positive part, $(G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma},0\}$, of $G_{\varepsilon,\sigma}$ are non-increasing in t for large p, so long as σ is sufficiently small.

4. The L^p estimates

The goal of this section is to prove the following proposition.

Proposition 4.1. *For all $\varepsilon > 0$ there exist constants $\ell \in (0,1)$ and $L > 1$, independent of σ and p, such that for all $p > L$ the $L^p\big(M,\mu(t)\big)$ norm of $\left(G_{\varepsilon,\sigma}(\cdot,t)\right)_+$ is non-increasing in t, so long as $\sigma < \ell p^{-\frac{1}{2}}$.*

To simplify notation, we denote $E := (G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma}, 0\}$. Then E^p is C^1 in the t variable for $p > 1$, with $\partial_t E^p = pE^{p-1}\partial_t G_{\varepsilon,\sigma}$. Recall that $\mu(t)$ denotes the Riemannian measure induced on M by the immersion $X(\cdot, t)$. Since μ is smooth in t, the integral $\int E^p \, d\mu$ is in $C^1(0, T)$. We will show that

$$\frac{d}{dt}\int E^p \, d\mu \leq 0$$

for large p and small σ (as in the statement of Proposition 4.1).

The evolution equation (3.3) for $G_{\varepsilon,\sigma}$ implies $\int E^p \, d\mu$ evolves under the flow according to

$$\begin{aligned}\frac{d}{dt}\int E^p \, d\mu = {} & p\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma} \, d\mu + p\int E^{p-1}F^\sigma R \, d\mu \\ & - 2\sigma p\int E^{p-1}\frac{\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F}{F} \, d\mu \\ & + p\sigma(\sigma+1)\int E^p \frac{|\nabla F|_F^2}{F^2} \, d\mu \\ & + \sigma p\int E^p |h|_F^2 \, d\mu - \int E^p H F \, d\mu \,,\end{aligned} \tag{4.1}$$

where $R := \big(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\big)\nabla_k h_{pq}\nabla_l h_{rs}$, and the final term comes from the evolution of $d\mu$ under the flow (Lemma 2.1, part (iii)). We integrate the first term by parts:

$$\begin{aligned}\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma} \, d\mu = {} & -(p-1)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2 \, d\mu \\ & - \int E^{p-1}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma} \, d\mu \,.\end{aligned}$$

Using the expression for the gradient, $\nabla G_{\varepsilon,\sigma} = F^\sigma \nabla G + \frac{\sigma}{F}G_{\varepsilon,\sigma}\nabla F$, we find

$$\begin{aligned}\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma} \, d\mu = {} & -(p-1)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2 \, d\mu \\ & - \int E^{p-1}F^\sigma \dot{G}^{pq}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq} \, d\mu \\ & - \sigma\int E^p F^{-1}\dot{F}^{pq}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq} \, d\mu \,.\end{aligned}$$

Therefore,

(4.2)

$$\begin{aligned}\frac{d}{dt}\int E^p\,d\mu = &-p(p-1)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu - p\int E^{p-1}F^\sigma Q,\,d\mu\\ &-\sigma p\int E^pF^{-1}\dot F^{pq}\ddot F^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq}\,d\mu\\ &-2\sigma p\int E^{p-1}\frac{\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F}{F}\,d\mu\\ &+p\sigma(\sigma+1)\int E^p\frac{|\nabla F|_F^2}{F^2}\,d\mu+\sigma p\int E^p|h|_F^2\,d\mu\\ &-\int E^pHF\,d\mu\,,\end{aligned}$$

where we have defined

$$Q := (\dot G^{pq}\ddot F^{kl,rs}+\dot F^{kl}\ddot G^{pq,rs}-\dot G^{kl}\ddot F^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\,.$$

It will be useful to compare ∇F with ∇h as follows:

Lemma 4.2. *There is a constant $c_3>0$ for which*

$$|\nabla F|_F^2\le c_3|\nabla h|^2$$

along the flow.

Proof. This is a simple application of Corollary 2.4. q.e.d.

The first term of (4.2) is manifestly non-positive, vanishing only if $G_{\varepsilon,\sigma}$ is non-positive or spatially constant. We can squeeze another good term out of Q as follows:

Lemma 4.3. *We have the following decomposition:*

$$Q=Q_1+Q_2\,,$$

where

$$\begin{aligned}Q_1 := &\dot f^1\ddot g^{11}\left(\frac{\nabla_1 G}{\dot g^1}\right)^2+\dot f^2\ddot g^{22}\left(\frac{\nabla_2 G}{\dot g^2}\right)^2\\ &+2\frac{f}{H^3}\left[(\nabla_1 h_{12})^2+(\nabla_2 h_{12})^2\right]\,,\end{aligned}$$

and

$$Q_2 := \left(\frac{\dot f^2-\dot f^1}{\kappa_2-\kappa_1}-2\frac{\dot f^1}{\kappa_2}\right)\nabla_1 G\,\nabla_1 h_{22}+\left(\frac{\dot f^2-\dot f^1}{\kappa_2-\kappa_1}-2\frac{\dot f^2}{\kappa_1}\right)\nabla_2 G\,\nabla_2 h_{11}\,,$$

from which we deduce that

$$-F^\sigma Q\le -\big(C_1-C_2p^{-\frac12}-C_3\sigma\big)E\frac{|\nabla h|_F^2}{F^2}+C_4p^{\frac12}\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E}\,,$$

wherever $G_{\varepsilon,\sigma}>0$, where C_1, C_2, C_3, and C_4 are positive constants that depend possibly on ε, but not on σ or p.

Proof. Recall that

$$Q := \left(\dot{G}^{pq}\ddot{F}^{kl,rs} + \dot{F}^{kl}\ddot{G}^{pq,rs} - \dot{G}^{kl}\ddot{F}^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs}\,.$$

We expand in an orthonormal frame of eigenvectors of $\mathcal{W}$. Using (2.5), we have

$$\begin{aligned}\dot{G}^{pq}\ddot{F}^{kl,rs}\nabla_k h_{pq}\nabla_l h_{rs} &= \ddot{F}^{kl,rs}\nabla_k G\nabla_l h_{rs}\\ &= \ddot{f}^{11}\nabla_1 h_{11}\nabla_1 G + \ddot{f}^{22}\nabla_2 h_{22}\nabla_2 G\\ &\quad + \ddot{f}^{12}\nabla_2 h_{11}\nabla_2 G + \ddot{f}^{12}\nabla_1 h_{22}\nabla_1 G\\ &\quad + \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}\nabla_1 G\,\nabla_2 h_{12} + \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}\nabla_2 G\,\nabla_1 h_{21}\,.\end{aligned}$$

Using $\nabla_k G = \dot{g}^1\nabla_k h_{11} + \dot{g}^2\nabla_k h_{22}$, and the homogeneity identities (2.6), this becomes

$$\begin{aligned}\dot{G}^{pq}\ddot{F}^{kl,rs}\nabla_k h_{pq}\nabla_l h_{rs} &= \frac{\ddot{f}^{11}}{\dot{g}^1}(\nabla_1 G)^2 + \frac{\ddot{f}^{22}}{\dot{g}^2}(\nabla_2 G)^2\\ &\quad + \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}\nabla_1 G\,\nabla_2 h_{12} + \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1}\nabla_2 G\,\nabla_1 h_{21}\,.\end{aligned}$$

The decomposition $Q = Q_1 + Q_2$ now follows from the definition of G and equation (2.7) from the proof of Lemma 2.2.

We will now show that there are positive constants, C_1, C_2, C_3, C_4, for which

$$-F^\sigma Q_1 \leq -C_1 E\frac{|\nabla h|_F}{F^2}\,, \tag{4.3}$$

$$\text{and}\quad -F^\sigma Q_2 \leq C_4 p^{\frac{1}{2}}\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E} + (C_2 p^{-\frac{1}{2}} + C_3\sigma)E\frac{|\nabla h|_F^2}{F^2}\,. \tag{4.4}$$

Consider first (4.3). Since $E = (G_{\varepsilon,\sigma})_+$ vanishes unless $G > \varepsilon$, we need only consider the points with $\kappa \in \Gamma_\varepsilon := \{x \in \Gamma : \varepsilon \leq g(x) \leq c_2\}$. Using the estimate $E \leq c_2 F^\sigma$, it suffices to show that $\widetilde{Q}_1 := |\nabla h|^{-1}F^2 Q_1$ has a positive lower bound when $\nabla h \neq 0$. The quantity $\widetilde{Q}_1$ is homogeneous of degree zero in the principal curvatures, so we only need to obtain a lower bound on the compact slice $K := \{x \in \bar{\Gamma}_\varepsilon : |x| = 1\}$. Now, since K is a compact subset of Γ, we have positive lower bounds for $f, \dot{f}^i$ and $\ddot{g}^{ij}$ for each $i, j = 1, 2$. Therefore, by the definition of Q_1, $\widetilde{Q}_1$ vanishes on K only if $\nabla G = \nabla_1 h_{12} = \nabla_2 h_{12} = 0$. Since $\nabla_k G = \dot{g}^1\nabla_k h_{11} + \dot{g}^2\nabla_k h_{22}$, this implies $\nabla_1 h_{11} = \kappa_1/\kappa_2\nabla_1 h_{22} = \kappa_1/\kappa_2\nabla_2 h_{12} = 0$, and similarly $\nabla_2 h_{22} = 0$. Therefore we must in fact have $\nabla h = 0$. The claim follows since $|\cdot|_F$ is equivalent to the usual norm.

We now show that (4.4) holds. Define

$$q_1 = \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1} - 2\frac{\dot{f}^1}{\kappa_2} \quad\text{and}\quad q_2 = \frac{\dot{f}^2 - \dot{f}^1}{\kappa_2 - \kappa_1} - 2\frac{\dot{f}^2}{\kappa_1}\,.$$

Recalling that $\nabla_k G_{\varepsilon,\sigma} = F^\sigma \nabla_k G + \frac{\sigma}{F} G_{\varepsilon,\sigma}\nabla_k F$, we have

$$
\begin{aligned}
F^\sigma Q_2 = {} & q_1 \nabla_1 G_{\varepsilon,\sigma} \nabla_1 h_{22} + q_2 \nabla_2 G_{\varepsilon,\sigma}\nabla_2 h_{11} \\
& - q_1 \frac{\sigma}{F} E \nabla_1 F \nabla_1 h_{22} - q_2 \frac{\sigma}{F} E \nabla_2 F \nabla_2 h_{11} \,.
\end{aligned}
\tag{4.5}
$$

Since the derivatives $\dot{f}^i$ are bounded above for $\kappa \in K$, and the denominators in the expressions for q_1 and q_2 are bounded away from zero for $\kappa \in K$, we have $Fq_i \leq C$ on K for each $i = 1, 2$, where $C := \max\{q_i : \kappa \in K, i = 1, 2\}$. Since Fq_i is homogeneous of degree zero in the principal curvatures, these bounds extend to Γ_ε.

We now apply Young's inequality, $|ab| \leq \frac{1}{2}(ra^2 + b^2/r)$, twice to equation (4.5) (with $r = p^{\frac{1}{2}}\frac{F}{E}$ for the first pair of terms, and $r = 1$ for the second pair). We find

$$
\begin{aligned}
F^\sigma Q_2 &\leq \frac{C}{F}\left[\frac{p^{\frac{1}{2}}F}{E}\frac{|\nabla G_{\varepsilon,\sigma}|^2}{2} + \frac{p^{-\frac{1}{2}}E}{F}\frac{|\nabla h|^2}{2} + \frac{\sigma}{F}E\left(\frac{|\nabla F|^2}{2} + \frac{|\nabla h|^2}{2}\right)\right] \\
&\leq \frac{c_1 C}{2} p^{\frac{1}{2}} \frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E} + \left(\frac{Cc_1}{2}p^{-\frac{1}{2}} + \frac{\sigma c_1 C}{2}(c_1 c_3 + 1)\right) E \frac{|\nabla h|_F^2}{F^2} \,.
\end{aligned}
$$

This completes the proof. q.e.d.

Corollary 4.4. *There are constants* $D_1, D_2, D_3, D_4, D_5, D_6 > 0$ *that are independent of* $\sigma \in (0, 1)$ *and* $p > 1$*, for which the following estimate holds:*

$$
\begin{aligned}
\frac{d}{dt}\int E^p \, d\mu \leq & -\left(p^2 - D_1 p^{\frac{3}{2}} - D_2 p\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2 \, d\mu \\
& - \left(D_3 p - D_4 p^{\frac{1}{2}} - D_5 \sigma p\right)\int E^p \frac{|\nabla h|_F^2}{F^2}\, d\mu \\
& + D_6(\sigma p + 1)\int E^p |h|_F^2 \, d\mu \,.
\end{aligned}
\tag{4.6}
$$

Proof. Recall equation (4.2). Apply Lemma 4.3 to the second term. The third term is estimated by noting that $F\dot{F}^{pq}\ddot{F}^{kl,rs}$ is homogeneous of degree zero in the principal curvatures, so that, estimating each of these terms above by some constant, we obtain

$$
-\sigma p \int E^p F^{-1} \dot{F}^{pq} \ddot{F}^{kl,rs} \nabla_k h_{rs} \nabla_l h_{pq} \, d\mu \leq C\sigma p \int E^p \frac{|\nabla h|_F^2}{F^2}\, d\mu
$$

for some $C > 0$. The next term is estimated as follows:

$$
\begin{aligned}
-2p\sigma \frac{E^{p-1}}{F}\left\langle \nabla G_{\varepsilon,\sigma}, \nabla F\right\rangle_F &\leq p\sigma E^p \left(\frac{|\nabla F|_F^2}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}\right) \\
&\leq p\sigma E^p \left(c_1 c_3 \frac{|\nabla h|_F^2}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}\right) .
\end{aligned}
$$

Finally, since $-\frac{HF}{|h|_F^2}$ is homogeneous of degree zero with respect to the principal curvatures, it may be estimated above by some constant D_6, which is sufficient to estimate the final term. q.e.d.

Notice that there are constants, c and C say, for which the first two terms of (4.6) become negative for p and C satisfying $p > C$ and $\sigma \leq cp^{-\frac{1}{2}}$. We now show that it is possible to estimate the final term of (4.6) in a similar manner. To achieve this, we integrate $\mathcal{L}G_{\varepsilon,\sigma}$ in conjunction with a Simons-type identity, inspired by the procedures carried out in [**24**, Lemma 5.4] and [**26**, Lemma 3.5]. In what follows, σ will always be restricted to the interval $(0,1)$.

Lemma 4.5 (Poincaré-type inequality)**.** *There exist constants A_i, $B_i > 0$, independent of $p > 1$ and $\sigma \in (0,1)$, such that*

$$\int E^p|h|_F^2 \leq \left(A_1p^{\frac{3}{2}} + A_2p + A_3p^{\frac{1}{2}} + A_4\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$+ \left(B_1p^{\frac{1}{2}} + B_2 + B_3p^{-\frac{1}{2}}\right)\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\,. \tag{4.7}$$

Recall the commutation formula (see, for example, [**2**, Proposition 5])

$$\nabla_k\nabla_l h_{pq} = \nabla_p\nabla_q h_{kl} + h_{kl}h_{pq}^2 - h_{pq}h_{kl}^2 + h_{kq}h_{pl}^2 - h_{pl}h_{kq}^2.$$

Contracting with $\dot{F}$ yields the following Simons-type identity:

$$\mathcal{L}h_{pq} = \dot{F}^{kl}\nabla_p\nabla_q h_{kl} + Fh_{pq}^2 - \dot{F}^{kl}h_{pq}h_{kl}^2 + \dot{F}^{kl}h_{kq}h_{pl}^2 - \dot{F}^{kl}h_{pl}h_{kq}^2\,.$$

Contracting this with $\dot{G}$ yields

$$\dot{G}^{pq}\mathcal{L}h_{pq} = \dot{G}^{pq}\dot{F}^{kl}\nabla_p\nabla_q h_{kl} + F\dot{G}^{pq}h_{pq}^2\,.$$

On the other hand, we have that

$$\dot{F}^{kl}\nabla_p\nabla_q h_{kl} = \nabla_p\nabla_q F - \ddot{F}^{kl,rs}\nabla_p h_{rs}\nabla_q h_{kl}\,,$$

so that

$$\dot{G}^{pq}\mathcal{L}h_{pq} = \dot{G}^{pq}\nabla_p\nabla_q F - \dot{G}^{pq}\ddot{F}^{kl,rs}\nabla_p h_{rs}\nabla_q h_{kl} + F\dot{G}^{kl}h_{kl}^2\,.$$

We now recall (3.4):

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &= F^\sigma\mathcal{L}G + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F + 2\frac{\sigma}{F}\left\langle\nabla G_{\varepsilon,\sigma},\nabla F\right\rangle_F \\ &\quad - \frac{\sigma(\sigma+1)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2 \\ &= F^\sigma(\dot{F}^{kl}\dot{G}^{pq}\nabla_k\nabla_l h_{pq} + \dot{F}^{kl}\ddot{G}^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs}) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F \\ &\quad + 2\frac{\sigma}{F}\left\langle\nabla G_{\varepsilon,\sigma},\nabla F\right\rangle_F - \frac{\sigma(\sigma+1)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.\end{aligned}$$

Putting this together, we obtain the following expression for $\mathcal{L}G_{\varepsilon,\sigma}$:

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} = {} & F^\sigma\big(\dot F^{kl}\ddot G^{pq,rs} - \dot G^{kl}\ddot F^{pq,rs}\big)\nabla_k h_{pq}\nabla_l h_{rs} + F^\sigma \dot G^{kl}\nabla_k\nabla_l F\\ & + F^\sigma F\dot G^{kl}h^2_{kl} + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F + \frac{2\sigma}{F}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F\\ & - \frac{\sigma(1+\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|^2_F\,.\end{aligned} \tag{4.8}$$

Note the appearance of $\dot G^{kl}h^2_{kl}$. Since $F\dot G$ is homogeneous of degree zero in the principal curvatures, and strictly negative definite wherever $G_{\varepsilon,\sigma} > 0$, we may estimate $F\dot G^{kl} \le -\gamma \dot F^{kl}$, for some $\gamma > 0$, whenever $\kappa \in \Gamma_\varepsilon := \{x \in \Gamma : \varepsilon \le g(x) \le c_2\}$. In particular, $F\dot G^{kl}h^2{}_{kl} \le -\gamma|h|^2_F$.

Return now to equation (4.8). Applying Young's inequality, we obtain, wherever $G_{\varepsilon,\sigma} > 0$,

$$\frac{2\sigma}{F}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F \le \sigma E\left(\frac{|\nabla F|^2_F}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|^2_F}{E^2}\right).$$

Note that the terms $F^2\big(\dot F^{kl}\ddot G^{pq,rs} - \dot G^{kl}\ddot F^{pq,rs}\big)$ are homogeneous of degree zero. Then we may estimate each of them above by some constant, $C/100$. Discarding the final term, recalling the estimates (2.8), (3.1), and Lemma 4.2, and using $\sigma < 1$, we arrive at

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} \le {} & (C + 2c_3 + \sigma c_3 c_2)F^\sigma\frac{|\nabla h|^2_F}{F^2} + F^\sigma\dot G^{kl}\nabla_k\nabla_l F - \gamma F^\sigma|h|^2_F\\ & + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F + \sigma c_2 F^\sigma\frac{|\nabla G_{\varepsilon,\sigma}|^2_F}{E^2}\,.\end{aligned}$$

Now put the $\gamma F^\sigma|h|^2_F$ term on the left, multiply the inequality by $E^pF^{-\sigma}$, and integrate over M to obtain

$$\begin{aligned}\gamma\int E^p|h|^2_F\,d\mu \le {} & -\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu + \int E^p\dot G^{kl}\nabla_k\nabla_l F\,d\mu\\ & + (C + 2c_3 + \sigma c_3 c_2)\int E^p\frac{|\nabla h|^2_F}{F^2}\,d\mu\\ & + \sigma\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu + c_2\sigma\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2_F\,d\mu\,.\end{aligned}$$

We estimate the first term as follows:

Lemma 4.6. *There are constants $a_1, a_2, b_1 > 0$, independent of $p > 1$ and $\sigma \in (0,1)$, for which*

$$\begin{aligned}-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu \le {} & \big(a_1 p + a_2\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2_F\,d\mu\\ & + b_1\int E^p\frac{|\nabla h|^2_F}{F^2}\,d\mu\,.\end{aligned}$$

Proof. Integrating by parts, we find

$$-\int E^p F^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu = p\int E^{p-1}F^{-\sigma}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$-\sigma\int E^p F^{-\sigma-1}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F\,d\mu$$
$$+\int E^p F^{-\sigma}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}\,d\mu\,.$$

Since the terms $F\ddot{F}^{kl,rs}$ are homogeneous of degree zero in the principal curvatures, they each have uniform upper bounds, so that

$$-\int E^p F^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu \le c_2 p\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$+\frac{c_2\sigma}{2}\int E^p\left(\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}+\frac{|\nabla F|_F^2}{F^2}\right)d\mu$$
$$+\frac{c_2 C}{2}\int E^p\left(\frac{|\nabla h|^2}{F^2}+\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}\right)d\mu$$

for some $C>0$. Therefore,

$$-\int E^p F^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu \le \left(c_2 p+\frac{c_2\sigma}{2}+\frac{c_2 C c_1}{2}\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$+\left(\frac{c_2 c_3\sigma}{2}+\frac{c_2 C c_1}{2}\right)\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\,.$$

q.e.d.

In a similar manner, we deduce the following:

Lemma 4.7. *There are constants $a_3, b_2, b_3 > 0$, independent of $p>1$ and $\sigma\in(0,1)$, for which*

$$\int E^p\dot{G}^{kl}\nabla_k\nabla_l F\,d\mu \le a_3 p^{\frac{3}{2}}\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$+\left(b_2 p^{\frac{1}{2}}+b_3\right)\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\,.$$

Proof. Integrating by parts, we find

$$\int E^p\dot{G}^{kl}\nabla_k\nabla_l F\,d\mu = -p\int E^{p-1}\dot{G}^{kl}\nabla_k G_{\varepsilon,\sigma}\nabla_l F\,d\mu$$
$$-\int E^p\dot{F}^{pq}\ddot{G}^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq}\,d\mu\,.$$

Again, each $F^2\dot{F}^{pq}\ddot{G}^{kl,rs}$ is homogeneous of degree zero in the principal curvatures, and, hence, uniformly bounded above. Thus

$$-\int E^p\dot{F}^{pq}\ddot{G}^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq}\,d\mu \le C\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu$$

for some $C > 0$.

We estimate the remaining term using $-F\dot{G}^{ij} \leq \gamma \dot{F}^{ij}$ and Young's inequality. We find

$$-p\int E^{p-1}\dot{G}^{kl}\nabla_k G_{\varepsilon,\sigma}\nabla_l F\,d\mu \leq \gamma p\int E^p\left(\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{rE^2}+\frac{r|\nabla F|_F^2}{F^2}\right)d\mu$$

for any $r > 0$. Choosing $r = p^{-1/2}$ and estimating $|\nabla F|_F^2 \leq c_3|\nabla h|_F^2$ implies the claim. q.e.d.

The final term to estimate is $\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu$.

Lemma 4.8. *There are constants a_4, a_5, b_4, b_5, b_6, independent of $p > 1$ and $\sigma \in (0,1)$, for which*

$$\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu \leq \left(a_4p^{\frac{3}{2}}+a_5p^{\frac{1}{2}}\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu$$
$$+\left(b_4p^{\frac{1}{2}}+b_5p^{-\frac{1}{2}}+b_6\right)\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\,.$$

Proof. We again integrate by parts. We find

$$\begin{aligned}\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu = &-(p+1)\int E^pF^{-1-\sigma}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F\,d\mu\\ &+(1+\sigma)\int E^{p+1}F^{-\sigma}\frac{|\nabla F|_F^2}{F^2}\\ &-\int E^{p+1}F^{-1-\sigma}\dot{F}^{pq}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l h_{pq}\,d\mu\,.\end{aligned}$$

The first term is estimated using the Young's inequality and the second by Lemma 4.2. The third may be estimated by observing that the terms $FF^{pq}F^{kl,rs}$ are homogeneous of degree zero in the curvature, and hence bounded above along the flow, and applying (3.2). We get, for some $C > 0$,

$$\begin{aligned}\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu \leq{}& \frac{c_2}{2}(p+1)\int E^p\left(\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{rE^2}+\frac{r|\nabla F|_F^2}{F^2}\right)d\mu\\ &+2c_2c_3\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu+C\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\,.\end{aligned}$$

Choosing $r = p^{-1/2}$, we arrive at

$$\begin{aligned}\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu \leq{}& \frac{c_2}{2}(p+1)p^{\frac{1}{2}}\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu\\ &+\left(c_2c_3(p+1)p^{-\frac{1}{2}}+2c_2c_3+C\right)\int E^p\frac{|\nabla h|_F^2}{F^2}\,d\mu\end{aligned}$$

as required. q.e.d.

This completes the proof of Lemma 4.5. We now complete the proof of Proposition 4.1.

Proof of Proposition 4.1. Recall the inequality (4.6) of Corollary 4.4. Combining this with Lemma 4.5 we find

$$\begin{aligned}\frac{d}{dt}\int E^p\,d\mu \le \alpha_7\Big(p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2\sigma p^2 - \alpha_3 p^{\frac{3}{2}} - \alpha_4 p \\ - \alpha_5 p^{\frac{1}{2}} - \alpha_6\Big)\int E^{p-2}|G_{\varepsilon,\sigma}|^2\,d\mu \\ + \beta_6\Big(p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}} \\ - \beta_4 - \beta_5 p^{-\frac{1}{2}}\Big)\int E^p\frac{|\nabla h|^2}{F^2}\,d\mu\end{aligned}$$

for some constants α_i, $\beta_i > 0$ that are independent of σ and p. The claim now follows easily. q.e.d.

5. Proof of Theorem 1.2

We are now able to proceed similarly as in [**24**, Section 5] and [**26**, Section 3], using Proposition 4.1 and the following lemma to derive the desired bound on $G_{\varepsilon,\sigma}$.

Lemma 5.1 (Stampacchia [**36**])**.** *Let $\varphi : [k_0,\infty) \to \mathbb{R}$ be a non-negative, non-increasing function satisfying*

$$\varphi(h) \le \frac{C}{(h-k)^\alpha}\varphi(k)^\beta, \qquad h > k > k_0\,, \tag{5.1}$$

for some constants $C > 0$, $\alpha > 0$, and $\beta > 1$. Then

$$\varphi(k_0 + d) = 0\,,$$

where $d^\alpha = C\varphi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

Given any $k \ge k_0$, where $k_0 := \sup_{\sigma\in(0,1)}\sup_M G_{\varepsilon,\sigma}(\cdot,0)$, set

$$v_k := \big(G_{\varepsilon,\sigma} - k\big)_+^{\frac{p}{2}} \quad \text{and} \quad A_{k,t} := \{x \in M : v_k(x,t) > 0\}.$$

We will show that $|A_{k,t}| := \int_0^T\int_{A_{k,t}} d\mu(t)\,dt$ satisfies the conditions of Stampacchia's Lemma for some $k_1 \ge k_0$. This provides us with a constant d for which the space-time measure $|A_{k_1+d,t}|$ vanishes. Theorem 1.2 then follows straightforwardly. Observe that $|A_{k,t}|$ is non-negative and non-increasing. Then we only need to demonstrate that an inequality of the form (5.1) holds.

We begin by noting that

Lemma 5.2. *There is a constant $L_1 \ge L$ such that, for all $p > L_1$, we have*

$$\frac{d}{dt}\int v_k^2 d\mu + \int |\nabla v_k|^2 d\mu \le c_4(\sigma p + 1)\int_{A_{k,t}} F^2 G_{\varepsilon,\sigma}^p d\mu \tag{5.2}$$

for some $c_4 > 0$.

Proof. We have

$$\frac{d}{dt}\int v_k^2 d\mu \leq \int \partial_t v_k^2\, d\mu = \int_{A_{k,t}} p(G_{\varepsilon,\sigma}-k)_+^{p-1}\partial_t G_{\varepsilon,\sigma}\, d\mu\,.$$

Proceeding as in Corollary 4.4, we obtain

$$\begin{aligned}\frac{d}{dt}\int v_k^2 d\mu \leq & -(p^2-\widetilde{D}_1 p^{\frac{3}{2}}-\widetilde{D}_2 p)\int_{A_{k,t}}(G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\, d\mu\\ & + \widetilde{D}_6(\sigma p+1)\int_{A_{k,t}}(G_{\varepsilon,\sigma})_+^p |h|_F^2\, d\mu\\ \leq & -4c_1(1-\widetilde{D}_1 p^{-\frac{1}{2}}-\widetilde{D}_2 p^{-1})\int |\nabla v_k|_F^2\, d\mu\\ & + c_4(\sigma p+1)\int_{A_{k,t}}(G_{\varepsilon,\sigma})_+^p F^2\, d\mu\end{aligned}$$

for some constants $\widetilde{D}_1$, $\widetilde{D}_2$, c_4, where we used

$$|\nabla v_k|^2 = \frac{p^2}{4}(G_{\varepsilon,\sigma}-k)_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,,$$

and estimated the homogeneous degree zero quantity $|h|_F^2/F^2$ above by $c_4/\widetilde{D}_6$. The claim now follows. q.e.d.

Now set $\sigma' = \sigma + \frac{2}{p}$. Then

$$\int F^2\,(G_{\varepsilon,\sigma})_+^p\, d\mu = \int \left(G_{\varepsilon,\sigma'}\right)_+^p d\mu\,,$$

so that

$$\begin{aligned}\int_{A_{k,t}} F^2 d\mu \leq \int_{A_{k,t}} F^2\frac{(G_{\varepsilon,\sigma})_+^p}{k^p}d\mu = k^{-p}\int_{A_{k,t}}\left(G_{\varepsilon,\sigma'}\right)_+^p d\mu\\ \leq k^{-p}\int \left(G_{\varepsilon,\sigma'}\right)_+^p d\mu\,.\end{aligned} \tag{5.3}$$

If we ensure

$$p \geq \max\left\{L_1, \frac{16}{\ell^2}\right\},\quad \sigma \leq \frac{\ell}{2}p^{-\frac{1}{2}}\,,$$

we have $p \geq L_1$ and $\sigma' \leq \ell p^{-\frac{1}{2}}$, so that, by Proposition 4.1,

$$\begin{aligned}\int_{A_{k,t}} F^2 d\mu \leq k^{-p}\int\left(G_{\varepsilon,\sigma'}\right)_+^p d\mu \leq k^{-p}\int\left(G_{\varepsilon,\sigma'}(\cdot,0)\right)_+^p d\mu_0\\ \leq \mu_0(M)\left(\frac{k_0}{k}\right)^p.\end{aligned} \tag{5.4}$$

For large enough k, we can make the right hand side of this inequality arbitrarily small. We will use this fact in conjunction with the following Sobolev inequality (see [**24**]) to exploit the good gradient term in (5.2).

Lemma 5.3. *There is a constant $c_5 \in [1/2, \infty)$, depending only on n and the initial datum, such that*

$$\left(\int v_k^{2q} d\mu\right)^{\frac{1}{q}} \leq c_5 \left(\int |\nabla v_k|^2 d\mu + \int F^2 d\mu \left(\int v_k^{2q} d\mu\right)^{\frac{1}{q}}\right), \tag{5.5}$$

for any $q > 0$.

Proof. Since we have the estimate $H^2 < CF^2$, where C only depends on the initial datum, this follows from the Michael-Simon Sobolev inequality [**33**] just as in [**24**]. q.e.d.

It follows from (5.5) and (5.4) that there is some $k_1 > k_0$ such that for all $k > k_1$ we have

$$\left(\int v_k^{2q} d\mu\right)^{\frac{1}{q}} \leq 2c_5 \int |\nabla v_k|^2 d\mu \,.$$

Therefore, from (5.2), we have for all $k > k_1$

$$\frac{d}{dt}\int v_k^2 d\mu + \frac{1}{2c_5}\left(\int v^{2q} d\mu\right)^{\frac{1}{q}} \leq c_4(\sigma p + 1)\int_{A_{k,t}} F^2 G^p_{\varepsilon,\sigma} d\mu \,.$$

Integrating this over time, and noting that $A_{k,0} = \emptyset$, we find

$$\sup_{t\in[0,T]} \int_{A_{k,t}} v_k^2 d\mu + \int_0^T \left(\int v^{2q} d\mu\right)^{\frac{1}{q}} dt \leq 2c_5 c_4(\sigma p + 1)\int_0^T \int_{A_{k,t}} F^2 G^p_{\varepsilon,\sigma} d\mu\, dt \,. \tag{5.6}$$

We now exploit the interpolation inequality for L^p spaces:

$$|f|_{q_0} \leq |f|_1^{1-\theta} |f|_q^{\theta} \,,$$

where $1 \leq q_0 \leq q$ and $\frac{1}{q_0} = 1 - \theta + \frac{\theta}{q}$. Setting $\theta = \frac{1}{q_0}$, we obtain

$$\int_{A_{k,t}} v_k^{2q_0} d\mu \leq \left(\int_{A_{k,t}} v_k^2 d\mu\right)^{q_0 - 1} \left(\int_{A_{k,t}} v^{2q} d\mu\right)^{\frac{1}{q}} .$$

Applying the Hölder inequality, we obtain

$$\left(\int_0^T \int_{A_{k,t}} v_k^{2q_0} d\mu\, dt\right)^{\frac{1}{q_0}} \leq \left(\sup_{t\in[0,T]} \int_{A_{k,t}} v_k^2 d\mu\right)^{\frac{q_0-1}{q_0}} \left(\int_0^T \left(\int_{A_{k,t}} v^{2q} d\mu\right)^{\frac{1}{q}} dt\right)^{\frac{1}{q_0}} .$$

We now use Young's inequality: $ab \leq \left(1 - \frac{1}{q_0}\right) a^{\frac{q_0}{q_0-1}} + \frac{1}{q_0} b^{q_0}$ on the right hand side to obtain

$$\left(\int_0^T \int_{A_{k,t}} v_k^{2q_0} d\mu\, dt\right)^{\frac{1}{q_0}} \leq \sup_{t\in[0,T]} \int_{A_{k,t}} v_k^2 d\mu + \int_0^T \left(\int_{A_{k,t}} v^{2q} d\mu\right)^{\frac{1}{q}} dt\,.$$

Recalling (5.6), we arrive at

$$\left(\int_0^T \int_{A_{k,t}} v_k^{2q_0} d\mu\, dt\right)^{\frac{1}{q_0}} \leq 2c_5 c_4(\sigma p + 1) \int_0^T \int_{A_{k,t}} F^2 G^p_{\varepsilon,\sigma} d\mu\, dt\,. \tag{5.7}$$

Now, using the Hölder inequality, we have

$$\begin{aligned} \int_0^T \int_{A_{k,t}} F^2 G^p_{\varepsilon,\sigma} d\mu\, dt &\leq |A_{k,t}|^{1-\frac{1}{r}} \left(\int_0^T \int_{A_{k,t}} F^{2r} G^{pr}_{\varepsilon,\sigma} d\mu\, dt\right)^{\frac{1}{r}} \\ &\leq c_6 |A_{k,t}|^{1-\frac{1}{r}} \end{aligned} \tag{5.8}$$

and

$$\int_0^T \int_{A_{k,t}} v_k^2 d\mu\, dt \leq |A_{k,t}|^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A_{k,t}} v_k^{2q_0} d\mu\, dt\right)^{\frac{1}{q_0}}, \tag{5.9}$$

where $c_6 := \mu_0(M)\left(\frac{k_0}{k_1}\right)^p$, and r is to be chosen. Finally, for $h > k \geq k_1$ we may estimate

$$\begin{aligned} |A_{h,t}| := \int_0^T \int_{A_{h,t}} d\mu\, dt &= \int_0^T \int_{A_{h,t}} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(G_{\varepsilon,\sigma} - k)_+^p} d\mu\, dt \\ &\leq \int_0^T \int_{A_{h,t}} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(h-k)^p} d\mu\, dt\,, \end{aligned}$$

so that, since $A_{h,t} \subset A_{k,t}$, and $v_k^2 = (G_{\varepsilon,\sigma} - k)_+^p$, we get

$$(h-k)^p |A_{h,t}| \leq \int_0^T \int_{A_{k,t}} v_k^2 d\mu\, dt\,. \tag{5.10}$$

Putting together estimates (5.7), (5.8), (5.9), and (5.10), we obtain

$$|A_{h,t}| \leq \frac{2c_4 c_5 c_6(\sigma p + 1)}{(h-k)^p} |A_{k,t}|^\gamma$$

for all $h > k \geq k_1$, where $\gamma := 2 - \frac{1}{q_0} - \frac{1}{r}$. Now fix $p > \max\left\{L_1, \frac{16}{\ell^2}\right\}$ and choose $\sigma < \ell p^{-\frac{1}{2}}$ sufficiently small that $\sigma p < 1$. Then, choosing $r > \frac{q_0}{q_0-1}$, so that $\gamma > 1$, we may apply Stampacchia's Lemma. We conclude

$$|A_{k,t}| = 0 \quad \forall\, k > k_1 + d,$$

where $d^p = c_4 c_6 2^{\frac{\gamma}{\gamma-1}+1} |A_{k_1,t}|^{\gamma-1}$. We note that d is finite, since T is finite and

$$\int_{A_{k_1}} d\mu \leq \int_{A_{k_1}} \frac{(G_{\varepsilon,\sigma})_+^p}{k_1^p}\, d\mu \leq k_1^{-p} \int (G_{\varepsilon,\sigma})_+^p\, d\mu$$
$$\leq k_1^{-p} \int (G_{\varepsilon,\sigma}(\cdot,0))_+^p\, d\mu_0\,,$$

where the final estimate follows from Proposition 4.1.

Therefore, from the definition of $A_{k,t}$, we obtain $G_{\varepsilon,\sigma} \leq k_1 + d < \infty$. Therefore,

$$\frac{-\kappa_1}{a\kappa_2 + \kappa_1} \leq \varepsilon + (k_1 + d) F^{-\sigma}\,.$$

Since the homogeneous degree zero quantity $\frac{ax_1+x_2}{f(x_1,x_2)}$ is bounded above on the compact slice $K := \bar{\Gamma}_{c_0} \cap \{\lambda \in \mathbb{R}^2 : \lambda_1 + \lambda_2 = 1\}$, we get bounds on the whole cone, and hence we can estimate $a\kappa_1 + \kappa_2 \leq c_7 F$ for some constant $c_7 > 0$ (which is independent of ε). It follows that

$$-\kappa_1 \leq \varepsilon C F + c_7 (k_1 + d) F^{1-\sigma}\,,$$

from which we easily obtain

$$-\kappa_1 \leq 2 c_7 \varepsilon F + C_\varepsilon$$

for some constant $C_\varepsilon > 0$. This completes the proof of Theorem 1.2.

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