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GROMOV–WITTEN THEORY OF ROOT GERBES I: STRUCTURE OF GENUS 0 MODULI SPACES

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Abstract

Let X be a smooth complex projective algebraic variety. Given a line bundle $\mathcal{L}$ over X and an integer $r > 1$, one defines the stack $\sqrt[r]{\mathcal{L}/X}$ of r-th roots of $\mathcal{L}$. Motivated by Gromov–Witten theoretic questions, in this paper we analyze the structure of moduli stacks of genus 0 twisted stable maps to $\sqrt[r]{\mathcal{L}/X}$. Our main results are explicit constructions of moduli stacks of genus 0 twisted stable maps to $\sqrt[r]{\mathcal{L}/X}$ starting from moduli stacks of genus 0 stable maps to X. As a consequence, we prove an exact formula expressing genus 0 Gromov–Witten invariants of $\sqrt[r]{\mathcal{L}/X}$ in terms of those of X.

1. Introduction

Orbifold Gromov–Witten theory, constructed in symplectic category by Chen-Ruan [**18**] and in algebraic category by Abramovich, Graber, and Vistoli [**3**], [**2**], has been an area of active research in recent years. Calculations of orbifold Gromov–Witten invariants in examples present numerous new challenges; see [**21**], [**19**], [**40**], and [**12**] for examples.

Étale gerbes over a smooth base provide interesting examples of smooth Deligne-Mumford stacks. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack and G a finite group scheme over $\mathcal{X}$. Intuitively one can think of a G-banded gerbe over $\mathcal{X}$ as a fiber bundle over $\mathcal{X}$ with its fiber the classifying stack BG. A detailed definition of gerbes can be found in, for example, [**28**], [**15**], [**24**]. We are interested in computing Gromov–Witten theory of G-banded gerbes.

Physics considerations have suggested that the geometry of étale gerbes possesses certain very intriguing structure. The so-called *decomposition conjecture* [**30**] in physics may be interpreted mathematically as a philosophy saying that the geometry of an étale gerbe is equivalent to the geometry of certain disconnected space twisted by a $U(1)$-gerbe. In-depth discussions on various mathematical aspects of this conjecture can be found in [**38**].

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The Gromov–Witten theoretic version of the decomposition conjecture, which can be formulated for arbitrary G-gerbes more general than G-banded gerbes, states that Gromov–Witten theory of the G-gerbe is equivalent to certain twists of the Gromov–Witten theory of some étale cover of the base. A detailed discussion of the conjecture in full generality can be found in [**38**]. For G-banded gerbes this conjecture states that the Gromov–Witten theory of a G-banded gerbe over $\mathcal{X}$ is equivalent to (certain twists of) the Gromov–Witten theory of the disjoint union of $|\mathrm{Conj}(G)|$ copies of $\mathcal{X}$ after a change of variables. Here $\mathrm{Conj}(G)$ is the set of conjugacy classes of G. Computations of Gromov–Witten invariants of étale gerbes are thus intimately connected to the decomposition conjecture.

The simplest examples of G-gerbes are *trivial gerbes*. The trivial G-gerbe over a Deligne-Mumford stack $\mathcal{X}$ is the product $\mathcal{X} \times BG$. In [**7**] the computation of Gromov–Witten invariants of $\mathcal{X} \times BG$ is handled as a special case of a general product formula for orbifold Gromov–Witten invariants of product Deligne-Mumford stacks $\mathcal{X} \times \mathfrak{Y}$. As a consequence the decomposition conjecture is proven for trivial G-gerbes.

An interesting class of non-trivial gerbes is provided by *root gerbes* associated to line bundles. This is the first of two papers in which we study Gromov–Witten theory of root gerbes of line bundles over smooth projective varieties, with the decomposition conjecture in mind. The present paper is devoted to studying the genus 0 Gromov–Witten theory of root gerbes.

Let X be a smooth complex projective variety and $\mathcal{L} \to X$ a line bundle. Given an integer $r > 0$, let

$$\mathcal{G} := \sqrt[r]{\mathcal{L}/X} \to X$$

be the stack of r-th roots of $\mathcal{L}$ over X. It can be shown that $\mathcal{G} \to X$ is a μ_r-banded gerbe over X. Such a gerbe is called a root gerbe. Constructions and properties of root gerbes are briefly reviewed in Section 2.2. In order to study the Gromov–Witten theory we consider moduli spaces $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$ of genus 0 twisted stable maps to $\mathcal{G}$. By composing a twisted stable map to $\mathcal{G}$ with the structure map $\mathcal{G} \to X$, one can define a morphism

$$\mathcal{K}_{0,n}(\mathcal{G}, \beta) \to \overline{M}_{0,n}(X, \beta), \tag{1}$$

where $\overline{M}_{0,n}(X, \beta)$ is a moduli space of genus 0 stable maps to X. The main idea used in our approach to Gromov–Witten theory of root gerbes is to compare Gromov–Witten invariants of $\mathcal{G}$ with Gromov–Witten invariants of the base X using the morphism (1). In the present paper, this idea is realized by our main results, Theorems 3.19 and 3.20, on the structures of the moduli spaces $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$. Roughly speaking, these structure results state that components of $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$ are μ_r-gerbes over certain base stacks constructed from $\overline{M}_{0,n}(X, \beta)$ using log geometry.

More details can be found in Sections 3.4 and 3.5. Our results extends a result of [**12**] for the gerbe $B\mu_r$. Our proofs are based on a detailed analysis of the moduli spaces $\mathcal{K}_{0,n}(\mathcal{G},\beta)$, and use heavily the results of [**35**] and [**37**].

As a consequence of our main structure results, Theorems 3.19 and 3.20, we prove a comparison result between virtual fundamental classes of $\mathcal{K}_{0,n}(\mathcal{G},\beta)$ and $\overline{M}_{0,n}(X,\beta)$; see Theorem 4.3. This comparison result yields an explicit computation of genus 0 Gromov–Witten invariants of $\mathcal{G}$ in terms of genus 0 Gromov–Witten invariants of X, which is Theorem 4.4. A reformulation of Theorem 4.4 in terms of generating functions confirms the decomposition conjecture for genus 0 Gromov–Witten theory of $\mathcal{G}$; see Theorem 4.6.

The paper is organized as follows. Section 2 contains discussions on some preparatory materials. In Section 3 we carry out the needed analysis on the structure of the moduli spaces of twisted stable maps to root gerbes. In Section 4 we prove results on virtual fundamental classes and Gromov–Witten invariants, in particular the decomposition conjecture in genus 0. In Appendix A we discuss extensions of our results to banded abelian gerbes.

Conventions. Unless otherwise mentioned, we work over $\mathbb{C}$ throughout this paper. By an *algebraic stack* we mean an algebraic stack over $\mathbb{C}$ in the sense of [**10**]. By a *Deligne-Mumford stack* we mean an algebraic stack over $\mathbb{C}$ in the sense of [**23**]. We assume moreover all stacks (and schemes) are quasi-separated, locally noetherian, locally of finite type. From time to time we use the notation $x \in X$ to indicate that x is a geometric point of X. Following [**33**], logarithmic structures are considered on the étale site of schemes. For the extension of logarithmic structures to stacks, see [**36**]. Given a scheme (or a stack) X, a geometric point x of X, and a sheaf of sets $\mathcal{F}$ on X, according to the standard notation we denote by $\mathcal{F}_{\overline{x}}$ the stalk of $\mathcal{F}$ at x in the étale topology. A gerbe is an algebraic stack as in [**34**], Definition 3.15.

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2. Preliminaries

2.1. Twisted stable maps. We recall the definition of twisted curves here; see [**3**], [**2**], [**5**] for more details.

Definition 2.1 ([**5**], Definition 4.1.2)**.** A twisted nodal n-pointed curve over a scheme S is a morphism $\mathcal{C} \to S$ together with n closed substacks $\sigma_i \subset \mathcal{C}$ such that

- $\mathcal{C}$ is a tame Deligne-Mumford stack, proper over S, and étale locally is a nodal curve over S;
- $\sigma_i \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \to S$;
- $\sigma_i \to S$ are étale gerbes;
- the map $\mathcal{C} \to C$ to the coarse moduli space C is an isomorphism away from marked points and nodes.

By definition the genus of a twisted curve $\mathcal{C} \to S$ is the genus of its coarse moduli space $C \to S$.

Throughout this paper we will always assume that twisted curves are *balanced*, i.e. at any twisted node, the local group acts on the two branches by opposite characters.

Let S be a noetherian scheme and let $\mathcal{X}/S$ be a proper Deligne-Mumford stack over S with projective coarse moduli space $X \to S$. We fix an ample invertible sheaf $\mathcal{O}_X(1)$ over X. Let $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ be the fibered category over S which to any S-scheme T associates the groupoid of the following data:

- a twisted n-pointed curve $(\mathcal{C}/T, \{\sigma_i\})$ over T;
- a representable morphism $f : \mathcal{C} \to \mathcal{X}$ such that the induced morphism $\bar{f} : C \to X$ between coarse moduli spaces is an n-pointed stable map of degree $\beta \in H_2^+(X,\mathbb{Z})$ (i.e. $\bar{f}_*[C] = \beta$).

According to [**5**], Theorem 1.4.1, the fibered category $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ is a Deligne-Mumford stack proper over S.

As discussed in [**2**], there exist evaluation maps

$$ev_i : \mathcal{K}_{0,n}(\mathcal{X},\beta) \to \bar{I}(\mathcal{X}), \quad 1 \le i \le n$$

taking values in the *rigidified inertia stack* $\bar{I}(\mathcal{X})$ of $\mathcal{X}$. These maps are obtained as follows. The rigidified inertia stack $\bar{I}(\mathcal{X})$ may be defined as the stack of cyclotomic gerbes in $\mathcal{X}$, i.e. representable morphisms from cyclotomic gerbes to $\mathcal{X}$. The evaluation map ev_i is defined to map a twisted stable map $f : (\mathcal{C}/T, \{\sigma_i\}) \to \mathcal{X}$ to its restriction to the i-th marked gerbe,

$$f|_{\sigma_i} : \sigma_i \to \mathcal{X},$$

which is an object of $\bar{I}(\mathcal{X})$.

The rigidified inertia stack $\bar{I}(\mathcal{X})$ has an alternative description. Define the *inertia stack* of $\mathcal{X}$ to be the fiber product over the diagonal:

$$I\mathcal{X} := \mathcal{X} \times_{\mathcal{X}\times_S\mathcal{X}} \mathcal{X}.$$

By definition, objects of $I\mathcal{X}$ are pairs (x, g) where x is an object of $\mathcal{X}$ and g is an element of the automorphism group of x. The rigidified inertia stack $\bar{I}(\mathcal{X})$ is obtained from $I\mathcal{X}$ by applying the rigidification procedure ([**1**], [**4**]). More details can be found e.g. in [**3**].

2.2. Root gerbes. We recall the notion of root gerbes. Let X be a smooth projective variety and let $\mathcal{L}$ be a line bundle over X corresponding to a morphism $\phi_{\mathcal{L}} : X \to B\mathbb{C}^*$. For an integer $r > 0$ let $\theta_r : B\mathbb{C}^* \to B\mathbb{C}^*$ be the morphism induced by the r-th power homomorphism $\mathbb{C}^* \overset{(\cdot)^r}{\to} \mathbb{C}^*$. The composite morphism $\theta_r \circ \phi_{\mathcal{L}} : X \to B\mathbb{C}^*$ corresponds to $\mathcal{L}^{\otimes r}$.

Definition 2.2. The stack $\sqrt[r]{\mathcal{L}/X}$ of r-th roots of $\mathcal{L}$ is defined as

$$\sqrt[r]{\mathcal{L}/X} := X \times_{\phi_{\mathcal{L}}, B\mathbb{C}^*, \theta_r} \mathrm{B}\mathbb{C}^*.$$

Explicitly it can be described as the X-groupoid whose objects over $(Y, f : Y \to X)$ are pairs (M, φ), with M a line bundle over Y and $\phi : M^{\otimes r} \to f^*\mathcal{L}$ an isomorphism. An arrow from (M, φ) to (N, ψ) lying over an X-morphism $h : (Y, f) \to (Z, g)$ is an isomorphism $\rho : M \to h^*N$ such that φ fits in the following commutative diagram:

$$\begin{tikzcd} M^{\otimes r} \arrow[r, "\rho^{\otimes r}"] \arrow[d, "\varphi"'] & h^*N^{\otimes r} \arrow[d, "h^*\psi"] \\ f^*\mathcal{L} \arrow[r, "\cong"'] & h^*g^*\mathcal{L}, \end{tikzcd}$$

where the bottom arrow is the canonical isomorphism.

The following proposition follows easily from the definition.

Proposition 2.3. *The stack $\sqrt[r]{\mathcal{L}/X}$ is the quotient stack $[\mathcal{L}^\times/\mathbb{C}^*]$, where $\mathcal{L}^\times$ is the principal $\mathbb{C}^*$-bundle obtained by deleting the zero section of $\mathcal{L}$, and $\mathbb{C}^*$ acts on $\mathcal{L}^\times$ via $\lambda \cdot z = \lambda^r z, \lambda \in \mathbb{C}^*, z \in \mathcal{L}^\times$. In particular $\sqrt[r]{\mathcal{L}/X}$ is a Deligne-Mumford stack.*

Proof. It is enough to observe that the following diagram is 2-cartesian:

$$\begin{tikzcd} \mathcal{L}^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \sqrt[r]{\mathcal{L}/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ pt \arrow[r] & B\mathbb{C}^* \arrow[r, "\theta_r"] & B\mathbb{C}^*. \end{tikzcd}$$

q.e.d.

Remark 2.4. The morphism $\theta_r : \mathrm{B}\mathbb{C}^* \to \mathrm{B}\mathbb{C}^*$ is a μ_r-gerbe, because of the Kummer exact sequence

$$1 \to \mu_r \to \mathbb{C}^* \overset{(\cdot)^r}{\to} \mathbb{C}^* \to 1.$$

Hence $\sqrt[r]{\mathcal{L}/X} \to X$ is a μ_r-gerbe.

Remark 2.5. The stack $\sqrt[r]{\mathcal{L}/X}$ may also be constructed as a toric stack bundle [**32**].

It is also possible to take roots of "line bundles with sections." Let $\mathcal{L}$ be a line bundle over X and let σ be a section of $\mathcal{L}$. The data $(\mathcal{L}, \sigma)$ correspond to a morphism $\phi_{\mathcal{L},\sigma} : X \to [\mathbb{A}^1/\mathbb{C}^*]$. Let $\theta_r : [\mathbb{A}^1/\mathbb{C}^*] \to [\mathbb{A}^1/\mathbb{C}^*]$ be the morphism induced by the r-th power morphisms on $\mathbb{A}^1$ and $\mathbb{C}^*$. The morphism $\theta_r \circ \phi_{\mathcal{L},\sigma}$ corresponds to the pair $(\mathcal{L}^{\otimes r}, \sigma^r)$. The stack $\sqrt[r]{(\mathcal{L},\sigma)/X}$ of r-th roots of $\mathcal{L}$ with the section σ is defined as

$$\sqrt[r]{(\mathcal{L},\sigma)/X} := X \times_{[\mathbb{A}^1/\mathbb{C}^*],\theta_r} [\mathbb{A}^1/\mathbb{C}^*].$$

The stack constructed in this way is isomorphic to X outside the vanishing locus $Z(\sigma) \subset X$ of σ, while the reduced substack of the closed substack mapping to $Z(\sigma)$ is a μ_r-gerbe over $Z(\sigma)$. Note that given a divisor $D \subset X$ there is an associated line bundle with a canonical section which vanishes on D. Therefore in the following we will also talk about roots of divisors.

2.3. Line bundles over twisted curves. We recall some results about line bundles over twisted curves. In [**16**] there is an explicit description of the Picard group of a smooth twisted curve. Let $\mathcal{C}$ be a smooth twisted curve over $\operatorname{Spec}\mathbb{C}$. Let C be the coarse curve and $D_i \in C, 1 \leq i \leq n$ the marked points. It is known that $\mathcal{C}$ can be constructed from its coarse curve C by applying the r_i-th root construction to the divisor D_i, for all $1 \leq i \leq n$. (Here $r_i \in \mathbb{N}$.) Let $\mathcal{T}_i, 1 \leq i \leq n$ be the tautological line bundles associated by the root construction and $\tau_i, 1 \leq i \leq n$ their tautological sections.

Lemma 2.6 ([**16**], Corollary 2.12)**.** *Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}$. Then there exists an invertible sheaf L on C and integers k_i satisfying $0 \leq k_i \leq r_i - 1$ such that*

$$\mathcal{L} \simeq \pi^* L \otimes \prod_{i=1}^{n} \mathcal{T}_i^{k_i}.$$

Moreover the integers k_i are unique, and L is unique up to isomorphism.

There is an analogous description for the global sections of invertible sheaves on $\mathcal{C}$.

Lemma 2.7 ([**16**], Corollary 2.13)**.** *Given the decomposition in Lemma* 2.6*, every global section of $\mathcal{L}$ is of the form $\pi^* s \otimes \tau_1^{k_1} ... \otimes \tau_n^{k_n}$ for a unique global section s of L, where τ_i is the tautological section of $\mathcal{T}_i$.*

Lemma 2.6 can be rephrased as saying that $\operatorname{Pic}\mathcal{C}$ is an extension of $\operatorname{Pic} C$ by a finite abelian group, namely

$$1 \to \operatorname{Pic} C \to \operatorname{Pic}\mathcal{C} \to \oplus_{i=1}^n \mathbb{Z}_{r_i} \to 1,$$

where r_i are the orders of the stabilizers of stack points.

Remark 2.8. The same description of $\operatorname{Pic}\mathcal{C}$ holds when $\mathcal{C}$ is not smooth but has only untwisted nodes.

The Picard groups of nodal twisted curves over $\operatorname{Spec}\mathbb{C}$ admit a similar description. This is shown e.g. in [**17**]. We sketch the argument for the reader's convenience.

Lemma 2.9 (see [**17**], Theorem 3.2.3)**.** *Let $\mathcal{C}$ be an unmarked twisted curve with nodes $e_1, \ldots, e_s$. Let γ_j be the order of the stabilizer of the node e_j. Then the following exact sequence holds:*

$$1 \to \operatorname{Pic} C \to \operatorname{Pic}\mathcal{C} \to \prod_{j=1}^{s} \mathbb{Z}/\gamma_j\mathbb{Z} \to 1.$$

Proof. Let $\pi : \mathcal{C} \to C$ be the map to the coarse curve. Consider the exact sequence of complexes over C given by

$$1 \to \pi_*\mathbb{G}_m \to R\pi_*\mathbb{G}_m \to R\pi_*\mathbb{G}_m/\pi_*\mathbb{G}_m \to 1.$$

Notice that $\pi_*\mu_r = \mu_r$ and $\pi_*\mathbb{G}_m = \mathbb{G}_m$. Therefore they are complexes concentrated in degree zero. The long hypercohomology exact sequence gives

$$1 \to H^1(C, \mathbb{G}_m) \to H^1(\mathcal{C}, \mathbb{G}_m) \to H^1(R\pi_*\mathbb{G}_m/\pi_*\mathbb{G}_m) \to 1.$$

This sequence is exact on the left because $E_2^{p,q} := H^p(C, H^q(R\pi_*\mathbb{G}_m/\mathbb{G}_m))$ abuts to $\mathbb{H}^{p+q}(C, R\pi_*\mathbb{G}_m/\mathbb{G}_m)$. The sheaf $H^q(R\pi_*\mathbb{G}_m/\mathbb{G}_m)$ is equal to $R^q\pi_*\mathbb{G}_m$ and does not vanish for $q > 0$. By [**1**], Proposition A.0.1, the stalk of $R^q\pi_*\mathbb{G}_m$ is canonically isomorphic to $H^q(Aut(p), \mathbb{G}_{m,p})$ where p is a geometric point of C. This sequence is exact on the right because $H^2(C, \mathbb{G}_m) = 0$ for C a genus zero nodal curve. The result follows by observing that $H^q(\mu_r, \mathbb{G}_m) = \mathbb{Z}/r\mathbb{Z}$ for q odd and is trivial for q even. q.e.d.

Remark 2.10. The above proof generalizes to nodal marked twisted curves.

Normalization of twisted curves. It is very useful to describe a twisted stable map over a point $\widetilde{f} : \mathcal{C} \to \mathcal{G}$ in terms of the induced morphism $\widetilde{f} \circ \nu : \widetilde{\mathcal{C}} \to \mathcal{G}$, where $\widetilde{\mathcal{C}}$ is the normalization of $\mathcal{C}$. This morphism is still a twisted stable map (with possibly disconnected domain). According to [**41**] the normalization of a reduced stack $\mathcal{X}$ is defined in the following way. Let $R \rightrightarrows U$ be a presentation of $\mathcal{X}$. Let $\widetilde{R}$ and $\widetilde{U}$ be the normalizations of R and U. It is possible to lift the structure morphisms of the groupoid $R \rightrightarrows U$ in such a way that $\widetilde{R} \rightrightarrows \widetilde{U}$ is also a groupoid. Moreover, the diagonal $\widetilde{R} \to \widetilde{U} \times \widetilde{U}$ is separated and quasi-compact. Therefore the groupoid defines an algebraic stack, which is the normalization of $\mathcal{X}$. In particular the normalization morphism $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$ is representable.

Smooth twisted curves admit line bundles whose fibers carry faithful representations of the stabilizer groups of the points in the special locus. Those are the tautological line bundles obtained from root constructions. Singular twisted curves over a point also admit line bundles with fibers carrying faithful representations of the stabilizer group of the nodes. This is the content of Lemma 2.9. In this case it is easy to describe those line bundles in terms of tautological line bundles on the normalization of the curve. Assume without loss of generality that $\mathcal{C}$ is a nodal twisted curve with only one node $\mathcal{E}$ of order γ. Let e be the image of the node in the coarse moduli space C. We have the following commutative diagram:

$$\begin{tikzcd}
 & 1 & & & \\
1 \arrow[r] & \operatorname{Pic} \widetilde{C} \arrow[r] \arrow[u] & \operatorname{Pic} \widetilde{\mathcal{C}} \arrow[r] & \oplus_{i=1}^{2} \langle \mathcal{T}_i \rangle \arrow[r] & 1 \\
1 \arrow[r] & \operatorname{Pic} C \arrow[r] \arrow[u, "\nu^*"] & \operatorname{Pic} \mathcal{C} \arrow[r] \arrow[u, "\widetilde{\nu}^*"] & \operatorname{Pic} \mathcal{E} \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \widetilde{\mathcal{O}}_e^*/\mathcal{O}_e^* \arrow[r] \arrow[u] & \widetilde{\mathcal{O}}_{\mathcal{E}}^*/\mathcal{O}_{\mathcal{E}}^* \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

where $\mathcal{O}_e$, resp. $\mathcal{O}_{\mathcal{E}}$, is the local ring at the node e, resp. at the twisted node $\mathcal{E}$, and $\widetilde{\mathcal{O}}_e$, resp. $\widetilde{\mathcal{O}}_{\mathcal{E}}$, is its integral closure. Note that $\mathcal{E} \simeq B\mu_\gamma$. Here $\langle \mathcal{T}_i \rangle$ is the group generated by $\mathcal{T}_i$ under tensor products. The line bundle carrying a representation of the stabilizers group of the node corresponding to an element ζ^k of μ_γ, where ζ is the standard generator, is mapped by the pullback along the normalization morphism $\widetilde{\nu} : \widetilde{\mathcal{C}} \to \mathcal{C}$ to the pair of line bundles $(\mathcal{T}_+^k, \mathcal{T}_-^{-k})$, where $\mathcal{T}_+$, $\mathcal{T}_-$ are the tautological line bundles associated to the preimages of the node in the normalization.

2.4. Logarithmic geometry and twisted curves. We recall here some basic facts about logarithmic geometry, which is the natural language to describe twisted curves. We will use logarithmic geometry to construct the auxiliary stack $\mathfrak{Y}_{0,n,\beta}^{\vec{g}}$ in Section 3.3.

Logarithmic structures have been introduced by Fontaine and Illusie and further studied by Kato [**33**]. A generalization to algebraic stacks can be found in [**36**]. We will consider log structures on the étale site of schemes and on the Lisse-Étale site ([**34**] 12.1.2 (i)) of algebraic stacks (see [**36**], Definition 5.1).

Given a scheme X, a *pre-logarithmic structure*, often called *pre-log structure*, consists of a sheaf of monoids M endowed with a morphism of monoids $\alpha : M \to \mathcal{O}_X$, where the structure sheaf is considered as a monoid with the multiplicative structure. Given a monoid or a sheaf of monoids M, we denote by M^* the submonoid or the subsheaf of invertible elements.

When the natural morphism $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism, a pre-log structure is called a *log structure*. The quotient $M/\alpha^{-1}(\mathcal{O}_X^*)$ is usually denoted by $\overline{M}$, and called the *characteristic* or the *ghost sheaf*. There is a canonical way to associate a log structure to a pre-log structure. Given a pre-log structure $\alpha : M \to \mathcal{O}_X$, the associated log structure, denoted M^a, is defined as the pushout in the category of sheaves of monoids as in the following diagram:

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d, "\alpha"'] & M \arrow[d] \\ \mathcal{O}_X^* \arrow[r] & M^a. \end{tikzcd}$$

The morphism to the structure sheaf $\alpha^a : M^a \to \mathcal{O}_X$ is induced by the pair of morphisms (α, ι), where $\iota : \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$ is the canonical inclusion. A scheme endowed with a log structure (X, M_X) is called a *log scheme*. Log schemes form a category. A morphism between two log schemes $(f, f^b) : (X, M_X) \to (Y, M_Y)$ is a pair consisting of a morphism of schemes $f : X \to Y$ and a morphism of sheaves of monoids $f^b : f^*M_Y \to M_X$ compatible with the morphisms to the structure sheaf. The pullback of a log structure is defined as the log structure associated to the pre-log structure obtained by taking the inverse image.

A log structure $\mathcal{M}_X$ over X is called *locally free* if for any geometric point $x \in X$ we have $\overline{\mathcal{M}}_{X,x} \simeq \mathbb{N}^r$ for some integer r, where $\overline{\mathcal{M}}_{X,x}$ denotes the stalk in the étale topology. A morphism between free monoids $\phi : P_1 \to P_2$ is called *simple* if P_1 and P_2 have the same rank, and for every irreducible element of $p_1 \in P_1$ there exists a unique element $p_2 \in P_2$ and an integer b such that $b \cdot p_2 = \phi(p_1)$. A morphism of locally free log structures is called simple if it induces simple morphisms on the stalks.

Let D be a reduced normal crossing divisor on a scheme X. According to [**33**], there is a locally free log structure canonically associated to D in the following way. Let $U := X \setminus D$ and let $i : U \hookrightarrow X$ be the inclusion. Then

$$\mathcal{M}_D := i_*(\mathcal{O}_U^*) \cap \mathcal{O}_X \to \mathcal{O}_X$$

defines a locally free log structure over X. Let x be a geometric point of X. The induced morphism

$$\overline{\mathcal{M}}_{D,x} \to \mathcal{O}_{X,x}$$

is of the form $\mathbb{N}^r \to \mathcal{O}_{X,x}$ for some integer r. In other words, every irreducible element of the monoid $\overline{\mathcal{M}}_{D,x}$ corresponds to an irreducible component of the pullback of D to $\operatorname{Spec}\mathcal{O}_{X,x}$. Roughly speaking, étale locally a normal crossing divisor becomes a *simple* normal crossing divisor; namely it is a union of smooth irreducible components. This construction generalizes to stacks.

The construction of Matsuki-Olsson. Let X be a smooth variety and let $D = \cup_{i\in I} D_i \subset X$ be an effective Cartier divisor with normal crossing support. Let $\{r_i\}_{i\in I}$ be a collection of positive integers. By [**35**], there exists a smooth Deligne-Mumford stack $\mathcal{X}$ with a normal crossing divisor $\mathcal{D} = \cup_{i\in I}\mathcal{D}_i \subset \mathcal{X}$ satisfying the following properties:

1) The smooth variety X is the coarse moduli space of $\mathcal{X}$.
2) The canonical map $\pi : \mathcal{X} \to X$ is quasi-finite and flat, and is an isomorphism over $X \setminus D$.
3) $\pi^*\mathcal{O}_X(-D_i) = \mathcal{O}_{\mathcal{X}}(-r_i\mathcal{D}_i)$.

Such a stack is defined as a category fibered in groupoids as follows. Objects over an X-scheme $f : T \to X$ are simple morphisms of log structures $\phi : f^*\mathcal{M}_D \to \mathcal{M}$ such that for any geometric point $t \in T$ with image $x = f(t) \in X$, the induced morphism on the stalks of the ghost sheaves is of the following form:

$$\begin{tikzcd} \overline{\mathcal{M}}_{D,\overline{x}} \arrow[r] \arrow[d, "\wr"'] & \overline{\mathcal{M}}_{\overline{t}} \arrow[d, "\wr"] \\ \bigoplus_{D_i} \mathbb{N} \arrow[r, "\oplus(\times r_i)"] & \bigoplus_{D_i} \mathbb{N} \end{tikzcd} \tag{2}$$

According to [**35**], if locally $X = \operatorname{Spec}(k[x_1, \cdots, x_n])$ and locally the divisor $D_i = Z(x_i)$ for $1 \le i \le m$, then $\mathcal{X}$ is canonically isomorphic to the quotient stack

$$[\operatorname{Spec}(k[y_1, \cdots, y_n])/\mu_{r_1} \times \cdots \times \mu_{r_m}],$$

where $k[y_1, \cdots, y_n]$ is a $k[x_1, \cdots, x_n]$-algebra via

$$x_i \mapsto \begin{cases} y_i^{r_i}, & i \le m \\ y_i, & i > m, \end{cases}$$

and the action of $\mu_{r_1} \times \cdots \times \mu_{r_m}$ is given by

$$(u_1, \cdots, u_m) \cdot y_i = \begin{cases} u_i y_i, & i \le m, \\ y_i, & i > m. \end{cases}$$

We compare this construction with the root construction. For a smooth scheme X, an effective Cartier divisor $D \subset X$, and a positive integer r, there exists (see [**3**], [**16**]) a smooth Deligne-Mumford stack $X_{(D,r)}$ satisfying the following properties:

1) The preimage of D is an infinitesimal neighborhood of the μ_r-gerbe $\mathcal{D}$ over D.
2) There is a canonical map $\pi : X_{(D,r)} \to X$ which is an isomorphism over $X \setminus D$. Every point in $X_{(D,r)}$ lying over D has stabilizer μ_r.

This is the r-th root construction of X with respect to the divisor D and r.

Let $\mathbb{D} := (D_1, \cdots, D_n)$ be an n-tuple of Cartier divisors and $\vec{r} = (r_1, \cdots, r_n)$ be an n-tuple of positive integers. Let $X_{(\mathbb{D},\vec{r})}$ be the stack obtained by iterating the root constructions over X and the sequence of divisors. One can see that if the divisor $D = \cup_i D_i$ has simple normal crossing, then $\mathcal{X} \simeq X_{(\mathbb{D},\vec{r})}$. However, if components of D have self-intersections, then along such self-intersections $\mathcal{X}$ has more automorphisms than $X_{(\mathbb{D},\vec{r})}$.

The stack of twisted curves. In [**37**] the stack of twisted curves $\mathfrak{M}^{tw}_{g,n}$ is constructed using logarithmic geometry. The stack $\mathfrak{M}^{tw}_{g,n}$ is a smooth Artin stack which has a natural map to the stack of prestable curves $\mathfrak{M}_{g,n}$ introduced in [**13**]. Such a map is defined by sending a marked twisted curve $(\mathcal{C}, \{\sigma_i\})$ to its coarse moduli space with marked points induced by the σ_i.

The notion of *log twisted curve* is introduced in [**37**].

Definition 2.11 ([**37**], Definition 1.7)**.** An *n-pointed log twisted curve* over a scheme S is a collection of data

$$(C/S, \{\sigma_i, a_i\}, l : \mathcal{M}_S \to \mathcal{M}'_S),$$

where C/S is an n-pointed prestable curve, $\sigma_i : S \to C$ are sections (marked points), a_i, $i = 1, \dots, n$ are integer-valued locally constant functions on S such that for each $s \in S$ the integer $a_i(s)$ is positive and invertible in the residue field $k(s)$, and $l : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ is a simple morphism of log structures over S, where $\mathcal{M}_S$ is the canonical log structure associated to C/S.

Log twisted curves turn out to be equivalent to usual twisted curves.

Theorem 2.12 ([**37**], Theorem 1.9)**.** *For any scheme S, there is a natural equivalence of groupoids between the groupoid of n-pointed twisted curves over S and the groupoid of n-pointed log twisted curves over S. Moreover, the equivalence is compatible with base change $S' \to S$.*

Using this equivalence $\mathfrak{M}^{tw}_{g,n}$ can be seen as the stack over $\mathbb{Z}$ which to any S associates the groupoid of n-marked genus g log twisted curves $(\mathcal{C}/S, \{\sigma_i, a_i\}, l : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$. For an n-tuple of integer numbers $\vec{b} = (b_1, \dots, b_n)$, let $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ be the substack of $\mathfrak{M}^{tw}_{g,n}$ classifying log twisted curves with $a_i = b_i$ for all i. There is a decomposition in open and closed components

$$\mathfrak{M}^{tw}_{g,n} \simeq \coprod_{\vec{b}} \mathfrak{M}^{tw}_{g,n}(\vec{b}). \tag{3}$$

All the components $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ are isomorphic with each other. The boundary of $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ is a normal crossing divisor D. Then there is an associated log structure that we denote by $\mathcal{M}_D$. For any $\vec{b}$, $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ is the stack over $\mathfrak{M}_{g,n}$ whose fiber over any $f : T \to \mathfrak{M}_{g,n}$ is the groupoid of simple extensions of log structures $f^*\mathcal{M}_D \hookrightarrow \mathcal{M}_T$ such that for any geometric point $t \in T$ with $f(t) = x$, $\mathrm{Coker}(\overline{\mathcal{M}}^{gp}_{D,x} \to \overline{\mathcal{M}}^{gp}_{T,t})$ is invertible in $k(t)$. From the construction just described, we see that $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ and $\mathfrak{M}_{g,n}$ are locally isomorphic outside of the boundary locus, while over the locus of singular curves $\mathfrak{M}^{tw}_{g,n}(\vec{b})$ acquires more automorphisms due to twisted nodes.

Given a log twisted curve $(C/S, \{\sigma_i, a_i\}, l : \mathcal{M}_S \to \mathcal{M}'_S)$, the corresponding twisted curve $\mathcal{C}/S$ can be reconstructed as follows. It is the category fibered in groupoids whose fiber over any $h : T \to S$ is the groupoid of data consisting of a morphism $s : T \to C$ over h together with a commutative diagram of locally free log structures on T:

$$\begin{tikzcd} h^*\mathcal{M}_S \arrow[r, "l"] \arrow[d] & h^*\mathcal{M}'_S \arrow[d, "\tau"] \\ s^*\mathcal{M}_C \arrow[r, "k"] & \mathcal{M}'_C, \end{tikzcd} \tag{4}$$

where

1) the morphism k is simple and for any geometric point t of T, the map $\overline{\mathcal{M}}'_{S,t} \to \overline{\mathcal{M}}'_{C,t}$ is either an isomorphism, or of the form $\mathbb{N}^r \to \mathbb{N}^{r+1}$ mapping e_i to e_i for $i < r$ and e_r to either e_r or $e_r + e_{r+1}$, and
2) for every $1 \leq i \leq n$ and geometric point t of T with image $s = s(t)$ in $\sigma_i(S) \subset C$, the group
$$Coker(\overline{\mathcal{M}}'^{gp}_{S,t} \oplus \overline{\mathcal{M}}^{gp}_{C,t} \to \overline{\mathcal{M}}'^{gp}_{C,t})$$
is a cyclic group of order a_i.

3. Moduli of twisted stable maps to root gerbes

Let X be a smooth projective variety over $\mathbb{C}$, $\mathcal{L}$ a line bundle over X, and $r \geq 1$ an integer. The purpose of this section is to study the structure of the moduli stack $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$ of genus 0 twisted stable maps to a root gerbe $\mathcal{G} := \sqrt[r]{\mathcal{L}/X}$. More precisely, we study components (i.e., unions of connected components) $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ of $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$ indexed by what we call *β-admissible vectors* (Definition 3.3). The main results of this section, Theorems 3.19 and 3.20, exhibit the structure of $\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$ over the moduli stack $\overline{M}_{0,n}(X,\beta)$ of stable maps to X. We always assume that $\overline{M}_{0,n}(X,\beta)$ is non-empty.

3.1. Components of moduli stack. We begin with some useful lemmas.

Lemma 3.1. *Let G be a finite group, and let $\mathcal{G}\to X$ be a G-banded gerbe. To give a lift $\mathcal{C}\to\mathcal{G}$ of a map to the coarse moduli space $C\to X$ is equivalent to giving a map $\mathcal{C}\to\mathcal{G}\times_X C$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\overline{f}"'] \arrow[rr, bend left, "\widetilde{f}"] \arrow[dr, "\pi"'] & \mathcal{G}\times_X C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G} \arrow[d] \\ & C \arrow[r, "f"] & X. \end{tikzcd} \tag{5}$$

To give a lift $\widetilde{f}$ of f means to give the outer square in diagram (5), namely the pair $(\widetilde{f},\pi)$. Due to the universal property of the fiber product, this is equivalent to giving a map $\overline{f}:\mathcal{C}\to\mathcal{G}\times_X C$. Moreover, $\overline{f}$ is representable if and only if $\widetilde{f}$ is. q.e.d.

Lemma 3.2 (c.f. [**16**])**.** *Let $(\mathcal{C},\{\sigma_i\})$ be an n-pointed smooth twisted curve with stack points $\sigma_i, 1\le i\le n$. Let μ_{r_i} be the isotropy group of the stack point σ_i. Denote by $\pi:(\mathcal{C},\{\sigma_i\})\to(C,\{p_i\})$ the coarse curve. Let $\widetilde{f}:\mathcal{C}\to\sqrt[r]{\mathcal{L}/X}$ be a morphism and $f:C\to X$ its induced map between coarse moduli spaces. Suppose $\widetilde{f}$ is given by a line bundle $M=\pi^*L\otimes\bigotimes_{i=1}^n\mathcal{T}_i^{m_i}$ over $\mathcal{C}$ (with $0\le m_i<r_i$) and an isomorphism $\psi:M^{\otimes r}\simeq\pi^*f^*\mathcal{L}$. Then $\widetilde{f}$ is representable if and only if for $1\le i\le n$, we have $r_i|r$ and m_i and r_i are co-prime.*

Proof. By [**5**], Lemma 4.4.3, it suffices to study the homomorphism

$$Aut(\sigma_i)\to Aut(\widetilde{f}(\sigma_i)), \tag{6}$$

induced by $\widetilde{f}$ on stack points. Here by σ_i we mean a morphism $\widetilde{h}_i:\operatorname{Spec}K\to\mathcal{C}$ from an algebraically closed field K to $\mathcal{C}$ with image in the special locus. By the root construction description of $\mathcal{C}$ (see e.g. [**16**], Example 2.7, and [**3**], Section 4.2), the stack point σ_i is equivalent to the data (h_i,M_i,t_i,ϕ_i), where $h_i:\operatorname{Spec}K\to C$ with image p_i, M_i is a line bundle over $\operatorname{Spec}K$, $\phi_i:M_i^{\otimes r_i}\xrightarrow{\sim}h_i^*\mathcal{O}(p_i)$, t_i is a section of M_i such that $\phi_i(t_i^{r_i})=h_i^*s_i$ (s_i is the section of $\mathcal{O}(p_i)$ defined by p_i). Hence $t_i=0$. The image $\widetilde{f}(\sigma_i)$ is given by $\widetilde{h}_i^*M$ and $\widetilde{h}_i^*\psi:\widetilde{h}_i^*M^{\otimes r}\simeq\widetilde{h}_i^*\pi^*f^*\mathcal{L}$. Note that $\widetilde{h}_i^*\mathcal{T}_i$ is naturally isomorphic to M_i. An automorphism $\epsilon\in Aut(\sigma_i)\simeq\mu_{r_i}$ is mapped to $\epsilon^{m_i}\in Aut(\widetilde{f}(\sigma_i)))\simeq\mu_r$ since $M=\pi^*L\otimes\bigotimes_{i=1}^n\mathcal{T}_i^{m_i}$. This homomorphism is injective if and only if $r_i|r$ and m_i and r_i are coprime. q.e.d.

Admissible vectors. The inertia stack $I\mathcal{G}$ admits a decomposition

$$I\mathcal{G} = \cup_{g\in\mu_r}\mathcal{G}_g$$

indexed by elements of μ_r. An object of $\mathcal{G}_g$ over $h : T \to X$ is a collection $((M, \phi), g)$ where (M, ϕ) is an object of $\mathcal{G}$ over T (i.e. M is a line bundle over T and $\phi : M^{\otimes r} \to h^*\mathcal{L}$ is an isomorphism) and g is an automorphism of (M, ϕ) defined by multiplying fibers of M by g. The identification of μ_r with the group of r-th roots of $1 \in \mathbb{C}^*$ allows us to identify $g \in \mu_r$ with complex numbers. We use this to make sense of the multiplication. In what follows we use this identification without explicit reference.

Definition 3.3. Let $\bar{I}(\mathcal{G})_g \subset \bar{I}(\mathcal{G})$ be the image of $\mathcal{G}_g$ under the natural map $I\mathcal{G} \to \bar{I}(\mathcal{G})$. Let $\vec{g} := (g_1, \dots, g_n) \in \mu_r^{\times n}$ be a vector of elements of μ_r. Set

$$\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} := \cap_{i=1}^n ev_i^{-1}(\overline{I}(\mathcal{G})_{g_i}). \tag{7}$$

The vector $\vec{g}$ is called *β-admissible* if $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ is nonempty.

Remark 3.4. Note that the definition of β-admissible vectors depends on a choice of the class β.

Let $[\widetilde{f} : (\mathcal{C}, \{\sigma_i\}) \to \mathcal{G}] \in \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}(\mathbb{C})$. By definition the morphism $\widetilde{f}|_{\sigma_i} : B\mu_{r_i} \simeq \sigma_i \to \mathcal{G}$ is equivalent to an injective homomorphism

$$\mu_{r_i} \hookrightarrow \mu_r, \quad \exp(2\pi\sqrt{-1}/r_i) \mapsto g_i.$$

The argument in the proof of Lemma 3.2, applied to the irreducible component of $\mathcal{C}$ containing σ_i, shows that we may write

$$g_i = \exp(2\pi\sqrt{-1}\frac{m_i}{r_i}), \quad \text{with } 0 \le m_i < r_i, \text{ and } (m_i, r_i) = 1. \tag{8}$$

Furthermore, if $\mathcal{L}^{1/r}$ is the universal r-th root of $\mathcal{L}$ over $\mathcal{G}$, then $\widetilde{f}|_{\sigma_i}^*\mathcal{L}^{1/r}$ is the μ_{r_i}-representation on which the standard generator $\exp(2\pi\sqrt{-1}/r_i) \in \mu_{r_i}$ acts by multiplication by $\exp(2\pi\sqrt{-1}m_i/r_i)$. In other words,

$$\mathrm{age}_{\sigma_i}(\widetilde{f}^*\mathcal{L}^{1/r}) = \frac{m_i}{r_i}. \tag{9}$$

Lemma 3.5. *Suppose $\vec{g} = (g_1, \dots, g_n) \in \mu_r^{\times n}$ is a β-admissible vector. Then*

$$\prod_{i=1}^n g_i = \exp\left(\frac{2\pi\sqrt{-1}}{r}\int_\beta c_1(\mathcal{L})\right). \tag{10}$$

Proof. Let $[\widetilde{f} : (\mathcal{C}, \sigma_1, \dots, \sigma_n) \to \sqrt[r]{\mathcal{L}/X}] \in \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}(\mathbb{C})$ be a twisted stable map. Let $\mathcal{L}^{1/r}$ be the universal r-th root of $\mathcal{L}$ over $\mathcal{G} = \sqrt[r]{\mathcal{L}/X}$. By Riemann-Roch for twisted curves (see e.g. [**3**], Theorem 7.2.1),

$$\chi(\widetilde{f}^*\mathcal{L}^{1/r}) = 1 + \deg \widetilde{f}^*\mathcal{L}^{1/r} - \sum_{i=1}^{n} \mathrm{age}_{\sigma_i}(\widetilde{f}^*\mathcal{L}^{1/r}), \tag{11}$$

which is an integer. Clearly

$$\deg \widetilde{f}^*\mathcal{L}^{1/r} = \frac{1}{r}\int_\beta c_1(\mathcal{L}).$$

By (8) and (9) we have

$$g_i = \exp(2\pi\sqrt{-1}\ \mathrm{age}_{\sigma_i}(\widetilde{f}^*\mathcal{L}^{1/r})).$$

The result follows. q.e.d.

Proposition 3.6. *Let $\mathcal{G} = \sqrt[r]{\mathcal{L}/X} \to X$ be a root gerbe. Let $[f : (C, p_1, \dots, p_n) \to X]$ be an object of $\overline{M}_{0,n}(X, \beta)(\mathbb{C})$. Then for a vector $\vec{g} = (g_1, \dots, g_n) \in \mu_r^{\times n}$ satisfying (10) there exists, up to isomorphisms, a unique twisted stable map $\widetilde{f} : (\mathcal{C}, \sigma_1, \dots, \sigma_n) \to \mathcal{G}$ in $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ lifting f.*

Proof. We first assume that C is smooth. Associate to $\vec{g}$ the numbers r_i and $m_i, 1 \le i \le n$ as in (8). Let $(\mathcal{C}, \sigma_1, \dots, \sigma_n)$ be the smooth twisted curve obtained by applying the r_i-th root construction to the divisor $p_i \subset C$ for $1 \le i \le n$. Denote by $\mathcal{T}_i, 1 \le i \le n$ the tautological sheaves and by $\pi : \mathcal{C} \to C$ the natural map. By (10) we have

$$\frac{1}{r}\deg \pi^* f^* \mathcal{L} - \sum_{1 \le i \le n} \frac{m_i}{r_i} \in \mathbb{Z}.$$

Pick $L \in \mathrm{Pic}(C)$ such that $\deg L = \frac{1}{r}\deg \pi^* f^* \mathcal{L} - \sum_{1 \le i \le n} \frac{m_i}{r_i}$. Set

$$M := \pi^* L \otimes \bigotimes_{1 \le i \le n} \mathcal{T}_i^{m_i}.$$

Then $\deg M^{\otimes r} = \deg \pi^* f^* \mathcal{L}$, so there exists an isomorphism $M^{\otimes r} \simeq \pi^* f^* \mathcal{L}$, which defines a map $\widetilde{f} : \mathcal{C} \to \mathcal{G}$. By construction $\widetilde{f}$ is a lifting of f. By Lemma 3.2 $\widetilde{f}$ is representable. Also, $\widetilde{f}$ is unique up to isomorphisms since the line bundle L on $C \simeq \mathbb{P}^1$ is determined up to isomorphisms by its degree. This proves the proposition in case C is smooth.

We treat the general case by induction on the number of irreducible components of C. The case of one irreducible component is proven above. We now establish the induction step. Let $C_1 \subset C$ be an irreducible component containing only one node x, and $C_2 := \overline{C \setminus C_1}$. In other words, C_1 is an irreducible component meeting the rest of the curve C_2 at the node x. Let $T \subset [n] := \{1, 2, \ldots, n\}$ be the marked points that are contained in C_1. Restrictions of f yield two stable maps

$$f_1 : (C_1, \{p_i | i \in T\} \cup \{x\}) \to X, \quad f_2 : (C_2, \{p_i | i \in T^C\} \cup \{x\}) \to X.$$

Here $T^C := [n] \setminus T$. Set $\beta_1 := f_{1*}[C_1]$ and define $m_x, r_x \in \mathbb{Z}$ by

$$\frac{m_x}{r_x} := \langle -\sum_{i \in T} \frac{m_i}{r_i} + \frac{1}{r} \int_{\beta_1} c_1(\mathcal{L}) \rangle, \quad (m_x, r_x) = 1.$$

Here $\langle - \rangle$ denotes the fractional part. Note that $r_x | r$ since $r_i | r$ for $i \in T$. Therefore we may define $g_x := \exp(2\pi\sqrt{-1}\frac{m_x}{r_x}) \in \mu_r$.

By the smooth case there exists a lifting

$$\widetilde{f_1} : (\mathcal{C}_1, \{\sigma_i | i \in T\} \cup \{\sigma_x\}) \to \mathcal{G}$$

of f_1 associated to the collection $\{g_i | i \in T\} \cup \{g_x\}$. By induction there exists a lifting

$$\widetilde{f_2} : (\mathcal{C}_2, \{\sigma_i | i \in T^C\} \cup \{\sigma'_x\}) \to \mathcal{G}$$

of f_2 associated to the collection $\{g_i | i \in T^C\} \cup \{g_x^{-1}\}$. By our choices of the actions of isotropy groups at σ_x and σ'_x we see that $\mathcal{C}_1$ and $\mathcal{C}_2$ glue along σ_x, σ'_x to form a balanced twisted curve $\mathcal{C}$, and the morphisms $\widetilde{f_1}, \widetilde{f_2}$ define a morphism $\widetilde{f} : (\mathcal{C}, \sigma_1, \ldots, \sigma_n) \to \mathcal{G}$ which is representable and is a lifting of f.

By restricting to irreducible components, we see that uniqueness of lifting follows from uniqueness of lifting in the smooth case. This completes the proof. q.e.d.

Remark 3.7. Lemma 3.5 and Proposition 3.6 combined show that β-admissible vectors (for a fixed class β) are completely characterized by the condition (10). This condition can be viewed as a generalization of the monodromy condition which is required to hold for genus 0 twisted stable maps to $B\mu_r$.

In what follows we prove that each open-and-closed component $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ of $\mathcal{K}_{0,n}(\mathcal{G}, \beta)$ is a gerbe over a base stack that can be constructed over $\overline{M}_{0,n}(X, \beta)$ by using logarithmic geometry. This base stack is isomorphic to $\overline{M}_{0,n}(X, \beta)$ over the (possibly empty) locus corresponding to twisted stable maps with smooth domain curve. Along the boundary it has more automorphisms, corresponding to the fact that singular twisted curves carry additional automorphisms associated to the stacky nodes. In order to describe the gerbe structure of $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ and its base stack we will need an auxiliary stack parametrizing weighted prestable curves.

3.2. The stacks $\mathfrak{M}_{g,n,\beta}$ and $\mathfrak{M}^{tw}_{g,n,\beta}$. We describe a stack introduced in [**22**], Section 2. It parametrizes weighted prestable curves with a stability condition. Denote by $\mathfrak{M}_{g,n}$ the stack of genus g prestable curves with n marked points.

Let $g, n \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2^+(X, \mathbb{Z})$. A triple (g, n, β) is called *stable* if either $\beta \neq 0$ or $\beta = 0$ and $2g - 2 + n > 0$. For a triple (g, n, β) the stack $\mathfrak{M}_{g,n,\beta}$ over $\mathfrak{M}_{g,n}$ is defined inductively in the following way:

1) If (g, n, β) is unstable, then $\mathfrak{M}_{g,n,\beta}$ is empty.
2) If (g, n, β) is stable, an object of $\mathfrak{M}_{g,n,\beta}$ over T is
   a) an object $(C, \{s_i\})$ of $\mathfrak{M}_{g,n}(T)$, namely a genus g prestable curve over T with n sections in the smooth locus;
   b) a constructible function $f : C_{gen} \to H_2^+(X, \mathbb{Z})$, where $C_{gen} \to T$ is the complement of the nodes and the sections in C. The function f must be locally constant on the geometric fibers of $C_{gen} \to T$.
3) If $T^0 \subset T$ is the open subscheme parametrizing nonsingular curves $C^0 \to T^0$, then $f : C^0_{gen} \to H_2^+(X, \mathbb{Z})$ must be constant with value β.
4) f has to satisfy two kinds of gluing conditions along the boundary of $\mathfrak{M}_{g,n}$:
   a) Suppose that there is a decomposition $g = g' + g''$ and $[n] = \{1, \ldots, n\} = T \coprod T^c$, with $|T| = n'$ and $|T^c| = n''$ and a map $S \to T$ such that the composite map $S \to \mathfrak{M}_{g,n}$ factors into
   $$S \to \mathfrak{M}_{g',T\coprod\{s'\}} \times \mathfrak{M}_{g'',T^C\coprod\{s''\}} \to \mathfrak{M}_{g,n},$$
   where the second map is obtained by gluing the marked sections s' and s''. Let $C'_V \in \mathfrak{M}_{g',n'}(S)$ and $C''_V \in \mathfrak{M}_{g'',n''}(S)$ be the associated families of curves. We require that the pulled back constructible functions $f' : C'_V \to H_2^+(X, \mathbb{Z})$ and $f'' : C''_V \to H_2^+(X, \mathbb{Z})$ define a morphism
   $$S \to \coprod_{\beta'+\beta''=\beta} \mathfrak{M}_{g',T,\beta'} \times \mathfrak{M}_{g'',T^c,\beta''}.$$
   b) Suppose that there is a map $S \to T$ such that the composite map $S \to \mathfrak{M}_{g,n}$ factors into
   $$S \to \mathfrak{M}_{g-1,n\coprod\{s',s''\}} \to \mathfrak{M}_{g,n}.$$
   Then the associated genus $g-1$ family of curves $C_S \to S$ with the pulled back constructible function $f : C_{Sgen} \to H_2^+(X, \mathbb{Z})$ has to define a morphism
   $$S \to \mathfrak{M}_{g-1,n\coprod\{s',s''\},\beta}.$$

The constructible function $f : C_{gen} \to H_2^+(X, \mathbb{Z})$ will be called the *weight*. An object (C, f) of $\mathfrak{M}_{g,n,\beta}$ is called an $H_2^+(X, \mathbb{Z})$-weighted prestable curve, and its *total weight* is by definition β.

Note that in this definition it is important that $H_2^+(X, \mathbb{Z})$ is a semi-group with indecomposable zero and any of its elements has a finite number of decompositions.

We quote some properties of the stack $\mathfrak{M}_{g,n,\beta}$.

Proposition 3.8 ([**22**], Proposition 2.0.2)**.** *The map* $\mathfrak{M}_{g,n,\beta} \to \mathfrak{M}_{g,n}$ *defined by forgetting the weights is étale, and relatively a scheme of finite type. Therefore the stack* $\mathfrak{M}_{g,n,\beta}$ *is a smooth algebraic stack.*

Proposition 3.9 ([**22**], Proposition 2.1.1)**.** *The natural morphism* $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ *defined by forgetting the* $(n+1)$*-st marked point is the universal family over* $\mathfrak{M}_{g,n,\beta}$*.*

Boundary of $\mathfrak{M}_{0,n,\beta}$**.** Boundary divisors of $\mathfrak{M}_{0,n}$ are indexed by subsets T of $[n] := \{1, 2, \ldots, n\}$. Each T corresponds to the boundary divisor D^T which parametrizes curves $C = C_1 \cup C_2$ meeting at a node such that marked points indexed by T are contained in C_1 and other marked points are contained in C_2.

Boundary divisors in $\mathfrak{M}_{0,n,\beta}$ can be similarly described.

Definition 3.10. Given (T, β'), where T is a not necessarily proper subset of $[n]$ and $\beta' \in H_2^+(X, \mathbb{Z})$ such that $\beta' \leq \beta$, define $D^T_{\beta'} \subset \mathfrak{M}_{0,n,\beta}$ to be the divisor which parametrizes curves $C = C_1 \cup C_2$ meeting at a node such that

1) marked points indexed by T are contained in C_1, and marked points indexed by $T^C := [n] \setminus T$ are contained in C_2;
2) $f|_{C_1} = \beta'$ and $f|_{C_2} = \beta - \beta'$.

Note that $D^T_{\beta'} = D^{T^C}_{\beta-\beta'}$. Let $h_n : \mathfrak{M}_{0,n+1,\beta} \to \mathfrak{M}_{0,n,\beta}$ be the map that forgets the $(n+1)$-th marked point. Then

$$h_n^{-1} D^T_{\beta'} = D^T_{\beta'} \cup D^{T\cup\{n+1\}}_{\beta'}. \tag{12}$$

Lemma 3.11. *The boundary divisors* $D^T_{\beta'}$ *are normal crossing divisors.*

Proof. Consider the natural map $l_n : \mathfrak{M}_{0,n,\beta} \to \mathfrak{M}_{0,n}$ that forgets the weights. The following relation holds:

$$l_n^{-1} D^T = \begin{cases} \cup_{0<\beta'\leq\beta} D^T_{\beta'} & \text{if } |T| < 2; \\ \cup_{0\leq\beta'\leq\beta} D^T_{\beta'} & \text{if } 2 \leq |T| \leq n-2; \\ \cup_{0\leq\beta'<\beta} D^T_{\beta'} & \text{if } |T| > n-2. \end{cases}$$

Since D^T are normal crossing divisors in $\mathfrak{M}_{0,n}$ and the morphism l_n is étale, the result follows. q.e.d.

Log structure on $\mathfrak{M}_{0,n,\beta}$**.** Let $\mathcal{I}_D$ be the set of pairs (T, β') with $T \subset [n], n \notin T$, such that one of the following holds:

1) $0 < \beta' < \beta$;
2) $\beta' = 0$ and $|T| \geq 2$;
3) $\beta' = \beta$ and $|T| \leq n - 2$.

The union of boundary divisors

$$\bigcup_{(T,\beta')\in\mathfrak{I}_D} D^T_{\beta'} \subset \mathfrak{M}_{0,n,\beta} \tag{13}$$

is a reduced normal crossing divisor. We want each divisor $D^T_{\beta'}$ to appear only once in this union i.e., the pairs $(T,\beta'),(T^C,\beta-\beta')$ shouldn't both occur. The requirement $n\notin T$ is a way to select only one of them. The divisor (13) defines a locally free log structure over $\mathfrak{M}_{0,n,\beta}$ which we denote by $\mathcal{M}^n_D$. This follows from a general construction we described in Section 2.4 (see page 9), following [**33**].

Let $\mathfrak{M}^{tw}_{g,n}$ be the stack of genus g twisted curves with n marked points introduced in Section 2.4. Define

$$\mathfrak{M}^{tw}_{g,n,\beta} := \mathfrak{M}_{g,n,\beta}\times_{\mathfrak{M}_{g,n}}\mathfrak{M}^{tw}_{g,n}.$$

The stack $\mathfrak{M}^{tw}_{g,n,\beta}$ parametrizes $H_2^+(X,\mathbb{Z})$-weighted genus g twisted curves with n marked points and total weight β.

3.3. The stack $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$. We will define a stack over $\mathfrak{M}_{0,n,\beta}$ using the log structure $\mathcal{M}^n_D$ and some additional data coming from a β-admissible vector $\vec{g}$, following [**35**].

Fix a β-admissible vector $\vec{g}=(g_1,\dots,g_n)\in\mu_r^{\times n}$. We define a collection of triples of integers $\{(\rho_i,r_i,m_i)|1\le i\le n\}$ as follows. Each $g_i, 1\le i\le n$ may be identified with a root of unity

$$g_i=\exp(2\pi\sqrt{-1}\theta_i),\quad \text{where } \theta_i\in\mathbb{Q}\cap[0,1),$$

which defines the rational numbers $\theta_i, 1\le i\le n$. The characterizing relation of β-admissible vectors (10) reads

$$\prod_{i=1}^n \exp\left(2\pi\sqrt{-1}\theta_i\right)=\exp\left(\frac{2\pi\sqrt{-1}}{r}\int_\beta c_1(\mathcal{L})\right).$$

For $1\le i\le n$, define

$$\rho_i := r\theta_i,\quad r_i:=\frac{r}{gcd(r,\rho_i)},\quad m_i:=\frac{\rho_i}{gcd(r,\rho_i)}. \tag{14}$$

Let (T,β') be an index of the boundary divisors of $\mathfrak{M}_{0,n,\beta}$ as in Definition 3.10. Define

$$\begin{aligned}\theta_{T,\beta'} &:= \left\langle \frac{1}{r}\int_{\beta'} c_1(\mathcal{L})-\sum_{i\in T}\theta_i\right\rangle,\quad r_{T,\beta'}:=\frac{r}{gcd(r,r\theta_{T,\beta'})},\\ & \qquad\qquad m_{T,\beta'}:=\frac{r\theta_{T,\beta'}}{gcd(r,r\theta_{T,\beta'})}.\end{aligned} \tag{15}$$

Here $\langle - \rangle$ again denotes the fractional part. This definition makes sense since $\int_{\beta'} c_1(\mathcal{L})-\sum_{i\in T} r\theta_i$ is an integer.

Definition 3.12. Let $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ be the stack obtained by applying the construction of [**35**], Theorem 4.1, recalled in Section 2.4, to the stack $\mathfrak{M}_{0,n,\beta}$, the normal-crossing divisor (13), and the collection of positive integers $\{r_{T,\beta'} | (T,\beta') \in \mathcal{I}_D\}$.

Let $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ be the stack obtained in the same way from the stack $\mathfrak{M}_{0,n+1,\beta}$ and the β-admissible vector $\vec{g} \cup \{1\} := (g_1, \ldots, g_n, 1) \in \mu_r^{\times n+1}$.

As in the proof of [**35**], Theorem 4.1, the stack $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ is defined as a category fibered in groupoids whose objects over a $\mathfrak{M}_{0,n,\beta}$-scheme $f : S \to \mathfrak{M}_{0,n,\beta}$ are simple morphisms of log structures

$$f^*\mathcal{M}^n_D \to \mathcal{M}_S$$

such that for every geometric point $\overline{s} \to S$ of S with $x = f(\overline{s})$ there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{(T_i,\beta_i)} \mathbb{N} \arrow[r, "\oplus(\times r_{T_i,\beta_i})"] \arrow[d, "\wr"'] & \bigoplus_{(T_i,\beta_i)} \mathbb{N} \arrow[d, "\wr"] \\ f^{-1}\overline{\mathcal{M}}^n_{D,\overline{x}} \arrow[r] & \overline{\mathcal{M}}_{S,\overline{s}}, \end{tikzcd} \tag{16}$$

where the sum is taken over pairs $(T_i, \beta_i) \in \mathcal{I}_D$ labelling the irreducible components of the pullback of D to $\operatorname{Spec} \mathcal{O}_{S,\overline{s}}$. Note that the (T_i, β_i) may have repetitions.

Remark 3.13. Note that property (2) listed in the construction of Matsuki-Olsson in Section 2.4 implies that the structure morphism $\mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}_{0,n,\beta}$ is quasi-finite and flat.

Proposition 3.14. *The stack* $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ *parametrizes* $H_2^+(X, \mathbb{Z})$-*weighted genus* 0 *twisted curves such that*
- *the* i*-th marked gerbe is banded by* μ_{r_i}*;*
- *the nodes are stack points of orders* $\{r_{T,\beta'}\}$*.*

In particular $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ *is an open substack of* $\mathfrak{M}^{tw}_{0,n,\beta}$.

Proof. By definition a morphism $\widetilde{f} : S \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ consists of the following data:

1) a morphism $f : S \to \mathfrak{M}_{0,n,\beta}$, corresponding to a weighted curve $(C/S, \{s_i\})$ with marked sections s_i;
2) a simple morphism of log structures $l : f^*\mathcal{M}^n_D \hookrightarrow \mathcal{M}_S$ inducing diagram (16), with coefficients associated to irreducible components of D as in (15);
3) an n-tuple of integer numbers $\{r_i\}_{i=1}^n$, determined by $\vec{g}$ and β according to (14).

The above data are equivalent by definition to a log twisted curve; see Definition 2.11. The corresponding twisted curve $\mathcal{C}$ is determined as in [**37**], Section 4 (see page 12 of this article). Consider a morphism $U \to \mathcal{C}$. Let $\overline{u} \to U$ be a geometric point mapping to the marked point s_i of C. Let t be an element in $\mathcal{O}_{C,\overline{u}}$ locally defining s_i. According to [**37**], Section 4.2, étale locally $\mathcal{C}$ is isomorphic to

$$\left[\operatorname{Spec}\left(\mathcal{O}_{C,\overline{u}}[z]/(z^{r_i}-t)\right)/\mu_{r_i}\right], \tag{17}$$

where μ_{r_i} acts by multiplication on z. Let $\overline{u} \to C$ map to a node. Such a node corresponds to an irreducible component $D^{T_i}_{\beta_i}$ of the boundary divisor D on $\mathfrak{M}_{0,n,\beta}$. Let us consider the pullback of $D^{T_i}_{\beta_i}$ to $\operatorname{Spec}\mathcal{O}_{U,\overline{u}}$. If the curve C has k nodes of type (T_i,β_i), the pulled back divisor will have k irreducible components corresponding to irreducible elements $e_{i_1},\dots,e_{i_k}$ of the monoid $f^{-1}\overline{\mathcal{M}}^n_{D,\overline{u}} \simeq \mathbb{N}^r$, where r is some integer number. The induced morphism $l_{\overline{u}} : f^{-1}\overline{\mathcal{M}}^n_{D,\overline{u}} \to \overline{\mathcal{M}}_{S,\overline{u}}$ acts on the submonoid generated by $e_{i_1},\dots,e_{i_k}$ as the multiplication by r_{T_i,β_i}. According to [**37**], Section 4.3, after choosing an étale morphism $C \to \operatorname{Spec}\mathcal{O}_{S,\overline{u}}[x,y]/(xy-t)$, the étale local description of $\mathcal{C}$ is

$$\left[\operatorname{Spec}\left(\mathcal{O}_{C,\overline{u}}[z,w]/(zw-t', z^{r_{T_i,\beta_i}}=x, w^{r_{T_i,\beta_i}}=y)\right)/\mu_{r_{T_i,\beta_i}}\right], \tag{18}$$

where the action of $\mu_{r_{T_i,\beta_i}}$ is the usual balanced action. q.e.d.

Lemma 3.15. *The natural morphism $\mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}^{tw,\vec{g}}_{0,n,\beta}$ is étale.*

Proof. This is immediate from the proof of Proposition 3.14 and [**37**], Lemma 5.3. q.e.d.

By Proposition 3.14 the stack $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ is the moduli stack of certain weighted twisted curves. Its universal family can be described as follows.

Proposition 3.16. *There exists a natural morphism*

$$\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}, \tag{19}$$

which is the universal family of genus 0 twisted curves with n marked gerbes over $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$. In particular for $1 \le i \le n$, the i-th marked gerbe is banded by the group μ_{r_i}.

Proof. By the construction of $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$, it has a normal crossing divisor

$$\mathcal{D} = \bigcup_{(T,\beta')\in\mathcal{I}_D} \mathcal{D}^T_{\beta'},$$

such that $\pi_n^*\mathcal{O}_{\mathfrak{M}_{0,n,\beta}}(-D^T_{\beta'}) = \mathcal{O}_{\mathfrak{Y}^{\vec{g}}_{0,n,\beta}}(-r_{T,\beta'}\mathcal{D}^T_{\beta'})$, where $\pi_n : \mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}_{0,n,\beta}$ is the natural map. Let $\mathcal{M}^n_{\mathcal{D}}$ be the locally free log structure associated to $\mathcal{D}$. By construction of $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ there is a universal simple morphism

$$\mathfrak{l}_n : \pi_n^*\mathcal{M}_D^n \to \mathcal{M}_{\mathcal{D}}^n \tag{20}$$

of log structures over $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$, where $\mathcal{M}_D^n$ is the log structure defined by the divisor (13).

Let $h_n : \mathfrak{C}_{0,n,\beta} \to \mathfrak{M}_{0,n,\beta}$ be the universal family of weighted curves over $\mathfrak{M}_{0,n,\beta}$. By Proposition 3.9, $\mathfrak{C}_{0,n,\beta}$ can be identified with $\mathfrak{M}_{0,n+1,\beta}$. Let $h'_n : \mathfrak{C}' \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ be the pullback of h_n via the natural map $\pi_n : \mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}_{0,n,\beta}$, i.e. there is a 2-cartesian diagram

$$\begin{tikzcd}
\mathfrak{C}' \arrow[r, "\pi'_{n+1}"] \arrow[d, "h'_n"'] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{0,n,\beta} \arrow[d, "h_n"] \\
\mathfrak{Y}^{\vec{g}}_{0,n,\beta} \arrow[r, "\pi_n"'] & \mathfrak{M}_{0,n,\beta}.
\end{tikzcd}$$

The data

$$(h'_n : \mathfrak{C}' \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}, \{r_i = \mathrm{order}(g_i)\}_{1\le i\le n}, \mathfrak{l}_n : \pi_n^*\mathcal{M}_D^n \to \mathcal{M}_{\mathcal{D}}^n) \tag{21}$$

is a log twisted curve. The universal *twisted* weighted curve $\mathfrak{C}^{\vec{g}}_{0,n,\beta} \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ over $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ is obtained from (21) by applying the construction of [**37**], Section 4 (see page 12 of this article for a summary).

Note that over $\mathfrak{C}'$ there is the following diagram of log structures:

$$\begin{tikzcd}
h_n'^*\pi_n^*\mathcal{M}_D^n \arrow[r, "h_n'^*\mathfrak{l}_n"] \arrow[d, "{\pi'}^*_{n+1}h_n^b"'] & h_n'^*\mathcal{M}_{\mathcal{D}}^n \\
\pi_{n+1}^*\mathcal{M}_D^{n+1}, &
\end{tikzcd} \tag{22}$$

where the vertical arrow is the pullback of the morphism $h_n^b : h_n^*\mathcal{M}_D^n \to \mathcal{M}_D^{n+1}$ induced by (12). Consider the morphism

$$\overline{\mathcal{M}}^n_{D,p} \to \overline{\mathcal{M}}^{n+1}_{D,q}, \tag{23}$$

which is induced by h_n^b for any geometric point $q \in \mathfrak{M}_{0,n+1,\beta}$ with $p = h_n(q)$. Let $C_p = h_n^{-1}(p)$. The morphism (23) has the following properties:

1) if $q \in C_p$ is a nodal point, i.e. $q \in D^{T_i}_{\beta_i} \cap D^{T_i\cup\{n+1\}}_{\beta_i}$ for some i, then, up to isomorphism, it is of the form $\mathbb{N}^r \to \mathbb{N}^{r+1}$, mapping e_i to e_i, $i < r$ and e_r to $e_r + e_{r+1}$;
2) if q is a marked point, i.e. $q \in D^{\{j,n+1\}}_0$ for some $1 \le j \le n$, it is of the form $\mathbb{N}^r \to \mathbb{N}^{r+1}$ mapping e_i to e_i for $i \le r$;

3) if q is a smooth point, it is of the form $\mathbb{N}^r \to \mathbb{N}^r$, mapping e_i to e_i for $i = 1, \dots, r$.

An object of $\mathfrak{C}^{\vec{g}}_{0,n,\beta}$ over S is given by a morphism $f : S \to \mathfrak{C}'$ and by a simple morphism $f^*\pi^*_{n+1}\mathcal{M}^{n+1}_D \to \mathcal{M}'_S$ completing (22) to a commutative diagram

$$\begin{tikzcd} f^*h'^*_n\pi^*_n\mathcal{M}^n_D \arrow[r] \arrow[d] & f^*h'^*_n\mathcal{M}^n_{\mathcal{D}} \arrow[d] \\ f^*\pi^*_{n+1}\mathcal{M}^{n+1}_D \arrow[r] & \mathcal{M}'_S \end{tikzcd} \tag{24}$$

satisfying conditions (1), (2) listed on page 12. Let $\overline{s}$ be a geometric point of S mapping to $q \in \mathfrak{M}_{0,n+1,\beta}$ and assume that $p = h_n(q)$ belongs to $D^{T_1}_{\beta_1} \cap \dots \cap D^{T_r}_{\beta_r}$. Consider the diagram obtained from (24) by taking the stalk at $\overline{s}$ of the associated ghost sheaves. There is a bijection between the divisors $D^{T_i}_{\beta_i}$ containing the point p and irreducible elements of the monoid $\overline{\mathcal{M}}^n_{D,\overline{s}}$. If $\overline{s}$ maps to the j-st marked point of C_p, i.e. if $q \in D^{T_1}_{\beta_i} \cap \dots \cap D^{T_r}_{\beta_r} \cap D^{T_{r+1}}_{\beta_{r+1}}$, with $T_{r+1} = \{j, n+1\}$ and $\beta_{r+1} = 0$, we get the following diagram:

$$\begin{tikzcd} \oplus^r_{i=1}\mathbb{N} \arrow[r, "\oplus(\times r_{T_i,\beta_i})"] \arrow[d] & \oplus^r_{i=1}\mathbb{N} \arrow[d] \\ \oplus^{r+1}_{i=1}\mathbb{N} \arrow[r, "\oplus(\times\alpha_i)"'] & \oplus^{r+1}_{i=1}\mathbb{N}, \end{tikzcd} \tag{25}$$

where the vertical arrows map e_i to e_i, $i = 1, \dots, r$. By condition (1) on page 12, we must have $\alpha_i = r_{T_i,\beta_i}$ for $i = 1, \dots, r$, and by condition (2) on page 12, $\alpha_{r+1} = r_j$, we must have the order of the j-th marked point of twisted curves parametrized by $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$.

Suppose that $\overline{s}$ maps to a nodal point of C_p. In this case q belongs to $D^{T_1}_{\beta_1} \cap \dots \cap D^{T_r}_{\beta_r} \cap D^{T_{r+1}}_{\beta_{r+1}}$, where $T_{r+1} = T_r \cup \{n+1\}$ and $\beta_{r+1} = \beta_r$ for some T_r, β_r. Note that we abuse the notation by using the same symbol for divisors in $\mathfrak{M}_{0,n,\beta}$ and $\mathfrak{M}_{0,n+1,\beta}$. In this case we again get a diagram as in (25), with vertical arrows mapping e_i to e_i, $i < r$ and mapping e_r to $e_r + e_{r+1}$. Commutativity of the diagram implies that $\alpha_r = \alpha_{r+1} = r_{T_r,\beta_r}$.

From the above discussion we see that an object of $\mathfrak{C}^{\vec{g}}_{0,n+1,\beta}$ over S encodes a morphism $f : S \to \mathfrak{M}_{0,n+1,\beta}$ and a simple morphism of log structures $f^*\mathcal{M}^{n+1}_D \to \mathcal{M}'_S$ as in [**35**], with coefficients $\alpha_{T,\beta'}$, $T \subset \{1, \dots, n\}$, $\beta' \le \beta$, associated to the irreducible components of the boundary divisor $D^T_{\beta'}$ as follows:

- $\alpha_{T,\beta'} = \alpha_{T\cup\{n+1\},\beta'} = r_{T,\beta'}, \quad (T, \beta') \in \mathcal{I}_D;$

- $\alpha_{\{i,n+1\},0} = r_i, \quad i = 1, \dots, n;$

where the $r_{T,\beta'}$ are the integer numbers determined as in (15) for the β-admissible vector $\vec{g}$ and the class β. Observe that the collection of the coefficients $\alpha_{T,\beta'}$ coincides with the collection of coefficients $r_{T,\beta'}$ associated to the β-admissible vector $\vec{g} \cup \{1\}$ and to the class β. In other words, there is a natural morphism $\mathfrak{C}^{\vec{g}}_{0,n,\beta} \to \mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$.

On the other hand, it is not hard to see that there is a morphism $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{C}^{\vec{g}}_{0,n,\beta}$. We start by showing that there is a morphism $\widetilde{h}_n : \mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}$ inducing a morphism $f_{n+1} : \mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{C}'$. Consider the universal simple morphism of log structures $\mathfrak{l}_{n+1} : \pi^*_{n+1}\mathcal{M}^{n+1}_D \to \mathcal{M}^{n+1}_{\mathcal{D}}$ over $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$, where $\pi_{n+1} : \mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{M}_{0,n+1,\beta}$ is the natural map, and $\mathcal{M}^{n+1}_{\mathcal{D}}$ is the log structure associated to the divisor $\mathcal{D}$ on $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ by construction of $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$. There is a natural sub-log structure $\mathcal{M}_{\mathcal{D}'^n}$ of $\mathcal{M}^{n+1}_{\mathcal{D}}$ which is associated to the divisor

$$\mathcal{D}'^n := \bigcup_{\substack{(T,\beta')\in\mathcal{I}_D \\ T\subseteq[n]}} \mathcal{D}^T_{\beta'}.$$

The composite morphism

$$\pi^*_{n+1}h^*_n\mathcal{M}^n_D \to \pi^*_{n+1}\mathcal{M}^{n+1}_D \to \mathcal{M}^{n+1}_{\mathcal{D}}$$

factors through $\mathcal{M}_{\mathcal{D}'^n}$ and $\pi^*_{n+1}h^*_n\mathcal{M}^n_D \to \mathcal{M}_{\mathcal{D}'^n}$ is a simple morphism of log structures. This defines the morphism $\widetilde{h}_n$. By construction $\mathcal{M}_{\mathcal{D}'^n}$ is the pullback of $\mathcal{M}^n_{\mathcal{D}}$ along $\widetilde{h}_n$. We conclude by observing that the pair $(f_{n+1}, \mathfrak{l}_{n+1})$ satisfies diagram (24) with f_{n+1} in place of f, giving the morphism $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{C}^{\vec{g}}_{0,n,\beta}$.

It is not hard to check that the two maps between $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ and $\mathfrak{C}^{\vec{g}}_{0,n,\beta}$ are inverse to each other.

We use [**37**], Lemma 5.3, to conclude that $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ is an open substack of $\mathfrak{M}^{tw}_{0,n+1,\beta}$. q.e.d.

Remark 3.17. An argument similar to the proof of Proposition 3.16 can be used to characterize the universal weighted twisted curve over $\mathfrak{M}^{tw}_{g,n,\beta}$ as an open substack of $\mathfrak{M}^{tw}_{g,n+1,\beta}$.

Let $[f : \mathcal{C} \to \mathcal{G}] \in \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$. By forgetting f and keeping degrees of the restrictions of f on irreducible components of $\mathcal{C}$, we obtain a morphism

$$\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \to \mathfrak{M}^{tw}_{0,n,\beta}.$$

Lemma 3.18. *There exists a natural morphism*

$$s : \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}.$$

Proof. Let $[f : (\mathcal{C}, \{\sigma_i\}) \to \mathcal{G}]$ be an object of $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}(\mathbb{C})$. The pushforward f_* defines the weight function $\mathcal{C}^{gen} \to H_2^+(X, \mathbb{Z})$. The domain $(\mathcal{C}, \{\sigma_i\})$ is a genus 0 twisted curve whose i-th marked gerbe $(1 \leq i \leq n)$ is banded by μ_{r_i}. Let $x \in \mathcal{C}$ be a node which separates $\mathcal{C}$ into two connected components $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Put $f_*[\mathcal{C}_1] = \beta'$ and let $T \subset [n]$ be the set of marked points contained in $\mathcal{C}_1$. Since f is representable, Lemma 3.2 and equation (15) imply that x is a stack point of order $r_{T,\beta'}$. This defines an object of $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$. Because indices of nodes are locally constant, extension to objects over general base schemes is straightforward. q.e.d.

3.4. Gerbe structures on components. We continue to fix a β-admissible vector $\vec{g} = (g_1, \dots, g_n) \in \mu_r^{\times n}$. Consider the following diagram:

$$
\begin{tikzcd}
\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \arrow[r, "t"'] \arrow[dr, "s"'] \arrow[rrr, bend left, "p"] & P_n^{\vec{g}} \arrow[r, "r'"'] \arrow[d] \arrow[dr, phantom, "\Box"] & P \arrow[r, "r"'] \arrow[d, "q'"'] \arrow[dr, phantom, "\Box"] & \overline{M}_{0,n}(X,\beta) \arrow[d, "q"] \\
& \mathfrak{Y}^{\vec{g}}_{0,n,\beta} \arrow[r] & \mathfrak{M}^{tw}_{0,n,\beta} \arrow[r] \arrow[d, "s'"'] \arrow[dr, phantom, "\Box"] & \mathfrak{M}_{0,n,\beta} \arrow[d, "s''"] \\
& & \mathfrak{M}^{tw}_{0,n} \arrow[r] & \mathfrak{M}_{0,n}.
\end{tikzcd}
\tag{26}
$$

Here the morphism $p : \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to \overline{M}_{0,n}(X, \beta)$ is defined by sending a twisted stable map to its associated map between coarse moduli spaces. The morphism $\mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}^{tw}_{0,n,\beta}$ is defined by Proposition 3.14. The stacks P and $P_n^{\vec{g}}$ are defined as fiber products. The morphism $t : \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to P_n^{\vec{g}}$ is evidently defined.

The goal of this subsection is to prove the following

Theorem 3.19. *Let $\mathcal{G} = \sqrt[r]{\mathcal{L}/X} \to X$ be an r-th root gerbe. Let $\vec{g}$ be a β-admissible vector for $\mathcal{G}$ and a choice of $\beta \in H_2^+(X, \mathbb{Z})$. Then the morphism $t : \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to P_n^{\vec{g}}$ exhibits $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ as a μ_r-gerbe over $P_n^{\vec{g}}$.*

Proof. We will prove that $t : \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to P_n^{\vec{g}}$ is a gerbe by showing that the structure morphism and the relative diagonal are epimorphisms in the sense of [**34**], Definition 3.6. We know by Proposition 3.6 that the morphism is bijective on geometric points; hence to prove the first claim it is enough to show that $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to P_n^{\vec{g}}$ is étale. For the notion of étale non-representable morphisms between algebraic stacks we refer to [**34**], Definition 4.14. Since the stacks involved are of finite type we can use Proposition 4.15 (ii) of [**34**]. Moreover, since we assume the stacks are noetherian, it is enough to prove that the lifting criterion of [**34**], Proposition 4.15, holds for morphisms from artinian local rings (cfr. [**26**], 17.5.4 and [**25**], **0**, Prop. 22.1.4.) Consider a square zero extension of artinian local rings:

$$1 \to I \to B \to A \to 1. \tag{27}$$

We need to show that given the outer commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \arrow[d] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & P_n^{\vec{g}}, \end{tikzcd}$$

the morphism $\operatorname{Spec} A \to \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$ factors through $\operatorname{Spec} B$. Given a twisted stable map $\widetilde{f}_A : \mathcal{C}_A \to \mathcal{G}$ over $\operatorname{Spec} A$ together with a lifting $\mathcal{C}_A \hookrightarrow \mathcal{C}_B$ of the domain curve to $\operatorname{Spec} B$ and a lift $f_B : C_B \to X$ of the coarse map $f_A : C_A \to X$ to $\operatorname{Spec} B$, we claim that $\widetilde{f}_A$ lifts to $\operatorname{Spec} B$ uniquely. To show this, first note that the exact sequence

$$H^1(I) \to H^1(\mathcal{O}^*_{\mathcal{C}_B}) \to H^1(\mathcal{O}^*_{\mathcal{C}_A}) \to H^2(I)$$

arising from the extension (27) gives an isomorphism $\operatorname{Pic} \mathcal{C}_A \simeq \operatorname{Pic} \mathcal{C}_B$. Indeed, $H^1(I)$ vanishes because the curves have arithmetic genus zero, and $H^2(I)$ vanishes by dimensional reasons. The morphism $\widetilde{f}_A$ is defined by a line bundle $\mathcal{N}_A$ satisfying $\mathcal{N}_A^{\otimes r} \simeq \pi_A^* f_A^* \mathcal{L}$, where $\pi_A : \mathcal{C}_A \to C_A$ is the map to the coarse curve. The isomorphism $\operatorname{Pic} \mathcal{C}_A \simeq \operatorname{Pic} \mathcal{C}_B$ yields a line bundle $\mathcal{N}_B$ on $\mathcal{C}_B$ which satisfies $\mathcal{N}_B^r \simeq \pi_B^* f_B^* \mathcal{L}$. (Again $\pi_B : \mathcal{C}_B \to C_B$ denotes the map to the coarse curve.) This defines the desired extension $\widetilde{f}_B : \mathcal{C}_B \to \mathcal{G}$.

It remains to show that the relative diagonal morphism

$$\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \to \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \times_{P_n^{\vec{g}}} \mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$$

is locally surjective, namely that any two local sections are locally isomorphic. Let $T \to P_n^{\vec{g}}$ be a morphism which gives rise to the twisted curve $\mathcal{C}_T$ over T with coarse curve $\pi_T : \mathcal{C}_T \to C_T$ and stable map $f_T : C_T \to X$. Consider the base-change

$$\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} \times_{P_n^{\vec{g}}} T \to T. \tag{28}$$

Sections of (28) are twisted stable maps $\mathcal{C}_T \to \mathcal{G}$ which induce f_T. Such maps are defined by line bundles $\mathcal{N}$ on $\mathcal{C}_T$ satisfying $\mathcal{N}^{\otimes r} \simeq \pi_T^* f_T^* \mathcal{L}$. Two sections are isomorphic if and only if their defining line bundles are isomorphic.

Let f and f' be two sections of (28) with line bundles $\mathcal{N}$ and $\mathcal{N}'$. Note that the line bundle $\mathcal{N} \otimes \mathcal{N}'^{\vee}$ is a pullback from the coarse moduli space C_T since it carries trivial representations of the special points of any geometric fiber of $\mathcal{C}_T$. This is due to the fact that $\mathcal{N}$ and $\mathcal{N}'$ induce twisted stable maps with the same β-admissible vector. Note moreover that $(\mathcal{N}\otimes\mathcal{N}'^{\vee})^{\otimes r}$ is trivial:

$$(\mathcal{N}\otimes\mathcal{N}'^{\vee})^{\otimes r}\simeq \pi_T^* f_T^*\mathcal{L}\otimes \pi_T^* f_T^*\mathcal{L}^{\vee}\simeq \mathcal{O}_{\mathcal{C}_T}.$$

As a consequence, $\mathcal{N}\otimes\mathcal{N}'^{\vee}$ restricts to a trivial line bundle over any irreducible component of any geometric fiber of $\mathcal{C}_T$. By the Theorem on Cohomology and Base Change, $\mathcal{N}\otimes\mathcal{N}'^{\vee}$ is isomorphic to the pullback of a line bundle from the base T. Every line bundle over a scheme is locally trivial; therefore up to base change by an étale morphism $T'\to T$ the two line bundles $\mathcal{N}$ and $\mathcal{N}'$ are isomorphic.

For any object ξ of $\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$ over some $x\in P_n^{\vec{g}}(T)$ the sheaf of relative automorphisms $Aut_x(\xi)$ is a sheaf of abelian groups. Therefore $\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}\to P_n^{\vec{g}}$ is a gerbe banded by a sheaf of abelian groups ([**28**], Proposition IV 1.2.3 (i)). For any ξ in $\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$, there is a natural identification $Aut_x(\xi)\simeq(\mu_r)_T$. Indeed automorphisms of ξ leaving x fixed are automorphisms of a line bundle over a twisted curve $p:\mathcal{C}\to T$ whose r-th power is the identity. The claim follows since for a family of twisted curves $p_*\mathcal{O}_{\mathcal{C}}\simeq\mathcal{O}_T$. Such a collection of natural identifications is compatible with restrictions and isomorphisms. This means by definition that the $\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}$ is banded by $(\mu_r)_{P_n^{\vec{g}}}$ ([**28**], IV, Definition 2.2.2). q.e.d.

3.5. Root gerbe structures on components. Consider the following diagram:

$$\begin{tikzcd} P_{n+1}^{\vec{g}\cup\{1\}} \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\square"] & \overline{M}_{0,n+1}(X,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{0,n}(X,\beta) \arrow[d] \\ \mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}} \arrow[r] & \mathfrak{M}_{0,n+1,\beta} \arrow[r] & \mathfrak{M}_{0,n,\beta}. \end{tikzcd} \tag{29}$$

The stack $P_{n+1}^{\vec{g}\cup\{1\}}$ is defined by the cartesian square on the left. The square on the right is cartesian by Proposition 3.9. The existence of the morphism $\mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}}\to\mathfrak{Y}_{0,n,\beta}^{\vec{g}}$ implies that the outer square in (29) is equivalent to the outer square of the following diagram:

$$\begin{tikzcd} P_{n+1}^{\vec{g}\cup\{1\}} \arrow[r, "\phi"] \arrow[d, "v"] & P_n^{\vec{g}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{0,n}(X,\beta) \arrow[d] \\ \mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}} \arrow[r] & \mathfrak{Y}_{0,n,\beta}^{\vec{g}} \arrow[r] & \mathfrak{M}_{0,n,\beta}. \end{tikzcd} \tag{30}$$

Since (29) and the right side of (30) are cartesian, the left part of (30) is also cartesian.

By construction the stack $P_n^{\vec{g}}$ parametrizes the data

$$((\mathcal{C},\sigma_1,\dots,\sigma_n),(f:(C,p_1,\dots,p_n)\to X))$$

where $(\mathcal{C}, \sigma_1, \dots, \sigma_n)$ is an n-pointed twisted curve with isotropy group at σ_i being μ_{r_i}, $(C, p_1, \dots, p_n)$ is the coarse curve, and $[f] \in \overline{M}_{0,n}(X, \beta)$ is a stable map. The universal twisted curve over $P_n^{\vec{g}}$ is given by the morphism $P_{n+1}^{\vec{g}\cup\{1\}} \to P_n^{\vec{g}}$, and the universal stable map is obtained from the composition

$$u : P_{n+1}^{\vec{g}\cup\{1\}} \to \overline{M}_{0,n+1}(X, \beta) \stackrel{ev_{n+1}}{\longrightarrow} X.$$

The purpose of this subsection is to prove the following refinement of Theorem 3.19:

Theorem 3.20. *$\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ is a root gerbe over the stack $P_n^{\vec{g}}$.*

Proof. We construct a line bundle over $P_n^{\vec{g}}$ and show that the stack of its r-th roots admits a representable morphism to $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ which covers the identity map on $P_n^{\vec{g}}$.

Recall that for a β-admissible vector $\vec{g} = (g_1, \dots, g_n)$ we have defined triples $(\rho_i, r_i, m_i), 1 \le i \le n$ of integers in (14). For $1 \le i \le n$ we choose $d_i \in \mathbb{Z}$ such that

$$g_i = \exp\left(\frac{2\pi\sqrt{-1}}{r_i} d_i\right) \quad \text{and} \quad \sum_{i=1}^{n} \frac{d_i}{r_i} = \frac{1}{r}\int_\beta c_1(\mathcal{L}). \tag{31}$$

This is possible because of (10). Note that d_i depends on β.

Associated to a pair (T, β') which indexes a boundary divisor of $\mathfrak{M}_{0,n,\beta}$, we have defined a triple $(\theta_{T,\beta'}, r_{T,\beta'}, m_{T,\beta'})$ of integers in (15). We define another integer $d_{T,\beta'}$ such that

$$\sum_{i\in T} \frac{d_i}{r_i} + \frac{d_{T,\beta'}}{r_{T,\beta'}} = \frac{1}{r}\int_{\beta'} c_1(\mathcal{L}). \tag{32}$$

Note that (32) implies $d_{T,\beta'} = -d_{T^C,\beta-\beta'}$ and $\langle \frac{d_{T,\beta'}}{r_{T,\beta'}} \rangle = \theta_{T,\beta'}$.

For $1 \le i \le n$, let $S_i \subset \mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}}$ denote the pullback of the i-th marked section divisor from $\mathfrak{M}_{0,n+1,\beta}$. Define a line bundle over $\mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}}$ as follows:

$$\mathcal{L}_{\mathfrak{Y}} := \mathcal{O}_{\mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}}}\left(\sum_{1\le i\le n} \frac{d_i}{r_i} S_i - \sum_{(T,\beta')\in\mathcal{I}_D} \frac{d_{T,\beta'}}{r_{T,\beta'}} D_{\beta'}^{T\cup\{n+1\}}\right). \tag{33}$$

Here $\mathcal{I}_D$ is the set of pairs (T, β') defined on page 18.

Lemma 3.21. *There exists a line bundle $\mathcal{L}_2$ over $P_n^{\vec{g}}$ such that*

$$(v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (u^*\mathcal{L})^{-1} \simeq \phi^*\mathcal{L}_2. \tag{34}$$

Proof. As in [**12**], it suffices to check that the degree of the line bundle $(v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (u^*\mathcal{L})^{-1}$ restricted to any component of any fiber of ϕ is zero. The argument works because $(v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (u^*\mathcal{L})^{-1}$ is in fact a pullback from $\mathfrak{M}_{0,n+1,\beta}$. Indeed $\mathcal{L}$ is a line bundle over a scheme and $(v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r}$ carries trivial representations of the automorphism groups of stacky points of $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ relative to $\mathfrak{M}_{0,n+1,\beta}$. Let $\mathcal{C}$ be a fiber of ϕ, with coarse curve C. Denote by $f : C \to X$ the corresponding stable map to X. Let $\mathcal{C}^0 \subset \mathcal{C}$ be an irreducible component with coarse curve C^0. Let $x_1, \ldots, x_m$ be nodes of $\mathcal{C}$ that are contained in $\mathcal{C}^0$. Let $T_j \subset [n], 1 \leq j \leq m$ be the marked points contained in the subcurves $\mathcal{C}^j \subset \mathcal{C}$ which are connected to $\mathcal{C}^0$ at x_j, and T_0 the marked points contained in $\mathcal{C}^0$. Then $[n] = T_0 \cup T_1 \cup \ldots \cup T_m$. Put $\beta_0 := f_*[C^0]$ and $\beta_j := f_*[C^j]$ (here C^j is the coarse curve of $\mathcal{C}^j$).

We need some properties about restrictions of these line bundles.

Claim. Consider the line bundle $\mathcal{L}^{(T,\beta')} := \mathcal{O}_{\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}}(\frac{1}{r_{T,\beta'}} D^{T\cup\{n+1\}}_{\beta'})$. Let $\mathcal{C}$ be a geometric fiber of $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta} \to \mathfrak{Y}^{\vec{g}}_{0,n,\beta}$.

- If there is no node e in $\mathcal{C}$ such that the two connected components of the normalization of $\mathcal{C}$ at e have degrees β', resp. β'' (such that $\beta' + \beta'' = \beta$), and contain marked points with indices in T, resp. T^C, then $\mathcal{L}^{(T,\beta')}|_{\mathcal{C}}$ is trivial.
- If there is such a node, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two connected components of the partial normalization at the node. Suppose that the preimages of the marked gerbes with indices in T are contained in $\mathcal{C}_1$. Let $\overline{\mathcal{C}}_1 \subset \mathcal{C}_1$ and $\overline{\mathcal{C}}_2 \subset \mathcal{C}_2$ be the two irreducible components in $\mathcal{C}_1$ and $\mathcal{C}_2$ containing the node e. Then
  1) $\mathcal{L}^{(T,\beta')}|_{\overline{\mathcal{C}}_1} \simeq \mathcal{O}_{\overline{\mathcal{C}}_1}(-\frac{1}{r_{T,\beta'}})$;
  2) $\mathcal{L}^{(T,\beta')}|_{\overline{\mathcal{C}}_2} \simeq \mathcal{O}_{\overline{\mathcal{C}}_2}(\frac{1}{r_{T,\beta'}})$.
  3) the restriction of $\mathcal{L}^{(T,\beta')}$ to any other component $\mathcal{C}'$ of $\mathcal{C}$ is trivial.

Proof of Claim. The first property follows since $\mathcal{C}$ misses $D^{T\cup\{n+1\}}_{\beta'}$. For the second property, first note that $\mathcal{C}_1 = \mathcal{C} \cap D^{T\cup\{n+1\}}_{\beta'}$ and $\mathcal{C}_2 = \mathcal{C} \cap D^{T^C}_{\beta''}$. Therefore $\mathcal{C}_2 \cap D^{T\cup\{n+1\}}_{\beta'}$ is one point and (2) above follows. (3) for any component $\mathcal{C}' \subset \mathcal{C}_2$ also follows. Moreover, since $\mathcal{O}(\frac{1}{r_{T,\beta'}}(D^T_{\beta'} \cup D^{T\cup\{n+1\}}_{\beta'}))$ over $\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}$ is the pullback of $\mathcal{O}(\frac{1}{r_{T,\beta'}} D^T_{\beta'})$ over $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$, its restriction to geometric fibers has to be trivial. This implies that $\mathcal{O}_{\mathfrak{Y}^{\vec{g}\cup\{1\}}_{0,n+1,\beta}}(\frac{1}{r_{T,\beta'}} D^T_{\beta'})|_{\overline{\mathcal{C}}_2} \simeq \mathcal{O}_{\overline{\mathcal{C}}_2}(-\frac{1}{r_{T,\beta'}})$. Statements (1) and (3) for $\mathcal{C}' \subset \mathcal{C}_1$ follow by symmetry. q.e.d.

Notice that the following diagram

$$
\begin{tikzcd}
P_{n+1}^{\vec{g}\cup\{1\}} \arrow[r, "v"] \arrow[d] & \mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}} \arrow[d] \\
P_n^{\vec{g}} \arrow[r] & \mathfrak{Y}_{0,n,\beta}^{\vec{g}}
\end{tikzcd} \tag{35}
$$

is cartesian; hence the fibers of $P_{n+1}^{\vec{g}\cup\{1\}} \to P_n^{\vec{g}}$ are isomorphic to the fibers of $\mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}} \to \mathfrak{Y}_{0,n,\beta}^{\vec{g}}$. Applying the above claim and the fact that $d_{T,\beta'} = -d_{T^C,\beta-\beta'}$, we find

$$
v^*\mathcal{L}_{\mathfrak{Y}}|_{\mathcal{C}^0} = \mathcal{O}_{\mathcal{C}^0}\left(\sum_{i\in T_0}\frac{d_i}{r_i}S_i - \sum_{1\le j\le m}\frac{d_{T_j,\beta_j}}{r_{T_j,\beta_j}}x_j\right). \tag{36}
$$

By (32) we have

$$
\sum_{i\in T_j}\frac{d_i}{r_i} + \frac{d_{T_j,\beta_j}}{r_{T_j,\beta_j}} = \frac{1}{r}\int_{\beta_j} c_1(\mathcal{L}).
$$

By (31) we have

$$
\sum_{i\in T_0}\frac{d_i}{r_i} + \sum_{1\le j\le m}\sum_{i\in T_j}\frac{d_i}{r_i} = \frac{1}{r}\int_{\beta} c_1(\mathcal{L}).
$$

Since $\beta = \beta_0 + \sum_{1\le j\le m}\beta_j$, we find that the degree of $v^*\mathcal{L}_{\mathfrak{Y}}|_{\mathcal{C}^0}$ is

$$
\begin{aligned}
&\sum_{i\in T_0}\frac{d_i}{r_i} - \sum_{1\le j\le m}\frac{d_{T_j,\beta_j}}{r_{T_j,\beta_j}} \\
&= \left(\frac{1}{r}\int_{\beta} c_1(\mathcal{L}) - \sum_{1\le j\le m}\sum_{i\in T_j}\frac{d_i}{r_i}\right) - \sum_{1\le j\le m}\left(\frac{1}{r}\int_{\beta_j} c_1(\mathcal{L}) - \sum_{i\in T_j}\frac{d_i}{r_i}\right) \\
&= \frac{1}{r}\int_{\beta_0} c_1(\mathcal{L}).
\end{aligned}
$$

Thus the degree of $(v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (u^*\mathcal{L})^{-1}|_{\mathcal{C}^0}$ is zero, as desired. q.e.d.

Let $\mathcal{P}_n^{\vec{g}} = \sqrt[r]{\mathcal{L}_2/P_n^{\vec{g}}}$ be the stack of r-th roots of $\mathcal{L}_2$. Denote by $\pi_{P_n} : \mathcal{P}_n^{\vec{g}} \to P_n^{\vec{g}}$. Consider the pullback of $P_{n+1}^{\vec{g}\cup\{1\}} \to P_n^{\vec{g}}$ to $\mathcal{P}_n^{\vec{g}}$ via π_{P_n}:

$$
\begin{tikzcd}
\mathcal{P}_{n+1}^{\vec{g}\cup\{1\}} \arrow[r, "\pi_{P_{n+1}}"] \arrow[d, "\phi'"] & P_{n+1}^{\vec{g}\cup\{1\}} \arrow[r, "u"] \arrow[d, "\phi"] & X \\
\mathcal{P}_n^{\vec{g}} \arrow[r, "\pi_{P_n}"] & P_n^{\vec{g}}.
\end{tikzcd} \tag{37}
$$

Lemma 3.22. *There exists a family of twisted stable maps*

$$\begin{tikzcd} \mathcal{P}_{n+1}^{\vec{g}\cup\{1\}} \arrow[r] \arrow[d] & \sqrt[r]{\mathcal{L}/X} \\ \mathcal{P}_n^{\vec{g}}. & \end{tikzcd} \tag{38}$$

Proof. Let $\mathcal{L}_2^{1/r}$ be the universal r-th root line bundle of $\pi_{P_n}^*\mathcal{L}_2$ over $\mathcal{P}_n^{\vec{g}}$. In (37) we calculate

$$\begin{aligned} \pi_{P_{n+1}}^* u^*\mathcal{L} \simeq & \pi_{P_{n+1}}^*((v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (\phi^*\mathcal{L}_2)^{-1}) \quad \text{by (34)} \\ \simeq & (\pi_{P_{n+1}}^* v^*\mathcal{L}_{\mathfrak{Y}})^{\otimes r} \otimes (\phi'^*\pi_{P_n}^*\mathcal{L}_2)^{-1} \\ \simeq & (\pi_{P_{n+1}}^* v^*\mathcal{L}_{\mathfrak{Y}} \otimes \phi'^*(\mathcal{L}_2^{1/r})^{-1})^{\otimes r}. \end{aligned}$$

The line bundle $\mathcal{L}_{\mathcal{P}} := \pi_{P_{n+1}}^* v^*\mathcal{L}_{\mathfrak{Y}} \otimes \phi'^*(\mathcal{L}_2^{1/r})^{-1}$ defines a morphism

$$\mathcal{P}_{n+1}^{\vec{g}\cup\{1\}} \to \sqrt[r]{\mathcal{L}/X}. \tag{39}$$

Since $\phi' : \mathcal{P}_{n+1}^{\vec{g}\cup\{1\}} \to \mathcal{P}_n^{\vec{g}}$ is a family of twisted curves, to show that the morphism (39) is a family of twisted stable maps we need to check that it is representable. For this purpose it suffices to work on geometric fibers of ϕ'.

Let $(\mathcal{C}, \{\sigma_i\})$ be a geometric fiber of ϕ', with coarse curve $(C, \{p_i\})$ and a given stable map $\bar{f} : (C, \{p_i\}) \to X$ of degree β. The restriction of (39) to $\mathcal{C}$ is given by the line bundle $\mathcal{L}_{\mathcal{P}}|_{\mathcal{C}}$, and fits into the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{f}"] \arrow[d] & \sqrt[r]{\mathcal{L}/X} \arrow[d] \\ C \arrow[r, "\bar{f}"] & X. \end{tikzcd}$$

By the choices of $d_i, 1 \leq i \leq n$ as in (31), the action of the stabilizer group of σ_i on $\mathcal{L}_{\mathcal{P}}|_{\sigma_i}$ is given by the element $g_i \in \mu_r$. This is due to the fact that by construction $Aut(\sigma_i)$ only acts non-trivially on the fibers of $\mathcal{L}_{\mathfrak{Y}}$. By the choice of $r_{T,\beta'}, d_{T,\beta'}$ as in (15) and (32), on the restriction of $\mathcal{L}_{\mathcal{P}}|_{\mathcal{C}}$ to a node $x \in \mathcal{C}$, the action of the stabilizer group of x is given by $\exp(2\pi\sqrt{-1}\frac{d_{T,\beta'}}{r_{T,\beta'}}) = \exp(2\pi\sqrt{-1}\frac{m_{T,\beta'}}{r_{T,\beta'}})$ (when $\mathcal{L}_{\mathcal{P}}|_{\mathcal{C}}$ is viewed as a line bundle on one of the branches meeting at x). Since by construction $m_{T,\beta'}$ and $r_{T,\beta'}$ are co-prime, it follows from Lemma 3.2 that the restriction of $\widetilde{f}$ to any irreducible component of $\mathcal{C}$ is representable. Therefore $\widetilde{f}$ is representable. Hence (39) gives the family (38) of twisted stable maps we want. q.e.d.

As observed in the proof of Lemma 3.22, the family (38) induces a morphism $\mathcal{P}_n^{\vec{g}} \to \mathcal{K}_{0,n}(\mathcal{G}, \beta)$. The choice of d_i ensures that actions of isotropy groups at marked points are given by the β-admissible vector $\vec{g}$. In other words, (38) gives a morphism

$$\mathcal{P}_n^{\vec{g}} \to \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}, \tag{40}$$

which fits into the following diagram:

$$\begin{tikzcd} \mathcal{P}_n^{\vec{g}} \arrow[r] \arrow[d] & \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \arrow[d] \\ P_n^{\vec{g}} \arrow[r, "id"] & P_n^{\vec{g}}. \end{tikzcd} \tag{41}$$

It is straightforward to check that the diagram is commutative. Indeed, an object $g : T \to \mathcal{P}_n^{\vec{g}}$ is given by

$$\pi_g : \mathcal{C} \to T, \quad f : C \to X, \quad \mathcal{N}, \tag{42}$$

where $\pi_g : \mathcal{C} \to T$ is a family of twisted curves, $f : C \to X$ is a stable map from the coarse moduli space of $\mathcal{C}$ to X, and $\mathcal{N}$ is a line bundle over T together with an isomorphism $\mathcal{N}^{\otimes r} \simeq g^* \pi_{P_n}^* \mathcal{L}_2$. Note that $\mathcal{C} \to T$ is obtained as the pullback of $\phi' : \mathcal{P}_{0,n+1,\beta}^{\vec{g} \cup \{1\}} \to \mathcal{P}_{0,n,\beta}^{\vec{g}}$ via $g : T \to \mathcal{P}_{0,n,\beta}^{\vec{g}}$; hence there is a morphism $g_{\mathcal{C}} : \mathcal{C} \to \mathcal{P}_{0,n+1,\beta}^{\vec{g} \cup \{1\}}$.

In (41), the morphism $\mathcal{P}_n^{\vec{g}} \to P_n^{\vec{g}}$ just forgets the line bundle $\mathcal{N}$. The morphism $\mathcal{P}_n^{\vec{g}} \to \mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ takes an object (42) to a twisted stable map $\mathcal{C} \to \mathcal{G}$ defined by the line bundle $g_{\mathcal{C}}^* v^* \mathcal{L}_{\mathfrak{Y}} \otimes \pi_g^* \mathcal{N}^{-1}$. The morphism $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}} \to P_n^{\vec{g}}$ retains $\pi_g : \mathcal{C} \to T$ and $f : C \to X$. Hence (41) is commutative.

An easy analysis on automorphism groups of (42) and twisted stable maps shows that the morphism (40) is representable. By [**28**], IV, Proposition 2.2.6, a representable morphism between two gerbes banded by the same group is an isomorphism. Since both $\mathcal{P}_n^{\vec{g}}$ and $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ are gerbes banded by the group μ_r (c.f. Theorem 3.19), (40) is an isomorphism. This completes the proof of Theorem 3.20. q.e.d.

4. Gromov–Witten invariants

4.1. Virtual fundamental class. The moduli space $\overline{M}_{0,n}(X, \beta)$ has a perfect obstruction theory relative to $\mathfrak{M}_{0,n}$ which is given by

$$E^\bullet := R\pi_*(f^* \Omega_X \otimes \omega_\pi) \to L_{\overline{M}_{0,n}(X,\beta)/\mathfrak{M}_{0,n}},$$

where

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \overline{M}_{0,n}(X,\beta) & \end{tikzcd}$$

is the universal stable map. Notice that since the morphism s'' in diagram (26) is étale, $E^\bullet$ is also a perfect obstruction theory relative to $\mathfrak{M}_{0,n,\beta}$.

Consider the universal twisted stable map to the gerbe $\mathcal{G}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\pi}"] & \mathcal{G} \\ \mathcal{K}_{0,n}(\mathcal{G},\beta). & \end{tikzcd}$$

According to [**2**] the moduli stack $\mathcal{K}_{0,n}(\mathcal{G},\beta)$ has a perfect obstruction theory $\widetilde{E}^\bullet$ relative to $\mathfrak{M}^{tw}_{0,n}$ given by

$$\widetilde{E}^\bullet := R\widetilde{\pi}_*(\widetilde{f}^*\Omega_{\mathcal{G}} \otimes \omega_{\widetilde{\pi}}) \to L_{\mathcal{K}_{0,n}(\mathcal{G},\beta)/\mathfrak{M}^{tw}_{0,n}}.$$

Since the morphism $\mathfrak{Y}^{\vec{g}}_{0,n,\beta} \to \mathfrak{M}^{tw}_{0,n,\beta}$ is étale (Lemma 3.15), we can view $\widetilde{E}^\bullet$ as a perfect obstruction theory relative to $\mathfrak{Y}^{\vec{g}}_{0,n,\beta}$.

The complex $\widetilde{E}^\bullet$ turns out to be the pullback of $E^\bullet$ as an object in $\mathcal{D}_{Coh}(\mathcal{K}_{0,n}(\mathcal{G},\beta))$.

Lemma 4.1. *There is a natural isomorphism of objects in $\mathcal{D}_{Coh}(\mathcal{K}_{0,n}(\mathcal{G},\beta))$,*

$$p^*E^\bullet \xrightarrow{\sim} \widetilde{E}^\bullet.$$

Proof. We will prove the statement for $\widetilde{E}^{\vee\bullet} = R\widetilde{\pi}_*\widetilde{f}^*T_{\mathcal{G}}$ and $E^{\vee\bullet} = R\pi_*f^*T_X$. Consider the complex $Lp^*R\pi_*f^*T_X$ in $\mathcal{D}_{coh}(\mathcal{K}_{0,n}(\mathcal{G},\beta))$. It suffices to show that $Lp^*R\pi_*f^*T_X \simeq R\widetilde{\pi}_*\widetilde{f}^*T_{\mathcal{G}}$. For this we consider the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rrr, "\widetilde{f}"] \arrow[dr, "\rho"] \arrow[drr] \arrow[ddr, "\widetilde{\pi}"'] & & & \mathcal{G} \arrow[dr, "\epsilon"] & \\ & p^*C \arrow[r, "p_1"] \arrow[d, "\pi_1"] & C \arrow[rr, "f"] \arrow[d, "\pi"] & & X \\ & \mathcal{K}_{g,n}(\mathcal{G},\beta) \arrow[r, "p"] & \overline{M}_{0,n}(X,\beta). & & \end{tikzcd}$$

Observe that $\epsilon^*T_X \simeq T_{\mathcal{G}}$. Also we have $R\rho_*L\rho^* \simeq Id$ because the map ρ is the relative coarse moduli space for the map $\widetilde{\pi}$. The arrow p is flat, as follows from Remark 3.13 and from the gerbe structure of $\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}$ over $P_n^{\vec{g}}$ for any β-admissible vector $\vec{g}$. The arrow π is flat because it is the structure morphism of the universal curve. Moreover, the square in the above diagram is cartesian; hence we calculate (based on [**34**], Proposition 13.1.9)

$$\begin{aligned} Lp^*R\pi_*f^*T_X &\simeq R\pi_{1*}Lp_1^*f^*T_X \\ &\simeq R\pi_{1*}R\rho_*L\rho^*Lp_1^*f^*T_X \\ &\simeq R\pi_{1*}R\rho_*\widetilde{f}^*\epsilon^*T_X \\ &\simeq R\pi_{1*}R\rho_*\widetilde{f}^*T_{\mathcal{G}} \\ &\simeq R\widetilde{\pi}_*\widetilde{f}^*T_{\mathcal{G}}. \end{aligned}$$

Since p^* is exact, we write p^* for Lp^*. q.e.d.

Remark 4.2. The composite morphism

$$p^*E^\bullet \to p^*L_{\overline{M}_{0,n}(X,\beta)/\mathfrak{M}_{0,n}} \xrightarrow{\sim} L_{\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}/\mathfrak{M}_{0,n}^{tw}}$$

in $D_{Coh}(\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}})$ is the same as $E^{\vee\bullet} \to L_{\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}/\mathfrak{M}_{0,n}^{tw}}$. This follows from functorial properties of the cotangent complex ([**31**], Ch. 2, Sect. 1 and 2). The relative obstruction theories $E^\bullet$ and $E^{\vee\bullet}$ are built from functorial morphisms between the cotangent complexes of the target scheme or stack and of the universal objects (cfr. [**14**], Sect. 6).

Theorem 4.3. *Let $\mathcal{G} \to X$ be the stack of r-th roots of a line bundle on X. Let $\vec{g}$ be a β-admissible vector. Then the following relation between the virtual fundamental classes holds:*

$$p_*[\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}]^{vir} = \frac{1}{r}[\overline{M}_{0,n}(X,\beta)]^{vir}. \tag{43}$$

Proof. Consider the diagram (26) again. Observe that the following hold:

- $\mathfrak{M}_{0,n,\beta}$ and $\mathfrak{Y}_{0,n,\beta}^{\vec{g}}$ are smooth Artin stacks of the same pure dimension.
- The morphism $\mathfrak{Y}_{0,n,\beta}^{\vec{g}} \to \mathfrak{M}_{0,n,\beta}$ is of Deligne-Mumford type and of pure degree.
- The morphism $r' \circ r$ is proper (because being proper is preserved by base change).
- $\overline{M}_{0,n}(X,\beta) \to \mathfrak{M}_{0,n,\beta}$ has a perfect relative obstruction theory $E^\bullet$ inducing a perfect relative obstruction theory on $P_n^{\vec{g}} \to \mathfrak{Y}_{0,n,\beta}^{\vec{g}}$.

Therefore we can apply Theorem 5.0.1 of [**22**] and conclude that $(r' \circ r)_*[P_n^{\vec{g}}]^{vir} = [\overline{M}_{0,n}(X,\beta)]^{vir}$, where the multiplicative factor is 1 because by construction ([**35**], Theorem 4.1) $\mathfrak{Y}_{0,n,\beta}^{\vec{g}} \to \mathfrak{M}_{0,n,\beta}$ is an isomorphism outside a locus of codimension 1, and hence its base change has virtual degree equal to 1. Since t is étale, $[\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}]^{vir} = deg(t)\cdot[P_n^{\vec{g}}]^{vir}$. Since, by Theorem 3.19 or 3.20, t is of degree $\frac{1}{r}$, (43) follows. q.e.d.

4.2. Genus 0 **invariants.** Let

$$\epsilon : \mathcal{G} = \sqrt[r]{\mathcal{L}/X} \to X$$

be a μ_r-root gerbe. Then the inertia stack admits the following decomposition,

$$I\mathcal{G} = \coprod_{g\in\mu_r} \mathcal{G}_g,$$

where $\mathcal{G}_g$ is a root gerbe isomorphic to $\mathcal{G}$. Let $\epsilon_g : \mathcal{G}_g \to X$ be the induced morphism. On each component there is an isomorphism between the rational cohomology groups

$$\epsilon_g^* : H^*(X,\mathbb{Q}) \xrightarrow{\simeq} H^*(\mathcal{G}_g,\mathbb{Q}).$$

Let $\vec{g} = (g_1,\dots,g_n)$ be a β-admissible vector. There are evaluation maps

$$ev_i : \mathcal{K}_{0,n}(\sqrt[r]{\mathcal{L}/X},\beta)^{\vec{g}} \to \bar{I}(\mathcal{G})_{g_i},$$

where $\bar{I}(\mathcal{G})_{g_i}$ is a component of the *rigidified inertia stack* $\bar{I}(\mathcal{G}) = \cup_{g\in\mu_r}\bar{I}(\mathcal{G})_g$. Although the evaluation maps ev_i do not take values in $I\mathcal{G}$, as explained in [**2**], Section 6.1.3, one can still define a pullback map at cohomology level,

$$ev_i^* : H^*(\mathcal{G}_{g_i},\mathbb{Q}) \to H^*(\mathcal{K}_{0,n}(\sqrt[r]{\mathcal{L}/X},\beta)^{\vec{g}},\mathbb{Q}).$$

Given $\delta_i \in H^*(\mathcal{G}_{g_i},\mathbb{Q})$ for $1\le i\le n$ and integers $k_i \ge 0, 1\le i\le n$, one can define descendant orbifold Gromov–Witten invariants

$$\langle \delta_1\bar{\psi}_1^{k_1},\cdots,\delta_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^{\mathcal{G}} := \int_{[\mathcal{K}_{0,n}(\sqrt[r]{\mathcal{L}/X},\beta)^{\vec{g}}]^{vir}} \prod_{i=1}^n ev_i^*(\delta_i)\bar{\psi}_i^{k_i},$$

where $\overline{\psi}_i$ are the pullback of the first Chern classes of the tautological line bundles over $\overline{M}_{0,n}(X,\beta)$ (which by abuse of notation we also denote by $\bar{\psi}_i$).

For classes $\delta_i \in H^*(\mathcal{G}_{g_i},\mathbb{Q})$, set $\overline{\delta}_i = (\epsilon_{g_i}^*)^{-1}(\delta_i)$. Descendant Gromov–Witten invariants $\langle\overline{\delta}_1\bar{\psi}_1^{k_1},\cdots,\overline{\delta}_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^X$ of X are similarly defined. Theorem 4.3 implies the following comparison result.

Theorem 4.4.

$$\langle \delta_1\bar{\psi}_1^{k_1},\dots,\delta_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^{\mathcal{G}} = \frac{1}{r}\langle\overline{\delta}_1\bar{\psi}_1^{k_1},\cdots,\overline{\delta}_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^X.$$

Moreover, if $\vec{g}$ is not β-admissible, then the Gromov–Witten invariants of $\mathcal{G}$ vanish.

Proof. Denote by $\overline{ev}_i : \overline{M}_{0,n}(X,\beta) \to X$ the i-th evaluation map. Using the definition of ev_i^* one can check that $ev_i^*(\delta_i) = p^*\overline{ev}_i^*(\overline{\delta}_i)$. Note also that $p^*\bar{\psi}_i = \bar{\psi}_i$. Thus using Theorem 4.3 we have

$$\begin{aligned}
\langle \delta_1\bar{\psi}_1^{k_1}, \ldots, \delta_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^{\mathcal{G}} &= \int_{[\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}]^{vir}} \prod_{i=1}^n ev_i^*(\delta_i)\bar{\psi}_i^{k_i} \\
&= \int_{[\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}]^{vir}} \prod_{i=1}^n p^*\overline{ev}_i^*(\overline{\delta}_i)\bar{\psi}_i^{k_i} \\
&= \int_{[\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}}]^{vir}} \prod_{i=1}^n p^*(\overline{ev}_i^*(\overline{\delta}_i)\bar{\psi}_i^{k_i}) \\
&= \frac{1}{r}\int_{[\overline{M}_{0,n}(X,\beta)]^{vir}} \prod_{i=1}^n \overline{ev}_i^*(\overline{\delta}_i)\bar{\psi}_i^{k_i} \\
&= \frac{1}{r}\langle \overline{\delta}_1\bar{\psi}_1^{k_1}, \cdots, \overline{\delta}_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^X.
\end{aligned}$$

q.e.d.

In the following we use complex numbers $\mathbb{C}$ as coefficients for the cohomology. For $\overline{\alpha} \in H^*(X,\mathbb{C})$ and an irreducible representation ρ of μ_r, we define

$$\overline{\alpha}_\rho := \frac{1}{r}\sum_{g\in\mu_r} \chi_\rho(g^{-1})\epsilon_g^*(\overline{\alpha}),$$

where χ_ρ is the character of ρ. The map $(\overline{\alpha},\rho) \mapsto \overline{\alpha}_\rho$ clearly defines an additive isomorphism

$$\bigoplus_{[\rho]\in\widehat{\mu_r}} H^*(X)_{[\rho]} \simeq H^*(I\mathcal{G},\mathbb{C}), \tag{44}$$

where $\widehat{\mu_r}$ is the set of isomorphism classes of irreducible representations of μ_r, and for $[\rho] \in \widehat{\mu_r}$ we define $H^*(X)_{[\rho]} := H^*(X,\mathbb{C})$.

Theorem 4.4 together with orthogonality relations of characters of μ_r implies the following

Theorem 4.5. *Given $\overline{\alpha}_1, \ldots, \overline{\alpha}_n \in H^*(X,\mathbb{Q})$ and integers $k_1, \ldots, k_n \geq 0$, we have*

$$\begin{aligned}
&\langle \overline{\alpha}_{1\rho_1}\bar{\psi}_1^{k_1}, \ldots, \overline{\alpha}_{n\rho_n}\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^{\mathcal{G}} \\
&= \begin{cases} \frac{1}{r^2}\langle \overline{\alpha}_1\bar{\psi}_1^{k_1}, \cdots, \overline{\alpha}_n\bar{\psi}_n^{k_n}\rangle_{0,n,\beta}^X & \\ \qquad \times\chi_\rho(\exp(\frac{-2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r})) & \textit{if } \rho_1 = \rho_2 = \ldots. = \rho_n =: \rho, \\ 0 & \textit{otherwise}. \end{cases}
\end{aligned}$$

Proof. By our definition we have

$$\begin{aligned}&\langle\overline{\alpha}_{1\rho_1}\bar{\psi}_1^{k_1},\dots,\overline{\alpha}_{n\rho_n}\bar{\psi}_n^{k_n}\rangle^{\mathcal{G}}_{0,n,\beta}\\ =&\frac{1}{r^n}\sum_{g_1,\dots,g_n\in\mu_r}\prod_{i=1}^n\chi_{\rho_i}(g_i^{-1})\langle\prod_{i=1}^n\epsilon^*_{g_i}(\overline{\alpha}_i)\bar{\psi}_i^{k_i}\rangle^{\mathcal{G}}_{0,n,\beta}.\end{aligned}$$

The term associated to $\vec{g} := (g_1,\dots,g_n)$ in the above sum vanishes unless $\vec{g}$ is a β-admissible vector. This implies that $\prod_{i=1}^n g_i = \exp(\frac{2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r})$. We rewrite this equation as $g_n^{-1} = \exp(\frac{-2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r})\prod_{i=1}^{n-1}g_i$. Substitute this into the above equation and use Theorem 4.4 to get

$$\begin{aligned}&\langle\overline{\alpha}_{1\rho_1}\bar{\psi}_1^{k_1},\dots,\overline{\alpha}_{n\rho_n}\bar{\psi}_n^{k_n}\rangle^{\mathcal{G}}_{0,n,\beta}\\ =&\frac{1}{r^n}\sum_{g_1,\dots,g_{n-1}\in\mu_r}\chi_{\rho_n}\left(\exp\left(\frac{-2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r}\right)\right)\\ &\left(\prod_{i=1}^{n-1}\chi_{\rho_i}(g_i^{-1})\chi_{\rho_n}(g_i)\right)\frac{1}{r}\left\langle\prod_{i=1}^n\overline{\alpha}_i\bar{\psi}_i^{k_i}\right\rangle^X_{0,n,\beta}.\end{aligned}$$

Applying the orthogonality condition

$$\frac{1}{r}\sum_{g\in\mu_r}\chi_\rho(g^{-1})\chi_{\rho'}(g)=\delta_{\rho,\rho'},$$

we find

$$\begin{aligned}&\langle\overline{\alpha}_{1\rho_1}\bar{\psi}_1^{k_1},\dots,\overline{\alpha}_{n\rho_n}\bar{\psi}_n^{k_n}\rangle^{\mathcal{G}}_{0,n,\beta}\\ =&\frac{1}{r}\chi_{\rho_n}\left(\exp\left(\frac{-2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r}\right)\right)\\ &\left(\prod_{i=1}^{n-1}\delta_{\rho_i,\rho_n}\right)\frac{1}{r}\left\langle\prod_{i=1}^n\overline{\alpha}_i\bar{\psi}_i^{k_i}\right\rangle^X_{0,n,\beta}.\end{aligned}$$

The result follows. q.e.d.

We now reformulate this in terms of generating functions. Let

$$\{\overline{\phi}_i\,|\,1\le i\le \mathrm{rank}H^*(X,\mathbb{C})\}\subset H^*(X,\mathbb{C})$$

be an additive basis. According to the discussion above, the set

$$\{\overline{\phi}_{i\rho}\,|\,1\le i\le \mathrm{rank}H^*(X,\mathbb{C}),[\rho]\in\widehat{\mu_r}\}$$

is an additive basis of $H^*(I\mathcal{G},\mathbb{C})$. Recall that the genus 0 descendant potential of $\mathcal{G}$ is defined to be

$$\begin{aligned}&\mathcal{F}^0_{\mathcal{G}}(\{t_{i\rho,j}\}_{1\le i\le\mathrm{rank}H^*(X,\mathbb{C}),\rho\in\widehat{\mu_r},j\ge 0};Q):=\\ &\sum_{\substack{n\ge0,\beta\in H_2(X,\mathbb{Z})\\ i_1,\dots,i_n;\rho_1,\dots,\rho_n;j_1,\dots,j_n}}\frac{Q^\beta}{n!}\prod_{k=1}^n t_{i_k\rho_k,j_k}\langle\prod_{k=1}^n\overline{\phi}_{i_k\rho_k}\bar{\psi}_k^{j_k}\rangle^{\mathcal{G}}_{0,n,\beta}.\end{aligned}\tag{45}$$

The descendant potential $\mathcal{F}^0_{\mathcal{G}}$ is a formal power series in variables $t_{i\rho,j}, 1 \leq i \leq \mathrm{rank} H^*(X,\mathbb{C}), \rho \in \widehat{\mu_r}, j \geq 0$ with coefficients in the Novikov ring $\mathbb{C}[[\overline{NE}(X)]]$, where $\overline{NE}(X)$ is the effective Mori cone of the coarse moduli space of $\mathcal{G}$. Here Q^β are formal variables labeled by classes $\beta \in \overline{NE}(X)$. See e.g. [**40**] for more discussion on descendant potentials for orbifold Gromov–Witten theory.

Similarly the genus 0 descendant potential of X is defined to be

$$\mathcal{F}^0_X(\{t_{i,j}\}_{1\leq i\leq \mathrm{rank} H^*(X,\mathbb{C}), j\geq 0}; Q) := \sum_{\substack{n\geq 0, \beta\in H_2(X,\mathbb{Z})\\ i_1,\ldots,i_n; j_1,\ldots,j_n}} \tag{46}$$
$$\frac{Q^\beta}{n!}\prod_{k=1}^n t_{i_k,j_k}\left\langle \prod_{k=1}^n \overline{\phi}_{i_k}\bar{\psi}_k^{j_k}\right\rangle^X_{0,n,\beta}.$$

The descendant potential $\mathcal{F}^0_X$ is a formal power series in variables $t_{i,j}, 1 \leq i \leq \mathrm{rank} H^*(X,\mathbb{C}), j \geq 0$ with coefficients in $\mathbb{C}[[\overline{NE}(X)]]$ and Q^β is (again) a formal variable. Theorem 4.5 may be restated as follows.

Theorem 4.6.

$$\begin{aligned}&\mathcal{F}^0_{\mathcal{G}}(\{t_{i\rho,j}\}_{1\leq i\leq rank H^*(X,\mathbb{C}), \rho\in\widehat{\mu_r}, j\geq 0}; Q)\\ &= \frac{1}{r^2}\sum_{[\rho]\in\widehat{\mu_r}} \mathcal{F}^0_X(\{t_{i\rho,j}\}_{1\leq i\leq rank H^*(X,\mathbb{C}), j\geq 0}; Q_\rho),\end{aligned}$$

where Q_ρ is defined by the following rule:

$$Q^\beta_\rho := Q^\beta \chi_\rho\left(\exp\left(\frac{-2\pi\sqrt{-1}\int_\beta c_1(\mathcal{L})}{r}\right)\right),$$

and χ_ρ is the character associated to the representation ρ.

Theorem 4.6 confirms the decomposition conjecture for genus 0 Gromov–Witten theory of $\mathcal{G}$.

We have another reformulation of Theorem 4.6. Consider a new set of variables

$$\{q_{i\rho,j} | 1 \leq i \leq \mathrm{rank} H^*(X,\mathbb{C}), [\rho] \in \widehat{\mu_r}, j \geq 0\}$$

defined by dilaton shifts

$$q_{i\rho,j} = \begin{cases} t_{i\rho,j} & \text{if } (i,j) \neq (1,1)\\ t_{1\rho,1} - 1 & \text{if } (i,j) = (1,1).\end{cases}$$

We view $\mathcal{F}^0_{\mathcal{G}}$ as functions in the variables $q_{i\rho,j}$:

$$\mathcal{F}^0_{\mathcal{G}} = \mathcal{F}^0_{\mathcal{G}}(\{q_{i\rho,j}\}; Q).$$

Each term in the right-hand side of Theorem 4.6 can also be viewed as a function of the new variables $q_{i\rho,j}$ for a fixed $\rho \in \widehat{\mu_r}$:

$$\mathcal{F}^0_X(\{q_{i\rho,j}\}; Q_\rho).$$

By dilaton equation, we have

$$\frac{1}{r^2}\mathcal{F}^0_X(\{q_{i\rho,j}\}; Q_\rho) = \mathcal{F}^0_X(\{\bar{q}_{i\rho,j}\}; Q_\rho)$$

where $\bar{q}_{i\rho,j} := rq_{i\rho,j}$. Let M_ρ denote the Frobenius structure on $H^*(X)_{[\rho]} = H^*(X)$ obtained using the potential function $\mathcal{F}^0_X(\{\bar{q}_{i\rho,j}\}; Q_\rho)$ and the Poincaré pairing on X. The following is a restatement of Theorem 4.6.

Theorem 4.7. *Under the isomorphism* (44)*, the Frobenius structure defined by the genus* 0 *Gromov–Witten theory of* $\mathcal{G}$ *is isomorphic to* $\oplus_\rho M_\rho$.

Remark 4.8. It is natural to ask for a generalization of Theorem 4.6 to higher genus Gromov–Witten theory. Suppose that the Frobenius structure associated to the genus 0 Gromov–Witten theory of X is generically semi-simple; then one can prove certain generalizations of Theorem 4.6 to higher genus *ancestor invariants* by using Givental's formula [**29**], [**39**] to reduce the question to genus 0. In [**8**] we will study the higher genus generalization of Theorem 4.6 in general (namely without assuming semi-simplicity).

Appendix A. Banded abelian gerbes

Let X be a smooth projective variety over $\mathbb{C}$. Let G be a finite abelian group. The purpose of this appendix is to explain (see Section A.1) how the results in the main part of the paper can be extended to banded G-gerbes $\mathcal{G}$ over X which are *essentially trivial*.

We begin with some preliminary materials. Recall the following well-known structure result for finite abelian groups:

Lemma A.1. *Let G be a finite abelian group of order N. Then there exists a decomposition*

$$G \simeq \prod_{j=1}^k \mu_{r^{(j)}}, \quad \textit{where } \prod_{j=1}^k r^{(j)} = N. \tag{47}$$

Throughout this appendix we fix such a decomposition (47) of G.

Observe that the inertia stack $I\mathcal{G}$ admits a decomposition

$$I\mathcal{G} = \cup_{g\in G}\mathcal{G}_g, \tag{48}$$

indexed by elements in G. Let $\bar{I}(\mathcal{G})_g \subset \bar{I}(\mathcal{G})$ be the image of $\mathcal{G}_g$ under the natural map $I\mathcal{G} \to \bar{I}(\mathcal{G})$ to the rigidified inertia stack. A vector of elements

$$\vec{g} := (g_1, \dots, g_n) \in G^{\times n}$$

is called β-*admissible* if the locus

$$\mathcal{K}_{0,n}(\mathcal{G},\beta)^{\vec{g}} := \cap_{i=1}^{n} ev_i^{-1}(\bar{I}(\mathcal{G})_{g_i})$$

is non-empty. Note that for $1 \leq i \leq n$ we may write

$$g_i := (g_i^{(1)},\ldots,g_i^{(k)}) \in \prod_{j=1}^{k} \mu_{r^{(j)}} = G.$$

A.1. Essentially trivial abelian gerbes. By definition a G-gerbe over X is *essentially trivial* if it becomes trivial after a contracted product with the trivial $\mathcal{O}_X^*$-gerbe. In this section, let $\mathcal{G} \to X$ be an essentially trivial G-banded gerbe over X. The following result is known (see e.g. [**27**], Proposition 6.9).

Lemma A.2. *Let $\mathcal{G} \to X$ be an essentially trivial G-banded gerbe over X, with G finite and abelian. Then there exist line bundles $\mathcal{L}^{(1)},\ldots,\mathcal{L}^{(k)}$ over X and positive integers $r^{(1)},\ldots,r^{(k)}$, such that*

$$\mathcal{G} \simeq \sqrt[r^{(1)}]{\mathcal{L}^{(1)}/X} \times_X \sqrt[r^{(2)}]{\mathcal{L}^{(2)}/X} \times_X \cdots \times_X \sqrt[r^{(k)}]{\mathcal{L}^{(k)}/X}. \tag{49}$$

Proof. Let $[\mathcal{G}] \in H^2_{et}(X,G)$ be the class of the gerbe $\mathcal{G}$. Fix a decomposition of G as in (47). Denote by $p_j : G \to \mu_{r^{(j)}}$ the projection to the j-th factor. For $1 \leq j \leq k$, the induced morphism $p_{j*} : H^2_{et}(X,G) \to H^2_{et}(X,\mu_{r^{(j)}})$ maps the class of a G-banded gerbe $\mathcal{G}$ to the class of the $\mu_{r^{(j)}}$-gerbe obtained from $\mathcal{G}$ by taking the contracted product with the trivial $\mu_{r^{(j)}}$-gerbe. This is the same as taking the rigidification of $\mathcal{G}$ by the subgroup of the inertia $\overline{G} := G/\mu_{r^{(j)}}$. The composition of p_j with the standard embedding $\mu_{r^{(j)}} \to \mathbb{C}^*$ yields a homomorphism $\phi_j : G \to \mathbb{C}^*$. Clearly the composition

$$G \xrightarrow{\phi_j} \mathbb{C}^* \xrightarrow{(\cdot)^{r^{(j)}}} \mathbb{C}^*$$

is trivial.

Associated to the Kummer sequence

$$1 \longrightarrow \mu_{r^{(j)}} \longrightarrow \mathbb{C}^* \xrightarrow{(\cdot)^{r^{(j)}}} \mathbb{C}^* \longrightarrow 1$$

there is a long exact sequence

$$\ldots \to \check{H}^1_{\acute{e}t}(X,\mathbb{C}^*) \to \check{H}^2_{\acute{e}t}(X,\mu_{r^{(j)}}) \to \check{H}^2_{\acute{e}t}(X,\mathbb{C}^*) \to \check{H}^2_{\acute{e}t}(X,\mathbb{C}^*) \to \ldots$$

The map ϕ_j induces a homomorphism $\phi_{j*} : \check{H}^2_{\acute{e}t}(X,G) \to \check{H}^2_{\acute{e}t}(X,\mathbb{C}^*)$ mapping the class $p_{j*}[\mathcal{G}]$ of the $\mu_{r^{(j)}}$-gerbe obtained from $\mathcal{G}$ by the homomorphism $p_j : G \to \mu_{r^{(j)}}$ to the class of its contracted product with the trivial $\mathcal{O}_X^*$-gerbe. Since $\mathcal{G}$ is essentially trivial, the class $\phi_{j*}([\mathcal{G}]) \in \check{H}^2_{\acute{e}t}(X,\mathbb{C}^*)$ is zero by definition. By the exact sequence above this means that there exists a line bundle $\mathcal{L}^{(j)}$ over X such that the $\mu_{r^{(j)}}$-gerbe of class $p_{j*}[\mathcal{G}]$ is isomorphic to the root gerbe $\sqrt[r^{(j)}]{\mathcal{L}^{(j)}/X}$.

We can prove the claim by induction on the number k of cyclic groups appearing in the decomposition of G. For $k=1$ the claim is true by definition. Assume it is true for $k=n-1$. Let $\mathcal{G}$ be a G-banded gerbe, where $G \simeq \prod_{j=1}^{n} \mu_{r^{(j)}}$. Consider the group homomorphisms induced by the rigidification

$$\check{H}^2_{\acute{e}t}(X, G) \to \check{H}^2_{\acute{e}t}(X, \overline{G}) \oplus \check{H}^2_{\acute{e}t}(X, \mu_{r^{(n)}}),$$

where $\overline{G} \simeq G/\mu_{r^{(n)}}$. We denote the corresponding gerbes by $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}}_k$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, bend right] & & \\
& \widetilde{\mathcal{G}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{G}} \arrow[d] \\
& \overline{\mathcal{G}}_k \arrow[r] & X,
\end{tikzcd} \tag{50}$$

where the dotted arrow is induced by the universal property of the fiber product. The morphism $\mathcal{G} \to \overline{\mathcal{G}} \times \overline{\mathcal{G}}_k$ is representable and factors through $\mathcal{G} \to \widetilde{\mathcal{G}}$, which is therefore representable. We conclude by observing that a representable morphism between two gerbes banded by the same group is an isomorphism. q.e.d.

In view of Lemma A.2 we assume that the gerbe $\mathcal{G}$ is of the form (49). We call this gerbe a *multi-root gerbe*. Recall that to give a morphism $Y \to \mathcal{G}$ is the same as giving a morphism $f : Y \to X$ and line bundles $M_1, \dots, M_k$ over Y together with isomorphisms $\phi_j : M_j^{\otimes r^{(j)}} \simeq f^*\mathcal{L}^{(j)}$, $1 \le j \le k$.

The constructions in Section 3 can be easily modified to treat the multi-root gerbe $\mathcal{G}$. Arguments proving Lemma 3.5 and Proposition 3.6 easily yield the following

Proposition A.3.

1) *A vector $\vec{g}$ is β-admissible (with respect to a class $\beta \in H_2^+(X, \mathbb{Z})$) if and only if*

$$\prod_{i=1}^{n} g_i^{(j)} = \exp\left(\frac{2\pi\sqrt{-1}}{r^{(j)}} \int_\beta c_1(\mathcal{L}^{(j)})\right), \quad 1 \le j \le k. \tag{51}$$

2) *Given a vector $\vec{g}$ satisfying (51) and a stable map $[f : (C, p_1, \dots, p_n) \to X] \in \overline{M}_{0,n}(X, \beta)(\mathbb{C})$, there exists, up to isomorphisms, a unique twisted stable map $\widetilde{f} : (\mathcal{C}, \sigma_1, \dots, \sigma_n) \to \mathcal{G}$ in $\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ lifting f.*

Remark A.4. The n-tuple $(g_1^{(j)}, \dots, g_n^{(j)})$ is a β-admissible vector for the root gerbe $\sqrt[r^{(j)}]{\mathcal{L}^{(j)}/X}$ as in Definition 3.3.

Next we define some numbers.

Definition A.5. 1) For $1 \leq i \leq n$, let r_i be the order of g_i in G. Each $g_i^{(j)}, 1 \leq i \leq n$ may be identified with a root of unity

$$g_i^{(j)} = \exp(2\pi\sqrt{-1}\theta_i^{(j)}), \quad \text{where } \theta_i^{(j)} \in \mathbb{Q} \cap [0,1),$$

which defines the rational numbers $\theta_i^{(j)}, 1 \leq i \leq n$. For $1 \leq i \leq n$ and $1 \leq j \leq k$, define

$$\rho_i^{(j)} := r^{(j)}\theta_i^{(j)}, \quad r_i^{(j)} := \frac{r^{(j)}}{gcd(r^{(j)}, \rho_i^{(j)})}, \quad m_i^{(j)} := \frac{\rho_i^{(j)}}{gcd(r^{(j)}, \rho_i^{(j)})}. \tag{52}$$

Note that $r_i^{(j)}$ divides r_i, and $r_i^{(j)}$ is the order of $g_i^{(j)}$ in $\mu_{r^{(j)}}$.

2) For a pair (T, β') indexing the boundary divisors of $\mathfrak{M}_{0,n,\beta}$ as in Definition 3.10, define

$$\theta_{T,\beta'}^{(j)} := \left\langle \frac{1}{r^{(j)}} \int_{\beta'} c_1(\mathcal{L}^{(j)}) - \sum_{i \in T} \theta_i^{(j)} \right\rangle, \quad r_{T,\beta'}^{(j)} := \frac{r^{(j)}}{gcd(r^{(j)}, r^{(j)}\theta_{T,\beta'}^{(j)})},$$

$$m_{T,\beta'}^{(j)} := \frac{r^{(j)}\theta_{T,\beta'}^{(j)}}{gcd(r^{(j)}, r^{(j)}\theta_{T,\beta'}^{(j)})}. \tag{53}$$

3) Define

$$g_{T,\beta'}^{(j)} := \exp(2\pi\sqrt{-1}\theta_{T,\beta'}^{(j)}) \in \mu_{r^{(j)}}, \quad g_{T,\beta'} := (g_{T,\beta'}^{(1)}, \ldots, g_{T,\beta'}^{(k)}) \in G.$$

And let $r_{T,\beta'}$ be the order of $g_{T,\beta'}$ in G.

With the numbers defined above, the constructions and results in Sections 3.3 and 3.4 are valid for the multi-root gerbe $\mathcal{G}$. The proofs are straightforward modifications. In particular we still have the diagram (26).

Moreover Theorem 3.20 admits a generalization to multi-root gerbes:

Theorem A.6. *$\mathcal{K}_{0,n}(\mathcal{G}, \beta)^{\vec{g}}$ is a multi-root gerbe over $P_n^{\vec{g}}$.*

To prove Theorem A.6 it suffices to repeat the arguments in the proof of Theorem 3.20 multiple times. The key point is to construct a collection of line bundles, generalizing the one in (33):

$$\mathcal{L}_{\mathfrak{Y}}^{(j)} := \mathcal{O}_{\mathfrak{Y}_{0,n+1,\beta}^{\vec{g}\cup\{1\}}} \left(\sum_{1 \leq i \leq n} \frac{d_i^{(j)}}{r_i^{(j)}} S_i - \sum_{(T,\beta') \in \mathcal{I}_D} \frac{d_{T,\beta'}^{(j)}}{r_{T,\beta'}^{(j)}} D_{\beta'}^{T\cup\{n+1\}} \right), \quad 1 \leq j \leq k. \tag{54}$$

Here the set $\mathcal{I}_D$ is defined on page 18, and we use the following definition:

Definition A.7. 1) For $1 \leq i \leq n$ and $1 \leq j \leq k$ we define $d_i^{(j)} \in \mathbb{Z}$ by requiring

$$g_i^{(j)} = \exp(\frac{2\pi\sqrt{-1}}{r_i^{(j)}} d_i^{(j)}) \quad \text{and} \quad \sum_{i=1}^{n} \frac{d_i^{(j)}}{r_i^{(j)}} = \frac{1}{r^{(j)}} \int_{\beta} c_1(\mathcal{L}^{(j)}). \tag{55}$$

2) To a pair (T, β') which indexes a boundary divisor of $\mathfrak{M}_{0,n,\beta}$, we associate integers $d_{T,\beta'}^{(j)}$ such that

$$\sum_{i\in T} \frac{d_i^{(j)}}{r_i^{(j)}} + \frac{d_{T,\beta'}^{(j)}}{r_{T,\beta'}^{(j)}} = \frac{1}{r^{(j)}} \int_{\beta'} c_1(\mathcal{L}^{(j)}), \quad 1 \leq j \leq k. \tag{56}$$

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