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THE HALF-SPACE PROPERTY AND ENTIRE POSITIVE MINIMAL GRAPHS IN $M \times \mathbb{R}$.

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Abstract

We show that a properly immersed minimal hypersurface in $M \times \mathbb{R}_+$ equals some $M \times \{c\}$ when M is a complete, recurrent n-dimensional Riemannian manifold with bounded curvature. If on the other hand, M is not necessarily recurrent but has nonnegative Ricci curvature with curvature bounded below, the same result holds for any positive entire minimal graph over M.

1. Introduction

A problem that has received considerable attention is to give conditions that force two minimal submanifolds S_1, S_2 of a Riemannian manifold N to intersect. If they do not intersect, does this determine the geometry of S_1, S_2 in N?

Perhaps the simplest example of this situation is when N is a strictly convex ovaloid (i.e., the 2-dimensional sphere with a metric of positive curvature) and S_1, S_2 are complete embedded geodesics of N. Then S_1 and S_2 must intersect. This generalizes to compact 3-manifolds N of positive sectional curvature: if S_1, S_2 are finite topology complete minimal surfaces embedded in N, they must intersect. This follows from the minimal lamination closure theorem [**15**]. There is also the classical theorem of Frankel [**7**] that states that if N is a closed n-dimensional manifold with positive Ricci curvature and S_1, S_2 are compact minimal $(n-1)$-dimensional submanifolds immersed in N, then they intersect. For some other results on this problem, see [**4**, **5**, **14**, **6**, **10**].

In this paper we consider this question when $N = M \times \mathbb{R}$ where M is a complete n-dimensional Riemannian manifold, $S_1 = M \times \{0\}$ and S_2 is a properly immersed minimal hypersurface in $M \times \mathbb{R}_+$. Our problem then becomes to determine what conditions on M imply that $S = S_2$ is the totally geodesic slice $M \times \{c\}$ for some positive c?

Perhaps the first result in this direction was the celebrated theorem of Bombieri, De Giorgi, and Miranda [**1**], who proved that an entire minimal positive graph over $\mathbb{R}^n$ is a totally geodesic slice. The hyperbolic

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plane $\mathbb{H}^2$ does not have this property; there are entire bounded minimal graphs that are not slices.

For a proper immersed minimal surface S in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}_+$, the foundational result was discovered by Hoffman and Meeks [**11**], who proved that $S = \mathbb{R}^2 \times \{c\}$, $c \geq 0$. They called this the *half-space theorem*.

Definition 1.1. We will say that M *has the half-space property* if a minimal hypersurface S properly immersed in $M \times \mathbb{R}_+$, equals a slice $M \times \{c\}$. Since there are rotationally invariant minimal hypersurfaces in $\mathbb{R}^{n+1}$, $n > 2$, that are bounded above and below (catenoids), $M = \mathbb{R}^n$, $n > 2$ does not have the half-space property, but entire minimal positive graphs over $\mathbb{R}^n$ are slices.

Hence it is interesting to find conditions on M that ensure that M has the half-space property or the property that positive entire minimal graphs over M are slices. Our contributions to these questions are the following two theorems.

Theorem 1.2. *Let M^n be a complete recurrent Riemannian manifold with bounded sectional curvatures $|K_\pi| \leq K_0$ for some constant K_0. Then M has the half-space property.*

Theorem 1.3. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature and sectional curvatures $K_\pi \geq -K_0$ for a nonnegative constant K_0. Let S be an entire minimal graph in $M \times \mathbb{R}$ with height function $u \geq 0$. Then $S = M \times \{c\}$ for some constant $c \geq 0$.*

In the same spirit, an interesting question is to study those complete embedded minimal hypersurfaces in $M \times \mathbb{R}$, whose angle function $\langle N, \frac{\partial}{\partial t}\rangle$ does not change sign; see [**6**].

Definition 1.4. That M in Theorem 1.2 is recurrent means that for any nonempty bounded open set U, every bounded harmonic function on $M \setminus U$ is determined by its boundary values. Furthermore, if $M \setminus U$ is quasi-isometric to $N \setminus V$, then M is recurrent if and only if N is recurrent. For a detailed discussion see [**9**, **13**].

Example 1.5. Some interesting examples of allowable M may be constructed as follows. Let N be a closed manifold and take $M = N \times \mathbb{R}^2$, or $M = N \times \mathbb{R}$, or $M = N \times S$, S a complete surface with quadratic area growth or finite total curvature. These examples have quadratic volume growth so they are recurrent. Thus, removing a bounded nonempty open set from M, what is left is parabolic, i.e., any bounded harmonic function is determined by its boundary values.

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2. Local formulas for minimal graphs

Let u be the height function of an n dimensional minimal graph $S = \{(x, u(x)) : x \in B_R(p)\}$ in $M^n \times \mathbb{R}$ where M is complete with nonnegative Ricci curvature and $B_R(p)$ is a geodesic ball of radius R about p. If $ds^2 = \sigma_{ij}dx_idx_j$ is a local Riemannian metric on M, then $M \times \mathbb{R}$ is given the product metric $ds^2 + dt^2$ where t is a coordinate for $\mathbb{R}$. Then the height function $u(x) \in C^2(\Omega)$ satisfies the divergence form equation

$$\text{div}^M\Big(\frac{\nabla^M u}{\sqrt{1+|\nabla^M u|^2}}\Big) = 0 \tag{2.1}$$

where the divergence and gradient $\nabla^M u$ are taken with respect to the metric on M. Equivalently, equation (2.1) can be written in nondivergence form

$$\frac{1}{W} g^{ij} D_i D_j u = 0\,, \qquad \text{where } W = \sqrt{1+|\nabla^M u|^2}\,, \tag{2.2}$$

D denotes covariant differentiation on M and

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}\ ,\ u^i = \sigma^{ij} D_j u\,.$$

This can be seen as follows. Let $x_1, \dots x_n$ be a system of local coordinates for M with corresponding metric σ_{ij}. Then the coordinate vector fields for S and the upward unit normal to S are given by

$$X_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial t} \tag{2.3}$$

and

$$N = \frac{1}{W}\Big(- u^j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial t}\Big)\ ,\ u^i = \sigma^{ij} u_j\ . \tag{2.4}$$

The induced metric on S is then

$$g_{ij} = \langle X_i, X_j\rangle = \sigma_{ij} + u_i u_j \tag{2.5}$$

with inverse

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}\ . \tag{2.6}$$

It is easily seen that

$$g = \det(g_{ij}) = \sigma W^2\ ,\ \sigma = \det(\sigma_{ij})\ . \tag{2.7}$$

The second fundamental form b_{ij} of S is given by ($\overline{D}$ is covariant differentiation on $M \times \mathbb{R}$)

$$\begin{aligned} b_{ij} &= \langle \overline{D}_{X_i} X_j, \nu\rangle = \Big\langle D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + u_{ij}\frac{\partial}{\partial t}, N\Big\rangle \\ &= \Big\langle \Gamma^k_{ij}\frac{\partial}{\partial x_k} + u_{ij}\frac{\partial}{\partial t}, \nu\Big\rangle = \frac{1}{W}\Big(- \Gamma^k_{ij} u^l \sigma_{kl} + u_{ij}\Big)\,. \end{aligned} \tag{2.8}$$

Hence,

$$b_{ij} = \frac{D_i D_j u}{W}, \tag{2.9}$$

and so the mean curvature H of S is then given by

$$nH = \frac{1}{W} g^{ij} D_i D_j u \ . \tag{2.10}$$

The area functional of S is given in local coordinates by

$$A(S) = \int W \sqrt{\sigma}\ dx \ .$$

As a functional of u, this gives the Euler-Lagrange equation

$$\mathrm{div}^M\left(\frac{\nabla^M u}{W}\right) = \frac{1}{\sqrt{\sigma}} D_i\left(\sqrt{\sigma}\frac{u^i}{W}\right) = 0\,. \tag{2.11}$$

It is easily seen that (2.2) is the nondivergence form of (2.11).

We will also need the well-known formulae

$$\Delta^S u = 0 \tag{2.12}$$

$$\Delta^S W^{-1} = -\big(|A|^2 + \widetilde{\mathrm{Ric}}(N,N)\big) W^{-1}\,, \tag{2.13}$$

where $|A|$ is the norm of the second fundamental form of S, $\widetilde{\mathrm{Ric}}$ is the Ricci curvature of $M \times \mathbb{R}$, and Δ^S is the Laplace-Beltrami operator of S given in local coordinates by

$$\Delta^S \equiv \mathrm{div}^S\big(\nabla^S \cdot\,\big) = \frac{1}{\sqrt{g}} D_i\big(\sqrt{g} g^{ij} D_j \cdot\,\big) = g^{ij} D_i D_j \ . \tag{2.14}$$

Since $\tau := \frac{d}{dt}$ is a Killing vector field on $M \times \mathbb{R}$, $W^{-1} = \langle N, \tau\rangle$ is a Jacobi field and so satisfies the Jacobi equation (2.13). For a clean derivation of (2.13) using moving frames, see [**17**, section 2] where M is three dimensional but the derivation is valid in all dimensions. Equation (2.12) is easily seen to be equivalent to (2.2).

From (2.14) follows the important formulae

$$\Delta^S \varphi(x) = g^{ij} D_i D_j\ \varphi \tag{2.15}$$

and

$$\Delta^S\ g(\varphi) = g'(\varphi)\Delta_S\ \varphi + g''(\varphi) g^{ij} D_i\varphi D_j\varphi \ . \tag{2.16}$$

This implies that

$$\Delta^S W = 2W^{-1}|\nabla^S W|^2 + W\big(|A|^2 + \widetilde{\mathrm{Ric}}(N,N)\big)\,.$$

Let us for the moment assume that at a point $p \in S$ the normal N is not equal to τ. We let

$$\gamma := \frac{p^{TM}(N)}{|p^{TM}(N)|}\,,$$

where p^{TM} is the projection to the tangent space of the horizontal plane through p in $M \times \mathbb{R}$. It then holds that

$$\widetilde{\mathrm{Ric}}(N,N) = \mathrm{Ric}^M(p^{TM}(N), p^{TM}(N)) = \big(1 - W^{-2}\big)\mathrm{Ric}^M(\gamma,\gamma)\,.$$

Noting that this is still trivially true if $N = \tau$, we arrive at

$$\Delta^S W = 2W^{-1}|\nabla^S W|^2 + W|A|^2 + W\big(1 - W^{-2}\big)\mathrm{Ric}^M(\gamma,\gamma)\ . \tag{2.17}$$

Note that by (2.4) we have $\widetilde{\mathrm{Ric}}(N,N) = W^{-2}\mathrm{Ric}^M(\nabla^M u, \nabla^M u)$, which is always well defined.

Now let $h(x) = \eta(x)W(x)$ with $\eta \geq 0$ smooth. In the following we introduce the auxiliary elliptic operator L. Using (2.17), a simple computation gives

$$\begin{aligned} Lh &:= \Delta^S h - 2g^{ij}\frac{D_i W}{W} D_j h \\ &= \eta\big(\Delta^S W - \frac{2}{W} g^{ij} D_i W D_j W\big) + W\Delta^S \eta \\ &= W\big(\Delta^S\eta + \eta\big(|A|^2 + (1 - W^{-2})\mathrm{Ric}^M(\gamma,\gamma)\big)\big)\ . \end{aligned} \tag{2.18}$$

3. The recurrent case

The original proof by Hoffman and Meeks of the half-space theorem in $\mathbb{R}^3$ used the family of minimal surfaces obtained from a catenoid by homothety. We will use a discrete family of minimal graphs in $M \times \mathbb{R}$, like the catenoids in $\mathbb{R}^3$.

Let $D_1 \subset M$ be open and bounded with ∂D_1 smooth. Since M has bounded sectional curvatures, we can apply Theorem 0.1 of Cheeger and Gromov [**2**] to assert the existence of an exhaustion of M, $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ by domains with smooth boundaries, such that the norm of the second fundamental form of the boundaries ∂D_i is uniformly bounded by C_1 and $\overline{D_i} \subset D_{i+1}$. We denote ∂D_n by ∂_n and by A_n the annular-type domain $D_n \setminus \overline{D_1}$, with $\partial A_n = \partial_1 \cup \partial_n$.

Note that $M \times \{0\}$ is a stable minimal hypersurface in $M \times \mathbb{R}$ (it is a leaf of the minimal foliation $M \times \{c\},\ c \in \mathbb{R}$). Since A_n is a strict subset of $M \times \{0\}$ it is strictly stable, so any sufficiently small smooth perturbation of ∂A_n to $\Gamma_{n,t}$ gives rise to a smooth family of minimal hypersurfaces $S_{n,t}$ with $\partial S_{n,t} = \Gamma_{n,t}$, and $S_{n,0} = A_n$. The $S_{n,t}$ are smooth up to their boundary (we will use C^2).

We apply this to the deformation of ∂A_n, which is the graph over ∂A_n given by $\partial_1 \cup (\partial_n \times \{t\})$, for $t \geq 0$. Then for t sufficiently small, $S_{n,t}$ is the graph of a smooth function $u_{n,t}$ defined on A_n, with boundary values 0 on ∂_1 and t on ∂_n. Note that $u_{n,t}$ satisfies the minimal surface equation on A_n and by the maximum principle we have $0 \leq u_{n,t} \leq t$. Furthermore, as long as $|\nabla^M u_{n,t}|$ is uniformly bounded, the DeGiorgi-Nash-Moser and Schauder estimates imply uniform estimates for all higher derivatives up to the boundary. Thus to apply the method of continuity, we need only show uniform gradient estimates.

We will first present a maximum principle for the function

$$W = \sqrt{1 + |\nabla^M u|^2}$$

on $S = \mathrm{graph}(u) \subset M \times \mathbb{R}$, where we assume that $u : \Omega \to \mathbb{R}$ is a solution to the minimal surface equation on $\Omega \subset M$. From (2.17), we see that if the Ricci curvature of M is nonnegative then W is bounded on S by its maximum on ∂S. To treat the case that the Ricci curvature of M is only bounded from below, we consider the function

$$h = \eta \cdot W\,, \quad \eta = e^{\alpha u}$$

where $\alpha > 0$. From (2.12) and (2.16) we find

$$\Delta^S \eta = \alpha^2 \eta |\nabla^S u|^2 = \alpha^2 (1 - W^{-2}) \eta\ .$$

Then, using (2.18), we have

$$Lh = h\Big(|A|^2 + \big(1 - W^{-2}\big)\big(\alpha^2 + \mathrm{Ric}^M(\gamma,\gamma)\big)\Big)\,. \tag{3.1}$$

This implies the following estimate.

Lemma 3.1. *Let $\Omega \subset M$ be open and bounded, and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the minimal surface equation in Ω. Then*

$$\sup_{\Omega} \sqrt{1 + |\nabla^M u|^2} \leq \sup_{\Omega} e^{-\alpha u} \cdot \sup_{\partial\Omega} \left(e^{\alpha u} \sqrt{1 + |\nabla^M u|^2} \right),$$

where $\alpha^2 = \sup\{\max\{-Ric^M(\gamma,\gamma), 0\} \mid \gamma \in T_pM,\ |\gamma| = 1,\ p \in \Omega\}$.

Proof. By our choice of $\alpha > 0$, we see from (2.18) that $Lh \geq 0$. The result now follows from the maximum principle. q.e.d.

Remark 3.2. In the case that S has constant mean curvature H, one can compute that

$$Lh = h\Big(\alpha \frac{nH}{W} + |A|^2 + \big(1 - W^{-2}\big)\big(\alpha^2 + \mathrm{Ric}^M(\gamma,\gamma)\big)\Big)\,.$$

By considering $-u$ instead of u if necessary, we can assume that $H \geq 0$ and arrive at the same gradient estimate as before.

Lemma 3.1 implies that to use the method of continuity for the surfaces $S_n(t)$ we only need a priori gradient bounds on ∂A_n.

For convenience of notation, assume the sectional curvatures of M are bounded from above by $K_0 = 1$. Then the Riccati comparison estimates imply that for any point p in M, the exponential map $\exp_p : T_pM \supset B_\pi(0) \to B_\pi(p)$ is a local diffeomorphism. Let us for the moment also assume that the injectivity radius of M is greater or equal to 1; i.e., the exponential map $\exp_p : T_pM \supset B_1(0) \to B_1(p)$ is actually a diffeomorphism.

We now almost explicitly construct a catenoid-like supersolution $w = w(r; r_0, p)$ of the minimal surface equation in an annulus $A(p) := B_{4r_0}(p) \setminus B_{2r_0}(p)$ of height $2\delta_0$ where $r = d(x,p)$ is the distance function from x to p. Here r_0 will be chosen sufficiently small depending on the bound $K_0 = 1$ for the sectional curvatures of M and the lower bound 1 for the injectivity radius of M.

Lemma 3.3. *For r_0 sufficiently small, there exists $w = \varphi(r) - \varphi(2r_0)$ satisfying*

$$\text{(3.2)} \qquad div^M\left(\frac{\nabla^M w}{\sqrt{1+|\nabla^M w|^2}}\right) < 0 \quad \textit{in } A(p)$$

$$\text{(3.3)} \qquad w = 0 \quad \textit{on } r = 2r_0$$

$$\text{(3.4)} \qquad w = 2\delta_0 := \varphi(4r_0) - \varphi(2r_0) \quad \textit{on } r = 4r_0$$

where $\varphi'(r) > 0$, $\varphi(r_0) = 0$, $\varphi'(r_0) = +\infty$ and the inverse function $r = \gamma(s)$ of $\varphi(r)$ is implicitly defined by

$$\text{(3.5)} \qquad s = \int_{r_0}^{\gamma} \frac{dt}{\sqrt{(\frac{t}{r_0})^{2n} - 1}}\ .$$

Proof. From (2.2) it suffices to show that in $A(p)$

$$\text{(3.6)} \quad Mw := \left(\sigma^{ij} - \frac{w^i w^j}{W^2}\right) D_i D_j w < 0\,, \quad \text{where } W = \sqrt{1+|\nabla^M w|^2}\,.$$

When $w = \varphi(r)$, we easily find from (3.6) that

$$\text{(3.7)} \qquad Mw = \varphi'(r)\Delta^M r + \frac{\varphi''(r)}{1+\varphi'^2(r)}.$$

We fix r_0 small enough that $\Delta^M r < \frac{n}{r}$ in $B_{4r_0}(p)$. Then from (3.7),

$$\text{(3.8)} \qquad Mw < \frac{\varphi''(r)}{1+\varphi'^2(r)} + \frac{n}{r}\varphi'(r),$$

and it suffices to solve

$$\text{(3.9)} \qquad \frac{\varphi''}{1+\varphi'^2} + \frac{n}{r}\varphi' = 0.$$

But (3.9) is the ode for the height function of the top half of the catenoid in $\mathbb{R}^{n+1} \times \mathbb{R}$ over $\{r > r_0\} \subset \mathbb{R}^{n+1}$ and its solution is well known to be given as described. q.e.d.

Remark 3.4. Using the continuity method, it is immediate that we can deform w to an exact solution of the minimal surface equation in $A(p)$.

Lemma 3.5. *For every $0 \le t \le \delta_0$ the surfaces $S_{n,t}$ exist and are smooth graphs of $u_{n,t}$ over $\overline{A}_n$ satisfying*

$$0 < u_{n,t} < t \quad \text{in } A_n \tag{3.10}$$

$$|\nabla^M u_{n,t}| \le C_3 \quad \text{on } \overline{A}_n \tag{3.11}$$

for all $0 \le t \le \delta_0$ and $n \in \mathbb{N}$, with C_3 independent of n and t.

Proof. By comparing with horizontal slices of height zero and δ_0, the height of the surfaces $S_{n,t}$ is bounded from below by zero and from above by δ_0.

We now use the barrier $Z_{r_0,p} = \text{graph}(w)$ near the boundary of A_n to obtain a gradient bound for $S_{n,t}$, provided $0 \le t \le \delta_0$. Let $p_0 \in \partial A_n$. Since the norm of the second fundamental form of each component of ∂A_n is bounded by C_1, there is a $p_1 \in M$ such for r_0 sufficiently small depending only on C_1, $B_{2r_0}(p_1)$ touches A_n from its exterior at p_0. Note that $B_{2r_0}(p_1)$ still might intersect A_n, but it touches A_n in p_0 from its exterior. See figure 1.

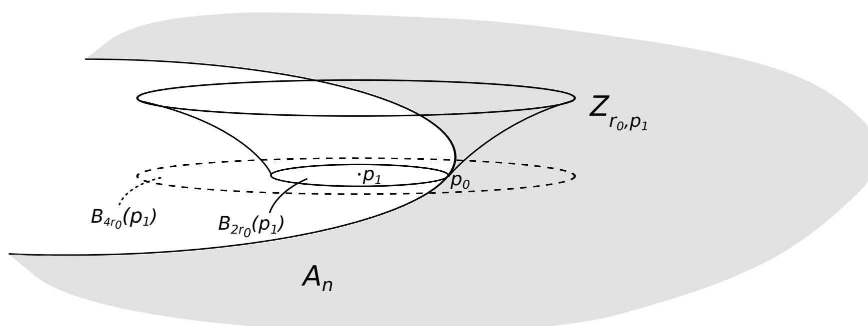

Figure 1. Barrier at the inner boundary ∂_1 of A_n.

We now consider the part of Z_{r_0,p_1} that is a graph over the connected component of $(B_{4r_0}(p_1) \setminus B_{2r_0}(p_1)) \cap A_n$ which has p_0 in its boundary. Suppose first $p_0 \in \partial_1$. Note that on its boundary Z_{r_0,p_1} always lies above $S_{n,t}$, as long as $0 \le t \le \delta_0$. By the maximum principle this implies that Z_{r_0,p_1} lies above $S_n(t)$, which in turn implies a gradient bound for $u_{n,t}$ at p_0. By reflecting $Z_{r_0,p}$ through the plane of height 0 in $M \times \mathbb{R}$ and translating up by t, we can do a similar construction at the outer boundary ∂_n of A_n for $S_{n,t}$ and obtain a gradient bound for $u_{n,t}$ that is uniform in n and t. See figure 2.

In the construction above, we have assumed that the injectivity radius of M is bounded from below by 1. In the case that there is no positive lower bound for the injectivity radius of M, we proceed as follows. As pointed out earlier, $\exp_p : T_pM \supset B_\pi(0) \to B_\pi(p)$ is a local diffeomorphism. Thus we can pull back the metric of M to $B_\pi(0) \subset T_pM$. It

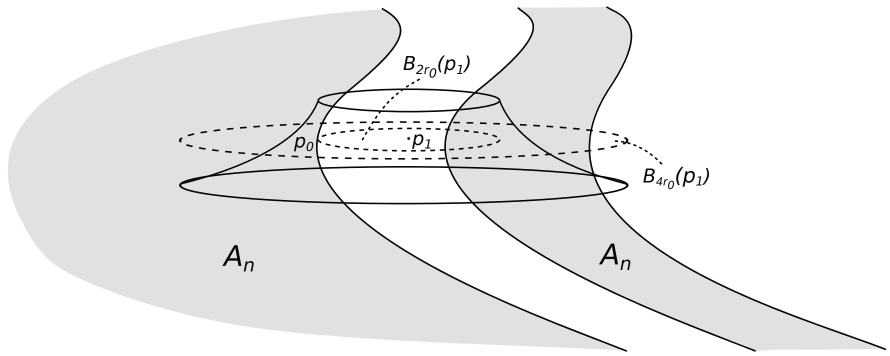

Figure 2. Reflected and translated barrier at the outer boundary ∂_n of A_n.

is then easy to see that $\exp_p : T_pM \supset B_2(0) \to B_2(p)$ is a local Riemannian covering map, and the injectivity radius at 0 of $B_\pi(0) \subset T_pM$ is π. To obtain the gradient bounds at $p_0 \in \partial A_n$ as discussed above, we can lift the whole construction, including A_n and $S_{n,t}$, locally to $B_1(0) \subset T_pM$ and again use $Z_{r_0,p}$ to obtain the same gradient bound for the lift of $u_{n,t}$. But this implies the gradient bound for $u_{n,t}$ itself.

The above construction of barriers at the boundary implies that

$$|\nabla^M u_{n,t}| \leq |\nabla^M w| = \varphi'(2r_0) = C_2$$

on ∂A_n, independent of n and t. By Lemma 3.1, this implies the stated a priori gradient bound for $u_{n,t}$ on $\overline{A}_n$. The DeGiorgi-Nash-Moser and Schauder estimates then imply a priori bounds of all higher derivatives of $u_{n,t}$ on $\overline{A}_n$. Thus we obtain existence by the method of continuity. q.e.d.

Remark 3.6. Note that to get a gradient bound at the inner boundary ∂_1 for the surfaces $S_{n,t}$ for $0 \leq t \leq \delta$, just for an implicit $0 < \delta \leq \delta_0$, one can argue that by the strict stability of $S_{1,0}$, the graphs $S_{1,t}$ exist for $t \in [0, \delta]$ and have bounded gradient. One can then use $S_{1,\delta}$ as an upper barrier for the surfaces $S_{n,t}$ on the inner boundary ∂_1 to obtain an a priori gradient estimate there.

By construction, we have that $S_{n,t}$ lies above $S_{m,t}$ on A_n for $m > n$. Since for $0 \leq t \leq \delta_0$ the surfaces have uniform gradient bounds, the DeGiorgi-Nash-Moser and Schauder estimates imply locally uniform estimates for all higher derivatives. We take the limit $n \to \infty$ of the surfaces S_{n,δ_0} to obtain a limit surface S, which is a minimal graph over $M \setminus D_1$ and has boundary value 0 on ∂_1. Furthermore, the height function u is bounded by δ_0 and the gradient of u by C.

Since the gradient of u is bounded, $S = \text{graph}(u)$ is quasi-isometric to $M \setminus D_1$, and hence it is parabolic. Thus the height function u on S is a bounded harmonic function on graph(u) and so must be constant, equal to zero. That is, $u \equiv 0$ and the graphs S_{n,δ_0} converge locally uniformly to zero.

Now we can prove the half-space theorem.

Proof of Theorem 1.2. Suppose S is a minimal hypersurface properly immersed in $M \times (-\infty, c)$. Lowering $M \times \{c\}$ until it "touches" S, we can suppose S is asymptotic to $M \times \{c\}$ at infinity. More precisely, if $M \times \{\tau\}$ touches S for the first time at some point of S, then $S = M \times \{\tau\}$ by the maximum principle and we are done. Otherwise, the first contact is at infinity so we can assume S is asymptotic to $M \times \{c\}$. By translating S vertically we can assume that $c = 0$.

Since S is proper, we can assume that there is a point $p_0 \in M$ and a cylinder $C = B_{r_0}(p_0) \times (-r_0, 0)$ for some $r_0 > 0$ such that $S \cap C = \emptyset$. We can assume that r_0 is less than the injectivity radius at p. In our construction of the surfaces S_{n,t_0}, we choose $D_1 = B_{r_0/2}(p_0)$ and $t_0 = \min\{\delta_0, r_0/2\}$. Note that translating S_{n,t_0} vertically downwards by an amount t_0 keeps the boundaries of the translates of S_{n,t_0} strictly above S. Thus by the maximum principle all the translates remain disjoint from S. We call S'_{n,t_0} this final translate. Note that all the surfaces S'_{n,t_0} lie above S and converge as $n \to \infty$ to the plane $M \times \{-t_0\}$. Thus S lies below $M \times \{-t_0\}$, which contradicts that S is asymptotic to $M \times \{0\}$. q.e.d.

4. The graphical case

Theorem 4.1. *Assume M is complete with nonnegative Ricci curvature and sectional curvatures $K_\pi \geq -K_0$ for a nonnegative constant K_0. Let $S = graph(u)$ be a minimal graph in $M \times \mathbb{R}$ over $B_R(p)$ with $u \geq 0$. Then*

$$|\nabla^M u(p)| \leq C_1 e^{C_2 u^2(p) \frac{\Psi(R)}{R^2}}$$

where $\Psi(R) = (n-1)\sqrt{K_0} R \coth(\sqrt{K_0} R) + 1$.

Proof. Let $h(x) = \eta(x) W(x)$ with $W = \sqrt{1 + |\nabla^M u|^2}$, $\eta(x) = g(\varphi(x))$, $g(t) = e^{Kt} - 1$ for some $K \geq 0$, and $\varphi(x) = (-u(x)/2u(p) + (1 - \frac{d(x,p)^2}{R^2}))^+$ where $+$ denotes the positive part. Note that $\eta(x)$ is nonnegative and equal to zero iff $\varphi(x) = 0$. Let $C(p)$ denote the cut locus of p and $\mathcal{U}(p) = B_R(p) \setminus C(p)$ be the set of points $q \neq p$ in $B_R(p)$ for which there is a unique minimal geodesic γ joining p and q with q not conjugate to p along γ. It is well known that $d(x,p)$ is smooth on $\mathcal{U}(p)$, which is open. Note that $d(x,p)^2$ and so $h(x)$ is smooth in a neighborhood of p.

Case 1: The max of h occurs at a point $q \in \mathcal{U}(p)$

From (2.18) we find since M has nonnegative Ricci curvature,

$$Lh := \Delta^S h - 2g^{ij}\frac{D_iW}{W}D_jh \geq Ke^{K\varphi}W\big(\Delta_S\varphi + Kg^{ij}D_i\varphi D_j\varphi\big)\,. \tag{4.1}$$

The point is now to choose K so that $\Delta^S\varphi + Kg^{ij}D_i\varphi D_j\varphi > 0$ on the set where $h > 0$ and W is large. We will need a standard comparison lemma [**12**].

Lemma 4.2. *Suppose M has sectional curvatures $K_\pi \geq -K_0$ for a nonnegative constant K_0. Let $q \in \mathcal{U}(p)$ Then the (nonzero) eigenvalues of $D^2d(p,x)$ at q (principal curvatures of the local distance sphere through q) are bounded above by those of the corresponding distance sphere in the hyperbolic space of curvature $-K_0$.*

We have $\Delta^S u = 0$, so

$$\Delta^S\varphi = -\frac{2}{R^2}\big(d(x,p)\Delta^S d(x,p) + g^{ij}D_id(x,p)D_jd(x,p)\big)\,.$$

Using Lemma 4.2 and (2.14) we see that

$$\Delta^S\varphi \geq -\frac{\Psi(R)}{R^2} \tag{4.2}$$

where $\Psi(R) = (n-1)\sqrt{K_0}R\coth\big(\sqrt{K_0}R\big) + 1$ at a point $q \in \mathcal{U}(p)$.

We next compute

$$\begin{aligned} g^{ij}D_i\varphi D_j\varphi &= g^{ij}D_i\left(\frac{u(x)}{2u(p)} + \frac{2d(x,p)}{R^2}D_id(x,p)\right)\\ &\qquad\cdot D_j\left(\frac{u(x)}{2u(p)} + \frac{2d(x,p)}{R^2}D_jd(x,p)\right)\\ &= \frac{|\nabla u|^2}{4u(p)^2W^2} + \frac{4d^2(x,p)}{R^4}\left(1 - \left\langle\frac{\nabla u}{W}, \nabla d(x,p)\right\rangle_M^2\right)\\ &\qquad + \frac{2d(x,p)}{u(p)R^2}\frac{\big\langle\nabla u, \nabla d(x,p)\big\rangle_M}{W^2}\,. \end{aligned}$$

Hence

$$g^{ij}D_i\varphi D_j\varphi \geq \left(\frac{|\nabla u|}{2u(p)W} - \frac{2}{RW}\right)^2. \tag{4.3}$$

Now assume that

$$W(q) \geq \max\left\{\frac{2}{\sqrt{3}}, \frac{16u(p)}{R}\right\}. \tag{4.4}$$

Then from (4.3) and (4.4),

$$g^{ij}D_i\varphi D_j\varphi \geq \frac{1}{64u(p)^2}\,. \tag{4.5}$$

Therefore from (4.2) and (4.5),

$$\big(\Delta^S\varphi + Kg^{ij}D_i\varphi D_j\varphi\big)(q) \geq -\frac{\Psi(R)}{R^2} + \frac{K}{64u^2(p)}. \tag{4.6}$$

Now choose

$$K = 64u^2(p)\frac{\Psi(R)}{R^2} + 2\ . \tag{4.7}$$

Then (4.2) and (4.6) imply $Lh(q) > 0$, contradicting the maximum principle. Hence (4.4) cannot hold, and so

$$W(q) \leq \max\left\{\frac{2}{\sqrt{3}}, \frac{16u(p)}{R}\right\}\ . \tag{4.8}$$

Therefore $h(p) = (e^{\frac{K}{2}}-1)W(p) \leq (e^K-1)\max\left\{\frac{2}{\sqrt{3}}, \frac{16u(p)}{R}\right\}$. After some manipulation we see that Theorem 4.1 follows.

Case 2: $q \notin \mathcal{U}(p)$.

Lemma 4.3. *(a) Suppose the maximum of h in $B_R(p)$ occurs at q. Then there is a unique minimal unit speed geodesic $\gamma(s)$ joining p and q.*
(b) For any $\varepsilon > 0$, let $p^\varepsilon = \gamma(\varepsilon)$. Then $d(x,p^\varepsilon)$ is smooth in a neighborhood of q.

Proof. (a) Suppose the maximum of h occurs at $q \neq p$. Then since $h(x) \leq h(q)$ and

$$d(x,p) = R\sqrt{1 - \frac{u(x)}{2u(p)} - \frac{1}{K}\log\left(1 + \frac{h(x)}{W(x)}\right)}\ ,$$

we see that

$$d(x,p) \geq \psi(x) := R\sqrt{1 - \frac{u(x)}{2u(p)} - \frac{1}{K}\log\left(1 + \frac{h(q)}{W(x)}\right)}$$

with equality at q. Note that $\psi(x)$ is possibly only well defined locally in a small neighborhood $B_{2\rho}(q)$. In this case, we can let $\overline{\psi}(x) = \lambda(x)\psi(x)$ where $0 \leq \lambda(x) \leq 1$ is a smooth cutoff function with

$$\lambda(x) = \begin{cases} 1 & x \in B_\rho(q), \\ 0 & x \in B_R(p) \setminus B_{2\rho}(q). \end{cases}$$

Then $d(x,p) \geq \overline{\psi}(x)$ in $B_R(p)$ with equality at q.

Hence we may assume $\psi(x)$ is smooth on $B_R(p)$, and so

$$\psi(x) - \psi(q) \leq d(x,p) - d(q,p) \leq d(x,q)\ ;$$

hence $|\nabla^M\psi(q)| \leq 1$.

Now let $\gamma(s)$ be a unit speed minimal geodesic joining p to q. Then

$$\psi(\gamma(s)) \leq s\ .$$

and

$$\psi(\gamma(d(q,p)) = \psi(q) = d(q,p) \ .$$

Hence $\nabla^M \psi(q) = \gamma'(d(q,p))$ and so there is only one minimal geodesic joining p and q.

(b) Clearly q is not conjugate to p^ε. Moreover, since $d(x,p^\varepsilon) + \varepsilon \geq d(x,p) \geq \psi(x)$ with equality at q, the argument of part (a) shows that γ is the unique minimal geodesic joining p^ε and q. Hence $q \in \mathcal{U}(p^\varepsilon)$, so $d(x,p^\varepsilon)$ is smooth in a neighborhood of q. q.e.d.

We now complete the proof of case 2. Define

$$\varphi^\varepsilon = -\frac{u(x)}{2u(p)} + \left(1 - \frac{(d(x,p^\varepsilon)+\varepsilon)^2}{R^2}\right)^+, \ \eta^\varepsilon = g(\varphi^\varepsilon), \ h^\varepsilon = \eta^\varepsilon W \ .$$

Then since $d(x,p^\varepsilon) + \varepsilon \geq d(x,p) \geq \psi(x)$ with equality at q, we have that

$$\varphi^\varepsilon \leq \varphi, \ \eta^\varepsilon \leq \eta, h^\varepsilon \leq h$$

with equality at $q \in \mathcal{U}(p^\varepsilon)$. Thus by Lemma 4.3 we may apply case 1 (and let $\varepsilon \to 0$) to complete the proof. q.e.d.

Corollary 4.4. *Let M be as in Theorem 4.1. If S is a complete minimal graph with height function $u \geq 0$, then $|\nabla^M u| \leq C_1$.*

Proof. Let $R \to \infty$ in Theorem 4.1. q.e.d.

Remark 4.5. As in [**18**], there is a version of Theorem 4.1 for graphs with constant or variable mean curvature $H(x)$ assuming the sectional curvatures of M are bounded below with no assumption on Ricci curvature. In particular, Corollary 4.4 holds for bounded solutions under these hypotheses. The method presented here sharpens the result of [**18**] in that no control of injectivity radius is needed.

Set $m(R) = \inf_{B_R(P)} u$. Then more generally we have the following corollary.

Corollary 4.6. *Let M be as in Theorem 4.1 and let S is a complete minimal graph with height function u.*
a) If $K_0 > 0$, assume $\limsup_{R\to\infty} \frac{m^2(R)}{R} = 0$.
b) if $K_0 = 0$, assume $\limsup_{R\to\infty} \frac{|m(R)|}{R} = 0$.
Then $|\nabla^M u| \leq C_1$.

Proof of Theorem 1.3. We will first give a proof, applying Theorem 7.4 in the work of Saloff-Coste [**16**]. Note that by Corollary 4.4 we have that $|\nabla^M u| \leq C_1$. Thus equation (2.1) implies that u is L-harmonic in the sense of Saloff-Coste (compare (8) in [**16**]), where the operator L is given by

$$Lv = -m^{-1}\mathrm{div}^M\big(m\nabla^M u\big)$$

with $m = W^{-1}$. Since the Ricci curvature of M is nonnegative, Theorem 7.4 in [**16**] implies that u is equal to a constant.

In the following we will give a more explanatory proof, using the Moser technique as developed by Saloff-Coste [**16**] and Grigor'yan [**8**].

Let L be a uniformly elliptic divergence form operator such that $L1 = 0$ (i.e. no zeroth order term) on a complete Riemannian manifold M. We say the classical Moser Harnack inequality holds, if for any $R > 0$, whenever u is a nonnegative solution of $Lu = 0$ on $B_{2R}(p)$, then

$$\sup_{B_R(p)} u \leq C \inf_{B_R(p)} u$$

for a constant C depending only on L and M.

It is now well understood that the classical Harnack inequality (in fact the stronger parabolic version) is equivalent to the following two properties:

$$|B_{2R}(p)| \leq C_1 |B_R(p)| \tag{1}$$

$$\int_{B_R(p)} |f - \overline{f}|^2 dV \leq C_2 R^2 \int_{B_{2R}(p)} |\nabla f|^2 dV,\ f \in C^\infty(M) \tag{2}$$

where $\overline{f} = \frac{1}{|B_R(p)|} \int_{B_R(p)} f\ dV$ and C_1, C_2 depend only on M; see, for example, Saloff-Coste [**16**] and Grigor'yan [**8**]. In particular under the assumption of nonnegative Ricci curvature, then property (1) follows from the classical comparison theorems and property (2) follows from the work of Buser [**3**].

We now prove our graphical half-space theorem. Assume that S is a complete minimal graph with height function $u \geq 0$. According to Corollary 4.6, $|\nabla u| \leq C_1$ globally on M. Thus the induced metric g_{ij} given by (2.5) is uniformly elliptic and the Laplacian Δ^S on S given by (2.14) is a divergence form uniformly elliptic operator. We may by translation assume $\inf_M u = 0$ Thus given any $\varepsilon > 0$ there is a point $p \in M$ with $u(p) \leq \varepsilon$. Applying the Moser Harnack inequality yields for all R

$$\sup_{B_R(P)} u \leq C \inf_{B_R(P)} u \leq C\varepsilon$$

for a uniform constant C independent of R. Letting $R \to \infty$ and then $\varepsilon \to 0$ gives $u \equiv 0$. q.e.d.

Remark 4.7. Theorem 1.3 can be improved somewhat to allow

$$\limsup_{R \to \infty} \frac{|m(R)|}{R^\alpha} = 0$$

for some controlled small $\alpha \in (0, \frac{1}{2})$.

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