CLASSIFICATION OF SEMISIMPLE SYMMETRIC SPACES WITH PROPER $SL(2,\mathbb{R})$-ACTIONS

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Abstract

We give a complete classification of irreducible symmetric spaces for which there exist proper $SL(2,\mathbb{R})$-actions as isometries, using the criterion for proper actions by T. Kobayashi [Math. Ann. '89] and combinatorial techniques of nilpotent orbits. In particular, we classify irreducible symmetric spaces that admit surface groups as discontinuous groups, combining this with Benoist's theorem [Ann. Math. '96].

1. Introduction

The aim of this paper is to classify semisimple symmetric spaces G/H that admit isometric proper actions of non-compact simple Lie group $SL(2,\mathbb{R})$, and also those of surface groups $\pi_1(\Sigma_g)$. Here, isometries are considered with respect to the natural pseudo-Riemannian structure on G/H.

We motivate our work in one of the fundamental problems on locally symmetric spaces, stated below:

Problem 1.1 (See [**20**]). Fix a simply connected symmetric space $\widetilde{M}$ as a model space. What discrete groups can arise as the fundamental groups of complete affine manifolds M which are locally isomorphic to the space $\widetilde{M}$?

By a theorem of É. Cartan, such M is represented as the double coset space $\Gamma\backslash G/H$. Here $\widetilde{M} = G/H$ is a simply connected symmetric space and $\Gamma \simeq \pi_1(M)$ a discrete subgroup of G acting properly discontinuously and freely on $\widetilde{M}$.

Conversely, for a given symmetric pair (G, H) and an abstract group Γ with discrete topology, if there exists a group homomorphism $\rho : \Gamma \to G$ for which Γ acts on G/H properly discontinuously and freely via ρ, then the double coset space $\rho(\Gamma)\backslash G/H$ becomes a C^∞-manifold such that the natural quotient map

$$G/H \to \rho(\Gamma)\backslash G/H$$

is a C^∞-covering. The double coset manifold $\rho(\Gamma)\backslash G/H$ is called a Clifford–Klein form of G/H, which is endowed with a locally symmetric structure through the covering. We say that G/H *admits* Γ *as a discontinuous group* if there exists such ρ.

Then Problem 1.1 may be reformalized as:

Problem 1.2. Fix a symmetric pair (G, H). What discrete groups does G/H admit as discontinuous groups?

For a compact subgroup H of G, the action of any discrete subgroup of G on G/H is automatically properly discontinuous. Thus our interest is in non-compact H, for which not all discrete subgroups Γ of G act properly discontinuously on G/H. Problem 1.2 is non-trivial, even when $\widetilde{M} = \mathbb{R}^n$ is regarded as an affine symmetric space, i.e. $(G, H) = (GL(n, \mathbb{R}) \ltimes \mathbb{R}^n, GL(n, \mathbb{R}))$. In this case, the long-standing conjecture (Auslander's conjecture) states that such discrete group Γ will be virtually polycyclic if the Clifford–Klein form M is compact (see [**1**, **3**, **11**, **43**]). On the other hand, as was shown by E. Calabi and L. Markus [**7**] in 1962, no infinite discrete subgroup of $SO_0(n+1, 1)$ acts properly discontinuously on the de Sitter space $SO_0(n+1, 1)/SO_0(n, 1)$. More generally, if G/H does not admit any infinite discontinuous group, we say that a Calabi–Markus phenomenon occurs for G/H.

For the rest of this paper, we consider the case that G is a linear semisimple Lie group. In this setting, a systematic study of Problem 1.2 for the general homogeneous space G/H was initiated in the late 1980s by T. Kobayashi [**15**, **16**, **17**]. One of the fundamental results of Kobayashi in [**15**] is a criterion for proper actions, including a criterion for the Calabi–Markus phenomenon on homogeneous spaces G/H. More precisely, he showed that the following four conditions on G/H are equivalent: the space G/H admits an infinite discontinuous group; the space G/H admits a proper $\mathbb{R}$-action; the space G/H admits the abelian group $\mathbb{Z}$ as a discontinuous group; and $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$. Furthermore, Y. Benoist [**5**] obtained a criterion for the existence of infinite non-virtually abelian discontinuous groups for G/H.

Obviously, such discontinuous groups exist if there exists a Lie group homomorphism $\Phi : SL(2, \mathbb{R}) \to G$ such that $SL(2, \mathbb{R})$ acts on G/H properly via Φ. We prove that the converse statement also holds when G/H is a semisimple symmetric space. More strongly, our first main theorem gives a characterization of symmetric spaces G/H that admit proper $SL(2, \mathbb{R})$-actions:

Theorem 1.3 (see Theorem 2.2)**.** *Suppose that G is a connected linear semisimple Lie group. Then the following five conditions on a symmetric pair (G, H) are equivalent*:

(i) *There exists a Lie group homomorphism $\Phi : SL(2, \mathbb{R}) \to G$ such that $SL(2, \mathbb{R})$ acts on G/H properly via Φ.*

(ii) *For some $g \geq 2$, the symmetric space G/H admits the surface group $\pi_1(\Sigma_g)$ as a discontinuous group, where Σ_g is a closed Riemann surface of genus g.*

(iii) *G/H admits an infinite discontinuous group Γ which is not virtually abelian (i.e. Γ has no abelian subgroup of finite index).*

(iv) *There exists a complex nilpotent orbit $\mathcal{O}_{nilp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ such that $\mathcal{O}_{nilp}^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}_{nilp}^{G_\mathbb{C}} \cap \mathfrak{g}^c = \emptyset$, where $\mathfrak{g}^c$ is the c-dual of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ (see (2.1) for definition).*

(v) *There exists a complex antipodal hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ (see Definition 2.3) such that $\mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{g}^c = \emptyset$.*

The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is straightforward and easy. The non-trivial part of Theorem 1.3 is the implication (iii) $\Rightarrow$ (i).

By using Theorem 1.3, we give a complete classification of semisimple symmetric spaces G/H that admit a proper $SL(2, \mathbb{R})$-action. As is clear for (iv) or (v) in Theorem 1.3, it is sufficient to work at the Lie algebra level. Recall that the classification of semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ was accomplished by M. Berger [6]. Our second main theorem is to single out which symmetric pairs among his list satisfy the equivalent conditions in Theorem 1.3:

Theorem 1.4. *Suppose G is a simple Lie group. Then the two conditions below on a symmetric pair (G, H) are equivalent:*

(i) *(G, H) satisfies one of (therefore, all of) the equivalent conditions in Theorem 1.3.*

(ii) *The pair $(\mathfrak{g}, \mathfrak{h})$ belongs to Table 6 in Appendix A.*

The existence problem for compact Clifford–Klein forms has been actively studied in the last two decades since Kobayashi's paper [15]. The properness criteria of Kobayashi and Benoist yield necessary conditions on (G, H) for the existence of compact Clifford-Klein forms [5, 15]. See also [24, 28, 30, 33, 45] for some other methods for the existence problem of compact Clifford–Klein forms. The recent developments on this topic can be found in [21, 22, 27, 31].

We go back to semisimple symmetric pair (G, H). By Kobayashi's criterion [15, Corollary 4.4], the Calabi–Markus phenomenon occurs for G/H if and only if $\text{rank}_\mathbb{R}\, \mathfrak{g} = \text{rank}_\mathbb{R}\, \mathfrak{h}$ holds (see Fact 2.6 for more details). In particular, G/H does not admit compact Clifford–Klein forms in this case unless G/H itself is compact. In Section 2, we give the list, as Table 2, of symmetric pair $(\mathfrak{g}, \mathfrak{h})$ with simple $\mathfrak{g}$ which does not appear in Table 6 and $\text{rank}_\mathbb{R}\, \mathfrak{g} > \text{rank}_\mathbb{R}\, \mathfrak{h}$, i.e. $(\mathfrak{g}, \mathfrak{h})$ does not satisfy the equivalent conditions in Theorem 1.3 with $\text{rank}_\mathbb{R}\, \mathfrak{g} > \text{rank}_\mathbb{R}\, \mathfrak{h}$. Applying a theorem of Benoist [5, Corollary 1], we see that G/H does not admit compact Clifford–Klein forms if $(\mathfrak{g}, \mathfrak{h})$ is in Table 2 (see Corollary 2.8). In this table, we find some "new" examples of semisimple symmetric

Table 1. Examples of G/H without compact Clifford–Klein forms

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{su}^*(4m+2)$	$\mathfrak{sp}(m+1,m)$
$\mathfrak{su}^*(4m)$	$\mathfrak{sp}(m,m)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(3,1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{so}(4m+2,\mathbb{C})$	$\mathfrak{so}(2m+2,2m)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(2)}$

spaces G/H that do not admit compact Clifford–Klein forms, which we cannot find in the existing literature, see Table 1.

We remark that Table 1 is the list of symmetric pairs $(\mathfrak{g},\mathfrak{h})$ in Table 2 which are neither in Benoist's examples [**5**, Example 1] nor in Kobayashi's examples [**17**, Example 1.7, Table 4.4], [**19**, Table 5.18].

The proof of the non-trivial implication (iii) $\Rightarrow$ (i) in Theorem 1.3 is given by reducing it to an equivalent assertion on complex adjoint orbits, namely, (v) $\Rightarrow$ (iv). The last implication is proved by using the Dynkin–Kostant classification of $\mathfrak{sl}_2$-triples (equivalently, complex nilpotent orbits) in $\mathfrak{g}_{\mathbb{C}}$. We note that the proof does not need Berger's classification of semisimple symmetric pairs.

The reduction from (iii) $\Rightarrow$ (i) to (v) $\Rightarrow$ (iv) in Theorem 1.3 is given by proving (i) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) as follows. We show the equivalence (i) $\Leftrightarrow$ (iv) by combining Kobayashi's properness criterion [**15**] and a result of J. Sekiguchi for real nilpotent orbits in [**38**] with some observations on complexifications of real hyperbolic orbits. The equivalence (iii) $\Leftrightarrow$ (v) is obtained from Benoist's criterion [**5**].

As a refinement of the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 1.3, we give a bijection between real nilpotent orbits $\mathcal{O}_{\mathrm{nilp}}^{G}$ in $\mathfrak{g}$ such that the complexifications of $\mathcal{O}_{\mathrm{nilp}}^{G}$ do not intersect another real form $\mathfrak{g}^c$ and Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to G$ for which the $SL(2,\mathbb{R})$-actions on G/H via Φ are proper, up to inner automorphisms of G (Theorem 10.1).

Concerning the proof of Theorem 1.4, for a given semisimple symmetric pair $(\mathfrak{g},\mathfrak{h})$, we give an algorithm to check whether or not the condition (v) in Theorem 1.3 holds, by using Satake diagrams of $\mathfrak{g}$ and $\mathfrak{g}^c$.

The paper is organized as follows. In Section 2, we set up notation and state our main theorems. The next section contains a brief summary of Kobayashi's properness criterion [**15**] and Benoist's criterion [**5**] as preliminary results. We prove Theorem 1.3 in Section 4. The proof is based on some theorems, propositions and lemmas that are proved in Section 5 to Section 8 (see Section 4 for more details). Section 9 is

about the algorithm for our classification. The last section establishes the relation between proper $SL(2, \mathbb{R})$-actions on G/H and real nilpotent orbits in $\mathfrak{g}$.

The main results of this paper were announced in [**34**] with a sketch of the proofs.

2. Main results

Throughout this paper, we shall work in the following:

Setting 2.1. G is a connected linear semisimple Lie group, σ is an involutive automorphism on G, and H is an open subgroup of $G^\sigma :=$ $\{\, g \in G \mid \sigma g = g \,\}$.

This setting implies that G/H carries a pseudo-Riemannian structure g for which G acts as isometries and G/H becomes a symmetric space with respect to the Levi-Civita connection. We call (G, H) a semisimple symmetric pair. Note that g is positive definite, namely $(G/H, g)$ is Riemannian, if and only if H is compact.

Since G is a connected linear Lie group, we can take a connected complexification, denoted by $G_\mathbb{C}$, of G. We write $\mathfrak{g}_\mathbb{C}$, $\mathfrak{g}$, and $\mathfrak{h}$ for Lie algebras of $G_\mathbb{C}$, G, and H, respectively. The differential action of σ on $\mathfrak{g}$ will be denoted by the same letter σ. Then $\mathfrak{h} = \{\, X \in \mathfrak{g} \mid \sigma X = X \,\}$, and we also call $(\mathfrak{g}, \mathfrak{h})$ a semisimple symmetric pair. Let us denote $\mathfrak{q} := \{\, X \in \mathfrak{g} \mid \sigma X = -X \,\}$, and write the c-dual of $(\mathfrak{g}, \mathfrak{h})$ for

$$(2.1) \qquad \mathfrak{g}^c := \mathfrak{h} + \sqrt{-1}\mathfrak{q}.$$

Then both $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of $\mathfrak{g}_\mathbb{C}$. We note that the complex conjugation corresponding to $\mathfrak{g}^c$ on $\mathfrak{g}_\mathbb{C}$ is the anti $\mathbb{C}$-linear extension of σ on $\mathfrak{g}_\mathbb{C}$, and the semisimple symmetric pair $(\mathfrak{g}^c, \mathfrak{h})$ is the same as $(\mathfrak{g}, \mathfrak{h})^{ada}$ (which coincides with $(\mathfrak{g}, \mathfrak{h})^{dad}$; see [**35**, Section 1] for the notation).

For an abstract group Γ with discrete topology, we say that G/H *admits Γ as a discontinuous group* if there exists a group homomorphism $\rho : \Gamma \to G$ such that Γ acts properly discontinuously and freely on G/H via ρ (then ρ is injective and $\rho(\Gamma)$ is discrete in G, automatically). For such Γ-action on G/H, the double coset space $\Gamma \backslash G/H$, which is called a *Clifford–Klein form* of G/H, becomes a C^∞-manifold such that the quotient map

$$G/H \to \rho(\Gamma) \backslash G/H$$

is a C^∞-covering. In our context, the freeness of the action is less important than the properness of it (see [**15**, Section 5] for more details).

Here is the first main result:

Theorem 2.2. *In Setting* 2.1, *the following ten conditions on a semisimple symmetric pair* (G, H) *are equivalent:*

(i) *There exists a Lie group homeomorphism* $\Phi : SL(2, \mathbb{R}) \to G$ *such that* $SL(2, \mathbb{R})$ *acts properly on* G/H *via* Φ.

(ii) *For any* $g \geq 2$, *the symmetric space* G/H *admits the surface group* $\pi_1(\Sigma_g)$ *as a discontinuous group, where* Σ_g *is a closed Riemann surface of genus* g.

(iii) *For some* $g \geq 2$, *the symmetric space* G/H *admits the surface group* $\pi_1(\Sigma_g)$ *as a discontinuous group.*

(iv) G/H *admits an infinite discontinuous group* Γ *which is not virtually abelian (i.e.* Γ *has no abelian subgroup of finite index).*

(v) G/H *admits a discontinuous group which is a free group generated by a unipotent element in* G.

(vi) *There exists a complex nilpotent adjoint orbit* $\mathcal{O}_{nilp}^{G_\mathbb{C}}$ *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}_{nilp}^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}_{nilp}^{G_\mathbb{C}} \cap \mathfrak{g}^c = \emptyset$.

(vii) *There exists a real antipodal hyperbolic adjoint orbit* $\mathcal{O}_{hyp}^{G}$ *of* G *in* $\mathfrak{g}$ *(defined below) such that* $\mathcal{O}_{hyp}^{G} \cap \mathfrak{h} = \emptyset$.

(viii) *There exists a complex antipodal hyperbolic adjoint orbit* $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{g}^c = \emptyset$.

(ix) *There exists an* $\mathfrak{sl}_2$-*triple* (A, X, Y) *in* $\mathfrak{g}$ *(i.e.* $A, X, Y \in \mathfrak{g}$ *with* $[A, X] = 2X$, $[A, Y] = -2Y$, *and* $[X, Y] = A$) *such that* $\mathcal{O}_A^G \cap \mathfrak{h} = \emptyset$, *where* $\mathcal{O}_A^G$ *is the real adjoint orbit through* A *of* G *in* $\mathfrak{g}$.

(x) *There exists an* $\mathfrak{sl}_2$-*triple* (A, X, Y) *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}_A^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}_A^{G_\mathbb{C}} \cap \mathfrak{g}^c = \emptyset$, *where* $\mathcal{O}_A^{G_\mathbb{C}}$ *is the complex adjoint orbit through* A *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$.

Theorem 1.3 is a part of this theorem.

The definitions of *hyperbolic* orbits and *antipodal* orbits are given here:

Definition 2.3. Let $\mathfrak{g}$ be a complex or real semisimple Lie algebra. An element X of $\mathfrak{g}$ is said to be hyperbolic if the endomorphism $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable with only real eigenvalues. We say that an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is hyperbolic if any (or some) element in $\mathcal{O}$ is hyperbolic. Moreover, an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is said to be antipodal if for any (or some) element X in $\mathcal{O}$, the element $-X$ is also in $\mathcal{O}$.

A proof of Theorem 2.2 will be given in Section 4. Here is a short remark on it. In (i) $\Rightarrow$ (ix), the homomorphism Φ associates an $\mathfrak{sl}_2$-triple (A, X, Y) by the differential of Φ (see Section 4.1). The complex adjoint orbits in (viii) and (x) are obtained by the complexification of the real adjoint orbits in (vii) and (ix), respectively (see Section 4.3). In (x) $\Rightarrow$ (vi), the $\mathfrak{sl}_2$-triple (A, X, Y) in (x) associates a complex nilpotent orbit in (vi) by $\mathcal{O}_X^{G_\mathbb{C}} := \mathrm{Ad}(G_\mathbb{C}) \cdot X$ (see Section 4.4). The implication (i)

$\Rightarrow$ (ii) is obvious if we take $\pi_1(\Sigma_g)$ inside $SL(2,\mathbb{R})$. The equivalence (iv) $\Leftrightarrow$ (vii) is a kind of paraphrase of Benoist's criterion [**5**, Theorem 1.1] on symmetric spaces (see Section 4.2). The key ingredient of Theorem 2.2 is the implication (iii) $\Rightarrow$ (i). We will reduce it to the implication (viii) $\Rightarrow$ (x). The condition (viii) will be used for a classification of (G,H) satisfying the equivalence conditions in Theorem 2.2 (see Section 9).

Remark 2.4. **(1):** K. Teduka [**40**] gave a list of (G,H) satisfying the condition (i) in Theorem 2.2 in the special cases where $(\mathfrak{g},\mathfrak{h})$ is a complex symmetric pair. He also studied proper $SL(2,\mathbb{R})$-actions on some non-symmetric spaces in [**41**].

(2): Y. Benoist [**5**, Theorem 1.1] proved a criterion for the condition (iv) in a more general setting than we treat here.

(3): The following condition on a semisimple symmetric pair (G,H) is weaker than the equivalent conditions in Theorem 2.2:

  • There exists a real nilpotent adjoint orbit $\mathcal{O}_{\mathrm{nilp}}^G$ of G in $\mathfrak{g}$ such that $\mathcal{O}_{\mathrm{nilp}}^G \cap \mathfrak{h} = \emptyset$.

For a discrete subgroup Γ of G, we say that a Clifford–Klein form $\Gamma \backslash G/H$ is *standard* if Γ is contained in closed reductive subgroup L of G (see Definition 3.1) acting properly on G/H (see [**14**]), and is *non-standard* if not. See [**13**] for an example of a Zariski-dense discontinuous group Γ for G/H, which gives a nonstandard Clifford–Klein form. We obtain the following corollary to the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 2.2.

Corollary 2.5. *Let $g \geq 2$. Then, in Setting 2.1, the symmetric space G/H admits the surface group $\pi_1(\Sigma_g)$ as a discontinuous group if and only if there exists a discrete subgroup Γ of G such that $\Gamma \simeq \pi_1(\Sigma_g)$ and $\Gamma \backslash G/H$ is standard.*

Theorem 2.2 may be compared with the fact below for proper actions by the abelian group $\mathbb{R}$ consisting of hyperbolic elements:

Fact 2.6 (Criterion for the Calabi–Markus phenomenon)**.** In Setting 2.1, the following seven conditions on a semisimple symmetric pair (G,H) are equivalent:

 (i) There exists a Lie group homomorphism $\Phi : \mathbb{R} \to G$ such that $\mathbb{R}$ acts properly on G/H via Φ.
 (ii) G/H admits the abelian group $\mathbb{Z}$ as a discontinuous group.
 (iii) G/H admits an infinite discontinuous group.
 (iv) G/H admits a discontinuous group which is a free group generated by a hyperbolic element in G.
 (v) $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h}$.
 (vi) There exists a real hyperbolic adjoint orbit $\mathcal{O}_{\mathrm{hyp}}^G$ of G in $\mathfrak{g}$ such that $\mathcal{O}_{\mathrm{hyp}}^G \cap \mathfrak{h} = \emptyset$.

(vii) There exists a complex hyperbolic adjoint orbit $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g}^c = \emptyset$.

The equivalence among (i), (ii), (iii), (iv), and (v) in Fact 2.6 was proved in a more general setting in T. Kobayashi [**15**, Corollary 4.4]. The real rank condition (v) serves as a criterion for the Calabi–Markus phenomenon (iii) in Fact 2.6 (cf. [**7**], [**15**]). We will give a proof of the equivalence among (v), (vi), and (vii) in Appendix B.

The second main result is a classification of semisimple symmetric pairs (G, H) satisfying one of (therefore, all of) the equivalent conditions in Theorem 2.2.

If a semisimple symmetric pair (G, H) is irreducible, but G is not simple, then G/H admits a proper $SL(2, \mathbb{R})$-action, since the symmetric space G/H can be regarded as a complex simple Lie group. Therefore, the crucial case is on symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with simple Lie algebra $\mathfrak{g}$.

To describe our classification, we denote

$$S := \{ (\mathfrak{g}, \mathfrak{h}) \mid (\mathfrak{g}, \mathfrak{h}) \text{ is a semisimple symmetric pair}$$
$$\text{with a simple Lie algebra } \mathfrak{g} \}.$$

The set S was classified by M. Berger [**6**] up to isomorphisms. We also put

$A := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid (\mathfrak{g}, \mathfrak{h}) \text{ satisfies one of the conditions in Theorem 2.2} \}$,

$B := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid \mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h} \} \setminus A$,

$C := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid \mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} = \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h} \}$.

Then $A \cap C = \emptyset$ by Fact 2.6, and we have

$$S = A \sqcup B \sqcup C.$$

One can easily determine the set C in S. Thus, to describe the classification of A, we only need to give the classification of B.

Here is our classification of the set B, namely, a complete list of $(\mathfrak{g}, \mathfrak{h})$ satisfying the following:

(2.2) $\mathfrak{g}$ is simple $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair with $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h}$

   but does not satisfy the equivalent conditions in Theorem 2.2.

Here, $k \geq 2$, $m \geq 1$, and $n \geq 2$.

Theorem 1.4, which gives a classification of the set A, is obtained by Table 2.

Concerning our classification, we will give an algorithm to check whether or not a given symmetric pair $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 2.2. More precisely, we will determine the set of complex antipodal hyperbolic orbits in a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (see Section 6.2) and introduce an algorithm to check whether or not a given such

Table 2. Classification of $(\mathfrak{g},\mathfrak{h})$ satisfying (2.2)

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(2k,\mathbb{R})$	$\mathfrak{sp}(k,\mathbb{R})$
$\mathfrak{sl}(2k,\mathbb{R})$	$\mathfrak{so}(k,k)$
$\mathfrak{sl}(2k-1,\mathbb{R})$	$\mathfrak{so}(k,k-1)$
$\mathfrak{su}^*(4m+2)$	$\mathfrak{sp}(m+1,m)$
$\mathfrak{su}^*(4m)$	$\mathfrak{sp}(m,m)$
$\mathfrak{su}^*(2k)$	$\mathfrak{so}^*(2k)$
$\mathfrak{so}(2k-1,2k-1)$	$\mathfrak{so}(i+1,i)\oplus\mathfrak{so}(j,j+1)$ $(i+j=2k-2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4,\mathbb{R})$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(3,1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{sl}(2k,\mathbb{C})$	$\mathfrak{sp}(k,\mathbb{C})$
$\mathfrak{sl}(2k,\mathbb{C})$	$\mathfrak{su}(k,k)$
$\mathfrak{so}(4m+2,\mathbb{C})$	$\mathfrak{so}(i,\mathbb{C})\oplus\mathfrak{so}(j,\mathbb{C})$ $(i+j=4m+2,\ i,j\text{ are odd})$
$\mathfrak{so}(4m+2,\mathbb{C})$	$\mathfrak{so}(2m+2,2m)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{sp}(4,\mathbb{C})$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{f}_{4,\mathbb{C}}$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(2)}$

orbit meets a real form $\mathfrak{g}$ [resp. $\mathfrak{g}^c$] (see Section 7). Table 2 is obtained by using this algorithm (see Section 9).

Remark 2.7. (1) Using [**5**, Theorem 1.1], Benoist gave a number of examples of symmetric pairs (G,H) which do not satisfy the condition (iv) in Theorem 2.2 with $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\,\mathfrak{h}$ (see [**5**, Example 1]). Table 2 gives its complete list.

(2) We take this opportunity to correct [**34**, Table 2.6], where the pair $(\mathfrak{sl}(2k-1,\mathbb{R}),\mathfrak{so}(k,k-1))$ was missing.

We discuss an application of the main result (Theorem 2.2) to the existence problem of compact Clifford–Klein forms. As we explained in the Introduction, a Clifford–Klein form of G/H is the double coset space $\Gamma\backslash G/H$ when Γ is a discrete subgroup of G acting on G/H properly discontinuously and freely. Recall that we say that a homogeneous space G/H *admits compact Clifford–Klein forms*, if there exists such Γ where $\Gamma\backslash G/H$ is compact. See also [**5**, **15**, **16**, **17**, **19**, **23**, **24**, **28**, **30**, **33**, **39**, **45**] for preceding results for the existence problem for compact Clifford–Klein forms. Among them, there are three methods that can be applied

to semisimple symmetric spaces to show the non-existence of compact Clifford–Klein forms:

- Using the Hirzebruch–Kobayashi–Ono proportionality principle [**15**, Proposition 4.10], [**23**].
- Using a comparison theorem of cohomological dimension [**17**, Theorem 1.5]. (A generalization of the criterion in [**15**] of the Calabi–Markus phenomenon.)
- Using a criterion for the non-existence of properly discontinuous actions of non-virtually abelian groups [**5**, Corollary 1].

As an immediate corollary of the third method and the description of the set B by Table 2, one concludes:

Corollary 2.8. *The simple symmetric space G/H does not admit compact Clifford–Klein forms if $(\mathfrak{g}, \mathfrak{h})$ is in Table 2.*

3. Preliminary results for proper actions

In this section, we recall results of T. Kobayashi [**15**] and Y. Benoist [**5**] in a form that we shall need. Our proofs of the equivalences (i) $\Leftrightarrow$ (x) and (iv) $\Leftrightarrow$ (viii) in Theorem 2.2 will be based on these results (see Section 4.1 and Section 4.2).

3.1. Kobayashi's properness criterion. Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for the Lie algebra of G. First, we fix a terminology as follows:

Definition 3.1. We say that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *reductive in* $\mathfrak{g}$ if there exists a Cartan involution θ of $\mathfrak{g}$ such that $\mathfrak{h}$ is θ-stable. Furthermore, we say that a closed subgroup H of G is *reductive in G* if H has only finitely many connected components and the Lie algebra $\mathfrak{h}$ of H is reductive in $\mathfrak{g}$.

For simplicity, we call $\mathfrak{h}$ [resp. H] a reductive subalgebra of $\mathfrak{g}$ [resp. a reductive subgroup of G] if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ [resp. H is reductive in G]. We call such (G, H) a reductive pair. Note that a reductive subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a reductive Lie algebra.

We give two examples relating to Theorem 2.2:

Example 3.2. In Setting 2.1, the subgroup H is reductive in G since there exists a Cartan involution θ on $\mathfrak{g}$, which is commutative with σ (cf. [**6**]).

Example 3.3. Let $\mathfrak{l}$ be a semisimple subalgebra of $\mathfrak{g}$. Then any Cartan involution on $\mathfrak{l}$ can be extended to a Cartan involution on $\mathfrak{g}$ (cf. G. D. Mostow [**32**]) and the analytic subgroup L corresponding to $\mathfrak{l}$ is closed in G (cf. K. Yosida [**44**]). Therefore, $\mathfrak{l}$ [resp. L] is reductive in $\mathfrak{g}$ [resp. G].

In the rest of this subsection, we follow the setting below:

Setting 3.4. G is a connected linear semisimple Lie group, H and L are reductive subgroups of G.

We denote by $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ the Lie algebras of G, H, and L, respectively. Take a Cartan involution θ of $\mathfrak{g}$ which preserves $\mathfrak{h}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ for the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{h}$ corresponding to θ, $\theta|_{\mathfrak{h}}$, respectively. We fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p}(\mathfrak{h})$ (i.e. $\mathfrak{a}_{\mathfrak{h}}$ is a maximally split abelian subspace of $\mathfrak{h}$), and extend it to a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ (i.e. $\mathfrak{a}$ is a maximally split abelian subspace of $\mathfrak{g}$). We write K for the maximal compact subgroup of G with its Lie algebra $\mathfrak{k}$, and denote the Weyl group acting on $\mathfrak{a}$ by $W(\mathfrak{g}, \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Since $\mathfrak{l}$ is also reductive in $\mathfrak{g}$, we can take a Cartan involution θ' of $\mathfrak{g}$ preserving $\mathfrak{l}$. We write $\mathfrak{l} = \mathfrak{k}'(\mathfrak{l}) + \mathfrak{p}'(\mathfrak{l})$ for the Cartan decomposition of $\mathfrak{l}$ corresponding to $\theta'|_{\mathfrak{l}}$, and fix a maximal abelian subspace $\mathfrak{a}'_{\mathfrak{l}}$ of $\mathfrak{p}'(\mathfrak{l})$. Then there exists $g \in G$ such that $\mathrm{Ad}(g) \cdot \mathfrak{a}'_{\mathfrak{l}}$ is contained in $\mathfrak{a}$, and we put $\mathfrak{a}_{\mathfrak{l}} := \mathrm{Ad}(g) \cdot \mathfrak{a}'_{\mathfrak{l}}$. The subset $W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}}$ of $\mathfrak{a}$ does not depend on a choice of such $g \in G$.

The following fact holds:

Fact 3.5 (T. Kobayashi [**15**, Theorem 4.1])**.** In Setting 3.4, L acts on G/H properly if and only if

$$\mathfrak{a}_{\mathfrak{h}} \cap W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}} = \{0\}.$$

The proof of Fact 2.6 is reduced to Fact 3.5 (see [**15**]). However, to prove the equivalences between (v), (vi), and (vii) in Fact 2.6, we need an additional argument which will be described in Appendix B.

3.2. Benoist's criterion. Let (G, H) be a reductive pair (see Definition 3.1). In this subsection, we use the notation $\mathfrak{g}$, $\mathfrak{h}$, θ, $\mathfrak{a}_{\mathfrak{h}}$, $\mathfrak{a}$, and $W(\mathfrak{g}, \mathfrak{a})$ as in the previous subsection.

Let us denote the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a})$. We fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of $\Sigma(\mathfrak{g}, \mathfrak{a})$, and put

$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(X) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \}.$$

Then $\mathfrak{a}_+$ is a fundamental domain for the action of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$. We write w_0 for the longest element in $W(\mathfrak{g}, \mathfrak{a})$ with respect to the positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then, by the action of w_0, every element in $\mathfrak{a}_+$ moves to $-\mathfrak{a}_+ := \{-A \mid A \in \mathfrak{a}_+\}$. In particular,

$$-w_0 : \mathfrak{a} \to \mathfrak{a}, \quad A \mapsto -(w_0 \cdot A)$$

is an involutive automorphism on $\mathfrak{a}$ preserving $\mathfrak{a}_+$. We put

$$\mathfrak{b} := \{ A \in \mathfrak{a} \mid -w_0 \cdot A = A \}, \quad \mathfrak{b}_+ := \mathfrak{b} \cap \mathfrak{a}_+.$$

Then the next fact holds:

Fact 3.6 (Y. Benoist [**5**, Theorem in Section 1.1])**.** The following conditions on a reductive pair (G, H) are equivalent:

(i) G/H admits an infinite discontinuous group which is not virtually abelian.
(ii) $\mathfrak{b}_+ \not\subset w \cdot \mathfrak{a}_\mathfrak{h}$ for any $w \in W(\mathfrak{g}, \mathfrak{a})$.
(iii) $\mathfrak{b}_+ \not\subset W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_\mathfrak{h}$.

Remark 3.7. Benoist showed (i) $\Leftrightarrow$ (ii) in Fact 3.6. The equivalence (ii) $\Leftrightarrow$ (iii) follows from the fact below (since $\mathfrak{b}_+$ is a convex set of $\mathfrak{a}$ and $w \cdot \mathfrak{a}_\mathfrak{h}$ is a linear subspace of $\mathfrak{a}$ for any $w \in W(\mathfrak{g}, \mathfrak{a})$).

Fact 3.8. Let $U_1, U_2, \ldots, U_n$ be subspaces of a finite dimensional real vector space V and Ω a convex set of V. Then Ω is contained in $\bigcup_{i=1}^{n} U_i$ if and only if Ω is contained in U_k for some $k \in \{1, \ldots, n\}$.

4. Proof of Theorem 2.2

We give a proof of Theorem 2.2 by proving the implications in the figure below:
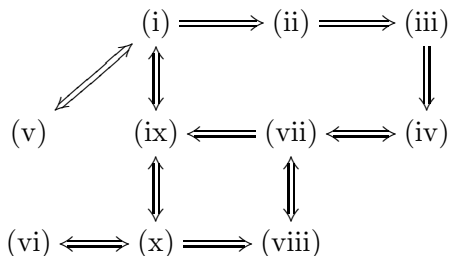
In this section, to show the implications, we use some theorems, propositions, and lemmas, which will be proved later in this paper.

Notation: Throughout this paper, for a complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$ and its real form $\mathfrak{g}$, we denote a complex [resp. real] nilpotent, hyperbolic, antipodal hyperbolic adjoint orbit in $\mathfrak{g}_\mathbb{C}$ [resp. $\mathfrak{g}$] simply by a complex [resp. real] nilpotent, hyperbolic, antipodal hyperbolic orbit in $\mathfrak{g}_\mathbb{C}$ [resp. $\mathfrak{g}$].

4.1. Proof of (i) $\Leftrightarrow$ (ix) **in Theorem 2.2.** Our proof of the equivalence (i) $\Leftrightarrow$ (ix) in Theorem 2.2 starts with the next theorem, which will be proved in Section 5:

Theorem 4.1 (Corollary to Fact 3.5)**.** *In Setting* 3.4*, the following conditions on* (G, H, L) *are equivalent*:

(i) *L acts on G/H properly;*
(ii) *there do not exist real hyperbolic orbits in $\mathfrak{g}$ (see Definition* 2.3*) meeting both $\mathfrak{l}$ and $\mathfrak{h}$ other than the zero-orbit,*

where $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ are Lie algebras of G, H, and L, respectively.

By using Theorem 4.1, we will prove the next proposition in Section 5:

Proposition 4.2. *Let (G, H) be a reductive pair* (*see Definition* 3.1). *Then there exists a bijection between the following two sets*:

- *The set of Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to G$ such that $SL(2,\mathbb{R})$ acts on G/H properly via Φ,*
- *The set of $\mathfrak{sl}_2$-triples (A, X, Y) in $\mathfrak{g}$ such that the real adjoint orbit through A does not meet $\mathfrak{h}$.*

In Setting 2.1, the subgroup H of G is reductive in G (see Example 3.2). Hence, we obtain the equivalence (i) $\Leftrightarrow$ (ix) in Theorem 2.2.

4.2. Proof of (iv) $\Leftrightarrow$ (vii) **in Theorem 2.2.** We will prove the next theorem in Section 5:

Theorem 4.3 (Corollary to Fact 3.6). *The following conditions on a reductive pair (G, H)* (*see Definition* 3.1) *are equivalent*:

(i) *G/H admits an infinite discontinuous group that is not virtually abelian.*

(ii) *There exists a real antipodal hyperbolic orbit in $\mathfrak{g}$ that does not meet $\mathfrak{h}$.*

In Setting 2.1, the equivalence (iv) $\Leftrightarrow$ (vii) in Theorem 2.2 holds as a special case of Theorem 4.3.

4.3. Proofs of (x) $\Leftrightarrow$ (ix), (viii) $\Leftrightarrow$ (vii), **and** (x) $\Rightarrow$ (viii) **in Theorem 2.2.** Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra. We use the following convention for hyperbolic elements (see Definition 2.3):

$\mathcal{H} := \{\, A \in \mathfrak{g}_{\mathbb{C}} \mid A \text{ is a hyperbolic element in } \mathfrak{g}_{\mathbb{C}} \,\}.$

$\mathcal{H}^a := \{\, A \in \mathcal{H} \mid \text{The complex adjoint orbit through } A \text{ is antipodal} \,\}.$

$\mathcal{H}^n := \{\, A \in \mathfrak{g}_{\mathbb{C}} \mid \text{There exist } X, Y \in \mathfrak{g}_{\mathbb{C}}$
$$\text{such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple} \,\}.$$

We also write $\mathcal{H}/G_{\mathbb{C}}$, $\mathcal{H}^a/G_{\mathbb{C}}$ for the sets of complex hyperbolic orbits and complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$, respectively. Let us denote by $\mathcal{H}^n/G_{\mathbb{C}}$ the set of complex adjoint orbits contained in $\mathcal{H}^n$.

The next lemma will be proved in Section 6.3:

Lemma 4.4. *For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, the element A of $\mathfrak{g}_{\mathbb{C}}$ is hyperbolic and the complex adjoint orbit through A in $\mathfrak{g}_{\mathbb{C}}$ is antipodal.*

By Lemma 4.4, we have

$$\mathcal{H}^n \subset \mathcal{H}^a \subset \mathcal{H}.$$

Hence, the implication (x) $\Rightarrow$ (viii) in Theorem 2.2 follows.

Further, for any subalgebra $\mathfrak{l}$ of $\mathfrak{g}_\mathbb{C}$, we also use the following convention:

$\mathcal{H}_\mathfrak{l} := \{\, A \in \mathcal{H} \mid \text{The complex adjoint orbit through } A \text{ meets } \mathfrak{l} \,\}.$

$\mathcal{H}_\mathfrak{l}^a := \mathcal{H}^a \cap \mathcal{H}_\mathfrak{l}.$

$\mathcal{H}_\mathfrak{l}^n := \mathcal{H}^n \cap \mathcal{H}_\mathfrak{l}.$

Let us write $\mathcal{H}_\mathfrak{l}/G_\mathbb{C}$, $\mathcal{H}_\mathfrak{l}^a/G_\mathbb{C}$, $\mathcal{H}_\mathfrak{l}^n/G_\mathbb{C}$ for the sets of complex adjoint orbits contained in $\mathcal{H}$, $\mathcal{H}^a$, $\mathcal{H}^n$ meeting $\mathfrak{l}$, respectively.

Here, we fix a real form $\mathfrak{g}$, and set

$\mathcal{H}(\mathfrak{g}) := \{\, A \in \mathfrak{g} \mid A \text{ is a hyperbolic element in } \mathfrak{g} \,\},$

$\mathcal{H}^a(\mathfrak{g}) := \{\, A \in \mathcal{H}(\mathfrak{g}) \mid \text{The real adjoint orbit through } A \text{ is antipodal} \,\},$

$\mathcal{H}^n(\mathfrak{g}) := \{\, A \in \mathfrak{g} \mid \text{There exist } X, Y \in \mathfrak{g}$

$\text{such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple} \,\}.$

We also write $\mathcal{H}(\mathfrak{g})/G$, $\mathcal{H}^a(\mathfrak{g})/G$, $\mathcal{H}^n(\mathfrak{g})/G$ for the sets of real adjoint orbits contained in $\mathcal{H}(\mathfrak{g})$, $\mathcal{H}^a(\mathfrak{g})$, $\mathcal{H}^n(\mathfrak{g})$, respectively.

Then the following proposition gives a one-to-one correspondence between real hyperbolic orbits and complex hyperbolic orbits with real points:

Proposition 4.5.   (i) *The following map gives a one-to-one correspondence between* $\mathcal{H}(\mathfrak{g})/G$ *and* $\mathcal{H}_\mathfrak{g}/G_\mathbb{C}$:

$$\mathcal{H}(\mathfrak{g})/G \to \mathcal{H}_\mathfrak{g}/G_\mathbb{C}, \quad \mathcal{O}_{hyp}^G \mapsto \mathrm{Ad}(G_\mathbb{C}) \cdot \mathcal{O}_{hyp}^G,$$

$$\mathcal{H}_\mathfrak{g}/G_\mathbb{C} \to \mathcal{H}(\mathfrak{g})/G, \quad \mathcal{O}_{hyp}^{G_\mathbb{C}} \mapsto \mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{g}.$$

(ii) *The bijection in* (i) *gives the one-to-one correspondence below:*

$$\mathcal{H}^a(\mathfrak{g})/G \xleftrightarrow{\;1:1\;} \mathcal{H}_\mathfrak{g}^a/G_\mathbb{C}.$$

(iii) *The bijection in* (i) *gives the one-to-one correspondence below:*

$$\mathcal{H}^n(\mathfrak{g})/G \xleftrightarrow{\;1:1\;} \mathcal{H}_\mathfrak{g}^n/G_\mathbb{C}.$$

The proof of Proposition 4.5 will be given in Section 7.

In Setting 2.1, recall that both $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of $\mathfrak{g}_\mathbb{C}$. In Section 8, we will prove the following proposition, which claims that a complex hyperbolic orbit meets $\mathfrak{h}$ if it meets both $\mathfrak{g}$ and $\mathfrak{g}^c$:

Proposition 4.6. *In Setting* 2.1, $\mathcal{H}_\mathfrak{g} \cap \mathcal{H}_{\mathfrak{g}^c} = \mathcal{H}_\mathfrak{h}.$

The equivalences (x) $\Leftrightarrow$ (ix) and (viii) $\Leftrightarrow$ (vii) in Theorem 2.2 follow from Proposition 4.5 and Proposition 4.6.

4.4. Proof of (vi) $\Leftrightarrow$ (x) **in Theorem 2.2.** The equivalence (vi) $\Leftrightarrow$ (x) in Theorem 2.2 can be obtained by the Jacobson–Morozov theorem and the lemma below (see Proposition 7.8 for a proof):

Lemma 4.7 (Corollary to J. Sekiguchi [**38**, Proposition 1.11]). *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex semisimple Lie algebra and* $\mathfrak{g}$ *a real form of* $\mathfrak{g}_{\mathbb{C}}$. *Then the following conditions on an* $\mathfrak{sl}_2$-*triple* (A, X, Y) *in* $\mathfrak{g}_{\mathbb{C}}$ *are equivalent*:
  (i) *The complex adjoint orbit through* A *in* $\mathfrak{g}_{\mathbb{C}}$ *meets* $\mathfrak{g}$.
  (ii) *The complex adjoint orbit through* X *in* $\mathfrak{g}_{\mathbb{C}}$ *meets* $\mathfrak{g}$.

4.5. Proof of (vii) $\Rightarrow$ (ix) **in Theorem 2.2.** Let $\mathfrak{g}$ be a semisimple Lie algebra. In this subsection, we use $\mathcal{H}(\mathfrak{g})$, $\mathcal{H}^a(\mathfrak{g})$, and $\mathcal{H}^n(\mathfrak{g})$ as in Section 4.3.

To prove the implication (vii) $\Rightarrow$ (ix), we use the next proposition and lemma:

Proposition 4.8. *We take*
$$\mathfrak{b} := \{\, A \in \mathfrak{a} \mid -w_0 \cdot A = A \,\}, \quad \mathfrak{b}_+ := \mathfrak{b} \cap \mathfrak{a}_+$$
as in Section 3.2. Then the following holds:
  (i) $\mathfrak{b} = \mathbb{R}$-$\mathrm{span}(\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}))$.
  (ii) $\mathcal{H}^a(\mathfrak{g}) = \mathrm{Ad}(G) \cdot \mathfrak{b}_+$.

Lemma 4.9. *Let* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *be a semisimple symmetric pair. We fix a Cartan involution* θ *on* $\mathfrak{g}$ *such that* $\theta\sigma = \sigma\theta$ *and denote by* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *the Cartan decomposition of* $\mathfrak{g}$ *with respect to* θ. *Let us take* $\mathfrak{a}$ *and* $\mathfrak{a}_{\mathfrak{h}} = \mathfrak{a} \cap \mathfrak{h}$ *as in Section* 3.1. *We fix an ordering on* $\mathfrak{a}_{\mathfrak{h}}$ *and extend it to* $\mathfrak{a}$, *and put* $\mathfrak{a}_+$ *to the closed Weyl chamber of* $\mathfrak{a}$ *with respect to the ordering. Then*
$$\mathfrak{a}_+ \cap \mathcal{H}_{\mathfrak{h}}(\mathfrak{g}) \subset \mathfrak{a}_{\mathfrak{h}},$$
where $\mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$ *is the set of hyperbolic elements in* $\mathfrak{g}$ *whose adjoint orbits meet* $\mathfrak{h}$.

Postponing the proof of Proposition 4.8 and Lemma 4.9 to later sections, we complete the proof of the implication (viii) $\Rightarrow$ (x) in Theorem 2.2.

Proof of (viii) $\Rightarrow$ (x) *in Theorem* 2.2. We shall prove that $\mathcal{H}^a(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$ under the assumption $\mathcal{H}^n(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$. By combining Proposition 4.8 (i) and Lemma 4.9 with the assumption, we have
$$\mathfrak{b} \subset \mathfrak{a}_{\mathfrak{h}}(\subset \mathfrak{h}).$$
Therefore, by Proposition 4.8 (ii), we obtain that $\mathcal{H}^a(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$. q.e.d.

We shall give a proof of Proposition 4.8 (i) in Section 7.5 by comparing Dynkin's classification of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ [**10**] with the Satake diagram of the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. The proof of Proposition 4.8 (ii) will be given in Section 5.1, and that of Lemma 4.9 in Section 8.

4.6. Proofs of (i) $\Rightarrow$ (ii)**,** (iii) $\Rightarrow$ (viii)**, and** (i) $\Leftrightarrow$ (v) **in Theorem 2.2.** The implication (i) $\Rightarrow$ (ii) in Theorem 2.2 is deduced from the lifting theorem of surface groups (cf. [**26**]). The implication (iii) $\Rightarrow$ (viii) follows by the fact that the surface group of genus g is not virtually abelian for any $g \geq 2$.

The equivalence (i) $\Leftrightarrow$ (v) can be proved by the observation below: Let Γ_0 be the free group generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $SL(2, \mathbb{R})$. Then, for any free group Γ generated by a unipotent element in a linear semisimple Lie group G, there exists a Lie group homomorphism $\Phi : SL(2, \mathbb{R}) \to G$ such that $\Phi(\Gamma_0) = \Gamma$ (by the Jacobson–Morozov theorem). Furthermore, by [**18**, Lemma 3.2], for any closed subgroup H of G, the $SL(2, \mathbb{R})$-action on G/H via Φ is proper if and only if the Γ-action on G/H is properly discontinuous.

5. Real hyperbolic orbits and proper actions of reductive subgroups

In this section, we prove Theorem 4.1, Proposition 4.2, Theorem 4.3, and Proposition 4.8 (ii).

5.1. Kobayashi's properness criterion and Benoist's criterion rephrased by real hyperbolic orbits. In this subsection, Theorem 4.1 and Theorem 4.3 are proved as corollaries to Fact 3.5 and Fact 3.6, respectively. We also prove Proposition 4.8 (ii) in this subsection.

Let $\mathfrak{g}$ be a semisimple Lie algebra. The next fact for real hyperbolic orbits in $\mathfrak{g}$ (see Definition 2.3) is well known:

Fact 5.1. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ and a maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ (i.e. $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$). Then any real hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^G$ in $\mathfrak{g}$ meets $\mathfrak{a}$, and the intersection $\mathcal{O}_{\mathrm{hyp}}^G \cap \mathfrak{a}$ is a single $W(\mathfrak{g}, \mathfrak{a})$-orbit, where $W(\mathfrak{g}, \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. In particular, we have a bijection

$$\mathcal{H}(\mathfrak{g})/G \to \mathfrak{a}/W(\mathfrak{g}, \mathfrak{a}), \quad \mathcal{O}_{\mathrm{hyp}}^G \mapsto \mathcal{O}_{\mathrm{hyp}}^G \cap \mathfrak{a},$$

where $\mathcal{H}(\mathfrak{g})/G$ is the set of real hyperbolic orbits in $\mathfrak{g}$ and $\mathfrak{a}/W(\mathfrak{g}, \mathfrak{a})$ the set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$.

Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ (see Definition 3.1). Take a maximally split abelian subspace $\mathfrak{a}_\mathfrak{h}$ of $\mathfrak{h}$ and extend it to a maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ in a similar way as in Section 3.1. Then the following lemma holds:

Lemma 5.2. *A real hyperbolic orbit $\mathcal{O}_{hyp}^G$ in $\mathfrak{g}$ meets $\mathfrak{h}$ if and only if it meets $\mathfrak{a}_\mathfrak{h}$. In particular, we have a bijection*

$$\mathcal{H}_\mathfrak{h}(\mathfrak{g})/G \to \{\, \mathcal{O}^{W(\mathfrak{g}, \mathfrak{a})} \in \mathfrak{a}/W(\mathfrak{g}, \mathfrak{a}) \mid \mathcal{O}^{W(\mathfrak{g}, \mathfrak{a})} \cap \mathfrak{a}_\mathfrak{h} \neq \emptyset \,\}, \quad \mathcal{O}_{hyp}^G \mapsto \mathcal{O}_{hyp}^G \cap \mathfrak{a},$$

where $\mathcal{H}_\mathfrak{h}(\mathfrak{g})/G$ is the set of real hyperbolic orbits in $\mathfrak{g}$ meeting $\mathfrak{h}$.

Sketch of the proof. Suppose that $\mathcal{O}_{\mathrm{hyp}}^{G}$ meets $\mathfrak{h}$; we shall prove that $\mathcal{O}_{\mathrm{hyp}}^{G}$ meets $\mathfrak{a}_{\mathfrak{h}}$. If $\mathfrak{h}$ is semisimple, then $\mathcal{O}_{\mathrm{hyp}}^{G} \cap \mathfrak{h}$ contains some hyperbolic orbits in $\mathfrak{h}$. Hence, our claim follows by Fact 5.1. For the cases where $\mathfrak{h}$ is reductive in $\mathfrak{g}$ with non-trivial center $Z(\mathfrak{h})$, we put

$$Z_{\mathfrak{k}(\mathfrak{h})}(\mathfrak{h}) := Z(\mathfrak{h}) \cap \mathfrak{k}, \quad Z_{\mathfrak{p}(\mathfrak{h})}(\mathfrak{h}) := Z(\mathfrak{h}) \cap \mathfrak{p},$$

where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ are Cartan decompositions of $\mathfrak{g}$, $\mathfrak{h}$ in Section 3.1. Then we have

$$\mathfrak{h} = Z_{\mathfrak{k}(\mathfrak{h})}(\mathfrak{h}) \oplus Z_{\mathfrak{p}(\mathfrak{h})}(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}].$$

Here, we fix any $X \in \mathcal{O}_{\mathrm{hyp}}^{G} \cap \mathfrak{h}$ and put

$$X = X_{\mathfrak{k}} + X_{\mathfrak{p}} + X'$$

for $X_{\mathfrak{k}} \in Z_{\mathfrak{k}(\mathfrak{h})}(\mathfrak{h})$, $X_{\mathfrak{p}} \in Z_{\mathfrak{p}(\mathfrak{h})}(\mathfrak{h})$, and $X' \in [\mathfrak{h}, \mathfrak{h}]$. Then one can prove that $X_{\mathfrak{k}} = 0$ and X' is hyperbolic in the semisimple subalgebra $[\mathfrak{h}, \mathfrak{h}]$ of $\mathfrak{g}$. Hence our claim follows from Fact 5.1. q.e.d.

We now prove Theorem 4.1 as a corollary to Fact 3.5.

Proof of Theorem 4.1. In Setting 3.4, by Fact 5.1 and Lemma 5.2, we have a bijection between the following two sets:

- the set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$ meeting both $\mathfrak{a}_{\mathfrak{h}}$ and $\mathfrak{a}_{\mathfrak{l}}$,
- the set of real hyperbolic orbits in $\mathfrak{g}$ meeting both $\mathfrak{h}$ and $\mathfrak{l}$.

Hence, our claim follows from Fact 3.5. q.e.d.

To prove Theorem 4.3, we shall show the next lemma:

Lemma 5.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then a real hyperbolic orbit in $\mathfrak{g}$ is antipodal if and only if it meets $\mathfrak{b}_{+}$ (see Section 3.2 for the notation). In particular, we have a bijection*

$$\mathcal{H}^{a}(\mathfrak{g})/G \to \{\, \mathcal{O}^{W(\mathfrak{g}, \mathfrak{a})} \in \mathfrak{a}/W(\mathfrak{g}, \mathfrak{a}) \mid \mathcal{O}^{W(\mathfrak{g}, \mathfrak{a})} \cap \mathfrak{b}_{+} \neq \emptyset \,\}, \ \mathcal{O}_{hyp}^{G} \mapsto \mathcal{O}_{hyp}^{G} \cap \mathfrak{a},$$

where $\mathcal{H}^{a}(\mathfrak{g})/G$ is the set of real antipodal hyperbolic orbits in $\mathfrak{g}$.

Proof of Lemma 5.3. By Fact 5.1, any real hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G}$ in $\mathfrak{g}$ meets $\mathfrak{a}_{+}$ with a unique element A_0 in $\mathcal{O}_{\mathrm{hyp}}^{G} \cap \mathfrak{a}_{+}$. It remains to prove that $-A_0$ is in $\mathcal{O}_{\mathrm{hyp}}^{G}$ if and only if $-w_0 \cdot A_0 = A_0$. First, we suppose that $-A_0 \in \mathcal{O}_{\mathrm{hyp}}^{G}$. Then the element $-A_0$ of $-\mathfrak{a}_{+}$ is conjugate to A_0 under the action of $W(\mathfrak{g}, \mathfrak{a})$ by Fact 5.1. Recall that both $\mathfrak{a}_{+}$ and $-\mathfrak{a}_{+}$ are fundamental domains of $\mathfrak{a}$ for the action of $W(\mathfrak{g}, \mathfrak{a})$, and $w_0 \cdot \mathfrak{a}_{+} = -\mathfrak{a}_{+}$. Hence, we obtain that $-w_0 \cdot A_0 = A_0$. Conversely, we assume that $-A_0 = w_0 \cdot A_0$. In particular, $-A_0$ is in $W(\mathfrak{g}, \mathfrak{a}) \cdot A_0$. This implies that $-A_0$ is also in $\mathcal{O}_{\mathrm{hyp}}^{G}$. q.e.d.

We are ready to prove Theorem 4.3.

Proof of Theorem 4.3. In the setting of Fact 3.6, by Fact 5.1, Lemma 5.2 and Lemma 5.3, we have a bijection between the following two sets:

- the set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$ which meet $\mathfrak{b}_+$ but not $\mathfrak{a}_\mathfrak{h}$,
- the set of real antipodal hyperbolic orbits in $\mathfrak{g}$ that do not meet $\mathfrak{h}$.

Hence, our claim follows from Fact 3.6.   q.e.d.

Proposition 4.8 (ii) is also obtained by Lemma 5.3 as follows:

Proof of Proposition 4.8 (ii). The first claim of Lemma 5.3 means that an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is real antipodal hyperbolic if and only if $\mathcal{O}$ is in $\mathrm{Ad}(G) \cdot \mathfrak{b}_+$. Thus we have $\mathcal{H}^a(\mathfrak{g}) = \mathrm{Ad}(G) \cdot \mathfrak{b}_+$.   q.e.d.

5.2. Lie group homomorphisms from $SL(2, \mathbb{R})$. In this subsection, we prove Proposition 4.2 by using Theorem 4.1.

Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for its Lie algebra. Then the next lemma holds:

Lemma 5.4. *Any Lie algebra homomorphism* $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ *can be uniquely lifted to* $\Phi : SL(2, \mathbb{R}) \to G$ (*i.e.* Φ *is the Lie group homomorphism with its differential* ϕ). *In particular, we have a bijection between the following two sets*:

- *the set of Lie group homomorphism from* $SL(2, \mathbb{R})$ *to* G,
- *the set of* $\mathfrak{sl}_2$-*triples in* $\mathfrak{g}$.

Proof of Lemma 5.4. The uniqueness follows from the connectedness of $SL(2, \mathbb{R})$. We shall lift ϕ. Let us denote by

$$\phi_\mathbb{C} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_\mathbb{C}$$

the complexification of ϕ. Recall that G is linear. Then we can take a complexification $G_\mathbb{C}$ of G. Since $SL(2, \mathbb{C})$ is simply connected, the Lie algebra homomorphism $\phi_\mathbb{C}$ can be lifted to

$$\Phi_\mathbb{C} : SL(2, \mathbb{C}) \to G_\mathbb{C}.$$

Then $\Phi_\mathbb{C}(SL(2, \mathbb{R}))$ is an analytic subgroup of $G_\mathbb{C}$ corresponding to the semisimple subalgebra $\phi(\mathfrak{sl}(2, \mathbb{R}))$ of $\mathfrak{g}$. In particular, $\Phi_\mathbb{C}(SL(2, \mathbb{R}))$ is a closed subgroup of G. Therefore, we can lift ϕ to $\Phi_\mathbb{C}|_{SL(2, \mathbb{R})}$.   q.e.d.

Let H be a reductive subgroup of G (see Definition 3.1) and denote by $\mathfrak{h}$ the Lie algebra of H. To prove Proposition 4.2, it remains to show the following corollary to Theorem 4.1:

Corollary 5.5. *Let* $\Phi : SL(2, \mathbb{R}) \to G$ *be a Lie group homomorphism, and denote its differential by* $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$. *We put*

$$A_\phi := \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}.$$

Then $SL(2, \mathbb{R})$ *acts on* G/H *properly via* Φ *if and only if the real adjoint orbit through* A_ϕ *in* $\mathfrak{g}$ *does not meet* $\mathfrak{h}$.

Proof of Corollary 5.5. Since $\mathfrak{sl}(2,\mathbb{R})$ is simple, we can assume that $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ is injective. We put

$$L := \Phi(SL(2,\mathbb{R})), \quad \mathfrak{l} := \phi(\mathfrak{sl}(2,\mathbb{R})).$$

Then L is a reductive subgroup of G (see Example 3.3). Since ϕ is injective and the center of $SL(2,\mathbb{R})$ is finite, the kernel $\operatorname{Ker}\Phi$ is also finite. Therefore, the action of $SL(2,\mathbb{R})$ on G/H via Φ is proper if and only if the action of L on G/H is proper. By Theorem 4.1, the action of L on G/H is proper if and only if there does not exist a real hyperbolic orbit in $\mathfrak{g}$ meeting both $\mathfrak{h}$ and $\mathfrak{l}$ apart from the zero-orbit. Here, we take $\mathfrak{a}_\mathfrak{l} := \mathbb{R}A_\phi$ as a maximally split abelian subspace of $\mathfrak{l}$. Then, by Lemma 5.2, for any real hyperbolic orbits in $\mathfrak{g}$, if it meets $\mathfrak{l}$, then it also meets $\mathfrak{a}_\mathfrak{l}$. Therefore, the action of $SL(2,\mathbb{R})$ on G/H via Φ is proper if and only if the real adjoint orbit through A_ϕ in $\mathfrak{g}$ does not meet $\mathfrak{h}$.   q.e.d.

6. Weighted Dynkin diagrams of complex adjoint orbits

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra. In this section, we recall some well-known facts for weighted Dynkin diagrams of complex hyperbolic orbits and complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$. We also prove Lemma 4.4, and determine weighted Dynkin diagrams of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

6.1. Weighted Dynkin diagrams of complex hyperbolic orbits.
In this subsection, we recall a parameterization of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ by weighted Dynkin diagrams.

Fix a Cartan subalgebra $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. Let us denote by $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, and define the real form $\mathfrak{j}$ of $\mathfrak{j}_\mathbb{C}$ by

$$\mathfrak{j} := \{\, A \in \mathfrak{j}_\mathbb{C} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \,\}.$$

Then $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ can be regarded as a subset of $\mathfrak{j}^*$. We fix a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then a closed Weyl chamber

$$\mathfrak{j}_+ := \{\, A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \,\}$$

is a fundamental domain of $\mathfrak{j}$ for the action of the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$.

In this setting, the next fact for complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ is well known.

Fact 6.1. Any complex hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}$, and the intersection $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}$ is a single $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$-orbit in $\mathfrak{j}$. In particular, we have one-to-one correspondences below:

$$\mathcal{H}/G_\mathbb{C} \xleftrightarrow{\ 1:1\ } \mathfrak{j}/W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \xleftrightarrow{\ 1:1\ } \mathfrak{j}_+,$$

where $\mathcal{H}/G_\mathbb{C}$ is the set of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{j}/W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ the set of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$-orbits in $\mathfrak{j}$.

Let Π denote the fundamental system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then, for any $A \in \mathfrak{j}$, we can define a map

$$\Psi_A : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(A).$$

We call Ψ_A the weighted Dynkin diagram corresponding to $A \in \mathfrak{j}$, and $\alpha(A)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{j}^*$, the correspondence

(6.1) $$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \ A \mapsto \Psi_A$$

is a linear isomorphism between real vector spaces. In particular, Ψ is bijective. Furthermore,

$$\Psi|_{\mathfrak{j}_+} : \mathfrak{j}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \ A \mapsto \Psi_A$$

is also bijective. We say that a weighted Dynkin diagram is trivial if all weights are zero. Namely, the trivial diagram corresponds to the zero of $\mathfrak{j}$ by Ψ.

The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is defined as the weighted Dynkin diagram corresponding to the unique element $A_\mathcal{O}$ in $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}_+$ (see Fact 6.1). Combining Fact 6.1 with the bijection $\Psi|_{\mathfrak{j}_+}$, the map

$$\mathcal{H}/G_\mathbb{C} \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad \mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \mapsto \Psi_{A_\mathcal{O}}$$

is also bijective.

6.2. Weighted Dynkin diagrams of complex antipodal hyperbolic orbits.
In this subsection, we determine complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ (see Definition 2.3) by describing the weighted Dynkin diagrams.

We consider the same setting as in Section 6.1. Let us denote by $w_0^\mathbb{C}$ the longest element of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ corresponding to the positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then, by the action of $w_0^\mathbb{C}$, every element in $\mathfrak{j}_+$ moves to $-\mathfrak{j}_+ := \{-A \mid A \in \mathfrak{j}_+\}$. In particular,

$$-w_0^\mathbb{C} : \mathfrak{j} \to \mathfrak{j}, \quad A \mapsto -(w_0^\mathbb{C} \cdot A)$$

is an involutive automorphism on $\mathfrak{j}$ preserving $\mathfrak{j}_+$. We put

$$\mathfrak{j}^{-w_0^\mathbb{C}} := \{A \in \mathfrak{j} \mid -w_0^\mathbb{C} \cdot A = A\}, \quad \mathfrak{j}_+^{-w_0^\mathbb{C}} := \mathfrak{j}_+ \cap \mathfrak{j}^{-w_0^\mathbb{C}}.$$

We recall that any complex hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}_+$ with a unique element $A_\mathcal{O}$ in $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}_+$ (see Fact 6.1). Then the lemma below holds:

Lemma 6.2. *A complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is antipodal if and only if the corresponding element $A_\mathcal{O}$ is in $\mathfrak{j}_+^{-w_0^\mathbb{C}}$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}^a/G_\mathbb{C} \xleftrightarrow{1:1} \mathfrak{j}_+^{-w_0^\mathbb{C}},$$

where $\mathcal{H}^a/G_\mathbb{C}$ is the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

Proof of Lemma 6.2. The proof parallels that of Lemma 5.3.   q.e.d.

Recall that the map

$$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \quad A \mapsto \Phi_A$$

is a linear isomorphism (see Section 6.1). Thus $-w_0^\mathbb{C}$ induces an involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$. By using this endomorphism, the following theorem gives a classification of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

Theorem 6.3. *Let ι denote the involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$ induced by $-w_0^\mathbb{C}$. Then the following holds*:

(i) *A complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is antipodal if and only if the weighted Dynkin diagram of $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ (see Section 6.1 for the notation) is held invariant by ι. In particular, we have a one-to-one correspondence*
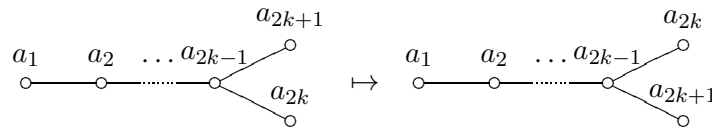
$$\mathcal{H}^a/G_\mathbb{C} \xleftrightarrow{1:1} \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}) \mid \Psi_A \text{ is held invariant by } \iota \,\}.$$

(ii) *Suppose $\mathfrak{g}_\mathbb{C}$ is simple. Then the endomorphism ι is non-trivial if and only if $\mathfrak{g}_\mathbb{C}$ is of type A_n, D_{2k+1}, or E_6 ($n \geq 2$, $k \geq 2$). In such cases, the forms of ι are:*
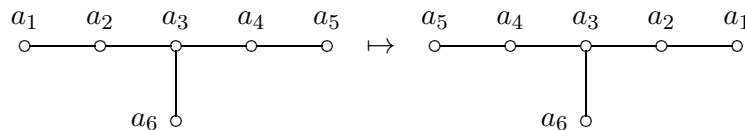   **For type A_n ($n \geq 2$, $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{sl}(n+1, \mathbb{C})$):**
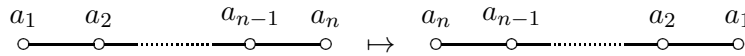
   For type D_{2k+1} ($k \geq 2$, $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{so}(4k+2, \mathbb{C})$):

   For type E_6 ($\mathfrak{g}_\mathbb{C} \simeq \mathfrak{e}_{6,\mathbb{C}}$):

It should be noted that for the cases where $\mathfrak{g}_\mathbb{C}$ is of type D_{2k} ($k \geq 2$), the involution ι on weighted Dynkin diagrams is trivial although the Dynkin diagram of type D_{2k} admits some involutive automorphisms.

Proof of Theorem 6.3. The first claim of the theorem follows from Lemma 6.2. One can easily show that the involutive endomorphism ι on $\mathrm{Map}(\Pi, \mathbb{R})$ is induced by the opposition involution on the Dynkin diagram with nodes Π, which is defined by

$$\Pi \to \Pi, \quad \alpha \mapsto -(w_0^{\mathbb{C}})^* \cdot \alpha.$$

Suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple. Then the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ is irreducible. It is known that the opposition involution is non-trivial if and only if $\mathfrak{g}_{\mathbb{C}}$ is of type A_n, D_{2k+1}, or E_6 ($n \geq 2$, $k \geq 2$) (see J. Tits [**42**, Section 1.5.1]), and the proof is complete.   q.e.d.

As a corollary to Theorem 6.3, we have the following:

Corollary 6.4. *If the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has no simple factor of type A_n, D_{2k+1}, or E_6 ($n \geq 2$, $k \geq 2$), then any complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ is antipodal. Namely, $\mathcal{H}/G_{\mathbb{C}} = \mathcal{H}^a/G_{\mathbb{C}}$.*

By Corollary 6.4, in Setting 2.1, if $\mathfrak{g}_{\mathbb{C}}$ has no simple factor of type A_n, D_{2k+1}, or E_6 ($n \geq 2$, $k \geq 2$), then the condition (viii) in Theorem 2.2 and the condition (vii) in Fact 2.6 are equivalent.

6.3. Weighted Dynkin diagrams of complex nilpotent orbits.
We consider the setting in Section 6.1, and use the notation $\mathcal{H}^n$ and $\mathcal{H}^n/G_{\mathbb{C}}$ as in Section 4.3. In this subsection, we prove Lemma 4.4, and recall weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$.

First, we prove Lemma 4.4, which claims that $\mathcal{H}^n \subset \mathcal{H}^a$, as follows:

Proof of Lemma 4.4. For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, it is well known that $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}(A) \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ is diagonalizable with only real integral numbers. Hence, A is hyperbolic in $\mathfrak{g}_{\mathbb{C}}$. We shall prove that the orbit $\mathcal{O}_A^{G_{\mathbb{C}}} := \mathrm{Ad}(G_{\mathbb{C}}) \cdot A$ is antipodal. We can easily check that the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathfrak{sl}(2, \mathbb{C})$$

are conjugate under the adjoint action of $SL(2, \mathbb{C})$. Then, for a Lie algebra homomorphism $\phi_{\mathbb{C}} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ with

$$\phi_{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A,$$

the elements A and $-A$ are conjugate under the adjoint action of the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\phi_{\mathbb{C}}(\mathfrak{sl}(2, \mathbb{C}))$. Hence, the orbit $\mathcal{O}_A^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ is antipodal.   q.e.d.

Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{N}/G_{\mathbb{C}}$ the set of nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$. For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, the element A is in $\mathcal{H}^n (\subset \mathcal{H}^a)$ and the elements X, Y are both in $\mathcal{N}$. Let us consider the map from the conjugacy classes of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ by inner

automorphisms of $\mathfrak{g}_{\mathbb{C}}$ to $\mathcal{N}/G_{\mathbb{C}}$ defined by

$$[(A,X,Y)] \mapsto \mathcal{O}_X^{G_{\mathbb{C}}}$$

where $[(A,X,Y)]$ is the conjugacy class of an $\mathfrak{sl}_2$-triple (A,X,Y) in $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{O}_X^{G_{\mathbb{C}}}$ the complex adjoint orbit through X in $\mathfrak{g}_{\mathbb{C}}$. Then, by the Jacobson–Morozov theorem, with a result in B. Kostant [25], the map is bijective. On the other hand, by A. I. Malcev [29], the map from the conjugacy classes of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$ to $\mathcal{H}^n/G_{\mathbb{C}}$ defined by

$$[(A,X,Y)] \mapsto \mathcal{O}_A^{G_{\mathbb{C}}}$$

is also bijective, where $\mathcal{O}_A^{G_{\mathbb{C}}}$ is the complex adjoint orbit through A in $\mathfrak{g}_{\mathbb{C}}$. Therefore, we have a one-to-one correspondence

$$\mathcal{N}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \mathcal{H}^n/G_{\mathbb{C}}.$$

In particular, by combining the argument above with Fact 6.1, we also obtain a bijection:

$$\mathcal{N}/G_{\mathbb{C}} \to \mathfrak{j}_+ \cap \mathcal{H}^n, \quad \mathcal{O}_{\text{nilp}}^{G_{\mathbb{C}}} \mapsto A_{\mathcal{O}},$$

where $A_{\mathcal{O}}$ is the unique element of $\mathfrak{j}_+$ with this property: there exist $X,Y \in \mathcal{O}_{\text{nilp}}^{G_{\mathbb{C}}}$ such that $(A_{\mathcal{O}},X,Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$.

Remark 6.5. It is known that the Jacobson–Morozov theorem and the result of Kostant in [25] also hold for any real semisimple Lie algebra $\mathfrak{g}$. Therefore, we have a surjective map from the set of real nilpotent orbits in $\mathfrak{g}$ to $\mathcal{H}^n(\mathfrak{g})/G$, where $\mathcal{H}^n(\mathfrak{g})/G$ is the notation in Section 4.3. However, in general, the map is not injective.

The weighted Dynkin diagram of a complex nilpotent orbit $\mathcal{O}_{\text{nilp}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ is defined as the weighted Dynkin diagram corresponding to $A_{\mathcal{O}} \in \mathfrak{j}_+ \cap \mathcal{H}^n$. Obviously, the weighted Dynkin diagram of $\mathcal{O}_{\text{nilp}}^{G_{\mathbb{C}}}$ is the same as the weighted Dynkin diagram of the corresponding orbit in $\mathcal{H}^n/G_{\mathbb{C}}$.

E. B. Dynkin [10] proved that any weight of a weighted Dynkin diagram of any complex adjoint orbit in $\mathcal{H}^n/G_{\mathbb{C}}$ is 0, 1, or 2. Hence, $\mathcal{H}^n/G_{\mathbb{C}}$ is (and therefore $\mathcal{N}/G_{\mathbb{C}}$ is) finite. Dynkin [10] gave a list of the weighted Dynkin diagrams of $\mathcal{H}^n/G_{\mathbb{C}}$ as the classification of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$. This also gives a classification of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ (see Bala–Cater [4] or Collingwood–McGovern [8, Section 3] for more details).

We remark that by combining Theorem 6.3 with Lemma 4.4, if $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{so}(4k+2,\mathbb{C})$, or $\mathfrak{e}_{6,\mathbb{C}}$ ($n \geq 2,\ k \geq 2$), then the weighted Dynkin diagram of any complex adjoint orbit in $\mathcal{H}^n/G_{\mathbb{C}}$ (and therefore the weighted Dynkin diagram of any complex nilpotent orbit) is invariant under the non-trivial involution ι.

Table 3. Classification of complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$

Partition	Weighted Dynkin diagram				
$[6]$	2	2	2	2	2
$[5,1]$	2	2	0	2	2
$[4,2]$	2	0	2	0	2
$[4,1^2]$	2	1	0	1	2
$[3^2]$	0	2	0	2	0
$[3,2,1]$	1	1	0	1	1
$[3,1^3]$	2	0	0	0	2
$[2^3]$	0	0	2	0	0
$[2^2,1^2]$	0	1	0	1	0
$[2,1^4]$	1	0	0	0	1
$[1^6]$	0	0	0	0	0

Example 6.6. It is known that there exists a bijection between complex nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$ and partitions of n (see [**8**, Section 3.1 and 3.6]). Table 3 presents the list of weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$ (i.e. the list of weighted Dynkin diagrams corresponding to $\mathfrak{j}_+ \cap \mathcal{H}^n$ for the case where $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(6,\mathbb{C})$)

7. Complex adjoint orbits and real forms

Let $\mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_\mathbb{C}$. Recall that, in Section 6, we have a parameterization of complex hyperbolic [resp. antipodal hyperbolic, nilpotent] orbits in $\mathfrak{g}_\mathbb{C}$ by weighted Dynkin diagrams. In this section, we also determine complex hyperbolic [resp. antipodal hyperbolic, nilpotent] orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$. For this, we give an algorithm to check whether or not a given complex hyperbolic [resp. nilpotent] orbit in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$. We also prove Proposition 4.5 and Proposition 4.8 (i) in this section.

7.1. Complex hyperbolic orbits and real forms. We give a proof of Proposition 4.5 (i) in this subsection.

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and use the following convention:

Definition 7.1. We say that a Cartan subalgebra $\mathfrak{j}_\mathfrak{g}$ of $\mathfrak{g}$ is split if $\mathfrak{a} := \mathfrak{j}_\mathfrak{g} \cap \mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$ (i.e. $\mathfrak{a}$ is a maximally split abelian subspace of $\mathfrak{g}$).

Note that such $\mathfrak{j}_\mathfrak{g}$ is unique up to the adjoint action of K, where K is the analytic subgroup of G corresponding to $\mathfrak{k}$.

Take a split Cartan subalgebra $\mathfrak{j}_\mathfrak{g}$ of $\mathfrak{g}$ in Definition 7.1. Then $\mathfrak{j}_\mathfrak{g}$ can be written as $\mathfrak{j}_\mathfrak{g} = \mathfrak{t} + \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{t}$ of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let us denote $\mathfrak{j}_\mathbb{C} := \mathfrak{j}_\mathfrak{g} + \sqrt{-1}\mathfrak{j}_\mathfrak{g}$ and $\mathfrak{j} := \sqrt{-1}\mathfrak{t} + \mathfrak{a}$. Then $\mathfrak{j}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{j}$ is a real form of it, with

$$\mathfrak{j} = \{A \in \mathfrak{j}_\mathbb{C} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})\},$$

where $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ is the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. We put

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})\} \setminus \{0\} \subset \mathfrak{a}^*$$

to the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. Then we can take a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ such that the subset

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})\} \setminus \{0\}$$

of $\Sigma(\mathfrak{g}, \mathfrak{a})$ becomes a positive system. In fact, if we take an ordering on $\mathfrak{a}$ and extend it to $\mathfrak{j}$, then the corresponding positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ satisfies the condition above. Let us denote by $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $W(\mathfrak{g}, \mathfrak{a})$ the Weyl groups of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $\Sigma(\mathfrak{g}, \mathfrak{a})$, respectively. We put the closed Weyl chambers

$$\mathfrak{j}_+ := \{A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})\},$$

$$\mathfrak{a}_+ := \{A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}.$$

Then $\mathfrak{j}_+$ and $\mathfrak{a}_+$ are fundamental domains of $\mathfrak{j}$, $\mathfrak{a}$ for the actions of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ and $W(\mathfrak{g}, \mathfrak{a})$, respectively. By the definition of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, we have $\mathfrak{a}_+ = \mathfrak{j}_+ \cap \mathfrak{a}$.

We recall that any complex hyperbolic orbit $\mathcal{O}_{\text{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}_+$ with a unique element $A_\mathcal{O}$ in $\mathcal{O}_{\text{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}_+$ (see Fact 6.1). Then the lemma below holds:

Lemma 7.2. *A complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$ if and only if the corresponding element $A_\mathcal{O}$ is in $\mathfrak{a}_+$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}_\mathfrak{g}/G_\mathbb{C} \overset{1:1}{\longleftrightarrow} \mathfrak{a}_+,$$

where $\mathcal{H}_\mathfrak{g}/G_\mathbb{C}$ is the set of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$.

Lemma 7.2 will be used in Section 7.2 to prove Theorem 7.4. We now prove Proposition 4.5 (i) and Lemma 7.2 simultaneously.

Proof of Proposition 4.5 (i) *and Lemma* 7.2. We show that for a complex hyperbolic orbit $\mathcal{O}_{\text{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$, the element $A_\mathcal{O}$ is in $\mathfrak{a}_+$ if $\mathcal{O}_{\text{hyp}}^{G_\mathbb{C}}$ meets $\mathfrak{g}$. Note that an element of $\mathfrak{g}$ is hyperbolic in $\mathfrak{g}$ (see Definition 2.3) if

and only if hyperbolic in $\mathfrak{g}_{\mathbb{C}}$. Thus any real adjoint orbit $\mathcal{O}'$ contained in $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is hyperbolic, and hence $\mathcal{O}'$ meets $\mathfrak{a}_+$ with a unique element $A_0 \in \mathcal{O}' \cap \mathfrak{a}_+$ by Fact 5.1. Since $\mathfrak{a}_+$ is contained in $\mathfrak{j}_+$, the element A_0 is in $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{j}_+$. Thus, $A_0 = A_{\mathcal{O}}$. Therefore, we obtain that $A_{\mathcal{O}}$ is in $\mathfrak{a}_+$ for any $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \in \mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}}$, which completes the proof of Lemma 7.2.

To prove Proposition 4.5 (i), it suffices to show that the intersection $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ becomes a single adjoint orbit. By the argument above, we have

$$\mathrm{Ad}(G) \cdot A_{\mathcal{O}} = \mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g},$$

and hence Proposition 4.5 (i) follows.   q.e.d.

7.2. Weighted Dynkin diagrams and Satake diagrams. Let us consider the setting in Section 7.1. In this subsection, we determine complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ by using the Satake diagram of $\mathfrak{g}$.

First, we recall briefly the definition of the Satake diagram of the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ (see [**2**, **36**] for more details). Let us denote by Π the fundamental system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then

$$\overline{\Pi} := \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \,\} \setminus \{0\}$$

is the fundamental system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. We write Π_0 for the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero. The Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ consists of the following data: the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π, black nodes Π_0 in S, and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same.

Second, we give the definition of weighted Dynkin diagrams *matching* the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 7.3. Let $\Psi_A \in \mathrm{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ (see Section 6.1 for the notation) and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π. We say that Ψ_A *matches* $S_{\mathfrak{g}}$ if all the weights on black nodes in Π_0 are zero and any pair of nodes joined by an arrow have the same weights.

Then the following theorem holds:

Theorem 7.4. *The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ matches the Satake diagram of $\mathfrak{g}$ if and only if $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}) \mid \Psi_A \ matches \ S_{\mathfrak{g}} \,\}.$$

Recall that Ψ is a linear isomorphism from $\mathfrak{j}$ to $\mathrm{Map}(\Pi, \mathbb{R})$ (see (6.1) in Section 6.1 for the notation), and there exists a one-to-one correspondence between $\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}}$ and $\mathfrak{a}_+$ (see Lemma 7.2). Therefore, to prove Theorem 7.4, it suffices to show the next lemma:

Lemma 7.5. *The linear isomorphism* $\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R})$ *induces a linear isomorphism*

$$\mathfrak{a} \to \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_A \ \text{matches} \ S_{\mathfrak{g}} \,\}, \quad A \mapsto \Psi_A.$$

Proof of Lemma 7.5. Let $A \in \mathfrak{j}$. By Definition 7.3, the weighted Dynkin diagram Ψ_A matches the Satake diagram of $\mathfrak{g}$ if and only if A satisfies the following condition $(\star)$:

$$(\star) \begin{cases} \alpha(A) = 0 \quad \text{(for any } \alpha \in \Pi_0), \\ \alpha(A) = \beta(A) \quad \text{(for any } \alpha, \beta \in \Pi \setminus \Pi_0 \text{ with } \alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}). \end{cases}$$

Thus, it suffices to show that the subspace

$$\mathfrak{a}' := \{\, A \in \mathfrak{j} \mid A \ \text{satisfies the condition} \ (\star) \,\}$$

of $\mathfrak{j}$ coincides with $\mathfrak{a}$. It is easy to check that $\mathfrak{a} \subset \mathfrak{a}'$. We now prove that $\dim_{\mathbb{R}} \mathfrak{a} = \dim_{\mathbb{R}} \mathfrak{a}'$. Recall that $\overline{\Pi}$ is a fundamental system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. In particular, $\overline{\Pi}$ is a basis of $\mathfrak{a}^*$. Thus, $\dim_{\mathbb{R}} \mathfrak{a} = \sharp\overline{\Pi}$. We define the element A'_ξ of $\mathfrak{a}'$ for each $\xi \in \overline{\Pi}$ by

$$\alpha(A'_\xi) = \begin{cases} 1 \quad \text{(if } \alpha|_{\mathfrak{a}} = \xi), \\ 0 \quad \text{(if } \alpha|_{\mathfrak{a}} \neq \xi), \end{cases}$$

for any $\alpha \in \Pi$. Then $\{\, A'_\xi \mid \xi \in \overline{\Pi} \,\}$ is a basis of $\mathfrak{a}'$ since

$$\overline{\Pi} = \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \,\} \setminus \{0\}.$$

Thus, $\dim_{\mathbb{R}} \mathfrak{a}' = \sharp\overline{\Pi}$, and hence $\mathfrak{a} = \mathfrak{a}'$. q.e.d.

7.3. Complex antipodal hyperbolic orbits and real forms. We consider the setting in Section 7.1 and Section 7.2. In this subsection, the proof of Proposition 4.5 (ii) is given. Related to the proof of Proposition 4.6 (i), which will be given in Section 7.5, we also determine the subset $\mathfrak{b}$ of $\mathfrak{a}$ (see Section 3.2 for the notation) by describing the weighted Dynkin diagrams in this subsection.

First, we prove Proposition 4.5 (ii), which gives a bijection between complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ and real antipodal hyperbolic orbits in $\mathfrak{g}$, as follows:

Proof of Proposition 4.5 (ii). Note that Proposition 4.5 (i) has been proved already in Section 7.1. Therefore, to prove Proposition 4.5 (ii), it remains to show that for any $\mathcal{O}^{G_{\mathbb{C}}} \in \mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}}$ and any element A of $\mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{g}$, the element $-A$ is also in $\mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{g}$. Since $\mathcal{O}^{G_{\mathbb{C}}}$ is antipodal, the element $-A$ is also in $\mathcal{O}^{G_{\mathbb{C}}}$. Hence, we have $-A \in \mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{g}$. q.e.d.

Recall that we have bijections between $\mathcal{H}^a/G_{\mathbb{C}}$ and $\mathfrak{j}_+^{-w_0^{\mathbb{C}}}$ (see Lemma 6.2) and between $\mathcal{H}^a(\mathfrak{g})/G$ and $\mathfrak{b}_+$ (see Lemma 5.3). By Proposition 4.5 (ii), which has been proved above, we have one-to-one correspondences

$$\mathfrak{b}_+ \xleftrightarrow{\text{1:1}} \mathcal{H}^a(\mathfrak{g})/G \xleftrightarrow{\text{1:1}} \mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}},$$

where $\mathcal{H}_{\mathfrak{g}}^a/G_{\mathbb{C}}$ is the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$.

To explain the relation between $\mathfrak{j}_+^{-w_0^{\mathbb{C}}}$ and $\mathfrak{b}_+$, we show the following lemma:

Lemma 7.6. *Let $w_0^{\mathbb{C}}$, w_0 be the longest elements of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, $W(\mathfrak{g}, \mathfrak{a})$ with respect to the positive systems $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, respectively. Then:*

$$\mathfrak{b} = \mathfrak{j}^{-w_0^{\mathbb{C}}} \cap \mathfrak{a}, \quad \mathfrak{b}_+ = \mathfrak{j}_+^{-w_0^{\mathbb{C}}} \cap \mathfrak{a},$$

where $\mathfrak{b} = \{ A \in \mathfrak{a} \mid -w_0 \cdot A = A \}$ and $\mathfrak{j}^{-w_0^{\mathbb{C}}} = \{ A \in \mathfrak{j} \mid -w_0^{\mathbb{C}} \cdot A = A \}$.

Proof of Lemma 7.6. We only need to show that $w_0^{\mathbb{C}}$ preserves $\mathfrak{a}$ and the action on $\mathfrak{a}$ is the same as w_0. Let us put τ to the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. Then we can easily check that both Π and $-\Pi$ are τ-fundamental systems of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ in the sense of [**36**, Section 1.1]. Since $(w_0^{\mathbb{C}})^* \cdot \Pi = -\Pi$, the endomorphism $w_0^{\mathbb{C}}$ is commutative with τ on $\mathfrak{j}$, and $w_0^{\mathbb{C}}$ induces on $\mathfrak{a}$ an element w_0' of $W(\mathfrak{g}, \mathfrak{a})$ by [**36**, Proposition A]. Recall that $\overline{\Pi} = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \}$. Then we have $(w_0')^* \cdot \overline{\Pi} = -\overline{\Pi}$, and hence $w_0' = w_0$.   q.e.d.

Recall that we have a bijection between $\mathfrak{a}$ and the set of weighted Dynkin diagrams matching the Satake diagram of $\mathfrak{g}$ (see Lemma 7.5). Combining with Lemma 7.6, we have a linear isomorphism

$$\mathfrak{b} \to \{ \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_A \text{ is held invariant by } \iota \text{ and matches } S_{\mathfrak{g}} \},$$
$$A \mapsto \Psi_A,$$

where ι is the involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$ defined in Section 6.2. Therefore, we can determine the subsets $\mathfrak{b}$ and $\mathfrak{b}_+$ of $\mathfrak{a}$. Here is an example of the isomorphism for the case where $\mathfrak{g} = \mathfrak{su}(4, 2)$.

Example 7.7. Let $\mathfrak{g} = \mathfrak{su}(4, 2)$. Then the complexification of $\mathfrak{su}(4, 2)$ is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C})$, and the involutive endomorphism ι on weighted Dynkin diagrams is described by

The Satake diagram of $\mathfrak{g} = \mathfrak{su}(4, 2)$ is here:
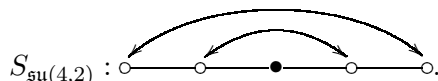
Therefore, we have a linear isomorphism

In particular, we have one-to-one correspondences below:

$$\mathcal{H}_{\mathfrak{g}}^a/G_{\mathbb{C}} \xleftrightarrow{\;1:1\;} \mathfrak{b}_+ \xleftrightarrow{\;1:1\;} \left\{ \begin{array}{c} \underset{a \quad\;\; b \quad\;\; 0 \quad\;\; b \quad\;\; a}{\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ} \end{array} \;\middle|\; a,b \in \mathbb{R}_{\geq 0} \right\}.$$

7.4. Complex nilpotent orbits and real forms.
Let us consider the setting in Section 7.1 and Section 7.2. In this subsection, we introduce an algorithm to check whether or not a given complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ meets the real form $\mathfrak{g}$. In this subsection, we also prove Proposition 4.5 (iii).

First, we show the next proposition:

Proposition 7.8 (Corollary to J. Sekiguchi [**38**, Proposition 1.11])**.** *Let (A, X, Y) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$. Then the following conditions on (A, X, Y) are equivalent*:

(i) *The complex adjoint orbit through X meets $\mathfrak{g}$.*
(ii) *The complex adjoint orbit through A meets $\mathfrak{g}$.*
(iii) *The complex adjoint orbit through X meets $\mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{p}_{\mathbb{C}}$ is the complexification of $\mathfrak{p}$.*
(iv) *The complex adjoint orbit through A meets $\mathfrak{p}_{\mathbb{C}}$.*
(v) *There exists an $\mathfrak{sl}_2$-triple (A', X', Y') in $\mathfrak{g}$ such that A' is in the complex adjoint orbit through A.*
(vi) *The weighted Dynkin diagram of the complex adjoint orbit through X matches the Satake diagram of $\mathfrak{g}$.*

Proof of Proposition 7.8. The equivalences between (i), (iii), and (iv) were proved by [**38**, Proposition 1.11]. The equivalence (iv) $\Leftrightarrow$ (ii) is obtained by the fact that $\mathcal{H}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{a}} = \mathcal{H}_{\mathfrak{p}_{\mathbb{C}}}$ (cf. Lemma 7.2 and the proof of [**38**, Proposition 1.11]). The equivalence (ii) $\Leftrightarrow$ (vi) is obtained by combining Theorem 7.4 with the observation that the weighted Dynkin diagram of the complex adjoint orbit through X is the same as the weighted Dynkin diagram of the complex adjoint orbit through A (see Section 6.3). The implication (ii) $\Rightarrow$ (v) can be obtained by the lemma below. q.e.d.

Lemma 7.9. *Let (A, X, Y) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$. Then the following holds*:

(i) *If A is in $\mathfrak{g}$, then there exists $g \in G_{\mathbb{C}}$ such that $\mathrm{Ad}(g) \cdot A = A$ and $\mathrm{Ad}(g) \cdot X$ is in $\mathfrak{g}$.*
(ii) *If both A and X are in $\mathfrak{g}$, then Y is automatically in $\mathfrak{g}$.*

Proof of Lemma 7.9. (i): See the proof of [**38**, Proposition 1.11]. (ii): Easy. q.e.d.

Here is a proof of Proposition 4.5 (iii), which gives a bijection between $\mathcal{H}_{\mathfrak{g}}^n/G_{\mathbb{C}}$ and $\mathcal{H}^n(\mathfrak{g})/G$ (see Section 4.3 for the notation):

Proof of Proposition 4.5 (iii)*.* We recall that Proposition 4.5 (i) has been proved already in Section 7.1. Then Proposition 4.5 (iii) follows from the implication (ii) $\Rightarrow$ (v) in Proposition 7.8.   q.e.d.

Recall that we have the one-to-one correspondence

$$\mathfrak{j}_+ \cap \mathcal{H}^n \xleftrightarrow{\ 1:1\ } \mathcal{N}/G_\mathbb{C},$$

where $\mathcal{N}/G_\mathbb{C}$ is the set of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$ (see Section 6.3). Combining Lemma 7.2 with Proposition 7.8, we also obtain

$$\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}) = (\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a} \xleftrightarrow{\ 1:1\ } \mathcal{N}_\mathfrak{g}/G_\mathbb{C},$$

where $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$ is the set of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$. Therefore, by Lemma 7.5, we obtain the theorem below:

Theorem 7.10. *Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_\mathbb{C}$. Then for a complex nilpotent orbit $\mathcal{O}_{nilp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$, the following two conditions are equivalent*:

(i) $\mathcal{O}_{nilp}^{G_\mathbb{C}} \cap \mathfrak{g} \neq \emptyset$ *(i.e. $\mathcal{O}_{nilp}^{G_\mathbb{C}} \in \mathcal{N}_\mathfrak{g}/G_\mathbb{C}$).*

(ii) *The weighted Dynkin diagram of $\mathcal{O}_{nilp}^{G_\mathbb{C}}$ matches the Satake diagram $S_\mathfrak{g}$ of $\mathfrak{g}$ (see Section 7.2 for the notation).*

Remark 7.11.   **(1):** The same concept as Definition 7.3 appeared earlier as "weighted Satake diagrams" in D. Z. Djokovic [**9**] and as the condition described in J. Sekiguchi [**37**, Proposition 1.16]. We call it a "match."

(2): J. Sekiguchi [**38**, Proposition 1.13] showed the implication (ii) $\Rightarrow$ (i) in Theorem 7.10. Our theorem claims that (i) $\Rightarrow$ (ii) is also true.

We give three examples of Theorem 7.10:

Example 7.12. Let $\mathfrak{g}$ be a split real form of $\mathfrak{g}_\mathbb{C}$. Then all nodes of the Satake diagram $S_\mathfrak{g}$ are white with no arrow. Thus, all weighted Dynkin diagrams match the Satake diagram of $\mathfrak{g}$. Therefore, all complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$ meet $\mathfrak{g}$.

Example 7.13. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_\mathbb{C}$. Then all nodes of the Satake diagram $S_\mathfrak{u}$ are black. Thus, no weighted Dynkin diagram matches the Satake diagram of $\mathfrak{u}$ except for the trivial one. Therefore, no complex nilpotent orbit in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{u}$ except for the zero-orbit.

Example 7.14. Let $(\mathfrak{g}_\mathbb{C}, \mathfrak{g}) = (\mathfrak{sl}(6, \mathbb{C}), \mathfrak{su}(4, 2))$. The Satake diagram of $\mathfrak{su}(4, 2)$ was given in Example 7.7. Then, by combining with Example 6.6, we obtain the list of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$ (i.e. the list of $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$) as follows:

$$\mathcal{N}_\mathfrak{g}/G_\mathbb{C} \xleftrightarrow{\ 1:1\ } \{\, [5, 1],\ [4, 1^2],\ [3^2],\ [3, 2, 1],\ [3, 1^3],\ [2^2, 1^2],\ [2, 1^4],\ [1^6] \,\}.$$

7.5. Proof of Proposition 4.8 (i)**.** In this subsection, we first explain the strategy of the proof of Proposition 4.8 (i), and then illustrate actual computations by an example.

By Lemma 5.3, we have

$$\mathfrak{b}_+ \supset \mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}).$$

Furthermore, in Section 7.4, we also obtained

$$\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}) = (\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}.$$

Therefore, the proof of Proposition 4.8 (i) is reduced to showing

(7.1) $$\mathfrak{b} \subset \mathbb{R}\text{-span}((\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a})$$

for all simple Lie algebras $\mathfrak{g}$.

In order to show (7.1), we recall the Dynkin–Kostant classification of weighted Dynkin diagrams corresponding to elements of $\mathfrak{j}_+ \cap \mathcal{H}^n$ (which gives a classification of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$; see Section 6.3) As its subset, we can classify the weighted Dynkin diagrams corresponding to elements in $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ by using the Satake diagram of $\mathfrak{g}$ (cf. Example 7.14). What we need to prove for (7.1) is that this subset contains sufficiently many in the sense that the $\mathbb{R}$-span of the weighted Dynkin diagrams corresponding to this subset coincides with the space of weighted Dynkin diagrams corresponding to elements in $\mathfrak{b}$. Recall that we can also determine such space corresponding to $\mathfrak{b}$ by the involution ι on weighted Dynkin diagrams (see Section 6.2 for the notation) with the Satake diagram of $\mathfrak{g}$ (cf. Example 7.7).

We illustrate this strategy by the following example:

Example 7.15. We give a proof of Proposition 4.8 (i) for the case where $\mathfrak{g} = \mathfrak{su}(4, 2)$, with its complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(6, \mathbb{C})$.

By Example 7.14, we have the list of weighted Dynkin diagrams corresponding to elements of $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ for $\mathfrak{g} = \mathfrak{su}(4, 2)$. Table 4 shows part of it.

By Example 7.7, we also have a linear isomorphism

$$\mathfrak{b} \xrightarrow{\sim} \left\{ \begin{array}{c} a \quad b \quad 0 \quad b \quad a \\ \circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ \end{array} \mid a, b \in \mathbb{R} \right\}.$$

Table 4. A part of $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ for $\mathfrak{g} = \mathfrak{su}(4, 2)$

Partition	Weighted Dynkin diagram
$[2^2, 1^2]$	$\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}$
$[2, 1^4]$	$\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$

Hence, we can observe that

$$\mathfrak{b} \subset \mathbb{R}\text{-span}((\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}).$$

This completes the proof of Proposition 4.8 (i) for the case where $\mathfrak{g} = \mathfrak{su}(4,2)$.

For the other simple Lie algebras $\mathfrak{g}$, we can find the Satake diagram of $\mathfrak{g}$ in [**2**] or [**12**, Chapter X, Section 6] and the classification of weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ in [**4**]. Then we can verify (7.1) in the spirit of case-by-case computations for other real simple Lie algebras. Detailed computations will be reported elsewhere.

8. Symmetric pairs

In this section, we prove Proposition 4.6 and Lemma 4.9.

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair and write σ for the involution on $\mathfrak{g}$ corresponding to $\mathfrak{h}$. First, we give Cartan decompositions on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{g}^c$ (see (2.1) in Section 2 for the notation), simultaneously.

Recall that we can find a Cartan involution θ on $\mathfrak{g}$ with $\sigma\theta = \theta\sigma$ (cf. [**6**]). Let us denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ the Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We set $\mathfrak{u} := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then $\mathfrak{u}$ becomes a compact real form of $\mathfrak{g}_{\mathbb{C}}$. We write τ, τ^c for the complex conjugations on $\mathfrak{g}_{\mathbb{C}}$ with respect to the real forms $\mathfrak{g}$, $\mathfrak{g}^c$, respectively. Then τ^c is the anti $\mathbb{C}$-linear extension of σ on $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$, and hence τ and τ^c are commutative. The compact real form $\mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ is stable under both τ and τ^c. We denote by $\overline{\theta}$ the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ corresponding to $\mathfrak{u}$, i.e. $\overline{\theta}$ is the anti $\mathbb{C}$-linear extension of θ. Then the restriction $\overline{\theta}|_{\mathfrak{g}^c}$ is a Cartan involution on $\mathfrak{g}^c$. We write

(8.1) $$\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$$

for the Cartan decomposition of $\mathfrak{g}^c$ with respect to $\overline{\theta}|_{\mathfrak{g}^c}$.

Let us fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p}(\mathfrak{h})$, and extend it to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ [resp. a maximal abelian subspace $\mathfrak{a}^c$ of $\mathfrak{p}^c$]. Obviously, $\mathfrak{a}_{\mathfrak{h}} = \mathfrak{a} \cap \mathfrak{a}^c$. We show the next lemma below:

Lemma 8.1. $[\mathfrak{a}, \mathfrak{a}^c] = \{0\}$.

The next proposition gives a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which contains split Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{g}^c$, and $\mathfrak{h}$ with respect to the Cartan decompositions.

Proposition 8.2. *There exists a Cartan subalgebra* $\mathfrak{j}_{\mathbb{C}}$ *of* $\mathfrak{g}_{\mathbb{C}}$ *with the following properties*:
- $\mathfrak{j}_{\mathfrak{g}} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{g}$ *is a split Cartan subalgebra of* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (*see Definition 7.1 for the notation*) *with* $\mathfrak{j}_{\mathfrak{g}} \cap \mathfrak{p} = \mathfrak{a}$.
- $\mathfrak{j}_{\mathfrak{g}^c} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{g}^c$ *is a split Cartan subalgebra of* $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$ *with* $\mathfrak{j}_{\mathfrak{g}^c} \cap \mathfrak{p}^c = \mathfrak{a}^c$.

- $\mathfrak{j}_{\mathfrak{h}} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{h}$ *is a split Cartan subalgebra of* $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ *with* $\mathfrak{j}_{\mathfrak{h}} \cap \mathfrak{p}(\mathfrak{h}) = \mathfrak{a}_{\mathfrak{h}}$.

Proof of Lemma 8.1 *and Proposition* 8.2. We put

$$\mathfrak{h}^a := \{\, X \in \mathfrak{g} \mid \theta\sigma X = X \,\}, \quad \mathfrak{q}^a := \{\, X \in \mathfrak{g} \mid \theta\sigma X = -X \,\}.$$

Then $(\mathfrak{g}, \mathfrak{h}^a)$ is the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h})$ (see [**35**, Section 1] for the notation). Note that $\mathfrak{q}^a = \mathfrak{p}^c \cap \mathfrak{g} + \sqrt{-1}(\mathfrak{p}^c \cap \sqrt{-1}\mathfrak{g})$ and $\mathfrak{p} \cap \mathfrak{q}^a = \mathfrak{p}(\mathfrak{h})$. Let us apply [**35**, Lemma 2.4 (i)] to the symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$. Then we have $[\mathfrak{a}, \mathfrak{a}^c] = \{0\}$, since the complexification of $\mathfrak{a}_c$ is a maximal abelian subspace of the complexification of $\mathfrak{q}^a$ containing $\mathfrak{a}_{\mathfrak{h}}$. This completes the proof of Lemma 8.1. Furthermore, let us extend $\mathfrak{a} + \mathfrak{a}^c$ to a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{j}_{\mathbb{C}}$ satisfies the properties in Proposition 8.2. q.e.d.

We fix such a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, and put

$$\mathfrak{j} := \mathfrak{j}_{\mathbb{C}} \cap \sqrt{-1}\mathfrak{u}.$$

Throughout this subsection, we denote the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ briefly by Δ, which is realized in $\mathfrak{j}^*$. Let us denote by Σ, Σ^c the restricted root systems of $(\mathfrak{g}, \mathfrak{a})$, $(\mathfrak{g}^c, \mathfrak{a}^c)$, respectively. Namely, we put

$$\Sigma := \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta \,\} \setminus \{0\} \subset \mathfrak{a}^*,$$

$$\Sigma^c := \{\, \alpha|_{\mathfrak{a}^c} \mid \alpha \in \Delta \,\} \setminus \{0\} \subset (\mathfrak{a}^c)^*.$$

Then we can choose a positive system Δ^+ of Δ with the properties below:

- $\Sigma^+ := \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+ \,\} \setminus \{0\}$ is a positive system of Σ.
- $(\Sigma^c)^+ := \{\, \alpha|_{\mathfrak{a}^c} \mid \alpha \in \Delta^+ \,\} \setminus \{0\}$ is a positive system of Σ^c.

In fact, if we take an ordering on $\mathfrak{a}_{\mathfrak{h}}$ and extend it stepwise to $\mathfrak{a}$, to $\mathfrak{a} + \mathfrak{a}^c$, and to $\mathfrak{j}$, then the corresponding positive system Δ^+ satisfies the properties above (see [**35**, Section 3] for more detail). Let us denote by

$$\mathfrak{j}_+ := \{\, A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+ \,\},$$

$$\mathfrak{a}_+ := \{\, A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+ \,\},$$

$$\mathfrak{a}_+^c := \{\, A \in \mathfrak{a}^c \mid \xi^c(A) \geq 0 \text{ for any } \xi^c \in (\Sigma^c)^+ \,\},$$

the closed Weyl chambers in $\mathfrak{j}$, $\mathfrak{a}$, and $\mathfrak{a}^c$ with respect to Δ^+, Σ^+, and $(\Sigma^c)^+$, respectively.

Combining Fact 6.1 with Lemma 7.2, we obtain the lemma below:

Lemma 8.3. *Let* $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ *be a complex hyperbolic orbit in* $\mathfrak{g}_{\mathbb{C}}$. *Then the following holds*:

(i) *There exists a unique element* $A_{\mathcal{O}}$ *in* $\mathcal{O}_{hyp}^{G_{\mathbb{C}}} \cap \mathfrak{j}_+$.

(ii) $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ *meets* $\mathfrak{g}$ *if and only if* $A_{\mathcal{O}}$ *is in* $\mathfrak{a}_+$.

(iii) $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ *meets* $\mathfrak{g}^c$ *if and only if* $A_{\mathcal{O}}$ *is in* $\mathfrak{a}_+^c$.

We now prove Proposition 4.6 by using Lemma 8.3.

Proof of Proposition 4.6. Let $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$ be a complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ meeting both $\mathfrak{g}$ and $\mathfrak{g}^c$. We shall prove that $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$ also meets $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}^c$. By Lemma 8.3, there exists a unique element $A_{\mathcal{O}} \in \mathfrak{a}_+ \cap \mathfrak{a}_+^c$ with $A_{\mathcal{O}} \in \mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$, and hence our claim follows.   q.e.d.

Lemma 4.9 is proved by using Lemma 8.3 as follows.

Proof of Lemma 4.9. Let us take $A \in \mathfrak{a}_+$ such that $\mathcal{O}_A^G$ meets $\mathfrak{h}$, where $\mathcal{O}_A^G$ is the adjoint orbit in $\mathfrak{g}$ through A. To prove our claim, we only need to show that A is in $\mathfrak{a}_{\mathfrak{h}}$. We denote by $\mathcal{O}_A^{G_{\mathbb{C}}}$ the complexification of $\mathcal{O}_A^G$. Then $\mathcal{O}_A^{G_{\mathbb{C}}}$ is a complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}^c$. Let us extend $\mathfrak{a}_{\mathfrak{h}}$ to a maximal abelian subspace $\mathfrak{a}^c$ of $\mathfrak{p}^c$ (see (8.1) for the notation of $\mathfrak{p}^c$) and take a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ in Proposition 8.2. We also extend the ordering on $\mathfrak{a}$ stepwise to $\mathfrak{a} + \mathfrak{a}^c$ and to $\mathfrak{j}$. Then by Lemma 8.3, the orbit $\mathcal{O}_A^{G_{\mathbb{C}}}$ intersects $\mathfrak{j}_+$ with a unique element $A_{\mathcal{O}}$, and $A_{\mathcal{O}}$ is in $\mathfrak{a}_+ \cap \mathfrak{a}_+^c \subset \mathfrak{a}_{\mathfrak{h}}$. Since A is also in $\mathcal{O}_A^{G_{\mathbb{C}}} \cap \mathfrak{j}_+$, we have $A = A_{\mathcal{O}}$. Hence A is in $\mathfrak{a}_{\mathfrak{h}}$.   q.e.d.

9. Algorithm for classification

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair (see Setting 2.1). In this section, we describe an algorithm to check whether or not $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 2.2, which coincides with the condition (v) in Theorem 1.3. More precisely, we give an algorithm to classify complex antipodal hyperbolic orbits $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g}^c = \emptyset$.

Recall that for any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, we can determine the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$, which is denoted by $\mathcal{H}^a/G_{\mathbb{C}}$, as ι-invariant weighted Dynkin diagrams by Theorem 6.3. Further, for any real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, we can classify complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ by using the Satake diagram of $\mathfrak{g}$ (see Section 7.3).

For a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we can specify another real form $\mathfrak{g}^c$ of $\mathfrak{g}_{\mathbb{C}}$ (see (2.1) in Section 2 for the notation) by the list of [**35**, Section 1], since the symmetric pair $(\mathfrak{g}^c, \mathfrak{h})$ is the same as $(\mathfrak{g}, \mathfrak{h})^{\mathrm{ada}}$. The Satake diagram of the real form $\mathfrak{g}$ [resp. $\mathfrak{g}^c$] of $\mathfrak{g}_{\mathbb{C}}$ can be found in [**2**] or [**12**, Chapter X, Section 6]. Therefore, we can classify the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ [resp. $\mathfrak{g}^c$], which is denoted by $\mathcal{H}_{\mathfrak{g}}^a/G_{\mathbb{C}}$ [resp. $\mathcal{H}_{\mathfrak{g}^c}^a/G_{\mathbb{C}}$]. This provides an algorithm to check whether the condition (viii) in Theorem 2.2 holds or not on $(\mathfrak{g}, \mathfrak{h})$.

Here, we give examples for the cases where $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$ or $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$.

Example 9.1. Let $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$ and $\mathfrak{g}^c = \mathfrak{su}^*(6)$. We shall determine both $\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}}$ and $\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}}$, and prove that $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 2.2.

The involutive endomorphism ι on weighted Dynkin diagrams of $\mathfrak{sl}(6,\mathbb{C})$ (see Section 6.2 for the notation) is given by

$$
\begin{array}{ccccc}
a & b & c & d & e \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\mapsto\;
\begin{array}{ccccc}
e & d & c & b & a \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}.
$$

Thus, by Theorem 6.3, we have the bijection below:

$$
\mathcal{H}^a/G_{\mathbb{C}} \xleftrightarrow{1:1}
\left\{
\begin{array}{ccccc}
a & b & c & b & a \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\Big|\; a,b,c \in \mathbb{R}_{\geq 0}
\right\}.
$$

Here are the Satake diagrams of $\mathfrak{g} = \mathfrak{su}(4,2)$ and $\mathfrak{g}^c = \mathfrak{su}^*(6)$:

$$
S_{\mathfrak{su}(4,2)} : \circ\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ\!\!-\!\!\circ \;,\;
S_{\mathfrak{su}^*(6)} : \bullet\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ\!\!-\!\!\bullet.
$$

Thus, by Theorem 7.4, we obtain the following one-to-one correspondences:

$$
\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}} \xleftrightarrow{1:1}
\left\{
\begin{array}{ccccc}
a & b & 0 & b & a \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\Big|\; a,b \in \mathbb{R}_{\geq 0}
\right\},
$$

$$
\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}} \xleftrightarrow{1:1}
\left\{
\begin{array}{ccccc}
0 & b & 0 & b & 0 \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\Big|\; b \in \mathbb{R}_{\geq 0}
\right\}.
$$

Therefore, the condition (viii) in Theorem 2.2 holds on the symmetric pair $(\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$.

Example 9.2. Let $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$ and $\mathfrak{g}^c = \mathfrak{su}(4,2)$. Thus, by the argument in Example 9.1, we have

$$
\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}} \xleftrightarrow{1:1}
\left\{
\begin{array}{ccccc}
0 & b & 0 & b & 0 \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\Big|\; b \in \mathbb{R}_{\geq 0}
\right\},
$$

$$
\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}} \xleftrightarrow{1:1}
\left\{
\begin{array}{ccccc}
a & b & 0 & b & a \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\;\Big|\; a,b \in \mathbb{R}_{\geq 0}
\right\}.
$$

Therefore, the condition (viii) in Theorem 2.2 does not hold on the symmetric pair $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. However, if we take a complex hyperbolic orbit $\mathcal{O}'$ in $\mathfrak{sl}(6,\mathbb{C})$ corresponding to

$$
\begin{array}{ccccc}
0 & b & 0 & b' & 0 \\
\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ
\end{array}
\qquad (\text{for some } b, b' \in \mathbb{R}_{\geq 0},\; b \neq b'),
$$

then $\mathcal{O}'$ meets $\mathfrak{g}$ but does not meet $\mathfrak{g}^c$. Note that $\mathcal{O}'$ is not antipo-dal. Thus the condition (vii) in Fact 2.6 holds on the symmetric pair $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. In particular, $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h}$.

Combining our algorithm with Berger's classification [**6**], we obtain Table 2 in Section 2. Concerning this, if $\mathfrak{g}_{\mathbb{C}}$ has no simple factor of type A_n, D_{2k+1}, or E_6 ($n \geq 2$, $k \geq 2$), then the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 2.2 if and only if $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h}$ (see Corollary 6.4 and Fact 2.6). Thus we need only consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type A_n, D_{2k+1}, or E_6.

We also remark that for a given semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, by using the Dynkin–Kostant classification [**10**] and Theorem 7.10, we can check whether the condition (vi) in Theorem 2.2 holds or not on $(\mathfrak{g}, \mathfrak{h})$, directly (see also Section 10).

10. Proper actions of $SL(2, \mathbb{R})$ and real nilpotent orbits

In this section, we describe a refinement of the equivalence (i) $\Leftrightarrow$ (vi) in Theorem 2.2, which provides an algorithm to classify proper $SL(2, \mathbb{R})$-actions on a given semisimple symmetric space G/H.

Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for its Lie algebra. By the Jacobson-Morozov theorem and Lemma 5.4, we have a one-to-one correspondence between Lie group homomorphisms $\Phi : SL(2, \mathbb{R}) \to G$ up to inner automorphisms of G and real nilpotent orbits in $\mathfrak{g}$. We denote by $\mathcal{O}_{\Phi}^G$ the real nilpotent orbit corresponding to $\Phi : SL(2, \mathbb{R}) \to G$. Then, by combining Proposition 4.2 and Proposition 4.6 with Lemma 4.7, we obtain the next theorem:

Theorem 10.1. *In Setting 2.1, the following conditions on a Lie group homomorphism $\Phi : SL(2, \mathbb{R}) \to G$ are equivalent:*

(i) *$SL(2, \mathbb{R})$ acts on G/H properly via Φ.*
(ii) *The complex nilpotent orbit $\mathrm{Ad}(G_{\mathbb{C}}) \cdot \mathcal{O}_{\Phi}^G$ in $\mathfrak{g}_{\mathbb{C}}$ does not meet $\mathfrak{g}^c$, where $\mathfrak{g}^c$ is the c-dual of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ (see (2.1) after Setting 2.1).*

In particular, we have the one-to-one correspondence

$$\{\, \Phi : SL(2, \mathbb{R}) \to G \mid SL(2, \mathbb{R}) \ \text{acts on } G/H \ \text{properly via } \Phi \,\}/G$$

$$\overset{1:1}{\longleftrightarrow} \{\, \text{Real nilpotent orbits } \mathcal{O}^G \ \text{in } \mathfrak{g} \mid (\mathrm{Ad}(G_{\mathbb{C}}) \cdot \mathcal{O}^G) \cap \mathfrak{g}^c = \emptyset \,\}.$$

Here is an example concerning Theorem 10.1:

Example 10.2. Let $(G, H) = (SU(4,2), Sp(2,1))$. Then $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}, \mathfrak{g}^c) = (\mathfrak{sl}(6, \mathbb{C}), \mathfrak{su}(4,2), \mathfrak{su}^*(6))$. Let us classify the following set:

(10.1) $\{$ Real nilpotent orbits $\mathcal{O}^G$ in $\mathfrak{su}(4,2)$

$\qquad\qquad \mid$ the complexifications of $\mathcal{O}^G$ do not meet $\mathfrak{su}^*(6)\}$.

Recall that complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$ are parameterized by partitions of 6 and these weighted Dynkin diagrams are listed in Example 6.6. By Theorem 7.10, we can classify the complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$ that meet $\mathfrak{su}(4,2)$ but not $\mathfrak{su}^*(6)$, by using these Satake diagrams (see Example 9.1 for Satake diagrams of $\mathfrak{su}(4,2)$ and $\mathfrak{su}^*(6)$), as follows:

$\{\,$Complex nilpotent orbits $\mathcal{O}^{G_\mathbb{C}}$ in $\mathfrak{sl}(6,\mathbb{C})$

$\mid \mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{su}(4,2) \neq \emptyset$ and $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{su}^*(6) = \emptyset\,\}$

$\xrightarrow{1:1} \{\,[5,1],\ [4,1^2],\ [3,2,1],\ [3,1^3],\ [2,1^4]\,\}.$

It is known that real nilpotent orbits in $\mathfrak{su}(4,2)$ are parameterized by signed Young diagrams of signature $(4,2)$, and the shape of the signed Young diagram corresponding to a real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{su}(4,2)$ is the partition corresponding to the complexification of $\mathcal{O}^G$ (see [**8**, Theorem 9.3.3 and a remark after Theorem 9.3.5] for more details). Therefore, we have a classification of (10.1) as shown in Table 5.

In particular, by Theorem 10.1, there are nine kinds of Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to SU(4,2)$ (up to inner automorphisms of $SU(4,2)$) for which the $SL(2,\mathbb{R})$-actions on $SU(4,2)/Sp(2,1)$ via Φ are proper.

Table 5. Classification of (10.1)

Partition	Signed Young diagram of signature $(4,2)$
$[5,1]$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline + & - & + & - & + \\\hline + \\\cline{1-1}\end{array}$
$[4,1^2]$	$\begin{array}{c}\boxed{+\,-\,+\,-}\\\boxed{+}\\\boxed{+}\end{array}\ ,\ \begin{array}{c}\boxed{-\,+\,-\,+}\\\boxed{+}\\\boxed{+}\end{array}$
$[3,2,1]$	$\begin{array}{c}\boxed{+\,-\,+}\\\boxed{+\,-}\\\boxed{+}\end{array}\ ,\ \begin{array}{c}\boxed{+\,-\,+}\\\boxed{-\,+}\\\boxed{+}\end{array}$
$[3,1^3]$	$\begin{array}{c}\boxed{+\,-\,+}\\\boxed{+}\\\boxed{+}\\\boxed{-}\end{array}\ ,\ \begin{array}{c}\boxed{-\,+\,-}\\\boxed{+}\\\boxed{+}\\\boxed{+}\end{array}$
$[2,1^4]$	$\begin{array}{c}\boxed{+\,-}\\\boxed{+}\\\boxed{+}\\\boxed{+}\\\boxed{-}\end{array}\ ,\ \begin{array}{c}\boxed{-\,+}\\\boxed{+}\\\boxed{+}\\\boxed{+}\\\boxed{-}\end{array}$

Appendix A. Classification

Table 6 presents a complete list of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with the following property:

(A.1) $\mathfrak{g}$ is simple, $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair

     satisfying one of (therefore, all of) the conditions in Theorem 2.2.

Table 6. Classification of $(\mathfrak{g}, \mathfrak{h})$ satisfying (A.1)

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(2k, \mathbb{R})$	$\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{so}(2)$
$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n-i, i)$
	$(2i < n)$
$\mathfrak{su}^*(2k)$	$\mathfrak{sp}(k-i, i)$
	$(2i < k-1)$
$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$
$\mathfrak{su}(2m-1, 2m-1)$	$\mathfrak{so}^*(4m-2)$
$\mathfrak{su}(p, q)$	$\mathfrak{su}(i, j) \oplus \mathfrak{su}(p-i, q-j) \oplus \mathfrak{so}(2)$
	$(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\})$
$\mathfrak{so}(p, q)$	$\mathfrak{so}(i, j) \oplus \mathfrak{so}(p-i, q-j)$
$(p+q$ is odd$)$	$(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\})$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{su}(n-i, i) \oplus \mathfrak{so}(2)$
$\mathfrak{sp}(2k, \mathbb{R})$	$\mathfrak{sp}(k, \mathbb{C})$
$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(i, j) \oplus \mathfrak{sp}(p-i, q-j)$
	$(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\})$
$\mathfrak{so}(p, q)$	$\mathfrak{so}(i, j) \oplus \mathfrak{so}(p-i, q-j)$
$(p+q$ is even$)$	$(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\},$
	unless $p = q = 2m+1$ and $\lvert i-j \rvert = 1)$
$\mathfrak{so}(2p, 2q)$	$\mathfrak{su}(p, q) \oplus \mathfrak{so}(2)$
$\mathfrak{so}^*(2k)$	$\mathfrak{su}(k-i, i) \oplus \mathfrak{so}(2)$
	$(2i < k-1)$
$\mathfrak{so}(k, k)$	$\mathfrak{so}(2k, \mathbb{C}) \oplus \mathfrak{so}(2)$
$\mathfrak{so}^*(4m)$	$\mathfrak{so}^*(4m-4i+2) \oplus \mathfrak{so}^*(4i-2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2, 2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{su}^*(6) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{sp}(3, 1)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(4, 4)$

Classification of $(\mathfrak{g}, \mathfrak{h})$ satisfying (A.1) *(continued)*

$\mathfrak{e}_{7(7)}$	$\mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6,2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6,2)$
$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{8(8)}$	$\mathfrak{so}^*(16)$
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(2,1) \oplus \mathfrak{su}(2)$
$\mathfrak{sl}(2k,\mathbb{C})$	$\mathfrak{su}^*(2k)$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n-i,i)$
	$(2i < n)$
$\mathfrak{so}(2k+1,\mathbb{C})$	$\mathfrak{so}(2k+1-i,i)$
	$(i < k)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n-i,i)$
$\mathfrak{so}(2k,\mathbb{C})$	$\mathfrak{so}(2k-i,i)$
	$(i < k$ unless $k = i+1 = 2m+1)$
$\mathfrak{so}(4m,\mathbb{C})$	$\mathfrak{so}(4m-2i+1,\mathbb{C}) \oplus \mathfrak{so}(2i-1,\mathbb{C})$
$\mathfrak{so}(2k,\mathbb{C})$	$\mathfrak{so}^*(2k)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-26)}$
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{8(-24)}$
$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$

Here, $k \geq 1$, $m \geq 1$, $n \geq 2$, $p,q \geq 1$ and $i,j \geq 0$. Note that $\mathfrak{so}(p,q)$ is simple if and only if $p + q \geq 3$ with $(p,q) \neq (2,2)$, and $\mathfrak{so}(2k,\mathbb{C})$ is simple if and only if $k \geq 3$.

Appendix B. The Calabi–Markus phenomenon and hyperbolic orbits

Here is a proof of the equivalence among (v), (vi), and (vii) in Fact 2.6:

Proof of (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) *in Fact* 2.6. We take $\mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{h}}$ in Section 3.1. The condition (v) means that $\mathfrak{a} \neq W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{h}}$. By Fact 5.1 and Lemma 5.2, we have a bijection between the following two sets:

- the set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$ that do not meet $\mathfrak{a}_{\mathfrak{h}}$,
- the set of real hyperbolic orbits in $\mathfrak{g}$ that do not meet $\mathfrak{h}$.

Then the equivalence (v) $\Leftrightarrow$ (vi) holds. Further, (vi) $\Leftrightarrow$ (vii) follows from Proposition 4.5 (i) and Proposition 4.6.   q.e.d.

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