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VOLUME OPTIMIZATION, NORMAL SURFACES, AND THURSTON'S EQUATION ON TRIANGULATED 3-MANIFOLDS

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Abstract

We propose a finite-dimensional variational principle on triangulated 3-manifolds so that its critical points are related to solutions to Thurston's gluing equation and Haken's normal surface equation. The action functional is the volume.

1. Introduction

1.1. The statement of the main theorem. Given a closed triangulated 3-manifold or pseudo 3-manifold, there are several linear and algebraic systems of equations associated to the triangulation. Beside the homology theory, the most prominent ones are Haken's theory of normal surfaces [**10**], Thurston's algebraic gluing equations for constructing hyperbolic metrics [**31**] using hyperbolic ideal tetrahedra, and the notion of angle structures [**16**, **26**]. The normal surface theory gives a parametrization of essential surfaces in the manifold, and solutions of Thurston's equation produce hyperbolic cone metrics. Thurston used a solution to Thurston's gluing equation to produce a complete hyperbolic metric on the figure-8 knot complement in the earlier stage of formulating his geometrization conjecture. The notion of (Euclidean) angle structures, introduced by Casson, Rivin, and Lackenby for 3-manifolds with torus boundary, is a linearized version of Thurston's equation.

The goal of this paper is to generalize the notion of angle structures introduced by Casson, Rivin [**26**], and Lackenby [**16**] to the circle-valued angle structure (or $\mathbf{S}^1$-angle structure, for short) on any closed triangulated pseudo 3-manifold $(M, \mathbf{T})$. These pseudo 3-manifolds include ideal triangulations of compact 3-manifolds even with non-torus boundary. Using the method introduced in [**19**], we show that circle-valued angle structures always exist on any $(M, \mathbf{T})$. Furthermore, the space of all $\mathbf{S}^1$-angle structures on $(M, \mathbf{T})$, denoted by $SAS(M, \mathbf{T})$, is shown to be a closed smooth manifold (proposition 2.6). Each $\mathbf{S}^1$-angle structure has a natural volume given by the Milnor–Lobachevsky function. This defines a continuous but not necessarily smooth volume function

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$\mathbf{V} : SAS(M, \mathbf{T}) \to \mathbf{R}$. Since the space $SAS(M, \mathbf{T})$ is compact, the volume function $\mathbf{V}$ has a maximum point. Our main result is the following.

Theorem 1.1. *Suppose $(M, \mathbf{T})$ is a triangulated closed orientable pseudo 3-manifold. Let p be a maximum point of the volume function $\mathbf{V} : SAS(M, \mathbf{T}) \to \mathbf{R}$.*

(a) If p is a smooth point for $\mathbf{V}$, then p produces a solution to the generalized Thurston gluing equation.

(b) If p is a non-smooth point for $\mathbf{V}$, then p produces a solution to Haken's normal surface equation with exactly one or two non-zero quadrilateral coordinates.

In this paper, we call a solution to Haken's equation in part (b) of theorem 1.1 a *2-quad-type solution.* Recall that a triangulation of M is called *minimal* if it has the smallest number of tetrahedra among all triangulations of M. It is conceivable that the existence of a 2-quad-type solution on a minimally triangulated 3-manifold puts constraints on the topology of the manifold. This is indeed the case. In our recent joint paper with S. Tillmann [**20**], using the work of Jaco and Rubinstein [**13**] and Futer and Guéritaud, we proved the following theorem

Theorem 1.2 (Luo and Tillmann [**20**])**.** *Suppose (M, T) is a minimally triangulated orientable closed 3-manifold so that the volume function $\mathbf{V} : SAS(M, \mathbf{T}) \to \mathbf{R}$ has a non-smooth maximum point. Then,*

(a) M is reducible, or

(b) M is toroidal, or

(c) M is a Seifert fibered space, or

(d) M contains the connected sum $\#_{i=1}^{3} RP^2$ of three copies of the projective plane.

Theorems 1.1 and 1.2 prompt us to make the following conjecture.

Conjecture 1.3. *Suppose $(M, \mathbf{T})$ is a minimally triangulated closed irreducible orientable 3-manifold so that all maximum points of $\mathbf{V} : SAS(\mathbf{T}) \to \mathbf{R}$ are smooth for $\mathbf{V}$. Then $(M, \mathbf{T})$ supports a solution to Thurston's gluing equation.*

We thank Ben Burton and Henry Segerman for providing data that helped us to formulate conjecture 1.3. As we will see in the next subsection, conjecture 1.3 for simply connected manifolds is equivalent to the Poincaré conjecture.

A weaker version of conjecture 1.3 is conjecture 5.1 in §5. It does not involve a maximization process and deals only with solutions to Thurston's equation and Haken's equation. It is shown in §5 that for a simply connected 3-manifolds, conjecture 5.1 is equivalent to the Poincaré conjecture.

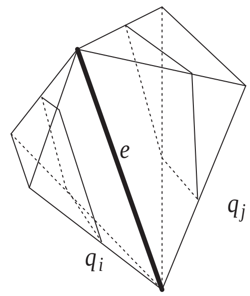

Figure 1.1

1.2. The Thurston equation. Recall that a closed pseudo 3-manifold is the quotient space of a disjoint union of tetrahedra so that their codimension-1 faces are identified in pairs by affine homeomorphisms. In particular, a closed 3-manifold is a pseudo 3-manifold. The *generalized Thurston gluing equation* associated to a triangulated oriented pseudo 3-manifold is defined as follows. Assign each edge in each tetrahedron in the triangulation $\mathbf{T}$ a complex number $z \in \mathbf{C} - \{0, 1\}$. The assignment is said to satisfy *the generalized Thurston equation* if

(a) opposite edges of each tetrahedron have the same assignment;
(b) the three complex numbers assigned to three pairs of opposite edges in each tetrahedron are z, $\frac{1}{1-z}$ and $\frac{z-1}{z}$ subject to an orientation convention; and
(c) for each edge e in the triangulation, if $\{z_1, \ldots, z_k\}$ is the set of all complex numbers assigned to the edge e in various tetrahedra adjacent to e, then

$$\prod_{i=1}^{k} z_i = \pm 1. \tag{1.1}$$

If the right-hand side of (1.1) is 1 for all edges, we say the assignment satisfies the *Thurston gluing equation* (or the *Thurston equation*, for short).

We would like to emphasize that the Thurston equation and its solutions are well defined on any closed oriented triangulated pseudo 3-manifold. The most investigated cases are solutions of the Thurston equation on an ideally triangulated 3-manifold with torus boundary so that the complex numbers z are in the upper-half plane (see, for instance [**31**, **30**], [**5**, **25**] and many others). We intend to study the Thurston equation in the most general setting. Even though a solution to Thurston equation in the general setting does not necessary produce a hyperbolic structure, one can still obtain some important information from it. For instance, it was observed in [**33**] (see also [**23**], [**28**]) that each solution of Thurston equation produces a representation of the fundamental group of the 3-manifold with vertices of the triangulation removed to $PSL(2, \mathbf{C})$. A simplified version of a theorem of Segerman and Tillmann [**28**] states that if $(M, \mathbf{T})$ is a one-vertex triangulation of a closed 3-manifold so that $\mathbf{T}$ supports a solution to the Thurston equation, then each edge in $\mathbf{T}$, considered as a loop in M, is homotopically essential in M. In particular, any one-vertex triangulation of a simply connected 3-manifold cannot support a solution to the Thurston equation.

Using this theorem of Segerman and Tillman and theorem 1.2, we can deduce the Poincaré conjecture from conjecture 1.3 as follows. Suppose M is a simply connected closed 3-manifold. By the Kneser–Milnor prime decomposition theorem, we may assume that M is irreducible. Take a minimal triangulation $\mathbf{T}$ of M. By the work of Jaco and Rubinstein on 0-efficient triangulations, we may assume that $\mathbf{T}$ has only one vertex, i.e., each edge is a loop. By Segerman and Tillmann's theorem above, we see that $(M, \mathbf{T})$ cannot support a solution to the Thurston equation. By conjecture 1.3, there exists a non-smooth maximum volume $\mathbf{S}^1$-angle structure. By theorem 1.2, the minimality of $\mathbf{T}$, and the irreducibility of M, we conclude that $M = \mathbf{S}^3$.

Theorem 1.1 is a special case of theorem 3.2 in §3 where one shows that under the same assumption as theorem 1.1 there exists either a solution to the generalized Thurston equation so that (a) and (b) of theorem 1.1 hold, and for any edge e,

$$\prod_{i=1}^{k} z_i = \pm k(e) \tag{1.2}$$

where $k(e) \in \mathbf{S}^1$ is a given function satisfying (2.16) and (2.17) or there exists a 2-quad-type solution to Haken's equation. Theorem 1.1 provides some evidence relating normal surface theory to representations.

A potential application of theorem 1.1 is to construct hyperbolic metrics on closed 3-manifolds. Namely, if a maximum-volume $\mathbf{S}^1$-angle structure produces a solution to the usual Thurston equation so that (i) the right-hand sides of Thurston equations (1.1) are 1 and (ii) the maximum volume is the Gromov norm of M multiplied by the volume of the ideal regular tetrahedron, then the associated representation will likely produce a hyperbolic metric on M ([**5**, **6**]). In our recent work with Tillmann and Yang [**21**], we have shown that for a closed hyperbolic 3-manifold, there exists a triangulation and a solution to Thurston equation so that the above conditions (i) and (ii) hold.

1.3. Remarks. We remark that all results in the paper can be generalized without difficulties to compact pseudo 3-manifolds with boundary. The simplest way to treat them is by taking the doubling construction. For simplicity, we will not state the corresponding theorems for pseudo manifolds with boundary.

Using volume optimization on the space of angle structures to find hyperbolic structures was carried out successfully by F. Guéritaud in [**9**] for punctured torus bundles over the circle. Our method is similar to that of [**9**] in a different setting.

The paper is organized as follows. In §2, we revisit the theory of normal surfaces and spun normal surfaces. In §3, we recall Thurston's work on gluing hyperbolic metrics and the volume of angle structures. Theorem 1.1 is proved in §4. Some open problems are discussed in §5.

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2. The theory of normal surfaces revisited

The normal surface theory, developed by Haken in the 1950s, is a beautiful chapter in 3-manifold topology. In the late 1970s, Thurston introduced the notion of spun normal surfaces and used it to study 3-manifolds. There are works by Tollefson, Kang and Rubinstein, Tillmann, Thurston, Jaco, and others that characterize spun normal surfaces using Haken's normal coordinates. It turns out a spun normal surface is most conveniently described in terms of the tangent vectors to $\mathbf{S}^1$-angle structures. In fact, the two systems of linear equations for the tangential angle structures and the spun normal surfaces are dual to each other. This observation, which is implicit in the work of [**32**, **14**], and [**29**], is very useful for us in §4 to relate the critical points of the volume functional with the normal surfaces.

We will revisit the normal surface theory and follow the expositions in [**11**] and [**29**] closely in this section. Some of the notation used in the section are new.

2.1. Triangulations of closed pseudo 3-manifolds and normal surfaces. Let X be a union of finitely many disjoint oriented Euclidean tetrahedra. The collection of all faces of tetrahedra in X is a simplicial complex $\mathbf{T}^*$ that is a triangulation of X. Identify codimension-1 faces in X in pairs by affine homeomorphisms. The quotient space M is a closed pseudo 3-manifold with a triangulation $\mathbf{T}$ whose simplexes are the quotients of simplexes in $\mathbf{T}^*$. See [**13**] for more information.

Note that in this definition of triangulation, we do not assume that simplexes in $\mathbf{T}$ are embedded in M. For instance, it may well be that $\mathbf{T}$ has only one vertex. Furthermore, the non-manifold points in M are either the vertices or the midpoints of the edges. If we require the affine homeomorphisms used in the gluing for M to be orientation-reversing, then the pseudo manifold M is oriented and non-manifold points of

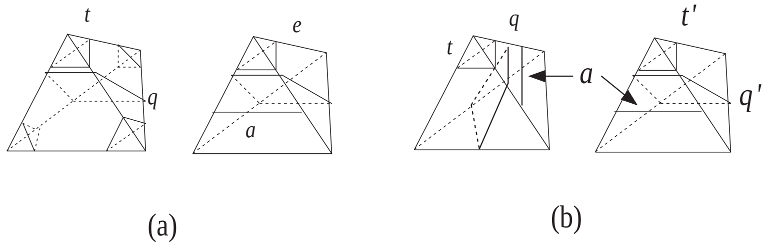

Figure 2.1

M are contained in the vertex set V. Let N be the compact 3-manifold obtained from M with a small open regular neighborhood of V removed. Then we call $\{\sigma \cap N | \sigma \in \mathbf{T}\}$ an *ideal triangulation* of N.

According to Haken [**10**], a *normal arc* in X is an embedded arc in a triangle face so that its end points are in different edges and a *normal disk* in X is an embedded disk in a tetrahedron so that its boundary consists of 3 or 4 normal arcs. These are called *normal triangles* and *normal quadrilaterals*, respectively.

The projections of normal arcs and normal disks from X to M constitute normal arcs and normal disks in the triangulated space $(M, \mathbf{T})$. An immersed surface S in M (or X) is called *normal* with respect to the triangulation $\mathbf{T}$ (or $\mathbf{T}^*$) if for each tetrahedron σ in the triangulation, the intersection $S \cap \sigma$ consists of normal disks. A *normal isotopy* is an isotopy of the ambient space X or M that leaves each simplex invariant. Normal arcs and disks will be considered up to normal isotopy. In each tetrahedron, there are four normal triangles and three normal quadrilaterals up to normal isotopy. We use $\triangle$, $\square$, and $\mathbf{A}$ to denote the sets of all normal isotopy classes of normal triangles, quadrilaterals and normal arcs in the triangulation $\mathbf{T}$. Since the set of all normal isotopy classes of normal quadrilaterals (and normal triangles) in $\mathbf{T}^*$ is the same as $\square$ (and $\triangle$), we will also use $\square$ and $\triangle$ to denote the sets of all normal isotopy classes of normal quadrilaterals and normal triangles in $\mathbf{T}^*$. In this paper, we will use both "triangle" and "quadrilateral" for the normal isotopy class of a triangle and a quadrilateral.

Let V, E, F, T be the sets of all vertices, edges, triangles, and tetrahedra in $\mathbf{T}$. The set of all edges and tetrahedra in $\mathbf{T}^*$ will be denoted by E^* and T^*. We consider E as the set of equivalence classes of elements in E^*, i.e., $E = \{[x] | x \in E^*$ where x and y in E^* are equivalent if they are identified in $X\}$.

If $x, y \in V \cup E \cup F \cup T$, we use $x > y$ to denote that y is a face of x. We use $|Y|$ to denote the cardinality of a set Y.

There are relationships among various sets $V, E, F, T, \triangle, \square, \mathbf{A}$. These incidence relations, which will be recalled below, are the basic ingredient for defining linear and algebraic equations on $\mathbf{T}$.

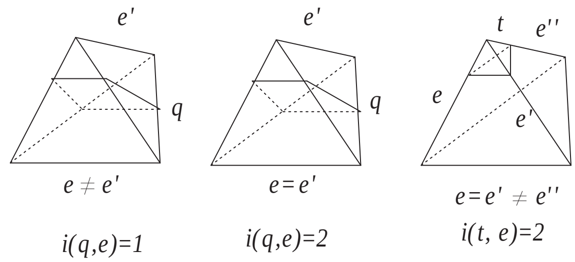

Figure 2.2

Take $t \in \triangle$, $a \in \mathbf{A}$, $q \in \square$, $e \in E$, and $\sigma \in T$. We use $a < t$ (and $a < q$) if there exist representatives $x \in a$, $y \in t$ (and $z \in q$) so that x is an edge of y (and z). We use $t \subset \sigma$ and $q \subset \sigma$ to denote that representatives of t and q are in the tetrahedron σ. In this case, we say the tetrahedron σ contains t and q.

The index $i(q, e)$ is an integer 0, 1, or 2 defined as follows. Given $q \in \square$ and $e^* \in E^*$, let $i(e^*, q)$ be 1 if q, e^* lie in the same tetrahedron so that $q \cap e^* = \emptyset$ and let $i(e^*, q) = 0$ in all other cases. If $e \in E$ and $q \in \square$, then the index $i(e, q) = \sum_{e^* \in e} i(e^*, q)$.

The index $i(t, e)$ is the number of edges $e^* \in e$ so that t has a vertex in e^*. See figure 2.2.

We remark that if $\mathbf{T}$ is a simplicial triangulation, then the indices $i(t, e)$ and $i(q, e)$ take only values $0, 1$.

As a convention, in this paper we will always use σ, e, and q to denote a tetrahedron, an edge, and a quadrilateral in the triangulation $\mathbf{T}$, respectively.

2.2. The Normal surface equation and Kang–Rubinstein basis. The normal surface equation is a system of linear equations defined in the space $\mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ introduced by W. Haken [**10**]. It is defined as follows. For each normal arc $a \in \mathbf{A}$, suppose σ, σ' are the two tetrahedra adjacent to the triangular face which contains a. Then there is a homogeneous linear equation for $x \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ associated to a:

$$x(t) + x(q) = x(q') + x(t') \tag{2.1}$$

where $t, q \subset \sigma$, $t', q' \subset \sigma'$, and $t, t', q, q' > a$. See figure 2.1(b). Let $\mathbf{S}_{ns}$ be the space of all solutions to (2.1) as a runs over all normal arcs.

A basis of the solution space $\mathbf{S}_{ns}$ to equations (2.1) was found by Kang and Rubinstein [**14**]. To state it, let us introduce one more notation. Given a finite set Z, the *standard basis* of the vector space $\mathbf{R}^Z$ will be denoted by $\{z^* | z \in Z\}$ where $z^*(z) = 1$ and $z^*(z') = 0$ if $z' \in Z - \{z\}$. We give $\mathbf{R}^Z$ the inner product so that $\{z^* | z \in Z\}$ forms an orthonormal basis. Now for each $\sigma \in T$ and $e \in E$, define the vectors $W_\sigma, W_e \in$

$\mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ as follows:

$$W_\sigma = \sum_{q \in \square, q \subset \sigma} q^* - \sum_{t \in \triangle, t \subset \sigma} t^* \tag{2.2}$$

and

$$W_e = \sum_{q \in \square} i(q,e) q^* - \sum_{t \in \triangle} i(t,e) t^*. \tag{2.3}$$

A basic theorem proved in [**14**] says:

Theorem 2.1 (Kang and Rubinstein)**.** *For any triangulated closed pseudo 3-manifold, the set* $\{W_x | x \in E \cup T\}$ *forms a basis of the solution space* $\mathbf{S}_{ns}$ *of the normal surface equation.*

For the convenience of the reader, an alternative interpretation of Kang and Rubinstein's proof is given in the appendix.

2.3. Spun normal surfaces and tangential angle structures. Given $x \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$, we will call $x(t)$ and $x(q)$ the t-coordinate and q-coordinate (triangle and quadrilateral coordinates) of x. Spun normal surface theory addresses the following question, first investigated by Thurston [**31**]. Given a vector $z \in \mathbf{R}^{\square}$, when does there exist a solution $x \in \mathbf{S}_{ns}$ to (2.1) whose projection to $\mathbf{R}^{\square}$ is z? Geometrically, it asks if a given finite set of normal quadrilaterals can be realized as the set of all normal quadrilaterals in a normal surface. The question was completely solved in [**32**, **14**, **29**] and [**12**]. The results of Kang and Rubinstein and of Tillmann are more general and give solutions to the projections of not necessarily closed normal surfaces.

The purpose of this section is to interpret a weak version of their work in terms of tangential angle structures.

Definition 2.1. A *tangential angle structure* on a triangulated pseudo 3-manifold $(M, \mathbf{T})$ is a vector $x \in \mathbf{R}^{\square}$ so that for each tetrahedron $\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = 0, \tag{2.4}$$

and for each edge $e \in E$,

$$\sum_{q \in \square} i(q,e) x(q) = 0. \tag{2.5}$$

The linear space of all tangential angle structures on $(M, \mathbf{T})$ is denoted by $TAS(M, \mathbf{T})$, or simply $TAS(\mathbf{T})$.

Recall that a (Euclidean type) angle structure, introduced by Casson, Rivin, and Lackenby, is a vector $x \in \mathbf{R}^{\square}_{>0}$ so that for each tetrahedron

$\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = \pi, \tag{2.6}$$

and for each $e \in E$,

$$\sum_{q \in \square} i(q,e)x(q) = 2\pi. \tag{2.7}$$

Thus one sees easily that a tangential angle structure is a tangent vector to the space of all angle structures. In [**19**], a *generalized angle structure* on $(M, \mathbf{T})$ is defined as a vector $x \in \mathbf{R}^{\square}$ so that (2.6) and (2.7) hold. It is proved in [**19**] that a generalized angle structure exists if and only if the euler characteristic of each link $lk(v)$ is zero for $v \in V$. We will consider in this paper those $x \in \mathbf{R}^{\square}$ so that the right-hand side of (2.6) is in $\pi + 2\pi\mathbf{Z}$ and the right-hand side of (2.7) is in $2\pi\mathbf{Z}$. These will be called $\mathbf{S}^1$*-valued angle structures* on $\mathbf{T}$ and will be shown to exist on any closed pseudo 3-manifold using the method introduced in [**19**]. Evidently, $TAS(\mathbf{T})$ is the tangent space to $SAS(\mathbf{T})$.

The following is a result proved by Tollefson for the closed 3-manifold case, and Kang and Rubinstein and Tillmann for all cases. The result was also known to Jaco [**12**].

Theorem 2.2 ([**32**, **14**, **29**])**.** *For a triangulated closed pseudo 3-manifold* $(M, \mathbf{T})$*, let* $Proj_{\square} : \mathbf{R}^{\triangle} \times \mathbf{R}^{\square} \to \mathbf{R}^{\square}$ *be the projection. Then*

$$Proj_{\square}(\mathbf{S}_{ns}) = TAS(\mathbf{T})^{\perp} \tag{2.8}$$

where $\mathbf{R}^{\square}$ *has the standard inner product so that* $\{q^* | q \in \square\}$ *is an orthonormal basis.*

We remark that theorem 2.2 is not stated in this form in the work of [**32**, **14**, **29**]. This interpretation is due to us.

Proof. Suppose $\mathbf{R}^n$ and $\mathbf{R}^m$ are Euclidean spaces with the standard inner product and $A : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with transpose $A^t : \mathbf{R}^m \to \mathbf{R}^n$. Then it is well known that $Im(A) = ker(A^t)^{\perp}$. Define a linear map

$$A : \mathbf{R}^E \times \mathbf{R}^T \to \mathbf{R}^{\square} \tag{2.9}$$

by

$$A(h) = Proj_{\square}(\sum_{e \in E} h(e)W_e + \sum_{\sigma \in T} h(\sigma)W_{\sigma}). \tag{2.10}$$

By definition, $Proj_{\square}(\mathbf{S}_{ns}) = Im(A)$.

By (2.2) and (2.3), we have

$$A(h)(q) = \sum_{\sigma \in T, q \subset \sigma} h(\sigma) + \sum_{e \in E} i(q,e)h(e). \tag{2.11}$$

To understand tangential angle structures, we define a linear map $B : \mathbf{R}^{\square} \to \mathbf{R}^E \times \mathbf{R}^T$ so that

$$B(x)(e) = \sum_{q \in \square} i(q,e)x(q) \tag{2.12}$$

and

$$B(x)(\sigma) = \sum_{q \in \square, q \subset \sigma} x(q). \tag{2.13}$$

By definition, we have $TAS(\mathbf{T}) = ker(B)$. We claim that $B = A^t$, i.e., $(B(x), h) = (x, A(h))$ for all $x \in \mathbf{R}^{\square}, h \in \mathbf{R}^E \times \mathbf{R}^T$ where $(,)$ is the standard inner product.

Indeed, by definition, we have

$$\begin{aligned}(B(x), h) &= \sum_{e \in E} h(e)B(x)(e) + \sum_{\sigma \in T} h(\sigma)B(x)(\sigma) \\ &= \sum_{e \in E, q \in \square} i(e,q)x(q)h(e) + \sum_{\sigma \in T, q \in \square, q \subset \sigma} h(\sigma)x(q) \\ &= \sum_{q \in \square} x(q) \sum_{e \in E} i(q,e)h(e) + \sum_{q \in \square} x(q) \sum_{\sigma \in T, q \subset \sigma} h(\sigma) \\ &= (x, A(h)).\end{aligned}$$

Therefore, $TAS(\mathbf{T})^{\perp} = ker(B)^{\perp} = Im(A) = Proj_{\square}(\mathbf{S}_{ns})$. This ends the proof. q.e.d.

Corollary 2.3 (Tillmann [**29**]). *(a)* $\dim(TAS(\mathbf{T})) = |V| - |E| + 2|T| = \chi(M) + |T|$.

(b) $\dim(Proj_{\square}(\mathbf{S}_{ns})) = -\chi(M) + 2|T|$.

Definition 2.2. Suppose $(M, \mathbf{T})$ is a triangulated closed pseudo 3-manifold. We say the triangulation $\mathbf{T}$ is *angle rigid* if there is $q \in \square$ so that $x(q) = 0$ for all $x \in TAS(\mathbf{T})$. We say $\mathbf{T}$ is *2-angle rigid* if there exists a non-zero vector $(c_1, c_2) \in \mathbf{R}^2$ and $q_1 \neq q_2 \in \square$ so that $c_1 x(q_1) + c_2 x(q_2) = 0$ for all $x \in TAS(\mathbf{T})$.

By definition, if $\mathbf{T}$ is angle rigid, then $x(q)$ is a constant for all $\mathbf{S}^1$-angle structures x, i.e., the angle at q cannot be deformed. If the triangulation $\mathbf{T}$ has an edge e of degree 1, then $\mathbf{T}$ is angle rigid at the quadrilateral q so that $i(q,e) \neq 0$. If $\mathbf{T}$ has an edge e of degree 2, then $\mathbf{T}$ is 2-angle rigid at the quadrilaterals q_1 and q_2 so that $i(q_j, e) \neq 0$ for $j = 1, 2$.

One simple consequence of Theorem 2.2 is the following corollary.

Corollary 2.4. *Under the same assumption as in theorem 2.2,*

(a) $(M, \mathbf{T})$ *is angle rigid if and only if there exists an embedded normal surface* Σ *in* $\mathbf{T}$ *so that the surface has exactly one normal quadrilateral type;*

(b) (M,T) is 2-angle rigid if and only if there exists a vector $v \in \mathbf{S}_{ns} \cap (\mathbf{Z}^{\triangle} \times \mathbf{Z}^{\square})$ so that $Proj_{\square}(v)$ is non-zero and has at most two non-zero coordinates.

To see part (a), if there exists a normal surface containing only one quadrilateral type $q \in \square$, then its normal coordinate $x \in \mathbf{R}^{\triangle} \times \mathbf{R}^{\square}$ is a vector so that $Proj_{\square}(x) = kq^* \in TAS(\mathbf{T})^{\perp}$ for some non-zero scalar k and some $q \in \square$. Thus, $z(q) = 0$ for all $z \in TAS(\mathbf{T})$. Conversely, if there exists $q \in \square$ so that $z(q) = 0$ for all $z \in TAS(\mathbf{T})$, then $q^* \in TAS(\mathbf{T})^{\perp}$. By theorem 2.2, $q^* = Proj_{\square}(v)$ for some $v \in \mathbf{S}_{ns}$. We may choose $v \in \mathbf{Q}^{\triangle} \times \mathbf{Q}^{\square}$ since the linear equations (2.1) have integer coefficients and q^* has integer coordinates. It follows that some integer multiple kv has non-negative integer q-coordinates. Now add to the vector kv a positive integer multiple of the normal coordinates of the normal surfaces $lk(v)$, the link of the vertex $v \in V$, so the resulting vector has positive t-coordinates. We obtain a vector $u \in \mathbf{S}_{ns} \cap (\mathbf{Z}^{\triangle}_{\geq 0} \times Z^{\square}_{\geq 0})$ with exactly one non-zero q-coordinate. By the work of Haken, this vector u is the normal coordinate of an embedded normal surface in $(M, \mathbf{T})$. The proof of (b) is similar and will be omitted. However, we are not able to conclude that $v \in \mathbf{Z}^{\triangle}_{\geq 0} \times \mathbf{Z}^{\square}_{\geq 0}$ in this case.

2.4. Existence of $\mathbf{S}^1$-valued angle structures. We begin with a definition which was also known to D. Futer and F. Gueritaud.

Definition 2.3. Suppose $(M, \mathbf{T})$ is a triangulated closed pseudo 3-manifold. Let $k : E \to \mathbf{S}^1$ be given. An *$\mathbf{S}^1$-valued angle structure with curvature k* on $\mathbf{T}$ is a function $x : \square \to \mathbf{S}^1$ so that for each tetrahedron $\sigma \in \mathbf{T}$,

$$\prod_{q \in \square, q \subset \sigma} x(q) = -1 \tag{2.14}$$

and for each edge $e \in E$,

$$\prod_{q \in \square} x(q)^{i(q,e)} = k(e). \tag{2.15}$$

The set of all $x \in (\mathbf{S}^1)^{\square}$ satisfying (2.14) and (2.15) will be denoted by $SAS(\mathbf{T}, k)$. The case that $k(e) = 1$ for all $e \in E$ is the most interesting one. We use $SAS(\mathbf{T})$ to denote $SAS(\mathbf{T}, 1)$ where $1(e) = 1$ for all $e \in E$.

For a complex number $w \in \mathbf{C}$, we use $\arg(w) \in [0, 2\pi)$ to denote its argument. If $x \in SAS(\mathbf{T}, k)$, by taking $\arg(x(q))$, we can interpret an $\mathbf{S}^1$-valued angle structure x as a map from $\square \to \mathbf{R}$ satisfying (2.6) and (2.7) so that the right-hand side of (2.6) is in $2\pi\mathbf{Z} + \pi$ and the right-hand side of (2.7) is in $2\pi\mathbf{Z} + \arg(k(e))$.

Lemma 2.5. *If $SAS(\mathbf{T}, k) \neq \emptyset$, then the function $k : E \to \mathbf{S}^1$ satisfies*

$$\prod_{e\in E} k(e) = 1, \tag{2.16}$$

and for each vertex $v \in V$,

$$\prod_{e>v} k(e) = 1. \tag{2.17}$$

Indeed, to see (2.16), using (2.14) and (2.15), we can write the left-hand side of (2.16) as

$$\prod_{q\in\square, e\in E} x(q)^{i(q,e)} = \prod_{\sigma\in T}\prod_{q\in\sigma, e<\sigma} x(q)^{i(q,e)} = \prod_{\sigma\in T}(\prod_{q\subset\sigma} x(q))^2 = 1$$

due to $\sum_{e\in E} i(q,e) = 2$ for each q.

To see (2.17), using (2.14), we can write the left-hand side of (2.17) as,

$$\prod_{e>v}\prod_{q\in\square} x(q)^{i(q,e)} = \prod_{\sigma\in T,\sigma>v}(\prod_{q\subset\sigma, e<\sigma, e>v} x(q)^{i(q,e)}) = (-1)^N$$

where N is the number of normal triangles at the vertex v. This number N is the same as the number of triangles in the link $lk(v)$. Since $lk(v)$ is a closed triangulated surface, N is an even number. Thus, (2.17) follows.

One can define the similar notion of an $\mathbf{S}^1$-valued angle structure on a closed triangulated surface by assigning each angle of a triangle a complex number of norm 1 so that the product of the complex numbers in each triangle is -1. The *curvature* at a vertex is the product of all complex numbers assigned to the angles at the vertex. For instance, if $(M^3, \mathbf{T})$ is a triangulated pseudo 3-manifold with an $\mathbf{S}^1$-valued angle structure, then the vertex link $lk(v)$ has the induced $\mathbf{S}^1$-angle structure. The identity (2.17) says that the product of its curvatures at all vertices is 1.

The main result in this section says that (2.16) and (2.17) are also sufficient. This generalizes our earlier work with Tillmann on 3-manifolds with torus boundary [**19**]. The method of the proof of the proposition below is that of [**19**].

Proposition 2.6 (See also [**19**])**.** *Given any triangulated closed pseudo 3-manifold $(M, \mathbf{T})$ and $k : E \to \mathbf{S}^1$ satisfying (2.16) and (2.17), then $SAS(\mathbf{T}, k) \neq \emptyset$. Furthermore, $SAS(\mathbf{T}, k)$ is a smooth not necessarily connected closed manifold of dimension $|\chi(M)| + |T|$.*

In a recent joint work with D. Futer, C. Hodgson, and H. Segerman, we are able to improve the above result by showing that any triangulated closed pseudo 3-manifold supports a $\mathbf{Z}_2$- angle structure.

Proof. We may assume without loss of generality that M is connected. Consider the Lie group homomorphism $F : (\mathbf{S}^1)^{\Box} \to (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$ given by

$$F(z)(e) = \prod_{q\in\Box} z(q)^{i(q,e)}$$

and

$$F(z)(\sigma) = \prod_{q\in\Box, q\subset\sigma} z(q)$$

where $z \in (\mathbf{S}^1)^{\Box}$, $e \in E$, and $\sigma \in T$. The goal is to show that the point $t : E \cup T \to \mathbf{S}^1$ given by $t(e) = k(e)$ for $e \in E$ and $t(\sigma) = -1$ for $\sigma \in T$ is in the image of F.

Suppose otherwise, that t is not in the image of F. Since F is a continuous group homomorphism from a torus to a torus, the image of F is a connected closed subgroup of $(\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$ that misses t. Thus there exists a continuous group homomorphism $h : (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T \to \mathbf{S}^1$ so that $h(t) \neq 1$ and $h \circ F$ is the trivial homomorphism.

Each homomorphism from $(\mathbf{S}^1)^n$ to $\mathbf{S}^1$ is given by a vector $(m_1, \ldots, m_n) \in \mathbf{Z}^n$, i.e., the homomorphism sends $(x_1, \ldots, x_n) \in (\mathbf{S}^1)^n$ to $x_1^{m_1} \ldots x_n^{m_n}$. Thus for the homomorphism h, there exists $\phi \in \mathbf{Z}^E \times \mathbf{Z}^T$ so that for all $x \in (\mathbf{S}^1)^E \times (\mathbf{S}^1)^T$,

$$h(x) = \prod_{e\in E} x(e)^{\phi(e)} \prod_{\sigma\in T} x(\sigma)^{\phi(\sigma)}.$$

By the choice of t, we have $h(t) = \prod_{\sigma\in T}(-1)^{\phi(\sigma)} \prod_{e\in E} k(e)^{\phi(e)}$. Thus, $h(t) \neq 1$ says that

$$\prod_{e\in E} k(e)^{\phi(e)} \neq (-1)^{\sum_{\sigma\in T}\phi(\sigma)}. \tag{2.18}$$

On the other hand, we will show that $\phi \circ F$ being trivial implies that (2.18) is an equality. The contradiction establishes the proposition. q.e.d.

Since the composition $h \circ F$ is trivial, for any $z \in (\mathbf{S}^1)^{\Box}$,

$$\begin{aligned} 1 = h(F(z)) &= \Big(\prod_{e\in E}\prod_{q} z(q)^{i(q,e)\phi(e)}\Big)\Big(\prod_{\sigma\in T}\prod_{q\subset\sigma} z(q)^{\phi(\sigma)}\Big) \\ &= \prod_{q\in\Box} z(q)^{\sum_{\sigma,q\subset\sigma}\phi(\sigma)+\sum_{e\in E}\phi(e)i(q,e)}. \end{aligned}$$

By the assumption, $h(F(z)) = 1$ for all choices of $z \in (\mathbf{S}^1)^{\Box}$. Thus, we obtain, for each $q \in \Box$,

$$\sum_{\sigma,q\subset\sigma}\phi(\sigma) + \sum_{e} i(q,e)\phi(e) = 0. \tag{2.19}$$

For a fixed tetrahedron $\sigma \in \mathbf{T}$, the above equation says that the sum of $\phi(e) + \phi(e')$ of the values of ϕ at two opposite edges e, e' in σ

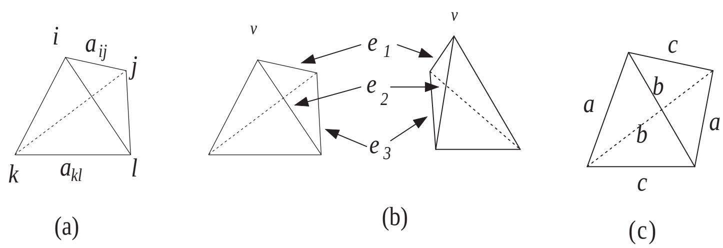

Figure 2.3: A topological interpretation of lemma 2.7

is independent of the choice of e, e'. We will need to use the following lemma.

Lemma 2.7. *Suppose $a_{ij} = a_{ji} \in \mathbf{Z}$ where $i \neq j \in \{1,2,3,4\}$ are six numbers so that*

$$a_{ij} + a_{kl} = c$$

is a constant independent of the choice of indices i, j, k, l where $\{i,j,k,l\} = \{1,2,3,4\}$. Then there exist $b_1, \ldots, b_4 \in \frac{1}{2}\mathbf{Z} = \{n/2 | n \in \mathbf{Z}\}$ so that

$$a_{ij} = b_i + b_j$$

for all $i \neq j \in \{1,2,3,4\}$.

Indeed, $b_i = \frac{a_{ij}+a_{ik}-a_{jk}}{2}$ is independent of the choices of $\{i,j,k\}$, $\{i,j,l\}$, or $\{i,k,l\}$ due to the assumption on $a_{ij} + a_{kl} = c$.

Thus, by the lemma, there exists a map $w : \{\text{vertice of } \sigma\} \to \frac{1}{2}\mathbf{Z}$ so that

$$\phi(e) = w(v,\sigma) + w(v',\sigma) \tag{2.20}$$

where v, v' are the end points of e. We claim that $w(v,\sigma)$ is independent of the choice of σ. Indeed, consider two tetrahedra σ, σ' sharing a common triangular face f (see figure 2.3(b)). Then for three edges e_1, e_2, e_3 in f, we solve (2.20) and obtain

$$w(v,\sigma) = w(v,\sigma') = \frac{\phi(e_1) + \phi(e_2) - \phi(e_3)}{2}$$

where v is the vertex opposite to the edge e_3 in f. It follows that $w(v,\sigma) = w(v,\sigma')$ is independent of the choice of tetrahedra σ and σ' since $(M, \mathbf{T})$ is a pseudo 3-manifold. Let $w : V \to \frac{1}{2}\mathbf{Z}$ be the map so that

$$\phi(e) = w(v) + w(v') \tag{2.21}$$

where v, v' are vertices of e. We claim that either all $w(v)$'s are integers, or all of $w(v)$ are half-integers (i.e., $k+1/2$ for some $k \in \mathbf{Z}$). Indeed, since $\phi(e)$ is an integer, it follows from (2.21) that either both $w(v), w(v')$ are in $\mathbf{Z}$, or both are in $\frac{1}{2}\mathbf{Z} - \mathbf{Z}$. Since the manifold M is connected, it follows that either $w(v) \in \mathbf{Z}$ for all v, or $w(v) \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$ for all v.

We now claim that the sum $\sum_{\sigma\in T}\phi(\sigma)$ has to be an even integer. Indeed, by (2.19) and (2.21), $\phi(\sigma) = -\sum_{v<\sigma} w(v)$. Thus,

$$\sum_{\sigma\in T}\phi(\sigma) = -\sum_{v\in V} w(v)(\sum_{\sigma>v} 1)$$

$$= -\sum_{v\in V} w(v)|\{\text{triangles in the link lk(v)}\}|.$$

For any triangulation of a closed surface, the number of triangles in the triangulation has to be even. Thus if all $w(v)\in \mathbf{Z}$, $\sum_{\sigma\in T}\phi(\sigma)$ is even. In the other case, all $w(v)\in \mathbf{Z}/2-\mathbf{Z}$. Thus,

$$\sum_{\sigma}\phi(\sigma) = -\sum_{v\in V}\frac{1}{2}|\{\text{triangles in the link lk(v)}\}| \quad mod(2)$$

$$= -\frac{1}{2}|\{\text{ normal triangles in T}\}| \quad mod(2).$$

Now each tetrahedron has four normal triangles, and thus the total number of normal triangles in T is divisible by 4. This implies again that $\sum_{\sigma\in T}\phi(\sigma)$ is an even number.

This implies that the right-hand side of (2.18) is 1. We claim that the left-hand side of (2.18) is also equal to 1. There are two cases to be considered. In the first case, all $w(v)$'s are in $\mathbf{Z}$. Then the left-hand side of (2.18) becomes

$$\prod_{e\in E} k(e)^{\sum_{v<e} w(v)} = \prod_{v\in V}(\prod_{e>v} k(e))^{w(v)}$$

which is 1 due to (2.17).

In the second case that $w(v) = W(v)+1/2$ where $W(v)\in\mathbf{Z}$ for all $v\in V$, we have $\phi(e) = 1+\sum_{v<e} W(v)$. Thus, the left-hand side of (2.18) becomes

$$\prod_{e\in E} k(e)^{1+\sum_{v<e} W(v)} = (\prod_{v\in V}\prod_{e>v} k(e)^{W(v)})\prod_{e\in E} k(e)$$

which is again 1 due to (2.16) and (2.17). This contradiction shows that $SAS(\mathbf{T},k)\neq\emptyset$.

Finally, since $SAS(\mathbf{T},k) = F^{-1}(t)$ where F is a Lie group homomorphism, one concludes that $SAS(\mathbf{T},k)$ is a closed smooth manifold that may be disconnected.

3. Thurston equation and volume

The Thurston equation mentioned in the introduction can be conveniently rephrased in terms of the normal quadrilaterals in the triangulation. It is based on the fact that a pair of opposite edges in a tetrahedron is the same as the normal isotopy class of the quadrilateral that is disjoint from the given edges. We will rewrite the Thurston equation in terms of quadrilaterals in this section. In order to do so, we first recall the Neumann–Zagier anti-symmetric bilinear form on $\mathbf{R}^{\square}$. This bilinear form appeared in the important work of Neumann–Zagier [**23**]. We assume that $(M, \mathbf{T})$ is an oriented closed pseudo 3-manifold in this section so that each tetrahedron in $\mathbf{T}$ has the induced orientation.

3.1. Neumann-Zagier Poisson structure. If σ is an oriented Euclidean tetrahedron with edges from one vertex labeled by a, b, c so that the opposite edges have the same labeling a, b, c (see figure 2.3(c)), then the cyclic order of edges a, b, c viewed from each vertex depends only on the orientation of the tetrahedron, i.e., is independent of the choice of the vertices. Now each pair of opposite edges in the tetrahedron corresponds to a normal isotopy class of quadrilateral q in σ via the relation $i(q, e) \neq 0$. Let q_1, q_2, q_3 be three quadrilaterals in σ so that $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order induced by the cyclic order on the opposite edges from a vertex. Let W be the vector space with a basis $\{q_1, q_2, q_3\}$. An anti-symmetric bilinear form $\omega : W \times W \to \mathbf{R}$ is defined by $\omega(q_i, q_j) = 1$ if and only if $(i, j) = (1, 2), (2, 3), (3, 1)$. In particular, $\omega(q_i, q_j) = -\omega(q_j, q_i)$. Given any two quadrilaterals $q, q' \in \square$, set $\omega(q, q')$ to be the value just defined if they are in the same tetrahedron, and $\omega(q, q') = 0$ if they are in different tetrahedra. In this way, one obtains the Neumann–Zagier anti-symmetric bilinear form

$$\omega : \mathbf{R}(\square) \times \mathbf{R}(\square) \to \mathbf{R}$$

where $\mathbf{R}(\square)$ is the vector space with a basis $\square$. More details of the form can be found [**23**, **4**], and [**29**]. See also [**18**].

The following was proved in [**23**].

Proposition 3.1 (Neumann-Zagier)**.** *. Suppose $(M, \mathbf{T})$ is a triangulated, oriented closed pseudo 3-manifold. Then*

(a) for any $q' \in \square$, $\sum_{q \in \square} \omega(q, q') = 0$.

(b) for any pair of edges $e, e' \in E$,

$$\sum_{q, q' \in \square} i(q, e) i(q', e') \omega(q, q') = 0.$$

Indeed, part (a) follows from the anti-symmetric property, i.e., for any $i = 1$, 2, or 3, $\sum_{j=1}^{3} \omega(q_j, q_i) = 0$. Part (b) is more complicated. First, anti-symmetry shows that the identity (b) holds if (1) $e = e'$, or (2) e and e' do not lie in a tetrahedron, or (3) e, e' lie in a tetrahedron and are opposite edges. Now if $e \neq e'$ and e, e' lie in a tetrahedron σ and are not opposite, then e, e' lie in a triangular face and there is a second tetrahedron σ' containing e, e'. In this case, due to the orientations on σ and σ', we have

$$\sum_{q, q' \subset \sigma} i(q, e) i(q', e') \omega(q, q') = - \sum_{q, q' \subset \sigma'} i(q, e) i(q', e') \omega(q, q'). \tag{3.1}$$

Thus part (b) follows. For more details of the proof, see [**23**], pp. 316–320.

It is known ([**23**]) that the restriction of the Neumann–Zagier 2-form to the subspace $\{x = \sum_{q\in\square} a_q q \in \mathbf{R}(\square) |$ for each $\sigma \in T$, $\sum_{q\subset\sigma} a_q = 0\}$ becomes non-degenerate. The 2-dimensional counterpart of the Neumann–Zagier Poisson structure is Thurston's anti-symmetric bilinear form on the space of measured laminations. It is very closely related to the Weil–Petersson symplectic form [**24**, **1**] on the Teichmüller spaces and plays a vital rule in Kontsevich's work [**15**] on Witten's conjecture and many other works. It is expected that the Neumann-Zagier Poisson structure will play an equally important role in $(2+1)$ TQFT.

3.2. The generalized Thurston equation with prescribed curvature. We begin with the following definition.

Definition 3.1. Suppose $(M, \mathbf{T})$ is an oriented closed pseudo 3-manifold with a triangulation and $k \in (\mathbf{S}^1)^E$. The Thurston equation (with curvature k) is defined for $z \in \mathbf{C}^\square$ so that for each $e \in E$,

$$\prod_{q\in\square} z(q)^{i(q,e)} = \pm k(z), \tag{3.2}$$

and if $q, q' \in \square$ so that $\omega(q, q') = 1$, then

$$z(q')(1 - z(q)) = 1. \tag{3.3}$$

By (3.3) and the fact that $f(t) = \frac{1}{1-t}$ satisfies $tf(t)f(f(t)) = -1$, we have, for each tetrahedron $\sigma \in T$,

$$\prod_{q\in\square, q\subset\sigma} z(q) = -1. \tag{3.4}$$

Note that we do not require that $Im(z(q)) > 0$, which corresponds to the positively oriented ideal tetrahedron ([**23**]). The work of Yoshida [**33**] (see also [**23**] and [**29**]) shows that each solution z so that the right-hand side of (3.2) is 1 produces a representation of $\pi_1(M - V)$ to $PSL(2, \mathbf{C})$.

Note that equation (3.2) is equivalent to

$$\prod_{q\in\square} z(q)^{2i(q,e)} = k(e)^2 \tag{3.5}$$

It is the solution to (3.3) and (3.5) that is addressed in theorem 1.1.

3.3. Volume of $\mathbf{S}^1$-valued angle structures. Recall that the Lobachevsky function $\Lambda(x) = -\int_0^x \ln|2\sin(u)|du$ is a continuous periodic function of period π defined on $\mathbf{R}$. It is real analytic on $\mathbf{R} - \pi\mathbf{Z}$ so that $\lim_{t\to 0} \Lambda'(t) = +\infty$. For more details, see Milnor [**22**]. Given $t = e^{\sqrt{-1}a} \in \mathbf{S}^1$, define $\lambda(t) = \Lambda(a)$. This is well defined since $\Lambda(a)$ has π as a period. Furthermore, $\lambda : \mathbf{S}^1 \to \mathbf{R}$ is real analytic on the subset

$\mathbf{S}^1 - \{\pm 1\}$. For an $\mathbf{S}^1$-valued angle structure $x : \square \to \mathbf{S}^1$ on $(M, \mathbf{T})$, define its *volume* $\mathbf{V}(x)$ to be

$$\mathbf{V}(x) = \sum_{q \in \square} \lambda(x(q)) = \sum_{q \in \square} \Lambda(\arg(x(q))).$$

The volume function $\mathbf{V}$ is continuous and, in particular, has a maximum and a minimum point. By definition the smooth points for $\mathbf{V} : SAS(\mathbf{T}, k) \to \mathbf{R}$ include those points x where $x(q) \neq \pm 1$ for all q.

The main theorem in the paper can be stated as follows.

Theorem 3.2. *Suppose $(M, \mathbf{T})$ is a triangulated oriented closed pseudo 3-manifold and $k \in (\mathbf{S}^1)^E$ satisfies (2.16) and (2.17). Let p be a maximum point of the volume function $\mathbf{V} : SAS(\mathbf{T}, k) \to \mathbf{R}$.*

(a) If $p(q) \neq \pm 1$ for all $q \in \square$, then p produces a solution to the generalized Thurston equation (3.3) and (3.5).

(b) If $p(q_0) = \pm 1$ for some $q_0 \in \square$, then p produces a 2-quad-type solution y to Haken's normal surface equation so that $y(q_0) \neq 0$.

3.4. Smooth critical point of the volume. The following lemma was known to Casson and Rivin.

Lemma 3.3. *Suppose $x \in SAS(\mathbf{T}, k)$ is a critical point of the volume $\mathbf{V} : SAS(\mathbf{T}) \to \mathbf{R}$ so that $x(q) \neq \pm 1$ for all $q \in \square$. Then the generalized Thurston equation (3.3) and (3.5) has a solution in $(\mathbf{C} - \mathbf{R})^{\square}$.*

Proof. Suppose q_1, q_2, q_3 are three quadrilaterals in a tetrahedron. Let $x_i = x(q_i)$ be the $\mathbf{S}^1$-valued angle at the quadrilateral. We define the associated complex values $z(q_i)$ by the formula.

$$z(q_i) = \frac{x_j - \bar{x}_j}{x_k - \bar{x}_k} x_i = \frac{\sin(\arg(x_j))}{\sin(\arg(x_k))} x_i$$

where $\omega(q_i, q_j) = 1$ and $\{i, j, k\} = \{1, 2, 3\}$. This is well defined since $x_k - \bar{x}_k \neq 0$ by the assumption that $x(q) \neq \pm 1$. More generally, for $x \in SAS(\mathbf{T})$ and $x(q) \neq \pm 1$ for all q, one defines $z \in \mathbf{C}^{\square}$ by

$$z(q) = x(q) \prod_{r \in \square} (\sin(\arg(x(r))))^{\omega(r,q)}.$$

We claim that z is a solution to the Thurston equation (3.3) and (3.5).

First, (3.3) follows by a direct calculation and the definition. Let us assume that $z_i = z(q_i)$ and that $\omega(q_1, q_2) = 1$. By definition, we have

$$z_1 = \frac{x_2 - \bar{x}_2}{x_3 - \bar{x}_3} x_1$$

and

$$z_2 = \frac{x_3 - \bar{x}_3}{x_1 - \bar{x}_1} x_2.$$

Due to $x_1x_2x_3 = -1$ and $x_i\bar{x}_i = 1$, then (3.3) says that

$$z_2(1 - z_1) = 1.$$

Indeed,

$$\begin{aligned} z_2(1 - z_1) &= (\frac{x_3 - \bar{x}_3}{x_1 - \bar{x}_1}x_2)\frac{x_3 - \bar{x}_3 - x_1x_2 + x_1\bar{x}_2}{x_3 - \bar{x}_3} \\ &= \frac{x_3 + x_1\bar{x}_2}{x_1 - \bar{x}_1}x_2 = \frac{x_3x_2 + x_1}{x_1 - \bar{x}_1} = 1. \end{aligned}$$

To see (3.5), we need to use the critical point equation for $\mathbf{V}$ at the smooth point x. By definition, we can identify the tangent space to a point of $SAS(\mathbf{T}, k)$ with $TAS(\mathbf{T})$. Indeed, for any $v \in TAS(\mathbf{T})$ and $x \in SAS(\mathbf{T}, k)$, the path $p(t) = xe^{tv} \in SAS(\mathbf{T}, k)$ given by

$$p(t)(q) = x(q)e^{tv(q)}$$

for $t \in (-\epsilon, \epsilon)$ has tangent vector v at $t = 0$ and all tangent vectors to $SAS(\mathbf{T}, k)$ at x are of this form. Now due to $x(q) \neq \pm 1$, $\frac{d\mathbf{V}(xe^{tv})}{dt}|_{t=0} = 0$ shows that,

$$\sum_{q \in \square} v(q) \ln |\sin(\arg(x(q)))| = 0. \tag{3.6}$$

Choose a specific $v \in TAS(\mathbf{T})$ as follows. Fix an edge $e \in E$, by proposition 3.1,

$$v_e = \sum_{q \in \square} \sum_{r \in \square} i(q, e)\omega(r, q)r^* \in TAS(\mathbf{T}). \tag{3.7}$$

Now substitute v_e for v in (3.6) and use the fact that $\sum_{q \in \square} i(q, e)v_e(q) = 0$, we obtain, for each $e \in E$,

$$\begin{aligned} \prod_{q \in \square} z(q)^{i(e,q)} &= \prod_{q \in \square} x(q)^{i(e,q)} \prod_{r \in \square} \sin(\arg(x(q)))^{i(e,q)\omega(r,q)} \\ &= k(e) \prod_{q,r \in \square} \sin(\arg(x(q)))^{i(e,q)\omega(r,q)} = \pm k(e) \end{aligned}$$

due to (3.6) and (3.7). This verifies (3.5) and ends the proof. q.e.d.

Furthermore, if z is a solution in $(\mathbf{C} - \mathbf{R})^{\square}$ to the generalized Thurston equation (3.3) and (3.5) over a closed 3-manifold, then $\frac{z}{|z|}$ is an $\mathbf{S}^1$-angle structure that is a smooth critical point of the volume $\mathbf{V}$ in $SAS(\mathbf{T})$. The proof uses the fact that for a closed manifold M, the tangent space $TAS(\mathbf{T})$ is generated by the vectors v_e's for $e \in E$ given by (3.7). We omit the details.

Corollary 3.4. *Under the same assumption as in lemma 3.3, if $x \in SAS(\mathbf{T}, k)$ is a smooth critical point of the volume $\mathbf{V}$ with $x(q) \neq \pm 1$ for all $q \in \square$, let $y \in \mathbf{R}^{\square}_{\geq 0}$ be the vector so that $y(q) = -\ln |(\sin(\arg(x(q)))|$. Then $y \in Proj_{\square}(\mathbf{S}_{ns})$.*

Indeed, (3.6) shows that $y \in TAS(\mathbf{T})^{\perp}$. Thus, by theorem 2.2, $y \in Proj_{\square}(\mathbf{S}_{ns})$.

It will be very interesting to see what topological information y contains.

4. Volume optimization and normal surfaces

A relationship between those smooth critical points p with $p(q) \neq \pm 1$ of the volume $\mathbf{V} : SAS(\mathbf{T}, k) \to \mathbf{R}$ and the normal surfaces is established in corollary 3.4. In this section, we will investigate the remaining critical points of the volume. Since the function $\mathbf{V}$ is not smooth, the definition of the critical points of $\mathbf{V}$ should be specified. First of all, we will show (corollary 4.3) that for any $p \in SAS(\mathbf{T})$ and $u \in TAS(\mathbf{T})$, the limit $\lim_{t\to 0} \frac{d\mathbf{V}(pe^{tu})}{dt}$ always exists as an element in $[-\infty, \infty] = \mathbf{R} \cup \{\infty, -\infty\}$. We say that a point $p \in SAS(\mathbf{T}, k)$ is a *critical point* of the volume $\mathbf{V}$ if for all u in $TAS(\mathbf{T})$,

$$\lim_{t\to 0} \frac{d\mathbf{V}(pe^{tu})}{dt} = 0. \tag{4.1}$$

Using this definition, one sees easily that the maximum and minimum points of $\mathbf{V}$ are critical points, i.e., the volume function $\mathbf{V}$ always has critical points.

The main theorem in the section, which implies theorem 1.1, is the following:

Theorem 4.1. *Suppose $(M, \mathbf{T})$ is an orientable closed triangulated pseudo 3-manifold with $SAS(M, \mathbf{T}, k) \neq \emptyset$. If the volume $\mathbf{V} : SAS(M, \mathbf{T}, k) \to \mathbf{R}$ has a critical point p so that $p(q_0) = \pm 1$ for some $q_0 \in \square$, then p produces a 2-quad-type solution y to Haken's normal surface equation so that $y(q_0) \neq 0$.*

Recall that by proposition 2.6, $SAS(M, \mathbf{T}, k) \neq \emptyset$ if and only if k satisfies (2.16) and (2.17).

4.1. Subderivatives of the volume function.

The volume function $\mathbf{V}$ is essentially composed by the function $W : P \to \mathbf{R}$ where $P = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_1 + x_2 + x_3 = \pi\}$ and $W(x_1, x_2, x_3) = \Lambda(x_1) + \Lambda(x_2) + \Lambda(x_3)$. The function W is not smooth on the subset defined by some $x_i \in \pi\mathbf{Z}$. However, we can obtain subderivative information of W at these points.

The function $h(t) = t \ln |t|$ can be extended to be a continuous function from $\mathbf{R} \to \mathbf{R}$ by declaring $h(0) = 0$. This extension, still denoted by $t \ln |t|$, will be used below.

Lemma 4.2. *Take a point $a = (a_1, a_2, a_3) \in P$ and $b = (b_1, b_2, b_3) \in \mathbf{R}^3$ so that $b_1 + b_2 + b_3 = 0$. Define $f(t) = \frac{dW(a+tb)}{dt}$. Then $\lim_{t\to 0} f(t)$ exists as an element in $\mathbf{R} \cup \{\pm\infty\}$ and* *(a) if $a_i \notin \pi\mathbf{Z}$ for all i,*

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|\sin(a_i)|; \tag{4.2}$$

(b) if $a_i \in \pi\mathbf{Z}$ for all i,

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|b_i|; \tag{4.3}$$

(c) if $a_1 \in \pi\mathbf{Z}$ and $a_2, a_3 \notin \pi\mathbf{Z}$, then

$$\lim_{t\to 0}(f(t) + b_1 \ln|t|) = -b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(a_i)|. \tag{4.4}$$

Proof. We have $f(t) = -\sum_{i=1}^{3} b_i \ln|2\sin(a_i + tb_i)| = -\sum_{i=1}^{3} b_i \ln|\sin(a_i + tb_i)|$ due to $\sum_{i=1}^{3} b_3 = 0$. Now part (a) follows from the definition.

For part (b), due to $\ln(|\sin(t+\pi)|) = \ln|\sin(t)|$, it follows that $f(t) = -\sum_{i=1}^{3} b_i \ln(|\sin(tb_i)|)$. The result is obvious if $b_i = 0$ for all i. Otherwise—say, $b_3 \neq 0$—then $b_3 = -b_1 - b_2$. Substitute it to $f(t)$, and we obtain

$$f(t) = -b_1 \ln|\frac{\sin(b_1 t)}{\sin(b_3 t)}| - b_2 \ln|\frac{\sin(b_2 t)}{\sin(b_3 t)}|.$$

By taking the limit as $t \to 0$, we obtain part (b).

For part (c), we write

$$\begin{aligned} f(t) &= -b_1 \ln|\frac{\sin(b_1 t)}{b_1 t}| - b_1 \ln|b_1 t| - \sum_{i=2}^{3} b_i \ln|\sin(a_i + tb_i)| \\ &= -b_1 \ln|t| - b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(a_i)| + o(t). \end{aligned}$$

where $o(t)$ is a quantity so that $\lim_{t\to 0} o(t) = 0$. This establishes part (c) and finishes the proof. q.e.d.

Note that due to $a_1 + a_2 + a_3 = \pi$, cases (a), (b), and (c) are the list of all cases up to symmetry. The limit $\lim_{t\to 0} f(t)$ in cases (b), (c) above is called the subderivative of the function W at the point a. The subderivative, considered as a function of the tangent vector b, is homogeneous of degree 1. However, due to the non-smoothness, the subderivative, as shown in (b) and (c), is not a linear function of b.

In the case of the $\mathbf{S}^1$-valued angle structure, consider $X = \{a = (a_1, a_2, a_3) \in (\mathbf{S}^1)^3 | a_1 a_2 a_3 = -1\}$ and the volume $\mathbf{V}(a) = \sum_{i=1}^{3} \lambda(a_i) = \sum_{i=1}^{3} \Lambda(\arg(a_i))$. Consider a tangent vector $b = (b_1, b_2, b_3) \in \mathbf{R}^3$ so that

$b_1 + b_2 + b_3 = 0$. Define $f(t) = \frac{d\mathbf{V}(ae^{tb})}{dt}$. Then $\lim_{t\to 0} f(t)$ exists as an element in $\mathbf{R} \cup \{\pm\infty\}$ and the above lemma says

(a) if $a_i \neq \pm 1$ for all i,

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|\sin(\arg(a_i))|; \tag{4.5}$$

(b) if $a_i = \pm 1$ for all i,

$$\lim_{t\to 0} f(t) = -\sum_{i=1}^{3} b_i \ln|b_i|; \tag{4.6}$$

(c) if $a_1 = \pm 1$ and $a_2, a_3 \neq \pm 1$, then

$$\lim_{t\to 0}(f(t) + b_1 \ln|t|) = -b_1 \ln|b_1| - \sum_{i=2}^{3} b_i \ln|\sin(\arg(a_i))|. \tag{4.7}$$

Corollary 4.3. *For any $a \in SAS(\mathbf{T}, k)$, there exists a unique linear function $g(b)$ of $b \in TAS(\mathbf{T})$ and a continuous function $f(b,t)$ of b and $t \in (-\epsilon, \epsilon)$ so that*

$$\frac{d\mathbf{V}(ae^{tb})}{dt} = g(b)\ln|t| + f(b,t).$$

In particular, the limit $\lim_{t\to 0} \frac{d\mathbf{V}(ae^{tb})}{dt}$ always exists as an element in $[-\infty, \infty]$. Furthermore, a local maximum or minimum point of $\mathbf{V}$ is a critical point.

4.2. A proof of theorem 4.1. Suppose $p \in SAS(\mathbf{T}, k)$ is a critical point of the volume function $\mathbf{V}$ so that $p(q_0) = \pm 1$ for some $q_0 \in \square$. By definition of critical points, we have

$$\lim_{t\to 0} \frac{d\mathbf{V}(pe^{tb})}{dt} = 0$$

for all b in $TAS(\mathbf{T})$. By the definition of $\mathbf{V}$, we have

$$\mathbf{V}(x) = \sum_{\sigma\in T} \sum_{q\in\square, q\subset\sigma} \Lambda(arg(x(q))).$$

Let $Y = \{q \in \square | p(q) = \pm 1\}$, which contains q_0, and $Y' = \{q \in Y |$ there exists $\sigma \in T$ so that $q \subset \sigma$ and for the other two $q', q'' \subset \sigma$, one has $p(q'), p(q'') \neq \pm 1\}$.

Thus, by (4.5)–(4.7), we can write

$$\lim_{t\to 0}(\frac{d\mathbf{V}(pe^{tb})}{dt} + \ln|t| \sum_{q\in Y'} b(q)) = -\sum_{q\in Y} b(q)\ln|b(q)|$$
$$- \sum_{q\notin Y} b(q) \ln|\sin(\arg(p(q)))|.$$

By the critical point condition (4.1), we obtain $\sum_{q\in Y'} b(q) = 0$ and

$$\sum_{q\in Y} b(q)\ln|b(q)| = -\sum_{q\notin Y} b(q)\ln|\sin(\arg(p(q)))|. \tag{4.8}$$

For each $q \in \square$, let $f_q : TAS(\mathbf{T}) \to \mathbf{R}$ be the linear function on $TAS(\mathbf{T})$ defined by $f_q(b) = b(q)$. Then the right-hand side of (4.8) is a linear function in b on $TAS(\mathbf{T})$ and the left-hand side of (4.8) is a sum of the functions $f_q(b)\ln|f_q(b)|$.

Lemma 4.4. *Suppose W is a finite-dimensional vector space over $\mathbf{R}$ and $f_1, \ldots, f_n$ and g are linear functions on W satisfying*

$$\sum_{i=1}^{n} f_i(x)\ln|f_i(x)| = g(x). \tag{4.9}$$

Then for each index i there exists $j \neq i$ and $\lambda_{ij} \in \mathbf{R}$ so that

$$f_i(x) = \lambda_{ij} f_j(x).$$

Proof. Note if one of f_i is the zero function $f_i(x) = 0$ for all x, then the lemma holds. Let us assume that all f_i's are non-zero functions. We may assume that $W = \mathbf{R}^m$ and $x = (x_1, \ldots, x_m) \in W$ after a linear change of variables. Write

$$f_i(x) = \sum_{j=1}^{m} a_{ij} x_j.$$

Now suppose the result does not hold—say, $f_1(x)$ is not proportional to $f_j(x)'s$ for $j \geq 2$. Then we can find a point $v \in ker(f_1)$ so that $v \notin \cup_{j=2}^{n} ker(f_j)$. Since $f_1 \neq 0$, for simplicity, let us assume that $a_{11} \neq 0$. Now take the partial derivative of (4.9) with respect to x_1. We obtain

$$\sum_{j=1} a_{1j}\ln|f_j(x)| = h(x) \tag{4.10}$$

where $h(x)$ is a linear function. Take a sequence of vectors x converging to v in (4.10), we obtain a contradiction since $a_{11} \neq 0$. This ends the proof. q.e.d.

Applying lemma 4.4 to (4.8) with f_i's being f_q's for $q \in Y$, we conclude that for f_{q_0}, there exist f_{q_1}, $q_1 \neq q_0$ and $\lambda \in \mathbf{R}$ so that $f_{q_0}(b) = \lambda f_{q_1}(b)$ for all $b \in TAS(\mathbf{T})$. This shows that $(b, q_0^* - \lambda q_1^*) = 0$ for all b. By theorem 2.2, there exists a solution y to Haken's equation so that $y(q_0) = 1, y(q_1) = -\lambda$, and $y(q) = 0$ for all $q \in \square - \{q_0, q_1\}$. This ends the proof.

4.3. A generalization. In [**7**], David Futer and François Guéritaud proved a very nice theorem concerning the non-smooth maximum points of the volume function. Given $x \in SAS(\mathbf{T})$, we say a tetrahedron $\sigma \in T$ is *flat* with respect to x if $x(q) = \pm 1$ for all $q \subset \sigma$ and *partially flat* if $x(q) = \pm 1$ for one $q \subset \sigma$.

Theorem 4.5 (Futer and Guéritaud)**.** *Suppose $(M, \mathbf{T})$ is an oriented triangulated closed pseudo 3-manifold. If x is a non-smooth maximum point of the volume function on $SAS(\mathbf{T})$, then there exists a non-smooth maximum volume point $y \in SAS(\mathbf{T})$ so that all partially flat tetrahedra in y are flat.*

A written proof of it, supplied by Futer and Guéritaud, can be found in [**18**]. Combining theorems 4.5 and 4.1, we obtain a stronger statement:

Theorem 4.6. *Suppose $(M, \mathbf{T})$ is a closed triangulated oriented pseudo 3-manifold so that it has a non-smooth maximum volume point in $SAS(\mathbf{T})$. Then there exist three 2-quad type solutions x_1, x_2, x_3 of Haken's normal surface equation so that there are three distinct quadrilaterals q_1, q_2, q_3 in a tetrahedron with $x_i(q_i) \neq 0$ for all i.*

We call the three 2-quad-type solutions that appear in theorem 4.6 a *cluster of 2-quad-type solutions.* In the joint work with Tillmann [**20**], we proved the following topological result.

Theorem 4.7 ([**20**])**.** *Suppose (M, T) is a minimally triangulated orientable closed 3-manifold which supports a cluster of three 2-quad-type solutions to Haken's equation. Then,*
(a) M is reducible, or
(b) M is toroidal, or
(c) M is a Seifert fibered space, or
(d) M contains the connected sum $\#_{i=1}^{3} RP^2$ of three copies of the projective plane.

Theorem 1.2 mentioned in the introduction is a consequence of theorems 4.6 and 4.7.

5. Open problems

Based on theorem 4.7, we propose the following conjecture, which is weaker than conjecture 1.3.

Conjecture 5.1. *Suppose $(M, \mathbf{T})$ is a minimally triangulated irreducible orientable closed 3-manifold so that the triangulation does not have a cluster of three 2-quad-type solutions to Haken's equation. Then there is a solution to the Thurston equation associated to $\mathbf{T}$.*

Note that by the same argument as in §1.2 and using theorem 4.7 instead of theorem 1.2 and Segerman and Tillmann's theorem, we see that conjecture 5.1 for simply connected manifold is equivalent to the Poincaré conjecture.

Conjecture 5.1 relates solutions of Thurston equation to solutions of Haken's equation and does not involve any volume optimization process. The minimality condition in conjecture 5.1 is necessary. This was shown to us by Ben Burton and Henry Segerman. Solutions to the Thurston equation have been found in many cases. For instance, Tillmann proved in [**30**] that if M is a non-compact finite volume hyperbolic 3-manifold and $\mathbf{T}$ is an ideal triangulation so that each edge is homotopically essential, then $\mathbf{T}$ supports a solution to the Thurston equation. However, a general existence theorem for solving the Thurston equation seems to be still lacking. Conjecture 5.1 is an attempt to address the issue.

Given a solution to the Thurston equation, by the work of Yoshida [**33**] one can produce a representation of the fundamental group of $M-V$ to $PSL(2,\mathbf{C})$ where V is the set of vertices. It is interesting to know when solutions to the Thurston equation produce irreducible representations of the fundamental group. See the work of [**6**] and [**8**].

Finally, solving the Thurston equation over the real numbers, i.e., $z \in \mathbf{R}^{\square}$, seems to be an attractive problem. The first step toward producing a real-valued solution to Thurston equation comes from the following definition.

Definition 5.1. Let $\mathbf{Z}_2$ be the multiplicative group of two elements $\{-1,1\}$. A $\mathbf{Z}_2$-*angle-taut structure* on a triangulated closed pseudo 3-manifold $(M,\mathbf{T})$ is a map $f : \square \to \{-1,1\}$ so that

(a) if q_1, q_2, q_3 are three quadrilaterals in each tetrahedron σ, then exactly one of $f(q_1), f(q_2), f(q_3)$ is -1; and

(b) for each edge e in $\mathbf{T}$, $\prod_{q\in\square} f(q)^{i(q,e)} = 1$.

The motivation for the definition comes from Lackenby's taut triangulations and real-valued solutions to thr Thurston equation. Indeed, if z is a real-valued solution to the Thurston equation, then $f(q) = sgn(z(q))$, i.e., the sign of $z(q)$, is a Z_2-angle-taut structure. A theorem of Baseilhac and Benedetti shows that every triangulated pseudo 3-manifold supports a Z_2-angle-taut structure.

6. Appendix

We give a new proof of the Kang–Rubinstein theorem in this section. First, one checks easily that both W_σ and W_e are in $\mathbf{S}_{ns}$. Next, by a simple dimension counting, one sees that $\dim(\mathbf{S}_{ns}) \leq |E|+|T|$. Indeed, a solution to the normal surface equation is determined by its intersection numbers with edges and the numbers of quads inside each tetrahedron. Thus, it suffices to prove that $\{W_\sigma, W_e | \sigma \in T, e \in E\}$ is an independent set. To this end, suppose otherwise that there exists $h \in \mathbf{R}^E \times \mathbf{R}^T$ so that

$$\sum_{e\in E} h(e)W_e + \sum_{\sigma\in T} h(\sigma)W_\sigma = 0.$$

We can write it as

$$\sum_{t\in\triangle}(-\sum_{e>t} h(e) - \sum_{\sigma>t} h(\sigma))t^* + \sum_{q\in\square}(\sum_{e\in E} h(e)i(q,e) + \sum_{\sigma\in T, q\subset\sigma} h(\sigma))q^* = 0$$

Since $\{t^*, q^*\}$ form a basis, we obtain for each $t \in \triangle$,

$$\sum_{e>t} h(e) + \sum_{\sigma>t} h(\sigma) = 0 \tag{6.1}$$

and for each $q \in \square$,

$$\sum_{e\in E} h(e)i(q,e) + \sum_{\sigma\in T, q\subset\sigma} h(\sigma) = 0. \tag{6.2}$$

Consider a fixed tetrahedron $\sigma \in T$. We claim that the system of linear equations (6.1) and (6.2) for the six edges of σ has only the trivial solution, i.e., $h(e) = h(\sigma) = 0$. In particular, this shows that $\{W_e, W_\sigma\}$ is independent.

To see the claim, let us label the vertices of σ by $1, 2, 3, 4$ and the six edges by e_{ij} where $i \neq j \in \{1, 2, 3, 4\}$. Let $h_{ij} = h(e_{ij})$ and $f = h(\sigma)$. Then (6.1) and (6.2) say that at the ith vertex

$$h_{ij} + h_{ik} + h_{il} = f \tag{6.3}$$

and

$$h_{ij} + h_{kl} = f \tag{6.4}$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Consider the sum of two equations (6.3) at the ith and jth vertices subtracting the sum of the two equations (6.3) at the kth and lth vertices. We obtain, $h_{ij} = h_{kl}$, i.e., $h(e) = h(e')$ when e, e' are opposite edges. Now by (6.4), we see that $h_{ij} = f/2$ for all $i \neq j$. Now substituting back to (6.3), we obtain $3f/2 = f$. Thus, $f = 0$ and $h_{ij} = 0$, i.e., $h(e) = h(\sigma) = 0$.

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