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VOLUME ESTIMATES FOR KÄHLER-EINSTEIN METRICS AND RIGIDITY OF COMPLEX STRUCTURES

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Abstract

This paper extends our earlier results to higher dimensions using a different approach, based on the rigidity of complex structures on certain domains. We prove a "low energy" result in all dimensions, in the sense that if normalized energy in a large ball is small enough, then the normalized energy in any interior ball must also be small.

1. Introduction

This is a continuation of our previous paper [**6**]. Let M be a compact Kähler-Einstein manifold with non-negative scalar curvature, and for $r > 0$, let Z_r be the r-neighbourhood of the points where $|\mathrm{Riem}| \geq r^{-2}$. Our purpose is to estimate the volume of $Z(r)$. In the previous paper we considered manifolds of complex dimension 3, and here we extend the results to all dimensions (under very slightly different hypotheses). We use a different approach, exploiting the rigidity of complex structures on quotient singularities. This also gives another approach to the three-dimensional case. The basic technique develops results of Tian [**10**], for the 3-dimensional case, with the difference that we work with complex domains rather than CR-structures. Meanwhile, as mentioned in [**6**], Cheeger and Naber have posted a preprint [**5**] which reaches the same general conclusions using different arguments. In addition, Tian has informed us that he obtained similar results some time ago.

Throughout the paper, we will make the following standing assumptions.

- (M, g) is a compact Kähler-Einstein manifold of complex dimension n with $\mathrm{Ric}(g) = \lambda g$ and $\lambda > 0$. As usual, we write ω for the metric 2-form.
- $\mathrm{Diam}(M) \leq D$.
- The class $[\omega/2\pi] \in H^2(M)$ is integral.

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Note that these hypotheses have the following standard consequences. First,

$$\mathrm{Vol}(M) \geq V_0 = (4\pi)^n/n! \tag{1}$$

(This follows from the integrality condition.) Second,

$$\mathrm{Vol}(B(x,r)) \geq \kappa(n,D) r^{2n}; \tag{2}$$

for metric balls $B(x,r) \subset M$. (This follows from Bishop-Gromov comparison.) Third,

$$\mathrm{Diam}(M) \geq D_0; \tag{3}$$

where we can take D_0 to be the radius of the Euclidean ball with volume V_0, again by Bishop comparison. Fourth,

$$0 < \lambda \leq \lambda_0 \tag{4}$$

follows from the diameter bound and Myers' Theorem.

Our main result is a "small energy" estimate. Recall that for a ball $B(x,r) \subset M$ with $r \leq \mathrm{Diam}(M)$, we define the normalized energy

$$E(x,r) = r^{4-2n} \int_{B(x,r)} |\mathrm{Riem}|^2.$$

Then we have:

Theorem 1. *There are $\epsilon_0 > 0, K$ such that if and $E(x,r) \leq \epsilon_0$, then $|\mathrm{Riem}| \leq K r^{-2}$ on $B(x, r/2)$.*

Given this, we obtain, using just the same line of argument as in [**6**]:

Corollary 1.

$$\mathrm{Vol}(Z_r) \leq C(n,D) E(M) r^4$$

where $E(M)$ is the square of the L^2 norm of the curvature.

Recall here that $E(M)$ is a topological invariant, determined by the Chern classes of M and the Kähler class.

For completeness we also mention that we get an "approximate monotonicity" property for the normalized energy.

Corollary 2. *For every $\epsilon > 0$, there is a $\delta > 0$ so that if $E(x,r) \leq \delta$, then for any $r' \leq r/2$ and $y \in B(x, r/2)$ we have $E(y, r') \leq \epsilon$.*

This follows easily from Theorem 1.

We will deduce Theorem 1 from the following result.

Theorem 2. *Let (M_i, g_i) satisfy the conditions above; let $x_i \in M_i$ for each i and let l_i be any sequence of numbers which tends to infinity. Suppose $(M_i, l_i^2 g_i, x_i)$ has based Gromov-Hausdorff limit M_∞. Then M_∞ is not a product $\mathbf{C}^{n-q} \times \mathbf{C}^q/\Gamma$ where $q > 2$ and $\Gamma \subset U(q)$ acts freely on S^{2q-1}.*

In turn, Theorem 2 will be proved entirely by complex geometry. The main ingredient is a result on rigidity of complex structures which may have independent interest and, as far as we are aware, is new (in the case when $n > q$). This uses some recent work of Chakrabarti and Shaw [**2**]. Given $q \leq n$ and a real number $a > 1$, let $V(a) \subset \mathbf{C}^n$ be the domain

$$\{(z, w) \in \mathbf{C}^{n-q} \times \mathbf{C}^q : |z| < a, a^{-1} < |w| < a\}.$$

Let $\Gamma \subset U(q)$ be as above (acting freely on S^{2q-1}) and write $V_\Gamma(a)$ for the quotient of $V(a)$ by Γ. Fix any a_1, b_1 with $b_1 < a_1$ so $V_\Gamma(b_1) \subset V_\Gamma(a_1)$.

Theorem 3. *Let $q \geq 3$ and J' be a deformation of the standard complex structure J on $V_\Gamma(a_0)$. If the deformation is sufficiently small in $C^{1,\alpha}$, then there is a diffeomorphism from $V_\Gamma(b_1)$ to a domain in $V_\Gamma(a_1)$ which pulls back J' to the standard complex structure on $V_\Gamma(b_1)$ and which is close in $C^{2,\alpha}$ to the inclusion map.*

Here of course we mean that the diffeomorphism can be forced as close to the inclusion map as we like by requiring that J' is sufficiently close to J.

In the case when $q = n$, the result is essentially covered by Hamilton's work in [**7**]. Alternatively, still in the case when $q = n$, the result is essentially the same as that proved by Tian in [**10**], using an approach through the rigidity of the CR structure on S^{2q-1}/Γ. However, the case $n > q$ seems to have essential new features, since the domain $V(a)$ does not then have a smooth boundary.

The "rigidity" expressed by Theorem 3 is related, in a more algebraic context, to the rigidity of quotient singularities proved by Schlessinger [**9**]. There is a notable distinction between the case $q \geq 3$ covered by the theorem and the case $q = 2$. In the latter case rigidity certainly does not hold. For example, when $\Gamma = \pm 1$, the singularity $\mathbf{C}^2/\Gamma$ is an affine quadric cone $Q(z) = 0$ which can be deformed into a nonsingular quadric $Q(z) = \epsilon$. Correspondingly, the complex structure on the quotient of the annulus in $\mathbf{C}^2$ is not rigid. The distinction between the cases $q = 2$ and $q \geq 3$ appears through the vanishing of 1-dimensional sheaf cohomology in the latter case but not in the former. In fact, it is very well known that products $\mathbf{C}^{n-2} \times \mathbf{C}^2/\Gamma$ can appear as Gromov-Hausdorff limits of blow-up sequences, under our hypotheses: the simplest example being when $n = 2$. See also the further discussion in Section 5.

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2. Theorem 2 implies Theorem 1

Suppose (M_i, g_i) is a sequence of manifolds satisfying our standing conditions (with fixed n, D) and that $B(x_i, r_i)$ are balls in M_i, so $r_i \leq D$. Suppose that the normalized energies $E(x_i, r_i)$ tend to zero as $i \to \infty$.

Then the pointed manifolds (M_i, x_i) with the rescaled metrics $r_i^{-2} g_i$ have a Gromov-Hausdorff convergent subsequence, which we may as well suppose is the full sequence. Let x_∞, M_∞ be the based limit. We claim that there are no singular points in the interior ball $B(x_\infty, 1/2)$. This statement implies Theorem 1. For suppose Theorem 1 is false, so there is a sequence of balls $B(x_i, r_i)$ as above but point $y_i \in B(x_i, r_i/2)$ with $|\mathrm{Riem}|(y_i) = K_i r_i^{-2}$ with $K_i \to \infty$. We get a contradiction to the fact that the rescaled metrics converge in C^∞ on the regular part of the limit M_∞.

To prove the claim above, we again argue by contradiction. Notice that the metric on the regular part of the unit ball $B(x_\infty, 1)$ in M_∞ is flat. Suppose that, contrary to the claim, $y \in B(x_\infty, 1/2)$ is a singular point. A tangent cone to M_∞ at y has the form $\mathbf{C}^{n-q_0} \times C(Y_0)$ for a length space Y_0. If Y_0 is itself singular, we take a tangent cone to $\mathbf{C}^{n-q_0} \times C(Y_0)$ at a singular point and by the general Cheeger-Colding-Tian theory this must have the form $\mathbf{C}^{n-q_1} \times C(Y_1)$ for some $q_1 < q_0$. After at most n steps we arrive at an iterated tangent cone of the form $\mathbf{C}^{n-q} \times C(Y)$ with Y smooth. Since the metric on the regular part is flat, we have $C(Y) = \mathbf{C}^q/\Gamma$, where Γ acts freely on the sphere. Passing to subsequences we can find a sequence of points x_i' in M_i and rescalings $l_i \to \infty$ such that $(M_i, x_i', l_i^2 g_i)$ have based limit $\mathbf{C}^{n-q} \times \mathbf{C}^q/\Gamma$. Thus we can deduce from Theorem 2 that $q \leq 2$ and in fact, by the result of Cheeger [**3**], the only possibility is $q = 2$.

To finish the proof, dealing with the singularities of complex codimension 2, we invoke the result of Cheeger, Colding, and Tian from [**4**] which was also crucial in our previous paper [**6**]. Given any $\alpha > 0$, we can find a ball $B(z, s) \subset M_\infty$ such that the Gromov-Hausdorff distance from $B(z, s)$ to the ball of radius s in the model $\mathbf{C}^q \times \mathbf{C}^q/\Gamma$ is less than $\alpha s/2$. Now fix s and choose i so large that for a suitable choice of $x_i'' \in M_i$ the Gromov-Hausdorff distance from $B(x_i'', s)$ to $B(z, s)$ is also less than $\alpha s/2$. Then Theorem 8.1 in [**4**] tells us that, for a suitable choice of α, we have a fixed $\eta > 0$ such that

$$\int_{B(x'', s)} |\mathrm{Riem}|^2 \geq \eta s^{n-2}.$$

This contradicts our hypothesis that $E(x_i, r_i) \to 0$.

3. Theorem 3 implies Theorem 2

Consider a compact differentiable submanifold Σ, of dimension $2q-1$ with $q > 2$, in a Kähler manifold M, ω. Suppose that $H^1(\Sigma, \mathbf{R}) = H^2(\Sigma, \mathbf{R}) = 0$. Thus we can write $\omega|_\Sigma = d\theta$ for a 1-form θ on Σ and the integral

$$I(\Sigma, \omega) = \int_\Sigma \omega^{q-1} \wedge \theta \tag{5}$$

does not depend on the choice of θ.

Let $S \subset V_\Gamma(a)$ be the quotient of the unit sphere in $\{0\} \times \mathbf{C}^q \subset \mathbf{C}^{n-q} \times \mathbf{C}^q$.

Proposition 1. *Let M, ω be a compact Kähler manifold of complex dimension n such that $[\omega/2\pi R]$ is an integral class. Suppose there is a holomorphic embedding $\iota : V_\Gamma(a) \to M$, for some $a > 1$ and let $\Sigma = \iota(S)$. Then $I(\Sigma, \omega) \geq (2\pi)^q R^q$.*

This is essentially standard complex geometry. By scaling, there is no loss in taking $R = 1$. Let $L \to M$ be a holomorphic line bundle with curvature form $-i\omega$, and choose a power $k > 0$ so that the sections of L^k give an embedding of M in $\mathbf{CP}^N$. For $a > 1$, let $W(a)$ be the annulus $\{w : a^{-1} < |w| < a\}$ in $\mathbf{C}^q$ and $W_\Gamma(a)$ be the quotient by the free action of $\Gamma \subset U(q)$. Let $\pi : W(a) \to W_\Gamma(a)$ be the quotient map. The pull-back $\pi^*\iota^*(L^k)$ is a holomorphic line bundle over $W(a)$. Since $H^1(W(a); \mathcal{O})$ vanishes, this line bundle is trivial. The line bundle $\iota^*(L^k)$ is determined by a character of Γ so, increasing k if necessary, we may suppose this is also trivial. Fix a trivialising section σ of $\iota^*(L^k)$. Thus the composite $W_\Gamma(a) \to M \to \mathbf{CP}^N$ is given by sections $s_i = f_i\sigma$ for $i = 0, \ldots, N$, where f_i are holomorphic functions on $W_\Gamma(a)$.

Let Δ be the ball $\{w : |w| < a^{-1}\}$ and Δ_Γ be the quotient by Γ. Lifting the f_i to $W(a)$ and applying Hartogs' theorem, we see that they extend to holomorphic functions on Δ_Γ (i.e. Γ-invariant functions on Δ). Thus ι extends to a holomorphic map $\iota^+ : \Delta_\Gamma \setminus T \to M \subset \mathbf{CP}^N$, where T is a discrete subset defined by the common zeros of the extended functions f_i. Let Z be the graph of ι^+ and $\overline{Z}$ be the closure of Z in $\Delta_\Gamma \times \mathbf{CP}^N$. Thus Z is an analytic variety and we have holomorphic maps $p : \overline{Z} \to \Delta_\Gamma$ and $j : \overline{Z} \to M$. Writing $\sigma = f_i^{-1}s_i$, we see that σ defines a meromorphic section of $j^*(L^k)$ with no zeros but with possible poles along a divisor supported in $\overline{Z} \setminus Z$, corresponding to the points of T.

We have to see that T is nonempty, so that σ does indeed have some poles. If T is empty, then $p : \overline{Z} \to \Delta_\Gamma$ is a holomorphic equivalence, so j can be viewed as a holomorphic map from Δ_Γ to M. We extend the argument in the obvious way to construct a holomorphic map J from $B \times \Delta_\Gamma$ to M, where B is a ball in $\mathbf{C}^{n-q}$, with J equal to the embedding ι on $B \times V_\Gamma(a)$. But it is clear that this is impossible if M is smooth, as we suppose.

Now we regard $c_1(j^*L^k)$ as a compactly supported cohomology class on $\overline{Z}$, using the trivialisation σ over the boundary. Since σ has poles, we have

$$\omega^{q-1} \wedge c_1(j^*L^k) < 0. \tag{6}$$

The integrality of the Chern class then implies that

$$\omega^{q-1} \wedge c_1(j^* L^k) \leq -(2\pi)^{q-1} k. \tag{7}$$

Let θ be the 1-form $k^{-1}\frac{i}{2}(\overline{\partial} - \partial)\log|\sigma|^2$ on Z. Then, regarding θ as a current on $\overline{Z}$, we have an equation of currents

$$d\theta = \omega + 2\pi k^{-1} E,$$

where E is the current of the divisor representing $c_1(j^* L^k)$, as a compactly supported cohomology class. If $\overline{Z}_0 \subset \overline{Z}$ is the region interior to Σ, in the obvious sense, then by Stokes' Theorem,

$$I(\Sigma, \omega) = \int_\Sigma \theta \wedge \omega^{q-2} = -k^{-1}\omega^{q-1} \wedge c_1(j^* L^k) + \int_{\overline{Z}_0} \omega^q \geq (2\pi)^q.$$

It is now easy to deduce Theorem 2 from Theorem 3 and the Proposition above. Suppose that $x_i, M_i, l_i^2 g_i$ is a sequence as considered in Theorem 2, with based Gromov-Hausdorff limit the length space $\mathbf{C}^{n-q} \times \mathbf{C}^q/\Gamma$. Recall that we have the non-collapsing condition (2) and a two-sided bound on the Ricci curvature of the M_i, g_i by (4). By standard theory (using results of Anderson [**1**]), this means that the metrics converge in C^∞ on the smooth part of $\mathbf{C}^{n-q} \times \mathbf{C}^q/\Gamma$. Regarding the complex structures as covariant constant tensors, we see that we can also suppose these converge. This means that if we fix any $a > 1$, we can find embeddings $\chi_i : V_\Gamma(a) \to M_i$ such that the pull-backs of the metrics and complex structures by χ_i converge to the standard structures on $V_\Gamma(a)$. Applying Theorem 3, we see that we can suppose the χ_i are holomorphic embeddings. So we are in the situation considered in Proposition 1 with submanifolds Σ_i. Applying Proposition 1, we see that $I(\Sigma_i, l_i^2\omega_i) \geq (2\pi)^q l_i^{2q} \to \infty$. But this is a contradiction, since $I(\Sigma_i, l_i^2\omega_i)$ is determined by the restriction of $l_i^2\omega_i$ to Σ_i, which converges to the standard model as $i \to \infty$.

4. Proof of Theorem 3: Complex rigidity

To simplify notation we will prove the result for some particular pair a_1, b_1, but it will be clear that the argument can be adjusted to any pair. We will work with the domains $V(a)$; equivariance under the action of Γ will allow us to deduce the result for the quotient spaces. We will consider various values of the parameter a, but all lying in some fixed interval, say $2 \leq a \leq 4$.

According to [**2**], any (0,1) form σ on $V(a)$ can be expressed as

$$\sigma = \overline{\partial} K(\sigma) + K\overline{\partial}\sigma$$

where $K(\sigma)$ is orthogonal to the L^2 holomorphic functions and $K(\overline{\partial}\sigma)$ is orthogonal to the image of $\overline{\partial}$. The "Kohn operator" K is bounded on L^2 (see [**2**], Section 2.3). It is easy to check that this bound can be taken independent of a. Notice that it is at this stage that the vanishing of $H^1(V(a);\mathcal{O})$ is fed into the proof.

Fix $\alpha \in (0,1)$ and let $\| \, \|_{k,\alpha,a}$ denote the $C^{k,\alpha}$ norm over $V(a)$. Now consider a different parameter $a^* < a$, so $V(a^*) \subset V(a)$.

Proposition 2. *There are fixed C, p such that*

- *If g is a function on $V(a)$, we have*
$$\|g\|_{2,\alpha,a^*} \leq C(a-a^*)^{-p}\left(\|\overline{\partial} g\|_{1,\alpha,a} + \|g\|_{L^2(V(a))}\right).$$
- *If τ is a $(0,1)$-form on $V(a)$ with $\overline{\partial}^* \tau = 0$, then*
$$\|\tau\|_{1,\alpha,a^*} \leq C(a-a^*)^{-p}\left(\|\overline{\partial}\tau\|_{0,\alpha,a} + \|\tau\|_{L^2(V(a))}\right).$$

To see this, we can cover $V(a^*)$ by balls of radius $(a-a^*)/10$ say, such that for each ball the twice-sized ball with the same center is contained in $V(a)$. On a unit-sized ball we have a standard elliptic estimate for functions
$$\|g\|_{C^{1,\alpha}(B/2)} \leq \text{const.}\left(\|\overline{\partial} g\|_{C^{0,\alpha}(B)} + \|g\|_{L^2(B)}\right)$$
and similarly for $(0,1)$ forms. Now the result follows by scaling.

For the rest of the proof we will use the standard convention that C, p are constants which may change from line to line.

We are now ready to begin our main construction. Consider a deformed complex structure on $V(a)$, defined by a tensor $\mu = \sum \mu_{ij} d\overline{z}_j \otimes \frac{\partial}{\partial z_i}$, smooth up to the boundary. Thus we have a deformed $\overline{\partial}$-operator $\overline{\partial}_\mu = \overline{\partial} + \mu\partial$. Let f be a holomorphic function on $V(a)$, for the standard complex structure. Thus $\overline{\partial}_\mu f = \beta$ where $\beta = \mu\partial f$. (In our application f will be one of the co-ordinate functions on $\mathbf{C}^n$.) Then $\beta = \overline{\partial} K(\beta) + K(\overline{\partial}\beta)$. Write $g = K(\beta)$ and $\beta' = K(\overline{\partial}\beta)$. Thus $\overline{\partial} g = \beta - \beta'$ while
$$\overline{\partial}\beta' = \overline{\partial}\beta \qquad \overline{\partial}^*\beta' = 0.$$
The integrability of the deformed complex structure gives
$$\overline{\partial}\beta = \overline{\partial}_\mu(\beta) - \mu\partial\beta = \overline{\partial}_\mu^2 f - \mu\partial\beta = -\mu\partial\beta.$$
Applying the second item in the proposition above, and the L^2-boundedness of the Kohn operator, we get
$$\|\beta'\|_{1,\alpha,a^*} \leq C(a-a^*)^{-p}\|\mu\partial\beta\|_{0,\alpha,a}. \tag{8}$$
Applying the second item to an intermediate region and then the first item, we obtain
$$\|g\|_{2,\alpha,a^*} \leq C(a-a^*)^{-p}\left(\|\beta\|_{1,\alpha,a} + \|\mu\partial\beta\|_{0,\alpha,a}\right). \tag{9}$$
Now write $f' = f - g$. We have $\overline{\partial}_\mu f' = \beta' - \mu\partial g$, so
$$\|\overline{\partial}_\mu f'\|_{1,\alpha,a^*} \leq C(a-a^*)^{-p}\left(\|\mu\|_{1,\alpha,a}\|\beta\|_{1,\alpha,a} + \|\mu\|^2_{1,\alpha,a}\|\beta\|_{1,\alpha,a}\right). \tag{10}$$

We make this construction starting with the n co-ordinate functions $f_i = z_i$, and getting new functions f'_i. Then $\beta_i = \mu \partial f_i$ are just the components of μ:

$$\beta_i = \sum_j \mu_{ij} d\overline{z}_j. \tag{11}$$

Let $\underline{f}' : V(a) \to \mathbf{C}^n$ be the map with components f'_i. Suppose that the restriction of $\underline{f}'$ is a diffeomorphism from $V(a^*)$ to its image in $\mathbf{C}^n$ and that the image contains a domain $V(a')$ where a' is slightly less than a^*. Let $F : V(a') \to V(a^*)$ be the inverse diffeomorphism. We transport the complex structure defined by μ to $V(a')$, using the map F. Write

$$\overline{\partial}_\mu f'_i = \sum \tau_{ij} d\overline{z}_j \qquad \overline{\partial}_\mu \overline{f'_i} = \sum D_{ij} d\overline{z}_j,$$

and suppose that the matrix (D_{ij}) is invertible at each point. A straightforward calculation shows that the "new" complex structure on $V(a')$ is defined by a tensor μ' which is given in matrix notation by

$$\mu'(z) = (D^{-1}\tau)(F(z)). \tag{12}$$

The upshot is that, provided the various conditions above are met, we get a complex structure defined by μ' on $V(a')$, given by the formula (12), and a diffeomorphism $F : V(a') \to V(a)$ which intertwines μ' and μ.

We want to iterate this procedure, provided always that the initial deformation is sufficiently small. We start by fixing a decreasing sequence of domains. Let $a_1 = 4$ and for integers $r \geq 2$ set

$$a_r = 4 - \sum_{i=2}^{r} \frac{1}{i^2},$$

which means that $a_r \geq 3$ for all r, so we take $b_1 = 3$. Let $a_r^* = \frac{1}{2}(a_r + a_1)$, so $a_r - a_r^* = 1/2(r+1)^2$. Suppose we start with a μ_1 on $V(a_1)$ and that at stage r we have constructed μ_r on $V(a_r)$ with a diffeomorphism $\mathcal{F}_r : V(a_r) \to V(a_1)$ which intertwines μ_r and μ_1. Then, provided the various conditions above are met, we perform the construction above to get μ_{r+1} on $V(a_{r+1})$ and a diffeomorphism $F_{r+1} : V(a_{r+1}) \to V(a_r)$, so we can continue the inductive construction with $\mathcal{F}_{r+1} = \mathcal{F}_r \circ F_{r+1}$.

We need to show that, if $\|\mu_1\|_{1,\alpha,a_1}$ is sufficiently small, then

- The construction can proceed at each stage.
- The restriction of the μ_r to the fixed interior domain $V(b_1)$ tends to zero in $\| \; \|_{1,\alpha,b_1}$.
- The restrictions of the diffeomorphisms $\mathcal{F}_r$ to $V(b_1)$ converge in $C^{2,\alpha}$ to a diffeomorphism $\mathcal{F} : V(b_1) \to V(a_1)$, which can be made as close as we please to the inclusion map by assuming μ_1 sufficiently small.

If we establish these facts, then we prove Theorem 3 as follows. Given a deformed complex structure on $V_\Gamma(a_1)$, we lift it to a Γ-invariant structure on $V(a_1)$. It is clear that the μ_r we construct at each stage are Γ-invariant and the diffeomorphisms are Γ-equivariant. Then $\mathcal{F}$ induces the desired diffeomorphism from $V_\Gamma(b_1)$ to $V_\Gamma(a_1)$.

Suppose we have constructed $\mu_r, \mathcal{F}_r$. Let $\underline{f}_r : V(a_r) \to \mathbf{C}^n$ be the map defined as above. Then the conditions for proceeding to the next stage will all be met if $\underline{f}_r$ is sufficiently close to the identity in $C^{2,\alpha}$. By (9) this will be the case if μ_r is sufficiently small in $C^{1,\alpha}$. More precisely, if we write $\eta_r = \|\mu_r\|_{1,\alpha,a_r}$, then we can proceed to the next stage if

$$\eta_r \leq \epsilon r^{-p_0}, \tag{13}$$

for some suitable fixed ϵ, p_0. Now we can estimate μ_{r+1} using (10), (12), and the behaviour of Hölder norms under compositions and products. We get

$$\eta_{r+1} \leq r^p \eta_r^2. \tag{14}$$

It is neater to express this as

$$\eta_{r+1} \leq C_1 \frac{r^{2p_1}}{(r+1)^{p_1}} \eta_r^2 \tag{15}$$

for some fixed C_1, p_1. For then if we write $w_r = C_1 r^{p_1} \eta_r$, we simply have $w_{r+1} \leq w_r^2$. Choose $k > 0$ so that

$$\exp(-k2^{s-1}) < C_1 s^{p_1} \epsilon s^{-p_0},$$

for all $s \geq 1$. Then if $w_1 \leq e^{-k}$, that is to say if η_1 is sufficiently small, it follows by induction that the condition (13) is met at each stage and $w_r \leq \exp(-k2^{r-1})$. Thus the iteration can proceed for all r and it is clear that the other conditions itemised above are met, because of the very rapid decay of the η_r.

5. Discussion

1) In this paper we have concentrated on proving what we need for our main result. However it seems likely that the arguments in the proof of Theorem 2 can be extended to obtain a precise description of the complex structure for a Kähler-Einstein manifold close to a singular limit $\mathbf{C}^{n-q} \times \mathbf{C}^q/\Gamma$, when we drop the integrality condition on the Kähler class. As Tian has suggested, one expects the complex structure in such a case to be a crepant resolution of the quotient singularity (at least when $\Gamma \subset SU(q)$), and one expects the metric to be modelled on Joyce's ALE metric. This seems significant because, in the general Cheeger-Colding-Tian theory, rather little is known about the structure of Einstein metrics close to a singular limit.

2) Our rigidity result, Theorem 3, can clearly be generalized to other settings. (For example, we could consider any domain V in $\mathbf{C}^n$ which has a suitable exhaustion by subsets V_a such that $H^1(V_a;\mathcal{O}) = 0$ and on which $\overline{\partial}$ has closed image.) It fits into a long line of similar statements, beginning with the Newlander-Nirenberg Theorem on the integrability of almost-complex structures. Our proof has some relation to the proof by Kohn [**8**] of this theorem, and also to the results of Hamilton [**7**] for more general domains. The solution of the "$\overline{\partial}$-problem" is an essential ingredient in all these results. However, there is a notable difference in our case. In [**8**], [**7**] it is first established that the $\overline{\partial}$-problem has a solution, obeying suitable uniform estimates, for all small deformations of the complex structure. In our situation we do not have such a statement: we only know that the problem can be solved for the unperturbed solution using (essentially) the Künneth formula. This is the reason why we have to introduce the "shrinking domains" in the problem.

   Note also that since we allow ourselves to shrink the domain we do not really need the full force of the result of Chakrabarti and Shaw, so the method may extend still further

References

[1] M.T. Anderson, *Convergence and rigidity of metrics under Ricci curvature bounds* Invent. Math. **102** 429–445 (1990) MR 1074481 Zbl 0711.53038.

[2] D. Chakrabarti & M-C Shaw, *The Cauchy-Riemann equations on product domains* Math. Ann. **349** No. 4 977–998 (2011) MR 2777041 Zbl 12223.32023.

[3] J. Cheeger, *Integral bounds on curvature, elliptic estimates and rectifiability of singular sets* Geom. Functional Analysis **13** 20–72 (2003) MR 1978491 Zbl 1086.53051.

[4] J. Cheeger, T.H. Colding & G. Tian, *On the singularities of spaces with bounded Ricci curvature* Geometric and Functional Analysis **12** 873–914 (2002) MR 1937830 Zbl 1030.53046.

[5] J. Cheeger & A. Naber, *Lower bounds on Ricci curvature and quantative behaviour of singular sets* arxiv 1103.1819 (To appear in Inventiones Math.)

[6] X-X Chen & S.K. Donaldson, *Volume estimates for Kahler-Einsten metrics: the three-dimensional case* arxiv 1104.0270.

[7] R.S. Hamilton, *Deformations of complex structures on manifolds with boundary, I; the stable case* Jour. Differential Geometry **12** 1–45 (1977) MR 0477158.

[8] J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds, I* Annals of Math. **78** 112–148 (1963) MR 0153030 Zbl. 0161.09302.

[9] M. Schlessinger, *Rigidity of quotient singularities* Inventiones Math. **14** 17–26 (1971) MR 0292830 Zbl 0232.14005.

[10] G. Tian, *Compactness theorems for Kähler-Einstein manifolds of dimension 3 and up* Jour. Differential Geometry **35** 535–558 (1992) MR 1163448.

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