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BOUNDING GEOMETRY OF LOOPS IN ALEXANDROV SPACES

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Abstract

For a path in a compact finite dimensional Alexandrov space X with curv $\geq \kappa$, the two basic geometric invariants are the length and the turning angle (which measures the closeness from being a geodesic). We show that the sum of the two invariants of any loop is bounded from below in terms of κ, the dimension, diameter, and Hausdorff measure of X. This generalizes a basic estimate of Cheeger on the length of a closed geodesic in a closed Riemannian manifold ([Ch], [GP1,2]). To see that the above result also generalizes and improves an analog of the Cheeger type estimate in Alexandrov geometry in [BGP], we show that for a class of subsets of X, the n-dimensional Hausdorff measure and rough volume are proportional by a constant depending on $n = \dim(X)$.

Introduction

Let X denote an Alexandrov space with curvature bounded from below, curv $\geq \kappa$, which is a length metric space such that each point has a neighborhood in which any geodesic triangle looks fatter than a comparison triangle in the 2-dimensional space form S^2_κ of constant curvature κ. A motivation for studying Alexandrov spaces is that the Gromov-Hausdorff limit of a sequence of Riemannian n-manifolds with sectional curvature sec $\geq \kappa$ is an Alexandrov space with curv $\geq \kappa$. A Riemannian manifold with sec $\geq \kappa$ is an Alexandrov space, but an Alexandrov space in general may have geometrical or topological singularities. A basic issue in Alexandrov geometry is to prove results whose counterparts to Riemannian geometry rely on the Toponogov triangle comparison theorem ([BGP]).

Let $\gamma : [0,1] \to X$ be a continuous curve. Given a partition, $P : 0 = t_1 < \cdots < t_{m+1} = 1$ with partition size $|P| = \delta$, let $p_i = \gamma(t_i)$, and let γ_m denote an m-broken geodesic, i.e., $\gamma_m|_{[t_i,t_{i+1}]} = [p_i p_{i+1}]$ is a minimal geodesic joint p_i and p_{i+1}. Let $\theta_i = \pi - \measuredangle p_{i-1} p_i p_{i+1}$. In particular, $\theta_1 = \pi - \measuredangle p_{m+1} p_1 p_2$ if $p_{m+1} = p_1$ (the loop case) and $\theta_1 = 0$ otherwise.

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Let $\Theta_P(\gamma)=\sum_{i=1}^m\theta_i$. We define the following number,

$$\Theta(\gamma)=\lim_{\delta\to 0}\sup_{|P|=\delta}\{\Theta_P(\gamma)\},$$

the *turning angle* of γ. For convenience, we assign 2π as the turning angle of a trivial loop. An m-broken geodesic γ_m has a finite turning angle $\Theta(\gamma_m)=\sum_{i=1}^m\theta_i$. $\Theta(\gamma)$ measures the closeness of a curve from a geodesic in the following sense: A curve γ is a geodesic if and only if $\Theta(\gamma)=0$. If M is a Riemannian manifold, then for any C^2-curve $\gamma\subset M$, $\Theta(\gamma)=\int_0^1|\nabla_{\gamma'}\gamma'|dt$ is the geodesic curvature (c.f. [AB]). Because a general Alexandrov space may contain no closed geodesic (nor an m-broken geodesic loop with a small turning angle; e.g., a flat cone), a loop with the minimal turning angle should be treated as a counterpart of a closed geodesic on a (closed) Riemannian manifold.

In this paper, Haus_n will denote the "normalized" n-dimensional Hausdorff measure such that $\mathrm{Haus}_n(I^n)=1$, where I^n is the unit n-cube in $\mathbb{R}^n$. In particular, if U is an open subset of an n-dimensional Riemannian manifold, $\mathrm{Haus}_n(U)=\mathrm{vol}(U)$. Let $\mathrm{Alex}^n(\kappa)$ be the collection of n-dimensional Alexandrov spaces with curvature bounded from below by κ and

$$\mathrm{Alex}^n(\kappa,D)=\{X\in\mathrm{Alex}^n(\kappa),\ \mathrm{diam}(X)\le D\}.$$

The purpose of this paper is to find an explicit upper bound for the volume of $X\in\mathrm{Alex}^n(\kappa,D)$ in terms of κ,D, $L(\gamma)$, and $\Theta(\gamma)$ for any given loop $\gamma\in X$ (Theorem A or Theorem 1.1).

When X is a closed Riemannian manifold, this generalizes a basic estimate of Cheeger on the length of a closed geodesic in [Ch] (see Theorem 0.3), as well as an overlap with a generalization of Cheeger's basic estimate in [GP1] (1.3 Main Lemma) and [GP2] (Lemma 1.5). As an application, we will present a local injectivity radius estimate (see Theorem B). To see that Theorem A also generalizes and improves an analog of the Cheeger type estimate in Alexandrov geometry ([BGP], Lemma 8.6), we show that for any open subset of X, the n-dimensional Hausdorff measure and rough volume are proportional by a constant depending on $n=\dim(X)$.

This implies that Theorem A generalizes and improves an analog of the Cheeger type estimate in [BGP] on the length of an almost closed geodesic in an Alexandrov space (see Theorem 0.5).

We now begin to state the main results of this paper. A more general form will be proposed in Theorem 1.1.

Theorem A. *Let X be a complete n-dimensional Alexandrov space $(n\ge 2)$ with curv $\ge\kappa$. If γ is a loop at $p\in X$ contained in an r-ball*

$B_r(p)$, then the length and turning angle of γ satisfy:

$$L(\gamma) + (n-1)r \cdot \Theta(\gamma) \geq \frac{(n-1)\mathrm{Haus}_n(B_r(p))}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(r_0)},$$

where S_1^m denotes a unit m-sphere, $r_0 = r$ for $\kappa \leq 0$ and $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$, and $sn_\kappa(r) = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} r$, r, $\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} r$ respectively for $\kappa > 0$, $\kappa = 0$, and $\kappa < 0$.

The lower bound on the left-hand side of the inequality in Theorem A is optimal in all dimensions; the inequality becomes an equality when γ is a great circle in an n-dimensional spherical κ-space form and $r = \frac{\pi}{\sqrt{\kappa}}$ (note that $\mathrm{vol}(S_1^n) = \frac{2\pi}{n-1} \cdot \mathrm{vol}(S_1^{n-2})$, $n \geq 2$). Furthermore, in the case when X contains no closed geodesic, the inequality is sharp modulo a constant depending only on n (see Example 2.9). Let $\mathrm{Alex}^n(\kappa, D, v) = \{X \in \mathrm{Alex}^n(\kappa, D),\ \mathrm{Haus}_n(X) \geq v\}$.

Corollary 0.1. *Let $X \in Alex^n(\kappa, D, v)$. For any loop γ on X,*

$$L(\gamma) + \Theta(\gamma) \geq c(n, k, D, v) > 0,$$

where $c(n, k, D, v) = \frac{v \cdot \min\{(n-1), D^{-1}\}}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D_0)}$, $D_0 = D$ for $\kappa \leq 0$, and $D_0 = \min\{D, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

Corollary 0.1 reveals a basic geometric property of the loop space over a compact Alexandrov space $X \in \mathrm{Alex}^n(\kappa, D, v)$: any short loop has a turning angle that is not small, or equivalently, any loop with a small turning angle is not short.

For $0 \leq \epsilon < 1$, we call a loop, γ, *ϵ-closed geodesic*, if $\Theta(\gamma) \leq \epsilon \cdot \frac{v}{D \cdot \mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D_0)}$, where $D_0 = D$ for $\kappa \leq 0$ and $D_0 = \min\{D, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$. A loop γ is a closed geodesic if and only if γ is a 0 geodesic.

For any ϵ-closed geodesic γ on X, its length can be bounded from below.

Corollary 0.2. *Let $X \in Alex^n(\kappa, D, v)$. If γ is a loop that is ϵ-closed geodesic, then*

$$L(\gamma) \geq (1-\epsilon) \cdot \frac{(n-1)v}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D_0)},$$

where $D_0 = D$ for $\kappa \leq 0$ and $D_0 = \min\{D, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

We will make a few comments on Theorem A:

(a) In Riemannian geometry, it is often important to bound the length of a closed geodesic from below. For instance, the following basic estimate of Cheeger on the length of closed geodesics plays a crucial role in the classical Cheeger's finiteness theorem ([Ch]).

Theorem 0.3 (Cheeger, [Ch]). *Let M be a closed n-manifold ($n \geq 2$) with sectional curvature $sec_M \geq \kappa$ ($\kappa \leq 0$) and diameter $D < \infty$. For any closed geodesic γ,*

$$L(\gamma) \geq \frac{(n-1)\mathrm{vol}(M)}{\mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D)}.$$

Corollary 0.2 reduces to Theorem 0.3 when restricting to a closed geodesic (i.e., $\epsilon = 0$) on a Riemannian manifold.

(b) We now state a special case of Theorem A.

Theorem B. *Let $X \in Alex^n(\kappa, D, v)$. For any $p, q \in X$ and any minimal geodesics γ_1, γ_2 from p to q, the distance between p and q satisfies*

$$|pq| \geq \frac{n-1}{2} \cdot \left[\frac{v}{\mathrm{vol}(S_1^{n-2}) sn_\kappa^{n-1}(D_0)} - D \cdot \Theta(\gamma_1 * \gamma_2^{-1})\right],$$

where $D_0 = D$ for $\kappa \leq 0$ and $D_0 = \min\{D, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

Let $a(n, \kappa, D, v) = \frac{v}{D \cdot \mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D_0)}$. Observe that if $\Theta(\gamma_1 * \gamma_2^{-1}) < a(n, \kappa, D, v)$, then $|pq| \geq c(n, \kappa, D, v) > 0$.

With a stronger assumption, Theorem B yields an explicit form comparing the Main Lemma in [GP1] (c.f. [GP2], Lemma 1.5), which generalizes Theorem 0.3. Consider a compact Riemannian n-manifold M. For $p, q \in M$, without loss of generality, let γ_1 and γ_2 be two minimal geodesics from p to q such that $\measuredangle(\dot\gamma_1(1), \dot\gamma_2(1)) \geq \measuredangle(\dot\gamma_1(0), \dot\gamma_2(0)) = \pi - 2\beta$. Then

$$\Theta(\gamma_1 * \gamma_2^{-1}) = \measuredangle(\dot\gamma_1(0), -\dot\gamma_2(0)) + \measuredangle(\dot\gamma_1(1), -\dot\gamma_2(1)) \leq 4\beta.$$

Applying Theorem B, we obtain an explicit lower bound for $|pq|$:

Corollary 0.4. *Let M be a closed n-manifold with $sec_M \geq \kappa$. Assume $\max\{\mathrm{diam}(\Gamma_{pq}), \mathrm{diam}(\Gamma_{qp})\} = \pi - 2\beta$, where $0 \leq \beta < \frac{a(n,\kappa,D,v)}{4}$. Then*

$$|pq| \geq \frac{(n-1)D}{2}\left(\frac{\mathrm{vol}(M)}{D \cdot \mathrm{vol}(S_1^{n-2}) \cdot sn_\kappa^{n-1}(D_0)} - 4\beta\right) > 0,$$

where $D_0 = D$ for $\kappa \leq 0$ and $D_0 = \min\{\mathrm{diam}(M), \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

Comparing to [GP1] and [GP2], let S_p be the unit tangent sphere, and let $\Gamma_{pq} \subseteq S_p$ (resp. $\Gamma_{qp} \subseteq S_q$) denote the subset of vectors tangent to minimal geodesics from p to q (resp. from q to p). For any $\theta > 0$, let $\Gamma_{pq}(\theta) = \{\vec{s} \in S_p, |\vec{s}\,\Gamma_{pq}|_{S_p} < \theta\}$, where $|\vec{s}\,\Gamma_{pq}|_{S_p}$ denotes the distance of $\vec{s}$ to Γ_{pq} on S_p. Then $\Gamma_{pq}(\frac{\pi}{2} + \beta) = S_p$ and $\Gamma_{qp}(\frac{\pi}{2} + \beta) = S_q$, by the Main Lemma in [GP1] (c.f. [GP2], Lemma 1.5, for an explicit estimate of β), $|pq| \geq r(n, \kappa, D)$, where $r(n, \kappa, D)$ is of an implicit form. If there is a closed geodesic through p and q, then $\beta = 0$, and Corollary 0.4 implies Theorem 0.3.

(c) Theorem A can be useful in analyzing local geometry concerning the *injectivity radius* of a point p (injrad_p) in a complete Riemannian manifold M. If $q \in M$ is a nearest cut point to p (consequently, $|pq| = \mathrm{injrad}_p < \infty$), then either q is a conjugate point to p or there is a geodesic loop γ at p passing through q. In the latter case, $2|pq| = L(\gamma)$ and $\Theta(\gamma)$ satisfy Theorem A. In the former case (e.g., no geodesic loop satisfying $L(\gamma) = 2|pq|$), a similar estimate can also be established (see Theorem B).

To extend the discussion to Alexandrov spaces, we introduce the following notions: we call a point $p \in X \in \mathrm{Alex}^n(\kappa)$ a regular point, if there is a non-trivial minimal geodesic along any direction in the space of directions at p. As in the Riemannian case, we define the *cut locus*, C_p, at a regular point as the collection of points $q \in X$ such that q is the furthest point on a radial curve from p with arc length equal to $|pq|$. Let $q \in C_p$ such that $|pq| = |pC_p|$, which is equal to the injectivity radius injrad_p. Clearly, the gradient-exponential map is a homeomorphism on the ball of radius $< \mathrm{injrad}_p$. Let $\mathrm{geod}(p, q) = \{[pq]\}$ denote the set of minimal geodesics, $[pq]$, from p to q. We call the following number in $[0, 2\pi]$,

$$\theta_p = \inf_{q\in C_p,\, |pq|=\mathrm{injrad}_p} \{\Theta(\gamma_1 * \gamma_2^{-1}),\ \gamma_1, \gamma_2 \in \mathrm{geod}(p, q)\},$$

the *geodesic angle* of p. Observe that $\theta_p = 0$ if and only if $2 \cdot \mathrm{injrad}_p$ is realized by the length of a closed geodesic at p and $\theta_p = 2\pi$ if and only if there is a unique minimal geodesic $[pq]$. (When X is a Riemannian manifold, $\theta_p = 2\pi$ implies that q is a conjugate point of p.) Hence, θ_p measures the existence of such a closed geodesic at p.

A consequence of Theorem A is:

Corollary 0.5. *Let X be a complete n-dimensional Alexandrov space ($n \geq 2$) with curv $\geq \kappa$. If $p \in X$ is a regular point, then for any $r > \mathrm{injrad}_p$,*

$$\mathrm{injrad}_p \geq \frac{n-1}{2} \cdot \left[\frac{\mathrm{Haus}_n(B_r(p))}{\mathrm{vol}(S_1^{n-2}) sn_\kappa^{n-1}(r_0)} - r \cdot \theta_p\right],$$

where $r_0 = r$ for $\kappa \leq 0$ and $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

Corollary 0.5 provides a local estimate for injrad_p in terms of local geometry when θ_p is relatively small (e.g., $\theta_p < \frac{\mathrm{Haus}_n(B_r(p))}{r\cdot \mathrm{vol}(S_1^{n-2})\cdot sn_\kappa^{n-1}(r)}$). On the other hand, θ_p that is not relatively small indicates that geodesics from p to q are confined to a narrow region.

(d) In [BGP], an analog of Theorem 0.3 in Alexandrov geometry was obtained, which implies a lower bound on the length of an *almost closed geodesic*, i.e., an m-broken geodesic loop $\gamma_m = \{[p_ip_{i+1}]\}_{i=1}^m$ ($p_{m+1} = p_1$), with $\Theta(\gamma_m)$ very small while m is fixed. To state the result, we recall two notions in [BGP]: the n-dimensional rough volume of a subset $K \subseteq X$ is the limit, $V_{r_n}(K) = \lim\limits_{\epsilon\to 0}\epsilon^n \cdot \beta_X(\epsilon)$, where $\beta_X(\epsilon) = \max\{|\{x_i\}|,\ \{x_i\} \subseteq K \text{ is an } \epsilon\text{-discrete net}\}$. Clearly, rough volume is easier to estimate than the Hausdorff measure and $\mathrm{Haus}_n(X) \le V_{r_n}(X)$. Consider the following function in κ and $d > 0$ defined in [BGP]:

$$\psi(\kappa, d) = \max_{q,p,r\in S^2_\kappa}\left\{\frac{|pr|}{\measuredangle pqr},\ |qp|, |qr|, |pr| \le d, |pr| \ge 2||qp| - |qr||\right\}.$$

Theorem 0.6 ([BGP])**.** *Let X be a compact n-dimensional Alexandrov space of curv $\ge \kappa$. If γ_m is an m-broken geodesic loop, then the n-dimensional rough volume,*

$$V_{r_n}(X) \le \chi_m(\delta_1, \delta) \cdot d \cdot \psi^{n-1}(\kappa, d),$$

where $d = \mathrm{diam}(X)$,

$$\delta_1 = \frac{1}{\mathrm{diam}(X)} \max\{|p_i p_{i+1}|,\ 1 \le i \le m\},$$

$\max_i\{\theta_i\} \le \delta$*, and $\chi_m(\delta_1, \delta)$ is a constant depending on m, δ_1, and δ such that $\chi_m(\delta_1, \delta) \to 0$ as $\delta_1, \delta \to 0$ (m fixed).*

Theorem 0.6 implies a lower bound on the length of an almost closed geodesic, implicitly in terms of n, κ, d, and $V_{r_n}(X)$ (when m is fixed and $\delta \to 0$, δ_1 must have a positive lower bound; see Remark 8.7 in [BGP]). However, because $\chi_m(\delta_1, \delta) \to \infty$ as $m \to \infty$, Theorem 0.6 fails to imply a lower bound on the length of an m-broken geodesic loop (of length, say, one) with m large while $m\delta$ are very small (so both δ_1 and δ are small).

In view of the above, it is natural to ask if the sharp estimate in Theorem A holds in terms of the rough volume. First, the rough volume is not equivalent to the Hausdorff measure in general. For example, the set of rational numbers in $[0, 1]$ has rough volume 1, while its complement and $[0, 1]$ both have rough volume 1. This also shows that the rough volume does not have additivity. However, we can establish the equivalency for the two measures on the bounded subset which is open or has lower dimensional boundary. Note that this includes the closed set whose Hausdorff measure is zero. Since we can't find this equivalency in literature, for completeness we give a proof for the following result.

Theorem C. *Let $U \subseteq X \in \mathrm{Alex}^n(\kappa)$ be a bounded subset. If U is open or the Hausdorff dimension $\dim_H(\partial U) < n$, then*

$$V_{r_n}(U) = c(n) \cdot \mathrm{Haus}_n(U),$$

where $c(n) = \frac{V_{r_n}(I^n)}{\mathrm{Haus}_n(I^n)} = V_{r_n}(I^n)$, and I^n denotes a Euclidean unit n-cube.

Theorem C can be useful in practice; if one wants to prove a result involving an estimate for $\mathrm{Haus}_n(X)$, then one reduces to prove it with $V_{r_n}(X)$, which is much easier to estimate. As for the value of $c(n)$, except $c(1) = 1$ and $c(2) \geq \frac{2}{\sqrt{3}}$, not much is known.

A consequence of Corollary 0.2 and Theorem C is:

Corollary 0.7. *Let X be a compact n-dimensional Alexandrov space ($n \geq 2$) with curv $\geq \kappa$. If γ is an ϵ-closed geodesic, then*

$$L(\gamma) \geq (1-\epsilon) \cdot \frac{V_{r_n}(X)}{C(n) \cdot sn_\kappa^{n-1}(D_0)},$$

where $D_0 = \mathrm{diam}(X)$ for $\kappa \leq 0$ and $D_0 = \min\{\mathrm{diam}(X), \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$, and $C(n) = \frac{c(n) \cdot \mathrm{vol}(S_1^{n-2})}{n-1}$ and $c(n)$ is the constant in Theorem C.

Corollary 0.7 generalizes and improves Theorem 0.6 by providing an explicit sharp estimate for any ϵ-closed geodesic (including all m-broken geodesic loops with $m\delta$ relatively small).

We conclude the introduction by giving an indication for the proof of Theorem A. First, it is worth noting that our arguments also imply a new (metric) proof for Theorem 0.3, which does not require a Riemannian structure. Our approach is very different from the proof of Theorem 0.6 in [BGP], which follows the lines of the proof of Theorem 0.3 in [Ch]. Indeed, we found Theorem A after an unsuccessful attempt to remove the dependence on m from $\chi_m(\delta_1, \delta)$ in Theorem 0.6.

We take an elementary approach to estimate $\mathrm{Haus}_n(X)$ (in the case that $r = \mathrm{diam}(X)$): expressing $\mathrm{Haus}_n(X)$ as a "Riemann sum," bounding each term, and evaluating the "Riemann sum" of the bounds by identifying a proper integrant. Let $\gamma_m = \{[p_i p_{i+1}]\}_{i=1}^m$ be an m-broken geodesic loop approximating to a loop c in Theorem A, and divide $X = \bigcup_{i=1}^m X_i$ such that $\mathrm{Haus}_n(X) = \sum_{i=1}^m \mathrm{Haus}_n(X_i)$, where $X_i = \{x \in X \ |xp_i| \leq |xp_j|, \text{ for all } 1 \leq j \neq i \leq m\}$. Observe that if γ_m is a closed geodesic and $|p_i p_{i+1}|$ is sufficiently small, then X_i is like the "union of normal slices" over $[p_i p_{i+1}]$ (when X is a Riemannian manifold). So in spirit, we are estimating $\mathrm{Haus}_n(X)$ via a Riemann sum of a double integral: first over a normal slice at $\gamma_m(t)$, followed by an integral over γ_m. To obtain a sharp estimate for $\mathrm{Haus}_n(X_i)$, we apply a basic Hausdorff measure estimate (see Corollary 1.6), which bounds the Hausdorff measure of any subset $A \subseteq X$ in terms of the Hausdorff measure of the space of directions at any point $p \in X$, $|pA|$, and $\mathrm{diam}(A \cup \{p\})$. The key point in our proof is an estimate of the upper and lower bound for $\measuredangle x p_i p_{i+1} - \frac{\pi}{2}$, $x \in X_i - \{p_i\}$, in terms of $|p_i p_{i+1}|, |xp_i|$, and θ_i (see Lemma 1.3).

The rest of the paper is organized as follows:

In Section 1, we will prove Theorem A.

In Section 2, we will prove Theorem C.

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1. Loops and Hausdorff Measure

Throughout this paper, we will freely use basic notions and properties (such as the space of directions, rough volume, etc.) in Alexandrov geometry. These can be found in [BGP].

The goal in this section is to prove the following volume estimate, which easily implies Theorem A.

Theorem 1.1. *Let $X \in \mathrm{Alex}^n(\kappa)$ $(n \geq 2)$. If γ is a loop at p with $\gamma \subset B_r(p)$, then*

$$\mathrm{Haus}_n(B_r(p)) \leq \mathrm{vol}(S_1^{n-2}) \left[\frac{sn_\kappa^{n-1}(r_0)}{n-1} L(\gamma) + \Theta(\gamma) \int_0^r sn_\kappa^{n-1}(t) dt \right],$$

where $r_0 = r$ for $\kappa \leq 0$ and $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

To prove Theorem 1.1, it's sufficient to consider the case that γ is a broken geodesic loop. Given an m-broken geodesic loop, $p \in \gamma_m = \{[p_i p_{i+1}]\}_{i=1}^m \subset B_r(p)$, let $\theta_i = \pi - \measuredangle p_{i-1} p_i p_{i+1}$; then the turning angle $\Theta(\gamma_m) = \sum_{i=1}^m \theta_i$. We divide $B_r(p)$ into m subsets "centered" at p_i,

$$X_i = \{x \in B_r(p), \;\; |xp_i| \leq |xp_j|, \text{for all } j \neq i\}, \qquad 1 \leq i \leq m.$$

Clearly, $B_r(p) = \bigcup_i X_i$ and thus $\mathrm{Haus}_n(B_r(p)) \leq \sum_i \mathrm{Haus}_n(X_i)$. We first introduce a volume estimation formula for certain subsets in an Alexandrov space.

Lemma 1.2. *Let $B_r(p) \subset X \in \mathrm{Alex}^n(\kappa)$, and let $[pq]$ denote a geodesic in X from p to q. Given $0 \leq \alpha \leq \pi$, $0 \leq \theta < \pi$, and $L_1, L_2 > 0$, for $\eta > 0$ arbitrarily small, let*

$$A([pq], \alpha, L_1, L_2, \theta) = \{x \in B_r(p) - \{p\},$$
$$\frac{L_2}{\tan_\kappa |xp|} \leq \measuredangle xpq - \alpha + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}} \leq \frac{L_1}{\tan_\kappa |xp|} + \theta\}.$$

Then

$$\mathrm{Haus}_n(A)$$
$$\leq \mathrm{vol}(S_1^{n-2}) \left[\frac{(L_1 + L_2)\mathrm{sn}_\kappa^{n-1}(r_0)}{n-1} + \theta \cdot \int_0^r \mathrm{sn}_\kappa^{n-1}(t) dt + O(\eta^{\frac{3}{2}}) \right],$$

where $r_0 = r$ for $\kappa \leq 0$ and $r_0 = \min\{r, \frac{\pi}{2\sqrt{\kappa}}\}$ for $\kappa > 0$.

In fact, the following lemma shows that each X_i is contained in a certain subset shaped as in Lemma 1.2.

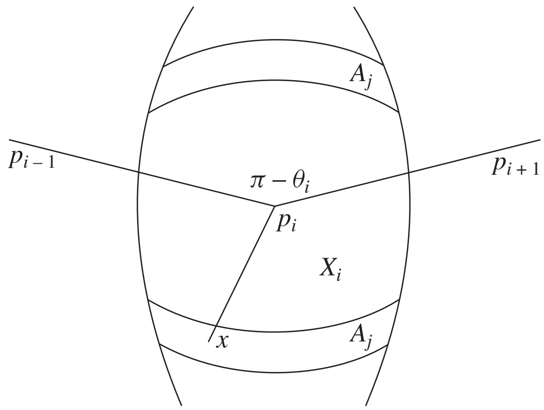

Figure 1

Lemma 1.3. *Let the assumptions be as in Theorem* 1.1 *and* θ_i, X_i *be defined as in Figure 1. For* $\epsilon > 0$, *there is* $\eta > 0$ *such that if* $\max_i\{|p_ip_{i+1}|\} < \eta$, *then for any* $x \in X_i - \{p_i\}$, *the following inequality holds:*

$$-\frac{e^{\epsilon}|p_ip_{i+1}|}{2\tan_{\kappa}|xp_i|} - \frac{36\eta^{\frac{3}{2}}}{|\tan_{\kappa}|xp_i||^{\frac{3}{2}}} \leq \measuredangle xp_ip_{i+1} - \frac{\pi}{2}$$
$$\leq \frac{e^{\epsilon}|p_ip_{i-1}|}{2\tan_{\kappa}|xp_i|} + \frac{36\eta^{\frac{3}{2}}}{|\tan_{\kappa}|xp_i||^{\frac{3}{2}}} + \theta_i,$$

where $\tan_{\kappa} t = \frac{sn_{\kappa}t}{sn'_{\kappa}(t)}$, *and when* $\kappa > 0$ *and* $|xp_i| = \frac{\pi}{2\sqrt{\kappa}}$, *the term* $\frac{36\eta^{\frac{3}{2}}}{|\tan_{\kappa}|xp_i||^{\frac{3}{2}}}$ *is defined to be zero.*

Assuming Lemmas 1.2 and 1.3, we can give a proof for Theorem 1.1.

Proof of Theorem 1.1. It's sufficient to prove for an m-broken geodesic γ_m, in which p is one of the vertices. For any $\epsilon > 0$, evenly adding $N(\epsilon)$ "broken" points, we may assume that the broken geodesic γ_m satisfies that $|p_ip_{i+1}| < \eta$ for all i, where η is given as in Lemma 1.3. Put $L_1^i = \frac{e^{\epsilon}|p_{i-1}p_i|}{2}$ and $L_2^i = \frac{e^{\epsilon}|p_ip_{i+1}|}{2}$. By Lemma 1.3, we see that $X_i \subseteq A([p_ip_{i+1}], \frac{\pi}{2}, L_1^i, L_2^i, \theta_i)$, and by Lemma 1.2,

$$\mathrm{Haus}_n(X_i) \leq \mathrm{Haus}_n(A([p_ip_{i+1}], \frac{\pi}{2}, L_1^i, L_2^i, \theta_i))$$
$$\leq \mathrm{vol}(S_1^{n-2}) \left[\frac{e^{\epsilon}(|p_{i-1}p_i| + |p_ip_{i+1}|)}{2} \cdot \frac{\mathrm{sn}_{\kappa}^{n-1}(r_0)}{n-1} + \theta_i \cdot \int_0^r \mathrm{sn}_{\kappa}^{n-1}(t)dt + O(\eta^{\frac{3}{2}})\right].$$

Then

$$\begin{aligned}
\mathrm{Haus}_n(B_r(p)) &\le \sum_{i=1}^{m+N(\epsilon)} \mathrm{Haus}_n(X_i) \\
&\le e^{\epsilon}\cdot \mathrm{vol}(S_1^{n-2}) \left[\left(\sum_{i=1}^{m+N(\epsilon)} \frac{|p_ip_{i+1}| + |p_1p_{i-1}|}{2} \right) \cdot \frac{\mathrm{sn}_\kappa^{n-1}(r_0)}{n-1} \right. \\
&\quad \left. + \sum_{i=1}^{m+N(\epsilon)} \theta_i \cdot \int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt + O(\eta^{\frac{1}{2}}) \right], \quad \left(\text{because } \eta \approx \frac{L(\gamma_m)}{m+N(\epsilon)}\right)
\end{aligned}$$

and the desired inequality follows when $\epsilon \to 0$, and thus $N(\epsilon) \to \infty$ and $\eta \to 0$. q.e.d.

To show Lemma 1.2, we need to divide A (or X_i in our context) into thin annulus A_j, and then apply an explicit volume formula for κ-cones (see Lemma 1.4).

For $\Sigma \in \mathrm{Alex}^{n-1}(1)$, one can construct an n-dimensional Alexandrov space $C_\kappa(\Sigma)$ with curv $\ge \kappa$ (cf. [BGP]): for $\kappa \le 0$, let $C_\kappa(\Sigma) = (\Sigma \times \mathbb{R})/(\Sigma \times \{0\})$ denote a cone over Σ, and for $\kappa > 0$, let $C_\kappa(\Sigma) = (\Sigma \times [0, \frac{\pi}{\sqrt{\kappa}}])/(\Sigma \times \{0\}, \Sigma \times \{\frac{\pi}{\sqrt{\kappa}}\})$ denote the suspension over Σ. We define a metric d on $C_\kappa(\Sigma)$ via the cosine law in the space form of constant sectional curvature κ. For instance, if $\kappa = 0$, then for $(x,t), (x',t') \in (\Sigma \times \mathbb{R})/(\Sigma \times \{0\})$,

$$d((x,t),(x',t'))^2 = t^2 + (t')^2 - 2tt' \cos |xx'|_\Sigma.$$

Note that for any $X \in \mathrm{Alex}^n(\kappa)$ and $p \in X$, the space of directions $\Sigma_p \in \mathrm{Alex}^{n-1}(1)$, and thus we get $C_\kappa(\Sigma_p) \in \mathrm{Alex}^n(\kappa)$ for a given κ. If $k > 0$, then $\mathrm{diam}(C_\kappa(\Sigma)) = \frac{\pi}{\sqrt{\kappa}}$.

Given $\Sigma \in \mathrm{Alex}^{n-1}(1)$ and $0 \le r_1 < r_2$, let

$$A_{r_1}^{r_2}(\Gamma) = \{x \in C_\kappa(\Sigma) : [px] \in \Gamma \text{ and } r_1 \le |px| \le r_2\},$$

where p is the vertex of the κ-cone $C_\kappa(\Gamma)$, which is a κ-suspension for $\kappa > 0$ (in particular, $r_2 \le \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$).

Lemma 1.4. *Let $A_{r_1}^{r_2}(\Gamma)$ be defined as in the above. Then*

$$\mathrm{Haus}_n(A_{r_1}^{r_2}(\Gamma)) = \mathrm{Haus}_{n-1}(\Gamma) \cdot \int_{r_1}^{r_2} sn_\kappa^{n-1}(t)dt.$$

Lemma 1.4 is clear if one assumes the co-area formula for Alexandrov spaces ([BGP], 10.6 in [BBI]). Since we do not find a proof in literature for the co-area formula, for completeness we will present an elementary proof using the cosine law in κ-space form.

Corollary 1.5.

$$\mathrm{Haus}_n(B_r(C_\kappa(\Gamma)) = \mathrm{Haus}_{n-1}(\Gamma)\cdot\int_0^r sn_\kappa^{n-1}(t)dt.$$

Corollary 1.6. *Let $X \in \mathrm{Alex}^n(\kappa)$. Given any bounded subset $A \subseteq X$, and $p \in X$, then*

$$\mathrm{Haus}_n(A) \le \mathrm{Haus}_{n-1}(\Gamma_p(A))\int_{r_1}^{r_2} sn_\kappa^{n-1}(t)dt, \tag{1.1}$$

where $\Gamma_p(A) = \{\uparrow_p^q \in \Sigma_p : q \in A\}$, $r_1 = \min_{x\in A}\{|px|\}$ and $r_2 = \max_{x\in A}\{|xp|\}$.

Corollary 1.6 may be viewed as an explicit (Hausdorff measure) version of the comparison theorem in [BGP] Lemma 8.2. One can also see it from Corollary 10.13 in [BGP], assuming the co-area formula for Alexandrov spaces.

Proof of Lemma 1.2. Let $A = A([p,q],\alpha,L_1,L_2,\theta)$. Given a partition for $[0,1] : 0 = a_0 < a_1 < \cdots < a_N = 1$, let $r_j = a_j r$, $A_j = \{x \in A,\ r_j \le |xp| \le r_{j+1}\}$, $1 \le j \le N$. If $\kappa > 0$ and $d > \frac{\pi}{2\sqrt{\kappa}}$, we will choose $\{a_j\}$ such that some $r_j = \frac{\pi}{2\sqrt{\kappa}}$ (note that some A_j may be an empty set; for instance, if $\theta = 0$, then $A_j = \varnothing$ when $r_j > \frac{\pi}{2\sqrt{\kappa}}$ because $\tan_\kappa |xp_i| < 0$). For $x \in A_j$,

$$-\frac{L_2}{\tan_\kappa |xp|} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}} \le \measuredangle xpq - \alpha \le \frac{L_1}{\tan_\kappa |xp|} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |xp||^{\frac{3}{2}}}$$

implies

$$\begin{aligned} -\frac{L_2}{\tan_\kappa(c_j)} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |c_j||^{\frac{3}{2}}} &\le \measuredangle xpq - \alpha \\ &\le \frac{L_1}{\tan_\kappa(c_j)} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |c_j||^{\frac{3}{2}}}, \end{aligned} \tag{1.2}$$

where $c_j = r_{j+1}$ when $\kappa \le 0$ or $\kappa > 0$ and $r_{j+1} \le \frac{\pi}{2\sqrt{\kappa}}$; otherwise $c_j = r_j$. Let $\Gamma_j = \{[xp] \in \Sigma_p(X),\ x \in A_j\}$. Because $\mathrm{curv}(\Sigma_{[pq]}(\Sigma_p)) \ge 1$, $\mathrm{vol}(\Sigma_{[pq]}(\Gamma_j)) \le \mathrm{vol}(S_1^{n-2})$, where $\Sigma_{[pq]}(\Gamma_j)$ denotes the space of directions of Γ_j at $[pq] \in \Gamma_j$. Applying Corollary 1.6 to Γ_j at $[pq]$, by $\mathrm{curv}(\Sigma_p) \ge 1$ and (1.2) we have

$$\begin{aligned} \mathrm{Haus}_{n-1}(\Gamma_j) &\le \mathrm{vol}(\Sigma_{[pq]}(\Gamma_j))\cdot\int_{\alpha_2}^{\alpha_1} \sin^{n-2}(t)dt \\ &\le \mathrm{vol}(S_1^{n-2})\cdot\left(\frac{L_1+L_2}{\tan_\kappa(c_j)} + \theta + \frac{72\eta^{\frac{3}{2}}}{|\tan_\kappa(c_j)|^{\frac{3}{2}}}\right), \end{aligned} \tag{1.3}$$

where $\alpha_1 = \alpha + \frac{L_1}{\tan_\kappa(c_j)} + \theta + \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |c_j||^{\frac{3}{2}}}$ and $\alpha_2 = \alpha - \frac{L_2}{\tan_\kappa(c_j)} - \frac{36\eta^{\frac{3}{2}}}{|\tan_\kappa |c_j||^{\frac{3}{2}}}$. For $\epsilon > 0$, when $\triangle_j = r_{j+1} - r_j$ is sufficiently small, we may assume that $\frac{sn_\kappa^{n-1}(r_{j+1})}{sn_\kappa(r_j)} \le e^\epsilon sn_\kappa^{n-2}(r_j)$.

Case 1. Assume $\kappa \le 0$ or $\kappa > 0$ and $d \le \frac{\pi}{2\sqrt{\kappa}}$. By applying Corollary 1.6 to A_j: from (1.3) we get

$$
\begin{aligned}
\mathrm{Haus}_n(A_j) &\le \mathrm{Haus}_{n-1}(\Gamma_j) \int_{r_j}^{r_{j+1}} \mathrm{sn}_\kappa^{n-1}(t)dt \\
&\le \mathrm{Haus}_{n-1}(\Gamma_j)(r_{j+1} - r_j)\mathrm{sn}_\kappa^{n-1}(c_j) \\
&\le \mathrm{vol}(S_1^{n-2}) \left(\frac{L_1 + L_2}{\tan_\kappa(c_j)} + \theta + \frac{72\eta^{\frac{3}{2}}}{|\tan_\kappa(c_j)|^{\frac{3}{2}}} \right) \mathrm{sn}_\kappa^{n-1}(c_j)\Delta_j \\
&\le e^\epsilon \cdot \mathrm{vol}(S_1^{n-2}) \Big[(L_1 + L_2)\mathrm{sn}_\kappa^{n-2}(c_j)\mathrm{sn}'_\kappa(c_j) + \theta \cdot \mathrm{sn}_\kappa^{n-1}(c_j) \\
&\qquad + 72\eta^{\frac{3}{2}} \mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j) \cdot |\mathrm{sn}'_\kappa(c_j)|^{\frac{3}{2}} \Big] \Delta_j. \qquad (1.4)
\end{aligned}
$$

Then

$$
\begin{aligned}
e^{-\epsilon} \cdot \mathrm{Haus}_n(A) &= e^{-\epsilon} \cdot \sum_{j=1}^{N} \mathrm{Haus}_n(A_j) \\
&\le \mathrm{vol}(S_1^{n-2})(L_1 + L_2) \sum_{j=0}^{N} sn_\kappa^{n-2}(c_j) sn'_\kappa(c_j)\Delta_j \\
&\quad + \theta \sum_{j=0}^{N} sn_\kappa^{n-1}(c_j)\Delta_j + 72\eta^{\frac{3}{2}} \sum_{j=0}^{N} \mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j) \cdot |\mathrm{sn}'_\kappa(c_j)|^{\frac{3}{2}} \Delta_j. \qquad (1.5)
\end{aligned}
$$

Finally, view (1.5) as the Riemann sum of some integrals and let $N \to \infty$. Note that for $n = 2$, $\int_0^r sn_\kappa^{-\frac{1}{2}}(t) \cdot |sn'_\kappa(t)|^{\frac{3}{2}} dt < \infty$ because $sn_\kappa^{-\frac{1}{2}}(t) = t^{-\frac{1}{2}} + o(t)$, we get

$$
\begin{aligned}
\mathrm{Haus}_n(A) &\le e^\epsilon \cdot \mathrm{vol}(S_1^{n-2}) \left[(L_1 + L_2) \int_0^{r_0} \mathrm{sn}_\kappa^{n-2}(t)\mathrm{sn}'_\kappa(t)dt \right. \\
&\qquad \left. + \theta \cdot \int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt + 72\eta^{\frac{3}{2}} \int_0^r \mathrm{sn}_\kappa^{n-\frac{5}{2}}(t) \cdot |\mathrm{sn}'_\kappa(t)|^{\frac{3}{2}} dt \right] \\
&= \mathrm{vol}(S_1^{n-2}) \left[e^\epsilon \cdot \frac{(L_1 + L_2)\mathrm{sn}_\kappa^{n-1}(r_0)}{n-1} + \theta \cdot \int_0^r \mathrm{sn}_\kappa^{n-1}(t)dt + O(\eta^{\frac{3}{2}}) \right].
\end{aligned}
$$

Letting $\epsilon \to 0$, we see the desired result.

Case 2. Assume $\kappa > 0$ and $d > \frac{\pi}{2\sqrt{\kappa}}$. For A_j with $c_j \le \frac{\pi}{2\sqrt{\kappa}}$, the estimate in (1.4) is still valid. If $c_j > \frac{\pi}{2\sqrt{\kappa}}$, then we modify the estimate

(1.3) by throwing out the negative term with "$\tan_\kappa(c_j) \le 0$", and obtain

$$\begin{aligned}&\mathrm{Haus}_n(A_j)\\ &\le e^\epsilon \cdot \mathrm{vol}(S_1^{n-2})[\theta \cdot sn_\kappa^{n-1}(c_j) + 72\eta^{\frac{3}{2}} \mathrm{sn}_\kappa^{n-\frac{5}{2}}(c_j)(\mathrm{sn}'_\kappa(c_j))^2]\triangle_i.\end{aligned} \tag{1.6}$$

Combining (1.4) for $c_j \le \frac{\pi}{2\sqrt{\kappa}}$ and (1.6), we derive

$$\begin{aligned}\mathrm{Haus}_n(A) &= \sum_{j=1}^{N} V_{r_n}(A_j)\\ &\le e^\epsilon \cdot \mathrm{vol}(S_1^{n-2})(L_1+L_2) \sum_{j=0}^{r_{j+1}\le\frac{\pi}{2\sqrt{\kappa}}} sn_\kappa^{n-2}(c_j) sn'_\kappa(c_j)\Delta_j\\ &\quad + \theta \sum_{j=0}^{N} sn_\kappa^{n-1}(r_j)\Delta_j + O(\eta^{\frac{3}{2}}).\end{aligned} \tag{1.7}$$

In (1.7), letting $N \to \infty$ and $\epsilon \to 0$, we get

$$\begin{aligned}&\mathrm{Haus}_n(A)\\ &\le \mathrm{vol}(S_1^{n-2})\left[(L_1+L_2)\int_0^{r_0} sn_\kappa^{n-2}(t) sn'_\kappa(t)dt + \theta\int_0^r sn_\kappa^{n-1}(t)dt\right]\\ &= \mathrm{vol}(S_1^{n-2})\left[\frac{(L_1+L_2)sn_\kappa^{n-1}(r_0)}{n-1} + \theta\int_0^r sn_\kappa^{n-1}(t)dt\right].\end{aligned}$$

q.e.d.

Proof of Lemma 1.3. For $\epsilon > 0$, we may choose η small so that for all i, $\frac{|p_ip_{i+1}|}{2} < \eta$ implies that $\tan_\kappa \frac{|p_ip_{i+1}|}{2} \le e^\epsilon \cdot \frac{|p_ip_{i+1}|}{2}$. We first claim that

$$\cos \tilde{\measuredangle} x p_i p_{i+1} \le \frac{e^\epsilon \cdot |p_ip_{i+1}|}{2\tan_\kappa(|xp_i|)}, \tag{1.8}$$

where $\tilde{\measuredangle} x p_i p_{i+1}$ denotes the corresponding angle in the comparison triangle $\tilde{\triangle} x p_i p_{i+1} \subset S_\kappa^2$. The proof of the claim relies on the cosine law in the κ-space form. We will give a proof for the case $\kappa = 0, \kappa = -1$, and $\kappa = 1$. The general case follows by an analog modification.

Case 1. Assume $\kappa = 0$. By the cosine law and by the fact that $|xp_i| \le |xp_{i+1}|$, we derive

$$\begin{aligned}\cos \tilde{\measuredangle} x p_i p_{i+1} &= \frac{|xp_i|^2 + |p_ip_{i+1}|^2 - |xp_{i+1}|^2}{2|xp_i| \cdot |p_ip_{i+1}|}\\ &\le \frac{|xp_i|^2 + |p_ip_{i+1}|^2 - |xp_i|^2}{2|xp_i| \cdot |p_ip_{i+1}|} = \frac{|p_ip_{i+1}|}{2|xp_i|} = \frac{|p_ip_{i+1}|}{2\tan_0(|xp_i|)}.\end{aligned} \tag{1.9}$$

Case 2. Assume $\kappa = -1$. By the cosine law and $|xp_i| \le |xp_{i+1}|$, we derive

$$\cos \tilde{\measuredangle} xp_ip_{i+1} = \frac{\cosh|xp_i|\cosh|p_ip_{i+1}| - \cosh|xp_{i+1}|}{\sinh|xp_i|\sinh|p_ip_{i+1}|}$$

$$\le \frac{\cosh|xp_i|}{\sinh|xp_i|}\cdot\frac{\cosh|p_ip_{i+1}|-1}{\sinh|p_ip_{i+1}|} = \frac{\tanh\frac{|p_ip_{i+1}|}{2}}{\tanh|xp_i|} \le \frac{|p_ip_{i+1}|}{2\tanh|xp_i|}. \tag{1.10}$$

Case 3. Assume $\kappa = 1$. Again by the cosine law and $|xp_i| \le |xp_{i+1}|$, we derive:

$$\cos \tilde{\measuredangle} xp_ip_{i+1} = \frac{\cos|xp_{i+1}| - \cos|xp_i|\cos|p_ip_{i+1}|}{\sin|xp_i|\sin|p_ip_{i+1}|}$$

$$\le \frac{\cos|xp_i| - \cos|xp_i|\cos|p_ip_{i+1}|}{\sin|xp_i|\sin|p_ip_{i+1}|}$$

$$= \frac{\cos|xp_i|2\sin^2\frac{|p_ip_{i+1}|}{2}}{\sin|xp_i|2\sin\frac{|p_ip_{i+1}|}{2}\cos\frac{|p_ip_{i+1}|}{2}} = \frac{\tan\frac{|p_ip_{i+1}|}{2}}{\tan|xp_i|} \le \frac{e^\epsilon\cdot|p_ip_{i+1}|}{2\tan|xp_i|}. \tag{1.11}$$

By now, (1.8) follows from (1.9)–(1.11). Next, we shall show that the inequality, $u \ge \cos\alpha$, implies

$$\alpha \ge \frac{\pi}{2} - u - 36|u|^{\frac{3}{2}}. \tag{1.12}$$

(This will give the left-hand side inequality in Lemma 1.3.) Note that, in our case, we may assume $0 \le \alpha \le \pi$. Thus, if $u \ge 1$ or $u \le -1$, then (1.12) holds. On the other hand, for $u \in (-1,1)$, it's sufficient to show $\cos^{-1} u \ge \frac{\pi}{2} - u - 36|u|^{3/2}$; equivalently, the function

$$f(u) = u + 36|u|^{3/2} - \frac{\pi}{2} + \cos^{-1} u \ge 0.$$

By direct calculation,

$$f'(u) = 1 + 54\cdot\mathrm{sign}(u)|u|^{1/2} - \frac{1}{\sqrt{1-u^2}}, \qquad f''(u) = \frac{27}{|u|^{1/2}} - \frac{u}{(1-u^2)^{3/2}}.$$

For $-1 < u < \frac{5\sqrt{13}-1}{18}$, it's easy to see that $f''(u) > 0$ and $u = 0$ is the only critical point for $f(u)$. Consequently, $f(0)$ is the global minimum for $0 < u < \frac{5\sqrt{13}-1}{18}$. For $\frac{5\sqrt{13}-1}{18} < u < 1$, $f''(u) < 0$ and thus the minimum of $f(u)$ is achieved at the end points. Note that $f(0) = 0$ and $f(1) > 0$; we get that $f(u) \ge 0$ for all $u \in (-1,1)$. Plugging in (1.12) with $\alpha = \measuredangle xp_ip_{i+1}$ and $u = \frac{e^\epsilon\cdot|p_ip_{i+1}|}{2\tan_\kappa|xp_i|}$, we obtain

$$\measuredangle xp_ip_{i+1} \ge \frac{\pi}{2} - \frac{e^\epsilon|p_ip_{i+1}|}{2\tan_\kappa|xp_i|} - 36\left(\frac{e^\epsilon|p_ip_{i+1}|}{2|\tan_\kappa|xp_i||}\right)^{3/2}$$

$$\ge \frac{\pi}{2} - \frac{e^\epsilon|p_ip_{i+1}|}{2\tan_\kappa|xp_i|} - \frac{36\eta^{3/2}}{|\tan_\kappa|xp_i||^{3/2}}. \tag{1.13}$$

Similarly applying $|xp_i| \le |xp_{i-1}|$ to the above 3 cases, we obtain

$$\measuredangle xp_ip_{i-1} \ge \frac{\pi}{2} - \frac{e^\epsilon |p_ip_{i-1}|}{2\tan_\kappa |xp_i|} - \frac{36\eta^{3/2}}{|\tan_\kappa |xp_i||^{3/2}}. \tag{1.14}$$

Plugging (1.13), (1.14), and $\measuredangle p_{i-1}p_ip_{i+1} = \pi - \theta_i$ into the condition (B) in [BGP]:

$$\measuredangle p_{i-1}p_ip_{i+1} + \measuredangle xp_ip_{i-1} + \measuredangle xp_ip_{i+1} \le 2\pi,$$

we get the right-hand side of the inequality in Lemma 1.3. q.e.d.

As mentioned in the Introduction (see Theorem 0.6 and comments following it), we did not succeed in an early attempt to modify the proof of Theorem 0.6 in [BGP] in order to remove the dependence on m from $\chi_m(\delta_1, \delta)$ and factor out $L(\gamma_m)$ from $\chi_m(\delta_1, \delta)$. We would like to conclude this section by explaining the reason for this failure. The proof in [BGP] is, following the idea in [Ch], to divide X into two parts and estimate their rough volumes: one part, U_{δ_1}, is like a δ_1-tube around γ_m, and the other part is $X - U_{\delta_1}$. Since points in $X - U_{\delta_1}$ are a definite distance away from $\{p_i\}$, this allowed [BGP] to have an estimate for the diameter of the directions pointing to points in $X - U_{\delta_1}$, in terms of δ_1, δ, and m. Unfortunately, the rough volumes of two parts in terms of δ_1 are in different order; that makes it impossible to remove the dependence on m, or to factor $L(\gamma_m)$ from $\chi_m(\delta_1, \delta)$.

2. Hausdorff Measure and Rough Volume

Our proof of Theorem C relies on the local structure of an Alexandrov space, which we briefly recall (see [BGP] for details). The notion of an (n, δ)-strainer may be viewed as a counterpart of a normal coordinate on a Riemannian manifold, defined as follows: for $p \in X$, the set of n-pairs of points $\{(p_i, q_i)\}_{i=1}^n$ is called an (n, δ)-strainer at p, if

$$\measuredangle p_ipp_j - \frac{\pi}{2} < \delta, \quad \measuredangle p_ipq_i - \pi < \delta, \quad \measuredangle q_ipq_j - \frac{\pi}{2} < \delta. \quad (1 \le i \ne j \le n)$$

We call the number, $\rho = \min\{|pp_i|, |pq_i|\}$, the radius of the (n, δ)-strainer. By continuity, the subset of points with an (n, δ)-strainer is open in X. Let S_δ denote the set of points admitting no (n, δ)-strainer. Then S_δ is a closed subset whose Hausdorff dimension $\dim_H(S_\delta) \le n-1$.

Given a bounded set $U \subseteq X \in \mathrm{Alex}^n(\kappa)$, we divide U into the "regular" part $U - S_\delta$ and the "singular" part S_δ. On the regular part, we have

Lemma 2.1 ([BGP] Theorem 9.4)**.** *Let $X \in Alex^n(\kappa)$. If $p \in X$ has an (n, δ)-strainer with radius $\rho > 0$, then there are $\epsilon = \epsilon(n, \delta, \rho) > 0$ and $\eta(n, \delta, \rho) > 0$ such that $B_\eta(p)$ is e^ϵ bi-Lipschitz to an open subset in $\mathbb{R}^n$. Moreover, $\epsilon \to 0$ as $\delta \to 0$.*

For our convenience, we call a subset U a region in a metric space with Hausdorff dimension n if the interior of U is non-empty and $\dim_H(\partial U) < n$. By Lemma 2.6, if $U \subseteq X \in \text{Alex}^n(\kappa)$ is a bounded region, then $V_{r_n}(U) = V_{r_n}(\overset{\circ}{U})$. In the following we show that Theorem C is true if $X = \mathbb{R}^n$. In particular, U has no singular point.

Lemma 2.2. *Let $U \subset \mathbb{R}^n$ be a bounded region. Then*

$$V_{r_n}(U) = c(n) \cdot Haus_n(U),$$

where $c(n) = \frac{V_{r_n}(I^n)}{Haus_n(I^n)}$ and I^n is a unit n-cube in $\mathbb{R}^n$.

Proof. By Lemma 2.6, it's sufficient to prove Lemma 2.2 for a bounded open set U. Note that $\text{Haus}_n(I^n(r)) = r^n \cdot \text{Haus}_n(I^n)$ and $V_{r_n}(I^n(r)) = r^n \cdot V_{r_n}(I^n)$, and thus for any $r > 0$,

$$V_{r_n}(I^n(r)) = c(n) \cdot \text{Haus}_n(I^n(r)). \tag{2.1}$$

It's clear that

$$V_{r_n}(I_1^n(r_1) \cup I_2^n(r_2)) = V_{r_n}(I_1^n(r_1)) + V_{r_n}(I_2^n(r_2)). \tag{2.2}$$

We approximate U by the finite union of n-cubes whose interiors have no overlap with each other. Let T_j and T_k be such approximations satisfying

$$T_1 \subset T_2 \subset \cdots \subset T_j \subset \cdots U \cdots \subset W_k \subset \cdots \subset W_2 \subset W_1$$

$$\text{and} \quad \bigcup_j T_j = U = \overset{k}{\cap} W_k.$$

By (2.2),

$$\begin{aligned} V_{r_n}(U) \geq V_{r_n}(T_j) &= \sum_{\alpha \in T_j} V_{r_n}(I_\alpha^n) \\ &= \sum_{\alpha \in T_j} c(n) \cdot \text{Haus}_n(I_\alpha^n) = c(n) \cdot \text{Haus}_n(T_j). \end{aligned}$$

Similarly,

$$V_{r_n}(U) \leq c(n) \cdot \text{Haus}_n(W_k).$$

Letting $j, k \to \infty$, we get the desired equality. q.e.d.

Using Lemma 2.1 and 2.2, one can get the equivalence for the regular part in U. As mentioned in the introduction, for any set S, $\text{Haus}_n(S) = 0$ may not imply $V_{r_n}(S) = 0$. We shall show that this is true in our context (see Lemma 2.6).

Lemma 2.4 will be used to improve the following rough volume estimate and get Corollary 2.5. This corollary will be used to deal with the singular part in U (i.e., show Lemma 2.6). Comparing Corollary 2.5 with Corollary 8.4 in [BGP], the latter one has the form $V_{r_n}(B_r(p)) \leq c(n, \kappa, r)$, which is inadequate in our approach for Lemma 2.6.

Lemma 2.3 ([BGP], Lemma 8.2)**.** *Let $X \in Alex^n(\kappa)$. Given any subset $A \subseteq X$, and $p \in M$,*

$$V_{r_n}(A) \leq 2d_1\psi^{n-1}(\kappa, d)V_{r_{n-1}}(\Gamma_p),$$

where $d_1 = diam(A \cup \{p\})$, $d = \max_{x\in A}\{|px|\} - \min_{x\in A}\{|px|\}$ and $\Gamma_p \subseteq \Sigma_p$ consists of geodesic $[pa]$ for every point $a \in A - \{p\}$.

Lemma 2.4. *The function $\psi(\kappa, d)$ satisfies the following inequalities:*

$$\frac{2}{3} \cdot sn_\kappa(d) \leq \psi(\kappa, d) \leq 2 \cdot sn_\kappa(d),$$

provided $d < \frac{\pi}{2\sqrt{\kappa}}$ when $\kappa > 0$, where $sn_\kappa(r)$ is defined in Theorem A.

We will leave the proof of Lemma 2.4 for the end of this section. Combining Lemmas 2.3 and 2.4, we get

Corollary 2.5. *Let $p \in X \in \mathrm{Alex}^n(\kappa)$. Then for any $r > 0$, $V_{r_n}(B_r(p)) \leq c(n, \kappa) \cdot r^n$, where $c(n, \kappa) > 0$ is a constant depending only on n and κ.*

Lemma 2.6. *Let $S \subset X \in \mathrm{Alex}^n(\kappa)$ be a compact subset with $\mathrm{Haus}_n(S) = 0$. Then*

(2.6.1) $V_{r_n}(S) = 0$,
(2.6.2) *there is a sequence $\mu_i \searrow 0$ such that $V_{r_n}(B_{\mu_i}(S)) \to 0$ as $i \to \infty$.*

Proof. We argue by contradiction for (2.6.1). If it is not so, then there is a sequence $\epsilon_i \to 0$, and ϵ_i-net $\{x_i^k\}_{k=1}^{\beta(\epsilon_i)} \subset S$ such that

$$\epsilon_i^n \cdot \beta(\epsilon_i) \to V_{r_n}(S) > 0. \tag{2.3}$$

Let $B_j(S) = \{x \in X : \text{there is } h \in S \text{ such that } |xh| < 1/j\}$ denote the j^{-1}-tubular neighborhood of S. Because S is closed, $S \subset \cdots \subset B_2 \subset B_1$, and $\bigcap_j B_j = S$. Consequently,

$$\mathrm{Haus}_n(B_j) \to \mathrm{Haus}_n(S) = 0. \tag{2.4}$$

Given any large j, choose $\epsilon_i \leq j^{-1}$, and we have

$$\bigcup_k B_{\frac{\epsilon_i}{2}}(x_i^k) \subseteq B_j, \qquad B_{\frac{\epsilon_i}{2}}(x_i^k) \cap B_{\frac{\epsilon_i}{2}}(x_i^l) = \varnothing, \quad k \neq l$$

and thus

$$\begin{aligned} &\beta(\epsilon_i) \cdot \min_k\{\mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k))\} \\ &\leq \sum_k \mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k)) \leq \mathrm{Haus}_n(B_j). \end{aligned} \tag{2.5}$$

By Bishop-Gromov relative volume comparison for Alexandrov spaces ([BGP]), we have that for any $p \in X$ and $r > 0$,

$$\mathrm{Haus}_n(B_r(p)) \geq \frac{\mathrm{Haus}_n(X)}{\mathrm{vol}(B^\kappa_{\mathrm{diam}(X)})} \cdot \mathrm{vol}(B^\kappa_r) = c(n, \kappa, X) \cdot r^n > 0.$$

In particular, $\mathrm{Haus}_n(B_{\frac{\epsilon_i}{2}}(x_i^k)) \geq c(n,\kappa,X)\cdot(\frac{\epsilon_i}{2})^n$, and thus (2.5) implies

$$\mathrm{Haus}_n(B_j) \geq \beta(\epsilon_i)\cdot c(n,\kappa,X)\cdot(\frac{\epsilon_i}{2})^n = \frac{c(n,\kappa,X)}{2^n}\cdot\epsilon_i^n\beta(\epsilon_i). \tag{2.6}$$

Let $\epsilon_i \to 0$; we get a contradiction with (2.3) and (2.4).

To prove (2.6.2), by (2.6.1), we may assume a sequence of $\epsilon_i \to 0$ and a sequence of finite ϵ_i-net $\{x_i^k\}_{i=1}^{\beta(\epsilon_i)} \subset S$ such that $\epsilon_i^n\cdot\beta(\epsilon_i) \leq i^{-1}$. Since $\{B_{\epsilon_i}(x_i^k)\}_{i=1}^{\beta(\epsilon_i)}$ is a finite open cover for S, we may assume $0 < \mu_i < \epsilon_i$ such that

$$B_{\mu_i}(S) \subseteq \bigcup_k B_{\epsilon_i}(x_i^k),$$

and thus

$$V_{r_n}(B_{\mu_i}(S)) \leq \sum_k V_{r_n}(B_{\epsilon_i}(x_i^k)) \leq \beta(\epsilon_i)\cdot\max_k\{V_{r_n}(B_{\epsilon_i}(x_i^k))\}.$$

By Corollary 2.5,

$$V_{r_n}(B_{\epsilon_i}(x_i^k)) \leq c(n,\kappa)\epsilon_i^n,$$

and thus

$$V_{r_n}(B_{\mu_i}(S)) \leq c(n,\kappa)\cdot(\epsilon_i^n\cdot\beta(\epsilon_i)) \leq i^{-1}\cdot c(n,\kappa).$$

q.e.d.

Since S_δ is closed and $\dim_H(S_\delta) \leq n-1$ for δ small, by Lemma 2.6, we have the following.

Corollary 2.7. *Let $X \in Alex^n(\kappa)$. Then for $\delta > 0$ small, $V_{r_n}(S_\delta) = 0$ and there is a sequence $\mu_i \searrow 0$ such that $V_{r_n}(B_{\mu_i}(S_\delta)) \to 0$ as $i \to \infty$.*

Now we are ready to prove Theorem C.

Proof of Theorem C. Due to Lemma 2.6, it's sufficient to prove for a bounded open set U. Fix small $\delta > 0$ and take a sequence $\mu_i \searrow 0$. The idea is to divide U into the disjoint union $B_{\mu_i}(S_\delta) \cup (U - B_{\mu_i}(S_\delta))$ and verify that

$$\lim_{i\to\infty} V_{r_n}(B_{\mu_i}(S_\delta)) = 0 \quad \text{and} \tag{2.7}$$

$$V_{r_n}(U - B_{\mu_i}(S_\delta)) = c(n)\cdot\mathrm{Haus}_n(U - B_{\mu_i}(S_\delta)). \tag{2.8}$$

By (2.7) and $V_{r_n}(U) \leq V_{r_n}(U - B_{\mu_i}(S_\delta)) + V_{r_n}(B_{\mu_i}(S_\delta))$, we get

$$V_{r_n}(U) \leq \lim_{i\to\infty} V_{r_n}(U - B_{\mu_i}(S_\delta)) \leq V_{r_n}(U).$$

Together with (2.8),

$$\begin{aligned} V_{r_n}(U) &= \lim_{i\to\infty} V_{r_n}(U - B_{\mu_i}(S_\delta)) \\ &= \lim_{i\to\infty} c(n)\cdot\mathrm{Haus}_n(U - B_{\mu_i}(S_\delta)) = c(n)\cdot\mathrm{Haus}_n(U). \end{aligned}$$

(2.7) is satisfied due to Corollary 2.7. It remains to show (2.8). For each μ_i, because the closure of $U - B_{\mu_i}(S_\delta)$ is compact, we can conclude that every point in $U - B_{\mu_i}(S_\delta)$ has an (n, δ)-strainer with radius $\rho = \rho(n, \delta, \mu_i) > 0$. (If not, then there is a sequence $x_j \in U - B_{\mu_i}(S_\delta)$ such that the (n, δ)-strainer at x_j has radius $\rho_i \to 0$. Passing to a subsequence, we may assume $x_j \to x \in U - B_{\mu_i}(S_\delta)$. Because the (n, δ)-strainer at x has radius $\rho > 0$, by definition we see that for large i, the (n, δ)-strainer at x_j has radius at least $\rho/2$, a contradiction.) By Lemma 2.1, we may assume that $\eta(\delta, \rho) > 0$ and $\epsilon > 0$ such that $B_\eta(p)$ is e^ϵ-bi-Lipschitz embedded to Euclidean space, and $\epsilon \to 0$ as $\delta \to 0$ and $\eta \to 0$ (equivalently, $\delta \to 0$ and $\mu_i \to 0$).

Now we decompose $U - B_{\mu_i}(S_\delta)$ into countable disjoint small regions: $U - B_{\mu_i}(S_\delta) = \bigcup_j U_j$, such that each U_j is contained in an $\frac{\eta}{10}$-ball. Let U_j^e be the corresponding subset in $\mathbb{R}^n$ (or equivalently, U_i^e denotes an Euclidean metric on U_j which is e^ϵ-bi-Lipschitz to U_j). In particular,

$$e^{-\epsilon} \le \frac{V_{r_n}(U_j)}{V_{r_n}(U_j^e)} \le e^\epsilon, \qquad e^{-\epsilon} \le \frac{\mathrm{Haus}_n(U_j)}{\mathrm{Haus}_n(U_j^e)} \le e^\epsilon.$$

Together with Lemma 2.2, we get

$$e^{-2\epsilon}c(n) = e^{-2\epsilon} \cdot \frac{V_{r_n}(U_j^e)}{\mathrm{Haus}_n(U_j^e)} \le \frac{V_{r_n}(U_j)}{\mathrm{Haus}_n(U_j)} \le e^{2\epsilon} \frac{V_{r_n}(U_j^e)}{\mathrm{Haus}_n(U_j^e)} = e^{2\epsilon}c(n).$$

Because V_{r_n} is finitely additive, we obtain

$$e^{-2\epsilon}c(n) \sum_j \mathrm{Haus}_n(U_j) \le \sum_j V_{r_n}(U_j) \le e^{2\epsilon}c(n) \sum_j \mathrm{Haus}_n(U_j),$$

and thus

$$\begin{aligned} e^{-2\epsilon}c(n) \cdot \mathrm{Haus}_n(B_{\mu_i}(S_\delta)) &\le V_{r_n}(U - B_{\mu_i}(S_\delta)) \\ &\le e^{2\epsilon}c(n) \cdot \mathrm{Haus}_n(U - B_{\mu_i}(S_\delta)). \end{aligned} \tag{2.9}$$

In (2.9), letting $\delta \to 0$ and $\mu_i \to 0$ (thus $\epsilon \to 0$), we get (2.8). q.e.d.

Remark 2.8. We see that both (2.7) and (2.8) rely on the Alexandrov structure.

Proof of Lemma 2.4. We will first reduce the proof to the case when $|qp| = |qr|$ (see (2.10) below). We may assume that $|qp| \ge |qr|$, and let s be a point on the geodesic from q to p such that $|qs| = |qr| = x$. From the condition that $2(|qp| - |qr|) \le |pr|$, we derive

$$|pr| - |rs| \le |ps| = |qp| - |qr| \le \frac{1}{2}|pr|,$$

and thus $|pr| \le 2|rs|$. From

$$|rs| \le |pr| + |ps| = |pr| + |qp| - |qr| \le |pr| + \frac{1}{2}|pr|,$$

we get that $|pr| \geq \frac{2}{3}|rs|$, and therefore

$$\frac{2}{3}\frac{|rs|}{\theta} \leq \frac{|pr|}{\theta} \leq 2\frac{|rs|}{\theta},$$

where $\theta = \measuredangle pqr$. In the above inequality, taking the maximum over $p, q, r \in S^2_\kappa$ under the conditions for $\psi(\kappa, d)$, we get

$$\begin{aligned}\frac{2}{3}\max_{q,r,s\in S^2_\kappa}&\left\{\frac{|rs|}{\theta},\ |qs| = |qr| \leq d\right\} \leq \psi(\kappa, d)\\ &\leq 2\max_{q,r,s\in S^2_\kappa}\left\{\frac{|rs|}{\theta},\ |qr| = |qs| \leq d\right\}. \end{aligned} \tag{2.10}$$

We claim that for each fixed x,

$$\max_{|rs|}\left\{\frac{|rs|}{\theta},\ |qr| = |qs| = x\right\} = sn_\kappa x. \tag{2.11}$$

Clearly, Lemma 2.4 follows from (2.10) and (2.11). In the rest of the proof, we will verify (2.11).

Case 1. For $k < 0$, applying the cosine law to the triangle $\triangle qrs$ we derive

$$\begin{aligned}\cosh(\sqrt{-\kappa}|rs|) &= \cosh^2(\sqrt{-\kappa}x) - \sinh^2(\sqrt{-\kappa}x)\cos\theta\\ &= 1 + \sinh^2(\sqrt{-\kappa}x)(1 - \cos\theta)\\ &= 1 + 2\sinh^2(\sqrt{-\kappa}x)\sin^2\frac{\theta}{2},\end{aligned}$$

and thus

$$\sinh\frac{\sqrt{-\kappa}|rs|}{2} = \sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x). \tag{2.12}$$

Since $\sin z \leq z$ and $z \leq \sinh z$ for $z > 0$, from (2.12) we get

$$\frac{\sqrt{-\kappa}|rs|}{2} \leq \sinh\frac{\sqrt{-\kappa}|rs|}{2} = \sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x) \leq \frac{\theta}{2}\sinh(\sqrt{-\kappa}x),$$

and thus

$$\frac{|rs|}{\theta} \leq \frac{\sinh(\sqrt{-\kappa}x)}{\sqrt{-\kappa}}.$$

On the other hand, $|rs| \to 0 \Leftrightarrow \theta \to 0$. Using (2.12), we derive

$$\lim_{\theta\to 0}\frac{|rs|}{\theta} = \lim_{\theta\to 0}\frac{|rs|}{\sinh\frac{\sqrt{-\kappa}|rs|}{2}}\cdot\frac{\sin\frac{\theta}{2}\sinh(\sqrt{-\kappa}x)}{\theta} = \frac{\sinh(\sqrt{-\kappa}x)}{\sqrt{-\kappa}}.$$

By now, we can conclude (2.11) for $k < 0$.

Case 2. For $k = 0$, applying the cosine law to $\triangle qrs$, we get that $|rs| = 2x\sin\frac{\theta}{2} \leq \theta x$ and thus $\frac{|rs|}{\theta} \leq x$. On the other hand,

$$\lim_{\theta\to 0}\frac{|rs|}{\theta} = \lim_{\theta\to 0}\frac{2x\sin\frac{\theta}{2}}{\theta} = x.$$

Similarly, we can conclude (2.11) for $k = 0$.

Case 3. For $\kappa > 0$, applying the cosine law to $\triangle qrs$, we get

$$\sin\frac{\sqrt{k}|rs|}{2} = \sin\frac{\theta}{2}\sin(\sqrt{k}x). \tag{2.13}$$

By (2.13), we get

$$\frac{|rs|}{\theta} = \frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\sqrt{\kappa}|rs|}{2}}{\sqrt{\kappa}\frac{\theta}{2}} = \frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \cdot \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}}. \tag{2.14}$$

We claim that

$$\frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \le 1.$$

Because $\theta \to 0$ if and only if $|rs| \to 0$,

$$\lim_{\theta\to 0}\frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = 1,$$

and consequently we conclude from (2.14) that (2.11) holds for $\kappa > 0$.

To see the claim, let $\lambda = \sin(\sqrt{\kappa}x)$, and rewrite (2.13) as

$$\sin\frac{\sqrt{\kappa}|rs|}{2} = \lambda\sin\frac{\theta}{2}, \qquad \frac{\sqrt{\kappa}|rs|}{2} = \sin^{-1}(\lambda\sin\frac{\theta}{2}).$$

Then

$$\frac{\frac{\sqrt{\kappa}|rs|}{2}}{\sin\frac{\sqrt{\kappa}|rs|}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = \frac{\sin^{-1}(\lambda\sin\frac{\theta}{2})}{\lambda\sin\frac{\theta}{2}} \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = \frac{\sin^{-1}(\lambda\sin\frac{\theta}{2})}{\lambda\frac{\theta}{2}} \le 1,$$

because for all $0 < \lambda \le 1$ and $0 \le \frac{\theta}{2} \le \frac{\pi}{2}$, $\lambda\sin\frac{\theta}{2} \le \sin(\lambda\frac{\theta}{2})$. q.e.d.

Example 2.9. We will calculate an example showing that when X contains neither a closed geodesic nor an almost closed geodesic, the inequality in Theorem A is sharp up to a constant depending only on n.

Consider a sector of angle θ ($0 < \theta < \pi$) in a flat 2-disk of radius d. We obtain a flat cone, X^2, by identifying the two sides of the sector. Then $\mathrm{vol}(X^2) = \frac{1}{2}\theta d^2$. Let c denote a geodesic loop at a point near the vertex. Then $L(c) << 1$ and $\Theta(c) = \theta$. In this case, the inequality in Theorem A reads:

$$L(c) + \Theta(c)\cdot d \ge \frac{(2-1)\cdot\mathrm{vol}(X^2)}{\mathrm{vol}(S_1^0)\cdot d} = \frac{\theta}{2}\cdot d.$$

Let B_d^m denote a closed ball of radius d in $\mathbb{R}^m$, and let $X^{m+1} = X^2 \times B_d^m$ be the metric product. Then X^{m+2} is a compact Alexandrov space of curv ≥ 0, and

$$\operatorname{diam}(X^{m+2}) = \sqrt{2}d,$$

$$\operatorname{vol}(X^{m+2}) = \operatorname{vol}(X^2)\cdot\operatorname{vol}(B_d^m) = \frac{\operatorname{vol}(S_1^{m-1})}{2(m+1)}\cdot\theta\cdot d^{m+2}.$$

Let $(p_i, x) \in X^{m+2} = X^2 \times B_d^m$ such that p_i converges to the vertex of X^2, and let $\gamma_i \subset X^2$ be a sequence of geodesic loops at p_i. Then $(\gamma_i, x) \subset X^{m+2}$ is a sequence of geodesic loops such that $L(\gamma_i, x) = L(\gamma_i) \to 0$ and $\Theta((\gamma_i, 0)) \equiv \theta$. Applying Theorem A to $(\gamma_i, 0)$ and taking the limit as $i \to \infty$, one gets (we also assume $m = 2s$ is even)

$$\begin{aligned}
\theta\cdot d &\geq \frac{(m+1)\cdot\operatorname{vol}(X^{m+2})}{(m-1)\cdot\operatorname{vol}(S_1^m)\cdot d^{m+1}}\\
&= \frac{\operatorname{vol}(S_1^{m-1})}{2(m-1)\cdot\operatorname{vol}(S_1^m)}\cdot\theta\cdot d\\
&= \frac{\frac{2^{\frac{m}{2}}\pi^{\frac{m-2}{2}}}{(m-1)!!}}{(m-1)\cdot\frac{\pi^{\frac{m}{2}}}{(\frac{m}{2})!}}\cdot\theta\cdot d\\
&= \frac{1}{\pi}\cdot\frac{1}{2s-1}\cdot\left[\frac{(2s)\cdot(2s-2)\cdots 4\cdot 2}{(2s-1)\cdot(2s-3)\cdots 3\cdot 1}\right]\cdot\theta\cdot d\\
&\geq \frac{1}{\pi(2s-1)}\cdot\theta\cdot d.
\end{aligned}$$

3. Appendix

In this section, we will give proofs for Lemma 1.4. The main ingredient in the proof is the cosine law in the κ-space form.

Proof of Lemma 1.4. Note that for $\kappa > 0$, $C_\kappa(\Gamma)$ is a κ-suspension over Γ. If $r_1 \geq \frac{\pi}{2\sqrt{\kappa}}$, by symmetry we see that

$$\operatorname{Haus}_n(A_{r_1}^{r_2}(\Gamma)) = \operatorname{Haus}_n(A_{\frac{\pi}{\sqrt{\kappa}}-r_2}^{\frac{\pi}{\sqrt{\kappa}}-r_1}(\Gamma)).$$

If $r_1 < \frac{\pi}{2\sqrt{\kappa}} < r_2$, then similarly we may identify

$$\operatorname{Haus}_n(A_{r_1}^{r_2}(\Gamma)) = \operatorname{Haus}_n(A_{r_1}^{\frac{\pi}{2\sqrt{\kappa}}}(\Gamma)) + \operatorname{Haus}_n(A_{\frac{\pi}{\sqrt{\kappa}}-r_2}^{\frac{\pi}{\sqrt{\kappa}}}(\Gamma)).$$

Hence, without loss of generality we may assume that $r_2 \leq \frac{\pi}{2\sqrt{\kappa}}$.

We will divide $A_{r_1}^{r_2}(\Gamma)$ into small annuli and express $\operatorname{Haus}_n(A_{r_1}^{r_2}(\Gamma))$ as a Riemannian sum of the Hausdorff measure of these small annuli. The key in the proof is an estimate of the Hausdorff measure of a small annulus in terms of the Hausdorff measure of a cross section and the width of the small annulus (one may view this as a local co-area formula estimate).

Let $\{t_i\}$ be an N-partition of $[r_1, r_2]$ and $\Delta t = \frac{r_2 - r_1}{N}$ be sufficiently small. By the above assumption, $sn_\kappa(t)$ is increasing in each $[t_1, t_{i+1}]$. Let $S_t = \{x \in A : |px| = t\}$ and $A_{t_i}^{t_{i+1}} = \{x \in A : t_i \le |px| \le t_{i+1}\}$. Define the product metric $|(a,u),(b,v)| = \sqrt{|a,b|^2 + |u,v|^2}$ over $S_{t_i} \times [t_i, t_{i+1}]$. Because S_{t_i} is an Alexandrov space and the normalized Haus_n has countable additivity, we have

$$\frac{\mathrm{Haus}_n(S_{t_i} \times [t_i, t_{i+1}])}{\mathrm{Haus}_{n-1}(S_{t_i}) \cdot (t_{i+1} - t_i)} = \frac{\mathrm{Haus}_n(I^n)}{\mathrm{Haus}_{n-1}(I^{n-1}) \cdot \mathrm{Haus}_1(I^1)} = 1. \tag{3.1}$$

Consider the map $f : A_{t_i}^{t_{i+1}} \to S_{t_i} \times [r_1, r_2]$ defined as the following: for $x \in A_{t_i}^{t_{i+1}}$, let $x' \in S_{t_i}$ be the point on geodesic $[px]$ such that $|px'| = t_i$, then $f(x) = (x', |px|)$ and $|f(x_1)f(x_2)|^2 = |x_1'x_2'|^2 + (|px_1| - |px_2|)^2$.

For any $x_1, x_2 \in A_{t_i}^{t_{i+1}}$, assume $|px_2| \ge |px_1|$. We will show that

$$\frac{|x_1x_2|}{|f(x_1)f(x_2)|} = 1 + O(\Delta t). \tag{3.2}$$

Applying the following version of cosine law (which can be easily derived) to the triangles $\triangle p x_1 x_2$ and $\triangle p x_1' x_2'$, we get that

$$\mathrm{sn}_\kappa^2 \frac{|x_1x_2|}{2} = \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + \sin^2 \frac{\measuredangle x_1 p x_2}{2} \cdot \mathrm{sn}_\kappa |px_1| \mathrm{sn}_\kappa |px_2|$$
$$\mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2} = \sin^2 \frac{\measuredangle x_1' p x_2'}{2} \cdot \mathrm{sn}_\kappa^2(t_i).$$

Since $\measuredangle x_1 p x_2 = \measuredangle x_1' p x_2'$,

$$\begin{aligned} \mathrm{sn}_\kappa^2 \frac{|x_1x_2|}{2} &= \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + \frac{\mathrm{sn}_\kappa |px_1| \mathrm{sn}_\kappa |px_2|}{\mathrm{sn}_\kappa^2(t_i)} \mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2} \\ &= \mathrm{sn}_\kappa^2 \frac{|px_1| - |px_2|}{2} + (1 + O(\Delta t)) \mathrm{sn}_\kappa^2 \frac{|x_1'x_2'|}{2}. \end{aligned}$$

By the Taylor expansion of $(\mathrm{sn}_\kappa^{-1}(\sqrt{\mathrm{sn}_\kappa^2(x) + (1 + O(\Delta t))\mathrm{sn}_\kappa^2(y)}))^2$, we get that

$$\begin{aligned} |x_1x_2|^2 &= (|px_1| - |px_2|)^2 + |x_1'x_2'|^2 + O(\Delta t)|x_1'x_2'|^2 \\ &= |f(x_1)f(x_2)|^2 + O(\Delta t)|x_1'x_2'|^2, \end{aligned}$$

which leads to (3.2). By the cosine law, it's easy to see that

$$\mathrm{Haus}_{n-1}(S_{t_i}) = sn_\kappa^{n-1}(t_i)\mathrm{Haus}_{n-1}(\Gamma_p).$$

Together with (3.1) and (3.2),

$$\begin{aligned} \mathrm{Haus}_n(A_{t_i}^{t_{i+1}}) &= (1 + O(\Delta t))^n \mathrm{Haus}_n(S_{t_i} \times [r_1, r_2]) \\ &= (1 + O(\Delta t))^n \mathrm{Haus}_{n-1}(S_{t_i}) \Delta t \\ &= (1 + O(\Delta t))^n \mathrm{Haus}_{n-1}(\Gamma_p) sn_\kappa^{n-1}(t_i) \Delta t. \end{aligned}$$

Summing up the above, for $i = 0, 1, \dots, N-1$ and let $\max\{\Delta t\} \to 0$, we get Lemma 1.4. q.e.d.

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