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YAU'S GRADIENT ESTIMATES ON ALEXANDROV SPACES

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Abstract

In this paper, we establish a Bochner-type formula on Alexandrov spaces with Ricci curvature bounded below. Yau's gradient estimate for harmonic functions is also obtained on Alexandrov spaces.

1. Introduction

The study of harmonic functions on Riemannian manifolds has been one of the basic topics in geometric analysis. Yau in [**50**] and Cheng–Yau in [**10**] proved the following well-known gradient estimate for harmonic functions on smooth manifolds (see also [**48**]).

Theorem 1.1. (Yau [**50**], Cheng–Yau [**10**]) *Let M^n be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K$, $(K \geqslant 0)$. Then there exists a constant C_n, depending only on n, such that every positive harmonic function u on M^n satisfies*

$$|\nabla \log u| \leqslant C_n(\sqrt{K} + \frac{1}{R})$$

in any ball $B_p(R)$.

A direct consequence of the gradient estimate is Yau's Liouville theorem, which states that a positive harmonic function on a complete Riemannian manifold of nonnegative Ricci curvature must be constant.

The main purpose of this paper is to extend the Yau's estimate to Alexandrov spaces. Roughly speaking, an Alexandrov space with curvature bounded below is a length space X with the property that any geodesic triangle in X is "fatter" than the corresponding one in the associated model space. The seminal paper [**6**] and the 10th chapter in the book [**2**] provide introductions to Alexandrov geometry.

Alexandrov spaces (with curvature bounded below) successfully generalize the notion of lower bounds of sectional curvature from Riemannian manifolds to metric spaces. In the last few years, several notions for

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the Ricci curvature bounded below on general metric spaces have appeared. Sturm [**45**] and Lott–Villani [**28**, **29**], independently, introduced a so-called curvature-dimension condition on metric measure spaces, denoted by $CD(K, n)$. The curvature-dimension condition implies a generalized Brunn–Minkowski inequality (hence also Bishop–Gromov comparison and Bonnet–Myer's theorem) and a Poincaré inequality (see [**45**, **28**, **29**]). Meanwhile, Sturm [**45**] and Ohta [**31**] introduced a measure contraction property, denoted by $MCP(K, n)$, which is a slight modification of a property introduced earlier by Sturm in [**46**] and in a similar form by Kuwae and Shioya in [**23**, **24**]. The condition $MCP(K, n)$ also implies Bishop–Gromov comparison, Bonnet–Myer's theorem, and a Poincaré inequality (see [**45**, **31**]).

In the framework of Alexandrov spaces, Kuwae–Shioya in [**22**] introduced an infinitesimal version of the Bishop–Gromov comparison condition, denoted by $BG(K, n)$. On an n-dimensional Alexandrov space with its Hausdorff measure, the condition $BG(K, n)$ is equivalent to $MCP(K, n)$ (see [**22**]). Under the condition $BG(0, n)$, Kuwae–Shioya in [**22**] proved a topological splitting theorem of Cheeger–Gromoll type. In [**52**], the authors introduced a notion of "Ricci curvature has a lower bound K", denoted by $Ric \geqslant K$, by averaging the second variation of arc-length (see [**37**]). On an n-dimensional Alexandrov space M, the condition $Ric \geqslant K$ implies that M (equipped with its Hausdorff measure) satisfies $CD(K, n)$ and $BG(K, n)$ (see [**38**] and Appendix in [**52**]). Therefore, Bishop–Gromov comparison and a Poincaré inequality hold on Alexandrov spaces with Ricci curvature bounded below. Furthermore, under this Ricci curvature condition, the authors in [**52**] proved an *isometric* splitting theorem of Cheeger–Gromoll type and the maximal diameter theorem of Cheng type. Note that all of these generalized notions of Ricci curvature bounded below are equivalent to the classical one on smooth Riemannian manifolds.

Let M be an Alexandrov space. In [**33**], Ostu–Shioya established a C^1-structure and a corresponding C^0-Riemannian structure on the set of regular points of M. Perelman in [**35**] extended it to a DC^1-structure and a corresponding BV^0_{loc}-Riemannian structure. By applying this DC^1-structure, Kuwae–Machigashira–Shioya in [**19**] introduced a canonical Dirichlet form on M. Under a DC^1 coordinate system and written the BV^0_{loc}-Riemannian metric by (g_{ij}), a harmonic function u is a solution of the equation

$$\sum_{i,j=1}^{n} \partial_i\big(\sqrt{g}g^{ij}\partial_j u\big) = 0 \tag{1.1}$$

in the sense of distribution, where $g = \det(g_{ij})$ and (g^{ij}) is the inverse matrix of (g_{ij}). By adapting the standard Nash–Moser iteration argument, one knows that a harmonic function must be locally Hölder continuous. More generally, in a metric space with a doubling measure and a Poincaré inequality for upper gradient, the same regularity assertion still holds for Cheeger-harmonic functions (see [**8**, **18**] for the details).

The classical Bernstein trick in PDEs implies that any harmonic function on smooth Riemannian manifolds is actually locally Lipschitz continuous. In the language of differential geometry, one can use the Bochner formula to bound the gradient of a harmonic function on smooth manifolds. The well-known Bochner formula states that for any C^3 function u on a smooth n-dimensional Riemannian manifold, there holds

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2\langle\nabla u, \nabla\Delta u\rangle + 2Ric(\nabla u, \nabla u). \tag{1.2}$$

But for singular spaces (including Alexandrov spaces), one meets serious difficulty in studying the Lipschitz continuity of harmonic function. First, due to the lack of the notion of second-order derivatives, the Bernstein trick does not work directly on singular spaces. Next, one notes that the singular set might be dense in an Alexandrov space. When one considers the partial differential equation (1.1) on an Alexandrov space, the coefficients $\sqrt{g}g^{ij}$ might be not well defined and not continuous on a dense subset. It seems that all PDE's approaches fail to give the Lipschitz continuity for the (weak) solutions of (1.1).

The first result for the Lipschitz continuity of harmonic functions on Alexandrov spaces was announced by Petrunin in [**41**]. In [**40**], Petrunin developed an argument based on the second variation formula of arc-length and Hamilton–Jacobi shift, and sketched a proof to the Lipschitz continuity of harmonic functions on Alexandrov spaces with nonnegative curvature, which is announced in [**41**]. In the present paper, a detailed exposition of Petrunin's proof is contained in Proposition 5.3 below. Furthermore, we will prove the Lipschitz continuity of solutions of the general Poisson equation; see Corollary 5.5 below. In [**21**], Koskela–Rajala–Shanmugalingam proved that the same regularity of Cheeger-harmonic functions on metric measure spaces, which supports an Ahlfors regular measure, a Poincaré inequality, and a certain heat kernel condition. In the same paper, they gave an example to show that, on a general metric supporting a doubling measure and a Poincaré inequality, a harmonic function might fail to be Lipschitz continuous. In [**53**], based on the Lipschitz continuity of harmonic functions and a representation of heat kernel in [**19**], we proved that every solution of the heat equation on an Alexandrov space must be Lipschitz continuous. Independently, in [**11**], by applying the contraction property of gradient flow of the relative entropy in L^2–Wasserstein space, Gigli–Kuwada–Ohta also obtained the Lipschitz continuity of solutions of the heat equation on Alexandrov spaces.

Yau's gradient estimate in Theorem 1.1 above is an improvement of the classical Bernstein gradient estimate. To extend Yau's estimates to

Alexandrov spaces, let us recall its proof in the smooth case. Consider a positive harmonic function u on an n-dimensional Riemannian manifold. By applying the Bochner formula (1.2) to $\log u$, one has

$$\Delta Q \geqslant \frac{2}{n}Q^2 - 2\langle \nabla \log u, \nabla Q\rangle - 2KQ,$$

where $Q = |\nabla \log u|^2$. Let ϕ be a cut-off function. By applying maximum principle to the smooth function ϕQ, one can get the desired gradient estimate in Theorem 1.1. In this proof, it is crucial that the positive quadratic term $\frac{2}{n}Q^2$ exists on the RHS of the above inequality.

Now let us consider an n-dimensional Alexandrov space M with $Ric \geqslant -K$. In [**11**], Gigli–Kuwada–Ohta proved a weak form of the Γ_2-condition

$$\Delta|\nabla u|^2 \geqslant 2\langle \nabla u, \nabla \Delta u\rangle - 2K|\nabla u|^2, \quad \text{for all} \quad u \in D(\Delta) \cap W^{1,2}(M).$$

This is a weak version of the Bochner formula. If we use the formula to $\log u$ for a positive harmonic function u, then

$$\Delta Q \geqslant -2\langle \nabla \log u, \nabla Q\rangle - 2KQ,$$

where $Q = |\nabla \log u|^2$. Unfortunately, this does not suffice to derive the Yau's estimate because the positive term $\frac{2}{n}Q^2$ vanishes. The first result in this paper is the following Bochner-type formula, which keeps the desired positive quadratic term.

Theorem 1.2. *Let M be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$, and Ω be a bounded domain in M. Let $f(x,s) : \Omega \times [0,+\infty) \to \mathbb{R}$ be a Lipschitz function and satisfy the following:*

(a) there exists a zero measure set $\mathcal{N} \subset \Omega$ such that for all $s \geqslant 0$, the functions $f(\cdot,s)$ are differentiable at any $x \in \Omega\backslash\mathcal{N}$;

(b) the function $f(x,\cdot)$ is of class C^1 for all $x \in \Omega$ and the function $\frac{\partial f}{\partial s}(x,s)$ is continuous, non-positive on $\Omega \times [0,+\infty)$.

Suppose that u is Lipschitz on Ω and

$$-\int_\Omega \langle \nabla u, \nabla \phi\rangle\, d\mathrm{vol} = \int_\Omega \phi \cdot f\big(x, |\nabla u|^2\big)\mathrm{vol}$$

for all Lipschitz functions ϕ with compact support in Ω.

Then we have $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega)$ and

$$-\int_\Omega \big\langle \nabla \varphi, \nabla|\nabla u|^2\big\rangle\, d\mathrm{vol} \geqslant 2\int_\Omega \varphi \cdot \Big(\frac{f^2(x,|\nabla u|^2)}{n} + \big\langle \nabla u, \nabla f(x,|\nabla u|^2)\big\rangle - K|\nabla u|^2\Big) d\mathrm{vol}$$

for all Lipschitz functions $\varphi \geqslant 0$ with compact support in Ω, provided $|\nabla u|$ is lower semi-continuous at almost all $x \in \Omega$. (That is, there exists *a representative of $|\nabla u|$, which is lower semi-continuous at almost all $x \in \Omega$.)*

Instead of the maximum principle argument in the above proof of Theorem 1.1, we will adapt a Nash–Moser iteration method to establish the following Yau's gradient estimate, the second result of this paper.

Theorem 1.3. *Let M be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ $(K \geqslant 0)$, and let Ω be a bounded domain in M. Then there exists a constant $C = C\big(n, \sqrt{K}\mathrm{diam}(\Omega)\big)$ such that every positive harmonic function u on Ω satisfies*

$$\max_{x \in B_p(\frac{R}{2})} |\nabla \log u| \leqslant C(\sqrt{K} + \frac{1}{R})$$

for any ball $B_p(R) \subset \Omega$. If $K = 0$, the constant C depends only on n.

We also obtain a global version of the above gradient estimate.

Theorem 1.4. *Let M be as above and u be a positive harmonic function on M. Then we have*

$$|\nabla \log u| \leqslant C_{n,K}$$

for some constant $C_{n,K}$ depending only on n, K.

The paper is organized as follows. In Section 2, we will provide some necessary materials for calculus, Sobolev spaces, and Ricci curvature on Alexandrov spaces. In Section 3, we will investigate a further property of Perelman's concave functions. Poisson equations and mean value inequality on Alexandrov spaces will be discussed in Section 4. Bochner-type formula will be established in Section 5. In the last section, we will prove Yau's gradient estimates on Alexandrov spaces.

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2. Preliminaries

2.1. Alexandrov spaces. Let $(X, |\cdot\cdot|)$ be a metric space. A rectifiable curve γ connecting two points p, q is called a geodesic if its length is equal to $|pq|$ and it has unit speed. A metric space X is called a geodesic space if every pair of points $p, q \in X$ can be connected by *some* geodesic.

Let $k \in \mathbb{R}$ and $l \in \mathbb{N}$. Denote by $\mathbb{M}^l_k$ the simply connected, l-dimensional space form of constant sectional curvature k. Given three points p, q, r in a geodesic space X, we can take a comparison triangle $\triangle\bar{p}\bar{q}\bar{r}$ in the model spaces $\mathbb{M}^2_k$ such that $|\bar{p}\bar{q}| = |pq|$, $|\bar{q}\bar{r}| = |qr|$, and

$|\bar{r}\bar{p}| = |rp|$. If $k > 0$, we add assumption $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$. Angles $\widetilde{\angle}_k pqr := \angle \bar{p}\bar{q}\bar{r}$ are called comparison angles.

A geodesic space X is called an Alexandrov space (of *locall* curvature bounded below) if it satisfies the following properties:

(i) it is locally compact;

(ii) for any point $x \in X$ there exists a neighborhood U_x of x and a real number κ such that, for any four different points p, a, b, c in U_x, we have

$$\widetilde{\angle}_\kappa apb + \widetilde{\angle}_\kappa bpc + \widetilde{\angle}_\kappa cpa \leqslant 2\pi.$$

The Hausdorff dimension of an Alexandrov space is always an integer. Let M be an n-dimensional Alexandrov space; we denote by vol the n-dimensional Hausdorff measure of M. Let $p \in M$; given two geodesics $\gamma(t)$ and $\sigma(s)$ with $\gamma(0) = \sigma(0) = p$, the angle

$$\angle \gamma'(0)\sigma'(0) := \lim_{s,t \to 0} \widetilde{\angle}_\kappa \gamma(t) p \sigma(s)$$

is well defined. We denote by Σ_p' the set of equivalence classes of geodesic $\gamma(t)$ with $\gamma(0) = p$, where $\gamma(t)$ is equivalent to $\sigma(s)$ if $\angle \gamma'(0)\sigma'(0) = 0$. The completion of metric space $(\Sigma_p', \angle)$ is called the space of directions at p, denoted by Σ_p. The tangent cone at p, T_p, is the Euclidean cone over Σ_p. For two tangent vectors $u, v \in T_p$, their "scalar product" is defined by (see Section 1 in [**39**])

$$\langle u, v \rangle := \frac{1}{2}(|u|^2 + |v|^2 - |uv|^2).$$

For each point $x \neq p$, the symbol $\uparrow_p^x$ denotes the direction at p corresponding to *some* geodesic px. We refer to the seminar paper [**6**] or the textbook [**2**] for the details.

Let $p \in M$. Given a direction $\xi \in \Sigma_p$, there possibly does not exist geodesic $\gamma(t)$ starting at p with $\gamma'(0) = \xi$. To overcome this difficulty, it is shown in [**36**] that for any $p \in M$ and any direction $\xi \in \Sigma_p$, there exists a *quasi-geodesic* $\gamma : [0, +\infty) \to M$ with $\gamma = p$ and $\gamma'(0) = \xi$ (see also Section 5 of [**39**]).

Let M be an n-dimensional Alexandrov space and $p \in M$. Denote by ([**33**])

$$W_p := \big\{x \in M \backslash \{p\} \ \big|\ \text{there exists } y \in M \text{ such that } y \neq x \\ \text{and } |py| = |px| + |xy|\big\}.$$

According to [**33**], the set W_p has full measure in X. For each $x \in W_p$, the direction $\uparrow_p^x$ is uniquely determined, since any geodesic in M does not branch ([**6**]). Recall that the map $\log_p : W_p \to T_p$ is defined by $\log_p(x) := |px| \cdot \uparrow_p^x$ (see [**39**]). We denote this by

$$\mathscr{W}_p := \log_p(W_p) \subset T_p.$$

The map $\log_p : W_p \to \mathscr{W}_p$ is one-to-one. After Petrunin in [**37**], the *exponential map* $\exp_p : T_p \to M$ is defined as follows: $\exp_p(o) = p$ and for any $v \in T_p \backslash \{o\}$, $\exp_p(v)$ is a point on some quasi-geodesic of length $|v|$ starting point p along direction $v/|v| \in \Sigma_p$. If the quasi-geodesic is not unique, we fix some one of them as the definition of $\exp_p(v)$. Then $\exp_p|_{\mathscr{W}_p}$ is the inverse map of $\log_p$, and hence $\exp_p|_{\mathscr{W}_p} :\ \mathscr{W}_p \to W_p$ is one-to-one. If M has curvature $\geqslant k$ on $B_p(R)$, then exponential map

$$\exp_p : B_o(R) \cap \mathscr{W}_p \subset T_p^k \to M$$

is a non-expending map ([**6**]), where T_p^k is the k-cone over Σ_p and o is the vertex of T_p.

A point p in an n-dimensional Alexandrov space M is called *regular* if its tangent cone T_p is isometric to Euclidean space $\mathbb{R}^n$ with standard metric. A point $p \in M$ is called a singular point if it is not regular. Denote by S_M the set of singular points of M. It is shown (in Section 10 of [**6**]) that the Hausdorff dimension of S_M is $\leqslant n-1$ (see [**6, 33**]). Note that the singular set S_M is possibly dense in M (see [**33**]). It is known that $M\backslash S_M$ is convex [**37**]. Let p be a regular point in M; for any $\epsilon > 0$ there is a neighborhood $B_p(r)$ which is bi-Lipschitz onto an open domain in $\mathbb{R}^n$ with bi-Lipschitz constant $1+\epsilon$ (see Theorem 9.4 of [**6**]). Namely, there exists a map F from $B_p(r)$ onto an open domain in $\mathbb{R}^n$ such that

$$(1+\epsilon)^{-1} \leqslant \frac{\|F(x)-F(y)\|}{|xy|} \leqslant 1+\epsilon \qquad \forall\ x, y \in B_p(r),\ x \neq y.$$

A (generalized) C^1-structure on $M\backslash S_M$ is established in [**33**] in the following sense: there is an open covering $\{U_\alpha\}$ of an open set containing $M\backslash S_M$, and a family of homeomorphism $\phi_\alpha :\ U_\alpha \to O_\alpha \subset \mathbb{R}^n$ such that if $U_\alpha \cap U_\beta \neq \varnothing$, then

$$\phi_\alpha \circ \phi_\beta^{-1} :\ \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^1 on $\phi_\beta\big((U_\alpha \cap U_\beta)\backslash S_M\big)$. A corresponding C^0-Riemannian metric g on $M\backslash S_M$ is introduced in [**33**]. In [**35**], this C^1-structure and the corresponding C^0-Riemannian metric has been extended to be a DC^1-structure and the corresponding BV^0_{loc}-Riemannian metric. Moreover, we have the following:

(1) The distance function on $M\backslash S_M$ induced from g coincides with the original one of M ([**33**]).

(2) The Riemannian measure on $M\backslash S_M$ coincides with the Haudorff measure of M; that is, under a coordinate system (U, ϕ), on the metric $g = (g_{ij})$, we have

$$d\mathrm{vol}(x) = \sqrt{\det(g(\phi(x)))}dx^1 \wedge \cdots \wedge dx^n \tag{2.1}$$

for all $x \in U\backslash S_M$ (Section 7 in [**33**]).

A point p is called a *smooth* point if it is regular and there exists a coordinate system (U, ϕ) around p such that

$$|g_{ij}(\phi(x)) - \delta_{ij}| = o(|px|), \tag{2.2}$$

where (g_{ij}) is the corresponding Riemannian metric (see [**33**]) near p and (δ_{ij}) is the identity $n \times n$ matrix. The set of smooth points has full measure [**35**].

Lemma 2.1. *Let $p \in M$ be a smooth point. We have*

$$\Big|\frac{d\mathrm{vol}(x)}{dH^n(v)} - 1\Big| = o(r), \qquad \forall\ v \in B_o(r) \cap \mathscr{W}_p, \tag{2.3}$$

where $x = \exp_p(v)$, and

$$H^n\big(B_o(r) \cap \mathscr{W}_p\big) \geqslant H^n\big(B_o(r)\big) \cdot (1 - o(r)) \tag{2.4}$$

where H^n is the n-dimensional Hausdorff measure on T_p.

Proof. Let (U, ϕ) be a coordinate system such that $\phi(p) = 0$ and $B_p(r) \subset U$. For each $v \in B_o(r) \cap \mathscr{W}_p \subset T_p$,

$$d\mathrm{vol}(x) = \sqrt{\det[g_{ij}(\phi(x))]}dx^1 \wedge \cdots \wedge dx^n,$$

where $x = \exp_p(v)$. Since p is regular, T_p is isometric to $\mathbb{R}^n$. We obtain that

$$dH^n(v) = dH^n(o) = dx^1 \wedge \cdots \wedge dx^n$$

for all $v \in T_p$. We get

$$\frac{d\mathrm{vol}(x)}{dH^n(v)} - 1 = \sqrt{\det[g_{ij}(\phi(x))]} - 1.$$

Now the estimate (2.3) follows from this and equation (2.2).

Now we want to show (2.4).

Equation (2.2) implies that (see [**35**]) for any $x, y \in B_p(r) \subset U$,

$$\big||xy| - \|\phi(x) - \phi(y)\|\big| = o(r^2).$$

In particular, the map $\phi : U \to \mathbb{R}^n$ satisfies

$$\phi\big(B_p(r)\big) \supset B_o\big(r - o(r^2)\big).$$

On one hand, from (2.2), we have

(2.5)

$$\begin{aligned}\mathrm{vol}(B_p(r)) = &\int_{\phi(B_p(r))} \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n \\ &\geqslant H^n\big(\phi(B_p(r))\big) \cdot \big(1 - o(r)\big) \geqslant H^n\big(B_o(r - o(r^2))\big) \cdot (1 - o(r)) \\ &= H^n(B_o(r)) \cdot (1 - o(r)).\end{aligned}$$

On the other hand, because $\exp_p : B_o(R) \cap \mathscr{W}_p \subset T_p^k \to M$ is a non-expending map ([**6**]), where T_p^k is the k-cone over Σ_p and o is the vertex of T_p, we have

$$\exp_p : B_o(R) \cap \mathscr{W}_p \subset T_p \to M$$

is a Lipschitz map with Lipschitz constant $1 + O(r^2)$. Hence we get

$$H^n(B_o(r) \cap \mathscr{W}_p) \cdot (1 + O(r^2)) \geqslant \mathrm{vol}(B_p(r)).$$

Therefore, by combining with equation (2.5), we have

$$H^n(B_o(r)\cap\mathscr{W}_p) \geqslant H^n\big(B_o(r)\big)\cdot(1-O(r^2))\cdot(1-o(r)) = H^n(B_o(r)\cdot(1-o(r)).$$

This is the desired estimate (2.4). q.e.d.

Remark 2.2. If M is a C^2-Riemannian manifold, then for sufficiently small $r > 0$, we have

$$\Big|\frac{d\mathrm{vol}(x)}{dH^n(v)} - 1\Big| = O(r^2), \quad \forall\, v \in B_o(r) \subset T_p \quad \text{and} \quad x = \exp_p(v).$$

Let M be an Alexandrov space without boundary and $\Omega \subset M$ be an open set. A locally Lipschitz function $f : \Omega \to \mathbb{R}$ is called λ-concave ([**39**]) if for all geodesics $\gamma(t)$ in Ω, the function

$$f \circ \gamma(t) - \lambda \cdot t^2/2$$

is concave. A function $f : \Omega \to \mathbb{R}$ is called semi-concave if for any $x \in \Omega$, there exists a neighborhood of $U_x \ni x$ and a number $\lambda_x \in \mathbb{R}$ such that $f|_{U_x}$ is λ_x-concave. In fact, it was shown that the term "geodesic" in the definition can be replaced by "quasi-geodesic" ([**36**, **39**]). Given a semi-concave function $f : M \to \mathbb{R}$, its differential d_pf and gradient ∇_pf are well-defined for each point $p \in M$ (see Section 1 in [**39**] for the basic properties of semi-concave functions).

From now on, we always consider Alexandrov spaces without boundary.

Given a semi-concave function $f : M \to \mathbb{R}$, a point p is called a f-*regular* point if p is smooth, d_pf is a linear map on T_p $(= \mathbb{R}^n)$, and there exists a quadratic form H_pf on T_p such that

$$f(x) = f(p) + d_pf(\uparrow_p^x) \cdot |xp| + \frac{1}{2}H_pf(\uparrow_p^x, \uparrow_p^x) \cdot |px|^2 + o(|px|^2) \tag{2.6}$$

for any direction $\uparrow_p^x$. We denote by Reg_f the set of all f-regular points in M. According to [**35**], Reg_f has full measure in M.

Lemma 2.3. *Let f be a semi-concave function on M and $p \in M$. Then we have*

$$\fint_{B_p(r)} \Big(f(x) - f(p)\Big) d\mathrm{vol}(x) = \frac{nr}{n+1} \cdot \fint_{\Sigma_p} d_pf(\xi) d\xi + o(r), \tag{2.7}$$

where $\fint_B f d\mathrm{vol} = \frac{1}{\mathrm{vol}(B)} \int_B f d\mathrm{vol}$. *Furthermore, if we add the assumption that* $p \in Reg_f$*, then*

$$\fint_{B_p(r)} \Big(f(x) - f(p)\Big) d\mathrm{vol}(x) = \frac{nr^2}{2(n+2)} \cdot \fint_{\Sigma_p} H_p f(\xi, \xi) d\xi + o(r^2). \tag{2.8}$$

Proof. According to Theorem 10.8 in [**6**], we have

$$\frac{d\mathrm{vol}(\exp_p(v))}{dH^n(v)} = 1 + o(1), \qquad \frac{\mathrm{vol}(B_p(r))}{H^n(B_o(r))} = 1 + o(1). \tag{2.9}$$

Similarly as in the proof of equation (2.4), we have

$$\mathrm{vol}(B_o(r) \cap \mathscr{W}_p) \geqslant H^n(B_o(r)) \cdot (1 - o(1)).$$

Since $f(x) - f(p) = d_p f(\uparrow_p^x) \cdot |px| + o(|px|)$, we get

$$\begin{aligned} &\int_{B_p(r)} \Big(f(x) - f(p)\Big) d\mathrm{vol}(x) \\ &\qquad = \int_{B_o(r) \cap \mathscr{W}_p} \Big(d_p f(v) + o(|v|)\Big)(1 + o(1)) dH^n(v). \end{aligned} \tag{2.10}$$

On the other hand, from (2.9), we have

$$\Big| \int_{B_o(r) \backslash \mathscr{W}_p} d_p f(v) dH^n(v) \Big| \leqslant O(r) \cdot H^n\big(B_o(r) \backslash \mathscr{W}_p\big) \leqslant o(r^{n+1}).$$

By combining this and (2.10), we obtain

$$\begin{aligned} \fint_{B_p(r)} \Big(f(x) - f(p)\Big) d\mathrm{vol}(x) &= \frac{H^n(B_o(r))}{\mathrm{vol}(B_p(r))} \fint_{B_o(r)} d_p f(v) dH^n(v) + o(r) \\ &= \fint_{B_o(r)} d_p f(v) dH^n(v)(1 + o(1)) + o(r) \\ &= \fint_{B_o(r)} d_p f(v) dH^n(v) + o(r) \\ &= \frac{nr}{n+1} \fint_{\Sigma_p} d_p f(\xi) d\xi + o(r). \end{aligned}$$

This is equation (2.7).

Now we want to prove (2.8). Assume that p is an f-regular point. From (2.6) and Lemma 2.1, we have

$$\begin{aligned} &\int_{B_p(r)} \Big(f(x) - f(p)\Big) d\mathrm{vol}(x) \\ &\qquad = \int_{B_o(r) \cap \mathscr{W}_p} \Big(d_p f(v) + \frac{1}{2} H_p f(v, v) + o(|v|^2)\Big) \cdot (1 + o(r)) dH^n(v). \end{aligned} \tag{2.11}$$

Using Lemma 2.1 again, we have

$$\Big|\int_{B_o(r)\backslash \mathscr{W}_p} d_pf(v)dH^n\Big| \leqslant O(r)\cdot H^n(B_o(r)\backslash \mathscr{W}_p) = O(r)\cdot o(r)\cdot H^n(B_o(r))$$
$$= o(r^{n+2}).$$

Noticing that $\int_{B_o(r)} d_pf(v)dH^n = 0$, we get

$$\int_{B_o(r)\cap \mathscr{W}_p} d_pf(v)dH^n = o(r^{n+2}). \tag{2.12}$$

Similarly, we have

$$\int_{B_o(r)\cap \mathscr{W}_p} H_pf(v,v)dH^n = \int_{B_o(r)} H_pf(v,v)dH^n + o(r^{n+3}). \tag{2.13}$$

From (2.11)–(2.13) and Lemma 2.1, we have

$$\begin{aligned}\fint_{B_p(r)} \Big(f(x)-f(p)\Big)d\mathrm{vol}(x) &= \frac{H^n(B_o(r))}{\mathrm{vol}(B_p(r))}\fint_{B_o(r)} H_pf(v,v)dH^n + o(r^2)\\ &= \fint_{B_o(r)} H_pf(v,v)dH^n(1+o(r)) + o(r^2)\\ &= \frac{nr^2}{2(n+2)}\fint_{\Sigma_p} H_pf(\xi,\xi)d\xi + o(r^2).\end{aligned}$$

This is the desired (2.8). q.e.d.

Given a continuous function g defined on $B_p(\delta_0)$, where δ_0 is a sufficiently small positive number, we have

$$\int_{\partial B_p(r)} g d\mathrm{vol} = \frac{d}{dr}\int_{B_p(r)} g d\mathrm{vol}$$

for almost all $r\in(0,\delta_0)$.

Lemma 2.3′ *Let f be a semi-concave function on M and $p\in M$. Assume δ_0 is a sufficiently small positive number. Then we have, for almost all $r\in(0,\delta_0)$,*

$$\fint_{\partial B_p(r)} \Big(f(x)-f(p)\Big)d\mathrm{vol}(x) = nr\cdot \fint_{\Sigma_p} d_pf(\xi)d\xi + o(r). \tag{2.14}$$

Furthermore, if we add the assumption that $p\in Reg_f$, then we have, for almost all $r\in(0,\delta_0)$,

$$\fint_{\partial B_p(r)} \Big(f(x)-f(p)\Big)d\mathrm{vol}(x) = \frac{r^2}{2}\cdot \fint_{\Sigma_p} H_pf(\xi,\xi)d\xi + o(r^2). \tag{2.15}$$

2.2. Sobolev spaces. Several different notions of Sobolev spaces have been established; see [**8**, **19**, **43**, **20**, **24**, **51**]. (In [**8**, **20**, **43**, **24**], Sobolev spaces are defined on metric measure spaces supporting a doubling property and a Poincaré inequality. Since Ω is bounded, it satisfies a doubling property and supports a weakly Poincaré inequality [**19**].) They coincide with each other on Alexandrov spaces.

Let M be an n-dimensional Alexandrov space and let Ω be a bounded open domain in M. Given $u \in C(\Omega)$, at a point $p \in \Omega$, the *pointwise Lipschitz constant* ([**8**]) and *subgradient norm* ([**30**]) of u at x are defined by:

$$\mathrm{Lip}u(x) := \limsup_{y\to x} \frac{|f(x)-f(y)|}{|xy|} \quad \text{and}$$
$$|\nabla^- u|(x) := \limsup_{y\to x} \frac{\big(f(x)-f(y)\big)_+}{|xy|},$$

where $a_+ = \max\{a, 0\}$. Clearly, $|\nabla^- u|(x) \leqslant \mathrm{Lip}u(x)$. It was shown in [**30**] for a locally Lipschitz function u on Ω,

$$|\nabla^- u|(x) = \mathrm{Lip}u(x)$$

for almost all $x \in \Omega$ (see Remark 2.27 in [**30**] and its proof).

Let $x \in \Omega$ be a regular point. We say that a function u is *differentiable* at x, if there exists a vector in T_x $(= \mathbb{R}^n)$, denoted by $\nabla u(x)$, such that for all geodesic $\gamma(t) : [0, \epsilon) \to \Omega$ with $\gamma(0) = x$ we have

$$u(\gamma(t)) = u(x) + t \cdot \big\langle \nabla u(x), \gamma'(0)\big\rangle + o(t). \tag{2.16}$$

Thanks to the Rademacher theorem, which was proved by Cheeger [**8**] in the framework of general metric measure spaces with a doubling measure and a Poincaré inequality for upper gradients and was proved by Bertrand [**3**] in Alexandrov space via a simply argument, a locally Lipschitz function u is differentiable almost everywhere in M. (See also [**32**].) Hence the vector $\nabla u(x)$ is well defined almost everywhere in M.

Note that any semi-concave function f is locally Lipschitz. The differential of u at any point x, $d_x u$, is well-defined. (See Section 1 in [**39**].) The gradient $\nabla_x u$ is defined as the maximal value point of $d_x u : B_o(1) \subset T_x \to \mathbb{R}$.

Proposition 2.4. *Let u be a semi-concave function on an open domain $\Omega \subset M$. Then for any $x \in \Omega \backslash S_M$, we have*

$$|\nabla_x u| \leqslant |\nabla^- u|(x).$$

Moreover, if u is differentiable at x, we have

$$|\nabla_x u| = |\nabla^- u|(x) = \mathrm{Lip}u(x) = |\nabla u(x)|.$$

Proof. Without loss of generality, we can assume that $|\nabla_x u| > 0$. (Otherwise, we are done.) Since x is regular, there exists direction $-\nabla_x u$. Take a sequence of point $\{y_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} y_j = x \quad \text{and} \quad \lim_{j\to\infty} \uparrow_x^{y_j} = -\frac{\nabla_x u}{|\nabla_x u|}.$$

By semi-concavity of u, we have

$$u(y_j) - u(x) \leqslant |xy_j| \cdot \left\langle \nabla_x u, \uparrow_x^{y_j} \right\rangle + \lambda |xy_j|^2/2, \quad j = 1, 2, \ldots$$

for some $\lambda \in \mathbb{R}$. Hence

$$-\left\langle \nabla_x u, \uparrow_x^{y_j} \right\rangle \leqslant \frac{\big(u(x) - u(y_j)\big)_+}{|xy_j|} + \lambda |xy_j|/2, \quad j = 1, 2, \ldots$$

Letting $j \to \infty$, we conclude $|\nabla_x u| \leqslant |\nabla^- u|(x)$.

Let us prove the second assertion. We need only to show $\mathrm{Lip}u(x) \leqslant |\nabla u(x)|$ and $|\nabla u(x)| \leqslant |\nabla_x u|$. Since u is differentiable at x, we have

$$u(y) - u(x) = |xy| \cdot \langle \nabla u(x), \uparrow_x^y \rangle + o(|xy|)$$

for all y near x. Consequently,

$$|u(y) - u(x)| = |xy| \cdot |\langle \nabla u(x), \uparrow_x^y \rangle| + o(|xy|) \leqslant |xy| \cdot |\nabla u(x)| + o(|xy|).$$

This implies that $\mathrm{Lip}u(x) \leqslant |\nabla u(x)|$.

Finally, let us show $|\nabla u(x)| \leqslant |\nabla_x u|$. Indeed, combining the differentiability and semi-concavity of u, we have

$$|xy| \cdot \langle \nabla u(x), \uparrow_x^y \rangle + o(|xy|) = u(y) - u(x) \leqslant |xy| \cdot \langle \nabla_x u, \uparrow_x^y \rangle + \lambda |xy|^2/2$$

for all y near x. Without loss of generality, we can assume that $|\nabla u(x)| > 0$. Take y such that direction $\uparrow_x^y$ is arbitrarily close to $\nabla u(x)/|\nabla u(x)|$. We get

$$|\nabla u(x)|^2 \leqslant \langle \nabla_x u, \nabla u(x) \rangle \leqslant |\nabla_x u| \cdot |\nabla u(x)|.$$

This is $|\nabla_x u| \leqslant |\nabla u(x)|$. q.e.d.

According to this proposition, we will not distinguish between two notations $\nabla_x u$ and $\nabla u(x)$ for any semi-concave function u.

We denote by $Lip_{loc}(\Omega)$ the set of locally Lipschitz continuous functions on Ω, and by $Lip_0(\Omega)$ the set of Lipschitz continuous functions on Ω with compact support in Ω. For any $1 \leqslant p \leqslant +\infty$ and $u \in Lip_{loc}(\Omega)$, its $W^{1,p}(\Omega)$-norm is defined by

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\mathrm{Lip}u\|_{L^p(\Omega)}.$$

Sobolev spaces $W^{1,p}(\Omega)$ are defined by the closure of the set

$$\{u \in Lip_{loc}(\Omega) |\ \|u\|_{W^{1,2}(\Omega)} < +\infty\},$$

under $W^{1,p}(\Omega)$-norm. Spaces $W_0^{1,p}(\Omega)$ are defined by the closure of $Lip_0(\Omega)$ under $W^{1,p}(\Omega)$-norm. (This coincides with the definition in [**8**];

see Theorem 4.24 in [**8**].) We say a function $u \in W^{1,p}_{loc}(\Omega)$ if $u \in W^{1,p}(\Omega')$ for every open subset $\Omega' \Subset \Omega$. According to Kuwae–Machigashira–Shioya [**19**] (see also Theorem 4.47 in [**8**]), the "derivative" ∇u is well-defined for all $u \in W^{1,p}(\Omega)$ with $1 < p < \infty$. Cheeger in Theorem 4.48 of [**8**] proved that $W^{1,p}(\Omega)$ is reflexive for any $1 < p < \infty$.

2.3. Ricci curvature. For an Alexandrov space, several different definitions of "Ricci curvature having lower bounds by K" have been given (see Introduction).

Here, let us recall the definition of lower bounds of Ricci curvature on Alexandrov space in [**52**].

Let M be an n-dimensional Alexandrov space. According to Section 7 in [**6**], if p is an interior point of a geodesic γ, then the tangent cone T_p can be isometrically split into

$$T_p = L_p \times \mathbb{R} \cdot \gamma', \qquad v = (v^\perp, t).$$

We set

$$\Lambda_p = \{\xi \in L_p : \ |\xi| = 1\}.$$

Definition 2.5. Let $\sigma(t) : (-\ell, \ell) \to M$ be a geodesic and $\{g_{\sigma(t)}(\xi)\}_{-\ell<t<\ell}$ be a family of functions on $\Lambda_{\sigma(t)}$ such that $g_{\sigma(t)}$ is continuous on $\Lambda_{\sigma(t)}$ for each $t \in (-\ell, \ell)$. We say that the family $\{g_{\sigma(t)}(\xi)\}_{-\ell<t<\ell}$ satisfies *Condition* (RC) on σ if for any two points $q_1, q_2 \in \sigma$ and any sequence $\{\theta_j\}_{j=1}^\infty$ with $\theta_j \to 0$ as $j \to \infty$, there exists an isometry $T : \Sigma_{q_1} \to \Sigma_{q_2}$ and a subsequence $\{\delta_j\}$ of $\{\theta_j\}$ such that

$$\tag{2.17}
\begin{aligned}
&|\exp_{q_1}(\delta_j l_1 \xi),\ \exp_{q_2}(\delta_j l_2 T\xi)| \\
&\quad \leqslant |q_1q_2| + (l_2 - l_1)\langle \xi, \gamma'\rangle \cdot \delta_j \\
&\qquad + \Big(\frac{(l_1 - l_2)^2}{2|q_1q_2|} - \frac{g_{q_1}(\xi^\perp)\cdot |q_1q_2|}{6}\cdot (l_1^2 + l_1\cdot l_2 + l_2^2)\Big)\cdot \Big(1 - \langle \xi, \gamma'\rangle^2\Big)\cdot \delta_j^2 \\
&\qquad + o(\delta_j^2)
\end{aligned}$$

for any $l_1, l_2 \geqslant 0$ and any $\xi \in \Sigma_{q_1}$.

If M has curvature bounded below by k_0 (for some $k_0 \in \mathbb{R}$), then by Theorem 1.1 of [**37**] (or see Theorem 20.2.1 of [**1**]), the family of functions $\{g_{\sigma(t)}(\xi) = k_0\}_{-\ell<t<\ell}$ satisfies *Condition* (RC) on σ. In particular, if a family $\{g_{\sigma(t)}(\xi)\}_{-\ell<t<\ell}$ satisfies *Condition* (RC), then the family $\{g_{\sigma(t)}(\xi) \vee k_0\}_{-\ell<t<\ell}$ satisfies *Condition* (RC) too.

Definition 2.6. Let $\gamma : [0, a) \to M$ be a geodesic. We say that M has *Ricci curvature bounded below by K along* γ, if for any $\epsilon > 0$ and any $0 < t_0 < a$, there exists $\ell = \ell(t_0, \epsilon) > 0$ and a family of continuous functions $\{g_{\gamma(t)}(\xi)\}_{t_0-\ell<t<t_0+\ell}$ on $\Lambda_{\gamma(t)}$ such that the family satisfies

Condition (RC) on $\gamma|_{(t_0-\ell,\ t_0+\ell)}$ and

$$(2.18)\qquad (n-1)\cdot \fint_{\Lambda_{\gamma(t)}} g_{\gamma(t)}(\xi)d\xi \geqslant K-\epsilon, \qquad \forall t\in(t_0-\ell,t_0+\ell),$$

where $\fint_{\Lambda_x} g_x(\xi)=\frac{1}{vol(\Lambda_x)}\int_{\Lambda_x} g_x(\xi)d\xi$.

We say that M has *Ricci curvature bounded below by* K, denoted by $Ric(M)\geqslant K$, if each point $x\in M$ has a neighborhood U_x such that M has Ricci curvature bounded below by K along every geodesic γ in U_x.

Remark 2.7. Let M be an n-dimensional Alexandrov space with curvature $\geqslant k$. Let $\gamma:[0,a)\to M$ be any geodesic. By [**37**], the family of functions $\{g_{\gamma(t)}(\xi):=k\}_{0<t<a}$ satisfies Condition (RC) on γ. According to Definition 2.6, we know that M has Ricci curvature bounded from below by $(n-1)k$ along γ. Because of the arbitrariness of geodesic γ, M has Ricci curvature bounded from below by $(n-1)k$.

Let M be an n-dimensional Alexandrov space M having Ricci curvature $\geqslant K$. In [**38**] and the Appendix of [**52**], it is shown that metric measure space (M,d,vol) satisfies Sturm–Lott–Villani curvature-dimension condition $CD(K,n)$, and hence measure contraction property $MCP(K,n)$ (see [**45**, **31**], since Alexandrov spaces are non-branching) and infinitesimal Bishop-Gromov condition $BG(K,n)$ ([**22**]; this is equivalent to $MCP(K,n)$ on Alexandrov spaces). Consequently, M satisfies a corresponding Bishop–Gromov volume comparison theorem [**45**, **22**] and a corresponding Laplacian comparison in the sense of distribution [**22**].

3. Perelman's concave functions

Let M be an Alexandrov space and $x\in M$. In [**34**], Perelman constructed a strictly concave function on a neighborhood of x. This implies that there exists a convex neighborhood for each point in M. In this section, we will investigate a further property of Perelman's concave functions.

In this section, we always assume that M has curvature bounded from below by k (for some $k\in\mathbb{R}$).

Let $f:\Omega\subset M\to\mathbb{R}$ be a semi-concave function and $x\in\Omega$. Recall that a vector $v_s\in T_x$ is said to be a supporting vector of f at x (see [**39**]) if

$$d_xf(\xi)\leqslant -\langle v_s,\xi\rangle \quad \text{for all } \xi\in\Sigma_x.$$

The set of supporting vectors of f at x is a non-empty convex set (see Lemma 1.3.7 of [**39**]). For a distance function $f=dist_p$, by the first variant formula (see, for example, [**2**]), any direction $\uparrow_x^p$ is a supporting vector of f at $x\neq p$.

Proposition 3.1. *Let $f : \Omega \subset M \to \mathbb{R}$ be a semi-concave function and $x \in \Omega$. Then we have*

$$\int_{\Sigma_x} d_x f(\xi) d\xi \leqslant 0.$$

Furthermore, if f is a distance function $f = dist_p$, $x \neq p$, and the "$=$" holds; then $\uparrow_x^p$ is uniquely determined and $\max_{\xi\in\Sigma_x} |\xi, \uparrow_x^p| = \pi$.

Proof. Let v_s be a support vector of f at x; then

$$d_x f(\xi) \leqslant -\langle v_s, \xi\rangle, \quad \forall\, \xi \in \Sigma_x.$$

Without loss of generality, we can assume $v_s \neq 0$. (If $v_s = 0$, then $d_x f(\xi) \leqslant 0$. We are done.) Setting $\eta_0 = \frac{v_s}{|v_s|} \in \Sigma_x$, we have

$$d_x f(\xi) \leqslant -\langle v_s, \xi\rangle = -|v_s| \cdot \cos(|\eta_0, \xi|) \qquad \forall\, \xi \in \Sigma_x.$$

Denote $D = \max_{\xi\in\Sigma_x} |\xi, \eta_0|$. By using the co-area formula, we have

$$I := \int_{\Sigma_x} d_x f(\xi) d\xi \leqslant -|v_s| \cdot \int_{\Sigma_x} \cos(|\eta_0, \xi|) d\xi = -|v_s| \cdot \int_0^D \cos t \cdot A(t) dt,$$

where $A(t) = \mathrm{vol}_{n-2}(\{\xi \in \Sigma_x :\ |\xi, \eta_0| = t\})$.

If $D \leqslant \pi/2$, then $I < 0$.

We consider the case $D > \pi/2$. Since Σ_x has curvature $\geqslant 1$, by Bishop–Gromov comparison, we have

$$A(\pi - t) \leqslant A(t) \cdot \frac{\mathrm{vol}_{n-2}(\partial B_o(\pi - t) \subset \mathbb{S}^{n-1})}{\mathrm{vol}_{n-2}(\partial B_o(t) \subset \mathbb{S}^{n-1})} = A(t)$$

for any $t \leqslant \pi/2$. Hence

$$\begin{aligned}
\frac{I}{|v_s|} &\leqslant -\int_0^{\pi/2} \cos t \cdot A(t) dt - \int_{\pi/2}^D \cos t \cdot A(t) dt \\
&\leqslant -\int_0^{\pi/2} \cos t \cdot A(\pi - t) dt - \int_{\pi/2}^D \cos t \cdot A(t) dt \\
&= \int_D^{\pi} \cos t \cdot A(t) dt \leqslant 0.
\end{aligned}$$

Moreover, if $I = 0$, then $D = \pi$.

If $f = dist_p$, then v_s can be chosen as any direction $\uparrow_x^p$. When $I = 0$, we have

$$d_x f(\xi) = -\langle \uparrow_x^p, \xi\rangle, \quad \forall\, \xi \in \Sigma_x, \tag{3.1}$$

and

$$\max_{\xi\in\Sigma_x} |\xi, \uparrow_x^p| = \pi.$$

The left-hand side of (3.1) does not depend on the choice of direction $\uparrow_x^p$. This implies that $\uparrow_x^p$ is determined uniquely. q.e.d.

Lemma 3.2. *Given any $n \in \mathbb{N}$ and any constant $C > 0$, we can find $\delta_0 = \delta_0(C, n)$ satisfying the following property: for any n-dimensional Alexandrov spaces Σ^n with curvature $\geqslant 1$, if there exist $0 < \delta < \delta_0$ and points $\{p_j\}_{j=1}^N \subset \Sigma^n$ such that*

$$|p_i p_j| > \delta \quad (i \neq j), \qquad N := \#\{p_j\} \geqslant C \cdot \delta^{-n} \tag{3.2}$$

and

$$\mathrm{rad}(p_j) := \max_{q \in \Sigma^n} |p_j q| = \pi \quad \text{for each} \quad 1 \leqslant j \leqslant N, \tag{3.3}$$

then Σ^n is isometric to $\mathbb{S}^n$.

Proof. We use an induction argument with respect to the dimension n. When $n = 1$, we take $\delta_0(C, 1) = C/3$. Then each 1-dimensional Alexandrov space Σ^1 satisfying the assumption of the lemma must contain at least three different points p_1, p_2, and p_3 with $\mathrm{rad}(p_i) = \pi$, $i = 1, 2, 3$. Hence Σ^1 is isometric to $\mathbb{S}^1$.

Now we assume that the lemma holds for dimension $n - 1$. That is, for any $\widetilde{C}$, there exists $\delta_0(\widetilde{C}, n-1)$ such that any $(n-1)$-dimensional Alexandrov space satisfying the condition of the lemma must be isometric to $\mathbb{S}^{n-1}$.

We want to prove the lemma for dimension n. Fix any constant $C > 0$ and let

$$\delta_0(C, n) := \min\Big\{\frac{10}{8} \cdot \delta_0\Big(\frac{C}{11\pi} \cdot (10/8)^{1-n}, n-1\Big), 1\Big\}. \tag{3.4}$$

Let Σ^n be an n-dimensional Alexandrov space with curvature $\geqslant 1$. Suppose that there exists $0 < \delta < \delta_0(C, n)$ and a set of points $\{p_\alpha\}_{\alpha=1}^N \subset \Sigma^n$ such that they satisfy (3.2) and (3.3).

Let $q_1 \in \Sigma^n$ be the point that $|p_1 q_1| = \pi$. Then Σ^n is a suspension over some $(n-1)$-dimensional Alexandrov space Λ of curvature $\geqslant 1$ and with vertex p_1 and q_1, denoted by $\Sigma^n = S(\Lambda)$. We divide Σ^n into pieces $A_1,\ A_2, \ldots, A_l, \ldots, A_{\bar{l}}$ as

$$A_l = \big\{x \in \Sigma^n :\ (\delta/10)\cdot l < |xp_1| \leqslant (\delta/10)\cdot(l+1)\big\}, \quad 0 \leqslant l \leqslant \bar{l} := [\frac{\pi}{\delta/10}],$$

where $[a]$ is the integer such that $[a] \leqslant a < [a] + 1$. Then there exists some piece, say A_{l_0}, such that

$$N_1 := \#\big(A_{l_0} \cap \{p_j\}_{j=1}^N\big) \geqslant \frac{N}{\bar{l}+1} \geqslant \frac{N}{10\pi/\delta + 1} \overset{(\delta<1)}{\geqslant} \frac{C}{11\pi} \cdot \delta^{1-n}. \tag{3.5}$$

Notice that

$$A_1 \cup A_2 \subset B_{p_1}(\delta/2) \quad \text{and} \quad A_{\bar{l}} \cup A_{\bar{l}-1} \subset B_{q_1}(\delta/2),$$

then we have $l_0 \notin \{1, 2, \bar{l}-1, \bar{l}\}$.

We denote the points $A_{l_0} \cap \{p_\alpha\}_{\alpha=1}^N$ as $(x_i, t_i)_{i=1}^{N_1} \subset S(\Lambda)$ $(= \Sigma^n)$, where $x_i \in \Lambda$ and $0 < t_i < \pi$ for $1 \leqslant i \leqslant N_1$. Let γ_i be the geodesic

$p_1(x_i,t_i)q_1$ and $\widetilde{p}_i = \gamma_i \cap \partial B_{p_1}\big((l_0+1)\cdot\delta/10\big)$. By triangle inequality, we have

$$|\widetilde{p}_i\widetilde{p}_j| \geqslant \frac{8}{10}\cdot\delta. \tag{3.6}$$

Applying cosine law, we have

$$\cos(|\widetilde{p}_i\widetilde{p}_j|) = \cos(|p_1\widetilde{p}_i|)\cdot\cos(|p_1\widetilde{p}_j|) + \sin(|p_1\widetilde{p}_i|)\cdot\sin(|p_1\widetilde{p}_j|)\cdot\cos(|x_ix_j|)$$

for each $i \neq j$. Since $|p_1\widetilde{p}_i| = |p_1\widetilde{p}_j|$, we get

$$|x_ix_j| \geqslant |\widetilde{p}_i\widetilde{p}_j|. \tag{3.7}$$

By the assumption (3.3), there exist points $(\bar{x}_i,\bar{t}_i) \in \Sigma^n \ \big(= S(\Lambda)\big)$ such that

$$|(x_i,t_i),(\bar{x}_i,\bar{t}_i)| = \pi$$

for each $1 \leqslant i \leqslant N_1$. By using the cosine law again, we have

$$\begin{aligned} -1 = \cos(|(x_i,t_i)(\bar{x}_i,\bar{t}_i)|) &= \cos t_i\cdot\cos\bar{t}_i + \sin t_i\cdot\sin\bar{t}_i\cdot\cos(|x_i\bar{x}_i|) \\ &= \cos(t_i+\bar{t}_i) + \sin t_i\cdot\sin\bar{t}_i\cdot\big(\cos(|x_i\bar{x}_i|)+1\big) \\ &\geqslant \cos(t_i+\bar{t}_i). \end{aligned}$$

By combining with $0 < t_i, \bar{t}_i < \pi$, we deduce

$$|x_i\bar{x}_i| = \pi \quad \text{and} \quad t_i+\bar{t}_i = \pi. \tag{3.8}$$

By the induction assumption and (3.4)–(3.8), we know Λ is isometric to $\mathbb{S}^{n-1}$. Hence Σ^n is isometric to $\mathbb{S}^n$. q.e.d.

Lemma 3.3. (Perelman's concave function.) *Let $p \in M$. There exists a constant $r_1 > 0$ and a function $h : B_p(r_1) \to \mathbb{R}$ satisfying:*

(i) h is (-1)-concave;

(ii) h is 2-Lipschitz, that is, h is Lipschitz continuous with a Lipschitz constant 2;

(iii) for each $x \in B_p(r_1)$, we have

$$\int_{\Sigma_x} d_xh(\xi)d\xi \leqslant 0. \tag{3.9}$$

Moreover, if "=" holds, then x is regular.

Proof. Let us recall Perelman's construction in [**34**]. Fix a small $r_0 > 0$ and choose a maximal set of points $\{q_\alpha\}_{\alpha=1}^N \subset \partial B_p(r_0)$ with $\widetilde{\angle} q_\alpha p q_\beta > \delta$ for $\alpha \neq \beta$, where δ is an arbitrarily (but fixed) small positive number $\delta \ll r_0$. By Bishop–Gromov volume comparison, there exists a constant C_1, which is independent of δ, such that

$$N \geqslant C_1\cdot\delta^{1-n}. \tag{3.10}$$

Consider the function

$$h(y) = \frac{1}{N}\cdot\sum_{\alpha=1}^{N}\phi(|q_\alpha y|)$$

on $B_p(r_1)$ with $0 < r_1 \leqslant \frac{1}{2}r_0$, where $\phi(t)$ is a real function with $\phi'(t) = 1$ for $t \leqslant r_0 - \delta$, $\phi'(t) = 1/2$ for $t \geqslant r_0 + \delta$, and $\phi''(t) = -1/(4\delta)$ for $t \in (r_0 - \delta, r_0 + \delta)$.

The assertions (i) and (ii) have been proved for some positive constant $r_1 \ll r_0$ in [**34**] (see also [**15**] for more details). The assertion (iii) is implicitly claimed in Petrunin's manuscript [**40**]. Here we provide a proof as follows.

Let x be a point near p. It is clear that (3.9) follows from Proposition 3.1 and the above construction of h. Thus we only need to consider the case of

$$\int_{\Sigma_x} d_x h(\xi) d\xi = 0. \tag{3.11}$$

We want to show that x is a regular point.

From $\angle q_\alpha p q_\beta \geqslant \widetilde{\angle} q_\alpha p q_\beta > \delta$ for $\alpha \neq \beta$ and the lower semi-continuity of angles (see Proposition 2.8.1 in [**6**]), we can assume $\angle q_\alpha x q_\beta \geqslant \delta/2$ for $\alpha \neq \beta$. Proposition 3.1 and (3.11) imply that

$$\int_{\Sigma_x} d_x dist_{q_\alpha}(\xi) d\xi = 0 \qquad \text{for each} \quad 1 \leqslant \alpha \leqslant N.$$

Using Proposition 3.1 again, we have

$$\max_{\xi \in \Sigma_x} |\uparrow_x^{q_\alpha} \xi| = \pi \qquad \text{for each} \quad 1 \leqslant \alpha \leqslant N. \tag{3.12}$$

From Lemma 3.2 and the arbitrarily small property of δ, the combination of (3.10) and (3.12) implies that Σ_x is isometric to $\mathbb{S}^{n-1}$. Hence x is regular. q.e.d.

4. Poisson equations and mean value inequality

4.1. Poisson equations. Let M be an n-dimensional Alexandrov space and Ω be a bounded domain in M. In [**19**], the canonical Dirichlet form $\mathscr{E} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ is defined by

$$\mathscr{E}(u, v) = \int_\Omega \langle \nabla u, \nabla v \rangle \, d\mathrm{vol} \qquad \text{for } u, v \in W_0^{1,2}(\Omega).$$

Given a function $u \in W_{loc}^{1,2}(\Omega)$, we define a functional $\mathscr{L}_u$ on $Lip_0(\Omega)$ by

$$\mathscr{L}_u(\phi) = -\int_\Omega \langle \nabla u, \nabla \phi \rangle \, d\mathrm{vol}, \qquad \forall \phi \in Lip_0(\Omega).$$

When a function u is λ-concave, Petrunin in [**38**] proved that $\mathscr{L}_u$ is a signed Radon measure. Furthermore, if we write its Lebesgue's decomposition as

$$\mathscr{L}_u = \Delta u \cdot \mathrm{vol} + \Delta^s u, \tag{4.1}$$

then $\Delta^s u \leqslant 0$ and

$$\Delta u(p) = n \int_{\Sigma_p} H_p u(\xi, \xi) d\xi \leqslant n \cdot \lambda \tag{4.2}$$

for almost all points $p \in M$, where $H_p u$ is the Perelman's Hessian (see (2.6) or [**35**]).

Nevertheless, to study harmonic functions on Alexandrov spaces, we can not restrict our attention only to semi-concave functions. We have to consider the functional $\mathscr{L}_u$ for general functions in $W^{1,2}_{loc}(\Omega)$.

Let $f \in L^2(\Omega)$ and $u \in W^{1,2}_{loc}(\Omega)$. If the functional $\mathscr{L}_u$ satisfies

$$\mathscr{L}_u(\phi) \geqslant \int_\Omega f\phi d\mathrm{vol} \qquad \Big(\text{or} \quad \mathscr{L}_u(\phi) \leqslant \int_\Omega f\phi d\mathrm{vol}\Big)$$

for all nonnegative $\phi \in Lip_0(\Omega)$, then, according to [**13**], the functional $\mathscr{L}_u$ is a signed Radon measure. In this case, u is said to be a subsolution (supersolution, resp.) of Poisson equation

$$\mathscr{L}_u = f \cdot \mathrm{vol}.$$

Equivalently, $u \in W^{1,2}_{loc}(\Omega)$ is a subsolution of $\mathscr{L}_u = f \cdot \mathrm{vol}$ if and only if it is a local minimizer of the energy

$$\mathcal{E}(v) = \int_{\Omega'} \big(|\nabla v|^2 + 2fv\big) d\mathrm{vol}$$

in the set of functions v such that $u \geqslant v$ and $u - v$ are in $W^{1,2}_0(\Omega')$ for every fixed $\Omega' \Subset \Omega$. It is known (see for example [**25**]) that every continuous subsolution of $\mathscr{L}_u = 0$ on Ω satisfies the Maximum Principle, which states that

$$\max_{x \in B} u \leqslant \max_{x \in \partial B} u$$

for any ball $B \Subset \Omega$.

A function u is a (weak) solution of Poisson equation $\mathscr{L}_u = f \cdot \mathrm{vol}$ on Ω if it is both a subsolution and a supersolution of the equation. In particular, a (weak) solution of $\mathscr{L}_u = 0$ is called a harmonic function.

Now note that u is a (weak) solution of Poisson equation $\mathscr{L}_u = f \cdot \mathrm{vol}$ if and only if $\mathscr{L}_u$ is a signed Radon measure and its Lebesgue's decomposition $\mathscr{L}_u = \Delta u \cdot \mathrm{vol} + \Delta^s u$ satisfies

$$\Delta u = f \qquad \text{and} \qquad \Delta^s u = 0.$$

Given a function $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$, we can solve the Dirichlet problem of the equation

$$\begin{cases} \mathscr{L}_u & = f \cdot \mathrm{vol} \\ u & = g|_{\partial\Omega}. \end{cases}$$

Indeed, by the Sobolev compact embedding theorem (see [**14**, **19**]) and a standard argument (see, for example, [**12**]), it is known that the solution of the Dirichlet problem exists and is unique in $W^{1,2}(\Omega)$. (See, for example, Theorem 7.12 and Theorem 7.14 in [**8**].) Furthermore, if we add the assumption $f \in L^s$ with $s > n/2$, then the solution is locally Hölder continuous in Ω (see [**18**, **19**]).

Definition 4.1. A function $u \in C(\Omega) \cap W^{1,2}_{loc}(\Omega)$ is called a λ-superharmonic (or λ-subharmonic, resp.) on Ω, if it satisfies the following comparison property: for every open subset $\Omega' \Subset \Omega$, we have

$$\widetilde{u} \leqslant u, \qquad (\text{or} \quad \widetilde{u} \geqslant u, \text{resp.}),$$

where $\widetilde{u}$ is the (unique) solution of the equation $\mathscr{L}_{\widetilde{u}} = \lambda \cdot \mathrm{vol}$ in Ω' with boundary value $\widetilde{u} = u$ on $\partial\Omega'$.

In particular, a 0-superharmonic (or 0-subharmonic, resp.) function is simply said a superharmonic (or subharmonic, resp.) function.

In partial differential equation theory, this definition is related to the notion of viscosity solution (see [**7**]).

According to the maximum principle, we know that a continuous supersolution of $\mathscr{L}_u = 0$ must be a superharmonic function. Notice that the converse is not true in general metric measure space (see [**16**]). Nevertheless, we will prove a semi-concave superharmonic function on M must be a supersolution of $\mathscr{L}_u = 0$ (see Corollary 4.6 below).

4.2. Mean value inequality for solutions of Poisson equations. Let $u \in W^{1,2}(\Omega)$ such that $\mathscr{L}_u$ is a signed Radon measure on Ω and $A \Subset \Omega$ be an open set. We define a functional $I_{u,A}$ on $W^{1,2}(A)$ by

$$I_{u,A}(\phi) = \int_A \langle \nabla u, \nabla \phi \rangle \, d\mathrm{vol} + \int_A \phi d\mathscr{L}_u. \tag{4.3}$$

Remark 4.2. (i) If $\phi_1, \phi_2 \in W^{1,2}(A)$ and $\phi_1 - \phi_2 \in W^{1,2}_0(A)$, then, by the definition of $\mathscr{L}_u$, we have $I_{u,A}(\phi_1) = I_{u,A}(\phi_2)$.
(ii) If M is a smooth manifold and ∂A is smooth, then $I_{u,A}(\phi) = \int_A \mathrm{div}(\phi \nabla u) d\mathrm{vol}$.

Lemma 4.3. *Let $0 < r_0 < R_0$ and $w(x) = \varphi(|px|)$ satisfy $\mathscr{L}_w \geqslant 0$ on some neighborhood of $\overline{B_p(R_0)\backslash B_p(r_0)}$, where $\varphi \in C^2(\mathbb{R})$. Consider a function $v \in W^{1,2}(B_p(R_0)\backslash \overline{B_p(r_0)}) \cap L^\infty(\overline{B(p,R_0)}\backslash B(p,r_0))$. Then for almost all $r, R \in (r_0, R_0)$, we have*

$$I_{w,A}(v) = \varphi'(R) \int_{\partial B_p(R)} v d\mathrm{vol} - \varphi'(r) \int_{\partial B_p(r)} v d\mathrm{vol},$$

where $A = B_p(R)\backslash \overline{B_p(r)}$.

Proof. Since $\mathscr{L}_w$ is a signed Radon measure, we have $\mathscr{L}_w\big(B_p(R_0)\backslash B_p(r_0)\big) < +\infty$. Hence, for almost all $r, R \in (r_0, R_0)$, $\mathscr{L}_w(A_j\backslash A) \to 0$ as $j \to \infty$, where $A_j = B_p(R+\frac{1}{j})\backslash B_p(r-\frac{1}{j})$. Now let us fix such r and R.

Let $v_j = v \cdot \eta_j(|px|) \in W_0^{1,2}(D)$, where $D = B_p(R_0)\backslash\overline{B_p(r_0)}$ and

$$\eta_j(t) := \begin{cases} 1 & \text{if} \quad t \in [r, R] \\ j\cdot(t-r)+1 & \text{if} \quad t \in [r-\frac{1}{j}, r] \\ -j\cdot(t-R)+1 & \text{if} \quad t\in[R, R+\frac{1}{j}] \\ 0 & \text{if} \quad t \in (-\infty, r-\frac{1}{j}) \cup (R+\frac{1}{j}, \infty). \end{cases}$$

By the definitions of $I_{w,A}(v)$ and $\mathscr{L}_w$, we have

(4.4)
$$\begin{aligned} I_{w,A}(v) = &\int_D \langle\nabla w, \nabla v_j\rangle \, d\mathrm{vol} - \int_{D\backslash A}\langle\nabla w, \nabla v_j\rangle\, d\mathrm{vol} + \int_D v_j d\mathscr{L}_w \\ &- \int_{D\backslash A} v_j d\mathscr{L}_w = -\int_{D\backslash A} v\,\langle\nabla w, \nabla\eta_j\rangle\, d\mathrm{vol} \\ &- \int_{D\backslash A}\eta_j\,\langle\nabla w, \nabla v\rangle\, d\mathrm{vol} - \int_{D\backslash A} v_j d\mathscr{L}_w \\ &:= -J_1 - J_2 - J_3. \end{aligned}$$

Notice that

$$|J_2| \leqslant \int_{A_j\backslash A}|\nabla w|\cdot|\nabla v| d\mathrm{vol} \quad \text{and} \quad |J_3| \leqslant \mathscr{L}_w(A_j\backslash A)\cdot|v|_{L^\infty(D)}.$$

Hence we have $J_2 \to 0$ and $J_3 \to 0$ as $j \to \infty$.

$$J_1 = j\cdot\int_{B_p(r)\backslash B_p(r-1/j)} v\varphi' d\mathrm{vol} - j\cdot\int_{B_p(R+1/j)\backslash B_p(R)} v\varphi' d\mathrm{vol}. \tag{4.5}$$

The assumption $v \in L^\infty(D)$ implies the function $h(t) = \int_{B_p(t)} v d\mathrm{vol}$ is Lipschitz continuous in (r_0, R_0). Indeed, for each $r_0 < s < t < R_0$,

$$|h(t)-h(s)| \leqslant \int_{B_p(t)\backslash B_p(s)}|v| d\mathrm{vol} \leqslant |v|_{L^\infty}\cdot \mathrm{vol}\big(B_p(t)\backslash B_p(s)\big) \leqslant c\cdot(t^n - s^n),$$

where constant c depends only on R_0, n and the lower bounds of curvature on $B_p(R_0)$. Then $h(t)$ is differentiable at almost all $t \in (r_0, R_0)$. By co-area formula, we have

$$h'(t) = \int_{\partial B_p(t)} v d\mathrm{vol}$$

for almost all $t \in (r_0, R_0)$.

Without loss of generality, we may assume that r and R are differentiable points of function h. Now

$$\Big|j\int_{B_p(r)\setminus B_p(r-1/j)}\varphi' v d\mathrm{vol}-\varphi'(r)\cdot j\Big(h(r)-h(r-1/j)\Big)\Big|$$
$$\leqslant \int_{B_p(r)\setminus B_p(r-1/j)}\max|\varphi''|\cdot|v|d\mathrm{vol}\to 0$$

as $j\to\infty$. A similar estimate also holds for $j\int_{B_p(R+1/j)\setminus B_p(R)}\varphi' v d\mathrm{vol}$. Therefore,

$$\begin{aligned}\lim_{j\to\infty}J_1&=\lim_{j\to\infty}\varphi'(r)\cdot j\Big(h(r)-h(r-1/j)\Big)\\&\quad-\lim_{j\to\infty}\varphi'(R)\cdot j\Big(h(R+1/j)-h(R)\Big)\\&=\varphi'(r)h'(r)-\varphi'(R)h'(R).\end{aligned}$$

By combining this and (4.4), we get the desired assertion. q.e.d.

If M has $Ric\geqslant(n-1)k$, then for a distance function $dist_p(x):=|px|$, Laplacian comparison (see Theorem 1.1 and Corollary 5.9 in [**22**]) asserts that $\mathscr{L}_{dist_p}$ is a signed Radon measure and

$$\mathscr{L}_{dist_p}\leqslant(n-1)\cdot\cot_k\circ dist_p\cdot\mathrm{vol}\qquad\text{on}\quad M\backslash\{p\}.$$

Moreover, $G(x):=\phi_k(|px|)$ is defined on $M\backslash\{p\}$ and

$$\mathscr{L}_G\geqslant 0\qquad\text{on}\quad M\backslash\{p\},$$

where $\phi_k(r)$ is the real value function such that $\phi\circ dist_o$ is the Green function on $\mathbb{M}^n_k$ with singular point o. That is, if $n\geqslant 3$,

$$\phi_k(r)=\frac{1}{(n-2)\cdot\omega_{n-1}}\int_r^\infty s_k^{1-n}(t)dt, \tag{4.6}$$

where $\omega_{n-1}=\mathrm{vol}(\mathbb{S}^{n-1})$ and

$$s_k(t)=\begin{cases}\sin(\sqrt{k}t)/\sqrt{k} & k>0\\ t & k=0\\ \sinh(\sqrt{-k}t)/\sqrt{-k} & k<0.\end{cases}$$

If $n=2$, the function ϕ_k can be given similarly.

By applying Lemma 4.3 to function G, we have the following mean value inequality for nonnegative supersolution of Poisson's equation.

Proposition 4.4. *Let M be an n-dimensional Alexandrov space with $Ric\geqslant(n-1)k$ and Ω be a bounded domain in M. Assume that $f\in L^\infty(\Omega)$ and u is a continuous, nonnegative supersolution of Poisson's*

equation $\mathscr{L}_u = f\cdot \mathrm{vol}$ *on* Ω*. Then for any ball* $B_p(R)\Subset\Omega$*, we have*
(4.7)

$$\frac{\mathrm{vol}(\Sigma_p)}{\omega_{n-1}}\bigg(\frac{1}{H^{n-1}(\partial B_o(R)\subset T^k_p)}\int_{\partial B_p(R)}u d\mathrm{vol}-u(p)\bigg)$$
$$\leqslant (n-2)\cdot\int_{B^*_p(R)}Gfd\mathrm{vol}-(n-2)\cdot\phi_k(R)\int_{B_p(R)}fd\mathrm{vol},$$

where $B^*_p(R) = B_p(R)\backslash\{p\}$ *and* T^k_p *is the* k*-cone over* Σ_p *(see* [**2**] *p. 354).*

Proof. For simplicity, we only give a proof for the case $n\geqslant 3$. A slight modification of the argument will prove the case $n = 2$.

Case 1: Assume that u is a solution of $\mathscr{L}_u = f\cdot\mathrm{vol}$.

Let $G(x) = \phi_k(|px|)$, where the real function ϕ_k is chosen such that $\phi_k(|ox|)$ is the Green function on $\mathbb{M}^n_k$ with singular point at o. Then, by Laplacian comparison theorem (see [**22**] or [**52**]), the signed Radon measure $\mathscr{L}_G$ is nonnegative on $M\backslash\{p\}$.

Since u is continuous on $B_p(R)$, the function $h(s) = \int_{B_p(s)} ud\mathrm{vol}$ is Lipschitz. From Lemma 4.3, we have

$$I_{G,A}(u) = \phi'_k(t)\cdot h'(t) - \phi'_k(s)\cdot h'(s)$$

for almost all $s, t\in(0,R)$ with $s < t$, where $A = B_p(t)\backslash\overline{B_p(s)}$. By the definition of supersolution of Poisson equation, we have

$$I_{G,A}(u) - I_{u,A}(G) = \int_A ud\mathscr{L}_G - \int_A Gd\mathscr{L}_u \geqslant -\int_A Gfd\mathrm{vol}.$$

On the other hand, letting

$$\bar G(x) = \begin{cases} G(x) & \text{if} \quad s\leqslant|px|\leqslant t\\ \phi_k(t) & \text{if} \quad |px|>t\\ \phi_k(s) & \text{if} \quad |px|<s,\end{cases}$$

we have

$$\int_A\langle\nabla G,\nabla u\rangle = \int_{B_p(t)}\big\langle\nabla(\bar G-\phi_k(t)),\nabla u\big\rangle - \int_{B_p(s)}\big\langle\nabla(\bar G-\phi_k(s)),\nabla u\big\rangle$$
$$= -\int_A Gd\mathscr{L}_u + \phi_k(t)\int_{B_p(t)}d\mathscr{L}_u - \phi_k(s)\int_{B_p(s)}d\mathscr{L}_u.$$

Hence, by $\mathscr{L}_u = f\cdot\mathrm{vol}$,

$$I_{u,A}(G) = \phi_k(t)\int_{B_p(t)}fd\mathrm{vol} - \phi_k(s)\int_{B_p(s)}fd\mathrm{vol}.$$

If we set

$$\psi(\tau) = \phi_k'(\tau) \cdot h'(\tau) - \phi_k(\tau) \int_{B_p(\tau)} f d\mathrm{vol},$$

then the function

$$\psi(\tau) + \int_{B_p^*(\tau)} G f d\mathrm{vol}$$

is nondecreasing with respect to τ (for almost all $\tau \in (0, R)$). Indeed, for almost all $s < t$,

$$\begin{aligned}\psi(t) + \int_{B_p^*(t)} G f d\mathrm{vol} - \psi(s) - \int_{B_p^*(s)} G f d\mathrm{vol} = I_{G,A}(u) - I_{u,A}(G) \\ + \int_A G f d\mathrm{vol} \geqslant 0.\end{aligned}$$

Thus by

$$\phi_k'(t) = -s_k^{1-n}(t) \cdot \frac{1}{(n-2)\omega_{n-1}} = -\frac{1}{n-2} \cdot \frac{\mathrm{vol}(\Sigma_p)}{\omega_{n-1}} \cdot \frac{1}{H^{n-1}(\partial B_o(t) \subset T_p^k)},$$

we have

$$\begin{aligned}\phi_k'(t) h'(t) - \phi_k(t) \int_{B_p(t)} f d\mathrm{vol} + \int_{B_p^*(t)} G f \mathrm{vol} \\ \geqslant \lim_{s \to 0} \Big(\psi(s) + \int_{B_p^*(s)} G f d\mathrm{vol} \Big) = -\frac{1}{n-2} \cdot \frac{\mathrm{Vol}(\Sigma_p)}{\omega_{n-1}} u(p).\end{aligned}$$

By combining this and $h'(s) = \int_{\partial B_p(r)} u d\mathrm{vol}$ a.e. in $(0, R)$, we obtain that (4.7) holds for almost all $r \in (0, R)$.

By combining the Bishop–Gromov inequality on spheres (see [**2**] or Lemma 3.2 of [**22**]), the assumption $u \geqslant 0$; and the continuity of u, we have

$$\liminf_{r \to R^-} \int_{\partial B_p(r)} u d\mathrm{vol} \geqslant \int_{\partial B_p(R)} u d\mathrm{vol}. \tag{4.8}$$

Therefore, we get the desired result for this case.

Case 2: u is a supersolution of $\mathscr{L}_u = f \cdot \mathrm{vol}$.

For each $R > 0$, let $\widetilde{u}$ be a solution of $\mathscr{L}_{\widetilde{u}} = f \cdot \mathrm{vol}$ on $B_p(R)$ with boundary value condition $\widetilde{u} = u$ on $\partial B_p(r)$. Since $\mathscr{L}_{\widetilde{u}-u} \geqslant 0$, by maximal principle, we get $\widetilde{u}(p) \leqslant u(p)$. Therefore, by applying *Case 1* to $\widetilde{u}$, we get the desired estimate. q.e.d.

Corollary 4.5. *Let M, Ω, u, and f be as above. If p is a Lebesgue point of f, i.e.,*

$$\fint_{B_p(R)} f d\mathrm{vol} = f(p) + o(1), \tag{4.9}$$

then

$$\frac{1}{H^{n-1}(\partial B_o(R)\subset T_p^k)}\int_{\partial B_p(R)} u(x)d\mathrm{vol}\leqslant u(p)+\frac{f(p)}{2n}\cdot R^2+o(R^2).$$

Proof. By using (4.7), we have

(4.10)

$$\frac{1}{H^{n-1}(\partial B_o(R)\subset T_p^k)}\int_{\partial B_p(R)} ud\mathrm{vol}-u(p)\leqslant (n-2)\cdot\frac{\omega_{n-1}}{\mathrm{vol}(\Sigma_p)}\cdot\varrho(R),$$

where

$$\begin{aligned}\varrho(R)&=\int_{B_p^*(R)} Gfd\mathrm{vol}-\phi_k(R)\int_{B_p(R)} fd\mathrm{vol}\\ &=\int_0^R\int_{\partial B_p(s)}\phi_k(s)f-\phi_k(R)\int_0^R\int_{\partial B_p(s)} f.\end{aligned}$$

Hence, by (4.9), we have

$$\begin{aligned}\varrho'(R)&=-\phi_k'(R)\int_{B_p(R)} fd\mathrm{vol}\\ &=\frac{\mathrm{vol}(\Sigma_p)}{(n-2)\omega_{n-1}}\cdot\frac{\int_0^R s_k^{n-1}(r)dr}{s_k^{n-1}(R)}\cdot\frac{\mathrm{vol}(B_p(R))}{H^n(B_o(R)\subset T_k^n)}\int\limits_{B_p(R)} fd\mathrm{vol}\\ &=\frac{\mathrm{vol}(\Sigma_p)}{(n-2)\omega_{n-1}}\cdot\big(\frac{R}{n}+o(R)\big)\cdot(1+o(1))\cdot(f(p)+o(1))\\ &=\frac{\mathrm{vol}(\Sigma_p)}{n(n-2)\omega_{n-1}}f(p)\cdot R+o(R).\end{aligned}$$

Hence, noting that $\rho(0)=0$, we get

$$\rho(R)=\frac{\mathrm{vol}(\Sigma_p)}{2n(n-2)\omega_{n-1}}f(p)\cdot R^2+o(R^2). \tag{4.11}$$

Therefore, the desired result follows from (4.10) and (4.11). q.e.d.

Corollary 4.6. *Let M be an n-dimensional Alexandrov space with $Ric\geqslant(n-1)k$ and Ω be a bounded domain in M. Let u be a semi-concave function on Ω and $f\in L^\infty(\Omega)$. Then u is a supersolution of $\mathscr{L}_u=f\cdot\mathrm{vol}$ provided it satisfies the following property: for each point $p\in Reg_u$ and every sufficiently small ball $B_p(R)\Subset\Omega$, we have*

$$\widetilde{u}_R-u\leqslant 0, \tag{4.12}$$

where the function $\widetilde{u}_R$ is the (unique) solution of the Dirichlet problem:

$$\begin{cases}\mathscr{L}_{\widetilde{u}_R}=f\cdot\mathrm{vol} & \text{in } B_p(R)\\ \widetilde{u}_R=u & \text{on } \partial B_p(R).\end{cases}$$

In particular, a semi-concave superharmonic function must be a supersolution of the equation $\mathscr{L}_u=0$.

Proof. Since the singular part of $\mathscr{L}_u$ is non-positive, we need only to consider its absolutely continuous part $\Delta u \cdot \mathrm{vol}$.

Fix a point $p \in Reg_u$ such that (4.2) holds and p is a Lebesgue point of f. Since the set of such points has full measure in Ω, we need only to show that $\Delta u(p) \leqslant f(p)$.

We set

$$g_R(x) = u(x) - \min_{x \in \overline{B_p(R)}} \widetilde{u}_R(x) \quad \text{and} \quad \widetilde{g}_R(x) = \widetilde{u}_R(x) - \min_{x \in \overline{B_p(R)}} \widetilde{u}_R(x).$$

Then $\widetilde{g}_R \leqslant g_R$ and $\widetilde{g}_R|_{\partial B_p(R)} = g_R|_{\partial B_p(R)}$. Noting that the function $\widetilde{g}_R$ is nonnegative and $\mathscr{L}_{\widetilde{g}_R} = f \cdot \mathrm{vol}$, by Corollary 4.5 and assuming p is regular, we have

(4.13)

$$\begin{aligned}\int_{\partial B_p(R)} g_R = \int_{\partial B_p(R)} \widetilde{g}_R &\leqslant H^{n-1}(\partial B_o(R) \subset T_p^k) \cdot \Big(\widetilde{g}_R(p) + \frac{f(p)}{2n}R^2 + o(R^2)\Big) \\ &\leqslant g_R(p) \cdot H^{n-1}(\partial B_o(R) \subset T_p^k) + \frac{f(p)}{2n}R^{n+1} \cdot \omega_{n-1} + o(R^{n+1}).\end{aligned}$$

On the other hand, since $p \in Reg_{g_R}$, from (2.15) and (4.2), we have

(4.14)

$$\int_{\partial B_p(R)} g_R = g_R(p) \cdot \mathrm{vol}(\partial B_p(R)) + \frac{\Delta g_R(p)}{2n} R^2 \cdot \mathrm{vol}(\partial B_p(R)) + o(R^{n+1})$$

for almost all $R \in (0, \delta_0)$, where δ_0 is a small positive number. Because p is a smooth point, Lemma 2.1 implies

$$H^{n-1}(\partial B_o(R) \subset T_p^k) - \mathrm{vol}(\partial B_p(R)) = o(R^n) \tag{4.15}$$

for almost all $R \in (0, \delta_0)$.

Now we want to show $g_R(p) = O(R)$. From (4.12) and the fact that u is locally Lipshitz (since u is semi-concave), we have

(4.16)

$$\begin{aligned}0 \leqslant g_R(p) &= u(p) - \min_{x \in \partial \overline{B_p(R)}} \widetilde{u}_R(x) + \min_{x \in \partial \overline{B_p(R)}} \widetilde{u}_R(x) - \min_{x \in \overline{B_p(R)}} \widetilde{u}_R(x) \\ &\leqslant C_1 R + \min_{x \in \partial \overline{B_p(R)}} \widetilde{u}_R(x) - \min_{x \in \overline{B_p(R)}} \widetilde{u}_R(x).\end{aligned}$$

Since R is sufficiently small, there exists the Perelman concave function h on $B_p(2R)$ given in Lemma 3.3. We have

$$\mathscr{L}_{\widetilde{u}_R + \|f\|_{L^\infty} \cdot h} \leqslant \mathscr{L}_{\widetilde{u}_R} - \|f\|_{L^\infty} \cdot \mathrm{vol} \leqslant 0.$$

Hence, by applying maximal principle, we have for any point $x \in \overline{B_p(R)}$,

$$\begin{aligned}\widetilde{u}_R(x) + \|f\|_{L^\infty} h(x) &\geqslant \min_{x \in \partial \overline{B_p(R)}} \Big(\widetilde{u}_R(x) + \|f\|_{L^\infty} h(x)\Big) \\ &\geqslant \min_{x \in \partial \overline{B_p(R)}} \widetilde{u}_R(x) + \|f\|_{L^\infty} \min_{x \in \partial \overline{B_p(R)}} h(x).\end{aligned}$$

Since h is Lipschitz continuous, this implies that

$$\min_{x\in\partial\overline{B_p(R)}}\widetilde{u}_R(x)-\widetilde{u}_R(x)\leqslant\|f\|_{L^\infty}\big(h(x)-\min_{x\in\partial\overline{B_p(R)}}h(x)\big)\leqslant C_2R$$

for any point $x\in\overline{B_p(R)}$. The combination of this and (4.16) implies

$$g_R(p)=O(R). \tag{4.17}$$

By combining (4.13)–(4.15) and (4.17), we have

$$\frac{\Delta g_R(p)}{2n}R^2\cdot\mathrm{vol}(\partial B_p(R))-\frac{f(p)}{2n}\omega_{n-1}R^{n+1}\leqslant O(R)\cdot o(R^n)+o(R^{n+1})=o(R^{n+1})$$

for almost all $R\in(0,\delta_0)$. Hence, $\Delta g_R(p)\leqslant f(p)$. Therefore, $\Delta u(p)\leqslant f(p)$, and the proof of the corollary is completed. q.e.d.

4.3. Harmonic measure. In this subsection, we basically follow Petrunin in [**40**] to consider harmonic measure.

Lemma 4.7. (Petrunin [**40**]) *Let M be an n-dimensional Alexandrov space with $Ric\geqslant(n-1)k$ and let Ω be a bounded domain in M. If u is a nonnegative harmonic function on Ω, then for any ball $B_p(R)\Subset\Omega$, we have*

$$u(p)\geqslant\frac{1}{\mathrm{vol}(\Sigma_p)\cdot s_k^{n-1}(R)}\int_{\partial B_p(R)}u\mathrm{dvol}. \tag{4.18}$$

Proof. By the definition, u is harmonic if and only if it is a solution of equation $\mathscr{L}_u=0$. Now the result follows from (4.7) with $f=0$. q.e.d.

Consider an n-dimensional Alexandrov space M and a ball $B_p(R)\subset M$. In order to define a new measure $\nu_{p,R}$ on $B_p(R)$, according to [**13**], we need only to define a positive functional on $Lip_0(B_p(R))$.

Now fix a nonnegative function $\varphi\in Lip_0(B_p(R))$. First we define a function $\mu:(0,R)\to\mathbb{R}$ as follows: for each $r\in(0,R)$, define

$$\mu(r):=u_r(p),$$

where u_r is the (unique) solution of Dirichlet problem $\mathscr{L}_u=0$ in $B_p(r)$ with boundary value $u=\varphi$ on $\partial B_p(r)$.

Lemma 4.8. *There exists $R>0$ such that $\mu(r)$ is continuous in $(0,R)$.*

Proof. From Lemma 11.2 in [**6**], we know that there exists $R>0$ such that, for all $x\in B_p(R)\backslash\{p\}$, we can find a point x_1 satisfying

$$\widetilde{\angle}pxx_1>\frac{99}{100}\pi\quad\text{and}\quad|px_1|\geqslant2|px|.$$

In particular, this implies, for each $r\in(0,R)$, that $B_p(r)$ satisfies an exterior ball condition in the following sense: there exists $C>0$ and $\delta_0>0$ such that for all $x\in\partial B_p(r)$ and $0<\delta<\delta_0$, the set $B_x(\delta)\backslash B_p(r)$ contains a ball with radius $C\delta$. Indeed, we can choose x_2 in geodesic

xx_1 with $|xx_2| = \delta/3$ (with $\delta \leqslant r/10$). The monotonicity of comparison angles says that $\widetilde{\angle}pxx_2 \geqslant \widetilde{\angle}pxx_1 \geqslant \frac{99}{100}\pi$. This concludes $|px_2| \geqslant |px| + \delta/6$. Therefore, $B_{x_2}(\delta/6) \subset B_x(\delta)\backslash B_p(r)$.

Since φ is Lipschitz continuous on $B_p(r)$, Björn in [**5**] (see Remark 2.15 in [**5**]) proved that u_r is Hölder continuous on $\overline{B_p(r)}$.

For any $0 < r_1 < r_2 < R$, by using the maximum principle, we have

$$|\mu(r_1) - \mu(r_2)| \leqslant \max_{x \in \partial B_p(r_1)} |u_{r_1}(x) - u_{r_2}(x)| = \max_{x \in \partial B_p(r_1)} |\varphi(x) - u_{r_2}(x)|.$$

By combining with the Hölder continuity of φ and u_{r_2}, we have that $|\mu(r_1) - \mu(r_2)| \to 0$ as $r_2 - r_1 \to 0^+$. Hence $\mu(r)$ is continuous. q.e.d.

Remark 4.9. If p is a regular point, then the constant R given in Lemma 4.8 can be chosen uniformly in a neighborhood of p.

Indeed, there exists a neighborhood of p, $B_p(R_0)$ and a bi-Lipschitz homeomorphism F mapping $B_p(R_0)$ to an open domain of $\mathbb{R}^n$ with bi-Lipschitz constant $\leqslant 1/100$. Then for each ball $B_q(r) \subset B_p(R_0/4)$ with $r \leqslant R_0/4$ and $x \in \partial B_q(r)$, let $x' \in \mathbb{R}^n$ such that

$$\begin{aligned} |x'F(x)| = |F(q)F(x)| \quad \text{and} \quad |x'F(q)| &= |F(q)F(x)| + |F(x)x'| \\ &= 2|F(q)F(x)|; \end{aligned}$$

we have

$$\begin{aligned} |qF^{-1}(x')| \geqslant \frac{99}{100}|F(q)x'| \geqslant 2\Big(\frac{99}{100}\Big)^2|xq| \quad \text{and} \quad &|x'F(x)| \\ \leqslant \Big(\frac{101}{100}\Big)^2|xq|. \end{aligned}$$

Hence, it is easy to check that $B_q(r)$ satisfies an exterior ball condition as above (in a similar way to that above). Therefore, the constant R in Lemma 4.8 can be chosen $R_0/4$ for all $q \in B_p(R_0/4)$.

Now we can define the functional $\nu_{p,R}$ by

$$\nu_{p,R}(\varphi) = \frac{\int_0^R s_k^{n-1}\mu(r)dr}{\int_0^R s_k^{n-1}dr}.$$

From Lemma 4.7, we have $\mu_r \geqslant 0$, and $\nu_{p,R}(\varphi) \geqslant 0$. Hence, it provides a Radon measure on $B_p(R)$. Moreover, it is a probability measure and by (4.18),

$$\nu_{p,R} \geqslant \frac{\mathrm{vol}}{H^n(B_o(R) \subset T_p^k)}. \tag{4.19}$$

Let u be a harmonic function on Ω. Then for any ball $B_p(R) \Subset \Omega$, we have

$$u(p) = \int_{B_p(R)} u(x)d\nu_{p,R}. \tag{4.20}$$

The following strong maximum principle was proved in an abstract framework of Dirichlet form by Kuwae in [**25**] and Kuwae–Machiyashira–Shioya in [**19**]. In metric spaces supporting a doubling measure and a Poincaré inequality, it was proved by Kinnunen–Shanmugalingan in [**18**]. Here, by (4.20), we give a short proof in Alexandrov spaces.

Corollary 4.10. (Strong Maximum Principle) *Let u be a subharmonic function on a bounded and connected open domain Ω. Suppose there exists a point $p \in \Omega$ for which $u(p) = \sup_{x\in\Omega} u$. Then u is constant.*

Proof. First, we consider u to be harmonic. By (4.19)–(4.20) and given that $\nu_{p,R}$ is a probability measure, we have $u(x) = u(p)$ in some neighborhood $B_p(R)$. Hence the set $\{x \in \Omega : \ u(x) = u(p)\}$ is open. On the other hand, the continuity of u implies that the set is closed. Therefore, it is Ω and u is a constant in Ω.

If u is a subharmonic function, the result follows from the definition of subharmonic and the above harmonic case. q.e.d.

The following lemma appeared in [**40**] (page 4). In this lemma, Petrunin constructed an auxiliary function, which is similar to Perelman's concave function.

Lemma 4.11. (Petrunin [**40**]) *For any point $p \in M$, there exists a neighborhood $B_p(r_2)$ and a function $h_0 : B_p(r_2) \to \mathbb{R}$ satisfying:*

(i) $h_0(p) = 0$;

(ii) $\mathscr{L}_{h_0} \geqslant 1 \cdot \mathrm{vol}$ *on* $B_p(r_2)$;

(iii) *there are* $0 < c < C < \infty$ *such that*

$$c \cdot |px|^2 \leqslant h_0(x) \leqslant C \cdot |px|^2.$$

Proof. A sketched proof was given in [**40**]. For completeness, we present a detailed proof as follows.

Without loss of generality, we may assume M has curvature $\geqslant -1$ on a neighborhood of p. Fix a small real number $r > 0$ and set

$$\phi(t) = \begin{cases} a + bt^{2-n} + t^2 & t \leqslant r \\ 0 & t > r, \end{cases}$$

where $a = -\frac{n}{n-2} r^2$ and $b = \frac{2}{n-2} r^n$.

Take a minimal set of points $\{q_\alpha\}_{\alpha=1}^N$ such that $|pq_\alpha| = r$ and $\min_{1\leqslant\alpha\leqslant N} \angle(\xi, \uparrow_p^{q_\alpha}) \leqslant \pi/10$ for each direction $\xi \in \Sigma_p$. Consider

$$h_0(x) = \sum_{\alpha=1}^{N} h_\alpha$$

where $h_\alpha = \phi(|q_\alpha x|)$. Clearly, $h_0(p) = 0$. Bishop–Gromov volume comparison of Σ_p implies that $N \leqslant c(n)$, for some constant depending only on the dimension n.

Fix any small $0 < \delta \ll r$. For each $x \in B_p(\delta)\backslash\{p\}$, there is some q_α such that $\angle(\uparrow_p^x, \uparrow_p^{q_\alpha}) \leqslant \pi/10$. When δ is small, the comparison angle $\widetilde{\angle}xq_\alpha p$ is small. Then $\widetilde{\angle}q_\alpha xp \geqslant \frac{3}{4}\pi$. This implies that $|\nabla_x dist_{q_\alpha}| \geqslant 1/\sqrt{2}$, when δ is sufficiently small.

Fix any α. Since the function $-h_\alpha$ is semi-concave near p, the singular part of $\mathscr{L}_{h_\alpha}$ is nonnegative. We only need to consider the absolutely continuous part Δh_α. By Laplacian comparison theorem (see [**52**] or [**22**]) and a direct computation, we have $\Delta h_\alpha(x) \geqslant -C\delta$ a.e. in $B_p(\delta)$ and $\Delta h_\alpha(x) \geqslant n - C\delta$ at almost all points x with $\widetilde{\angle}q_\alpha xp \geqslant \frac{3}{4}\pi$, where C denotes the various positive constants depending only on n and r. Indeed, since $r - \delta \leqslant |q_\alpha x| \leqslant r$ (the fact $\widetilde{\angle}q_\alpha xp \geqslant \frac{3}{4}\pi$ implies $|q_\alpha x| \leqslant r$),

$$\begin{aligned}
\Delta h_\alpha(x) &= \phi'(|q_\alpha x|)\cdot \Delta dist_{q_\alpha}(x) + \phi''(|q_\alpha x|)|\nabla dist_{q_\alpha}|^2\\
&= 2|q_\alpha x|\cdot\Big(1 - \frac{r^n}{|q_\alpha x|^n}\Big)\cdot \Delta dist_{q_\alpha}(x)\\
&+ 2\Big(1 + (n-1)\frac{r^n}{|q_\alpha x|^n}\Big)\cdot|\nabla_x dist_{q_\alpha}|^2\\
&\geqslant 2|q_\alpha x|\cdot\Big(1 - \frac{r^n}{|q_\alpha x|^n}\Big)\cdot\Big(\frac{n-1}{|q_\alpha x|} + C|q_\alpha x|\Big)\\
&\quad + 2\Big(1 + (n-1)\frac{r^n}{|q_\alpha x|^n}\Big)\cdot|\nabla_x dist_{q_\alpha}|^2\\
&\geqslant -C_n\frac{\delta}{r} + 2n\cdot|\nabla_x dist_{q_\alpha}|^2.
\end{aligned}$$

On the other hand, at the points x where $\angle(\uparrow_p^x, \uparrow_p^{q_\alpha}) \leqslant \pi/10$ and $|px| \leqslant |pq_\alpha|/10$, we have

$$r - |px| \leqslant |q_\alpha x| \leqslant r - |px|/2.$$

Hence, by applying $\phi'(r) = 0$ and $2n \leqslant \phi''(t) \leqslant 2n\cdot 2^n$ for all $r/2 \leqslant t \leqslant r$, it is easy to check that there exist two positive numbers c_1, C_1 depending only on n and r such that

$$c_1\cdot|px|^2 \leqslant h_\alpha(x) = \phi(|q_\alpha x|) \leqslant C_1\cdot|px|^2$$

if $r - |px| \leqslant |q_\alpha x| \leqslant r - |px|/2$.

Therefore, we have (since for each $x \in B_p(\delta)\backslash\{p\}$, there is some q_α such that $\Delta h_\alpha(x) \geqslant n - C\delta$)

$$\Delta h_0 \geqslant n - N\cdot C\delta \quad \text{on} \quad B_p(\delta)$$

and

$$c_1\cdot|px|^2 \leqslant h_\alpha(x) = \phi(|q_\alpha x|) \leqslant N\cdot C_1\cdot|px|^2.$$

By $N \leqslant c(n)$ for some constant $c(n)$ depending only on the dimension n, if $\delta < (C\cdot c_n)^{-1}$, the function h_0 satisfies all of the conditions in the lemma. q.e.d.

REMARK 4.12. If p is a regular point, then the constant r_2 given in Lemma 4.11 can be chosen uniformly in a neighborhood of p. Indeed, in this case, there exists a neighborhood of p which is bi-Lipschitz homeomorphic to an open domain of $\mathbb{R}$ with a bi-Lipschitz constant close to 1. The constant r and δ in the above proof can be chosen to have a lower bound depending only on the bi-Lipschitz constant.

Proposition 4.13. (Petrunin [**40**]) *Given any $p \in \Omega$ and $\lambda \geqslant 0$, there exist constants R_p and $c(p, \lambda)$ such that, for any $u \in W^{1,2}(\Omega) \cap C(\Omega)$ satisfing $\mathscr{L}_u \leqslant \lambda \cdot \mathrm{vol}$ on Ω, we have*

$$\int_{B_p(R)} u d\nu_{p,R} \leqslant u(p) + c(p, \lambda) \cdot R^2 \tag{4.21}$$

for any ball $B_p(R) \Subset \Omega$ with $0 < R < R_p$, where the constant $c(p, \lambda) = 0$ if $\lambda = 0$.

Proof. This proposition was given by Petrunin in [**40**] (page 5). For completeness, we give a detailed proof as follows.

Case 1: $\lambda = 0$.

For each $r \in (0, R)$, let u_r be the harmonic function on $B_p(r)$ with boundary value $u_r = u$ on $\partial B_p(r)$. Then $\mathscr{L}_{u-u_r} \leqslant 0$ and $(u-u_r)|_{\partial B_p(r)} = 0$. By applying maximum principle, we know that $u - u_r \geqslant 0$ on $B_p(r)$; that is, by the definition of $\mu(r)$, $\mu(r) \leqslant u(p)$. Therefore, by the definition of $\nu_{p,R}$, we have

$$\int_{B_p(R)} u d\nu_{p,R} \leqslant u(p).$$

Case 2: $\lambda > 0$.

Let h_0 be the function given in Lemma 4.11; we have $\mathscr{L}_{u-\lambda h_0} \leqslant 0$ on $B_p(r_2)$, where r_2 is the constant given in Lemma 4.11. Hence, we can use the case above for function $u - \lambda h_0$. This gives us, by Lemma 4.11,

$$u(p) = u(p) - \lambda h_0(p) \geqslant \int_{B_p(R)} (u - \lambda h_0) d\nu_{p,R} \geqslant \int_{B_p(R)} u d\nu_{p,R} - C \cdot \lambda \cdot R^2$$

for all $0 < R < r_2$, where C is the constant given in Lemma 4.11. q.e.d.

REMARK 4.14. If p is regular, according to Remark 4.9 and Remark 4.12, the constant R_p can be chosen uniformly in a neighborhood of p.

The following lemma is similar to one that appeared in [**40**] (page 10).

Lemma 4.15. (Petrunin [**40**]) *Let h be the Perelman concave function given in Lemma 3.3 on a neighborhood $U \subset M$. Assume that f is a semi-concave function defined on U. And suppose that $u \in W^{1,2}(U) \cap C(U)$ satisfies $\mathscr{L}_u \leqslant \lambda \cdot \mathrm{vol}$ on U for some constant $\lambda \in \mathbb{R}$.*

We assume that point $x^* \in U$ is a minimal point of function $u+f+h$; then x^* has to be regular. Moreover, f is differentiable at x^* (in the sense of Taylor expansion (2.16)*).*

Proof. Without loss of generality, we may assume that $\lambda \geqslant 0$. In the proof, we denote $B_{x^*}(R)$ $(\subset U)$ by B_R. From the minimum property of x^*, we have

$$\int_{B_R} (u+f+h)d\nu_{p,R} \geqslant u(x^*) + f(x^*) + h(x^*). \tag{4.22}$$

By Proposition 4.13, we get

$$\int_{B_R} ud\nu_{p,R} \leqslant u(x^*) + cR^2 \tag{4.23}$$

for some constant $c = c(p, \lambda)$ and for all sufficiently small R.

On the other hand, setting $\bar{h} = f + h$, we have

$$\begin{aligned} \int_{B_R} \bar{h}d\nu_{p,R} &= \bar{h}(x^*) + \int_{B_R} (\bar{h} - \bar{h}(x^*))d\Big(\nu_{p,R} - \frac{\text{vol}}{H^n(B_o^k(R))}\Big) \\ &\quad + \frac{1}{H^n(B_o^k(R))}\int_{B_R} (\bar{h} - \bar{h}(x^*))d\text{vol} \\ &:= \bar{h}(x^*) + J_1 + J_2, \end{aligned} \tag{4.24}$$

where $B_o^k(R)$ is the ball in T_p^k.

Because $\bar{h} = f + h$ is a Lipschitz function and $\frac{\text{vol}(B_R)}{H^n(B_o^k(R))} = 1 + o(1)$, we have

$$|J_1| \leqslant O(R)\cdot\int_{B_R} 1d\Big(\nu_{p,R} - \frac{\text{vol}}{H^n(B_o^k(R))}\Big) = O(R)\cdot\Big(1 - \frac{\text{vol}(B_R)}{\text{vol}(B_o^k(R))}\Big) = o(R). \tag{4.25}$$

Since $\bar{h} = f + h$ is semi-concave, according to equation (2.7), we have

$$\begin{aligned} J_2 &= \frac{\text{vol}(B_p(R))}{H^n(B_o^k(R))}\int_{B_R} (\bar{h} - \bar{h}(x^*))d\text{vol} \\ &= \frac{\text{vol}(B_R)}{H^n(B_o^k(R))}\cdot\Big(\frac{nR}{n+1}\int_{\Sigma_{x^*}} d_{x^*}\bar{h}(\xi)d\xi + o(R)\Big) \\ &= \frac{nR}{n+1}\int_{\Sigma_{x^*}} d_{x^*}\bar{h}(\xi)d\xi + o(R). \end{aligned} \tag{4.26}$$

By combining (4.22)–(4.26), we have

$$\frac{nR}{n+1}\int_{\Sigma_{x^*}} d_{x^*}\bar{h}(\xi)d\xi + o(R) + cR^2 \geqslant 0.$$

By combining with Proposition 3.1,

$$\int_{\Sigma_{x^*}} d_{x^*}\bar{h}(\xi)d\xi = \int_{\Sigma_{x^*}} d_{x^*}f(\xi)d\xi + \int_{\Sigma_{x^*}} d_{x^*}h(\xi)d\xi \leqslant 0,$$

we have

$$\int_{\Sigma_{x^*}} d_{x^*}f(\xi)d\xi = \int_{\Sigma_{x^*}} d_{x^*}h(\xi)d\xi = 0.$$

Then by using Lemma 3.3 (iii), we conclude that x^* is regular.

Next we want to show that f is differentiable at x^*.

Since x^* is regular, we have

$$\int_{\Sigma_{x^*}} \langle \nabla_{x^*}f, \xi\rangle\, d\xi = \int_{\mathbb{S}^{n-1}} \langle \nabla_{x^*}f, \xi\rangle\, d\xi = 0.$$

Hence

$$\int_{\Sigma_{x^*}} \big(d_{x^*}f(\xi) - \langle \nabla_{x^*}f, \xi\rangle\big)d\xi = \int_{\Sigma_{x^*}} d_{x^*}f(\xi)d\xi = 0.$$

On the other hand, by the definition of $\nabla_{x^*}f$ (see Section 1.3 of [**39**]), we have

$$d_{x^*}f(\xi) \leqslant \langle \nabla_{x^*}f, \xi\rangle \qquad \forall\ \xi \in \Sigma_{x^*}.$$

According to the combination of the above two equations, we have

$$d_{x^*}f(\xi) = \langle \nabla_{x^*}f, \xi\rangle \qquad \forall\ \xi \in \Sigma_{x^*}.$$

By combining with the fact that x^* is regular, we get that f is differentiable at x^*. q.e.d.

We now follow Petrunin in [**40**] to introduce a perturbation argument. Let $u \in W^{1,2}(D) \cap C(\overline{D})$ satisfy $\mathscr{L}_u \leqslant \lambda \cdot \mathrm{vol}$ on a bounded domain D. Suppose that x_0 is the unique minimum point of u on D and $u(x_0) < \min_{x\in\partial D} u$. Suppose also that x_0 is regular and $g = (g_1,\ g_2, \ldots, g_n) : D \to \mathbb{R}^n$ is a coordinate system around x_0 such that g satisfies the following:

(i) g is an almost isometry from D to $g(D) \subset \mathbb{R}^n$ (see [**6**]). Namely, there exists a sufficiently small number $\delta_0 > 0$ such that

$$\Big|\frac{\|g(x) - g(y)\|}{|xy|} - 1\Big| \leq \delta_0, \qquad \text{for all} \quad x, y \in D,\ x \neq y;$$

(ii) all of the coordinate functions $g_j,\ 1 \leqslant j \leqslant n$, are concave ([**34**]).

Then there exists $\epsilon_0 > 0$ such that, for each vector $V = (v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$ with $|v^j| \leqslant \epsilon_0$ for all $1 \leqslant j \leqslant n$, the function

$$G(V, x) := u(x) + V \cdot g(x)$$

has a minimum point in the interior of D, where $\cdot$ is the Euclidean inner product of $\mathbb{R}^n$ and $V \cdot g(x) = \sum_{j=1}^n v^j g_j(x)$.

Let

$$\mathscr{U} = \{V \in \mathbb{R}^n :\ |v^j| < \epsilon_0,\ 1 \leqslant j \leqslant n\} \subset \mathbb{R}^n.$$

We define $\rho : \mathscr{U} \to D$ by setting $\rho(V)$ to be one of the minimum points of $G(V,x)$. Note that the map ρ might not be uniquely defined.

The following was given by Petrunin in [**40**] (page 8). For completeness, a detailed proof is given here.

Lemma 4.16. (Petrunin [**40**]) *Let* $u,\ x_0,\ \{g_j\}_{j=1}^n$ *and* ρ *be as above. There exists some* $\epsilon \in (0,\epsilon_0)$ *and* $\delta > 0$ *such that*

$$|\rho(V)\rho(W)| \geqslant \delta \cdot \|V-W\| \qquad \forall\ V,W \in \mathscr{U}_\epsilon^+. \tag{4.27}$$

where

$$\mathscr{U}_\epsilon^+ := \{V=(v_1,v_2,\ldots,v_n) \in \mathbb{R}^n :\ 0<v^j<\epsilon \ \text{ for all }\ 1\leqslant j\leqslant n\}.$$

In particular, for arbitrary $\epsilon' \in (0,\epsilon)$*, the image* $\rho(\mathscr{U}_{\epsilon'}^+)$ *has nonzero Hausdorff measure.*

Proof. Without loss of generality, we can assume that $\lambda \geqslant 0$.

Since x_0 is a regular point, according to Remark 4.14, the mean value inequality in Proposition 4.13 holds uniformly on some neighborhood of x_0. Namely, there exists neighborhood $U_{x_0} \ni x_0$ and two constants R_0, c_0 such that for any $w \in W^{1,2}(D)\cap C(D)$ satisfying $\mathscr{L}_w \leqslant \lambda \cdot \mathrm{vol}$, we have

$$\int_{B_q(R)} w d\nu_{q,R} \leqslant w(q) + c_0\cdot R^2 \tag{4.28}$$

for all $q \in U_{x_0}$ and all $R \in (0,R_0)$.

Noting that $G(V,x) = u(x) + V\cdot g$ converges to u as $V \to 0$, and that x_0 is the *uniquely* minimal value point of $u(x)$, we can conclude that $\rho(V)$ converges to x_0 as $V\to 0$. Hence, there exists a positive number $\epsilon > 0$ such that $\rho(V) \in U_{x_0}$ provided $V=(v^1,\ldots,v^n)$ satisfies $|v^j| \leqslant \epsilon$ for all $1\leqslant j\leqslant n$. From now on, we fix such ϵ and let

$$\mathscr{U}_\epsilon^+ := \{V=(v_1,v_2,\ldots,v_n) \in \mathbb{R}^n :\ 0<v^j<\epsilon \ \text{ for all }\ 1\leqslant j\leqslant n\}.$$

Let $V,W \in \mathscr{U}_\epsilon^+$. Denote by $\rho := \rho(V)$ and $\hat\rho := \rho(W)$. That means

$$G(V,\rho) \leqslant G(V,x) \quad \text{and} \quad G(W,\hat\rho) \leqslant G(W,x)$$

for any $x\in D$. Hence, we have

$$\begin{aligned}(W-V)\cdot g(\hat\rho) - (W-V)\cdot g(x) &= G(W,\hat\rho) - G(V,\hat\rho)\\ &\quad - G(W,x) + G(V,x)\\ &\leqslant G(V,x) - G(V,\hat\rho)\\ &\leqslant G(V,x) - G(V,\rho).\end{aligned} \tag{4.29}$$

Notice that $v^j > 0$ and g_j are concave for $1\leqslant j\leqslant n$. We know that $G(V,x) = u(x) + V\cdot g(x)$ also satisfies $\mathscr{L}_{G(V,x)} \leqslant \lambda\cdot\mathrm{vol}$. By the mean value inequality (4.28), we have

$$\int_{B_\rho(R)} \big(G(V,x) - G(V,\rho)\big) d\nu_{\rho,R} \leqslant c_0\cdot R^2 \tag{4.30}$$

for any $0 < R < R_0$. We denote $\phi_+ := \max\{\phi, 0\}$ for a function ϕ. It is clear that $(\phi + a)_+ \leqslant \phi_+ + |a|$ for any $a \in \mathbb{R}$. By combining this and the assumption that g is an almost isometry, we have

$$
\begin{aligned}
&\int_{B_\rho(R)} \big((W-V)\cdot g(\rho) - (W-V)\cdot g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad \leqslant \int_{B_\rho(R)} \big((W-V)\cdot g(\hat\rho) - (W-V)\cdot g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad\qquad + |(W-V)\cdot g(\rho) - (W-V)\cdot g(\hat\rho)| \\
&\qquad \leqslant \int_{B_\rho(R)} \big((W-V)\cdot g(\hat\rho) - (W-V)\cdot g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad\qquad + \|g(\rho) - g(\hat\rho)\| \cdot \|W-V\| \\
&\qquad \leqslant \int_{B_\rho(R)} \big((W-V)\cdot g(\hat\rho) - (W-V)\cdot g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad + c_1 \cdot |\rho\hat\rho| \cdot \|W-V\|,
\end{aligned}
\tag{4.31}
$$

where constant c_1 depends only on δ_0.

Consider the set

$$
\begin{aligned}
K := \Big\{X \in \mathbb{R}^n \;\Big|\; \frac{R}{4} \leqslant \|X - g(\rho)\| \leqslant \frac{R}{2},\ \ (X - g(\rho))\cdot(W-V) \\
\leqslant -\frac{1}{2}\|X-g(\rho)\|\cdot\|V-W\|\Big\}.
\end{aligned}
$$

In fact, K is a trunked cone in $\mathbb{R}^n$ with vertex $g(\rho)$, central direction $-W + V + g(\rho)$, cone angle $\frac{\pi}{3}$, and radius from $\frac{R}{4}$ to $\frac{R}{2}$.

Since $K \subset B_{g(\rho)}(R/2)$ and g is an almost isometry with δ sufficiently small, it is obvious that $g^{-1}(K) \subset B_\rho(R)$. Hence, we have

$$
\begin{aligned}
&\int_{B_\rho(R)} \big((W-V)\cdot g(\rho) - (W-V)\cdot g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad \geqslant \int_{g^{-1}(K)} \big((W-V)\cdot (g(\rho) - g(x)\big)_+ d\nu_{\rho,R} \\
&\qquad \geqslant \frac{1}{2}\|W-V\| \cdot \int_{g^{-1}(K)} \|g(\rho)-g(x)\| d\nu_{\rho,R} \\
&\qquad \geqslant \frac{R}{8}\|W-V\| \cdot \nu_{\rho,R}\big(g^{-1}(K)\big).
\end{aligned}
\tag{4.32}
$$

By the estimate (4.18) and that g is a δ_0-almost isometry, we have

$$
\nu_{\rho,R}\big(g^{-1}(K)\big) \geqslant \frac{\mathrm{vol}(g^{-1}(K))}{H^n(B_o(R) \subset T^k_\rho)} \geqslant c_2 \tag{4.33}
$$

for some constant c_2 depending only on δ_0 and the dimension n, the lower bound k of curvature.

By combining (4.29)–(4.33), we obtain

$$\frac{c_2 \cdot R}{8} \cdot \|W - V\| \leqslant c_1 |\rho\hat{\rho}| \cdot \|W - V\| + c_0 R^2$$

for any $0 < R < R_0$. We set

$$N = \frac{n\epsilon \cdot c_2}{c_0 R_0} + 1. \tag{4.34}$$

Since $\|V - W\| \leqslant n\epsilon$, we get

$$R' := \frac{c_2 \cdot \|V - W\|}{10c_0 \cdot N} \leqslant R_0/10.$$

Then we have

$$c_1 \cdot |\rho\hat{\rho}| \cdot \|V - W\| \geqslant \frac{c_2 R'}{8} \cdot \|V - W\| - c_0 R'^2 = \frac{c_2^2 \cdot \|V - W\|^2}{10c_0 N}\Big(\frac{1}{8} - \frac{1}{10N}\Big).$$

Now the desired estimate (4.27) follows from the choice of

$$\delta := \frac{c_2^2}{400c_0 \cdot c_1 \cdot N}. \tag{4.35}$$

Therefore, the proof of this lemma is completed. q.e.d.

5. Hamilton–Jacobi semigroup and Bochner-type formula

5.1. Hamilton–Jacobi semigroup. Let M be an n-dimensional Alexandrov space and Ω be a bounded domain of M. Given a continuous and bounded function u on Ω, the *Hamilton–Jacobi* semigroup is defined by

$$Q_t u(x) = \inf_{y \in \Omega} \Big\{ u(y) + \frac{|xy|^2}{2t} \Big\}, \qquad t > 0$$

and $Q_0 u(x) := u(x)$. Clearly, $Q_t u$ is semi-concave for any $t > 0$, since $u(y) + |\cdot y|^2/(2t)$ is semi-concave, for each $y \in \Omega$. In particular, $Q_t u$ is locally Lipschitz for any $t > 0$.

If $|xy| > \sqrt{4t\|u\|_{L^\infty}}$, then

$$u(y) + \frac{|xy|^2}{2t} > u(y) + 2\|u\|_{L^\infty} \geqslant \|u\|_{L^\infty}.$$

On the other hand, $Q_t u(x) \leqslant u(x) \leqslant \|u\|_{L^\infty}$. We conclude that

$$Q_t u(x) = \inf_{y \in \overline{B_x(C)} \cap \Omega} \Big\{ u(y) + \frac{|xy|^2}{2t} \Big\},$$

where $C = \sqrt{4t\|u\|_{L^\infty}}$. Therefore, for any $\Omega' \Subset \Omega$, there exists $\bar{t} = \bar{t}(\Omega', \|u\|_{L^\infty})$ such that

$$Q_t u(x) = \min_{y \in \Omega} \Big\{ u(y) + \frac{|xy|^2}{2t} \Big\} \tag{5.1}$$

for all $x \in \Omega'$ and $0 < t < \bar{t}$.

For convenience, we always set $u_t := Q_t u$ in this section.

The following was shown in [**30**] in the framework of length spaces.

Lemma 5.1. (Lott–Villani [**30**])
(*i*) *For each* $x \in \Omega'$, *we have* $\inf u \leqslant u_t(x) \leqslant u(x)$.
(*ii*) $\lim_{t\to 0^+} u_t = u$ *in* $C(\Omega')$.
(*iii*) *For any* $t, s > 0$ *and any* $x \in \Omega'$, *we have*

$$0 \leqslant u_t(x) - u_{t+s}(x) \leqslant \frac{s}{2} \cdot \mathbf{Lip}^2 u_t, \tag{5.2}$$

where $\mathbf{Lip} u_t$ *is the Lipschitz constant of* u_t *on* Ω' (*see* [**8**] *for this notation*).
(*iv*) *For any* $t > 0$ *and almost all* $x \in \Omega'$, *we have*

$$\lim_{s\to 0^+} \frac{u_{t+s}(x) - u_t(x)}{t} = -\frac{|\nabla u_t(x)|^2}{2}. \tag{5.3}$$

The following lemma is similar to Lemma 3.5 in [**3**].

Lemma 5.2. *Let* $t > 0$. *Assume* u_t *is differentiable at* $x \in \Omega'$. *Then there exists a unique point* $y \in \Omega$ *such that*

$$u_t(x) = u(y) + \frac{|xy|^2}{2t}. \tag{5.4}$$

Furthermore, the direction $\uparrow_x^y$ *is determined uniquely and*

$$|xy| \cdot \uparrow_x^y = -t \cdot \nabla u_t(x). \tag{5.5}$$

Proof. Now fix a regular point x. We arbitrarily choose y such that (5.4) holds. Taking any geodesic $\gamma(s) : [0, \epsilon) \to M$ with $\gamma(0) = x$, by the definition of u_t and (5.4), we have

$$u_t(\gamma(s)) - u_t(x) \leqslant \frac{|y\gamma(s)|^2}{2t} - \frac{|xy|^2}{2t}. \tag{5.6}$$

If $x = y$, we have $\nabla u_t(x) = 0$. Hence equation (5.5) holds.

If $x \neq y$, by using the differentiability of u_t at x and the first variant formula, we have

$$u_t(\gamma(s)) = u_t(x) + d_x u_t(\gamma'(0)) \cdot s + o(s) \tag{5.7}$$

and

$$\frac{|y\gamma(s)|^2}{2t} - \frac{|xy|^2}{2t} \leqslant -\frac{|xy|}{t} \cdot \left\langle \uparrow_x^y, \gamma'(0) \right\rangle \cdot s + o(s) \tag{5.8}$$

for any direction $\uparrow_x^y$ from x to y. By combining (5.6)–(5.8), we have

$$d_x u_t(\gamma'(0)) \leqslant -\frac{|xy|}{t} \cdot \left\langle \uparrow_x^y, \gamma'(0) \right\rangle$$

for all geodesic γ with $\gamma(0) = x$. For each $\xi \in \Sigma_x$, we take a sequence of geodesics $\gamma(t)$ starting from x such that $\gamma'(0)$ converges to ξ. Then we have

$$d_x u_t(\xi) \leqslant -\frac{|xy|}{t} \cdot \langle \uparrow_x^y, \xi \rangle \tag{5.9}$$

for all $\xi \in \Sigma_x$.

Since u_t is differentiable at x, we know that the direction $-\xi$ exists and $d_x u(-\xi) = -d_x u(\xi)$. By replacing ξ by $-\xi$ in the above inequality, we obtain

$$\nabla u_t(x) = -\frac{|xy|}{t} \cdot \uparrow_x^y .$$

The left-hand side does not depend on the choices of point y and direction of $\uparrow_x^y$. This gives the desired assertion. q.e.d.

For each $t > 0$, we define a map $F_t : \Omega' \to \Omega$ by $F_t(x)$ to be one of points such that

$$u_t(x) = u\big(F_t(x)\big) + \frac{|xF_t(x)|^2}{2t}. \tag{5.10}$$

According to Lemma 5.2 and the Rademacher theorem ([**8**, **3**]), we have, for almost all $x \in \Omega'$,

$$|xF_t(x)| = t \cdot |\nabla u_t(x)|. \tag{5.11}$$

By Lemma 5.2 again, F_t is continuous at x, where u_t is differentiable (since the point y satisfying (5.4) is unique). Then F_t is measurable.

In [**40**], Petrunin sketched a proof of his key lemma, which states that, on an Alexandrov space with nonnegative curvature, u_t is superharmonic on Ω' for each $t > 0$, provided u is a supersolution of $\mathscr{L}_u = 0$ on Ω. The following proposition is an extension.

Proposition 5.3. *Let M be an n-dimensional Alexandrov space with $Ric \geqslant -K$ and let Ω be a bounded domain of M. Assume that $u \in W^{1,2}(\Omega) \cap C(\Omega)$, $f \in L^\infty(\Omega)$ is upper semi-continuous for almost all $x \in \Omega$ and*

$$\mathscr{L}_u \leqslant f \cdot \mathrm{vol}$$

in the sense of measure. Then, for any $\Omega' \Subset \Omega$, there exists some $t_0 > 0$ such that for all $0 < t < t_0$, we have

$$a^2 \cdot \mathscr{L}_{u_t} \leqslant \Big[f \circ F_t + \frac{n(a-1)^2}{t} + \frac{Kt}{3}(a^2 + a + 1)|\nabla u_t|^2\Big] \cdot \mathrm{vol} \tag{5.12}$$

on Ω' for all $a > 0$.

Proof. We divide the proof into the following four steps.

Step 1. Setting up a contradiction argument.

Since, for almost all $x \in \Omega$, f is upper semi-continuous and $|xF_t(x)| = t|\nabla u_t(x)|$, it is sufficient to prove that there exists some $t_0 > 0$ such that for all $0 < t < t_0$, we have

$$a^2 \cdot \mathscr{L}_{u_t} \leqslant \Big[\sup_{z \in B_{F_t(x)}(\theta)} f(z) + \frac{n(a-1)^2}{t} + \frac{K}{3t}\big(a^2 + a + 1\big) \cdot |xF_t(x)|^2 + \theta\Big] \cdot \mathrm{vol} \tag{5.13}$$

on Ω' for all $a > 0$ and all $\theta > 0$.

For each $t > 0$, $a > 0$, and $\theta > 0$, we set

(5.14)

$$a^2 \cdot w_{t,a,\theta}(x) = \sup_{z \in B_{F_t(x)}(\theta)} f(z) + \frac{n(a-1)^2}{t} + \frac{K}{3t}\big(a^2+a+1\big) \cdot |xF_t(x)|^2 + \theta.$$

For each $t > 0$, $a > 0$, and $\theta > 0$, since u_t is semi-concave, $|\nabla u_t| \in L^\infty(\Omega')$ and hence we have $w_{t,a,\theta} \in L^\infty(\Omega')$. Noting that u_t is semi-concave again, it is sufficient to prove that u_t satisfies the corresponding comparison property in Corollary 4.6 for all sufficiently small $t > 0$.

Let us argue by contradiction. Suppose that there exists a sequence of $t_j \to 0^+$ as $j \to \infty$, a sequence $a_j > 0$, and a sequence $\theta_j > 0$ satisfying the following: for each t_j, a_j, and θ_j, we can find p_j and $R_j > 0$ with $a_j R_j + R_j \to 0^+$ and $B_{p_j}(R_j) \Subset \Omega'$, such that the corresponding comparison property in Corollary 4.6 is false. That is, if the function v_j is the solution of equation

$$\mathscr{L}_{v_j} = -w_{t_j,a_j,\theta_j} \cdot \mathrm{vol}$$

in $B_{p_j}(R_j)$ with boundary value $v_j = -u_{t_j}$ on $\partial B_{p_j}(R_j)$, then the function $u_{t_j} + v_j$ has a minimum point in the interior of $B_{p_j}(R_j)$ and

$$\min_{x \in B_{p_j}(R_j)} (u_{t_j} + v_j) < \min_{x \in \partial B_{p_j}(R_j)} (u_{t_j} + v_j).$$

We say in this case that $u_{t_j} + v_j$ has a strict minimum in the interior of $B_{p_j}(R_j)$.

Since Ω' is bounded, we can assume that some subsequence of $\{p_j\}_{j=1}^\infty$ converges to a limit point p_∞. Denote the subsequence by $\{p_j\}_{j=1}^\infty$ again. So we can choose a convex neighborhood $U \Subset \Omega$ of p_∞ and a Perelman concave function h on U given in Lemma 3.3. Since u is bounded, by $|xF_t(x)|^2 \leqslant 4t\|u\|_{L^\infty(\Omega)}$, we have $|xF_{t_j}(x)| \to 0$ as $j \to \infty$ uniformly on Ω'. Now we fix some j^* so large that

$$B_{p_{j^*}}\big(a_{j^*}R_{j^*} + R_{j^*}\big) \cup B_{F_{t_{j^*}}(p_{j^*})}\big(a_{j^*}R_{j^*} + R_{j^*}\big) \subset U$$

and $F_{t_{j^*}}(x) \in U$ for all $x \in B_{p_{j^*}}\big(a_{j^*}R_{j^*} + R_{j^*}\big)$.

Step 2. Perturbing the functions to achieve the minimums at smooth points.

From now on, we omit the index j^* to simplify the notations.

Let x_1 be a minimum of $u_t + v$ in the interior of $B_p(R)$. Because h is 2-Lipschitz on U, for any sufficiently small positive number ϵ_0, the function

$$u_t + v + \epsilon_0 h$$

also achieves a strict minimum at some point $\bar{x}$ in the interior of $B_p(R)$. Noting that u_t is semi-concave and $w_{t,a,\theta}$ is bounded and $\mathscr{L}_v \leqslant -w_{t,a,\theta} \cdot \mathrm{vol}$, according to Lemma 4.15, we know $\bar{x}$ is regular and that u_t is differentiable at $\bar{x}$. Now we fix such a sufficiently small ϵ_0.

On the other hand, according to the condition $Ric \geqslant -K$ and Laplacian comparison (see [**52**] or [**22**]), we have $\mathscr{L}_{|x\bar{x}|^2} \leqslant c(n, K, \mathrm{diam}\Omega)$. Thus, by the fact that h is (-1)-concave, we can choose some sufficiently small positive number ϵ_0' such that

$$\mathscr{L}_{\epsilon_0 h+\epsilon_0'|x\bar{x}|^2} \leqslant 0.$$

Setting $v_0 = v + \epsilon_0 h + \epsilon_0'|x\bar{x}|^2$, we have that the function

$$u_t + v_0 = u_t + v + \epsilon_0 h + \epsilon_0'|x\bar{x}|^2$$

achieves a *unique* minimum at $\bar{x}$ and

$$\mathscr{L}_{v_0} = \mathscr{L}_v + \mathscr{L}_{\epsilon_0 h+\epsilon_0'|x\bar{x}|^2} \leqslant \mathscr{L}_v = -w_{t,a,\theta} \cdot \mathrm{vol}.$$

Consider function

$$H(x, y) = v_0(x) + u(y) + \frac{|xy|^2}{2t}, \qquad (x, y) \in \Omega \times \Omega.$$

Then it achieves a *unique* strict minimum at $(\bar{x}, F_t(\bar{x})) \in B_p(R) \times U$. Indeed,

$$\begin{aligned} H(x, y) &\geqslant u_t(x) + v_0(x) \geqslant u_t(\bar{x}) + v_0(\bar{x}) = u(F_t(\bar{x})) + \frac{|\bar{x}F_t(\bar{x})|}{2t} + v_0(\bar{x}) \\ &= H(\bar{x}, F_t(\bar{x})). \end{aligned}$$

Since $\bar{x}$ is a regular point and u_t is differentiable at $\bar{x}$, by Lemma 5.2, the point pair $(\bar{x}, F_t(\bar{x}))$ is the unique minimum of H in $B_p(R) \times U$.

Applying the fact that h is 2-Lipschitz on U, we know that, for any sufficiently small positive number ϵ_1,

$$H_1(x, y) := v_1(x) + u_1(y) + \frac{|xy|^2}{2t}$$

also achieves its strict minimum in the interior of $B_p(R) \times U$, where

$$v_1(x) = v_0(x) + \epsilon_1 h(x) \quad \text{and} \quad u_1(y) = u(y) + \epsilon_1 h(y).$$

Let (x^*, y^*) denote one minimal point of H_1.

By the condition $Ric \geqslant -K$ and Laplacian comparison (see [**52**] or [**22**]), we have

$$\mathscr{L}_{|xx^*|^2} \leqslant c(n, K, \mathrm{diam}\Omega) \quad \text{and} \quad \mathscr{L}_{|yy^*|^2} \leqslant c(n, K, \mathrm{diam}\Omega).$$

Since

$$H_1(x, y^*) = v_0(x) + u_1(y^*) + \frac{|xy^*|^2}{2t} + \epsilon_1 h(x)$$

is continuous and $w_{t,a,\theta}$ is bounded, we know that

$$\mathscr{L}_{v_0+u_1(y^*)+\frac{|xy^*|^2}{2t}} \leqslant \Big(-w_{t,a,\theta} + \frac{c(n, K, \mathrm{diam}\Omega)}{2t}\Big) \cdot \mathrm{vol} \leqslant \lambda \cdot \mathrm{vol}$$

on $B_p(R)$ for some constant $\lambda \in \mathbb{R}$ and $H_1(x, y^*)$ has a minimum at x^*. By Lemma 4.15, we know that x^* is regular. The point y^* is also regular, by the boundedness of f and the same argument.

Let $v_2(x) = v_1(x) + \epsilon_2|xx^*|^2$ and $u_2(y) = u_1(y) + \epsilon_2|yy^*|^2$ with any positive number ϵ_2. Then

$$H_2(x,y) := v_2(x) + u_2(y) + \frac{|xy|^2}{2t}$$

achieves a unique minimum point (x^*, y^*).

Since (x^*, y^*) is regular in $M \times M$, now we choose one almost orthogonal coordinate system near x^* by concave functions $g_1, g_2, \ldots, g_n$ and another almost orthogonal coordinate system near y^* by concave functions $g_{n+1}, g_{n+2}, \ldots, g_{2n}$. Using Lemma 4.16, there exist arbitrarily small positive numbers $b_1, b_2, \ldots, b_{2n}$ such that

$$H_2(x,y) + \sum_{i=1}^{n} b_i g_i(x) + \sum_{i=n+1}^{2n} b_i g_i(y)$$

achieves a minimal point (x^o, y^o) near point (x^*, y^*), where (x^o, y^o) satisfies the following properties:

1) $x^o \neq y^o$;
2) x^o is a $dist_{y^o}$-regular point and y^o is a $dist_{x^o}$-regular point (hence, they are smooth);
3) geodesic $x^o y^o$ can be extended beyond x^o and y^o;
4) y^o is a Lebesgue point of f;
5) x^o is a Lebesgue point of $w_{t,a,\theta}$;
6) x^o is a Lebesgue point of $\Delta(|xy^o|^2)$ and y^o is a Lebesgue point of $\Delta(|x^o y|^2)$,

where $\Delta(|xy^o|^2)$ (or $\Delta(|x^o y|^2)$) is the density of the absolutely continuous part of $\mathscr{L}_{|xy^o|^2}$ (or $\mathscr{L}_{|x^o y|^2}$, resp.).

Indeed, let $\mathcal{A}$ be the set of points satisfying all of conditions (1)–(6) above. It is easy to check that $H^{2n}\big((B_p(R) \times U)\backslash\mathcal{A}\big) = 0$. By applying Lemma 4.16, we can find the desired (x^o, y^o).

Set

$$v_3(x) = v_2(x) + \sum_{i=1}^{n} b_i g_i(x) \qquad \text{and} \qquad u_3(y) = u_2(y) + \sum_{i=n+1}^{2n} b_i g_i(y).$$

Then

$$H_3(x,y) := v_3(x) + u_3(y) + \frac{|xy|^2}{2t}$$

has a minimal value at (x^o, y^o).

Step 3. Ricci curvature and second variation of arc-length.

Let $\gamma : [0, \bar{s}] \to U$ be a geodesic with $x^o, y^o \in \gamma\backslash\{\gamma(0), \gamma(\bar{s})\}$. Put $x^o = \gamma(t_x)$ and $y^o = \gamma(t_y)$ with $0 < t_x < t_y < \bar{s}$. Assume that some neighborhood of γ has curvature $\geqslant k_0$, for some $k_0 \in \mathbb{R}$. For each $t \in$

$(0, \bar{s})$, the tangent cone $T_{\gamma(t)}$ can be split isometrically into $T_{\gamma(t)} = \mathbb{R} \times L_{\gamma(t)}$. We denote

$$\Lambda_{\gamma(t)} = \Sigma_\gamma \cap L_{\gamma(t)} = \{\xi \in \Sigma_{\gamma(t)} \mid \langle \xi, \gamma' \rangle = 0\}.$$

Fix an arbitrarily small positive number ϵ_3. According to the definition of M having Ricci curvature $\geqslant -K$ along geodesic γ (see Definition 2.6), for each $t_0 \in [t_x, t_y]$, there exists an open neighborhood $I_{t_0} \ni t_0$ and a family of function $\{g_{\gamma(t)}\}_{t \in I_{t_0}}$ such that $\{g_{\gamma(t)}\}_{t \in I_{t_0}}$ satisfies *Condition* (RC) and

$$(n-1) \cdot \int_{\Lambda_{\gamma(t)}} g_{\gamma(t)}(\xi) d\xi \geqslant -K - \epsilon_3, \qquad \forall t \in I_{t_0}. \tag{5.15}$$

It is shown in [**52**] that

$$|g_{\gamma(t)}| \leqslant C, \qquad \forall t \in I_{t_0} \tag{5.16}$$

for some constant C depends only on the distance $|x^o\gamma(0)|, |y^o\gamma(\bar{s})|, |I_{t_0}|$ and the lower bound k_0 of curvature on some neighborhood of γ. For completeness, we recall its proof as follows. Since the family $\{\underline{g}_{\gamma(t)} = k_0\}$ satisfies *Condition* (RC) (see Remark 2.7), we can assume that $g_{\gamma(t)} \geqslant k_0$. Otherwise, we replace $g_{\gamma(t)}$ by $g_{\gamma(t)} \vee k_0$. On the other hand, for any $q_1, q_2 \in \gamma|_{I_{t_0}}$ with $|q_1 q_2| \geqslant |I_{t_0}|/2$, letting isometry $T : \Sigma_{q_1} \to \Sigma_{q_2}$ and sequence δ_j be in the definition of *Condition* (RC) (see Definition 2.5), by applying equation (2.17) with $l_1 = l_2 = 1$ and $\langle \xi, \gamma' \rangle = 0$, we have

$$|\exp_{q_1}(\delta_j \xi)\ \exp_{q_2}(\delta_j T\xi)| \leqslant |q_1 q_2| - g_{q_1}(\xi) \cdot |q_1 q_2| \cdot \delta_j^2/2 + o(\delta_j^2).$$

By the semi-concavity of distance functions $dist_{\gamma(0)}$ and $dist_{\gamma(\bar{s})}$, we get

$$|\gamma(0)\ \exp_{q_1}(\delta_j \xi)| \leqslant |\gamma(0)\ q_1| + C_{k_0, |\gamma(0)x^o|} \cdot \delta_j^2$$

and

$$|\gamma(\bar{s})\ \exp_{q_2}(\delta_j T\xi)| \leqslant |\gamma(\bar{s})\ q_2| + C_{k_0, |\gamma(\bar{s})y^o|} \cdot \delta_j^2.$$

Combining with triangle inequality

$$\begin{aligned} |\exp_{q_1}(\delta_j \xi)\ \exp_{q_2}(\delta_j T\xi)| \geqslant{}& |\gamma(0)\ \gamma(\bar{s})| - |\gamma(0)\ \exp_{q_1}(\delta_j \xi)| - |\gamma(\bar{s}) \\ & \exp_{q_2}(\delta_j T\xi)|, \end{aligned}$$

we can obtain

$$g_{q_1}(\xi) \leqslant \frac{2}{|q_1 q_2|} \cdot (C_{k_0, |\gamma(0)x^o|} + C_{k_0, |\gamma(\bar{s})y^o|}) \leqslant \frac{4}{|I_{t_0}|} \cdot (C_{k_0, |\gamma(0)x^o|} + C_{k_0, |\gamma(\bar{s})y^o|}).$$

All of such neighborhood I_{t_0} forms an open covering of $[t_x, t_y]$. Then there exists a sub-covering $I_1, I_2, \ldots, I_S$. Now we divide $[t_x, t_y]$ into N-equal parts by

$$x_0 = x^o, x_1, \ldots, x_m, \ldots, x_N = x_N.$$

We can assume that any pair of adjacent x_m, x_{m+1} is lying into some same $I_\alpha,\ \alpha \in \{1, 2, \ldots, S\}$.

By *Condition* (RC), we can find a sequence $\{\delta_j\}$ and an isometry $T_0 : \Sigma_{x_0} \to \Sigma_{x_1}$ such that equation (2.17) holds. Next, we can find a further subsequence $\{\delta_{1,j}\} \subset \{\delta_j\}$ and an isometry $T_1 : \Sigma_{x_1} \to \Sigma_{x_2}$ such that equation (2.17) holds. After finite steps of these procedures, we get a subsequence $\{\delta_{N-1,j}\} \subset \cdots \subset \{\delta_{1,j}\} \subset \{\delta_j\}$ and a family of isometries $T_m : \Sigma_{x_m} \to \Sigma_{x_{m+1}}$ such that, for each $m = 0, 1, \ldots, N-1$,

$$\begin{aligned}
&|\exp_{x_m}(\delta_{N-1,j} l_{1,m}\xi_m),\ \exp_{x_{m+1}}(\delta_{N-1,j} l_{2,m} T_m \xi_m)| \\
&\quad \leqslant |x_m x_{m+1}| + (l_{2,m} - l_{1,m}) \langle \xi_m, \gamma' \rangle \cdot \delta_{N-1,j} \\
&\qquad + \Big(\frac{(l_{1,m} - l_{2,m})^2}{2|x_m x_{m+1}|} - \frac{g_{x_m}(\xi_m^{\perp}) \cdot |x_m x_{m+1}|}{6} \\
&\qquad\quad \cdot (l_{1,m}^2 + l_{1,m} \cdot l_{2,m} + l_{2,m}^2) \Big) \\
&\qquad\quad \cdot \Big(1 - \langle \xi_m, \gamma' \rangle^2 \Big) \cdot \delta_{N-1,j}^2 \\
&\qquad + o(\delta_{N-1,j}^2)
\end{aligned}$$

for any $l_{1,m}, l_{2,m} \geqslant 0$ and any $\xi_m \in \Sigma_{x_m}$.

Denote the isometry $T : \Sigma_{x^o} \to \Sigma_{y^o}$ by

$$T = T_{N-1} \circ \cdots \circ T_1 \circ T_0.$$

It can be extended naturally to an isometry $T : T_{x^o} \to T_{y^o}$.

We fix $a \geqslant 0$ and

$$a_m = \frac{m}{N} \cdot (1 - a) + a, \qquad m = 0, 1, \ldots, N-1.$$

We have $a_m \geqslant 0$, and $a_0 = a, a_N = 1$.

To simplify notations, we put $\{\delta_j\} = \{\delta_{N-1,j}\}$ and denote

$$\mathscr{W} = \{ v \in T_{x_0} \ \big|\ av \in \mathscr{W}_{x^o} \quad \text{and} \quad Tv \in \mathscr{W}_{y^o} \}.$$

Claim 1: We have

$$\begin{aligned}
&\int_{B_o(\delta_j) \cap \mathscr{W}} \Big(|\exp_{x^o}(a\eta)\ \exp_{y^o}(T\eta)|^2 - |x^o y^o|^2 \Big) dH^n(\eta) \\
&\quad \leqslant \frac{\omega_{n-1}}{(n+2)} \cdot \delta_j^{2+n} \cdot \Big((1-a)^2 + \frac{(K + \epsilon_3) \cdot |x^o y^o|^2}{3n} \cdot \big(a^2 + a + 1 \big) \Big) \\
&\quad + o(\delta_j^{n+2}).
\end{aligned} \tag{5.17}$$

By applying *Condition* (RC), we have

$$\begin{aligned}
&|\exp_{x_m}(\delta_j a_m\cdot b\xi_m),\ \exp_{x_{m+1}}(\delta_j a_{m+1}\cdot b\xi_{m+1})|\\
&\quad\leqslant\frac{\ell}{N}+(a_{m+1}-a_m)\cdot b\left\langle\xi,\gamma'\right\rangle\cdot\delta_j\\
&\qquad+b^2\cdot\Big(\frac{N\cdot(a_m-a_{m+1})^2}{2\ell}-\frac{g_{x_m}(\xi_m^\perp)\cdot\ell}{6N}\\
&\qquad\quad\cdot(a_m^2+a_m\cdot a_{m+1}+a_{m+1}^2)\Big)\\
&\qquad\quad\cdot\left(1-\left\langle\xi,\gamma'\right\rangle^2\right)\cdot\delta_j^2\\
&\qquad+o(\delta_j^2)
\end{aligned}$$

for any $b\in[0,1]$ and any $\xi\in\Sigma_{x_0}$, where $\ell=|x_0x_N|=|x^oy^o|$ and

$$\xi_m:=T_m\circ T_{m-1}\circ\cdots\circ T_0\xi.$$

Hence, by combining with triangle inequality, we have

$$\begin{aligned}
&|\exp_{x_0}(\delta_j a_0\cdot b\xi),\ \exp_{x_N}(\delta_j a_N\cdot b\xi_N)|\\
&\quad\leqslant\sum_{m=0}^{N-1}|\exp_{x_m}(\delta_j a_m\cdot b\xi_m),\ \exp_{x_{m+1}}(\delta_j a_{m+1}\cdot b\xi_{m+1})|\\
&\quad\leqslant\ell+(a_N-a_0)\left\langle\xi,\gamma'\right\rangle b\cdot\delta_j\\
&\qquad+b^2\cdot\sum_{m=0}^{N-1}\Big(\frac{N\cdot(a_m-a_{m+1})^2}{2\ell}-\frac{g_{x_m}(\xi_m^\perp)\cdot\ell}{6N}\\
&\qquad\quad\cdot(a_m^2+a_m\cdot a_{m+1}+a_{m+1}^2)\Big)\\
&\qquad\qquad\cdot\left(1-\left\langle\xi,\gamma'\right\rangle^2\right)\cdot\delta_j^2\\
&\qquad+o(\delta_j^2)
\end{aligned}$$

for any $b\in[0,1]$. That is, by setting $v=b\xi$,

$$\begin{aligned}
&|\exp_{x^o}(\delta_j av),\ \exp_{y^o}(\delta_j Tv)|^2-|x^oy^o|^2\\
&\quad\leqslant2\ell\cdot(1-a)\left\langle v,\gamma'\right\rangle\cdot\delta_j+(1-a)^2\left\langle v,\gamma'\right\rangle^2\cdot\delta_j^2\\
&\qquad+\sum_{m=0}^{N-1}\Big(N\cdot(a_m-a_{m+1})^2-\frac{g_{x_m}(\xi_m^\perp)\cdot\ell^2}{3N}\\
&\qquad\quad\cdot(a_m^2+a_m\cdot a_{m+1}+a_{m+1}^2)\Big)\\
&\qquad\quad\cdot\left(|v|^2-\left\langle v,\gamma'\right\rangle^2\right)\cdot\delta_j^2\\
&\qquad+o(\delta_j^2)
\end{aligned}\tag{5.18}$$

for any vector $v\in B_o(1)\subset T_{x_0}$.

Let $\mathscr{F}_j(v)$ be the function defined on $B_o(1) \subset T_{x_0}$ by

$$\begin{aligned}\mathscr{F}_j(v) := &\,|\exp_{x^o}(\delta_j av),\ \exp_{y^o}(\delta_j Tv)|^2 - |x^o y^o|^2\\ &- 2\ell \cdot (1-a)\left\langle v, \gamma'\right\rangle \cdot \delta_j - (1-a)^2 \left\langle v, \gamma'\right\rangle^2 \cdot \delta_j^2\\ &- \sum_{m=0}^{N-1}\Big(N\cdot(a_m - a_{m+1})^2 - \frac{g_{x_m}(\xi_m^{\perp})\cdot \ell^2}{3N}\\ &\quad \cdot (a_m^2 + a_m \cdot a_{m+1} + a_{m+1}^2)\Big)\\ &\quad \cdot \Big(|v|^2 - \left\langle v, \gamma'\right\rangle^2\Big)\cdot \delta_j^2.\end{aligned}$$

For any $v \in B_o(1)$, we rewrite (5.18) as

$$\limsup_{j\to\infty} \mathscr{F}_j(v)/\delta_j^2 \leqslant 0.$$

Next, we will prove that $\mathscr{F}_j(v)/\delta_j^2$ has a uniformly upper bound on $B_o(1)$. Take the midpoint z of x^o and y^o. By the semi-concavity of distance function $dist_z$, we have

$$|z\ \exp_{x^o}(\delta_j \cdot av)| \leqslant |zx^o| - a\left\langle v, \gamma'\right\rangle \delta_j + C_{k_0, |x^o y^o|}\cdot \delta_j^2$$

and

$$|z\ \exp_{y^o}(\delta_j \cdot Tv)| \leqslant |zy^o| + \left\langle Tv, \gamma'\right\rangle \delta_j + C_{k_0, |x^o y^o|}\cdot \delta_j^2.$$

By applying triangle inequality, we get

$$|\exp_{x^o}(\delta_j \cdot av)\ \exp_{y^o}(\delta_j \cdot Tv)| \leqslant |x^o y^o| + (1-a)\left\langle v, \gamma'\right\rangle \delta_j + 2C_{k_0, |x^o y^o|}\cdot \delta_j^2.$$

Hence

$$\begin{aligned}&|\exp_{x^o}(\delta_j \cdot av)\ \exp_{y^o}(\delta_j \cdot Tv)|^2 - |x^o y^o|^2\\ &\qquad \leqslant 2\ell\cdot(1-a)\left\langle v, \gamma'\right\rangle \delta_j + (4C^2 + (1-a)^2)\cdot \delta_j^2.\end{aligned}$$

By combining with the boundedness of g_{x_m} (i.e., equation (5.16)), we conclude that $\mathscr{F}_j(v)/\delta_j^2 \leqslant C$.

Now, by applying Fatou's Lemma, we have

$$\limsup_{j\to\infty} \int_{B_o(1)} \frac{\mathscr{F}_j(v)}{\delta_j^2} dH^n(v) \leqslant \int_{B_o(1)} \limsup_{j\to\infty} \frac{\mathscr{F}_j(v)}{\delta_j^2} dH^n(v) \leqslant 0.$$

That is,

(5.19)

$$
\begin{aligned}
&\int_{B_o(1)}\Big(|\exp_{x^o}(\delta_j av),\ \exp_{y^o}(\delta_j Tv)|^2-|x^oy^o|^2\Big)dH^n(v)\\
&\quad\leqslant 2\ell\cdot(1-a)\int_{B_o(1)}\big\langle v,\gamma'\big\rangle\,dH^n(v)\cdot\delta_j+(1-a)^2\\
&\qquad\qquad\int_{B_o(1)}\big\langle v,\gamma'\big\rangle^2\,dH^n(v)\cdot\delta_j^2\\
&\qquad+\sum_{m=0}^{N-1}\Big(N\cdot(a_m-a_{m+1})^2\cdot\int_{B_o(1)}\Big(|v|^2-\big\langle v,\gamma'\big\rangle^2\Big)dH^n(v)\cdot\delta_j^2\\
&\qquad-\frac{\ell^2}{3N}\cdot\sum_{m=0}^{N-1}(a_m^2+a_m\cdot a_{m+1}+a_{m+1}^2)\\
&\qquad\qquad\cdot\int_{B_o(1)}g_{x_m}(\xi_m^\perp)\cdot\Big(|v|^2-\big\langle v,\gamma'\big\rangle^2\Big)dH^n(v)\cdot\delta_j^2\\
&\qquad+o(\delta_j^2).
\end{aligned}
$$

Since x^o is regular, we have

$$
\begin{aligned}
&\int_{B_o(1)}\big\langle v,\gamma'\big\rangle\,dH^n(v)=0,\\
&\int_{B_o(1)}\big\langle v,\gamma'\big\rangle^2\,dH^n(v)=\frac{1}{n}\int_{B_o(1)}|v|^2dH^n(v)=\frac{\omega_{n-1}}{n(n+2)}
\end{aligned}
$$

and

$$
\int_{B_o(1)}\Big(|v|^2-\big\langle v,\gamma'\big\rangle^2\Big)dH^n(v)=\frac{n-1}{n}\int_{B_o(1)}|v|^2dH^n(v)=\frac{(n-1)\omega_{n-1}}{n(n+2)},
$$

where $\omega_{n-1}=\mathrm{Vol}(\mathbb{S}^{n-1})$.

By equation (5.15), and denoting $\xi_m=(\xi_m^\perp,\theta)\subset\Sigma_{x_m}$, the spherical suspension over Λ_{x_m}, we have

$$
\begin{aligned}
&\int_{\Sigma_{x_m}}g_{x_m}(\xi_m^\perp)\cdot\Big(|\xi_m|^2-\big\langle\xi_m,\gamma'\big\rangle^2\Big)dH^{n-1}(\xi_m)\\
&\qquad=\int_{\Sigma_{x_m}}(1-\cos^2\theta)g_{x_m}(\xi_m^\perp)dH^{n-1}(\xi)\\
&\qquad=\int_0^\pi\int_{\Lambda_{x_m}}\sin^2\theta g_{x_m}(\xi_m^\perp)\cdot\sin^{n-2}\theta dH^{n-2}(\xi_m^\perp)d\theta\\
&\qquad=\int_0^\pi\sin^n\theta d\theta\int_{\Lambda_{x_m}}g_{x_m}(\xi_m^\perp)dH^{n-2}(\xi_m^\perp)\\
&\qquad\geqslant\int_0^\pi\sin^n\theta d\theta\cdot\frac{-K-\epsilon_3}{n-1}\omega_{n-2}=-\frac{K+\epsilon_3}{n}\omega_{n-1}.
\end{aligned}
$$

Hence, we have

$$\begin{aligned}\int_{B_o(1)} & g_{x_m}(\xi_m^{\perp})\cdot\Big(|v|^2-\big\langle v,\gamma'\big\rangle^2\Big)dH^n(v)\\ &=\int_0^1 r^2\int_{\Sigma_{x_m}} g_{x_m}(\xi_m^{\perp})\cdot\Big(|\xi_m|^2-\big\langle \xi_m,\gamma'\big\rangle^2\Big)\cdot r^{n-1}dH^{n-1}(\xi_m)dr\\ &=\frac{1}{n+2}\int_{\Sigma_{x_m}} g_{x_m}(\xi_m^{\perp})\cdot\Big(|\xi_m|^2-\big\langle \xi_m,\gamma'\big\rangle^2\Big)dH^{n-1}(\xi_m)\\ &\geqslant -\frac{K+\epsilon_3}{n(n+2)}\omega_{n-1}.\end{aligned}$$

Putting these into (5.19), and combining with $a_{m+1}-a_m=\frac{1-a}{N}$, we have

$$\begin{aligned}\int_{B_o(1)} & \Big(|\exp_{x^o}(\delta_j av),\ \exp_{y^o}(\delta_j Tv)|^2-|x^oy^o|^2\Big)dH^n(v)\\ \leqslant\,& (1-a)^2\frac{\omega_{n-1}}{n(n+2)}\cdot\delta_j^2\\ &+\frac{(n-1)\omega_{n-1}}{n(n+2)}\cdot\delta_j^2\\ &\cdot\sum_{m=0}^{N-1}\Big(N\cdot(a_m-a_{m+1})^2+\frac{\ell^2(K+\epsilon_3)}{3N(n-1)}\cdot\big(a_m^2+a_m\cdot a_{m+1}+a_{m+1}^2\big)\Big)\\ &+o(\delta_j^2)\\ =\,&(1-a)^2\frac{\omega_{n-1}}{n(n+2)}\cdot\delta_j^2\\ &+\frac{(n-1)\omega_{n-1}}{n(n+2)}\cdot\delta_j^2\cdot\sum_{m=0}^{N-1}\Big(\frac{(a-1)^2}{N}+\frac{\ell^2(K+\epsilon_3)}{3N(n-1)}\cdot\frac{a_{m+1}^3-a_m^3}{a_{m+1}-a_m}\Big)\\ &+o(\delta_j^2)\\ =\,&\frac{\omega_{n-1}}{(n+2)}\cdot\delta_j^2\cdot\Big((1-a)^2+\frac{\ell^2(K+\epsilon_3)}{3n}\cdot\big(a^2+a+1\big)\Big)+o(\delta_j^2).\end{aligned}$$

By setting $\eta=v\delta_j$, we have

(5.20)
$$\begin{aligned}\int_{B_o(\delta_j)} & \Big(|\exp_{x^o}(a\eta),\ \exp_{y^o}(T\eta)|^2-|x^oy^o|^2\Big)dH^n(\eta)\\ &\leqslant\frac{\omega_{n-1}}{(n+2)}\cdot\delta_j^{2+n}\cdot\Big((1-a)^2+\frac{\ell^2(K+\epsilon_3)}{3n}\cdot\big(a^2+a+1\big)\Big)+o(\delta_j^{n+2}).\end{aligned}$$

Since x^o and y^o are smooth, by (2.4) in Lemma 2.1, we have

$$H^n\big(B_o(\delta_j)\backslash\mathscr{W}\big)=o(\delta_j^{n+1}).$$

On the other hand, by triangle inequality, we have

$$\begin{aligned}
\big||\exp_{x^o}(a\eta),\ \exp_{y^o}(T\eta)|^2 - |x^oy^o|^2\big|
&\leqslant (|\exp_{x^o}(a\eta),\ \exp_{y^o}(T\eta)| + |x^oy^o|)\cdot(a|\eta| + |T\eta|)\\
&\leqslant C\delta_j
\end{aligned}$$

for all $\eta \in B_o(\delta_j)$.

Now the desired estimate (5.17) in **Claim 1** follows from the above two inequalities and equation (5.20).

Step 4. Integral version of maximum principle.

Let us recall that in Step 2, the point pair (x^o, y^o) is a minimum of $H_3(x, y)$ on $B_p(R)\times U$. Then we have

$$\begin{aligned}
0 &\leqslant \int_{B_o(r)\cap\mathscr{W}} \Big(H_3\big(\exp_{x^o}(a\eta), \exp_{y^o}(T\eta)\big) - H_3(x^o, y^o)\Big)dH^n(\eta)\\
&= \int_{B_o(r)\cap\mathscr{W}} \Big(v_3\big(\exp_{x^o}(a\eta)\big) - v_3(x^o)\Big)dH^n(\eta)\\
&\quad + \int_{B_o(r)\cap\mathscr{W}} \Big(u_3\big(\exp_{y^o}(T\eta)\big) - u_3(y^o)\Big)dH^n(\eta)\\
&\quad + \int_{B_o(r)\cap\mathscr{W}} \frac{|\exp_{x^o}(a\eta)\exp_{y^o}(T\eta)|^2 - |x^oy^o|^2}{2t}dH^n(\eta)\\
&:= I_1(r) + I_2(r) + I_3(r),
\end{aligned} \tag{5.21}$$

where $\mathscr{W} = \{v \in T_{x_0} \mid av \in \mathscr{W}_{x^o} \quad \text{and} \quad Tv \in \mathscr{W}_{y^o}\}$.

By the condition $Ric \geqslant -K$ and Laplacian comparison (see [**52**] or [**22**]), we have

$$\mathscr{L}_{|xx^o|^2} \leqslant c(n, K, \mathrm{diam}\Omega) \quad \text{and} \quad \mathscr{L}_{|yy^o|^2} \leqslant c(n, K, \mathrm{diam}\Omega).$$

Claim 2: We have

$$I_1(r) \leqslant \frac{-\epsilon_1 + c\cdot\epsilon_2 - w_{t,a,\theta}(x^o)}{2n(n+2)}\cdot a^2\cdot\omega_{n-1}r^{n+2} + o(r^{n+2}) \tag{5.22}$$

and

$$I_2(r) \leqslant \frac{-\epsilon_1 + c\cdot\epsilon_2 + f(y^o)}{2n(n+2)}\cdot\omega_{n-1}r^{n+2} + o(r^{n+2}) \tag{5.23}$$

for all small $r > 0$, where $c = c(n, K, \mathrm{diam}\Omega)$.

Let

$$\alpha(x) = v_3(x) + \frac{|xy^o|^2}{2t} \quad \text{and} \quad \beta = \frac{|xy^o|^2}{2t}.$$

Since x^o is a smooth point, by Lemma 2.1, we have

$$\int_{B_o(r)\cap\mathscr{W}_{x^o}}\Big(\alpha\big(\exp_{x^o}(a\eta)\big)-\alpha(x^o)\Big)dH^n(\eta)$$
$$=a^{-n}\cdot\int_{B_{x^o}(ar)}\big(\alpha(x)-\alpha(x^o)\big)(1+o(r))d\mathrm{vol}(x).$$

Note that $\alpha(x)-\alpha(x^o)\geqslant 0$ and

$$\begin{aligned}\mathscr{L}_{v_3}\leqslant\mathscr{L}_{v_2}&\leqslant\big(-\epsilon_1+c(n,K,\mathrm{diam}\Omega)\cdot\epsilon_2\big)\cdot\mathrm{vol}+\mathscr{L}_{v_0}\\&\leqslant(-w_{t,a,\theta}-\epsilon_1+c\cdot\epsilon_2)\cdot\mathrm{vol},\\ \mathscr{L}_{\alpha-\alpha(x^o)}=\mathscr{L}_{v_3}+\mathscr{L}_\beta&\leqslant\big(-w_{t,a,\theta}-\epsilon_1+c\cdot\epsilon_2+\Delta\beta\big)\cdot\mathrm{vol}.\end{aligned}$$

Since x^o is a Lebesgue point of $-w_{t,a,\theta}+\Delta\beta$, by Corollary 4.5, we get

$$\int_{\partial B_{x^o}(s)}\big(\alpha(x)-\alpha(x^o)\big)d\mathrm{vol}(x)$$
$$\leqslant\frac{-w_{t,a,\theta}(x^o)-\epsilon_1+c\cdot\epsilon_2+\Delta\beta(x^o)}{2n}\cdot s^2\cdot H^{n-1}\big(\partial B^k_o(s)\big)+o(r^{n+1})$$

for all $0<s<ar$. By combining with the fact that x^o is regular, we have

$$\int_{B_{x^o}(ar)}\big(\alpha(x)-\alpha(x^o)\big)d\mathrm{vol}(x)$$
$$\leqslant\frac{-w_{t,a,\theta}(x^o)-\epsilon_1+c\cdot\epsilon_2+\Delta\beta(x^o)}{2n(n+2)}\cdot\omega_{n-1}\cdot(ar)^{n+2}+o(r^{n+2}).$$

Therefore, we obtain (since $\alpha(x)-\alpha(x^o)\geqslant 0$),

(5.24)

$$\int_{B_o(r)\cap\mathscr{W}}\Big(\alpha\big(\exp_{x^o}(a\eta)\big)-\alpha(x^o)\Big)dH^n(\eta)$$
$$\leqslant\int_{B_o(r)\cap\mathscr{W}_{x^o}}\Big(\alpha\big(\exp_{x^o}(a\eta)\big)-\alpha(x^o)\Big)dH^n(\eta)$$
$$\leqslant\frac{-w_{t,a,\theta}(x^o)-\epsilon_1+c\cdot\epsilon_2+\Delta\beta(x^o)}{2n(n+2)}\cdot a^2\cdot\omega_{n-1}r^{2+n}+o(r^{2+n}).$$

On the other hand, since β is Lipschitz (since it is semi-concave) and equation (2.4)

$$H^n\big(B_o(r)\backslash\mathscr{W}\big)=o(r^{n+1}),$$

we have

$$\int_{B_o(r)\cap\mathscr{W}}\Big(\frac{|\exp_{x^o}(a\eta)y^o|^2}{2t}-\frac{|x^oy^o|^2}{2t}\Big)dH^n(\eta)$$
$$=\int_{B_o(r)\cap\mathscr{W}_{x^o}}\big(\beta(\exp_{x^o}(a\eta))-\beta(x^o)\big)dH^n(\eta)+o(r^{n+2}).$$

Since $x^o \in Reg_\beta$, by applying equation (2.3) in Lemma 2.1, the Lipschitz continuity of β, and Lemma 2.3, we get

$$\begin{aligned}\int_{B_o(r)\cap \mathscr{W}_{x^o}} &\big(\beta(\exp_{x^o}(a\eta)) - \beta(x^o)\big)dH^n(\eta)\\ &= a^{-n}\int_{B_{x^o}(ar)}\big(\beta(x)-\beta(x^o)\big)d\mathrm{vol} + o(r^{n+2})\\ &= \frac{\Delta\beta(x^o)}{2n(n+2)}\cdot a^2\cdot\omega_{n-1}r^{n+2} + o(r^{n+2}).\end{aligned}$$

By combining above two equalities, we have

$$\begin{aligned}\int_{B_o(r)\cap\mathscr{W}} &\Big(\frac{|\exp_{x^o}(a\eta)y^o|^2}{2t} - \frac{|x^oy^o|^2}{2t}\Big)dH^n(\eta)\\ &= \frac{\Delta\beta(x^o)}{2n(n+2)}\cdot a^2\cdot\omega_{n-1}r^{n+2} + o(r^{n+2}).\end{aligned}\tag{5.25}$$

Therefore, the desired estimate (5.22) follows from equations (5.24), (5.25), and $v_3 = \alpha - \beta$.

The estimate for (5.23) is similar. Let

$$\widetilde{\alpha}(y) = u_3(y) + \frac{|x^oy|^2}{2t} \quad\text{and}\quad \widetilde{\beta} = \frac{|x^oy|^2}{2t}.$$

By a similar argument to (5.24) and (5.25), we have, for all small $r>0$,

$$\begin{aligned}\int_{B_o(r)\cap\mathscr{W}} &\Big(\widetilde{\alpha}\big(\exp_{y^o}(T\eta)\big) - \widetilde{\alpha}(y^o)\Big)dH^n(\eta)\\ &\leqslant \frac{f(y^o) - \epsilon_1 + c\cdot\epsilon_2 + \Delta\widetilde{\beta}(y^o)}{2n(n+2)}\cdot\omega_{n-1}r^{2+n} + o(r^{2+n})\end{aligned}$$

and

$$\begin{aligned}\int_{B_o(r)\cap\mathscr{W}} &\Big(\frac{|\exp_{y^o}(T\eta)x^o|^2}{2t} - \frac{|x^oy^o|^2}{2t}\Big)dH^n(\eta)\\ &= \frac{\Delta\widetilde{\beta}(y^o)}{2n(n+2)}\cdot\omega_{n-1}r^{n+2} + o(r^{n+2}).\end{aligned}$$

Thus the combination of these two estimates and $u_3(y) = \widetilde{\alpha} - \widetilde{\beta}$ implies (5.23). The proof of **Claim 2** is finished.

By combining (5.21), **Claim 1** (5.17) and **Claim 2** (5.22)–(5.23), we have

$$\begin{aligned}&\Big[\frac{-\epsilon_1 + c\cdot\epsilon_2}{2n}(a^2+1) - \frac{a^2\cdot w_{t,a,\theta}(x^o)}{2n} + \frac{f(y^o)}{2n} + \frac{(a-1)^2}{2t}\Big]\cdot\delta_j^{n+2}\\ &+\Big[\frac{(K+\epsilon_3)|x^oy^o|^2}{6nt}(1+a+a^2)\Big]\cdot\delta_j^{n+2} + o(\delta_j^{n+2}) \geqslant 0\end{aligned}$$

for all $j \in \mathbb{N}$. Thus,

$$\frac{-\epsilon_1 + c \cdot \epsilon_2}{2n}(a^2+1) - \frac{a^2 \cdot w_{t,a,\theta}(x^o)}{2n} + \frac{f(y^o)}{2n} + \frac{(a-1)^2}{2t} + \frac{(K+\epsilon_3)|x^o y^o|^2}{6nt}(1+a+a^2) \geqslant 0.$$

Combining with the definition of function $w_{t,a,\theta}$, (5.14), we have

(5.26)

$$\begin{aligned} 0 \leqslant & (a^2+1)\frac{-\epsilon_1 + c\epsilon_2}{2n} + \frac{(a^2+a+1)}{6nt}\big((K+\epsilon_3)|x^o y^o|^2 - K|x^o F(x^o)|^2\big) \\ & - \frac{1}{2n}\Big(\sup_{z \in B_{F_t(x^o)}(\theta)} f(z) - f(y^o)\Big) - \frac{\theta}{2n}. \end{aligned}$$

In Step 2, we have known that $(\bar{x}, F_t(\bar{x}))$ is the *unique* minimum point of $H(x,y)$. Because $H_3(x,y)$ converges to $H(x,y)$ as ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$, tend to 0^+, we know that (x^o, y^o) converges to $(\bar{x}, F_t(\bar{x}))$, as ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$, tend to 0^+.

On the other hand, because $\bar{x}$ is regular and x^o converges to $\bar{x}$ as ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$, tend to 0^+, function

$$u(y) + \frac{|x^o y|^2}{2t}$$

converges to function

$$u(y) + \frac{|\bar{x} y|^2}{2t}$$

as ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$, tend to 0^+. $F_t(x^o)$ is a minimum of $u(y) + |x^o y|^2/(2t)$. u_t is differentiable at $\bar{x}$ (see Step 2). So $F_t(\bar{x})$ is the *unique* minimum point of $u(y) + |\bar{x}y|^2/(2t)$. Therefore, $F_t(x^o)$ converges to $F_t(\bar{x})$ as ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$, tend to 0^+.

Hence, when we choose ϵ_1, ϵ_2 and b_j, $1 \leqslant j \leqslant 2n$ sufficiently small, we have that $|y^o F_t(x^o)| \ll \theta$. This implies

$$y^o \in B_{F_t(x^o)}(\theta) \quad \text{and} \quad \big||x^o y^o| - |x^o F_t(x^o)|\big| \ll \theta.$$

Now we can choose ϵ_1, ϵ_2, and ϵ_3 so small that

$$(a^2+1)\frac{-\epsilon_1 + c\epsilon_2}{2n} + \frac{(a^2+a+1)}{6nt}\big((K+\epsilon_3)|x^o y^o|^2 - K|x^o F(x^o)|^2\big) \leqslant \frac{\theta}{4n}$$

and $y^o \in B_{F_t(x^o)}(\theta)$. This contradicts (5.26). Therefore we have completed the proof of the proposition. q.e.d.

Lemma 5.4. *Let Ω be a bounded open domain in an n-dimensional Alexandrov space. Assume that a $W^{1,2}(\Omega)$-function u satisfies $\mathscr{L}_u \geqslant f \cdot \mathrm{vol}$ for some $f \in L^\infty(\Omega)$. Then, for any $\Omega' \Subset \Omega$, we have*

$$\sup_{x \in \Omega'} u \leqslant C\|u\|_{L^1(\Omega)} + C\|f\|_{L^\infty(\Omega)},$$

where the constant C depends on lower bounds of curvature, Ω and Ω'.

Proof. If $f = 0$ and $u \geqslant 0$, this lemma has been shown in Theorem 8.2 of [**4**] for any metric measure space supporting a doubling property and a weak Poincaré inequality. According to volume comparison and Theorem 7.2 of [**19**], it holds for Alexandrov spaces.

On the other hand, according to Lemma 6.4 of [**4**] (see also Lemma 3.10 of [**17**]), we know that u_+ is also a subsolution of $\mathscr{L}_u = 0$, that is, $\mathscr{L}_{u_+} \geqslant 0$.

Therefore, if $f = 0$, we have

$$\sup_{x\in\Omega'} u \leqslant \sup_{x\in\Omega'} u_+ \leqslant C\|u_+\|_{L^1(\Omega)} \leqslant C\|u\|_{L^1(\Omega)}.$$

In fact, the proof in [**4**] works for general $f \in L^\infty(\Omega)$. In the following, we give a simple argument for the general case on Alexandrov spaces.

For each $p \in \Omega$, we choose a Perelman concave function h defined on some neighborhood $B_p(r_p)$, which is given in Lemma 3.3, such that $-1 \leqslant h \leqslant 0$. Then we have

$$\mathscr{L}_{u-\|f\|_{L^\infty(\Omega)}h} \geqslant (f + \|f\|_{L^\infty(\Omega)}) \cdot \mathrm{vol} \geqslant 0 \quad \text{on } B_p(r_p).$$

Applying the above estimate (in case $f = 0$), we have

$$\begin{aligned}\sup_{B_p(r_p/2)} u \leqslant \sup_{B_p(r_p/2)} (u - \|f\|_{L^\infty(\Omega)}h) &\leqslant C\|u - \|f\|_{L^\infty(\Omega)}h\|_{L^1(B_p(r_p))} \\ &\leqslant C\|u\|_{L^1(B_p(r_p))} + C\|f\|_{L^\infty(\Omega)} \cdot \mathrm{vol}(B_p(r_p)).\end{aligned}$$

Since $\overline{\Omega'}$ is compact, there are finite balls $B_{p_i}(r_i)$ such that the above estimate holds on each $B_{p_i}(r_i)$ and that $\Omega' \subset \cup_i B_{p_i}(r_i/2)$. Therefore, we have

$$\sup_{\Omega'} u \leqslant C\|u\|_{L^1(\Omega)} + C\|f\|_{L^\infty(\Omega)} \cdot \mathrm{vol}(\Omega).$$

The proof of the lemma is finished. q.e.d.

In [**40**, **41**], by using his key lemma, Petrunin proved that any harmonic function on an Alexandrov space with nonnegative curvature is locally Lipschitz continuous. Very recently, this Lipschitz continuity result on compact Alexandrov spaces was also obtained by Gigli–Kuwada–Ohta in [**11**] via probability method. We can now establish the locally Lipschitz continuity for solutions of general Poisson equations.

Corollary 5.5. *Let M be an n-dimensional Alexandrov space and Ω be a bounded domain of M. Assume that u satisfies $\mathscr{L}_u = f \cdot \mathrm{vol}$ on Ω and $f \in Lip(\Omega)$. Then u is locally Lipschitz continuous.*

Proof. Since Ω is bounded, we may assume that M has Ricci curvature $\geqslant -K$ on Ω with some $K \geqslant 0$.

By applying Lemma 5.4 to both $\mathscr{L}_u = f \cdot \mathrm{vol}$ and $\mathscr{L}_{-u} = -f \cdot \mathrm{vol}$, we can conclude that $u \in L^\infty(\Omega')$ for any $\Omega' \Subset \Omega$. Without loss of generality, we may assume

$$-1 \leqslant u \leqslant 0$$

on Ω'. Otherwise, we replace u by $(u - \sup_{\Omega'} u)/(\sup_{\Omega'} u - \inf_{\Omega'} u)$.

Fix any open subset $\Omega_1 \Subset \Omega'$ and let $(u_t)_{0 \leqslant t \leqslant \bar{t}}$ be its Hamilton–Jacobi semigroup defined on Ω_1. By Lemma 5.1, we know

$$-1 \leqslant u_t \leqslant 0$$

on Ω_1, for all $0 \leqslant t \leqslant \bar{t}$.

By Proposition 5.3, there is $t_0 > 0$ such that (5.12) holds for all $t \in (0, t_0)$ and all $a > 0$. By putting $a = 1$ in (5.12), we have

$$\mathscr{L}_{u_t} \leqslant (f \circ F_t + Kt|\nabla u_t|^2) \cdot \mathrm{vol}, \qquad \forall\ 0 < t < t_0$$

on Ω_1.

Set

$$\bar{K} = K + 1 \qquad \text{and} \qquad \Phi_t(x) = \frac{\exp(-\bar{K} t u_t) - 1}{t}$$

for all $0 < t < t_0 (\leqslant 1)$. Then we have

$$0 \leqslant \Phi_t \leqslant \bar{K} e^{\bar{K}}, \quad 1 \leqslant \exp(-\bar{K} t u_t) \leqslant e^{\bar{K}}$$

and, for each $t \in (0, t_0)$,

$$\begin{aligned} \mathscr{L}_{\Phi_t} &= -\bar{K} \exp(-\bar{K} t u_t) \cdot (\mathscr{L}_{u_t} - \bar{K} t |\nabla u_t|^2) \cdot \mathrm{vol} \\ &\geqslant -\bar{K} \exp(-\bar{K} t u_t) \cdot (f \circ F_t + Kt|\nabla u_t|^2 - \bar{K} t |\nabla u_t|^2) \cdot \mathrm{vol} \\ &\geqslant -\bar{K} \exp(-\bar{K} t u_t) \cdot \|f\|_{L^\infty(\Omega)} \cdot \mathrm{vol} \\ &\geqslant -C \cdot \mathrm{vol} \end{aligned} \tag{5.27}$$

in the sense of measure on Ω_1. Here and in the following, C will denote various positive constants that do not depend on t (while they might depend on K, t_0, $\Omega, \Omega_1, \Omega_2, \Omega_3$, $\|f\|_{L^\infty(\Omega)}$ and the Lipschitz constant of f, $\mathbf{Lip} f$, on Ω).

By applying Caccioppoli inequality (see Proposition 7.1 of [**4**]), or by choosing test function $\varphi \Phi_t$ for some suitable cut-off φ on Ω_1, we have

$$\|\nabla \Phi_t\|_{L^2(\Omega_2)} \leqslant C \|\Phi_t\|_{L^2(\Omega_1)} \leqslant C$$

for any open subset $\Omega_2 \Subset \Omega_1$.

Noting that $-\bar{K} u_t \geqslant 0$ and

$$|\nabla \Phi_t| = \bar{K} \exp(-\bar{K} t u_t) |\nabla u_t| \geqslant \bar{K} |\nabla u_t|,$$

we have

$$\|\nabla u_t\|_{L^2(\Omega_2)} \leqslant C. \tag{5.28}$$

By using inequalities $\exp(-\bar{K}tu_t) \leqslant e^{\bar{K}}$ and $|1-e^{\gamma}+\gamma\cdot e^{\gamma}| \leqslant C\cdot\gamma^2/2$ for any $0 \leqslant \gamma \leqslant \bar{K}t_0$, we have, for each $t \in (0,t_0)$ and $x \in \Omega_1$,

(5.29)

$$\begin{aligned}\big|\Phi_{t+s}(x)-\Phi_t(x)\big| \leqslant & \Big|\frac{\exp\big(-\bar{K}(t+s)u_{t+s}\big)-1}{t+s}-\frac{\exp\big(-\bar{K}tu_{t+s}\big)-1}{t}\Big| \\ &+\Big|\frac{\exp\big(-\bar{K}tu_{t+s}\big)-1}{t}-\frac{\exp\big(-\bar{K}tu_t\big)-1}{t}\Big| \\ &\leqslant s\cdot\max_{t\leqslant t'\leqslant t+s}\Big|\frac{\exp(-t'\bar{K}u_{t+s})(-\bar{K}u_{t+s})t'-\exp(-t'\bar{K}u_{t+s})+1}{(t')^2}\Big| \\ &+\bar{K}|u_{t+s}-u_t|\cdot\max_{u_{t+s}\leqslant a\leqslant u_t}\exp(-\bar{K}ta) \\ &\leqslant Cs+C|u_{t+s}-u_t|\end{aligned}$$

for all $0<s<t_0-t$.

By applying the Dominated Convergence Theorem, (5.28), (5.29), and Lemma 5.1(iii)–(iv), we have

$$\begin{aligned}\frac{\partial^+}{\partial t}\|\Phi_t\|_{L^1(\Omega_2)} &= \limsup_{s\to 0^+}\int_{\Omega_2}\frac{\Phi_{t+s}(x)-\Phi_t(x)}{s}d\mathrm{vol} \\ &\leqslant C\mathrm{vol}(\Omega_2)+C\limsup_{s\to 0^+}\int_{\Omega_2}\frac{|u_{t+s}-u_t|}{s}d\mathrm{vol} \\ &= C\mathrm{vol}(\Omega_2)+\frac{C}{2}\int_{\Omega_2}|\nabla u_t|^2 d\mathrm{vol}\leqslant C.\end{aligned}$$

This implies that

$$\|\Phi_t\|_{L^1(\Omega_2)} \leqslant \|\Phi_{t'}\|_{L^1(\Omega_2)}+C(t-t') \tag{5.30}$$

for any $0<t'<t<t_0$. Since $0 \leqslant \Phi_{t'} \leqslant \bar{K}e^{\bar{K}}$ and $\lim_{t'\to 0^+}\Phi_{t'}(x) = -\bar{K}u(x)$, we have

$$\lim_{t'\to 0^+}\|\Phi_{t'}\|_{L^1(\Omega_2)} = \int_{\Omega_2}(-\bar{K}u)d\mathrm{vol}.$$

By combining with (5.30), we have

$$\int_{\Omega_2}\frac{\Phi_t+\bar{K}u}{t}d\mathrm{vol} = \frac{1}{t}(\|\Phi_t\|_{L^1(\Omega_1')}-\lim_{t'\to 0^+}\|\Phi_{t'}\|_{L^1(\Omega_1')}) \leqslant C.$$

On the other hand, for each $t \in (0,t_0)$, since f is Lipschitz and

$$|xF_t(x)| = t|\nabla u_t(x)|,$$

for almost all $x \in \Omega_1$, we have

$$\begin{aligned}
\mathscr{L}_{\Phi_t+\bar{K}u} &= -\bar{K}\exp(-\bar{K}tu_t)\big(\mathscr{L}_{u_t} - \bar{K}t|\nabla u_t|^2\big)\cdot \mathrm{vol} + \bar{K}f\cdot \mathrm{vol}\\
&= -\bar{K}\exp(-\bar{K}tu_t)\cdot\big(\mathscr{L}_{u_t} - \bar{K}t|\nabla u_t|^2 - f\big)\cdot \mathrm{vol}\\
&\quad -\bar{K}f\cdot\big(\exp(-\bar{K}tu_t)-1\big)\cdot \mathrm{vol}\\
&\geqslant -\bar{K}\exp(-\bar{K}tu_t)\cdot\big(f\circ F_t + Kt|\nabla u_t|^2 - \bar{K}t|\nabla u_t|^2 - f\big)\cdot \mathrm{vol}\\
&\quad -\bar{K}f\cdot\big(\exp(-\bar{K}tu_t)-1\big)\cdot \mathrm{vol}\\
&\geqslant -\bar{K}\exp(-\bar{K}tu_t)\cdot(\mathbf{Lip}f\cdot|xF_t(x)| - t|\nabla u_t|^2)\cdot \mathrm{vol}\\
&\quad - Ct\cdot\|f\|_{L^\infty(\Omega)}\cdot \mathrm{vol}\\
&= -\bar{K}\exp(-\bar{K}tu_t)\cdot t\cdot(\mathbf{Lip}f\cdot|\nabla u_t| - |\nabla u_t|^2)\cdot \mathrm{vol}\\
&\quad - Ct\cdot\|f\|_{L^\infty(\Omega)}\cdot \mathrm{vol}\\
&\geqslant -Ct\cdot\Big(\frac{\mathbf{Lip}^2 f}{4} + \|f\|_{L^\infty(\Omega)}\Big)\cdot \mathrm{vol}\\
&\geqslant -Ct\cdot \mathrm{vol}
\end{aligned}$$

in the sense of measure on Ω_2. Note that $\Phi_t + \bar{K}u \geqslant -\bar{K}u_t + \bar{K}u \geqslant 0$ (because of Lemma 5.1(i)). According to Lemma 5.4, we get

$$\max_{\Omega_3}\Big|\frac{\Phi_t+\bar{K}u}{t}\Big| \leqslant C\|\frac{\Phi_t+\bar{K}u}{t}\|_{L^1(\Omega_2)} + C = C\int_{\Omega_2}\frac{\Phi_t+\bar{K}u}{t}d\mathrm{vol} + C \leqslant C$$

for any open subset $\Omega_3 \Subset \Omega_2$. Hence, we have (since $\Phi_t \geqslant -\bar{K}u_t$)

$$\frac{-u_t+u}{t} \leqslant \bar{K}^{-1}\frac{\Phi_t+\bar{K}u}{t} \leqslant C$$

on Ω_3, for each $t \in (0, t_0)$.

Therefore, by the definition of u_t, we obtain

$$u(x) \leqslant u_t(x) + Ct \leqslant u(y) + \frac{|xy|^2}{2t} + Ct$$

for all $x, y \in \Omega_3$ and $t \in (0, t_0)$. Now fix x and y in Ω_3 such that $|xy| < t_0$. By choosing $t = |xy|$, we get

$$u(x) - u(y) \leqslant C|xy|.$$

Hence, by replacing x and y, we have

$$|u(x) - u(y)| \leqslant C|xy|, \quad \text{for all } |xy| < t_0.$$

This implies that u is Lipschitz continuous on Ω_3.

By the arbitrariness of $\Omega_3 \Subset \Omega_2 \Subset \Omega_1 \Subset \Omega' \Subset \Omega$, we get that u is locally Lipschitz continuous on Ω, and complete the proof. q.e.d.

5.2. Bochner-type formula. Bochner formula is one of the important tools in differential geometry. In this subsection, we will extend it to Alexandrov space with Ricci curvature bounded below.

Lemma 5.6. *Let $u \in Lip(\Omega)$ with Lipschitz constant* $\mathbf{Lip}u$*, and let u_t be its* Hamilton–Jacobi *semigroup defined on $\Omega' \Subset \Omega$, for $0 \leqslant t < \bar{t}$. Then we have the following properties:*

(i) For any $t > 0$, we have

$$|\nabla^- u|(F_t(x)) \leqslant |\nabla u_t(x)| \leqslant \mathrm{Lip}u(F_t(x)) \tag{5.31}$$

for almost all $x \in \Omega'$, where F_t is defined in (5.10)*.*

In particular, the Lipschitz constant of u_t, $\mathbf{Lip}u_t \leqslant \mathbf{Lip}u$.

(ii) For almost all $x \in \Omega'$, we have

$$\lim_{t \to 0^+} \frac{u_t(x) - u(x)}{t} = -\frac{1}{2}|\nabla u(x)|^2. \tag{5.32}$$

Furthermore, for each sequence t_j converging to 0^+, we have

$$\lim_{t_j \to 0^+} \nabla u_{t_j}(x) = \nabla u(x)$$

for almost all $x \in \Omega'$.

Proof. (i) Lipschitz function u_t is differentiable at almost all points $x \in \Omega'$. For such a point x, we first prove $|\nabla^- u|(F_t(x)) \leqslant |\nabla u_t(x)|$.

Assume $|\nabla^- u|(F_t(x)) > 0$. (If not, we are done.) This implies $y := F_t(x) \neq x$. Indeed, if $F_t(x) = x$, we have

$$u(x) \leqslant u(z) + \frac{|xz|^2}{2t}$$

for all $z \in \Omega'$. Hence $\big(u(x) - u(z)\big)_+ \leqslant |xz|^2/(2t)$. This concludes $|\nabla^- u|(F_t(x)) = 0$.

Take a sequence of points y_j converging to y such that

$$\lim_{y_j \to y} \frac{u(y) - u(y_j)}{|yy_j|} = |\nabla^- u|(y).$$

Let x_j be points in geodesic xy such that $|xx_j| = |yy_j|$. By

$$u_t(x_j) \leqslant u(y_j) + \frac{|x_j y_j|^2}{2t} \quad \text{and} \quad u_t(x) = u(y) + \frac{|xy|^2}{2t},$$

we have

$$u_t(x_j) - u_t(x) \leqslant u(y_j) - u(y) + \frac{1}{2t}(|x_j y_j|^2 - |xy|^2). \tag{5.33}$$

Since u_t is differentiable at x,

$$u_t(x_j) - u_t(x) = |xx_j| \cdot \big\langle \nabla u_t(x), \uparrow_x^{x_j} \big\rangle + o(|xx_j|).$$

Triangle inequality implies

$$|x_j y_j| \leqslant |x_j y| + |yy_j| = |x_j y| + |xx_j| = |xy|.$$

Therefore, by combining with (5.33), we have

$$\begin{aligned}u(y)-u(y_j)&\leqslant -|xx_j|\cdot\left\langle\nabla u_t(x),\uparrow_x^{x_j}\right\rangle+o(|xx_j|)\\&\leqslant |xx_j|\cdot|\nabla u_t(x)|+o(|xx_j|)\\&=|yy_j|\cdot|\nabla u_t(x)|+o(|yy_j|).\end{aligned}$$

Letting $y_j\to y$, this implies $|\nabla^- u|(y)\leqslant|\nabla u_t(x)|$.

Now let us prove $|\nabla u_t(x)|\leqslant \mathrm{Lip}u(F_t(x))$ at a point x, where u_t is differentiable. Assume $|\nabla u_t(x)|>0$. (If not, we are done.) This implies $y:=F_t(x)\neq x$. Indeed, if $y=x$, we have

$$u_t(z)\leqslant u(x)+\frac{|xz|^2}{2t}=u_t(x)+\frac{|xz|^2}{2t},\quad \forall\ z\in\Omega'.$$

On the other hand, u_t is differentiable at x,

$$u_t(z)=u_t(x)+\langle\nabla u_t(x),\uparrow_x^z\rangle\cdot|xz|+o(|xz|).$$

Hence, we obtain

$$\langle\nabla u_t(x),\uparrow_x^z\rangle\leqslant|xz|/(2t)+o(1)$$

for all z near x. Hence $|\nabla u_t(x)|=0$.

Let the sequence $x_j\in\Omega'$ converge to x and

$$\lim_{x_j\to x}\left\langle\nabla u_t(x),\uparrow_x^{x_j}\right\rangle=|\nabla u_t(x)|. \tag{5.34}$$

Take y_j to be points in geodesic xy with $|yy_j|=|xx_j|$. By triangle inequality, we have

$$|x_jy_j|\leqslant|xx_j|+|xy_j|=|yy_j|+|xy_j|=|xy|.$$

Combining with

$$u_t(x_j)\leqslant u(y_j)+\frac{|x_jy_j|^2}{2t}\quad\text{and}\quad u_t(x)=u(y)+\frac{|xy|^2}{2t},$$

we have

$$u_t(x_j)-u_t(x)\leqslant u(y_j)-u(y)\leqslant|u(y_j)-u(y)|. \tag{5.35}$$

Since u_t is differentiable at x,

$$u_t(x_j)-u_t(x)=\left\langle\nabla u_t(x),\uparrow_x^{x_j}\right\rangle\cdot|xx_j|+o(|xx_j|).$$

Hence, by combining with (5.34), (5.35), and $|x_jx|=|y_jy|$, we get

$$|\nabla u_t(x)|\leqslant\limsup_{y_j\to y}\frac{|u(y_j)-u(y)|}{|yy_j|}\leqslant \mathrm{Lip}u(y).$$

The assertion (i) is proved.

(ii) The equation (5.32) was proved by Lott–Villani in [**30**] (see also Theorem 30.30 in [**49**]).

Now let us prove the second assertion. The functions u and u_{t_j} are Lipschitz on Ω'. Then they are differentiable at almost all points $x \in \Omega'$. For such a point x, according to (5.5) in Lemma 5.2, we have, for each t_j,

$$u_{t_j}(x) = u(y_{t_j}) + \frac{|xy_{t_j}|^2}{2t_j} = u(y_{t_j}) + t_j \cdot \frac{|\nabla u_{t_j}(x)|^2}{2},$$

where y_{t_j} is the (unique) point such that (5.4) holds, and

$$\begin{aligned} u(y_{t_j}) &= u(x) + |xy_{t_j}| \left\langle \nabla u(x), \uparrow_x^{y_{t_j}} \right\rangle + o(t_j) \\ &= u(x) - t_j \left\langle \nabla u(x), \nabla u_{t_j}(x) \right\rangle + o(t_j). \end{aligned}$$

The combination of the above two equations and (5.32) implies that

$$\lim_{t_j \to 0^+} \Big(- \left\langle \nabla u(x), \nabla u_{t_j}(x) \right\rangle + \frac{|\nabla u_{t_j}(x)|^2}{2} \Big) = - \frac{|\nabla u(x)|^2}{2}.$$

This is

$$\lim_{t_j \to 0^+} \Big(|\nabla u(x)|^2 - 2 \left\langle \nabla u(x), \nabla u_{t_j}(x) \right\rangle + |\nabla u_{t_j}(x)|^2 \Big) = 0,$$

which implies

$$\lim_{t_j \to 0^+} \nabla u_{t_j}(x) = \nabla u(x).$$

Now the proof of this lemma is completed. q.e.d.

Next we have the following Bochner-type formula.

Theorem 5.7 (Bochner-type formula)**.** *Let M be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ and Ω be a bounded domain in M. Let $f(x, s) : \Omega \times [0, +\infty) \to \mathbb{R}$ be a Lipschitz function and satisfy the following:*

(*a*) *there exists a zero measure set $\mathcal{N} \subset \Omega$ such that for all $s \geqslant 0$, the functions $f(\cdot, s)$ are differentiable at any $x \in \Omega \backslash \mathcal{N}$;*

(*b*) *the function $f(x, \cdot)$ is of class C^1 for all $x \in \Omega$ and the function $\frac{\partial f}{\partial s}(x, s)$ is continuous, non-positive on $\Omega \times [0, +\infty)$.*

Suppose that $u \in Lip(\Omega)$ and

$$\mathscr{L}_u = f(x, |\nabla u|^2) \cdot \mathrm{vol}.$$

Then we have $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega)$ and

$$\mathscr{L}_{|\nabla u|^2} \geqslant 2 \Big(\frac{f^2(x, |\nabla u|^2)}{n} + \left\langle \nabla u, \nabla f(x, |\nabla u|^2) \right\rangle - K |\nabla u|^2 \Big) \cdot \mathrm{vol} \tag{5.36}$$

in the sense of measure on Ω, provided $|\nabla u|$ is lower semi-continuous at almost all $x \in \Omega$, namely, there exists a representative of $|\nabla u|$ which is lower semi-continuous at almost all $x \in \Omega$.

Proof. Recalling the pointwise Lipschitz constant Lip$u$ of u in Section 2.2, we defined a function

$$g(x) := \max\{\mathrm{Lip}^2 u, |\nabla u(x)|^2\}, \quad \forall\, x \in \Omega.$$

Noting the fact that Lip$u = |\nabla u|$ for almost all $x \in \Omega$, we have $g = |\nabla u|^2$ for almost all $x \in \Omega$, and hence

$$\mathscr{L}_u = f\big(x, g(x)\big) \cdot \mathrm{vol}$$

in the sense of measure on Ω.

The function g is lower semi-continuous at almost all $x \in \Omega$. Indeed, by the definition of g, we have $g(x) \geqslant |\nabla u(x)|^2$ at any $x \in \Omega$. On the other hand, $g(x) = |\nabla u(x)|^2$ at almost all $x \in \Omega$. Combining with the fact that $|\nabla u|$ is lower semi-continuous at almost all $x \in \Omega$, we can get the desired lower semi-continuity of g at almost all $x \in \Omega$.

The combination of the assumption $\frac{\partial f}{\partial s} \leqslant 0$ and the lower semi-continuity of g at almost everywhere in Ω implies that $f = f\big(x, g(x)\big)$ is upper semi-continuous at almost all $x \in \Omega$.

Fix any open subset $\Omega' \Subset \Omega$. Let u_t be the Hamilton–Jacobi semi-group of u, defined on Ω', and let F_t be the map defined in (5.10). By applying Proposition 5.3, there exists some $t_0 > 0$ such that for each $t \in (0, t_0)$, we have

$$a^2 \cdot \mathscr{L}_{u_t} \leqslant \Big[f \circ F_t + \frac{n(a-1)^2}{t} + \frac{Kt}{3}(a^2 + a + 1)|\nabla u_t|^2 \Big] \cdot \mathrm{vol}$$

for all $a > 0$. Hence, the absolutely continuous part Δu_t satisfies

$$a^2 \cdot \Delta u_t(x) \leqslant f \circ F_t(x) + \frac{n(a-1)^2}{t} + \frac{Kt}{3}(a^2 + a + 1)|\nabla u_t(x)|^2$$

for all $a > 0$ and almost all $x \in \Omega'$. By setting

$$D = -\frac{K}{3}|\nabla u_t(x)|^2$$

and

$$A_1 = -\Delta u_t(x) + \frac{n}{t} - tD, \quad A_2 = -\frac{2n}{t} - tD, \quad A_3 = f \circ F_t(x) + \frac{n}{t} - tD,$$

we can rewrite this equation as

$$A_1 \cdot a^2 + A_2 \cdot a + A_3 \geqslant 0$$

for all $a > 0$ and almost all $x \in \Omega'$.

By taking $a = 1$, we have

$$\Delta u_t(x) \leqslant f \circ F_t(x) - 3tD. \tag{5.37}$$

Because u is Lipschitz, by Lemma 5.6(i), we have

$$|D| = |K| \cdot |\nabla u_t|^2/3 \leqslant |K| \cdot \mathbf{Lip}^2 u/3, \qquad g \leqslant \mathbf{Lip}^2 u,$$

and then $f = f\big(x, g(x)\big)$ is bounded in Ω'.

The combination of equation (5.37) and the boundedness of D, f implies that $A_1 > 0$ and $A_2 < 0$, when t is sufficiently small. By choosing $a = -\frac{A_2}{2A_1}$, we obtain

(5.38)

$$\big(\Delta u_t(x) - f\circ F_t(x)\big)\cdot\big(\frac{n}{t} - tD\big) \leqslant -\Delta u_t(x)\cdot f\circ F_t(x) - 3nD + \frac{3}{4}t^2D^2.$$

Therefore, (by writing $f = f(x, g(x))$ and $f\circ F_t = f\circ F_t(x) = f(F_t(x), g\circ F_t(x)))$,

$$\begin{aligned}
\frac{\Delta u_t(x) - f\big(x, g(x)\big)}{t} &\leqslant \frac{(n - t^2D)\big(f\circ F_t - f\big)/t - f\cdot f\circ F_t - 3nD + 3t^2D^2/4}{n - t^2D + tf\circ F_t}\\
&= \frac{f\circ F_t - f}{t} - \frac{f^2 + 3nD}{\mathcal{A}} + \frac{f^2 - f^2\circ F_t}{\mathcal{A}} + \frac{3t^2D^2}{4\mathcal{A}}\\
&= \frac{f\circ F_t - f\big(F_t(x), |\nabla u_t(x)|^2\big)}{t} + \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f}{t} - \frac{f^2 + 3nD}{\mathcal{A}}\\
&\quad + \frac{f^2 - f^2\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} + \frac{f^2\big(F_t(x), |\nabla u_t(x)|^2\big) - f^2\circ F_t}{\mathcal{A}} + \frac{3t^2D^2}{4\mathcal{A}}\\
&= \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f}{t} + \frac{f^2 - f^2\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} - \frac{f^2 + 3nD}{\mathcal{A}}\\
&\quad + \Big(f\circ F_t - f\big(F_t(x), |\nabla u_t(x)|^2\big)\Big)\cdot\Big(\frac{1}{t} - \frac{f\circ F_t + f\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}}\Big)\\
&\quad + \frac{3t^2D^2}{4\mathcal{A}}
\end{aligned}$$

for almost all $x\in\Omega'$, where

$$\mathcal{A} = n - t^2D + tf\circ F_t.$$

From Lemma 5.6(i) and the definition of function g, we have

$$g\circ F_t(x) \geqslant \mathrm{Lip}^2 u(F_t(x)) \geqslant |\nabla u_t(x)|^2, \quad a.e.,\ x\in\Omega'.$$

Combining with the assumption $\frac{\partial f}{\partial s}\leqslant 0$, we have, for almost all $x\in\Omega'$,

$$f\circ F_t - f\big(F_t(x), |\nabla u_t(x)|^2\big) = f\big(F_t(x), g\circ F_t(x)\big) - f\big(F_t(x), |\nabla u_t(x)|^2\big) \leqslant 0.$$

On the other hand, by the boundedness of D and f, we have

$$\mathcal{A} = n - t^2D + tf\circ F_t \geqslant \frac{n}{2}$$

when t is sufficiently small. By combining with the boundedness of f, we have

$$\frac{1}{t} - \frac{f\circ F_t + f\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} \geqslant 0$$

when t is sufficiently small.

When t is sufficiently small, by using $\mathcal{A} \geqslant n/2$ and the boundedness of D again, we have

$$\begin{aligned}\frac{\Delta u_t(x) - f\big(x, g(x)\big)}{t} \leqslant & \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f}{t} \\ & + \frac{f^2 - f^2\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} \\ & - \frac{f^2 + 3nD}{\mathcal{A}} + C \cdot t.\end{aligned}$$

Here and in the following proof, C will denote various positive constants that do not depend on t.

Note that $\mathscr{L}_{u_t} \leqslant \Delta u_t \cdot \mathrm{vol}$ and $\mathscr{L}_u = f \cdot \mathrm{vol}$. The above inequality implies that

$$\begin{aligned}\frac{1}{t}\mathscr{L}_{u_t - u} \leqslant & \Big[\frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f}{t} \\ & + \frac{f^2 - f^2\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} - \frac{f^2 + 3nD}{\mathcal{A}} + C \cdot t\Big] \cdot \mathrm{vol}\end{aligned}$$

in the sense of measure on Ω'.

Fix arbitrary $0 \leqslant \phi \in Lip_0(\Omega')$. We have

$$\begin{aligned}\frac{1}{t}\mathscr{L}_{u_t - u}(\phi) \leqslant & \int_{\Omega'} \phi \cdot \Big(\frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f}{t}\Big) d\mathrm{vol} \\ & + \int_{\Omega'} \phi \cdot \frac{f^2 - f^2\big(F_t(x), |\nabla u_t(x)|^2\big)}{\mathcal{A}} d\mathrm{vol} \\ & - \int_{\Omega'} \phi \cdot \frac{f^2 + 3nD}{\mathcal{A}} d\mathrm{vol} + Ct \sup|\phi| \\ := & I_1(t) + I_2(t) - I_3(t) + Ct \sup|\phi|.\end{aligned} \tag{5.39}$$

We want to take the limit in the inequality above. So we have to estimate the limits of $I_1(t)$, $I_2(t)$, and $I_3(t)$, as $t \to 0^+$.

Since for almost all $x \in \Omega'$,

$$g = \mathrm{Lip}u(x) = |\nabla u(x)|,$$

we have

$$
\begin{aligned}
I_1(t) &= \int_{\Omega'} \phi \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f\big(x, g(x)\big)}{t} d\mathrm{vol} \\
&= \int_{\Omega'} \phi \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f(x, |\nabla u(x)|^2)}{t} d\mathrm{vol} \\
&= \int_{\Omega'} \phi \frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f\big(F_t(x), |\nabla u(x)|^2\big)}{t} d\mathrm{vol} \\
&\quad + \int_{\Omega'} \phi \frac{f\big(F_t(x), |\nabla u(x)|^2\big) - f\big(x, |\nabla u(x)|^2\big)}{t} d\mathrm{vol} \\
&:= J_1(t) + J_2(t).
\end{aligned} \tag{5.40}
$$

In order to calculate $\lim_{t\to 0^+} J_1(t)$, we need the following:

Claim: For any $\Omega_1 \Subset \Omega'$, there exists constant $C > 0$ such that

$$\int_{\Omega_1} \Big|\nabla\Big(\frac{u_t - u}{t}\Big)\Big|^2 d\mathrm{vol} \leqslant C$$

for all $t \in (0, t_0)$.

Proof of the Claim. For each $t \in (0, t_0)$, by combining equation (5.37) and semi-concavity of u_t, we have

$$
\begin{aligned}
\mathscr{L}_{\frac{u_t - u}{t}} &\leqslant \Big(\frac{f \circ F_t - f}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\
&= \Big(\frac{f\big(F_t(x), g \circ F_t(x)\big) - f(x, g)}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol}
\end{aligned} \tag{5.41}
$$

in the sense of measure on Ω'. Noting that $\frac{\partial f}{\partial s} \leqslant 0$, and that, for almost all $x \in \Omega'$,

$$g \circ F_t(x) \geqslant \mathrm{Lip}^2 u(F_t(x)) \geqslant |\nabla u_t(x)|^2, \qquad g(x) = |\nabla u(x)|^2,$$

(see Lemma 5.6(i)) we have, for each $t \in (0, t_0)$,

$$
\begin{aligned}
\mathscr{L}_{\frac{u_t - u}{t}} &\leqslant \Big(\frac{f\big(F_t(x), |\nabla u_t(x)|^2\big) - f(x, |\nabla u|^2)}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\
&\leqslant \Big(2\mathbf{Lip} f \cdot \frac{|xF_t(x)| + \big||\nabla u_t|^2 - |\nabla u|^2\big|}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\
&\leqslant \Big(2\mathbf{Lip} f \cdot \frac{\big||\nabla u_t|^2 - |\nabla u|^2\big|}{t} + 2\mathbf{Lip} f \cdot |\nabla u_t| + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\
&\qquad \text{because } |xF_t(x)| = t \cdot |\nabla u_t(x)| \text{ for a.e. } x \in \Omega' \text{(see (5.11))} \\
&\leqslant \Big(C \cdot \frac{\big||\nabla u_t|^2 - |\nabla u|^2\big|}{t} + C\Big) \cdot \mathrm{vol} \\
&\qquad \text{because } |\nabla u_t(x)| \leqslant \mathbf{Lip} u \text{ (see Lemma 5.6(i))}
\end{aligned}
$$

$$= \Big(C\cdot\Big\langle\nabla\Big(\frac{u_t-u}{t}\Big),\nabla(u_t+u)\Big\rangle+C\Big)\cdot\mathrm{vol}$$
$$\leqslant \Big(C\cdot\Big|\nabla\Big(\frac{u_t-u}{t}\Big)\Big|+C\Big)\cdot\mathrm{vol}$$

in the sense of measure on Ω'.

Since $u_t-u\leqslant 0$, according to Caccioppoli inequality, Theorem 7.1 in [**4**] (or by choosing test function $-\varphi(u_t-u)/t$ for some suitable nonnegative cut-off φ on Ω'), for any $\Omega_1\Subset\Omega'$, there exists positive constant C, independent of t, such that

$$\int_{\Omega_1}\Big|\nabla\Big(\frac{u_t-u}{t}\Big)\Big|^2 d\mathrm{vol}\leqslant C\int_{\Omega'}\Big(\frac{u_t-u}{t}\Big)^2 d\mathrm{vol}+C. \tag{5.42}$$

On the other hand, for almost all $x\in\Omega'$, according to Eq. (2.6) in [**29**], we have

$$\frac{|u(x)-u_t(x)|}{t}\leqslant\frac{\mathbf{Lip}^2 u}{2}.$$

Consequently,

$$\int_{\Omega_1}\Big(\frac{u_t-u}{t}\Big)^2 d\mathrm{vol}\leqslant C.$$

The desired estimate follows from the combination of this and (5.42). Now the proof of the claim is finished. q.e.d.

Let us continue the proof of Theorem 5.7.

Let $\Omega_1=\mathrm{supp}\phi\Subset\Omega'$. By combining (5.32), the above **Claim**, and the reflexivity of $W^{1,2}(\Omega)$ (see Theorem 4.48 of [**8**]), we can conclude the following **facts**:

(i) u_t converges (strongly) to u in $W^{1,2}(\Omega_1)$ as $t\to 0^+$;

(ii) there exists some sequence t_j converging to 0^+, such that $(u_{t_j}-u)/t_j$ converges weakly to $-|\nabla u|^2/2$ in $W^{1,2}(\Omega_1)$, as $t_j\to 0^+$.

Let us estimate $J_1(t)$. For each $t\in(0,t_0)$,

$$\begin{aligned}
J_1(t) &= \int_{\Omega'}\phi\frac{f\big(F_t(x),|\nabla u_t(x)|^2\big)-f\big(F_t(x),|\nabla u(x)|^2\big)}{t}d\mathrm{vol}\\
&= \int_{\Omega'}\phi\frac{f\big(F_t(x),|\nabla u_t(x)|^2\big)-f\big(F_t(x),|\nabla u(x)|^2\big)}{|\nabla u_t|^2-|\nabla u|^2}\\
&\qquad\cdot\Big\langle\nabla(u_t+u),\nabla\Big(\frac{u_t-u}{t}\Big)\Big\rangle d\mathrm{vol}\\
&= \int_{\Omega'}\phi\cdot\frac{\partial f}{\partial s}\big(F_t(x),s_t\big)\cdot\Big\langle\nabla(u_t+u),\nabla\Big(\frac{u_t-u}{t}\Big)\Big\rangle d\mathrm{vol}
\end{aligned}$$

for some s_t between $|\nabla u_t(x)|^2$ and $|\nabla u(x)|^2$.

Let t_j be the sequence coming from fact (ii) above. According to Lemma 5.6(ii),

$$\lim_{t_j\to 0^+} |\nabla u_{t_j}(x)| = |\nabla u(x)|$$

for almost all $x \in \Omega'$; combining with the continuity of $\frac{\partial f}{\partial s}$, we get

$$\lim_{t_j\to 0^+} \frac{\partial f}{\partial s}\big(F_{t_j}(x), s_{t_j}\big) = \frac{\partial f}{\partial s}\big(x, |\nabla u(x)|^2\big).$$

On the other hand, by facts (i), (ii) above and the boundedness of

$$\Big|\frac{\partial f}{\partial s}\big(F_t(x), s_t\big)\Big| \leqslant \mathbf{Lip} f,$$

we have

$$\begin{aligned} \lim_{t_j\to 0^+} J_1(t_j) &= \int_{\Omega'} \phi \cdot \frac{\partial f}{\partial s}(x, |\nabla u|^2) \cdot \Big\langle 2\nabla u, \nabla\Big(\frac{-|\nabla u|^2}{2}\Big)\Big\rangle \, d\mathrm{vol} \\ &= -\int_{\Omega'} \phi \cdot \frac{\partial f}{\partial s}(x, |\nabla u|^2) \cdot \big\langle \nabla u, \nabla |\nabla u|^2\big\rangle \, d\mathrm{vol}. \end{aligned} \tag{5.43}$$

Let us calculate the limit $J_2(t_j)$, where the sequence comes from fact (ii) above.

For each $t \in (0, t_0)$, if $x \in \Omega' \backslash \mathcal{N}$ and u_t is differentiable at point x, by Lemma 5.2, we have

$$\begin{aligned} &f(F_t(x), |\nabla u(x)|^2) - f(x, |\nabla u(x)|^2) \\ &\qquad = |xF_t(x)| \Big\langle \nabla_1 f(x, |\nabla u(x)|^2), \uparrow_x^{F_t(x)} \Big\rangle + o(|xF_t(x)|) \\ &\qquad = -t \cdot \big\langle \nabla_1 f(x, |\nabla u(x)|^2), \nabla u_t(x)\big\rangle + o(|xF_t(x)|) \end{aligned}$$

where $\nabla_1 f(x, s)$ means the differential of function $f(\cdot, s)$ at point x (see equation (2.16)). For the sequence t_j, the combination of this, equation (5.11), and Lemma 5.6(ii),

$$\lim_{t_j\to 0^+} \nabla u_{t_j}(x) = \nabla u(x),$$

implies that

$$\begin{aligned} \lim_{t_j\to 0^+} &\frac{f(F_{t_j}(x), |\nabla u(x)|^2) - f(x, |\nabla u(x)|^2)}{t_j} \\ &= -\big\langle \nabla_1 f(x, |\nabla u(x)|^2), \nabla u(x)\big\rangle \end{aligned}$$

for almost all $x \in \Omega'$. Note that

$$\Big|\frac{f(F_{t_j}(x), |\nabla u(x)|^2) - f(x, |\nabla u(x)|^2)}{t_j}\Big| \leqslant \mathbf{Lip} f \cdot \frac{|xF_{t_j}(x)|}{t_j} \leqslant \mathbf{Lip} f \cdot \mathbf{Lip} u$$

for almost everywhere in Ω'. Dominated Convergence Theorem concludes that

$$\lim_{t_j\to 0^+} J_2(t_j) = \lim_{t_j\to 0^+}\int_{\Omega'}\phi\frac{f(F_{t_j}(x),|\nabla u(x)|^2)-f(x,|\nabla u(x)|^2)}{t_j}d\mathrm{vol}$$
$$= -\int_{\Omega'}\phi\left\langle\nabla_1 f(x,|\nabla u(x)|^2),\nabla u(x)\right\rangle d\mathrm{vol}.$$

By combining with equations (5.40) and (5.43), we have

(5.44)
$$\lim_{t_j\to 0^+} I_1(t_j) \leqslant \lim_{t_j\to 0^+} J_1(t_j) + \lim_{t_j\to 0^+} J_2(t_j)$$
$$= -\int_{\Omega'}\phi\cdot\left\langle\nabla u,\frac{\partial f}{\partial s}(x,|\nabla u|^2)\cdot\nabla|\nabla u|^2+\nabla_1 f(x,|\nabla u(x)|^2)\right\rangle d\mathrm{vol}$$
$$= -\int_{\Omega'}\phi\cdot\left\langle\nabla u,\nabla f(x,|\nabla u|^2)\right\rangle d\mathrm{vol}.$$

Let us calculate $\lim_{t_j\to 0} I_2(t_j)$ for the sequence $t_j\to 0^+$ coming from fact (ii) above.

From Lemma 5.6(ii),

$$\lim_{t_j\to 0^+}|\nabla u_{t_j}(x)|^2 = |\nabla u(x)|^2 = g(x)$$

at almost all $x\in\Omega'$. Combining with the Lipschitz continuity of $f(x,s)$ and $\mathcal{A}\geqslant n/2$ for sufficiently small t, we have

$$\lim_{t_j\to 0^+}\frac{f^2\big(F_{t_j}(x),|\nabla u_{t_j}|^2\big)-f^2(x,g(x))}{\mathcal{A}} = 0$$

at almost all $x\in\Omega'$. On the other hand, using that $\mathcal{A}\geqslant n/2$ again (when t is sufficiently small) and that f is bounded, we have

$$\Big|\frac{f^2\big(F_{t_j}(x),|\nabla u_{t_j}|^2\big)-f^2(x,g(x))}{\mathcal{A}}\Big|\leqslant C,$$
$$\text{for almost all }\ x\in\Omega',\quad j=1,2,\ldots,$$

for some constant C. Dominated Convergence Theorem concludes that

(5.45)
$$\lim_{t_j\to 0^+} I_2(t_j) = \lim_{t_j\to 0^+}\int_{\Omega'}\frac{-f^2\big(F_{t_j}(x),|\nabla u_{t_j}|^2\big)+f^2\big(x,g(x)\big)}{\mathcal{A}}d\mathrm{vol} = 0.$$

Let us calculate $\lim_{t_j\to 0} I_3(t_j)$ for the sequence t_j coming from fact (ii) above.

According to Lemma 5.6 (i) and (ii), we get

$$|\nabla u_{t_j}|\leqslant \mathbf{Lip}u \quad\text{and}\quad \lim_{t_j\to 0^+}|\nabla u_{t_j}| = |\nabla u|.$$

By combining with the boundedness of D and f, and applying Dominated Convergence Theorem, we conclude that

$$\begin{aligned}\lim_{t_j\to 0^+} I_3(t_j) &= \int_{\Omega'} \phi \frac{f^2 - nK|\nabla u|^2}{\mathcal{A}} d\mathrm{vol}\\ &= \int_{\Omega'} \phi \frac{f^2\big(x, g(x)\big) - nK|\nabla u|^2}{n} d\mathrm{vol}.\end{aligned}$$

By the fact that

$$g(x) = \mathrm{Lip}u = |\nabla u|$$

for almost everywhere in Ω', we get

$$\lim_{t_j\to 0^+} I_3(t_j) = \int_{\Omega'} \phi\Big(\frac{f^2(x, |\nabla u|^2)}{n} - K|\nabla u|^2\Big) d\mathrm{vol}. \tag{5.46}$$

By applying the above **Claim** again,

$$\frac{u_{t_j} - u}{t_j} \longrightarrow -\frac{|\nabla u|^2}{2} \quad \text{weakly in } \; W^{1,2}(\Omega_1),$$

as $t_j \to 0$. By combining with the definition of $\mathscr{L}_{u_{t_j}-u}$ (see the first paragraph of Section 4.1), we have

$$\begin{aligned}\lim_{t_j\to 0^+} \frac{1}{t_j}\mathscr{L}_{u_{t_j}-u}(\phi) &= -\lim_{t_j\to 0^+} \int_{\Omega'} \Big\langle \nabla\phi, \nabla\Big(\frac{u_{t_j} - u}{t_j}\Big)\Big\rangle\\ &= \frac{1}{2}\int_{\Omega'} \big\langle \nabla\phi, \nabla|\nabla u|^2\big\rangle\, d\mathrm{vol}.\end{aligned} \tag{5.47}$$

The combination of equations (5.39) and (5.44)–(5.47) shows that, for any $\phi \in Lip_0(\Omega')$,

$$\begin{aligned}&\frac{1}{2}\int_{\Omega'} \big\langle \nabla\phi, \nabla|\nabla u|^2\big\rangle\, d\mathrm{vol}\\ &\quad\leqslant -\int_{\Omega'} \phi\Big(\big\langle \nabla u, \nabla f(x, |\nabla u|^2)\big\rangle + \frac{f^2(x, |\nabla u|^2)}{n} - K|\nabla u|^2\Big) d\mathrm{vol}.\end{aligned}$$

The desired result follows from this and the definition of $\mathscr{L}_{|\nabla u|^2}$. Now the proof of Theorem 5.7 is completed. q.e.d.

If $f(x, s) = f(x)$, then we can remove the technical condition that $|\nabla u|$ is lower semi-continuous at almost everywhere in Ω. That is,

Corollary 5.8. *Let M be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$, and let Ω be a domain in M. Assume function $f \in Lip(\Omega)$ and $u \in W^{1,2}(\Omega)$ satisfying*

$$\mathscr{L}_u = f \cdot \mathrm{vol}.$$

Then we have $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega)$ and $|\nabla u|$ is lower semi-continuous on Ω. Consequently, we get

$$\mathscr{L}_{|\nabla u|^2} \geqslant 2\Big(\frac{f^2}{n} + \langle \nabla u, \nabla f\rangle - K|\nabla u|^2\Big) d\mathrm{vol}$$

in the sense of measure on Ω.

Proof. At first, by Corollary 5.5, we conclude that $u \in Lip_{loc}(\Omega)$. Fix any $\Omega^* \Subset \Omega$. Then $u \in Lip(\Omega^*)$ and $f(x,s) = f(x)$ satisfies the conditions (a), (b) on Ω^* in Theorem 5.7.

Let us recall that in the proof of Theorem 5.7, the technique condition that $|\nabla u|$ is lower semi-continuous (with $\frac{\partial f}{\partial s} \leqslant 0$) is only used to ensure the upper semi-continuity of $f = f\big(x, g(x)\big)$ on Ω^* so that Proposition 5.3 is applicable. Now, since $f(x)$ is Lipschitz, Proposition 5.3 still works for equation

$$\mathscr{L}_u = f \cdot \mathrm{vol}.$$

Using the same notations as in the above proof (with $f(x,s) = f(x)$) of Theorem 5.7, we get the corresponding equation

$$\mathscr{L}_{\frac{u_t - u}{t}} \leqslant \Big(\frac{f \circ F_t - f}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} = \Big(\frac{f\big(F_t(x)\big) - f(x)}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol}$$

in the sense of measure on any $\Omega' \Subset \Omega^*$ (see equation (5.41) in the proof of the **Claim** above). Then we get, by (5.11), $|xF_t(x)| = t|\nabla u_t(x)|$ at almost all $x \in \Omega^*$,

$$\begin{aligned} \mathscr{L}_{\frac{u_t - u}{t}} &\leqslant \Big(\mathbf{Lip} f \frac{|xF_t(x)|}{t} + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\ &= \Big(\mathbf{Lip} f \cdot |\nabla u_t| + K|\nabla u_t|^2\Big) \cdot \mathrm{vol} \\ &\leqslant C \cdot \mathrm{vol} \qquad (\text{because} \quad |\nabla u_t| \leqslant \mathbf{Lip} u) \end{aligned} \tag{5.48}$$

in the sense of measure on Ω'. Here and in the following, C denotes various positive constants independent of t.

Using the same argument as in the proof of **Claim** above, we obtain that the $W^{1,2}$-norm of $\frac{u_t - u}{t}$ is uniformly bounded on any $\Omega_1 \Subset \Omega'$. Hence there exists a sequence $t_j \to 0^+$ such that

$$\frac{u_{t_j} - u}{t_j} \longrightarrow -\frac{|\nabla u|^2}{2} \quad \text{weakly in} \quad W^{1,2}(\Omega_1),$$

as $t_j \to 0^+$. Combining with (5.48), we have $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega_1)$ and

$$\mathscr{L}_{|\nabla u|^2} \geqslant -2C \cdot \mathrm{vol}$$

in the sense of measure on Ω_1.

By setting

$$w = |\nabla u|^2 + 2C,$$

we have $w \geqslant 2C$ and

$$\mathscr{L}_w \geqslant -2C \cdot \mathrm{vol} \geqslant -w \cdot \mathrm{vol}.$$

Consider the product space $M \times \mathbb{R}$ (with direct product metric) and the function $v(x,t) : \Omega' \times \mathbb{R} \to \mathbb{R}$ as

$$v(x,t) := w(x) \cdot e^t.$$

Then v satisfies $\mathscr{L}_v \geqslant 0$ in $\Omega_1 \times \mathbb{R}$. Hence it has a lower semi-continuous representative (see Theorem 5.1 in [**16**]). Therefore, w is lower semi-continuous on Ω_1. So is $|\nabla u|$.

Because of the arbitrariness of $\Omega_1 \Subset \Omega' \Subset \Omega^* \Subset \Omega$, we obtain that $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega)$ and $|\nabla u|$ is lower semi-continuous on Ω.

It is easy to check that $f(x,s) = f(x)$ satisfies the conditions (a), (b) on Ω in Theorem 5.7 (since f is Lipschitz and $\partial f/\partial s = 0$). We can apply Theorem 5.7 to equation

$$\mathscr{L}_u = f \cdot \mathrm{vol}$$

and conclude the last assertion of the corollary. q.e.d.

As a direct application of the Bochner-type formula, we have the following Lichnerowicz estimate, which was earlier obtained by Lott–Villani in [**29**] by a different method. Further applications have been given in [**42**].

Corollary 5.9. *Let M be an n-dimensional Alexandrov space with Ricci curvature bounded below by a positive constant $n-1$. Then we have*

$$\int_M |\nabla u|^2 d\mathrm{vol} \geqslant n \int_M u^2 d\mathrm{vol}$$

for all $u \in W^{1,2}(M)$ with $\int_M u d\mathrm{vol} = 0$.

Proof. Let u be a first eigenfunction and λ_1 be the first eigenvalue. It is clear that $\lambda_1 \geqslant 0$ and $u(x)e^{\sqrt{\lambda_1}t}$ is a harmonic function on $M \times \mathbb{R}$. According to Corollary 5.5, we know that u is locally Lipschitz continuous.

(Generalized) Bonnet–Myers' theorem implies that M is compact (see Corollary 2.6 in [**45**]). By using the Bochner-type formula Corollary 5.8 for equation

$$\mathscr{L}_u = -\lambda_1 u,$$

and choosing test function $\phi = 1$, we get the desired estimate immediately. q.e.d.

6. Gradient estimates for harmonic functions

Let Ω be a bounded domain of an n-dimensional Alexandrov space with Ricci curvature $\geqslant -K$ and $K \geqslant 0$.

In this section, we always assume that u is a positive harmonic function on Ω. According to Corollary 5.8, we know that $|\nabla u|$ is lower semi-continuous in Ω and $|\nabla u|^2 \in W^{1,2}_{loc}(\Omega)$.

Remark 6.1. In the previous version of this paper, by using some complicated pointwise C^1-estimate of the elliptic equation (see, for example, [**7**, **27**]), we can actually show that $|\nabla u|$ is continuous in almost all Ω. Nevertheless, in this new version, we avoid using this continuity result.

Now, let us prove the following estimate.

Lemma 6.2. *Let M be an n-dimensional Alexandrov space with Ricci curvature $\geqslant -K$ and $K \geqslant 0$. Suppose that u is a positive harmonic function in $B_p(2R)$. Then we have*

$$\|Q\|_{L^s(B_p(R))} \leqslant \Big(2nK + \frac{8ns}{R^2}\Big) \cdot \Big(\mathrm{vol}\big(B_p(2R)\big)\Big)^{1/s} \tag{6.1}$$

for $s \geqslant 2n+4$, where $Q = |\nabla \log u|^2$.

Proof. Since $u > 0$ in $B_p(2R)$, setting $v = \log u$, we have

$$\mathscr{L}_v = -|\nabla v|^2 \cdot \mathrm{vol} = -Q \cdot \mathrm{vol}.$$

For simplicity, we denote $B_p(2R)$ by B_{2R}.

Let $\psi(x)$ be a nonnegative Lipschitz function with support in B_{2R}. By choosing test function $\psi^{2s}Q^{s-2}$ and using the Bochner-type formula (5.36) to v (with function $f(x,s) = -s$, which satisfies the conditions (a) and (b) in Theorem 5.7), we get

$$\begin{aligned} -\int_{B_{2R}} \big\langle \nabla Q, \nabla(\psi^{2s}Q^{s-2})\big\rangle\, d\mathrm{vol} \\ \geqslant \frac{2}{n}\int_{B_{2R}} \psi^{2s}Q^s d\mathrm{vol} - 2\int_{B_{2R}} \psi^{2s}Q^{s-2}\langle \nabla v, \nabla Q\rangle\, d\mathrm{vol} \\ - 2K\int_{B_{2R}} \psi^{2s}Q^{s-1} d\mathrm{vol}. \end{aligned} \tag{6.2}$$

Hence we have

$$\begin{aligned} &\frac{2}{n}\int_{B_{2R}} \psi^{2s}Q^s d\mathrm{vol} - 2K\int_{B_{2R}} \psi^{2s}Q^{s-1} d\mathrm{vol} \\ &\leqslant -2s\int_{B_{2R}} \psi^{2s-1}Q^{s-2}\langle \nabla Q, \nabla \psi\rangle\, d\mathrm{vol} \\ &\quad -(s-2)\int_{B_{2R}} \psi^{2s}Q^{s-3}|\nabla Q|^2 d\mathrm{vol} + 2\int_{B_{2R}} \psi^{2s}Q^{s-2}\langle \nabla v, \nabla Q\rangle\, d\mathrm{vol} \\ &= s\cdot I_1 - (s-2)\cdot I_2 + I_3. \end{aligned} \tag{6.3}$$

We now estimate I_1, I_2, and I_3. By Cauchy–Schwarz inequality, we have

$$I_1 \leqslant \frac{1}{2}\int_{B_{2R}} \psi^{2s}Q^{s-3}|\nabla Q|^2 d\mathrm{vol} + 2\int_{B_{2R}} Q^{s-1}\psi^{2s-2}|\nabla\psi|^2 d\mathrm{vol}$$

and

$$I_3 \leqslant n\int_{B_{2R}} \psi^{2s}Q^{s-3}|\nabla Q|^2 d\mathrm{vol} + \frac{1}{n}\int_{B_{2R}} \psi^{2s}Q^s d\mathrm{vol}.$$

By combining with (6.3), we obtain

$$\frac{1}{n}\int_{B_{2R}}\psi^{2s}Q^s d\mathrm{vol} - 2K\int_{B_{2R}}\psi^{2s}Q^{s-1}d\mathrm{vol}$$
$$\leqslant \Big(\frac{s}{2}-(s-2)+n\Big)\cdot I_2 + 2s\int_{B_{2R}}Q^{s-1}\psi^{2s-2}|\nabla\psi|^2 d\mathrm{vol}.$$

If we choose $s \geqslant 2n+4$, then we can drop the first term in RHS.

Set

$$\tau = \Big(\int_{B_{2R}}\psi^{2s}Q^s d\mathrm{vol}\Big)^{\frac{1}{s}}.$$

Then by $K \geqslant 0$ and Hölder inequality, we have

$$\frac{\tau^s}{n} \leqslant 2K\Big(\int_{B_{2R}}\psi^{2s}d\mathrm{vol}\Big)^{1/s}\cdot\tau^{s-1} + 2s\Big(\int_{B_{2R}}|\nabla\psi|^{2s}d\mathrm{vol}\Big)^{1/s}\cdot\tau^{s-1}.$$

Therefore, when we choose ψ such that $\psi = 1$ on B_R, $\psi = 0$ outside B_{2R}, and $|\nabla\psi| \leqslant 2/R$, we get the desired estimate (6.1). q.e.d.

Corollary 6.3. *Let u be a positive harmonic function on an n-dimensional complete noncompact Alexandrov space with Ricci curvature $\geqslant -K$ and $K \geqslant 0$. Then we have*

$$|\nabla \log u| \leqslant C_{n,K}.$$

Proof. Without loss of generality, we may assume $K > 0$. From Lemma 6.2 above, setting $s = R^2$ for R large enough, we have

$$\big\||\nabla\log u|^2\big\|_{L^{R^2}(B_p(R))} \leqslant \Big(2nK + 8n\Big)\cdot\Big(\mathrm{vol}\big(B_p(2R)\big)\Big)^{\frac{1}{R^2}}.$$

According to Bishop–Gromov volume comparison theorem (see [**22**] or [**45**]), we have

$$\mathrm{vol}\big(B_p(2R)\big) \leqslant H^n\big(B_o(2R)\subset \mathbb{M}^n_{K/(n-1)}\big) \leqslant C_1e^{C_2R},$$

where both constants C_1 and C_2 depend only on n and K. Combining the above two inequalities, we get

$$\big\||\nabla\log u|^2\big\|_{L^{R^2}(B_p(R))} \leqslant C_{n,K}\cdot C_1^{1/R^2}e^{C_2/R}.$$

Letting $R\to\infty$, we obtain the desired result. q.e.d.

In order to get a local L^∞ estimate of $|\nabla\log u|$, let us recall the local version of Sobolev inequality.

Let $D = D(\Omega)$ be a doubling constant on Ω; i.e., we have

$$\mathrm{vol}(B_p(2R)) \leqslant 2^D\cdot\mathrm{vol}(B_p(R))$$

for all balls $B_p(2R)\Subset\Omega$. According to Bishop–Gromov volume comparison, it is known that if M has nonnegative Ricci curvature, the constant D can be chosen $D = n$. For the general case, if M has $Ric \geqslant -K$ for some $K \geqslant 0$, then the constant can be chosen to depend on n and

$\sqrt{K}\cdot\mathrm{diam}(\Omega)$, where $\mathrm{diam}(\Omega)$ is the diameter of Ω. Here and in the following, without loss of generality, we always assume that the doubling constant $D \geqslant 3$.

Let $C_P = C_P(\Omega)$ be a (weak) Poincaré constant on Ω; i.e., we have

$$\int_{B_p(R)} |\varphi - \varphi_{p,R}|^2 d\mathrm{vol} \leqslant C_P \cdot R^2 \cdot \int_{B_p(2R)} |\nabla\varphi|^2 d\mathrm{vol}$$

for all balls $B_p(2R) \Subset \Omega$ and $\varphi \in W^{1,2}(\Omega)$, where $\varphi_{p,R} = ⨍_{B_p(R)} \varphi d\mathrm{vol}$. By Bishop–Gromov volume comparison and Cheeger–Colding's segment inequality, it is known that if M has nonnegative Ricci curvature, the constant C_P can be chosen to depend only on n. For the general case, if M has $Ric \geqslant -K$ for some $K \geqslant 0$, then the constant can be chosen to depend on n and $\sqrt{K}\cdot\mathrm{diam}(\Omega)$.

It is well known that the doubling property and a Poincaré inequality imply a Sobolev inequality in length spaces (see, for example, [**44**, **47**, **14**]). Explicitly, there exists a constant $C_S = C_S(\Omega)$, which depends only on D and C_P, such that

$$\Big(\int_{B_p(R)} |\varphi|^{\frac{2D}{D-2}} d\mathrm{vol}\Big)^{\frac{D-2}{D}} \leqslant C_S \cdot \frac{R^2}{\mathrm{vol}(B_p(R))^{2/D}} \cdot \int_{B_p(2R)} \big(|\nabla\varphi|^2 + R^{-2}\cdot\varphi^2\big) d\mathrm{vol} \tag{6.4}$$

for all balls $B_p(2R) \Subset \Omega$ and $\varphi \in W_0^{1,2}(\Omega)$.

Now by combining Lemma 6.2 and the standard Nash–Moser iteration method, we can get the following local estimate.

Theorem 6.4. *Let M be an n-dimensional Alexandrov space with $Ric \geqslant -K$, for some $K \geqslant 0$. Suppose that $\Omega \subset M$ is a bounded open domain. Then there exists a constant $C = C\big(n, \sqrt{K}\mathrm{diam}(\Omega)\big)$ such that*

$$\max_{x\in B_p(R/2)} |\nabla \log u| \leqslant C \cdot (\sqrt{K} + \frac{1}{R})$$

for every positive harmonic function u on Ω and $B_p(2R) \Subset \Omega$.

If $K = 0$, the constant $C = C(n)$.

Proof. Let v and Q be as in Lemma 6.2. We choose test function $\psi^2 Q^{s-1}$, where ψ has support in ball $B_R := B_p(R)$. By using the Bochner-type formula (5.36) for function v (with function $f(x,s) = -s$), we have

$$\begin{aligned}(6.5)\qquad \frac{2}{n}\int_{B_R}\psi^2 Q^{s+1}d\mathrm{vol} \leqslant\ & 2\int_{B_R}\psi^2 Q^{s-1}\left\langle\nabla v,\nabla Q\right\rangle d\mathrm{vol}\\ &-2\int_{B_R}\psi Q^{s-1}\left\langle\nabla\psi,\nabla Q\right\rangle d\mathrm{vol}\\ &-(s-1)\int_{B_R}\psi^2 Q^{s-2}|\nabla Q|^2 d\mathrm{vol}+2K\int_{B_R}\psi^2 Q^s d\mathrm{vol}.\end{aligned}$$

Note that

$$\begin{aligned}2\int_{B_R}\psi^2 Q^{s-1}\left\langle\nabla v,\nabla Q\right\rangle d\mathrm{vol} \leqslant\ & \frac{n}{2}\int_{B_R}\psi^2 Q^{s-2}|\nabla Q|^2 d\mathrm{vol}\\ &+\frac{2}{n}\int_{B_R}\psi^2 Q^s|\nabla v|^2 d\mathrm{vol}\end{aligned}$$

and

$$-2\int_{B_R}\psi Q^{s-1}\left\langle\nabla\psi,\nabla Q\right\rangle d\mathrm{vol}\leqslant\int_{B_R}\psi^2 Q^{s-2}|\nabla Q|^2 d\mathrm{vol}+\int_{B_R}Q^s|\nabla\psi|^2 d\mathrm{vol}.$$

By combining with (6.5), we get

(6.6)

$$(s-2-\frac{n}{2})\int_{B_R}\psi^2 Q^{s-2}|\nabla Q|^2 d\mathrm{vol}\leqslant\int_{B_R}Q^s|\nabla\psi|^2 d\mathrm{vol}+2K\int_{B_R}\psi^2 Q^s d\mathrm{vol}.$$

Taking $s\geqslant 2n+4$, then, $s-2-n/2\geqslant s/2$. Let $\frac{R}{2}\leqslant\rho<\rho'\leqslant R$. Choose ψ such that $\psi(x)=1$ if $x\in B_p(\rho)$, $\psi(x)=0$ if $x\notin B_p(\rho')$ and $|\nabla\psi|\leqslant 2/(\rho'-\rho)$. Then by (6.4) and (6.6), we have

$$\left(\int_{B_p(\rho)}Q^{s\theta}d\mathrm{vol}\right)^{1/\theta}\leqslant\Big(\mathscr{A}\cdot(2sK+\frac{1}{R^2}+\frac{8s}{(\rho'-\rho)^2})\Big)\cdot\int_{B_p(\rho')}Q^s d\mathrm{vol},$$

where $\theta=D/(D-2)$ and

$$(6.7)\qquad \mathscr{A}=C_S\cdot\frac{R^2}{\mathrm{vol}(B_p(R))^{2/D}}.$$

Let l_0 be an integer such that $\theta^{l_0}\geqslant 2n+4$. Taking $s_l=\theta^l$, $\rho_l=R(1/2+1/2^l)$ with $l\geqslant l_0$, we have

$$\log J_{l+1}-\log J_l\leqslant\frac{1}{\theta^l}\cdot\log\left(\mathscr{A}\cdot(2\theta^l K+\frac{1}{R^2}+\frac{2\cdot\theta^l\cdot 4^{l+2}}{R^2})\right),$$

where

$$J_l=\Big(\int_{B_p(\rho_l)}Q^{s_l}d\mathrm{vol}\Big)^{1/s_l}=\|Q\|_{L^{\theta^l}(B_p(\rho_l))}.$$

Hence, we have

$$\begin{aligned}\log J_\infty - \log J_{l_0} &\leqslant \log \mathscr{A} \cdot \sum_{l=l_0}^{\infty} \theta^{-l} + \sum_{l=l_0}^{\infty} \theta^{-l} \cdot \log(2\theta^l K + \frac{33(4\theta)^l}{R^2}) \\ &\leqslant \theta^{-l_0} \cdot \log \mathscr{A}^{D/2} + \sum_{l=l_0}^{\infty} \theta^{-l} \cdot \big(l \cdot \log(4\theta) + \log(K + \frac{33}{R^2})\big).\end{aligned}$$

On the other hand, by Lemma 6.2, we have

$$\log J_{l_0} \leqslant \log(2nK + \frac{8n\theta^{l_0}}{R^2}) + \theta^{-l_0} \log \operatorname{vol}(B_p(2R)).$$

Hence, we obtain

(6.8)

$$\begin{aligned}\log J_\infty \leqslant \log(2nK + \frac{8n\theta^{l_0}}{R^2}) + \theta^{-l_0}\Big(\log \operatorname{vol}(B_p(2R)) + \log \mathscr{A}^{D/2}\Big) \\ + \log(4\theta) \cdot \sum_{l=l_0}^{\infty} l \cdot \theta^{-l} + \log(K + \frac{33}{R^2}) \sum_{l=l_0}^{\infty} \theta^{-l}.\end{aligned}$$

From (6.7) and (6.8), we have

$$\begin{aligned}\log J_\infty &\leqslant \log(2nK + \frac{8n\theta^{l_0}}{R^2}) + \frac{D}{2}\theta^{-l_0} \log\Big(4C_S R^2\Big) \\ &\quad + \log(4\theta) \cdot \sum_{l=l_0}^{\infty} l \cdot \theta^{-l} + \log(K + \frac{33}{R^2}) \sum_{l=l_0}^{\infty} \theta^{-l} \\ &\leqslant \log(2nK + \frac{8n\theta^{l_0}}{R^2}) + \frac{D}{2}\theta^{-l_0} \log\Big(4C_S(KR^2 + 33)\Big) + C(\theta, l_0).\end{aligned}$$

Taking l_0 such that $\theta^{l_0} \leqslant 8n$, we get

$$\log J_\infty \leqslant \log(2nK + \frac{64n^2}{R^2}) + C(n, \sqrt{K}\operatorname{diam}(\Omega)).$$

This gives the desired result. q.e.d.

The gradient estimate shows that any sublinear growth harmonic function on an Alexandrdov space with nonnegative Ricci curvature must be a constant. Explicitly, we have the following:

Corollary 6.5. *Let M be an n-dimensional complete non-compact Alexandrov space with nonnegative Ricci curvature. Assume that u is a harmonic function on M. If*

$$\lim_{r\to\infty} \frac{\sup_{x\in B_p(r)} |u(x)|}{r} = 0$$

for some $p \in M$, then u is a constant.

Proof. Clearly, for any $q \in M$, we still have

$$\lim_{r\to\infty} \frac{\sup_{x\in B_q(r)} |u(x)|}{r} = 0.$$

Let $\overline{u_r} = \sup_{x\in B_q(r)} |u(x)|$. Then $2\overline{u_r} - u$ is a positive harmonic on $B_q(r)$, unless u is identically zero. By Theorem 6.4, we have

$$|\nabla u(q)| \leqslant C(n) \frac{\sup_{x\in B_q(r)}(2\overline{u_r} - u)}{r} \leqslant C(n)\frac{3\overline{u_r}}{r}.$$

Letting $r \to \infty$, we get $|\nabla u(q)| = 0$. This completes the proof. q.e.d.

As another application of the gradient estimate, we have the following mean value property, by using Cheeger–Colding–Minicozzi's argument in [**9**]. In the smooth case, it was first proved by Peter Li in [**26**] via a parabolic method.

Corollary 6.6. *Let M be an n-dimensional complete non-compact Alexandrov space with nonnegative Ricci curvature. Suppose that u is a bounded superharmonic function on M. Then*

$$\lim_{r\to\infty} \fint_{\partial B_p(r)} u d\mathrm{vol} = \inf u.$$

Proof. Without loss of generality, we can assume that $\inf u = 0$.

Fix any $\epsilon > 0$. Then there exists some $R(\epsilon)$ such that $\inf_{B_p(R)} u < \epsilon$ for all $R > R(\epsilon)$. For any $R > 4R(\epsilon)$, we consider the harmonic function h_R on $B_p(R)$ with boundary value $h_R = u$ on $\partial B_p(R)$. By maximum principle and the gradient estimate of h_R, we have

$$\sup_{B_p(R/2)} h_R \leqslant C(n) \cdot \inf_{B_p(R/2)} h_R < C(n) \cdot \epsilon.$$

On the other hand, from the monotonicity of $r^{1-n} \cdot \int_{\partial B_p(r)} h_R d\mathrm{vol}$ on $(0, R)$ (see the proof of Proposition 4.4), we have

$$\int_{\partial B_p(R)} h_R d\mathrm{vol} \leqslant C(n) \int_{\partial B_p(R/2)} h_R d\mathrm{vol}.$$

Then we get

$$\int_{\partial B_p(R)} u d\mathrm{vol} = \int_{\partial B_p(R)} h_R d\mathrm{vol} \leqslant C(n) \cdot \epsilon \cdot \mathrm{vol}(\partial B_p(R/2)).$$

Therefore, the desired result follows from Bishop–Gromov volume comparison and the arbitrariness of ϵ. q.e.d.

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