THE WEIL-PETERSSON HESSIAN OF LENGTH ON TEICHMÜLLER SPACE

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Abstract

We present a brief but nearly self-contained proof of a formula for the Weil-Petersson Hessian of the geodesic length of a closed curve (either simple or not simple) on a hyperbolic surface. The formula is the sum of the integrals of two naturally defined positive functions over the geodesic, proving convexity of this functional over Teichmuller space (due to Wolpert (1987)). We then estimate this Hessian from below in terms of local quantities and distance along the geodesic. The formula extends to proper arcs on punctured hyperbolic surfaces, and the estimate to laminations. Wolpert's result that the Thurston metric is a multiple of the Weil-Petersson metric directly follows on taking a limit of the formula over an appropriate sequence of curves. We give further applications to upper bounds of the Hessian, especially near pinching loci, recover through a geometric argument Wolpert's result on the convexity of length to the half-power, and give a lower bound for growth of length in terms of twist.

1. Introduction

One of the foundations of modern Teichmüller theory is Wolpert's [**Wol87**] theorem that the function on Teichmüller space that records the geodesic length of a closed curve is convex with respect to the Weil-Petersson metric. That paper provided a lower bound for the Hessian of the length function; our purpose here is to present a brief derivation of a concise formula for the Hessian in terms of natural objects on the surface associated to the curve and the tangent vectors to Teichmüller space.

To explain this result and add some context, we fix some terminology and notation. Let S be a smooth closed surface of genus g, and let $\mathcal{T}(S)$ be the Teichmüller space of (isotopy classes of) marked hyperbolic structures on S. Let $[\gamma]$ be a free homotopy class of closed curves, not necessarily simple. Typically γ will denote the representative of $[\gamma]$ that is geodesic with respect to a given metric g, with the context making it

clear whether γ is the immersed curve on the surface or an immersion from a circle into the surface.

For each hyperbolic surface (S, g), there is a geodesic representative $\gamma = \gamma_g$ of $[\gamma]$. By the uniformization theorem, as each point in $\mathcal{T}(S)$ is represented by a unique hyperbolic metric, say g, the length of the geodesic representative γ of $[\gamma]$ defines a function $\ell = \ell_\gamma = \ell_\gamma([g])$ on $\mathcal{T}(S)$. We investigate the second derivative of that function.

The Hessian of a function is well-defined once there is a background metric. There are many metrics on $\mathcal{T}(S)$; among the more basic is the Weil-Petersson metric. Representing the tangent space to Teichmüller space as the space of Beltrami differentials which are harmonic with respect to the hyperbolic metric g representing a point in $\mathcal{T}(S)$, the Weil-Petersson metric is the L^2 metric on that space with respect to the hyperbolic area form dA_g.

The first goal of this paper is to write a formula for the Weil-Petersson Hessian of the function.

1.1. Statement of the formula. In this subsection, we describe our formula in the simplest case; we discuss the situation of the second derivative of the length function ℓ of a closed curve, with the derivative taken along a Weil-Petersson geodesic Γ. Even in this case, we require some notation.

Let $\Gamma = \Gamma(t)$ a Weil-Petersson geodesic arc; the class $[\gamma]$ is represented by the $\Gamma(t)$-geodesic γ_t. The tangent vector to Teichmüller space at $\Gamma(0)$ is given by a harmonic Beltrami differential, say $\mu = \frac{\bar{\Phi}}{g_0}$.

We can extend $\Gamma(0)$-Fermi coordinates along the curve γ_0 to complex coordinates in a neighborhood of (a subarc of) γ_0. In terms of those coordinates, the quantity $-\frac{\operatorname{Im}\Phi}{g_0} = \operatorname{Im}\mu$ is well-defined. Let U^Φ denote the solution to the ordinary differential equation

$$(1.1) \qquad U_{yy} - U = -\frac{\operatorname{Im}\Phi}{g_0},$$

where here the geodesic is represented by a vertical line in the Fermi coordinate patch with a parametrization given by arclength.

This is enough terminology so that we may state our main result as

Theorem 1.1. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the length $\ell(t) = L(\Gamma(t), [\gamma])$ is given by*

(1.2)
$$\frac{d^2}{dt^2}\ell(t) = \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}ds + \int_{\gamma_0} [U_y^\Phi]^2 + [U^\Phi]^2 ds$$
$$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}ds$$
$$+ \frac{1}{2\sinh(\frac{\ell}{2})}\iint_{\gamma_0 \times \gamma_0} \operatorname{Im}\mu(p)[\cosh(d(p,q) - \frac{\ell}{2})]\operatorname{Im}\mu(q)ds(p)ds(q).$$

While this paper was under review, Axelsson and Schumacher [**AS**] posted an analogous formula for the second variation of geodesic length on a holomorphically varying family of hyperbolic surfaces.

1.2. Applications and extensions. The formula (1.2) lends itself to applications. Using well-known techniques for solving and estimating solutions of the relevant differential equations, we obtain results in a number of different directions.

Of course, it is immediately apparent that this Hessian is positive definite, even for curves which are not simple. The first summand is written in terms of $2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$, a ubiquitous term in Teichmüller theory whose definition involves the global geometry of (S, g). In Lemma 5.1, we find a locally defined lower bound for this quantity. Thus, when combined with the second expression for the second term, we obtain an estimate defined only in terms of local quantities or distance along the geodesic.

In addition to providing a means to estimate the Hessian from below, we also obtain an improvement on the basic convexity result: from an easily derived formula for the gradient of length, we will see (Corollary 8.1) immediately that $\ell^{\frac{2}{3}}$ is convex on all of Teichmüller space. A recent theorem [**Wol08**] of Wolpert is that this may be improved to $\ell^{\frac{1}{2}}$ being convex on all of Teichmüller space; we provide a proof for that as well. The proof is geometric in the sense that it hinges on a comparison of two harmonic diffeomorphisms of a cylinder.

It is straightforward to extend our derivations both to laminations and to proper geodesic arcs which connect cusps on a hyperbolic cusped surface. The explicit nature of formula (1.2) also lends itself to estimates from above, leading to estimates on the Weil-Petersson connection near the pinching locus. These were also recently obtained (and announced some time ago) by Wolpert [**Wol08**]. We also give a bound from below on the Hessian of length in terms of the infinitesimal twist.

Our final application is a new proof of Wolpert's [**Wol86b**] proof that the Thurston metric is (a multiple of) the Weil-Petersson metric. We consider a sequence $\{\gamma_n\}$ of curves whose geodesic representatives are becoming equidistributed in the unit tangent bundle $T^1(S, g)$. Thurston observed that $\lim_{n \to \infty} \text{Hess}\ell_{\gamma_n}$ would be a positive definite quadratic form on $T_{(S,g)}\mathcal{T}(S)$: by taking a limit of the right-hand side of formula (1.2), we see that this Thurston metric is a multiple of the Weil-Petersson metric. Wolpert's argument followed a more quasiconformal analytic tradition, while this derivation is more Riemannian in perspective.

1.3. Organization of the paper. We organize the paper as follows. In Part 1, we derive Theorem 1.1. Computations in Teichmüller theory often require fixing a gauge; here we find it convenient to vary hyperbolic

structures (S, g_t) under the condition that the identity mapping id :
$(S, g_0) \to (S, g_t)$ is harmonic. This requirement gives the prescribed
curvature equation a particularly convenient form, and our derivation
begins with a sketch of a useful computation from [**Wol89**]. The rest
of the derivation is self-contained, occupying sections 2–4.

Part 2 of the paper contains the applications and extensions. In
section 5, we find a lower bound for the integrand in the first term;
this leads to an estimate (Corollary 5.2) for the Hessian in terms of an
integral along the curve of local quantities. Section 6 is devoted to a
derivation of our results in the setting of a curve which connects ends
of a (complete) hyperbolic punctured surface. Here, while the length
of a geodesic arc is infinite, the Hessian of a regularized version of the
length is positive and finite. Section 7 extends our work from curves to
laminations. In section 8, we use the formulae to give some geometric
estimates: we quickly prove that not only is the length ℓ of curves
convex, but so is $\ell^{\frac{2}{3}}$. We then give a longer geometric argument that $\ell^{\frac{1}{2}}$
is also convex. We give a general upper bound for the Hessian, as well as
estimates for the Weil-Petersson connection near the Deligne-Mumford
compactification divisor, and a lower bound on the Hessian in terms
of the twist. Finally, in section 9, we take a limit of formula (1.2) in
Theorem 1.1 over curves that are becoming equidistributed to recover
the result that the Thurston metric is a multiple of the Weil-Petersson
metric.

Part I. A formula for the Weil-Petersson Hessian of length

We begin with a brief background discussion of some of the theory of
Teichmüller space and the Weil-Petersson metric we will need. Tangent
vectors to Teichmüller space at a point (S, g) are represented by 'har-
monic Beltrami differentials' of the form $\mu = \frac{\overline{\Phi}}{g}$, where g is a hyperbolic
metric on S and Φ is a holomorphic quadratic differential on (S, g). The
Weil-Petersson inner product of two such tangent vectors is the L^2 inner

product

$$(1.3) \qquad \left\langle \frac{\overline{\Phi}}{g}, \frac{\overline{\Psi}}{g} \right\rangle = \mathrm{Re} \int_S \frac{\Phi}{g} \frac{\overline{\Psi}}{g} dArea_g.$$

Much is now known about this metric: by means of an introduction to the subject, the Weil-Petersson metric on Teichmüller space is not complete [**Chu76**] [**Wol75**]; it is Kähler [**Ahl61**], negatively curved with good expressions [**Roy**] [**Tro86**] [**Wol86a**] for and estimates [**Hua05**] [**LSY04**] [**Wolb**] [**Wola**] of the curvatures; it is quasi-isometric to the pants complex [**Bro03**], and the isometry group is exactly the (extended) mapping class group [**MW02**]. More background on the Weil-Petersson metric on Teichmüller space may be found in [**Wol10**]. Of course, the stimulus for this article and an important ingredient in some of the results above is that the Weil-Petersson metric is geodesically convex [**Wol87**].

2. The second derivative of length in space and time

We are interested in computing the second variation of geodesic length of a curve along a Weil-Petersson geodesic. We imagine the setting as a fixed differentiable surface S equipped with a family of metrics g_t, and on this surface there is a family of curves γ_t. The curves γ_t are all freely homotopic and may or may not be simple. The defining equation is that the curves γ_t are g_t-geodesics; we shall shortly write that equation out in coordinates.

To begin, though, we separate the overall second variation of length into a term that refers only to the second variation of the metric g_t and a term that refers only to the second variation of the curve γ_t. This separation is quite standard for a variational functional. Write the length of γ_s in the metric g_t as $L(g_t, \gamma_s)$. Then, in this language, the geodesic equation takes the form, for all t,

$$(2.1) \qquad \frac{\partial}{\partial s}\bigg|_{s=s_0} L(g_t, \gamma_s)\left[\frac{\partial}{\partial s}\gamma_s\right] = 0,$$

if γ_{s_0} is a g_t-geodesic, and $\frac{\partial}{\partial s}\gamma_s$ is an infinitesimal variation of curves through γ_{s_0}.

The second variation of length of the g_t-geodesics γ_t is given by
(2.2)
$$\frac{d^2}{dt^2} L(g_t, \gamma_t) = D_{11}^2 L(g_0, \gamma_0)[\dot g, \dot g] + 2D_{12}^2 L(g_0, \gamma_0)[\dot g][\dot \gamma] + D_{22}^2 L(g_0, \gamma_0)[\dot \gamma, \dot \gamma],$$

where $\dot g = \frac{d}{dt} g_t$ and $\dot \gamma = \frac{d}{dt} \gamma_t$. Of course, if γ_t is a g_t-geodesic, then we write the geodesic equation (2.1) above in this notation as

$$D_2 L(g_t, \gamma_t)[\dot \gamma] = 0$$

and so

$$\begin{aligned} 0 &= \frac{d}{dt} D_2 L(g_t, \gamma_t)[\dot{\gamma}] \\ &= D_1 D_2 L(g_0, \gamma_0)[\dot{g}][\dot{\gamma}] + D_2 D_2 L(g_0, \gamma_0)[\dot{\gamma}, \dot{\gamma}]. \end{aligned}$$

Thus,

$$(2.3) \qquad D_{12}^2 L(g_0, \gamma_0)[\dot{g}][\dot{\gamma}] = -D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}, \dot{\gamma}].$$

Substituting (2.3) into (2.2) yields that

$$(2.4) \qquad \frac{d^2}{dt^2} L(g_t, \gamma_t) = D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] - D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}].$$

Some remarks on this equation (2.4) are in order. First, note that the term $D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}]$ is non-negative, as the surface (S, g_0) is negatively curved; indeed, this second variation term is positive unless the vector field $\dot{\gamma}$ is tangent to the curve γ_0. Thus our task is to prove that the first term $D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}]$ is larger than the second term.

In the next sections, we evaluate the terms $D_{11}^2 L$ and $D_{22}^2 L$ via different methods.

3. Second variation of arclength of γ_0 in a family of metrics

We briefly recall the computational scheme of [**Wol89**]. Let $\Phi \in \mathrm{QD}(g_0)$ denote a quadratic differential, holomorphic with respect to a conformal metric g_0. Then we may consider a family of metrics on S decomposed by type as

$$(3.1) \qquad g_t = t\Phi dz^2 + g_0 \left(\mathcal{H}(t) + \frac{t^2|\Phi|^2}{g^2\mathcal{H}(t)} \right) dz d\bar{z} + t\overline{\Phi} d\bar{z}^2.$$

Here z is a conformal coordinate for (S, g_0). It is straightforward to check [**SY78**] that the metric g_t is hyperbolic if

$$(3.2) \qquad \Delta_{g_0} \log \mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2t^2|\Phi|^2}{g_0^2\mathcal{H}(t)} - 2.$$

(Of course, the pullback of a hyperbolic metric by a diffeomorphism is hyperbolic, and so we might imagine that if we pullback g_t by a family of diffeomorphisms ψ_t, then the result $\psi_t^* g_t$ would also be hyperbolic. Here we have chosen a gauge by requiring that the identity map id : $(S, g_0) \to (S, g_t)$ is harmonic.)

We are interested in second variations. Differentiating twice and applying the maximum principle to the first derivative (see [**Wol89**] for

an expanded description) yields

$$\dot{\mathcal{H}} = \frac{d}{dt}\bigg|_{t=0} \mathcal{H}(t) \equiv 0$$

$$(3.3) \qquad \ddot{\mathcal{H}} = \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{H}(t) = -2(\Delta - 2)^{-1}\frac{2|\Phi|^2}{g_0^2}.$$

We observe that $-2(\Delta - 2)^{-1}$ is a positive operator and so $\ddot{\mathcal{H}} \geq 0$. Combining (3.1) and (3.3), we conclude that

(3.4)

$$g_t = g(t) = g_0 dz d\bar{z} + t(\Phi dz^2 + \overline{\Phi} d\bar{z}^2)$$

$$+ t^2/2 \left(\frac{2|\Phi|^2}{g_0^2} + -2(\Delta - 2)^{-1}\frac{2|\Phi|^2}{g_0^2} \right) g_0 dz d\bar{z} + O(t^4).$$

Now use that

$$D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] = \frac{d^2}{dt^2}\bigg|_0 L(g_t, \gamma_0) = \frac{d^2}{dt^2}\bigg|_{t=0} \int_{\gamma_0} \sqrt{g_t}.$$

Substituting (3.4) into this last integral with a choice of coordinate so that γ_0 is a line $\{\operatorname{Re} z = \text{const}\}$ and differentiating under the integral symbol then yields

$$(3.5) \quad D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] = \int_{\gamma_0} -\frac{1}{4}(g_0)^{-3/2}(2\operatorname{Re}\Phi)^2$$

$$+ \frac{1}{2}\sqrt{g_0}\left(\frac{2|\Phi|^2}{g_0^2} - 2(\Delta - 2)^{-1}\frac{2|\Phi|^2}{g_0^2} \right)$$

$$= \int_{\gamma_0} \left\{ \frac{(\operatorname{Im}\Phi)^2}{g_0^2} - \left[2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} \right] \right\} \sqrt{g_0},$$

since $|\Phi|^2 - (\operatorname{Re}\Phi)^2 = (\operatorname{Im}\Phi)^2$. Both terms are positive, and so we see that this expression is positive, as we expected (and needed if the expression $D_{11}^2 L - D_{22}^2 L$ is to be positive).

REMARK. As an easy model of this method, we quickly reproduce a formula for the first variation of length. (We'll have use of this expression in later sections.)

We compute the first derivative of length ℓ along $\Gamma(t)$ to be

$$(3.6) \qquad \frac{d}{dt}\ell_\gamma(\Gamma(t)) = D_1 L(g_t, \gamma_t)[\dot{g}] + D_2 L(g_t, \gamma_t)[\dot{\gamma}].$$

Of course, as γ_0 is a geodesic, the second term $D_2 L(g_t, \gamma_t)[\dot{\gamma}] = 0$ and from (3.1) and (3.2), we find

$$(3.7) \qquad \frac{d}{dt}\ell_\gamma(\Gamma(t)) = \frac{d}{dt}\int_{\gamma_0} \sqrt{g_t}$$

$$= \int_{\gamma_0} \frac{\frac{d}{dt}g_t}{2g_0}$$

$$= \int_{\gamma_0} \frac{\operatorname{Re}\Phi}{g_0} ds,$$

by (3.4), concluding the computation.

4. Second variation of g_0-arclength of the family γ_t of g_t-geodesics

Our next step is to evaluate the term $D_{22}^2 L(g_0, \gamma_0) = \frac{d^2}{dt^2}L(g_0, \gamma_t)[\dot{\gamma}, \dot{\gamma}]$ in (2.4).

It is of course standard (see for example [**Spi79b**]) that if V is the variational field of a family of curves through a geodesic γ_0, then

$$(4.1) \qquad \frac{d^2}{dt^2}L(g_0, \gamma_t) = \int_{\gamma_0} \left| \frac{\partial V}{\partial s} \right|^2 - K|V|^2 ds$$

where $K = K(s)$ denotes the Gaussian curvature of the surface at the point $\gamma_0(s)$. To compare this formula (4.1) to (3.5), we will need to find an expression for V in terms of the quadratic differential Φ. That is the main goal of this section.

Of course, the defining equation of γ_t is that it is a geodesic, or equivalently that its geodesic curvature vanishes. We write this schematically, in a similar way that we write the length $L = L(g_t, \gamma_t)$, as a function $\kappa = \kappa(g_t, \gamma_t)$ of a metric and a curve:

$$(4.2) \qquad\qquad \kappa(g_t, \gamma_t) = 0.$$

Differentiating in t at $t = 0$, we find that

$$(4.3) \qquad \frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{\gamma}] = -\frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{g}].$$

As expected, the left-hand side of (4.3) is the classical Jacobi operator $\left(\frac{d^2}{ds^2} + K\right)$, but the right-hand side will involve the first derivatives of g_t, i.e. the metric g_0 and the quadratic differential Φ. Our next task will be to find an expression for the solution $\dot{\gamma}$ to (4.3).

4.1. The inhomogeneous Jacobi equation. We first expand the right-hand side of (4.3). We pick conformal (Fermi) coordinates $z = x + iy$ so that the geodesic γ_0 is described by $\{x = \text{const}\}$. These coordinates are a bit unusual in that the geodesic may repeatedly visit the same points on the surface: it's possibly better to regard the geodesic as

embedded in the unit tangent bundle $T^1 M$ with $z = x + iy$ its projection to the surface S.

Then, invoking a coordinate expression for κ (see [**Opr04**], consistent with definitions in [**Spi79a**] and [**Spi79b**]), we have

$$\frac{d}{dt}\kappa(g_0, \gamma_0)[\dot{g}] = -\frac{d}{dt}\left\{\Gamma^1_{22}(t)\frac{\sqrt{\det g(t)}}{g_{22}(t)^{3/2}}\right\}$$

where $g(t) = g_{ij}(t)$ is defined by (2.1) as

$$(4.4) \qquad g(t) = \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{22}(t) & g_{22}(t) \end{pmatrix}$$
$$= \begin{pmatrix} g_0 + 2t\operatorname{Re}\Phi & -2t\operatorname{Im}\Phi \\ -2t\operatorname{Im}\Phi & g_0 - 2t\operatorname{Re}\Phi \end{pmatrix} + O(t^2) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

We include the very classical notation for the first fundamental form at the end as it actually simplifies some of our notation; for example, we write

$$(4.5) \qquad \kappa(t) = -\Gamma^1_{22}(t)\frac{\sqrt{EG - F^2}}{G^{3/2}},$$

where here of course the variables $E = E(t)$, $F = F(t)$, and $G = G(t)$ all depend on t. Now, in this language, suppressing some of the dependence on t, we have

$$(4.6) \qquad \Gamma^1_{22}(t) = \frac{2GF_y - GG_x - FG_y}{2(EG - F^2)}.$$

Since $\kappa(0) = 0$ and $F(0) \equiv 0$, we find that

$$(4.7) \qquad \frac{\partial}{\partial x}g_{22}(0) = \frac{\partial}{\partial x}G = 0 \quad \text{on} \quad \gamma_0.$$

We differentiate (4.6) in t and use (4.7) and $F(0) \equiv 0$ to find that

$$\frac{d}{dt}\Gamma^1_{22}(t) = \frac{1}{2g_0^2}\{-4g_0(\operatorname{Im}\Phi)_y + 2g_0(\operatorname{Re}\Phi)_x + 2(\operatorname{Im}\Phi)(g_0)_y\}$$
$$= \frac{1}{2g_0^2}\{-2g_0(\operatorname{Im}\Phi)_y + 2(\operatorname{Im}\Phi)(g_0)_y\};$$

here the last equality follows from the Cauchy-Riemann equations for the real and imaginary parts of the holomorphic quadratic differential Φ. We conclude that

$$(4.8) \qquad \frac{d}{dt}\Gamma^1_{22}(t) = -\frac{\partial}{\partial y}\left\{\frac{\operatorname{Im}\Phi}{g_0}\right\}.$$

Combining (4.3), (4.5), and (4.8) yields the equation we will focus on:

$$(4.9) \qquad \frac{\partial^2}{\partial y^2}V - V = -\frac{\partial}{\partial y}\left\{\frac{\operatorname{Im}\Phi}{g_0}\right\}.$$

REMARK. It is easy to compute that the Beltrami differential tangent to our deformation is given by $\mu = \frac{\bar{\Phi}}{g_0}$. In that language, our equation (4.9) becomes

$$(4.10) \qquad V_{yy} - V = \frac{\partial}{\partial y} \operatorname{Im} \mu.$$

4.2. The primitive of the variation field. The right-hand side of (4.3) is the derivative of the basic quantity $-\frac{\operatorname{Im}\Phi}{g_0}$ appearing in (3.5). This term provides a link between the two different terms $D^2_{11}L$ and $D^2_{22}L$ of the basic expression (2.4) for the Hessian of length. To find our first version of a formula for the Hessian of the length function, we consider the primitive of $V = \dot{\gamma}$ along γ and use this to relate the expressions for $D^2_{11}L$ and $D^2_{22}L = \int_{\gamma_0} V'^2 + V^2$.

In particular, we begin with the equation (4.9) and then start by defining a particular primitive U of V. The procedure is in two steps, as we need to correctly choose the constant for the primitive. So first we set

$$(4.11) \qquad u(y) = \int_a^y V(s) ds.$$

Note that we need to check the well-definedness of u on γ, as it is a closed loop; on the other hand, it is enough to check that the period $u(2\pi) - u(0) = \int_\gamma V$ vanishes (here using the obvious notation of $\{0, 2\pi\}$ for a pair of endpoints for the loop).

For convenience in the sequel, set

$$(4.12) \qquad \mathcal{F} = \frac{\operatorname{Im}\Phi}{g_0}$$

so that equation (4.9) becomes

$$(4.13) \qquad V_{yy} - V = -\mathcal{F}_y.$$

Then, for well-definedness of u, we observe that (letting subscripts indicate differentiation in the variable)

$$u(2\pi) - u(0) = \int_\gamma V$$
$$= \int_\gamma V_{yy} + \mathcal{F}_y$$
$$= \int_\gamma (V_y + \mathcal{F})_y dy$$
$$(4.14) \qquad\qquad\qquad = 0.$$

Thus u (and $u + c$, for any constant c) is well-defined along γ.

Next begin again with the equation

$$(4.15) \qquad\qquad V_{yy} - V = -\mathcal{F}_y,$$

and then note that

$$(u_{yy} - u + \mathcal{F})_y = V_{yy} - V + \mathcal{F}_y$$
$$(4.16) \qquad\qquad\qquad\qquad = 0.$$

Thus we have that $u_{yy} - u + \mathcal{F} = c_0$, where c_0 is a constant. In particular, if we set

$$(4.17) \qquad\qquad\qquad U = u + c_0$$

to be another primitive of V, then

$$(4.18) \qquad\qquad\qquad U_{yy} - U = -\mathcal{F}.$$

The choice of constant c_0 here compensates for the implicit choice of a constant made when we passed from $\mathcal{F}_y$ in (4.13) to $\mathcal{F}$ in (4.18).

The point of all of this is that the positive part of the second variation of length integral (4.1) will turn out to be the energy of U, while the negative part will once again be the L^2 norm of $\mathcal{F}$ along ℓ_0 (cancelled out by a term in the metric variation contribution $D^2_{11}L$).

We compute the contribution $-D^2_{22}L$ from the second variation of length along the surface through

$$(4.19)$$
$$-D^2_{22}L = -\int_{\gamma_0} V_y^2 + V^2 dy$$
$$= \int_{\gamma_0} V_{yy}V - V^2 dy \qquad \text{by parts}$$
$$= \int_{\gamma_0} (V_{yy} - V)V \, dy$$
$$= \int_{\gamma_0} (-\mathcal{F}_y)V \, dy \qquad \text{by (4.9)}$$
$$= \int_{\gamma_0} \mathcal{F}V_y dy \qquad \text{by parts}$$
$$= \int_{\gamma_0} \mathcal{F}U_{yy} dy \qquad \text{from the definition of } U \text{ as a primitive of } V$$

$$= \int_{\gamma_0} \mathcal{F}(U - \mathcal{F})dy \qquad \text{from (4.18)}$$

$$= -\int_{\gamma_0} \mathcal{F}^2 dy + \int_{\gamma_0} U\mathcal{F}dy$$

$$= -\int_{\gamma_0} \mathcal{F}^2 dy + \int_{\gamma_0} U\{-(U_{yy} - U)\}dy \qquad \text{from (4.18)}$$

(4.20) $$= -\int_{\gamma_0} \mathcal{F}^2 dy + \int_{\gamma_0} U_y^2 + U^2 dy \qquad \text{by parts.}$$

Combining this last equation with (3.5) and (4.1) (and recalling that we are parametrizing the curve γ_0 by arclength $s = y$), we find that

$$\frac{d^2}{dt^2}L(g_t, \gamma_t) = D_{11}^2 L(g_0, \gamma_0)[\dot{g}, \dot{g}] - D_{22}^2 L(g_0, \gamma_0)[\dot{\gamma}][\dot{\gamma}]$$

$$= \int_{\gamma_0} \mathcal{F}^2 - \left[2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}\right] dy - \int_{\gamma_0} V_y^2 + V^2 dy$$

(4.21) $$= \int_{\gamma_0} \mathcal{F}^2 - \left[2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}\right] dy$$

$$- \int_{\gamma_0} \mathcal{F}^2 dy + \int_{\gamma_0} U_y^2 + U^2 dy \qquad \text{from (4.2)}$$

(4.22) $$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} ds + \int_{\gamma_0} U_y^2 + U^2 dy.$$

In summary, the Weil-Petersson Hessian of length can be expressed as the sum of two integrals along the curve, each of which has a positive function as an integrand. The first integrand is the restriction to the curve of a solution of a differential equation on the surface, and the second is the energy density of a solution of a differential equation along the curve.

We record this formula as a theorem, extending Theorem 1.1 from the introduction. To set the notation, let $[\gamma]$ be the free homotopy class of a closed curve (simple or not) on the surface, and $\Gamma(t)$ a Weil-Petersson geodesic arc; the class $[\gamma]$ is represented by the $\Gamma(t)$-geodesic γ_t. The tangent vector to Teichmüller space at $\Gamma(0)$ is given by a harmonic Beltrami differential, say $\frac{\bar{\Phi}}{g_0}$. Let $\frac{\bar{\Psi}}{g_0}$ denote a second harmonic Beltrami differential on $\Gamma(0)$.

Let U^Φ and U^Ψ denote the respective solutions to the ordinary differential equations (see (1.1))

(4.23) $$U_{yy} - U = -\frac{\text{Im}\,\Phi}{g_0}$$

and

$$(4.24) \qquad\qquad U_{yy} - U = -\frac{\operatorname{Im}\Psi}{g_0}.$$

This is enough terminology so that we may summarize our discussion as

Theorem 4.1. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the length $\ell(t) = L(\Gamma(t),[\gamma])$ is given by*

$$(4.25) \qquad \frac{d^2}{dt^2}\ell(t) = \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}ds + \int_{\gamma_0} [U_y^\Phi]^2 + [U^\Phi]^2 ds.$$

More generally, the Weil-Petersson Hessian $HessL[\frac{\bar\Phi}{g_0}, \frac{\bar\Psi}{g_0}]$ is given by

$$(4.26)$$
$$HessL[\frac{\bar\Phi}{g_0}, \frac{\bar\Psi}{g_0}] = \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{\operatorname{Re}\Phi\bar\Psi}{g_0^2}ds + \int_{\gamma_0} U_y^\Phi U_y^\Psi + U^\Phi U^\Psi ds.$$

Proof. The solutions U^Φ and U^Ψ to (4.23) and (4.24) are unique, and the equations are linear in the unknown and the parameters Φ and Ψ. Thus the unique solution $U^{\Phi+\Psi}$ to

$$(4.27) \qquad\qquad U_{yy} - U = -\frac{\operatorname{Im}(\Phi + \Psi)}{g_0}$$

satisfies

$$(4.28) \qquad\qquad U^{\Phi+\Psi} = U^\Phi + U^\Psi.$$

Then a straightforward polarization of (1.2), together with our understanding (4.28), yields (4.26).   q.e.d.

REMARK. In terms of our previous notation for the Beltrami differential $\mu = \frac{\bar\Phi}{g_0}$, the equation (4.23) takes the form

$$(4.29) \qquad\qquad U_{yy} - U = \operatorname{Im}\mu.$$

4.3. A geometric kernel representation. The second term of equation (1.2) is expressed as the energy of the solution of a differential equation. We wish to provide a more geometric interpretation; not only do we hope that this version is more appealing on its own, but it will be important in section 9 where we treat the Thurston metric via the Hessian of the length function, and in section 8 when we estimate the Hessian of length from below in terms of infinitesimal twist. Indeed, the kernel representation will exhibit the dependence of the Hessian on the "infinitesimal twist" component of the deformation.

To begin, note that the second term of (1.2) (=(4.25)) may be written

$$\int_{\gamma_0} U_y^2 + U^2 ds = -\int_{\gamma_0} U(U_{yy} - U)ds$$

$$= \int_{\gamma_0} U(s)\mathcal{F}(s)ds$$

in the notation where $\mathcal{F}(s) = \text{Im}\,\frac{\Phi}{g_0} = -\text{Im}\,\mu$, since

$$U_{yy} - U = -\mathcal{F}$$

by (4.18). It is well-known that we can represent the solution $U(s)$ to (4.18) by

$$U(s) = -\int \mathcal{F}(t)K(s,t)dt$$

for kernels $K(s,t)$ which satisfy

(4.30) $$\frac{d^2}{dt^2}K(s,t) - K(s,t) = \delta_s(t).$$

It is easy to find the solutions to (4.30) using that the solutions to $\frac{d^2}{dt^2}K(s,t) - K(s,t) = 0$ are linear combinations of $\sinh(t)$ and $\cosh(t)$. (Indeed, if we represent γ_0 as the interval $[-L/2, L/2]$ with endpoints identified, set $s_0 = \pm L/2$ to be an endpoint, and look to solve (4.30) on that interval, then it is evident that setting $K(s,t) = -\cosh(t)$ is correct up to an easily computed multiplicative constant.) In general, for γ parametrized by an interval of length L (so that we may choose $|t - s| < L/2$), we have that

$$K(s,t) = \begin{cases} -\frac{1}{2}\frac{\cosh(s-t-L/2)}{\sinh(\frac{L}{2})}, & t < s \\ -\frac{1}{2}\frac{\cosh(t-s-L/2)}{\sinh(\frac{L}{2})}, & t > s \end{cases}$$

solves (4.30) (where we require that $|t - s| < L/2$).

Of course, the variables s and t parametrize the curve γ_0 with respect to arclength, and so, for $|s - t| < L/2$, we have $|s - t| = d(\gamma_0(s), \gamma_0(t))$. Thus $K(s,t)$ admits the description in terms of $p = \gamma_0(s), q = \gamma_0(t)$ as

(4.31) $$K(p,q) = -\frac{1}{2}\frac{\cosh(d(p,q) - L/2)}{\sinh(L/2)}.$$

This leads to the representation

$$\frac{d^2}{dt^2}\ell(t) = \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} + \int_{\gamma_0} U_y^2(s) + U^2(s)ds$$

$$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} - \int_{\gamma_0} U(s)\operatorname{Im}\mu(s)ds$$

$$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} - \int_{\gamma_0} \operatorname{Im}\mu(s)\int_{\gamma_0} K(s,t)\operatorname{Im}\mu(t)dtds$$

$$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} - \iint_{\gamma_0 \times \gamma_0} \operatorname{Im}\mu(s)K(s,t)\operatorname{Im}\mu(t)dtds$$

$$= \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2} - \iint_{\gamma_0 \times \gamma_0} \operatorname{Im}\mu(p)K(p,q)\operatorname{Im}\mu(q)ds(p)ds(q)$$

$$(4.32) \quad = \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}$$

$$(4.33) \qquad + \frac{1}{2\sinh(\frac{L}{2})}\iint_{\gamma_0 \times \gamma_0} \operatorname{Im}\mu(p)[\cosh(d(p,q) - \frac{L}{2})]\operatorname{Im}\mu(q)ds(p)ds(q),$$

where $ds(p)$ and $ds(q)$ refer to arclength measure.

Part II. Extensions and applications of the formula for the Hessian

5. A lower bound expressed in terms of pointwise quantities

We claim

Lemma 5.1. *Let $v = 1/3\frac{|\Phi|^2}{g_0^2}$. Then v is a subsolution of $(\Delta - 2)f = -\frac{2|\Phi|^2}{g_0^2}$ and in particular $0 \leq v \leq -2(\Delta - 2)^{-1}(\frac{|\Phi|^2}{g_0^2})$.*

We begin by noting that the curvature of a metric expressed as $G|dz|^2$ is given by

$$K(G|dz|^2) = -\frac{1}{2}\frac{1}{G}\Delta_0 \log G$$

where $\Delta_0 = \partial_x^2 + \partial_y^2$.

Then using that $K(g_0|dz|^2) \equiv -1$ and that $|\Phi_0||dz|^2$ is a flat metric with concentrated (Dirac function type) curvature singularities at the zeroes $\Phi^{-1}(0)$ of Φ, we see that

$$\Delta_0 \log \frac{|\Phi|^2}{g_0^2} = \Delta_0 \log |\Phi|^2 - \Delta_0 \log g_0^2$$

$$= -4|\Phi|K(|\Phi||dz|^2) + 4g_0 K(g_0)$$

$$= 4|\Phi| \sum_{p \in \Phi^{-1}(0)} \pi\delta_p \deg_p \Phi - 4g_0,$$

where δ_p indicates a delta function at p. On the other hand, using that $\Delta_0 \log F = \frac{\Delta_0 F}{F} - \frac{|\nabla_0 F|^2}{F^2}$, we see we may write

$$\Delta_0 \log \frac{|\Phi|^2}{g_0^2} = \frac{\Delta_0 \frac{|\Phi|^2}{g_0^2}}{\frac{|\Phi|^2}{g_0^2}} - \frac{|\nabla_0 \left(\frac{|\Phi|^2}{g_0^2}\right)|^2}{\left(\frac{|\Phi|^2}{g_0^2}\right)^2}.$$

Putting the last two of these displayed equations together yields

$$\frac{1}{g_0}\Delta_0 \frac{|\Phi|^2}{g_0^2} = \frac{|\Phi|^2}{g_0^2}\left(4\frac{|\Phi|}{g_0}\sum_{p\in\Phi^{-1}(0)}\pi\delta_p \deg_p \Phi\right) - 4\frac{|\Phi|^2}{g_0^2} + \frac{|\nabla_0 \left(\frac{|\Phi|^2}{g_0^2}\right)|^2}{\frac{|\Phi|^2}{g_0^2}g_0}$$

In particular, writing $\Delta = \frac{1}{g_0}\Delta_0$ for the g_0-Laplace Beltrami operator on S, and noting the vanishing of the first term on the right-hand side (in particular, notice that at zeroes p of Φ, where $\delta_p \neq 0$, the sum is multiplied by the factor $\Phi(p)$, which vanishes because p is a zero of Φ), we conclude that

$$\Delta\frac{|\Phi|^2}{g_0^2} \geq -4\frac{|\Phi|^2}{g_0^2}.$$

We are of course interested in the operator $\Delta - 2$, so we note the obvious implication that

$$(\Delta - 2)\frac{|\Phi|^2}{g_0^2} \geq -6\frac{|\Phi|^2}{g_0^2}$$

so that $v = 1/3\frac{|\Phi|^2}{g_0^2}$ is a subsolution for the equation $(\Delta - 2)f = -2\frac{|\Phi|^2}{g_0^2}$.

It is obvious that $v = \frac{1}{3}\frac{|\Phi|^2}{g_0^2} \geq 0$, and if f satisfies $(\Delta - 2)f = -\frac{2|\Phi|^2}{g_0^2}$, then $\Delta(f - v) \leq 2(f - v)$ and so the minimum principle guarantees that at a minimum of $(f-v)$, we have $f - v \geq 0$; hence $f - v \geq 0$ everywhere, concluding the proof of the lemma.   q.e.d.

Combining Lemma 5.1 with Theorem 1.1 we obtain

Corollary 5.2. *Let $\Phi \in \mathrm{QD}(g_0)$ be a holomorphic quadratic differential in (Σ, g_0) and let $\Gamma(t)$ denote a Weil-Petersson geodesic arc with initial tangent vector given by the harmonic Beltrami differential $\overline{\Phi}g_0^{-1}$. Let $\ell(t)$ denote the geodesic length of a representative γ_t of a curve class $[\gamma]$ on S. Then, for γ_0 the geodesic represented of $[\gamma]$ on $\Gamma(0)$, we have*

$$\frac{d^2}{dt^2}\bigg|_{t=0}\ell(t) \geq \frac{1}{3}\int_{\gamma_0}\frac{|\Phi|^2}{g_0^2}ds.q.e.d.$$

We will apply this estimate in section 8.

6. The second variation of the length of an arc

In this section, we adapt our derivation to the case where S is a surface of finite genus with a finite number of punctures, and we are interested in the variation of length of an arc α that runs between two of the punctures (or a puncture itself). Naturally, the length of such an arc is infinite, so we will be discussing the variation of some regularization of its length; nevertheless, all of the basic considerations will extend to this case with only minor modifications.

6.1. Notation and preliminaries. Let α_t be the geodesic on (S, g_t) that connects punctures p and q in a fixed homotopy class (rel p and q). Consider a sequence of points $p_n, q_n \in \alpha_0$ with $p_n \to p$ and $q_n \to q$. These points p_n and q_n pass through horocycles linking p and q (respectively) that vary in length and position with t. Consider a family of g_t-horocycles near p_n whose g_t-length agrees with the g_0-length ℓ_n^+ of the horocycle through p_n. There is of course an analogous family near q_n of horocycles whose g_t-lengths are required to be a fixed constant ℓ_n^-. Let $\{\alpha_{t,n}\}$ be the homotopy of g_t-geodesic arcs connecting those horocycles.

Of course, a consequence of this construction and the uniqueness of geodesic arcs in hyperbolic space connecting two points at infinity is that for $n > m$, we have that $\alpha_{t,m} \subset \alpha_{t,n}$.

Our plan is to derive a formula for

$$\frac{d^2}{dt^2} L(g_t, \alpha_{t,n}),$$

and show that the limit exists and is independent of the choice of the sequence $\{p_n, q_n\}$.

We learned while preparing this manuscript that Wolpert [**Wol09**] recently found an inequality in the analogous case of finite length arcs between horocycles. (Indeed, in the course of our derivation, we will also treat this case.)

It is easy to check that the formal preliminaries remain the same as in the derivation of (2.4), and so we conclude

$$(6.1) \qquad \frac{d^2}{dt} L(g_t, \alpha_{t,n}) = D_{11}^2 L(g_0, \alpha_{0,n})[\dot{g}, \dot{g}] - D_{22}^2 L(g_0, \alpha_{0,n})[\dot{\alpha}_{0,n}, \dot{\alpha}_{0,n}].$$

6.2. The second variation of arclength of $\alpha_{0,n}$ in g_t. As in the case of a closed curve, the first term in (6.1) is relatively straightforward to compute; the only new issue to consider is the dependence of the term on the choice of endpoints p_n, q_n of $\alpha_{t,n}$. Indeed, exactly as in the derivation of (3.5), we formally compute

$$(6.2) \qquad D_{11}^2 L(g_0, \alpha_{0,n}) = \int_{\alpha_{0,n}} \left\{ \frac{|\operatorname{Im}\Phi|^2}{g_0^2} - 2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} \right\} \sqrt{g_0}$$

where the principal issue is to determine the meaning of $(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$. We discuss this in the next subsection.

6.3. Variations of metrics of finite area. The basic point here in understanding $-2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$ is to construe it as $\frac{1}{2}\ddot{\mathcal{H}}$ for the family of pullback metrics g_t in (3.1). As these maps id $: (S, g_0) \to (S, g_t)$ are harmonic, we can apply some results from the theory of harmonic maps between cusped hyperbolic surfaces. We show in this subsection that the positive function $\ddot{\mathcal{H}}$ (what we have written in the previous subsection as $-2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2}$) is integrable on proper arcs, so that $\int_{\alpha_{0,n}} \frac{1}{2}\ddot{\mathcal{H}} ds$ is well-defined.

In this direction, results in [**Wol91**] (Theorem 5.1) and of Lohkamp (see the remark after Theorem 4 in [**Loh91**], especially with Lemma 12 informed by Proposition 3.13 in [**Wol91**]) proved that $\mathcal{H}(t) \in C^{k,\alpha}(S, g_0)$ is analytic in t on the compactified surface $\bar{M}$. In particular, $\ddot{\mathcal{H}}$ is bounded.

Indeed, we can easily show from this that $\ddot{\mathcal{H}} = O(\frac{1}{(\log \frac{1}{r})^\alpha}) = O(y^{-\alpha})$ for some $\alpha \in (0,1)$ as $r \to 0$. The basic elements of this argument are that $\frac{1}{(\log \frac{1}{r})^\alpha} = y^{-\alpha}$ for some $\alpha \in (0,1)$ is a supersolution of the equation $(\Delta - 2)\ddot{\mathcal{H}} = -2\|\Phi\|^2$ on the cusp, as well as the point that the kernel of $(\Delta - 2)$ on a half-infinite cylinder $\mathcal{C} = \{\operatorname{Im} z > 1, |\operatorname{Re} z| < 1/2\}$ (with the standard identifications) is spanned by the pair of functions $k_1(z) = y^2$ and $k_2(z) = y^{-1}$. With that background, consider, on the finite cylinder $\{1 < \operatorname{Im} z < y_n\}$, a function $H_j(z)$ of the form $H_j(z) = C_0 y^{-\alpha} + C_1 y^{-1} + \epsilon_j y^2$.

Then, for appropriate choices of $\epsilon_j \to 0$, we find that $H_j(z)$ majorizes $\ddot{\mathcal{H}}$; letting $j \to \infty$ and $\epsilon_j \to 0$ while C_0 and C_1 stay bounded (as the boundary values $\ddot{\mathcal{H}}(z)$ for $\{\operatorname{Im} z = 1\}$ are fixed independently of j) allows us to conclude that $\ddot{\mathcal{H}}(z)$ decays like $C_0 y^{-\alpha} + C_1 y^{-1}$. Thus $\ddot{\mathcal{H}}(z) = O(y^{-\alpha}) = O(\frac{1}{(\log \frac{1}{r})^\alpha})$.

Looking ahead to the final form of the second variation of length, it is worth recording the

Proposition 6.1. *Let $p_n \to p$ and $q_n \to q$, and let $\alpha_{0,n}$ be the geodesic arc connecting p_n to q_n as in the introduction to the section. Then*

$$\int_{\alpha_{0,n}} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g_0^2} ds = \int_{\alpha_{0,n}} \frac{1}{2} \ddot{\mathcal{H}} ds$$

converges as $n \to \infty$.

Proof. In the upper half plane coordinates, we have that for each end of an $\alpha_{0,n}$,

$$(6.3) \qquad \int_{\alpha_{0,n}} \ddot{\mathcal{H}} ds = \int_a^{y_n} \ddot{\mathcal{H}} \frac{dy}{y} = \int_a^{y_n} O(\frac{1}{y^\alpha}) \frac{dy}{y} = O(1) \quad \text{as} \ \ n \to \infty.$$

The proposition then follows from the integrals being positive.   q.e.d.

6.4. The Jacobi Field for an Arc. The next term we must address is the second term in (6.1). The variational vector field, defined geometrically, also satisfies (4.9). Here, of course the variational field $V = V_n$ depends on n, as it is defined in terms of $\alpha_{t,n}$; our notation is meant to reflect that.

Our curves $\alpha_{t,n}$ have endpoints $p_{t,n}$ and $q_{t,n}$ on horocycles of fixed length ℓ_n^+ and ℓ_n^-, respectively, which we will abbreviate as $\ell_n^\pm$. Naturally, these endpoints move on the fixed manifold M with t. It is well-known that the second variation of arclength for a family of geodesics connecting points $p_{t,n}$ and $q_{t,n}$ is given by

$$(6.4) \qquad\qquad D_{22}^2 L = \int_{\alpha_{t,n}} V'^2 + V^2 ds + \nabla_V V |_{q_{0,n}}^{p_{0,n}}$$

Our plan is to evaluate this expression in four steps: first we show that the term that depends on the endpoints is negligible in n. Then we show that the variational field V, which satisfies equation (4.9) with boundary terms given by $\frac{d}{dt} p_{t,n}$ and $\frac{d}{dt} q_{t,n}$, is close to a solution of (4.9) with vanishing boundary terms; in particular, using the latter variational field in $D_{22}^2 L = \int_{\alpha_{t,n}} V'^2 + V^2 ds$ would result in a negligible change. Our third step is to evaluate the integral, finding some error terms. Finally, we will make the easy observation that the error terms vanish rapidly with n.

6.5. The variation of the endpoints. The arcs $\alpha_{t,n}$ are defined to be arcs perpendicular to g_t-horocycles of g_t-length $\ell_n^{\pm}$. (Of course, $\ell_n^{\pm} \to 0$ as $n \to \infty$.) In this subsection, we find that the endpoints $p_{t,n}$ and $q_{t,n}$ of these arcs vary only slightly if n is large. First we look for how the point $p_{0,n}$ might vary in a path $\hat{p}_{t,n}$ in the direction normal to the horocycle, while still staying on the g_t-horocycle of length $\ell_n^{\pm}$. Then we consider how far $\hat{p}_{t,n}$ is from the actual endpoint $p_{t,n}$ of the geodesic arc $\alpha_{t,n}$.

To begin, we first wish to observe that the horocycle itself does not vary much. To see this, note that the defining conditions for the horocycle $h = h(s) = h_t(s)$ are

$$\sqrt{g}h_{ss}^k + \Gamma_{ij}^k h_s^i h_s^j = -J_i^k h_s^i (g_{mn} h_s^m h_s^n)^{\frac{1}{2}}$$

$$\int_h (g_{ij} h_s^i h_s^j)^{\frac{1}{2}} ds = \ell_n^{\pm}$$

where $g_{ij} = {}^t g_{ij}, \Gamma_{ij}^k = {}^t \Gamma_{ij}^k$, and $J_i^k = {}^t J_i^k$ are the time t metric, its Christoffel symbols, and its almost complex structure. It's useful to imagine $h_t(s)$ as parametrized by arclength. Note that in these coordinates, the length of $\frac{\partial}{\partial x^1}$ varies in x^2; thus so do $g_{ij}(x)$ and $J_i^k(x)$.

Differentiating in time both the condition of the horocycle being constant curvature and being of length $\ell_n^{\pm}$, we find conditions for the normal variation field $W^k = \frac{d}{dt} h^k$. Looking ahead, we choose coordinates (x^1, x^2) on M so that $x^1 = s$ is tangential to the horocycle while $x^2 = J\frac{d}{ds}$ is normal to the horocycle. Our differentiated pair of equations then take the form

$$(6.5) \qquad \sqrt{g}W_{ss}^2 + \{\frac{\partial}{\partial x^2}\Gamma_{11}^2(h_s^1)^2 + \frac{1}{2}\frac{\partial}{\partial x^2}g_{11}(h_s^1)^2\{J_1^2 h_s^1\}[g_{11}(h_s^1)^2]^{-\frac{1}{2}}\}W^2$$

$$= -(\frac{d}{dt}\Gamma_{11}^{2,t}) - \frac{1}{2}(\frac{d}{dt}g_{mn})h_s^m h_s^n J_1^2 h_s^1 [g_{11}(h_s^1)^2]^{-\frac{1}{2}}$$

$$\int_h [g_{11}(h_s^1)^2]^{-\frac{1}{2}}(\frac{\partial}{\partial x^2}g_{11})(h_s^1)^2 W^2 = -\int_h [g_{11}(h_s^1)^2]^{-\frac{1}{2}}\{\frac{d}{dt}g_{11}\}(h_s^1)^2$$

As the horocycle is a loop, we may write $W^2(s)$ as a Fourier series

$$W^2(s) = \sum a_m e^{2\pi i m s}.$$

The second equation above, using that in our coordinates $\frac{\partial}{\partial x^2}g_{11} = -2(g_{11})^{\frac{3}{2}}$, shows that $a_0 = \frac{1}{2}\int \frac{d}{dt}g_{11}(h(s))ds$. However, as $|\dot{g}| \asymp |\Phi|$, we see that $|\dot{g}| \asymp e^{-\frac{1}{\ell_n^{\pm}}}$; hence $a_0 = O(e^{-\frac{1}{\ell_n^{\pm}}})$, and so the first equation in (6.5) then controls the remaining Fourier coefficients. After integrating the first equation for W^2 against W^2, we obtain

$$\sum (1 + 4\pi^2 m^2)a_m^2 = \sum a_m^2 + \sum b_m a_m,$$

where $\{b_m\}$ are the Fourier coefficients for the terms on the right-hand side of the first equation in (6.5): these terms depend on time derivatives of the metric and its Christoffel symbols. Our estimates on $\frac{d}{dt}g_{ij}$ and $\frac{d}{dt}\Gamma_{ij}^k$ imply that $(\sum b_m^2)^{\frac{1}{2}} \le O(e^{-\frac{1}{\ell_n^{\pm}}})$. So applying Cauchy-Schwarz and our estimate on a_0, we find that

$$(\sideset{}{'}\sum m^2 a_m^2)^{\frac{1}{2}} \le O(e^{-\frac{1}{\ell_n^{\pm}}}).$$

The geometric conclusion of this analysis is that if we allow initial points $p_n \in \alpha_{0,n}$ and $q_n \in \alpha_{0,n}$ to flow along with the horocycles H_n^+ and H_n^+ of g_t-length $\ell_n^{\pm}$ but restricted to stay in an arc normal to the original horocycle, then the resulting paths $\hat{p}_{t,n}$ and $\hat{q}_{t,n}$ of points (normal to H_n^+ and H_n^-) are of size $O(e^{-\frac{1}{\ell_n^{\pm}}})$ in C^{∞}, as are the horocycles H_n^t themselves. In particular, the angle made between a geodesic arc connecting the pair of g_t-horocycles of length $\ell_n^{\pm}$ at $\hat{p}_{t,n}$ and $\hat{q}_{t,n}$ with the horocycles H_n^t is $\frac{\pi}{2} - O(e^{-\frac{1}{\ell_n^{\pm}}})$. Elementary hyperbolic geometry then says that, if the shortest geodesic between those horocycles H_n^t meets those same horocycles at the points $p_{t,n}$ and $q_{t,n}$, then the paths $\hat{p}_{t,n}$ and $p_{t,n}$ (respectively, the paths $\hat{q}_{t,n}$ and $q_{t,n}$) differ by $O(e^{-2\frac{1}{\ell_n^{\pm}}})$ in C^2. We summarize this analysis as

Lemma 6.2. *Let Γ_n^t be a family of shortest geodesic arcs connecting g_t-horocycles H_n^t of g_t-length $\ell_n^{\pm}$. Let $V_n = \frac{d}{dt}\Gamma_n^t$. Then $\|\nabla_{V_n} V_n\| \to 0$ as $n \to \infty$.*

Of course the vector field V is defined as $V_n = \frac{d}{dt}\alpha_{t,n}$, and so has possibly non-vanishing boundary values $\frac{d}{dt}p_{t,n}$ and $\frac{d}{dt}q_{t,n}$. It is convenient in the computations which follow to replace this vector field by one, say for the moment W_n, whose boundary values vanish. We observe that this replacement will have only a negligible effect on the second derivative of length.

Lemma 6.3. *For variation vector fields V_n and W_n defined by $V_n = \frac{d}{dt}\alpha_{t,n}$ and W_n satisfying the variational equation* (4.9) *with vanishing boundary values, we have that*

$$\int_{\alpha_{0,n}} V_n'^2 + V_n^2 - W_n'^2 - W_n^2 = o(1)$$

as $n \to \infty$.

Proof. The two vector fields satisfy the same equation, but differ in their boundary conditions. Yet, at the points $p_{0,n}$ and $q_{0,n}$, the change in boundary conditions is $O(e^{-\frac{1}{\ell_n^{\pm}}})$, by the analysis in the proof of the previous lemma. Thus the difference $Y_n = W_n - V_n$ satisfies the equation $Y_n'' - Y_n = 0$ with boundary values $Y_n = O(e^{-\frac{1}{\ell_n^{\pm}}})$: we conclude that on the interval $x \in [-\frac{L}{2}, \frac{L}{2}]$, we can estimate the difference $|W_n - V_n| = |Y_n| \leq \frac{o(n)}{\cosh(\frac{L}{2})} \cosh x$. Using this estimate and the boundedness of W_n (maximum principle), we find that

$$\int_{\alpha_{0,n}} V_n'^2 + V_n^2 - W_n'^2 - W_n^2 = o(e^{-\frac{1}{\ell_n^{\pm}}}),$$

as desired.   q.e.d.

6.6. Formulas for the variation of finite arcs. Our next step is to compute the second variation of length of the finite arcs. In view of the previous lemma, for the resulting integral, we may consider the vector field V_n as satisfying the basic equation (4.9) with vanishing boundary conditions, as long as we carry along the error term of $o(e^{-\frac{1}{\ell_n^{\pm}}})$—we will follow this expositional course, rather than change the notation for the variational field at this stage.

As in our discussion of the closed curve case, we express the vector field V_n as an integral of the inhomogeneous term of (4.9) against the kernel for the operator $\frac{d^2}{dy^2} - 1$ in (4.9).

It is straightforward to check that the kernel for the operator $\frac{d^2}{dy^2} - 1$ on the segment $[-\frac{L_n}{2}, \frac{L_n}{2}]$ is given by

$$(6.6)\quad K_n(y,s) = \begin{cases} K_n^-(y,s) = -\frac{\sinh(\frac{L_n}{2}+s)\sinh(\frac{L_n}{2}-y)}{\sinh(L_n)}, & -\frac{L_n}{2} \le s \le y \le \frac{L_n}{2} \\ K_n^+(y,s) - \frac{\sinh(\frac{L_n}{2}-s)\sinh(\frac{L_n}{2}+y)}{\sinh(L_n)}, & -\frac{L_n}{2} \le y \le s \le \frac{L_n}{2}. \end{cases}$$

This gives the representations

$$V_n(y) = \int_{p_n}^{q_n} K_n(y,s)(\operatorname{Im}\mu)'(s)ds,$$

or in terms of the quantity $\mathcal{F} = \operatorname{Im}\mu = -\operatorname{Im}\Phi/g_0$ defined in (4.12), we have

$$(6.7)\qquad V_n(y) = -\int_{p_n}^{q_n} K_n(y,s)\mathcal{F}'(s)ds.$$

Now, we return to computing $D_{22}^2 L$, which, by the classical formula (6.4) and our computations in Lemmas 6.2 and 6.3, we may write as

$$D_{22}^2 L = \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_n'' - V_n)V_n ds + o(e^{-\frac{1}{\ell_n^{\pm}}}).$$

Taking the error term to the right-hand side so as to focus on the integrated terms, we rewrite this expression in terms of the representation (6.7) as

$$D_{22}^2 L + o(e^{-\frac{1}{\ell_n^{\pm}}}) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}'(y)\int_{-\frac{L}{2}}^{y} K_n^-(y,s)\mathcal{F}'(s)dsdy$$
$$+ \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}'(y)\int_{y}^{\frac{L}{2}} K_n^+(y,s)\mathcal{F}'(s)dsdy$$
$$= I_1 + I_2.$$

We treat the integrals in turn. We begin with

$$I_1 \equiv \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}'(y)\int_{-\frac{L}{2}}^{y} K_n^-(y,s)\mathcal{F}'(s)dsdy$$
$$= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\frac{\partial}{\partial y}\int_{-\frac{L}{2}}^{y} K_n^-(y,s)\mathcal{F}'(s)dsdy + \mathcal{F}\left(\frac{L}{2}\right)\int_{-\frac{L}{2}}^{\frac{L}{2}} K_n^-\left(\frac{L}{2},s\right)\mathcal{F}'(s)ds$$
$$- \mathcal{F}\left(-\frac{L}{2}\right)\cdot 0$$

$$= -\int_{\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)K_n^-(y,y)\mathcal{F}'(y)dy - \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\int_{-\frac{L}{2}}^{y} \frac{\partial}{\partial y}K_n^-(y,s)\mathcal{F}'(s)dsdy$$
$$+ o_L(1)$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \{\mathcal{F}(y)\}^2 \frac{\partial}{\partial y}K_n^-(y,y)dy - \frac{1}{2}\left\{\mathcal{F}\left(\frac{L}{2}\right)\right\}^2 K_n^-\left(\frac{L}{2},\frac{L}{2}\right)$$
$$+ \frac{1}{2}\left\{\mathcal{F}\left(-\frac{L}{2}\right)\right\}^2 K_n^-\left(-\frac{L}{2},-\frac{L}{2}\right)$$
$$+ \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\int_{-\frac{L}{2}}^{y} \frac{\partial}{\partial s}\frac{\partial}{\partial y}K_n^-(y,s)\mathcal{F}(s)dsdy - \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)$$
$$\left[\mathcal{F}(y)\frac{\partial}{\partial y}K_n^-(y,s)|_{s=y} - \mathcal{F}\left(-\frac{L}{2}\right)\frac{\partial}{\partial y}K_n^-\left(y,-\frac{L}{2}\right)\right]dy + o_L(1).$$

In this succession of equalities, the first step is via integration by parts on the outer integral, the second is through an application of the fundamental theorem of calculus, and the third step is via another integration by parts.

Next use the following estimates and fact. First, both $\frac{\partial}{\partial y}K_n^-(y,y)$ and $\mathcal{F}(\pm\frac{L}{2})$ are rapidly decreasing in L; moreover, $\frac{\partial}{\partial s}\frac{\partial}{\partial y}K_n^-(y,s)$ differs from $-K_n^-(y,s)$ by a term that is rapidly decreasing in L. Also, $\mathcal{F}(\frac{L}{2})\frac{\partial}{\partial y}K_n^-(y,\frac{L}{2})$ is rapidly decreasing in L, and $\frac{\partial}{\partial y}K_n^-(y,-\frac{L}{2}) = 0$. Finally, note that $\frac{\partial}{\partial y}K_n^-(y,y) \to 0$ but $\frac{\partial}{\partial y}K_n^-(y,s) = \frac{1}{2} + o_L(1)$.

Similarly,

$$I_2 \equiv \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}'(y)\int_{y}^{\frac{L}{2}} K_n^+(y,s)\mathcal{F}'(s)dsdy$$
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\int_{y}^{\frac{L}{2}} \frac{\partial}{\partial s}\frac{\partial}{\partial y}K_n^+(y,s)\mathcal{F}(s)dsdy$$
$$+ \int_{-\frac{L}{2}}^{\frac{L}{2}} [\mathcal{F}(y)]^2 \frac{\partial}{\partial y}K_n^+(y,s)|_{s=y}dy + o_L(1).$$

Thus,

$$D_{22}^2 L = I_1 + I_2$$
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\int_{\frac{L}{2}}^{\frac{L}{2}} -K_n(y,s)\mathcal{F}(s)dsdy + \int_{-\frac{L}{2}}^{\frac{L}{2}} [\mathcal{F}(y)]^2 [\frac{\partial}{\partial y}K_n^+(y,s)|_{s=y}$$
$$- \frac{\partial}{\partial y}K_n^-(y,s)|_{s=y}]dy + o_L(1)$$
$$= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{F}(y)\int_{\frac{L}{2}}^{\frac{L}{2}} K_n(y,s)\mathcal{F}(s)dsdy + \int_{-\frac{L}{2}}^{\frac{L}{2}} [\mathcal{F}(y)]^2 dy.$$

Finally, we wish to take a limit, sending $L \to \infty$, or alternatively, for our horocycle of length $\ell_n^\pm$, letting $n \to \infty$ or the length to zero. To do this we

observe that the kernels K_n tend to limits

$$K(y,s) = \begin{cases} -\frac{e^{s-y}}{2}, & s \le y \\ -\frac{e^{y-s}}{2}, & y \le s \end{cases}$$

which we can write succinctly as

(6.8)
$$K(y,s) = -\frac{1}{2}e^{-d(y,s)}.$$

We conclude that the limit of the term $D_{22}^2 L$ (as $L \to \infty$) may be written as

(6.9)
$$D_{22}^2 L = -\frac{1}{2}\iint_{\alpha_0 \times \alpha_0} e^{-d(s,y)}\operatorname{Im}\mu(s)\operatorname{Im}\nu(y)dsdy + \int_{\alpha_0}[\mathcal{F}(y)]^2 dy.$$

Finally, we combine the original formal equation (6.1) with the equation (6.2) for the first term and what we have now in (6.9) for the second term to find—after an important cancellation of the terms $\int_{\alpha_0}[\mathcal{F}(y)]^2 dy$—the formula

$$\frac{d^2}{dt}L(g_t, \alpha_{t,n}) = \int_{\alpha_0} -2(\Delta-2)^{-1}\frac{|\Phi|^2}{g_0^2}\sqrt{g_0}$$
$$+ \frac{1}{2}\iint_{\alpha_0 \times \alpha_0} e^{-d(s,y)}\operatorname{Im}\mu(s)\operatorname{Im}\nu(y)dsdy.$$

Here, of course, we have made use of Proposition 6.1 to allow us to pass from the finite integral in equation (6.2) to the integral over the entire arc α_0.

We summarize our discussion in this section with formulae for the second variations of an open arc α analogous to those in Theorem 1.1 for the second variations of length of a simple closed curve γ.

Theorem 6.4. *Along the Weil-Petersson geodesic arc $\Gamma(t)$, the second variation $\frac{d^2}{dt^2}\ell$ of the $\Gamma(t)$-length $\ell(t) = L(\Gamma(t), \alpha)$ of a (class of an) arc α is given by the (convergent) expression*
(6.10)
$$\frac{d^2}{dt^2}\ell(t) = \int_{\alpha} -(\Delta-2)^{-1}\frac{2|\Phi|^2}{g_0^2}ds + \frac{1}{2}\iint_{\alpha \times \alpha} e^{-d(s,y)}\operatorname{Im}\mu(s)\operatorname{Im}\mu(y)dsdy.$$

More generally, the Weil-Petersson Hessian $HessL[\frac{\bar\Phi}{g_0}, \frac{bar\Psi}{g_0}]$ is given by the (convergent) expression

(6.11)
$$HessL[\frac{\bar\Phi}{g_0}, \frac{\bar\Psi}{g_0}] = \int_{\alpha} -(\Delta-2)^{-1}\frac{2\operatorname{Re}\Phi\bar\Psi}{g_0^2}ds$$
$$+ \frac{1}{2}\iint_{\alpha \times \alpha} e^{-d(s,y)}\operatorname{Im}\mu(s)\operatorname{Im}\nu(y)dsdy,$$

where $\mu = \frac{\bar\Phi}{g_0}$ and $\nu = \frac{\bar\Psi}{g_0}$ are the harmonic Beltrami differential representatives of two tangent directions at $[g_0]$.

7. Convexity for laminations

We have already seen in Theorem 1.1 and Corollary 5.2 that if γ is a simple closed curve, then on a Weil-Petersson ray $\Gamma = \Gamma(t)$, we have

$$(7.1) \qquad \frac{d^2}{dt^2}\ell_\gamma(\Gamma(t)) \geq \frac{1}{3}\int_\gamma \|\Phi_t\|^2 ds$$

where $\Phi(t)$ is the holomorphic quadratic differential tangent to Γ at $\Gamma(t)$, and $\|\Phi\| = \frac{|\Phi|}{g_0}$. In this section, we extend that result to prove the

Proposition 7.1. *Let* $\Gamma = \Gamma(t)$ *be a Weil-Petersson ray and* λ *a measured lamination on* S. *Then*

$$(7.2) \qquad \frac{d^2}{dt^2}\ell_\lambda(\Gamma(t)) \geq \frac{1}{3}\int_\lambda \|\Phi_t\|^2 ds$$

where Φ_t *is the holomorphic quadratic differential tangent to* Γ *at* $\Gamma(t)$.

Our definition of $\int_\lambda \|\Phi\|^2 ds$ is straightforward and parallels the definition of length of the lamination λ. See [**Bon01**] for a background discussion. In particular, a measured lamination λ is defined as a measure $\lambda(k) = \int_k d\lambda$ on transverse arcs k. Choose arcs $k_1, \ldots, k_J$ which are transverse to λ and construct flow boxes $\{F_i\}$ for λ bounded by the k_j and parallel to λ.

Then if G_λ is the geodesic lamination underlying λ, then $\lambda - \cup_j k_j$ is a (possibly infinite) collection of finite length arcs. The length of λ is then the integral, with respect to the transverse measure $d\lambda$ of λ, of the lengths of the finite arc components. More precisely, we lift each of these components λ to $\hat{\lambda}_a \subset T^1 M$, endow $\hat{\lambda}_a$ with the natural arclength measure ds, and then integrate the product to get

$$\ell(\lambda) =: \int_\lambda ds =: \iint_{\lambda_a} ds\, d\lambda(a).$$

In order to define $\int_\lambda \|\Phi\|^2 ds$, we proceed analogously, except that we note that the function $\|\Phi\|^2$ on S then naturally defines a measure $\|\Phi\|^2 ds$ on $T^1 M$. In other words, we set

$$\int_\lambda \|\Phi\|^2 ds = \iint_{\lambda_a} \|\Phi\|^2 ds\, d\lambda(a).$$

Proof of Proposition 7.1. Let γ_n be a sequence of simple closed curves converging to λ. The idea is to apply (7.1) to γ_n and then take a limit in n to find (7.2).

Now ℓ_γ and ℓ_λ are real analytic functions on Γ [**Ker85**], and thus since $\ell_{\gamma_n} \to \ell_\lambda$, so does $\frac{d^2}{dt^2}\ell_{\gamma_n} \to \frac{d^2}{dt^2}\ell_\lambda$. Thus the left-hand sides of (7.1) converge to the left-hand side of (7.2).

For the right-hand side, the argument is virtually tautological. We first note that the arclength measure $\|\Phi\|^2 ds$ is continuous on $T^1 M$. Then, choose n sufficiently large so that the flow boxes $\{F_i\}$ described above also serve as flow boxes for γ_n. The definition of convergence in $\mathcal{ML}$ then easily implies that the right-hand side of (7.1) converges to the right-hand side of (7.2). q.e.d.

8. Convexity of $\ell^{\frac{1}{2}}$; upper and lower bounds on the Hessian

8.1. The function $\ell^{\frac{1}{2}}$.

8.1.1. The function $\ell^{\frac{2}{3}}$. In light of (7.1), we can quickly refine the basic convexity result for the length ℓ to a convexity result for a concave function of ℓ, namely $\ell^{\frac{2}{3}}$. We will see in the next subsection that this is not sharp, but at this stage this particular bound is elementary, so we include it.

To begin, recall from (3.7) that the first variation of length may be expressed as

$$\frac{d}{dt}\ell_\gamma(\Gamma(t)) = \int_{\gamma_0} \frac{\operatorname{Re}\Phi}{g_0} ds.$$

We can then estimate this derivative as

$$|\frac{d}{dt}\ell_\gamma(\Gamma(t))| \leq \int_{\gamma_0} |\frac{\operatorname{Re}\Phi}{g_0}| ds$$

$$\leq \big(\int_{\gamma_0} \frac{|\Phi|^2}{g_0^2} ds\big)^{\frac{1}{2}} \ell_{\gamma_0}^{\frac{1}{2}}.$$

Squaring and combining with (7.1) yields

(8.1) $$\ell\frac{d^2}{dt^2}\ell_\gamma(\Gamma(t)) \geq \frac{1}{3}\big(\frac{d}{dt}\ell_\gamma(\Gamma(t))\big)^2.$$

We compute $\frac{d^2}{dt^2}\ell^{\frac{2}{3}}(\Gamma_t)$ and substitute in the above inequality to conclude the

Corollary 8.1. *The function $\ell^{\frac{2}{3}}$ is Weil-Petersson convex on Teichmüller space.*

8.1.2. The length of an annulus. In this subsection, we compute in two ways the second variation of the length of the core geodesic in a hyperbolic annulus; the basic result is well-known (see Example 3.6 in [**Wol08**]).

Let $\mathcal{C}(\ell)$ denote a complete hyperbolic annulus whose core geodesic has length ℓ. This cylinder may be parametrized as

$$\mathcal{C}(\ell) = [-\frac{\pi}{2\ell}, \frac{\pi}{2\ell}] \times [0, 1],$$

where top and bottom edges are identified, and we consider the metric cylinder as equipped with the hyperbolic metric $g = ds_\ell^2 = \ell^2 \sec^2 \ell x|dz|^2$.

Consider the rotationally harmonic map $R(t) : \mathcal{C}(\ell) \to \mathcal{C}(\ell + t)$ which does not twist the boundary; in other words, this map may be expressed in coordinates as $R(t) = u(t) + iv(t)$ where $u(t)(z) = u(t)(x)$ and $v(t)(z) = y$. Now the rotationally invariant holomorphic quadratic differentials on a cylinder have a particularly simple form: we may write one as $\Phi = cdz^2$ in the complex coordinates above. We may then compute

$$\nu = \frac{R(t)_{\bar{z}}}{R(t)_z} = \frac{u' - 1}{u' + 1}$$

and so $u' = \frac{1+\nu}{1-\nu}$. On the other hand, since $c = \Phi = g(\ell(t))\mathcal{H}(\ell(t))\nu(t)$, we see since $\nu(0) = 0$ and $\mathcal{H}(\ell(0)) = 1$, that $\dot{c} = g\dot{\nu} = g\frac{d}{dt}[\frac{u'-1}{u'+1}] = \frac{\dot{u}'}{2}g$. We conclude that $\frac{d}{dt}u_t' = 2\frac{\dot{c}}{g}$.

If $c(t)$ is the factor so that the Hopf differential is parametrizing a family of harmonic maps $R(t) : \mathcal{C}(\ell) \to \mathcal{C}(\ell + t)$ whose targets are progressing through Teichmüller space $\mathcal{T}(\mathcal{C})$ at unit Weil-Petersson speed, then the choice of $c(t)$ provides for $c(t)dz^2$ to be the Hopf differential for the map $R(t)$.

Now, for $w(t)$ to have image $\mathcal{C}(\ell + t)$, we must have the boundary of $\mathcal{C}(\ell)$ map to the boundary of $\mathcal{C}(\ell + t)$, i.e.

$$\frac{\pi}{2(\ell + t)} = u(\frac{\pi}{2\ell}) = \int_0^{\frac{\pi}{2\ell}} u'(x)dx + u(0).$$

Upon differentiating in t, we obtain

$$-\frac{\pi}{2\ell^2} = \int_0^{\frac{\pi}{2\ell}} \dot{u}'$$

$$= \int_0^{\frac{\pi}{2\ell}} 2\dot{c}g^{-1}$$

$$= 2\dot{c}\int_0^{\frac{\pi}{2\ell}} \ell^{-2}\cos^2\ell x\, dx$$

$$= \frac{\pi\dot{c}}{2\ell^3}$$

after finding the resulting elementary integrals. We obtain $\dot{c} = -\ell$. Therefore

$$\|\frac{\partial}{\partial\ell}\|_{WP}^2 = \|\dot{c}g_\ell^{-1}\|_{WP}^2 = \|(-\ell)\ell^{-2}\cos^2\ell x\|_{WP}^2 = \frac{\pi}{2\ell}$$

after another explicit integration. Thus $\|\frac{\partial}{\partial\ell}\|_{WP} = (\frac{\pi}{2\ell})^{\frac{1}{2}}$ and so $ds_{WP}^2 = \frac{\pi}{2\ell}d\ell^2$ on the Teichmüller space $\mathcal{T}(\mathcal{C})$. This implies that on this space, $\frac{d\ell}{ds} = (\frac{2\ell}{\pi})^{\frac{1}{2}}$ and so $\ell = (2\pi)^{-1}s^2$. Thus the length ℓ of the core geodesic satisfies that $\ell^{\frac{1}{2}}$ is convex, but not convex to any lower power.

8.1.3. $\ell^{\frac{1}{2}}$ is convex. We now offer a geometric proof of Wolpert's recent result [**Wol08**] that $\ell_\gamma^{\frac{1}{2}}$ is Weil-Petersson convex. In effect, we will compare the general case to the rotationally symmetric case we treated explicitly just above, finding that the rotationally invariant case is an extremum in the space of cases.

Comparison of lifted harmonic map to rotationally invariant map. The essential point is best understood in the setting of the annular covers $(\mathcal{C}, \tilde{g}_t)$ of the family of surfaces (S, g_t). Consider the harmonic maps $w_t : (S, g_0) \to (S, g_t)$ and their lifts $\tilde{w}_t : (S, \tilde{g}_0) \to (S, \tilde{g}_t)$. These lifts are in the homotopy class of the rotationally invariant harmonic map $R_t : \mathcal{C}(\ell_0) \to \mathcal{C}(\ell_t)$, where $\ell_\gamma(g_t) = \ell_t$. Now w_t is conformal only at the (isolated) zeroes of the Hopf differential, and so, off of small neighborhoods of the zeroes of the lifted Hopf differential, the harmonic map $\tilde{w}_t$ has quasi-isometric constant uniformly bounded away from 1. By contrast, one can either compute or reason geometrically that the rotationally invariant harmonic map $R_t : \mathcal{C}(\ell_0) \to \mathcal{C}(\ell_t)$ has quasi-isometric constant tending uniformly to 1 as one leaves compacta in $\mathcal{C}(\ell_0)$: the image curves are growing exponentially in length, so because the longitudinal curves are stretching at incomparably lower rates, the requirement of energy efficiency forces the map to be increasingly close to an isometry as one leaves compact sets.

Let $\mathcal{H}^R(t)$ be the holomorphic energy (see equations (3.1)–(3.3)) of the rotationally invariant map $R(t)$, and $\mathcal{H}(t)$ be the holomorphic energy of $\tilde{w}_t$. Since both $R(t)$ and $\tilde{w}_t$ are the identity when $t = 0$, and since, as we have just seen, $R(t)$ is asymptotically an isometry while $\tilde{w}_t$ is boundedly away from being the identity off small sets, then we find

$$\ddot{\mathcal{H}}(t) \geq \ddot{\mathcal{H}}^R(t)$$

outside some large compact set (at least away from small neighborhoods of the zeroes of the lift $\tilde{\Phi}$ of Φ). [Here we are using that the real analytic functions $\mathcal{H}(t), \mathcal{H}^R(t) \geq 1$, that $\mathcal{H}(0) = \mathcal{H}^R(0) = 1$ everywhere, and that $\mathcal{H}^R(t)(z) \to 1$ as $z \to \partial \mathcal{C}(\ell_0)$.] In particular, parametrizing $\mathcal{C}(\ell)$ as in subsection 8.1.2, we see that

$$(8.2) \qquad \int_{x=\pm\frac{\pi}{2\ell}\mp\delta} \ddot{\mathcal{H}}(t) \geq \int_{x=\pm\frac{\pi}{2\ell}\mp\delta} \ddot{\mathcal{H}}^R(t).$$

A comparison of ODEs. The rest of the proof follows by applying other inequalities that reflect that $R(t)$ is a harmonic map of lower (regularized in some way) energy than $\tilde{w}_t$. In particular, consider the Fourier expansion $\Phi = \sum b_n(x)e^{2\pi i n y}$ of the quadratic differential Φ (where the map $\tilde{w}_t$ has Hopf differential Φ).

Because

$$\dot{\ell} = \int_{x=0} \frac{\operatorname{Re}\Phi}{g}\sqrt{g}\,dy = \ell\operatorname{Re}b_0$$

we know that the Hopf differential Φ^R for the rotationally invariant map $R(t)$ must be $\Phi^R = \operatorname{Re}b_0 dz^2$ on $\mathcal{C}(\ell)$: this is because the targets $\mathcal{C}(\ell + t)$ agree for the two maps w_t and $R(t)$, and hence the change in core-curve length is the same.

The upshot is that, for an arbitrary constant curvature circle $\{x = \xi\}$, we have

$$(8.3) \qquad \int_{x=\xi} \frac{|\Phi|^2}{g^2}ds = g^{-\frac{3}{2}}(x)\sum |b_n|^2 \geq g^{-\frac{3}{2}}(x)(\operatorname{Re}b_0)^2 = \int_{x=\xi} \frac{|\Phi^R|^2}{g^2}ds.$$

Of course, we know from formula (1.2) that

$$\ddot{\ell} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g^2}ds = \frac{1}{2}\int_{\gamma_0} \ddot{\mathcal{H}}ds.$$

To estimate this last integral, let u be the solution of

$$\Delta_g u - 2u = \frac{-2|\Phi|^2}{g^2}.$$

If we were to integrate this equation along the vertical parameter curves $\{x = const\}$, we would obtain an ordinary differential equation for the function $\int_x u\,dy = \int_x \frac{1}{2}\ddot{\mathcal{H}}dy$ in the single variable $x \in (-\frac{\pi}{2\ell}, \frac{\pi}{2\ell})$, i.e.

$$(8.4) \qquad \frac{1}{g}\partial_x^2\int_x u\,dy - 2\int_x u\,dy = -2\int_x \frac{|\Phi|^2}{g^2}dy.$$

Of course, a similar equation holds for the integrals $\int_x \frac{1}{2}\ddot{\mathcal{H}}^R dy$ and $\int_x \frac{|\Phi^R|^2}{g^2}dy$ associated to the rotationally invariant map. Indeed, inequality (8.2) says that there are boundary points $x = \pm\frac{\pi}{2\ell}\mp\delta$ at which $\int_x \frac{1}{2}\ddot{\mathcal{H}}^R dy \leq \int_x \frac{1}{2}\ddot{\mathcal{H}}dy$;

moreover, inequality (8.3) asserts that on the interval $(-\frac{\pi}{2\ell}, \frac{\pi}{2\ell})$, the right-hand side of (8.4) is less in the lifted case than in the rotationally invariant case.

The upshot is that the comparison principle for ordinary differential equations implies that

$$
\begin{aligned}
\int_x -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2} dy &= \int_x \frac{1}{2} \ddot{\mathcal{H}}^R dy \\
&\geq \int_x \frac{1}{2} \ddot{\mathcal{H}} dy \\
&= \int_x -2(\Delta - 2)^{-1} \frac{(\operatorname{Re} b_0)^2}{g^2} dy.
\end{aligned}
$$

In particular, specializing to the curve $\{x = 0\}$, and recalling the implication above of (1.2), we find

$$
\ddot{\ell} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{|\Phi|^2}{g^2} \geq \int_{\gamma_0} -2(\Delta - 2)^{-1} \frac{(\operatorname{Re} b_0)^2}{g^2} = \ddot{\ell^R} \geq 2 \frac{\dot{\ell_R}^2}{\ell_R} = 2 \frac{\dot{\ell}^2}{\ell}.
$$

Here the last inequality is inherited from the rotationally invariant case of the last subsection. For the last equality, we simply recall that our choice that $\operatorname{Re} \int_{\gamma_0} \Phi = \int_{\gamma_0} \Phi^R$ implies that the infinitesimal change of lengths agrees between the lifted and rotationally invariant maps. We conclude

Corollary 8.2. *(Wolpert [**Wol08**]) The function $\ell^{\frac{1}{2}}$ is Weil-Petersson convex in the Teichmüller space $\mathcal{T}(S)$.*

8.2. A general upper bound for the Hessian. We have already seen a lower bound for the Hessian of length in Corollary 5.2 and Proposition 7.1. In this passage, we note an easy upper bound as well.

We begin by noting that if

$$
(\Delta - 2)h = -2 \frac{|\Phi|^2}{g_0^2}
$$

on a surface S, then the maximum principle implies

$$
\text{(8.5)} \qquad\qquad\qquad h \leq \| \frac{|\Phi|^2}{g_0^2} \|_\infty
$$

where the right-hand side is the maximum of the function $\frac{|\Phi|^2}{g_0^2}$ on S. In the formula (1.2), this will estimate the first term.

To estimate the second term, we consider equation (4.18) (combined with (4.12))

$$
\text{(8.6)} \qquad\qquad\qquad U_{yy} - U = -\frac{\operatorname{Im} \Phi}{g_0}.
$$

The maximum principle then implies that

$$
\text{(8.7)} \qquad\qquad\qquad U \leq \max_\gamma \left| \frac{\operatorname{Im} \Phi}{g_0} \right|.
$$

Thus the second term in formula (1.2) is estimated as

$$(8.8) \quad \int_\gamma U_y^2 + U^2 = -\int_\gamma (U_{yy} - U)U$$

$$= \int_\gamma (\frac{\operatorname{Im}\Phi}{g_0})U$$

$$\leq \int_\gamma \big(\max_\gamma |\frac{\operatorname{Im}\Phi}{g_0}|\big)\big(\max_\gamma |\frac{\operatorname{Im}\Phi}{g_0}|\big) \quad \text{after a substitution}$$

$$= \ell_\gamma(\max_\gamma |\frac{\operatorname{Im}\Phi}{g_0}|)^2.$$

We conclude, taking into account Corollary 5.2, that

Corollary 8.3.

$$(8.9) \quad \frac{1}{3}\int_\gamma \|\Phi_t\|^2 ds \leq \frac{d^2}{dt^2}\ell_\gamma(\Gamma(t)) \leq \ell_\gamma(\max_S \|\frac{|\Phi|^2}{g_0^2}\| + (\max_\gamma |\frac{\operatorname{Im}\Phi}{g_0}|)^2).$$

8.3. Estimates for the Weil-Petersson connection near the compactification divisor. In this passage we refine the method above to estimate the Weil-Petersson connection on a codimension two distribution $\mathcal{P} \subset T\mathcal{M}$ of the tangent bundle near the Deligne-Mumford compactification divisor (i.e. for surfaces with small injectivity radius) which is in some sense "parallel" to the tangent bundle of the compactification divisor. Roughly, we prove that $\mathcal{P}$ is quite flat, in the sense that for $X, Y \in \mathcal{P}$, we will have that the normal component $(\nabla_X Y)^\perp$ of $(\nabla_X Y)$ satisfies $(\nabla_X Y)^\perp = O(\ell^2)$. (Here ℓ signifies the length of the curve which vanishes on the nearby component of the compactification divisor.)

To state this precisely, we choose a simple closed curve $\gamma \subset S$ and a small number $\ell > 0$; we consider the level set $L_\gamma(\ell)$ of hyperbolic surfaces for which $L(\cdot, \gamma) = \ell$. The set $L_\gamma(\ell)$ is a submanifold of the Teichmüller space $\mathcal{T}$ of real codimension one, and it is orthogonal to the vector $\operatorname{grad}\ell_\gamma$, the Weil-Petersson gradient of ℓ_γ. Let J be the almost complex structure of $\mathcal{T}$ and consider the projection $\tau \in TL_\gamma(\ell)$ of $J\operatorname{grad}\ell_\gamma$ into $TL_\gamma(\ell)$. Let $\mathcal{P} \subset TL_\gamma(\ell)$ denote the distribution of $(\dim \mathcal{T} - 2)$-planes in $TL_\gamma(\ell)$ orthogonal to the span of $\operatorname{grad}\ell_\gamma$ and τ; note that $\mathcal{P}$ is Weil-Petersson orthogonal to both $\operatorname{grad}\ell_\gamma$ and $J\operatorname{grad}\ell_\gamma$.

REMARK. One does not expect $\mathcal{P}$ to be integrable. In particular, one does not expect that $J\operatorname{grad}\ell_\gamma$ is parallel to the Fenchel-Nielsen twist vector field. (See [**Wol82**].) Nevertheless, it will be a consequence of equation (8.11) of the early part of the next proof that, as $\ell_\gamma \to 0$, the distribution $\mathcal{P}$ converges to the tangent bundle $T\mathcal{C}_\gamma$ of the compactification divisor $\mathcal{C}_\gamma = \{\ell_\gamma = 0\}$ of the augmented Teichmüller space $\bar{\mathcal{T}}$. (Compare [**Wol91**].)

Now, for $X \in \mathcal{P}$ and Y a section of $\mathcal{P} \to \mathcal{T}$ (i.e. a vector field on $L_\gamma(\ell)$), we can consider the Weil-Petersson covariant derivative $\nabla_X Y$. Of course, the vector $\nabla_X Y$ has components both in $\mathcal{P}$ and in the orthogonal complement $\mathcal{P}^\perp$ of $\mathcal{P}$; we focus here on the component of this vector in the orthogonal complement.

REMARK. It might be difficult to formulate general results on the full vector $\nabla_X Y$ in useful and incisive ways. For example, if we were to "lift" a curve α

from the compactification divisor $\mathcal{C}_\gamma$ to an almost parallel curve $\hat{\alpha}$ tangent to $\mathcal{P}$, then since $\nabla_{\dot{\alpha}}\dot{\alpha} \subset T\mathcal{C}_\gamma$ can be arbitrary, so might we expect $\nabla_{\dot{\alpha}}\dot{\hat{\alpha}}$ to have an arbitrary non-orthogonal component.

Our main result in this section is

Theorem 8.4. *In the notation above, for $X \in \mathcal{P}$ and $Y \subset \mathcal{P}$ of unit norm, we have*

$$(8.10) \qquad\qquad (\nabla_X Y)^\perp = < \nabla_X Y, \mathcal{P}^\perp > = O(\ell_\gamma^2).$$

In particular, $< \nabla_X Y, \operatorname{grad} \ell_\gamma > = O(\ell_\gamma^2)$ and $< \nabla_X Y, J \operatorname{grad} \ell_\gamma > = O(\ell_\gamma^2)$.

REMARK. Similar results were obtained recently by Wolpert [**Wol08**] and [**Wol09**], using his estimates on the Hessian. From the formula for the Hessian presented in (1.2), we see in the present derivation the elementary nature of the expansion of $\nabla_X Y$ in ℓ_γ. The flatness of order $O(\ell_\gamma^2)$ occurs because the (explicit) rotationally invariant even solution $u(x) = x\tan \ell x + 1/\ell$ of the Jacobi equation has the following property: at the core geodesic of the cylinder, this function $u(x)$ is smaller by a factor comparable to $O(\ell_\gamma)$ than it is on the boundary of the cylinder. Since the geodesic has length $O(\ell_\gamma)$, the dominant term of the integral of $u(x)$ over the geodesic decays like $O(\ell_\gamma^2)$.

Proof. We begin the proof of Theorem 8.4 with a preliminary proposition characterizing the quadratic differentials which represent elements of $\mathcal{P}$.

Proposition 8.5. *Let $X \in T_{[M,g]}\mathcal{T}$ with $X \in \mathcal{P}$ and $\Phi = \Phi_X$ be a holomorphic quadratic differential on the surface (S, g) for which $\bar{\Phi}/g$ is a harmonic Beltrami differential representing X. Then for γ the geodesic on S used to define $L_\gamma(\ell)$, we have*

$$(8.11) \qquad\qquad \int_\gamma \frac{\Phi}{g} ds = 0.$$

Proof. Since $X \in \mathcal{P}$, we have $< X, \operatorname{grad} \ell_\gamma > = 0$. But from (3.7), we find that

$$\begin{aligned} 0 &= < X, \operatorname{grad} \ell_\gamma > \\ &= X(\ell_\gamma) \\ &= \int_\gamma \frac{\operatorname{Re}\Phi}{g} ds. \end{aligned}$$

Of course the other defining condition of $X \in \mathcal{P}$ is that $< X, J \operatorname{grad} \ell_\gamma > = 0$. But since the Weil-Petersson metric is Kähler, this implies

$$\begin{aligned} 0 &= < X, J \operatorname{grad} \ell_\gamma > \\ &= - < JX, \operatorname{grad} \ell_\gamma > . \end{aligned}$$

But JX is represented by $i\bar{\Phi}/g$, and so we derive as above that

$$0 = \int_\gamma \frac{\operatorname{Re} i\Phi}{g} ds,$$

proving the result.

$$\text{q.e.d.}$$

REMARK. The covectors in $T^*\mathcal{C}_\gamma$ are represented by holomorphic quadratic differentials which have at worst simple poles at the nodes. On the other hand, a generic (real) covector in $T^*_\Sigma \bar{\mathcal{T}}$ (written as $(\frac{dt_j}{t_j}, ds_k)$ in the customary $(\vec{s}, \vec{t})$ plumbing notation) described in terms of a meromorphic quadratic differential with a second order pole and other data (see e.g. [**Wol91**]), at an element $\Sigma \in \mathcal{C}_\gamma$ has a second order pole with a non-vanishing residue at the node obtained by pinching γ.

Continuation of the Proof of Theorem 8.4. The heart of the matter is a computation of $\mathrm{Hess}\ell_\gamma(X, X)$, in particular to prove

Lemma 8.6. *For $X \subset \mathcal{P}$ as above, we have*

$$(8.12) \qquad\qquad Hess\ell_\gamma(X, X) = O(\ell_\gamma^2).$$

To see that this is enough, note first that polarization will imply then that $\mathrm{Hess}\ell_\gamma(X, Y) = O(\ell_\gamma^2)$ for $X, Y \subset \mathcal{P}$ of unit norm. But then

$$
\begin{aligned}
- < \nabla_X Y, \mathrm{grad}\,\ell_\gamma > &= -(\nabla_X Y)\ell_\gamma \\
&= (XY - \nabla_X Y)\ell_\gamma \qquad \text{since } X, Y \text{ are tangent to } L_\gamma(\ell) \\
&= \mathrm{Hess}\ell_\gamma(X, Y) \qquad \text{by definition} \\
(8.13) \qquad &= O(\ell_\gamma^2) \qquad \text{by Lemma 8.6.}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
< \nabla_X Y, J\,\mathrm{grad}\,\ell_\gamma > &= - < J\nabla_X Y, \mathrm{grad}\,\ell_\gamma > \\
&\qquad \text{as } J \text{ is a Weil-Petersson isometry} \\
&= - < \nabla_X JY, \mathrm{grad}\,\ell_\gamma > \\
&\qquad \text{as Weil-Petersson is a Kähler metric} \\
&= - < \nabla_X Z, \mathrm{grad}\,\ell_\gamma >
\end{aligned}
$$

for some $Z \in \mathcal{P}$ as $\mathcal{P}$ is J-invariant, being orthogonal to a J-invariant subspace of a Kähler manifold. Then $< \nabla_X Y, J\,\mathrm{grad}\,\ell_\gamma >= O(\ell_\gamma^2)$ follows in the manner of (8.13). This concludes the proof of Theorem 8.4, pending the proof of the main lemma.   q.e.d.

Proof of Lemma 8.6. The basic idea of the proof is to estimate the terms in the formula (1.2) for the Hessian, where we take X to be represented by a harmonic Beltrami differential $\mu = \bar{\Phi}/g$, and we apply the features of $X \subset \mathcal{P} \subset TL_\gamma(\ell)$ to prove that those terms are small. In particular, because the length $L(g, \gamma)$ is small, the geodesic γ is embedded in a wide, thin collar. Then, by Proposition 8.5, we learn that $|\Phi|$ must decay rapidly toward the center of the collar. Those facts together are enough to conclude that each of the pair of terms in (1.2) is small.

We carry out the plan in steps.

Step 0. The collar Consider the collar $\mathcal{C} = [-\frac{1}{\ell}\sec^{-1}\frac{1}{\ell}, \frac{1}{\ell}\sec^{-1}\frac{1}{\ell}] \times [0, 1]$ with horizontal edges $[-\frac{1}{\ell}\sec^{-1}\frac{1}{\ell}, \frac{1}{\ell}\sec^{-1}\frac{1}{\ell}] \times \{0, 1\}$ identified, equipped with the hyperbolic metric $g_0 = \ell^2 \sec^2 \ell x |dz^2|$. This collar $\mathcal{C}$ embeds in a neighborhood of the geodesic γ, with $\{0\} \times [0, 1]$ mapping onto γ.

Step 1. Decay of $|\Phi|$ On this annular collar $\mathcal{C}$, we may regard the quadratic differential Φ as a function (or more formally, we divide Φ by the nonvanishing holomorphic quadratic differential dz^2 to obtain a function in the quotient). Then, by the rotational invariance of the collar (or by working in Fermi coordinates), we see that g may be taken as constant along γ, and so the conditions in Proposition 8.5 imply that Φ has no period in the collar. Finally, we estimate boundary conditions.

Since, for ℓ small, every $X \subset \mathcal{P} \subset TL_\gamma(\ell)$ can be approximated on compacta away from γ by an integrable meromorphic quadratic differential on a noded Riemann surface in the compactification divisor $\mathcal{C}_\gamma$, and the collection of such quadratic differentials of unit Weil-Petersson norm is compact, we see that we may take $|\Phi|$ as bounded on the horocycle of length one on $\partial\mathcal{C}$.

Because Φ is holomorphic and hence harmonic, the vanishing of the period together with the fixed boundary conditions is enough to show that $|\Phi|$ decays rapidly on the interior of the cylinder.

We can obtain an estimate of this decay through a Fourier analysis of Φ. On the cylinder $\mathcal{C}$, set $\Phi = \sum a_n(x)e^{2\pi i n y}$. Then the harmonicity of Φ implies

$$0 = \triangle\Phi = \sum(a_n''(x) - 4\pi n^2 a_n(x))e^{2\pi i n y}$$

and in particular the equations

$$a_n''(x) - 4\pi n^2 a_n(x) = 0.$$

Now as $a_0(0) = \int_{x=0}\Phi = 0$ by (8.11), we find that $\int_{\gamma^*}\Phi = 0$ along any cycle γ^* in $\mathcal{C}_\gamma$ homologous to γ; hence $a_0(x) = 0$.

Of course, $\int_{\partial\mathcal{C}_\gamma}|\Phi|^2 \leq C_0$ by our argument on limits above, and so

$$(8.14) \qquad \sum_{n\neq 0} a_n^2(\pm\frac{1}{\ell}\sec^{-1}\frac{1}{\ell}) = \int_{\partial\mathcal{C}_\gamma}|\Phi|^2 \leq C_0.$$

We note that

$$(8.15) \qquad (\sum_{n\neq 0} a_n^2(x))'' = \sum_{n\neq 0} 2a_n''a_n + 2(a_n')^2$$
$$= \sum_{n\neq 0} 8\pi^2 n^2 a_n^2 + 2(a_n')^2 \qquad \text{by (8.3)}$$
$$\geq 8\pi^2\sum_{n\neq 0} a_n^2.$$

Thus, by the maximum principle applied to the differential inequality (8.15) with boundary conditions (8.14), we have

$$(8.16) \qquad \int_{x=x_0}|\Phi|^2 \leq \frac{C_0\cosh\sqrt{8}\pi x_0}{\cosh\sqrt{8}\pi(\frac{1}{\ell}\sec^{-1}\frac{1}{\ell})} := D\cosh\sqrt{8}\pi x_0.$$

Finally, consider $\Phi(z_0)$ for $z_0 = x_0 + iy_0$, a fixed point of the collar. Note that on our parametrization of the collar, the balls of radius $\frac{1}{2}$ inject into the

parameter domain. As Φ is harmonic,

$$\begin{aligned}
|\Phi(z_0)| &\leq \frac{4}{\pi} |\int_{B_{\frac{1}{2}}(z_0)} \Phi| \\
&\leq \frac{4}{\sqrt{\pi}} (\int_{B_{\frac{1}{2}}(z_0)} |\Phi|^2)^{\frac{1}{2}} \\
&< \frac{4}{\sqrt{\pi}} (\int_{|\operatorname{Re}(z-z_0)| \leq \frac{1}{2}} |\Phi|^2)^{\frac{1}{2}} \\
&= \frac{4}{\sqrt{\pi}} (\int_{\operatorname{Re} z_0 - \frac{1}{2}}^{\operatorname{Re} z_0 + \frac{1}{2}} \int_{x=t} |\Phi|^2 dy dt)^{\frac{1}{2}} \\
&\leq \frac{4}{\sqrt{\pi}} (\int_{\operatorname{Re} z_0 - \frac{1}{2}}^{\operatorname{Re} z_0 + \frac{1}{2}} D \cosh \sqrt{8}\pi t dt)^{\frac{1}{2}} \qquad \text{by (8.16)} \\
&= D_1 (\cosh \sqrt{8}\pi x_0)^{\frac{1}{2}}.
\end{aligned}$$

We conclude that

$$|\Phi(z_0)|^2 < D_2 \cosh \sqrt{8}\pi x_0,$$

where (using (8.16))

(8.17) $$D_2 = O(e^{-\frac{\sqrt{8}\pi}{\ell}})$$

in ℓ, for ℓ small, justifying our remark about the decay of $|\Phi(z)|$ into the collar.

Step 2. The function U There are two terms in the formula (1.2) for the Hessian; here we estimate the one involving the energy of the function U. In particular, we already know from (8.8) that

$$\begin{aligned}
\int_{\gamma_0} U'^2 + U^2 &\leq \ell (\max \frac{|\Phi|}{g})^2 \\
&\leq \ell g^{-2}|_{x=0} D_2 \\
&= O(\ell^{-3} e^{\frac{-\sqrt{8}\pi}{\ell}})
\end{aligned}$$

using $g|_{x=0} = \ell^2$ and (8.17). This term is then consistent with the statement of the lemma that $\operatorname{Hess}\ell(X, X) \leq O(\ell^2)$.

Step 3. The function $(\triangle - 2)^{-1}\frac{|\Phi|^2}{g^2}$ We are left to estimate the second term in the expression (1.2) for $\operatorname{Hess}\ell(X, X)$, namely

$$\int_{\gamma} -2(\triangle - 2)^{-1}\frac{|\Phi|^2}{g^2}.$$

In particular, we need to estimate the solution u_0 to the equation

(8.18) $$(\triangle - 2)u_0 = -2\frac{|\Phi|^2}{g^2}$$

evaluated on the core geodesic.

We again estimate the solution to this by estimating it on the cylinder $\mathcal{C}_\gamma$, on which we have good control on $\frac{|\Phi|^2}{g^2}$. Of course we already know from the maximum principle (see (8.5)) that

$$u_0 \leq \sup \frac{|\Phi|^2}{g^2}.$$

Now the latter is bounded on $\mathcal{C}_\gamma$, and the complement of $\mathcal{C}_\gamma$ has bounded geometry on which $\int_{M \sim \mathcal{C}_\gamma} \frac{|\Phi|^2}{g^2} dA_g \leq 1$ (because Φ is of unit Weil-Petersson norm). We conclude that there is a C_0 for which

$$\frac{|\Phi|^2}{g^2}\Big|_{\partial \mathcal{C}_\gamma} \leq C_0.$$

Thus by the maximum principle, it is enough to estimate the solution u of the boundary value problem

$$\frac{1}{g} u''(x) - 2u(x) = \frac{D_2 \cosh \sqrt{8}\pi x}{g^2}$$

$$u(\pm \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}) = C_0.$$

Using that $g = \ell^2 \sec^2 \ell x$, we rewrite the equation above as

$$(8.19) \qquad u''(x) - 2\ell^2 (\sec^2 \ell x) u(x) = D_3 \cosh(\sqrt{8}\pi x) \cos^2 \ell x$$

$$u(\pm \frac{1}{\ell} \sec^{-1} \frac{1}{\ell}) = C_0,$$

where $D_3 = O(\ell^{-2} e^{-\sqrt{8}\pi/\ell})$. The homogeneous equation

$$u''(x) - 2\ell^2 (\sec^2 \ell x) u(x) = 0$$

has the two solutions

$$(8.20) \qquad\qquad\qquad u_1(x) = \tan \ell x$$

$$u_2(x) = x \tan \ell x + \frac{1}{\ell}.$$

Using these, one can solve for a particular solution of the form $u_0 = u_1 v_1 + u_2 v_2$ by elementary integrations, namely

$$v_1 = \int u_2 D_3 \cosh(\sqrt{8}\pi x) \cos^2 \ell x$$

$$v_2 = -\int u_1 D_3 \cosh(\sqrt{8}\pi x) \cos^2 \ell x.$$

An asymptotic expansion shows that $u_1 v_1 + u_2 v_2|_{\partial \mathcal{C}_\gamma} = O(1)$; this is actually quite remarkable, as both $u_1 v_1$ and $u_2 v_2$ are separately comparable to ℓ^{-3} on $\partial \mathcal{C}_\gamma$. Moreover, $u_1 v_1 + u_2 v_2|_{x=0} = O(e^{-\sqrt{8}\pi/\ell} \ell^{-2})$. With these computations in mind, we observe that the general solution to (8.19) is given by

$$u = c_1 u_1 + c_2 u_2 + u_1 v_1 + u_2 v_2.$$

By our estimates on $u_1 v_1 + u_2 v_2|_{\partial \mathcal{C}_\gamma}$ and the definitions of u_1 and u_2, we see that we may take $c_1 = O(\ell)$ and $c_2 = O(\ell^2)$. Thus we compute that

$$
\begin{aligned}
u(0) &= c_1 u_1(0) + c_2 u_2(0) + u_1(0) v_1(0) + u_2(0) v_2(0) \\
&= c_2 u_2(0) + u_2(0) v_2(0) \qquad \text{since } u_1(0) = 0 \\
&= O(\ell) + O(e^{-\sqrt{8}\pi/\ell} \ell^{-3}) \\
&= O(\ell).
\end{aligned}
$$

We conclude that $-2(\triangle - 2)^{-1} \frac{|\Phi|^2}{g^2}\Big|_\gamma = u_0\Big|_\gamma \le u(0) = O(\ell)$ and so

$$
\int_\gamma -2(\triangle - 2)^{-1} \frac{|\Phi|^2}{g^2} = \int_\gamma u\, ds \le O(\ell) \int_\gamma ds
$$
$$
= O(\ell^2).
$$

Combining the estimate for this term of the Hessian with the estimate for the other term of the Hessian discussed in Step 2 concludes the proof of the lemma.

<div align="right">q.e.d.</div>

8.4. Upper bounds on twisting.

We have seen in (3.7) that the period $\operatorname{Re} \int_{\gamma_0} \frac{\Phi}{g_0} ds = \operatorname{Re} \int_{\gamma_0} \frac{\Phi}{\sqrt{g_0}} dy$ (where the latter integral is expressed as the period of the one-form $\frac{\Phi}{\sqrt{g_0}}$ around a parameter loop $\{x = const\}$) records the infinitesimal change in the length of the curve $[\gamma_0]$ under the deformation determined by Φ.

We turn our attention in this section to the imaginary part $\operatorname{Im} \int_{\gamma_0} \frac{\Phi}{g_0} ds = \operatorname{Im} \int_{\gamma_0} \frac{\Phi}{\sqrt{g_0}} dy$ of the period of the one-form $\frac{\Phi}{\sqrt{g_0}}$, which we may regard as reflecting the 'twist' of the surface about $[\gamma_0]$ or the twist of the Weil-Petersson geodesic Γ about the locus $\{\ell_{\gamma_0} = 0\}$ in the augmented Teichmüller space.

In particular, immediately from our main formula (1.2) we have

$$
\begin{aligned}
\frac{d^2}{dt^2} \ell(t) &\ge \frac{1}{2 \sinh(\frac{\ell}{2})} \iint_{\gamma_0 \times \gamma_0} \operatorname{Im} \mu(p)[\cosh(d(p,q) - \frac{\ell}{2})] \operatorname{Im} \mu(q) ds(p) ds(q) \\
&\ge \frac{1}{2 \sinh(\frac{\ell}{2})} \int_{\gamma_0} \operatorname{Im} \frac{\Phi(p)}{g_0(p)} ds(p) \int_{\gamma_0} \operatorname{Im} \frac{\Phi(q)}{g_0(q)} ds(q),
\end{aligned}
$$

since $\cosh(d(p,q) - \frac{\ell}{2}) \ge 1$. Thus defining

$$
\operatorname{Per}_{\gamma_0} \Phi = \int_{\gamma_0} \frac{\Phi}{g_0} ds,
$$

we conclude immediately the

Proposition 8.7. *If $\frac{d}{dt} g_t = 2 \operatorname{Re} \Phi$, then $\frac{d^2}{dt^2} \ell_{\gamma_0}(t) \ge (\operatorname{Im} \operatorname{Per}_{\gamma_0} \Phi)^2$.*

Example 1. We imagine a Weil-Petersson geodesic arc parametrized at unit speed with a large component of 'twist' near the locus $\{\ell_{\gamma_0} = 0\}$. Let $\mathcal{C}$ denote a collar around the short geodesic γ_0, so that in the collar $\mathcal{C} = [-a, a] \times [0, 1]$, the metric g_0 and the quadratic differential Φ may be expressed

as $g_0 = \ell^2 \sec^2 \ell x |dz|^2$ and $\Phi = cdz^2 + l.o.t.$ Then these conditions translate to

$$
\begin{aligned}
1 = \|\frac{\Phi}{g_0}\|^2 &= \iint_{\mathcal{R}} \frac{|\Phi|^2}{g_0^2} dA \\
&\geq \iint_{\mathcal{C}} \frac{|\Phi|^2}{g_0^2} dA \\
&\asymp \int_0^1 \int_{-a}^{a} \frac{c^2}{\ell^2 \sec^2 \ell x} dx dy \\
&= c^2 O(\ell^3)
\end{aligned}
$$

after an explicit integration. Thus $c \leq O(\ell^{\frac{3}{2}})$. In this example, we study a geodesic path where we assume this residue term c to be as large as these considerations allow, i.e. we assume that along Γ, we have the constant term cdz^2 for the deformation term Φ to satisfy $c \geq k_0 \ell^{\frac{3}{2}}$, for some fixed constant k_0. Of course, we are continuing to imagine this geodesic arc Γ to be near the locus $\{\ell_{\gamma_0} = 0\}$.

We next apply the above proposition in this setting to obtain a bound on the behavior of the Weil-Petersson geodesic arc Γ. As we are assuming ℓ to be small, we have that $\sinh \frac{\ell}{2} \asymp \frac{\ell}{2}$, and thus the proposition yields

$$
\begin{aligned}
\frac{d^2}{ds^2} \ell(s) &\geq \frac{k_1}{\ell} \left(\int_0^1 \frac{\Phi}{\ell^2 \sec^2 \ell x} \ell \sec \ell x dy|_{\{x=0\}} \right)^2 \\
&= \frac{k_1}{\ell} \left(\int_0^1 \frac{c}{\ell} dy \right)^2 \\
&= \frac{k_1}{\ell} \left(\frac{O(\ell^{\frac{3}{2}})}{\ell} \right)^2 \\
&= O(1).
\end{aligned}
$$

Now for a geodesic progressing at unit speed, we have seen in (3.7) that $|\frac{d\ell_{\gamma_0}}{ds}| = |\operatorname{Re} \int \frac{\Phi}{g_0} ds|$, and so, for small ℓ, is also bounded by a computation similar to the one just above. Putting together the estimates

$$
|\frac{d\ell(s)}{ds}| < O(\ell^{\frac{1}{2}})
$$

$$
|\frac{d^2}{ds^2} \ell(s)| \geq O(1)
$$

shows that such a twisting geodesic can only stay near the locus $\{\ell_{\gamma_0} = 0\}$ for a brief $(o(1))$ length.

9. The Thurston metric and the Weil-Petersson metric

¿From the formula (1.2), we can easily derive the result [**Wol86b**] (see also [**McM08**] and [**Bon88**]) that the Thurston metric is a multiple of the Weil-Petersson metric.

9.1. The Thurston metric. We begin by recalling the Thurston metric. To define this, imagine a sequence $\{\gamma_n\}$ of closed curves which are becoming equidistributed in the sense that if B is a ball in the unit tangent bundle

T^1S, and we lift γ_n to its representative in T^1S, then

$$\lim_{n \to \infty} \frac{\ell(\gamma_n \cap B)}{\ell(\gamma_n)} = \frac{\text{Volume}(B)}{\text{Volume}(T^1S)}.$$

Thurston noted that since, for such a sequence of curves, $\frac{d\ell(\gamma_n)}{\ell(\gamma_n)} \to 0$ as $n \to \infty$, then $\frac{\text{Hess}\ell(\gamma_n)}{\ell(\gamma_n)}$ would tend to a symmetric quadratic form on $T(S)$; by the convexity of the length function, this tensor would be positive semi-definite, hence a (pseudo)-metric. Wolpert showed that

Theorem 9.1. [**Wol86b**]. *The Thurston metric is a multiple of the Weil-Petersson metric.*

The goal of the present section is to give a proof of this result that proceeds from evaluating formula (1.2) on a sequence $\{\gamma_n\}$ of curves that are becoming equidistributed in T^1S.

9.2. First variation. We begin by first showing that $d\ell(\gamma_n)/\ell(\gamma_n) \to 0$ as $n \to \infty$. Recall from (3.7) that

$$\frac{d}{dt}L(g_t, \gamma_{n,t}) = \int_{\gamma_n(0)} \frac{\text{Re}\,\Phi}{g_0} ds.$$

Now, the curves γ_n have unit tangent vectors $\frac{d}{ds}\gamma_n(s)$ which equidistribute themselves in T^1S, and so

$$\begin{aligned}
\frac{1}{\ell(\gamma_n)}\frac{d}{dt}L(g_t, \gamma_{n,t}) &= -\frac{1}{\ell(\gamma_n)}\int_{\gamma_n} \frac{\text{Re}\,\Phi}{g_0} ds \\
&= -\int_{\gamma_n} \frac{\text{Re}\,\Phi}{g_0} \frac{ds}{\ell(\gamma_n)} \\
&\to \iiint_{T^1S} \frac{\text{Re}\,\Phi(p,\theta)}{g_0} \frac{d\,\text{vol}_{T^1S}(p,\theta)}{\text{vol}(T^1S)}
\end{aligned}$$

where we interpret the meaning of the notation as follows. In the discussion so far, we have written $\text{Re}\,\Phi$ to denote the value of the expression $\text{Re}\,\varphi$, when the quadratic differential $\Phi = \varphi dz^2$ was written in coordinates $z = x + iy$ with $\frac{\partial}{\partial y}$ being tangent to the geodesic. Now the vector field $\frac{\partial}{\partial y}$ lifts to the canonical vector fields in T^1S tangent to the geodesic flow. We let the expression $\text{Re}\,\Phi(p,\theta)$ denote the value of $\text{Re}\,\Phi$ on the surface in terms of a coordinate $z = x + iy$ in which the geodesic direction described by $(p,\theta) \in T^1S$ is in the coordinate direction $\frac{\partial}{\partial y}$. Of course, if we change surface coordinates so that $z_\theta = e^{-i\theta}z$, then for $\Phi = \varphi_\theta dz_\theta^2$, we have $\varphi_\theta = e^{2i\theta}\varphi_0$. Thus, when we integrate along the fiber of $T^1S \to S$ (with respect to θ in the coordinates (z, θ) for T^1S), we find $\int \frac{\text{Re}\,\varphi(p,\theta)}{g_0}d\theta = 0$.

9.3. The second variation. We recall the (second) formula (1.2) for the second variation of length:

$$\frac{1}{\ell(\gamma_n)}\frac{d^2}{dt^2}\ell(\gamma_n(t))$$

$$= \frac{1}{\ell(\gamma_n)}\int -(\Delta - 2)^{-1}\frac{2|\Phi|^2}{g_0^2}ds$$

$$+ \frac{1}{\ell(\gamma_n)}\iint_{\gamma_n\times\gamma_n}\operatorname{Im}\mu(p)\frac{\cosh(d(p,q)-\frac{\ell(\gamma_n)}{2})}{2\sinh(\frac{\ell(\gamma_n)}{2})}\operatorname{Im}\mu(q)ds(p)ds(q)$$

$$= \frac{1}{\ell(\gamma_n)}\int -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}ds$$

$$+ \frac{1}{\ell(\gamma_n)}\int_{\gamma_n}\frac{\operatorname{Im}\Phi(p)}{g_0(p)}\int_{\gamma_n}\frac{\cosh(d(p,q)-\ell(\gamma_n)/2)}{2\sinh(\ell(\gamma_n)/2)}\frac{\operatorname{Im}\Phi(q)}{g_0(q)}dqdp$$

$$= I_1 + I_2.$$

We examine the two integrals I_1 and I_2 separately. The first integral is immediate, as

$$I_1 = \int_{\gamma_n} -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}\frac{ds}{\ell(\gamma_n)} \longrightarrow 2\pi\iint_M -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}\frac{d\operatorname{Area}}{2\pi\operatorname{Area}}$$

by the equidistribution property, and

$$\iint_M -2(\Delta - 2)^{-1}\frac{|\Phi|^2}{g_0^2}\frac{d\operatorname{Area}}{\operatorname{Area}} = \iint\{-2(\Delta - 2)^{-1}(1)\}\frac{|\Phi|^2}{g_0^2}\frac{d\operatorname{Area}}{\operatorname{Area}}$$

as the operator $(\Delta - 2)^{-1}$ is self-adjoint. Then as $-2(\Delta - 2)^{-1}(1) = 1$, we find that

$$(9.1) \qquad\qquad I_1 \longrightarrow \iint \frac{|\Phi|^2}{g_0^2}\frac{d\operatorname{Area}}{\operatorname{Area}}.$$

9.4. The integral I_2. We turn next to I_2, where the computation is a bit more involved. Our basic plan mirrors our discussion of the first variation; we extend the terms in the integrand of I_2 to all of T^1S, and then integrate over the circular fiber to be left with an integral over the surface.

We begin our more detailed discussion of the second (energy) integral by considering the version (4.33) of it in terms of a geometric kernel, i.e.

$$I_2 = \frac{1}{\ell(\gamma_n)}\int_{\gamma_0}\frac{\operatorname{Im}\Phi(p)}{g_0(p)}\int_{\gamma_0}\frac{\cosh(d(p,q)-L/2)}{2\sinh(L/2)}\frac{\operatorname{Im}\Phi(q)}{g_0(q)}dqdp$$

where $L = \ell(\gamma_0)$.

Now considering γ_0 as an embedded curve in the unit tangent bundle $T^1(S, g_0)$, if we fix the point $p = (\bar{p}, v) \in T^1S$ as representing a point $\bar{p} \in S$ and a unit vector $v \in T_p^1S$, then a point $q = (\bar{q}, w)$ along γ_0 at distance t from p could be written

$$q = \exp_{\bar{p}} tv = G_t p$$

where G_t denotes the geodesic flow in T^1S for distance t.

Thus we see that as $L \to \infty$, the integral I_2 converges to

$$\lim_{L\to\infty} I_2 = \int_{T^1 S} \frac{\operatorname{Im}\Phi(p)}{g_0(p)} \int_0^\infty \frac{e^{-t}}{2} \Big(\frac{\operatorname{Im}\Phi(G_t p)}{g_0(G_t(p))} + \frac{\operatorname{Im}\Phi(G_{-t}p)}{g_0(G_{-t}(p))}\Big) \frac{dtdp}{\operatorname{vol}(T^1 S)}.$$

Of course, as p varies in the fiber $\{(\bar{p}, e^{i\theta}v)\}$ over $\bar{p} \in S$, we observe the points $G_t(\bar{p}, v)$ also arising as $G_{-t}(\bar{p}, -v)$, and so we may rewrite the above limit integral as

$$\lim_{L\to\infty} I_2 = \int_{T^1 S} \frac{\operatorname{Im}\Phi(p)}{g_0(p)} \int_0^\infty e^{-t} \frac{\operatorname{Im}\Phi(G_t p)}{g_0(G_t(p))} \frac{dtdp}{\operatorname{vol}(T^1 S)}.$$

To evaluate this last integral, imagine $\bar{p} = 0$ in the disk $\{|z| < 1\}$, and we represent the hyperbolic metric as

$$g_0 = \frac{4|dz|^2}{(1-r^2)^2} = g_0(0)(1-r^2)^{-2}|dz|^2.$$

An important matter here (as it was in the calculation of the first variation) is the question of how to interpret the meaning of $\operatorname{Im}\Phi(q)$ in these coordinates: recall that we understood $\operatorname{Im}\frac{\Phi}{g_0}$ to be the value of $\operatorname{Im}\frac{\Phi}{g_0}$ when we defined the geodesic γ_0 as a vertical line in the coordinate system. In the fixed coordinate system of the disk $\{|z| < 1\}$, write $\Phi = \phi(z)dz^2$; then since the hyperbolic geodesic through the origin and the point $z = re^{i\theta}$ is given by the ray $te^{i\theta}$, we see that $\operatorname{Im}\Phi|_{re^{i\theta}} = \operatorname{Im} e^{2i\theta}\phi(re^{i\theta})$. With this notation, our integral becomes

$$\lim_{L\to\infty} I_2$$
$$= \int_S \int_0^{2\pi} \frac{\operatorname{Im} e^{2i\theta}\phi(0)}{g_0(p)} e^{-t} \frac{\operatorname{Im} e^{2i\theta}\phi(r(t)e^{i\theta})}{g_0(p)(1-r(t)^2)^{-2}} \frac{dtd\theta d\operatorname{Area}}{2\pi \operatorname{Area}(S)}$$
$$= \int_S \int_0^\infty \frac{e^{-t}(1-r(t)^2)^2}{g_0^2(p)} \int_0^{2\pi} \operatorname{Im}(e^{2i\theta}\phi(0))\operatorname{Im}(e^{2i\theta}\phi(r(t)e^{i\theta})) \frac{dtd\theta d\operatorname{Area}}{2\pi \operatorname{Area}(S)}.$$

Using the change-of-coordinates formula $e^{-t}dt = 2(1+r)^{-2}dr$ and the mean value theorem for harmonic functions (together with a half-angle formula to simplify the averaging), we find

$$\lim_{L\to\infty} I_2 = \int_S \frac{1}{g_0(p)^2} \Big(\int_0^1 2(1-r)^2 dr\Big) \frac{|\phi(0)|^2}{2} \frac{d\operatorname{Area}}{\operatorname{Area}(S)}$$

(9.2)
$$= \frac{1}{3} \int_S \frac{|\Phi(p)|^2}{g_0(p)^2} \frac{d\operatorname{Area}(p)}{\operatorname{Area}(S)}.$$

Combining (9.1) and (9.2), we verify Theorem 9.1. In particular,

$$\lim_{n\to\infty} \frac{\operatorname{Hess}\ell(\gamma_n)}{\ell(\gamma_n)} = \frac{4}{3} \int_S \frac{|\Phi(p)|^2}{g_0(p)^2} \frac{d\operatorname{Area}(p)}{\operatorname{Area}(S)}$$
$$= \frac{4}{3\operatorname{Area}(S)} \|\mu\|_{\mathrm{WP}}^2.$$

REMARK. The constant $\frac{4}{3}$ found here agrees with that found by Wolpert [**Wol86b**] and McMullen [**McM08**]. See the comments ([**McM08**], p. 376) of McMullen on the consistency of conventions.

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