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A CHARACTERIZATION OF HARMONIC SPACES

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Abstract

The authors proved in [**5**] that in a complete, connected, and simply connected Riemannian manifold, the volume of the intersection of two small geodesic balls depends only on the distance between the centers and the radii if and only if the space is harmonic. In this paper, we show that this proposition remains true, if the condition is restricted to balls of equal radii.

1. Introduction

Z. I. Szabó proved in [**12**] (corollary 2.1) that in a complete, connected, simply connected, and harmonic Riemannian manifold, the volume of the intersection of two geodesic balls depends only on the distance between the centers and the radii of the balls. As this property is related to the extendability of the Kneser-Poulsen conjecture to Riemannian manifolds, it was denoted by KP_2 in [**6**] and [**5**], where the reader can find more details about this connection. The converse of Szabó's theorem holds as well ([**5**]); namely, if a complete, connected, and simply connected Riemannian manifold has the KP_2 property, then the manifold is harmonic. In this paper, we prove that this statement is also true under a weaker condition $KP_2^=$, which requires the KP_2 condition only for balls of equal radii (Theorem 1). This proposition was conjectured in [**5**], where it was verified for symmetric spaces.

In general, we say that a Riemannian manifold has the KP_k property if the volume of the intersection of k geodesic balls depends only on the distances between the centers and the radii of the balls. The KP_1 property is quite closely related to the notion of ball-homogeneity, introduced in [**11**]. Recall that a Riemannian manifold is called ball-homogeneous if the volume of *small* geodesic balls depends only on the radius. Ball-homogeneous spaces have been studied by many authors; see, e.g., the papers [**3**], [**4**], [**7**] and the references therein. KP_1 manifolds are ball-homogeneous, and ball-homogeneous spaces are known to be of constant scalar curvature. The first author and D. Kunszenti-Kovács [**6**] proved that if a complete connected Riemannian manifold has the KP_3 property, then it is one of the simply connected spaces

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of constant curvature. Conversely, every complete, connected, simply connected space of constant curvature obviously has the KP_k property for all k.

The present paper is an organic continuation of [**5**]. For the sake of the reader, Section 2 gives a summary of those results from [**5**] that will be needed for the proof of our main theorem given in Section 3. A key step of the proof is Theorem 2, which claims that $KP_2^=$ manifolds are D'Atri spaces.

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2. Preliminaries

In this section, we recall some notations and propositions from [**5**] that will be used later.

Consider a unit speed geodesic γ in a Riemannian manifold. Choose a parallel orthonormal frame $E_1, \dots, E_n$ along γ such that $E_n = \gamma'$. If $\mathbf{v} \in \gamma'(t)^\perp = \{\mathbf{x} \in T_{\gamma(t)}M \mid \mathbf{x} \perp \gamma'(t)\}$ is a tangent vector at a given point, then we shall denote by $[\mathbf{v}]$ the column vector of its coordinates with respect to the basis $E_1(t), \dots, E_{n-1}(t)$. Denote by $R(t)$ the matrix of the restriction to $\gamma'(t)^\perp$ of the Jacobi operator for $\gamma'(t)$ with respect to the basis $E_1(t), \dots, E_{n-1}(t)$. For a real r, let $J(r,.)\colon \mathbb{R} \to \mathbb{R}^{(n-1)\times(n-1)}$ be the solution of the following matrix differential equation:

(i) $\partial_2^2 J(r,t) + R(t)J(r,t) = 0$,
(ii) $J(r,r) = 0$,
(iii) $\partial_2 J(r,r) = I$,

where ∂_i stands for the partial derivative of a multivariable function with respect to its ith variable.

For an arbitrary tangent vector $\mathbf{v} \in \gamma'(r)^\perp$, consider the vector field $J_{r,\mathbf{v}}$ along γ defined uniquely by $[J_{r,\mathbf{v}}(t)] = J(r,t)[\mathbf{v}]$. This is the Jacobi field along γ satisfying the initial conditions $J_{r,\mathbf{v}}(r) = \mathbf{0}$ and $J'_{r,\mathbf{v}}(r) = \mathbf{v}$. If $\gamma(t)$ is close to $\gamma(r)$, i.e., $\gamma(t)$ is in the neighborhood onto which the exponential map at $\gamma(r)$ is a diffeomorphism, then $\gamma(t)$ is not conjugate to $\gamma(r)$ along γ, so $J(r,t)$ is invertible.

Denote by $\Sigma(r,t)$ the geodesic sphere of radius $|r|$ centered at $\gamma(t)$. If $|r|$ is sufficiently small, then $\Sigma(r,t)$ is a smooth hypersurface passing through $\gamma(t+r)$. In such a case, consider the Weingarten map of $\Sigma(r,t-r)$ at $\gamma(t)$ with respect to the normal vector $\gamma'(t)$, and denote by $L(r,t)$ its matrix with respect to the basis $E_1(t), \dots, E_{n-1}(t)$. We know that $L(r,t)J(t-r,t) = -\partial_2 J(t-r,t)$ for small $|r| > 0$ (see lemma 3 in [**8**] or proposition 1 in [**5**]), that is,

$$L(r,t) = -\partial_2 J(t-r,t)J(t-r,t)^{-1}. \tag{1}$$

We also have (equation (10) in [**5**])

$$L(r,t) = I\frac{1}{r} + \frac{R(t)}{3}r + O\left(r^2\right). \tag{2}$$

A recursive formula to compute higher order terms of the Laurent series can be found in [**9**]. Equation (2) shows that we cannot define L for $r = 0$, so we sometimes consider the function $L^0(r,t) = rL(r,t)$, which we can extend to $(0,t)$ smoothly with the value 0 for all t. Obviously we have

$$L^0(r,t) = I + \frac{R(t)}{3}r^2 + O\left(r^3\right). \tag{3}$$

As R is symmetric, the Wronskian

$$J(r_1,t)^{\mathrm{T}}\partial_2 J(r_2,t) - \partial_2 J(r_1,t)^{\mathrm{T}} J(r_2,t) \tag{4}$$

is constant along γ. This means that its values at $t = r_1$ and $t = r_2$ are equal, that is,

$$-J(r_2,r_1) = J(r_1,r_2)^{\mathrm{T}}. \tag{5}$$

We also use that (4) is equal to its value at $t = r_2$. Thus,

$$J(r_1,t)^{\mathrm{T}}\partial_2 J(r_2,t) - \partial_2 J(r_1,t)^{\mathrm{T}} J(r_2,t) = J(r_1,r_2)^{\mathrm{T}}.$$

Rearranging, we get

$$\begin{aligned} &\partial_2 J(r_2,t)J(r_2,t)^{-1} - \left(\partial_2 J(r_1,t)J(r_1,t)^{-1}\right)^{\mathrm{T}} \\ &\qquad\qquad = {J(r_1,t)^{-1}}^{\mathrm{T}} J(r_1,r_2)^{\mathrm{T}} J(r_2,t)^{-1}. \end{aligned} \tag{6}$$

Equations (1), (6), and $L^{\mathrm{T}} = L$ give

$$-L(t-r_2,t) + L(t-r_1,t) = {J(r_1,t)^{-1}}^{\mathrm{T}} J(r_1,r_2)^{\mathrm{T}} J(r_2,t)^{-1}. \tag{7}$$

Define the function $D_\gamma(r,t) = \det(L(r,t) - L(-r,t))$. The function $r \mapsto D_\gamma(r,t)$ has a pole singularity at 0; nevertheless, its germ at 0 can be defined in the usual way. We will use the following proposition.

Proposition 1 ([**5**]). *If in a Riemannian manifold, the volume of the intersection of two geodesic balls of the same radius depends only on the common value of the radii and the distance between the centers, then the germ of the function $r \mapsto D_\gamma(r,t)$ at $r = 0$ does not depend on t and γ.*

As a consequence, in a manifold of type $KP_2^=$, we can define the function D on a small punctured neighborhood of 0 by $D(r) = D_\gamma(r,t)$.

It was also proved in [**5**] that every $KP_2^=$ manifold is Einstein, so the metric is analytic in normal coordinates by the Kazdan-DeTruck theorem ([**2**]). In this case, the Jacobi operator is analytic, and by the Cauchy-Kowalevski theorem, J and consequently L and L^0 are also analytic.

3. Equivalence of the $KP_2^=$ property and harmonicity

In this section, we prove the main result of the paper.

Theorem 1. *A complete, connected, and simply connected Riemannian manifold is harmonic if and only if the volume of the intersection of two geodesic balls of the same radius depends only on the distance between the centers and the common value of the radii of the balls.*

The "only if" part is known ([**12**]); the proof of the "if" part consists of two steps. First we prove that every $KP_2^=$ manifold is a D'Atri space. For the definition and basic properties of D'Atri spaces, see [**10**] or §2.7 [**1**].

Theorem 2. *Every Riemannian manifold having the $KP_2^=$ property is a D'Atri space.*

Proof. We shall use that a Riemannian manifold is a D'Atri space if and only if $h_p(q) = h_q(p)$ holds for all sufficiently close points p, q of the manifold, where h_m denotes the mean curvature function of small geodesic spheres centered at m ([**1**]). We can rewrite this condition with our notation so that for any geodesic curve γ,

$$\mathrm{tr}L(a-b,a) + \mathrm{tr}L(b-a,b) = 0$$

holds for small distinct values of a and b. Fix a geodesic γ and let J and L be the matrix valued functions defined in Section 2. Choose $\varepsilon > 0$ such that both $D(r)$ and $L(r,t)$ are defined for $0 < |r| \le 2\varepsilon$ and $|t| \le 2\varepsilon$. Assuming $0 < |r| \le \varepsilon$ and $|t| \le \varepsilon$, we can express the function D with the help of equation (7):

$$\begin{aligned} D(r) &= \det(L(r,t) - L(-r,t)) \\ &= \det\left(J(t-r,t)^{-1\,\mathrm{T}} J(t-r,t+r)^{\mathrm{T}} J(t+r,t)^{-1}\right) \\ &= \frac{\det J(t-r,t+r)}{\det J(t-r,t)\det J(t+r,t)}. \end{aligned}$$

Rearranging, we get

$$\det J(t-r,t)\det J(t+r,t) D(r) = \det J(t-r,t+r). \tag{8}$$

Taking the logarithmic derivative of (8) with respect to t gives

$$\begin{aligned} &\mathrm{tr}(\partial_1 J(t-r,t)J(t-r,t)^{-1}) + \mathrm{tr}(\partial_2 J(t-r,t)J(t-r,t)^{-1}) + \\ &\mathrm{tr}(\partial_1 J(t+r,t)J(t+r,t)^{-1}) + \mathrm{tr}(\partial_2 J(t+r,t)J(t+r,t)^{-1}) \\ &\qquad = \mathrm{tr}(\partial_1 J(t-r,t+r)J(t-r,t+r)^{-1}) + \\ &\qquad\quad \mathrm{tr}(\partial_2 J(t-r,t+r)J(t-r,t+r)^{-1}). \end{aligned} \tag{9}$$

We can transform those terms in (9) which contain the differentiation ∂_1 into terms containing ∂_2 with the help of equation

$$\partial_1 J(r_1,r_2) = -(\partial_2 J(r_2,r_1))^{\mathrm{T}},$$

which follows from (5). Using the symmetry relation (5) a few more times, (9) can be brought to the form

$$\begin{aligned}&\operatorname{tr}(\partial_2 J(t,t-r)J(t,t-r)^{-1}) + \operatorname{tr}(\partial_2 J(t-r,t)J(t-r,t)^{-1}) +\\ &\operatorname{tr}(\partial_2 J(t,t+r)J(t,t+r)^{-1}) + \operatorname{tr}(\partial_2 J(t+r,t)J(t+r,t)^{-1})\\ &\qquad = \operatorname{tr}(\partial_2 J(t+r,t-r)J(t+r,t-r)^{-1}) +\\ &\qquad\quad \operatorname{tr}(\partial_2 J(t-r,t+r)J(t-r,t+r)^{-1}).\end{aligned}$$

Applying (1) yields

$$\begin{aligned}&\operatorname{tr}(L(-r,t-r)) + \operatorname{tr}(L(r,t)) + \operatorname{tr}(L(r,t+r)) + \operatorname{tr}(L(-r,t))\\ &\qquad = \operatorname{tr}(L(-2r,t-r)) + \operatorname{tr}(L(2r,t+r)).\end{aligned} \tag{10}$$

Define the function $f\colon [-\varepsilon,+\varepsilon]\times[-\varepsilon,+\varepsilon] \to \mathbb{R}$ by

$$f(a,b) = \operatorname{tr}(L(a-b,a)) + \operatorname{tr}(L(b-a,b)).$$

L has singularity at $(0,t)$ for all t; thus, $f(a,b)$ is defined initially only when $a \neq b$. However, as shown by (2), f has an analytic extension to the diagonal $a = b$ such that $f(a,a) = 0$. Now we prove that the function f is identically 0, which will complete the proof. Equation (10) says that

$$f(t-r,t) + f(t,t+r) = f(t-r,t+r),$$

or equivalently,

$$f(a,b) = f\left(a, \frac{a+b}{2}\right) + f\left(\frac{a+b}{2}, b\right).$$

If we apply this k times, we get

$$f(a,b) = \sum_{i=0}^{2^k-1} f\left(a + \frac{i}{2^k}(b-a), a + \frac{i+1}{2^k}(b-a)\right). \tag{11}$$

According to (3), the Taylor polynomial of $\operatorname{tr}L^0(r,t)$ with Lagrange remainder term with respect to r is

$$\operatorname{tr}L^0(r,t) = n - 1 + \frac{\operatorname{tr}R(t)}{3}r^2 + \frac{\operatorname{tr}\partial_1^3 L^0(\tilde{r},t)}{6}r^3,$$

where $|\tilde{r}| < |r|$. The function $|\operatorname{tr}\partial_1^3 L^0|$ has a maximal value C on the compact set $[-2\varepsilon,+2\varepsilon]\times[-\varepsilon,+\varepsilon]$. As the manifold is Einstein, $\operatorname{tr}R$ is constant. So we have $|f(a',b')| \leq \frac{C}{3}|a'-b'|^2$ for every $a',b' \in [-\varepsilon,+\varepsilon]$. Applying this upper bound to the summands on the right-hand side of (11), we obtain

$$\begin{aligned}|f(a,b)| &\leq \sum_{i=0}^{2^k-1} \left| f\left(a + \frac{i}{2^k}(b-a), a + \frac{i+1}{2^k}(b-a)\right)\right|\\ &\leq 2^k \frac{C}{3}\left|\frac{|b-a|}{2^k}\right|^2 = \frac{C}{3}\frac{|b-a|^2}{2^k}\end{aligned}$$

for arbitrary $a, b \in [-\varepsilon, +\varepsilon]$. If k tends to infinity, we get $f(a,b) = 0$, as we wanted to prove. q.e.d.

Now we are ready to prove the main result.

Proof of Theorem 1. Let ε be as in the proof of Theorem 2 and assume $0 < |r| \le \varepsilon$ and $|t| \le \varepsilon$. Differentiating the logarithm of (8) with respect to r yields

(12)
$$\begin{aligned} &-\mathrm{tr}(\partial_1 J(t-r,t)J(t-r,t)^{-1})+\mathrm{tr}(\partial_1 J(t+r,t)J(t+r,t)^{-1})+(\log D)'(r) \\ &= -\mathrm{tr}(\partial_1 J(t-r,t+r)J(t-r,t+r)^{-1})+\mathrm{tr}(\partial_2 J(t-r,t+r)J(t-r,t+r)^{-1}). \end{aligned}$$

In the same way as equation (10) was obtained from equation (9), we can transform equation (12) to the form

$$\begin{aligned} \mathrm{tr}(L(-r,t-r)) - \mathrm{tr}(L(r,t+r)) + (\log D)'(r) \\ = \mathrm{tr}(L(-2r,t-r) - \mathrm{tr}(L(2r,t+r)). \end{aligned}$$

Since the manifold is D'Atri, $h_m(\exp_m(\mathbf{v})) = h_m(\exp_m(-\mathbf{v}))$ holds for any sufficiently small tangent vector $\mathbf{v}$ (see [**1**]), which means

$$\mathrm{tr}(L(-r,t-r)) = -\mathrm{tr}(L(r,t+r))$$

with our notation. We also have $\mathrm{tr}(L(-2r,t-r)) = -\mathrm{tr}(L(2r,t+r))$; consequently,

$$-2\mathrm{tr}(L(r,t+r)) + (\log D)'(r) = -2\mathrm{tr}(L(2r,t+r)). \tag{13}$$

As (13) holds for every $t, r \in \mathbb{R}$ such that $0 < |r| \le \varepsilon$ and $|t| \le \varepsilon$, equation

$$(\log D)'(r) = -2\mathrm{tr}(L(2r,t)) + 2\mathrm{tr}(L(r,t)) \tag{14}$$

is valid for $0 < |r| \le \frac{\varepsilon}{2}$ and $|t| \le \frac{\varepsilon}{2}$. Since the manifold is analytic, $\mathrm{tr}L^0(r,t)$ is an analytic function. Thus we can write the trace of the Weingarten map as the sum of a Laurent series:

$$\mathrm{tr}(L(r,t)) = \frac{n-1}{r} + \sum_{i=1}^{\infty} a_i(t) r^i.$$

With this substitution, equation (14) takes the form

$$(\log D)'(r) = \frac{n-1}{r} - 2\sum_{i=1}^{\infty}(2^i - 1)a_i(t)r^i.$$

Thus, the coefficients of the Laurent series of the function $(\log D)'(r)$ determine the coefficients $a_i(t)$, from which we conclude that $\mathrm{tr}(L(r,t))$ depends only on r (but neither on t nor on the geodesic γ). This means that the mean curvature of small geodesic balls is a constant depending only on the radius, which is one of the equivalent definitions of harmonicity. q.e.d.

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