J. DIFFERENTIAL GEOMETRY
88 (2011) 395-482

DIRICHLET DUALITY AND THE NONLINEAR DIRICHLET PROBLEM ON RIEMANNIAN MANIFOLDS

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Abstract

In this paper we study the Dirichlet problem for fully nonlinear second-order equations on a riemannian manifold. As in our previous paper [HL_4], we define equations via closed subsets of the 2-jet bundle where each equation has a natural dual equation. Basic existence and uniqueness theorems are established in a wide variety of settings. However, the emphasis is on starting with a constant coefficient equation as a model, which then universally determines an equation on every riemannian manifold which is equipped with a topological reduction of the structure group to the invariance group of the model. For example, this covers *all branches* of the homogeneous complex Monge-Ampère equation on an *almost complex* hermitian manifold X.

In general, for an equation F on a manifold X and a smooth domain $\Omega \subset\subset X$, we give geometric conditions which imply that the Dirichlet problem on Ω is uniquely solvable for all continuous boundary functions. We begin by introducing a weakened form of comparison which has the advantage that local implies global. We then introduce two fundamental concepts. The first is the notion of a monotonicity cone M for F. If X carries a global M-subharmonic function, then weak comparison implies full comparison. The second notion is that of boundary F-convexity, which is defined in terms of the asymptotics of F and is used to define barriers. In combining these notions the Dirichlet problem becomes uniquely solvable as claimed.

This article also introduces the notion of *local affine jet-equivalence* for subequations. It is used in treating the cases above, but gives results for a much broader spectrum of equations on manifolds, including inhomogeneous equations and the Calabi-Yau equation on almost complex hermitian manifolds.

A considerable portion of the paper is concerned with specific examples. They include a wide variety of equations which make sense on any riemannian manifold, and many which hold universally on almost complex or quaternionic hermitian manifolds, or topologically calibrated manifolds.

Received 10/8/2009.

Contents

1. Introduction

In a recent article [HL_4] the authors studied the Dirichlet problem for fully nonlinear equations of the form $\mathbf{F}(\mathrm{Hess}\, u) = 0$ on smoothly bounded domains in $\mathbf{R}^n$. Our approach employed a duality which enabled us to geometrically characterize domains for which one has existence and uniqueness of solutions for all continuous boundary data. These results covered, for example, all branches of the real, complex, and quaternionic Monge-Ampère equations, and all branches of the special Lagrangian potential equation.

Here we shall extend these results in several ways. First, all results in [HL_4] are shown to carry over to riemannian manifolds with an appropriate *topological* reduction of the structure group. For example, we treat the complex Monge-Ampère equation on almost complex manifolds with hermitian metric. Second, the results are extended to equations involving the full 2-jet of functions. There still remains a basic notion of duality, and a geometric form of F-boundary convexity. Existence and uniqueness theorems are established, and a large number of examples are examined in detail.

In [HL$_4$] our approach was to replace the function $\mathbf{F}(\mathrm{Hess}\, u)$ by a closed subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ of the symmetric $n \times n$ matrices subject only to the *positivity condition*

$$F + \mathcal{P} \ \subset \ F \tag{1.1}$$

where $\mathcal{P} \equiv \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0\}$ (cf. [K]). Such an F was called a *Dirichlet set* in [HL$_4$] but will be called a *subequation* here. A C^2-function u on a domain Ω is *F-subharmonic* if $\mathrm{Hess}_x u \in F$ for all $x \in \Omega$, and it is *F-harmonic* if $\mathrm{Hess}_x u \in \partial F$ for all x. (The reader might note the usefulness of this approach in treating other branches of the equation $\det(\mathrm{Hess}\, u) = 0$.) The important point is to extend these notions to upper semicontinuous $[-\infty, \infty)$-valued functions. We will follow the approach in [HL$_4$] by using the *dual subequation*

$$\widetilde{F} \ \equiv \ -(\sim F) \ = \sim (-F). \tag{1.2}$$

The class of subaffine functions, or $\widetilde{\mathcal{P}}$-subharmonic functions, played a key role in [HL$_4$] since the positivity condition $F + \mathcal{P} \subset F$ is equivalent to $F + \widetilde{F} \subset \widetilde{\mathcal{P}}$.

Each subequation F has an associated *asymptotic interior* $\overrightarrow{F}$, and using this we introduced a notion of strictly $\overrightarrow{F}$*-convex* boundaries. For the qth branch of the complex Monge-Ampère equation, for example, this is just classical q-pseudo-convexity. It is then shown in [HL$_4$] that for any domain Ω whose boundary is strictly $\overrightarrow{F}$- and $\overrightarrow{\widetilde{F}}$-convex, solutions to the Dirichlet problem exist for all continuous boundary data. Furthermore, uniqueness holds on arbitrary domains Ω.

In this paper the results from [HL$_4$] will be generalized in several ways. To begin we shall work on a general manifold X and consider subequations given by closed subsets

$$F \subset J^2(X)$$

of the 2-jet bundle which satisfy three conditions. The first is the *positivity condition*

$$F + \mathcal{P} \ \subset \ F, \tag{P}$$

where $\mathcal{P}_x$ is the set of 2-jets of non-negative functions with critical value zero at x. The second is the *negativity condition*

$$F + \mathcal{N} \ \subset \ F, \tag{N}$$

where $\mathcal{N}$ = the jets of non-positive constant functions. The third is a mild *topological condition* (T), which is satisfied in all interesting cases. (See Section 2.) We always assume our subequations F satisfy these conditions. For the pure second-order constant coefficient subequations considered in [HL_4], the positivity conditions (P) and (1.1) are equivalent and automatically imply the other two conditions (N) and (T), as long as F is a closed set.

Here the dual subequation $\widetilde{F}$ is defined by (1.2) exactly as in [HL_4]. While duality continues to be important, it is not used to define F-subharmonic functions in the upper semicontinuous setting. This is because subaffine (i.e., co-convex) functions do not satisfy the maximum principle on general riemannian manifolds. So we use instead the equivalent [HL_4, Remark 4.9], viscosity definition (cf. [CIL]). The F-subharmonic functions have most of the important properties of classical subharmonic functions, such as closure under taking maxima, decreasing limits, uniform limits, and upper envelopes. This is discussed in Section 2 and Appendices A and B.

The notion of a strictly $\overrightarrow{F}$-convex boundary can be extended to this general context. This is discussed below in the introduction and treated in Section 11.

The focus of this paper is an important class of subequations constructed on riemannian manifolds, with an appropriate reduction of structure group, from constant coefficient equations on $\mathbf{R}^n$. It is a basic fact that on riemannian manifolds X, there is a canonical bundle splitting

$$J^2(X) \;=\; \mathbf{R} \oplus T^*X \oplus \text{Sym}^2(T^*X) \tag{1.3}$$

given by the riemannian hessian. This enables us, for example, to carry over any O_n-invariant constant coefficient subequation $F \subset \text{Sym}^2(\mathbf{R}^n)$ (the so-called *Hessian equations* in [CNS]) to all riemannian manifolds. That is, any purely second-order subequation on $\mathbf{R}^n$ which depends only on the eigenvalues of the hessian carries over to general riemannian manifolds.

Much more generally, however, consider a constant coefficient subequation, i.e., a subset

$$\mathbf{F} \;\subset\; \mathbf{R} \oplus \mathbf{R}^n \oplus \text{Sym}^2(\mathbf{R}^n)$$

satisfying (P), (N), and (T), and its compact invariance group

$$G \;\equiv\; \{g \in \text{O}_n : g(\mathbf{F}) = \mathbf{F}\}.$$

Suppose X is a riemannian manifold with a *topological G-structure.* This means that X is provided with an open covering $\{U_\alpha\}_\alpha$ and a tangent frame field $e^\alpha = (e^\alpha_1, \dots, e^\alpha_n)$ on each U_α so that each change of framing $G_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subset \text{O}_n$ is valued in the subgroup G. Then from the splitting (1.3), the subequation $\mathbf{F}$ can be transplanted to a globally defined subequation F on X. Such subequations are called *G-universal* or *riemannian G-subequations* and make sense on any riemannian manifold with a topological G-structure.

An important new ingredient is the concept of a convex *monotonicity cone* for F (Definition 9.4). This is a subset $M \subset J^2(X)$ which is a convex cone (with vertex 0) at each point and satisfies

$$F + M \ \subset\ F.$$

The subequation P defined by requiring that the riemannian hessian be ≥ 0 is a monotonicity cone for all purely second-order subequations. The corresponding P-subharmonic functions are just the riemannian convex functions. (Note that these exist globally on $\mathbf{R}^n$ and, in fact, on any complete simply-connected manifold of non-negative sectional curvature.) Similarly, on an almost complex hermitian manifold X the subequation $P^{\mathbf{C}}$, defined by requiring the hermitian symmetric part of $\mathrm{Hess}\, u$ to be ≥ 0, is a monotonicity cone for any U_n-universal subequation depending only on the hermitian symmetric part. The corresponding $P^{\mathbf{C}}$-subharmonic functions are *hermitian plurisubharmonic functions* on X.

Our first main result is the following.

Theorem 13.1. *Suppose F is a riemannian G-subequation on a riemannian manifold X provided with a topological G-structure as above. Suppose there exists a C^2 strictly M-subharmonic function on X where M is a monotonicity cone for F.*

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem. That is, for every $\varphi \in C(\partial\Omega)$ there exists a unique F-harmonic function $u \in C(\overline{\Omega})$ with $u\big|_{\partial\Omega} = \varphi$.

The simplest case of this, where F is a constant coefficient equation in euclidean $\mathbf{R}^n$ (and $G = \{1\}$), already generalizes the main result in [HL$_4$]. Here comparison holds for any subequation which is gradient independent since the squared distance to a point can be used to construct the required M-subharmonic function. (See Theorem 13.4.) A similar comment holds for any complete simply-connected manifold X of non-positive sectional curvature.

In general, requiring the existence of a strictly M-subharmonic function (or something similar) is an intuitively necessary global hypothesis; for example, in the case where $G = \mathrm{SO}_n$, one is free to arbitrarily change the riemannian geometry inside a domain Ω while preserving the F-convexity of $\partial\Omega$. However, even for quite regular metrics—domains in $S^3 \times S^3$—examples in Appendix D show that uniqueness fails without this hypothesis.

Theorem 13.1 applies in a quite broad context. We point out some examples here. They will be discussed in detail in the latter sections of the paper.

Example A. (Parallelizable manifolds). Even when F admits absolutely no symmetries, i.e., $G = \{1\}$, the theorem has broad applicability. Suppose X is a riemannian manifold which is parallelizable, that is, on which there exist global vector fields $e_1, \ldots, e_n$ which form a basis of T_xX at every point. The single open set $U = X$ with $e = (e_1, \ldots, e_n)$ is a $G = \{1\}$-structure, and *every constant coefficient subequation F in $\mathbf{R}^n$ can be carried over to a subequation F on X.*

Note that many manifolds are indeed parallelizable. For example, all orientable 3-manifolds have this property.

Example B. (General riemannian manifolds). Any constant coefficient subequation $\mathbf{F} \subset \mathbf{R} \oplus \mathbf{R}^n \oplus \mathrm{Sym}^2(\mathbf{R}^n)$ which is invariant under the action of O_n carries over to all riemannian manifolds. There are, for example, many invariant functions of the riemannian hessian which yield universal "purely second-order" equations. Associated to each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ is its set of ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. These eigenvalues have the property that $\lambda_q(A + P) \geq \lambda_q(A)$ for any $P \in \mathcal{P}$. Thus, for example, the set $\mathbf{P}_q = \{A : \lambda_q(A) \geq 0\}$ gives a subequation on any riemannian manifold. This subequation P_q is the qth branch of the Monge-Ampère equation.

There are many other equations of this type. In fact, let $\Lambda \subset \mathbf{R}^n$ be any subset which is invariant under permutations of the coordinates and satisfies the positivity condition: $\Lambda + \mathbf{R}^n_+ \subset \Lambda$ where $\mathbf{R}_+ \equiv \{t \geq 0\}$. Then this set, considered as a relation on the eigenvalues of the riemannian hessian, determines a universal subequation on every riemannian manifold.

A classical case is $\mathbf{F}(\sigma_k) = \{\lambda : \sigma_1(\lambda) \geq 0, \ldots, \sigma_k(\lambda) \geq 0\}$ (the principal branch of $\sigma_k(\lambda) = 0$ where σ_k denotes the kth elementary symmetric function). One can compute that the convexity of $\partial\Omega$, for this equation and its dual, corresponds to the $\mathbf{F}(\sigma_{k-1})$-convexity of its second fundamental form. For domains in $\mathbf{R}^n$ this result was proved in [CNS]. It is generalized here in two ways. We establish the result for *all branches* of the equation and we carry it over to general riemannian manifolds. (See Theorem 14.4.)

We are also able to treat inhomogeneous subequations such as $\mathrm{Hess}\, u \geq 0$ and $\det\mathrm{Hess}\, u \geq f(x)$ for a positive function f on X (and the analogous subequations for the other σ_k as above). This follows from Theorem 10.1 on local affine jet-equivalence.

Example C. (Almost complex hermitian manifolds). Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ where $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ represents multiplication by $\sqrt{-1}$. To any $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$ one can associate the hermitian symmetric part

$A^{\mathbf{C}} \equiv \frac{1}{2}(A - JAJ)$ which has n real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ occurring with multiplicity 2 on n complex lines. The entire discussion in Example B now applies. Any permutation-invariant subset $\Lambda \subset \mathbf{R}^n$ satisfying the $\mathbf{R}^n_+$-positivity condition $\Lambda + \mathbf{R}^n_+ \subset \Lambda$ gives a natural subequation on any manifold with U_n-structure, i.e., any almost complex manifold with a compatible riemannian metric. This includes, for example, all branches of the homogeneous complex Monge-Ampère equation. We note that almost complex hermitian manifolds play an important role in modern symplectic geometry.

One can also treat the Dirichlet problem for Calabi-Yau–type equations in this context of almost complex hermitian manifolds. (See, e.g., Example 6.15.)

One can also consider U_n-invariant functions of the skew-hermitian part $A^{\mathrm{sk}} = \frac{1}{2}(A + JAJ)$. An important case of this is the Lagrangian equation discussed below.

Example D. (Almost quaternionic hermitian manifolds). A discussion parallel to that in Example C holds with the complex numbers $\mathbf{C}$ replaced by the quaternions $\mathbf{H} = (\mathbf{R}^{4n}, I, J, K)$. In particular, Theorem 13.1 applies to the Dirichlet problem for all branches of the quaternionic Monge-Ampère equation on almost quaternionic hermitian manifolds.

Example E. (Grassmann structures). Fix any closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$ of the Grassmannian of p-planes in $\mathbf{R}^n$, and consider the subequation $F(\mathbb{G})$ defined for $(r, p, A) \in \mathbf{R} \oplus \mathbf{R}^n \oplus \mathrm{Sym}^2(\mathbf{R}^n)$ by the condition

$$\mathrm{tr}_\xi A \;\equiv\; \mathrm{tr}\left(A\big|_\xi\right) \;\geq\; 0 \quad \text{for all } \xi \in \mathbb{G}.$$

Here the $F(\mathbb{G})$-subharmonic functions are more appropriately called $\mathbb{G}$-*plurisubharmonic*. Now $F(\mathbb{G})$ carries over to a subequation on any riemannian manifold with G-structure where

$$G \equiv \{g \in \mathrm{O}_n : g(\mathbb{G}) = \mathbb{G}\}.$$

For example, if $\mathbb{G} = G(p, \mathbf{R}^n)$, then $G = \mathrm{O}_n$ and the corresponding subequation, which makes sense on any riemannian manifold, states that the sum of any p eigenvalues of $\mathrm{Hess}\, u$ must be ≥ 0. This is called *geometric p-convexity* or *p-plurisubharmonicity.*

A particularly interesting example is given by the U_n-invariant set $\mathrm{LAG} \subset G(n, \mathbf{C}^n)$ of Lagrangian subspaces of $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$. On any almost complex hermitian manifold, this gives rise to a notion of *Lagrangian subharmonic* and *Lagrangian harmonic* functions with an associated Dirichlet problem which is solvable on Lagrangian convex domains. This is discussed in more detail below.

Another rich set of examples comes from *almost calibrated manifolds* such as manifolds with topological G_2 and Spin_7 structures. These are also discussed at the end of the introduction.

For all such structures, Theorem 13.1 gives the following general result. Call a domain $\Omega \subset X$ *strictly* $\mathbb{G}$*-convex* if it has a strictly $\mathbb{G}$-psh global defining function. This holds if $\partial\Omega$ is strictly $\mathbb{G}$-convex (cf. (1.5) below) and there exists $f \in C^2(\overline{\Omega})$ which is strictly $\mathbb{G}$-psh.

Theorem 16.1. *Let X be a riemannnian manifold with topological G-structure so that the riemannian G-subequation $F(\mathbb{G})$ is defined on X. Then on any strictly $\mathbb{G}$-convex domain $\Omega \subset\subset X$ the Dirichlet problem for $F(\mathbb{G})$-harmonic functions is uniquely solvable for all continuous boundary data.*

Jet-Equivalence of Subequations. Although it is not our main focus, one of the important results of this paper is that Theorem 13.1 actually holds for a much broader class of subequations, namely those which are locally affinely jet-equivalent to constant coefficient subequations. The notion of (ordinary) jet-equivalence is expressed in terms of automorphisms of the 2-jet bundle. A (linear) automorphism of $J^2(X)$ is a smooth bundle isomorphism $\Phi : J^2(X) \to J^2(X)$ which has certain natural properties with respect to the short exact sequence $0 \to \text{Sym}^2(T^*X) \to J^2(X) \to J^1(X) \to 0$. (See Definition 6.1 for details.) The automorphisms form a group, and two subequations $F, F' \subset J^2(X)$ are called *jet-equivalent* if there exists an automorphism Φ with $\Phi(F) = F'$.

A subequation on a general manifold X is said to be *locally jet-equivalent to a constant coefficient subequation* if each point x has a coordinate neighborhood U so that $F\big|_U$ is jet-equivalent to a constant coefficient subequation $U \times \mathbf{F}$ in those coordinates.

Any riemannian G-subequation on a riemannian manifold with topological G-structure is locally jet-equivalent to a constant coefficient subequation. Theorem 13.1 remains true if one drops the riemannian metric on X and replaces the "G-universal assumption" with the assumption that F is locally jet-equivalent to a constant coefficient subequation. This is a strictly broader class of equations. For example, a generic subequation defined by a smoothly varying linear inequality in the fibers of $J^2(X)$ is locally jet-equivalent to constant coefficients.

The notion of jet-equivalence can be further substantially broadened by using the *affine* automorphism group. This is the fiberwise extension of the linear automorphisms by the full group of translations in the fibers $J^2_x(X)$.

Theorem 13.1′. *Theorem 13.1 holds for subequations which are locally affinely jet-equivalent to riemannian G-subequations.*

This allows one to treat inhomogeneous equations with variable right hand side. For example, one can establish existence and uniqueness of solutions to the Dirichlet problem for the Calabi-Yau–type equation

$$\left(\frac{1}{i}\partial\overline{\partial}u + \omega\right)^n \;=\; e^u f(x)\omega^n, \tag{1.4}$$

or the same equation with e^u replaced by any non-decreasing positive function $F(u)$. Here the results hold on any **almost complex** hermitian manifold (see Section 19). One also gets existence and uniqueness results for equations of the type

$$\sigma_1(\operatorname{Hess} u) \;\geq\; 0, \;\ldots\;, \sigma_k(\operatorname{Hess} u) \;\geq\; 0 \qquad \text{and}$$

$$\sigma_k(\operatorname{Hess} u) \;=\; f(x)$$

where $f > 0$. This is also true for equations such as

$$\lambda_q(\operatorname{Hess} u) \;=\; f(x)$$

for any smooth function f. Examples are given in Section 18.

Note. The notion of jet-equivalence is a quite weak relation. A jet-equivalence of subequations $\Phi : F \to F'$ does *not* induce a correspondence between F-subharmonic functions and F'-subharmonic functions. In fact, for a C^2-function u, $\Phi(J^2u)$ is almost never the 2-jet of a function.

Comparison Results. This paper contains a number of other possibly quite useful existence and uniqueness results which lead to Theorem 13.1 above.

A central concept in this subject is that of comparison for upper semicontinuous functions, which is treated in Section 8.

Definition. We say that *comparison holds* for F on X if for all compact subsets $K \subset X$ and functions

$$u \in F(K) \quad \text{and} \quad v \in \widetilde{F}(K),$$

the **Zero Maximum Principle** holds for $u + v$ on K, that is,

$$u + v \;\leq\; 0 \quad \text{on} \quad \partial K \qquad \Rightarrow \qquad u + v \;\leq\; 0 \quad \text{on} \quad K. \tag{ZMP}$$

If comparison holds for F on X, then one easily deduces the uniqueness of solutions to the Dirichlet problem for F-harmonic functions on every compact subdomain.

Obviously, comparison for small compact sets K does not imply that comparison holds for arbitrary compact sets. However, this is true for a weakened form of comparison involving a notion of strictly F-subharmonic functions.

Consider a subequation F on a riemannian manifold X. For each constant $c > 0$ we define $F^c \subset F$ to be the subequation whose fiber at x is

$$F^c_x \;\equiv\; \{J \in F_x : \operatorname{dist}(J, \sim F) \geq c\}$$

where "dist" denotes distance in the fiber $J_x(X)$. We define an upper semicontinuous function u on X to be *strictly F-subharmonic* if for each point $x \in X$ there is a neighborhood B of x and a $c > 0$ such that u is F^c-subharmonic on B.

Definition. We say that *weak comparison holds* for F on X if for all compact subsets $K \subset X$ and functions

$$u \in F^c(K) \text{ (some } c > 0) \quad \text{and} \quad v \in \widetilde{F}(K),$$

the Zero Maximum Principle holds for $u+v$ on K. We say that *local weak comparison holds* for F on X if every $x \in X$ has some neighborhood on which weak comparison holds.

It is proved (see Theorem 8.3) that:

Local weak comparison implies weak comparison.

Then in Section 10 we prove that:

Local weak comparison holds for any subequation which is locally affinely jet-equivalent to a constant coefficient subequation.

Taken together we have that global weak comparison holds for all such subequations. This constitutes a very broad class. In particular, we have that

Weak comparison holds for riemannian G-subequations on a riemannian manifold.

We then establish, under certain global conditions, that weak comparison implies comparison.

Theorem 9.7. *Suppose F is a subequation for which local weak comparison holds. Suppose there exists a C^2 strictly M_F-subharmonic function on X where M_F is a monotonicity cone for F. Then comparison holds for F on X.*

Combined with our notions of boundary convexity, we prove the following.

Theorem 13.3. *Suppose comparison holds for a subequation F on X. Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem.*

In Section 9 we introduce the concept of strict approximation for F on X and show that if X admits a C^2 strictly M_F-subharmonic function (where M_F is a monotonicity cone for F as above), then strict approximation holds for F on X. Furthermore, we show that weak comparison plus strict approximation implies comparison.

Boundary Convexity. An important part of this paper is the formulation and study of the notion of boundary convexity for a general fully nonlinear second-order equation. This concept is presented in Section 11. It strictly generalizes the boundary convexity defined in [CNS] and in [HL_4].

In the Grassmann examples this boundary convexity condition is particularly transparent and geometric. Suppose $F(\mathbb{G})$ is a purely second-order equation, defined as in Example E by a subset $\mathbb{G} \subset G(p, TX)$ of the Grassmann bundle. Then boundary convexity for a domain Ω becomes the requirement that the second fundamental form $II_{\partial\Omega}$ of $\partial\Omega$ satisfy

$$\mathrm{tr}_\xi II_{\partial\Omega} \ \geq\ 0 \tag{1.5}$$

for all $\xi \in \mathbb{G}$ such that $\xi \subset T(\partial\Omega)$. (When there are no $\mathbb{G}$-planes in $T_x\partial\Omega$, $\mathbb{G}$-convexity automatically holds at x.) This convexity automatically implies convexity for the dual subequation.

Domains with convex boundaries in the riemannian sense are F-convex for any purely second-order subequation.

For subequations which do not involve the dependent variable, F-convexity is defined in terms of the *asymptotic interior* of F. This is an open, point-wise conical set consisting of rays with conical neighborhoods which eventually lie in F. For general subequations, we freeze the dependent variable to be a constant λ (e.g. replace $\Delta u \geq e^u$ with $\Delta u \geq e^\lambda$) and use the associated asymptotic interiors to define convexity.

We then derive a simple, geometric criterion for F-convexity in terms of the second fundamental form of $\partial\Omega$. In almost all interesting cases, the boundary convexity condition is straightforward to compute.

Some subequations F have the property that every boundary is F-convex. These include the p-Laplace-Beltrami subequation,

$$\|\nabla u\|^2 \Delta u + (p-2)(\nabla u)^t (\mathrm{Hess}\, u)(\nabla u) \geq\ 0$$

for $1 < p < \infty$, and the infinite Laplace-Beltrami subequation

$$(\nabla u)^t (\mathrm{Hess}\, u)(\nabla u) \ \geq\ 0.$$

For the general minimal surface subequation $(1 + \|\nabla u\|^2)\Delta u - (\nabla u)^t (\mathrm{Hess}\, u)(\nabla u) \geq 0$, strict boundary convexity is equivalent to strictly positive mean curvature with respect to the interior normal. For the Calabi-Yau equation, and also for the homogeneous complex Monge-Ampère equation, F-convexity of the boundary is simply standard pseudo-convexity (in the hermitian almost-complex case).

Finally we note that for a general F *all sufficiently small balls in any local coordinate system are strictly F-convex* under the mild condition that F contains critical jets (see Proposition 11.9).

Existence without Comparison. Certain methods taken from complex analysis, which go back to Bremermann/Perron and then Walsh ([B], [W]), enable us to prove existence in the constant coefficient setting with no assumptions about a monotonicity cone. This includes many cases where uniqueness does not hold. More generally, we prove in Section 12 that if $X = K/G$ is a riemannian homogeneous space and $F \subset J^2(X)$ is a K-invariant subequation, then existence holds for all continuous boundary data on any domain which is strictly F- and $\widetilde{F}$-convex. This applies in particular to the p-Laplace-Beltrami and infinite Laplace-Beltrami subequations mentioned above, where all domains are strictly F- and $\widetilde{F}$-convex.

The euclidean version of this result is Theorem 12.7. It establishes existence for any constant coefficient subequation $\mathbf{F}$ on all strictly $\mathbf{F}$ and $\widetilde{\mathbf{F}}$-convex domains (when they exist). We follow this with an Example 12.8 of a second-order equation where uniqueness does in fact fail.

Further Examples. There are many geometrically interesting subequations which are covered by the results above. We examine a few more examples here. We start with a general observation.

Inhomogeneous Equations. The methods above apply to any subequation which is locally *affinely* jet-equivalent to a constant coefficient equation. This greatly extends the equations that one can treat. For example, suppose that F is a riemannian G-subequation with a monotonicity cone M on a manifold X, and let J be any smooth section of the 2-jet bundle $J^2(X)$. Then $F_J \equiv F + J$ (fiber-wise translation) is also a subequation having the same asymptotic interior $\overrightarrow{F_J} = \overrightarrow{F}$ and also having M as a monotonicity cone.

As a simple but interesting example, suppose that F is one of the branches of the homogeneous Monge-Ampère equation (in the real, complex, or quaternionic case), and choose $J_x = f(x)I$ where f is an arbitrary smooth function on X (and I is the identity section of $\mathrm{Sym}^2(T^*X)$). Then one can treat the inhomogeneous equation

$$\lambda_k(\mathrm{Hess}\, u) \;=\; f(x)$$

under the same conditions that one can treat the homogeneous equation $\lambda_k(\mathrm{Hess}\, u) = 0$. For the principal branch $\mathcal{P}$, $\mathcal{P}^{\mathbf{C}}$ etc., and $f \equiv -1$, this yields functions which are quasi-convex, quasi-psh, etc. The higher branches with variable f are more interesting.

Similar remarks hold for any purely second-order subequation, such as those below. However, many other interesting subequations arise from local affine jet-equivalence (cf. Example 6.15).

Example F. (Almost calibrated manifolds). An important class of Grassmann structures (discussed in Example E) are given by calibrations. A constant coefficient calibration is a p-form ϕ with the property that $\phi(\xi) \leq 1$ for all $\xi \in G(p, \mathbf{R}^n)$. The associated set is

$$\mathbb{G}(\phi) \equiv \{\xi \in G(p, \mathbf{R}^n) : \phi(\xi) = 1\}$$

and the associated group is

$$G(\phi) = \{g \in \mathrm{O}_n : g^*(\phi) = \phi\}.$$

Any riemannian manifold with a topological $G(\phi)$-structure will carry a global p-form $\widetilde{\phi}$, called an *almost calibration*, which is of type ϕ at every point but is not necessarily closed under exterior differentiation. Some of the subequations already referred to can be defined in this way. For example, almost complex hermitian geometry arises from $\phi = \omega$, the standard (not necessarily closed) Kähler form. There are, however, many others. Several interesting examples are given next.

Example G. (Almost Hyperkahler Manifolds). Here we consider a $4n$-dimensional manifold X equipped with a subbundle $\mathcal{Q} \subset \mathrm{Hom}\,(TX, TX)$ generated by global sections I, J, K satisfying the standard quaternion relations: $I^2 = J^2 = K^2 = -1$, and $IJ = K$, etc. and equipped with a compatible riemannian metric g. The topological structure group is Sp_n. This is a special case of Example D above so the quaternionic plurisubharmonic functions are defined, and when the Hyperkahler structure is integrable, they coincide with the ones used by Alesker and Verbitsky to study the Quaternionic Monge-Ampere equation in the principal-branch case. (See [$\mathrm{A}_{1,2}$], [AV])

However, a manifold with a topological Sp_n-structure carries other almost calibrations such as the generalized Cayley calibration $\widetilde{\Omega} = \frac{1}{2}\left\{\omega_I^2 + \omega_J^2 - \omega_K^2\right\}$ introduced in [BH].

Example H. (Almost Calabi-Yau Manifolds). This is an almost complex hermitian manifold with a global section of $\Lambda^{n,0}(X)$ whose real part Φ has comass $\equiv 1$. This is equivalent to having topological structure group SU_n. In addition to the structures discussed in Case 2, these manifolds carry functions associated with the *Special Lagrangian calibration* Φ. The Φ-submanifolds are called *Special Lagrangian submanifolds* and the Φ-subharmonic functions are said to be *Special Lagrangian subharmonic.*

Example I. (Almost $\mathbf{G}_2$ manifolds). Let $\mathrm{Im}\mathbf{O} = \mathbf{R}^7$ denote the imaginary Cayley numbers. This space is acted on by the group G_2 of automorphisms of $\mathbf{O}$ which preserves the 3-form

$$\varphi(x, y, z) = \langle x \cdot y, z\rangle$$

called the *associative calibration.* Any 7-dimensional manifold X with a topological G_2-structure carries an associated riemannian metric and a global (non-closed) calibration φ. There also exists a degree 4 calibration $\psi = *\varphi$ on X. One then has φ and ψ subharmonic and harmonic functions on X and one can consider the Dirichlet problem on bounded domains. We note that these structures exist in abundance. In [LM, page 348] it is shown that for any 7-manifold X

$$X \text{ has a topological } \mathrm{G}_2 \text{ structure} \qquad \iff \qquad X \text{ is spin.}$$

Example J. (Almost Spin_7 manifolds). On the Cayley numbers $\mathbf{O} = \mathbf{R}^8$ there is a 4-form of comass 1 defined by

$$\Phi(x, y, z, w) \;=\; \langle (x \cdot y) \cdot z - x \cdot (y \cdot z), w \rangle$$

and preserved by the subgoup $\mathrm{Spin}_7 \subset \mathrm{SO}_8$ (cf. [HL_1], [H], [LM]). This determines a non-closed calibration Φ on any 8-manifold X with a topological Spin_7-structure. In [LM, page 349] it is shown that for any 8-manifold X

$$X \text{ has a topological } \mathrm{Spin}_7 \text{ structure} \qquad \iff$$

$$X \text{ is spin and } p_1(X)^2 - 4p_2(X) + 8\chi(X) = 0$$

for an appropriate choice of orientation on X. Here $p_k(X)$ is the kth Pontrjagin class and $\chi(X)$ denotes the Euler class of X.

Example K. (Lagrangian subharmonicity). Suppose (X, J) is an almost complex hermitian manifold of real dimension $2n$. Then, as mentioned in Example E, there is a natural Grassmann structure $\mathrm{LAG} \subset G(n, TX)$ consisting of the Lagrangian n-planes. This gives rise to the Lagrangian subequation defined in terms of the riemannian hessian by the condition that

$$\mathrm{tr}\left\{\mathrm{Hess}\, u\big|_\xi\right\} \;\geq\; 0 \quad \text{for all } \xi \in \mathrm{LAG}.$$

Interestingly, there is a beautiful polynomial operator which vanishes on the Lagrangian harmonic functions, and so the Dirichlet problem here can be considered to be for this operator. Furthermore, just as in the Monge-Ampère case, this operator has many branches, each of which is another U_n-invariant subequation on the manifold. This follows from Gårding's theory of hyperbolic polynomials ([G], [HL_4], [HL_7]).

Example L. (Equations involving the dependent variable but independent of the gradient). Many purely second-order equations can be enhanced to ones which involve the dependent variable u and our theory continues to apply. For example, consider the O_n-universal subequation F defined by requiring

$$f(u, \lambda_q(A)) \;\geq\; 0$$

where $f(x, y)$ is non-increasing in x and non-decreasing in y. As a special case, consider

$$\lambda_q(A) - \varphi(u) \;\geq\; 0$$

where φ is monotone non-decreasing. The dual subequation $\widetilde{F}$ is given by

$$\lambda_{n-q+1} + \varphi(-u) \;\geq\; 0.$$

If $\varphi(0) = 0$, then the required convexity of the boundary is that

$$\min\{\lambda_q(II_{\partial\Omega}), \lambda_{n-q}(II_{\partial\Omega})\} \ > \ 0.$$

Another interesting O_n-universal subequation is

$$F \ \equiv \ \{(r, p, A) : A \geq 0 \text{ and } \det A - e^r \geq 0\}$$

which is discussed in Remark 12.9. Its dual equation is

$$\widetilde{F} \ = \ \{(r, p, A) : -A \not< 0 \text{ or } \ -|\det A| + e^{-r} \geq 0\}.$$

Gårding hyperbolic polynomials. Gårding's beautiful theory of hyperbolic polynomials [G], when applied to homogeneous polynomials M on $\mathrm{Sym}^2(\mathbf{R}^n)$, fits perfectly into this paper. It unifies and generalizes many of our constructions. We give here a brief sketch of how this works, and refer to [HL$_7$] for full details.

By definition a homogeneous polynomial M of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$ is *hyperbolic with respect to the identity* I if for all $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ the polynomial $s \mapsto M(sI + A)$ has exactly m real roots. The negatives of these roots are called the *M-eigenvalues of A*. It is useful to order these eigenvalues

$$\lambda_1^M(A) \ \leq \ \cdots \ \leq \lambda_m^M(A)$$

and normalize so that $M(I) = 1$. Then the polynomial factors as $M(sI + A) \ = \ \prod_{k=1}^m \left(s + \lambda_k^M(A)\right)$. Using the ordered eigenvalues, we can define branches

$$\mathbf{F}_k^M \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda_k^M(A) \geq 0\}$$

which satisfy $\mathbf{F}_1^M \subset \mathbf{F}_2^M \subset \cdots$. The *principal branch* $\mathbf{F}^M \equiv \mathbf{F}_1^M$ is the connected component of $\{M \neq 0\}$ containing I. Gårding proves that:

(1) The principal branch $\mathbf{F}^M$ is a convex cone.

(2) $\mathbf{F}_k^M + \mathbf{F}^M \ \subset \ \mathbf{F}_k^M$ for all k.

Under the positivity assumption $\mathbf{F}^M + \mathcal{P} \subset \mathbf{F}^M$, we then have that:

Each $\mathbf{F}_k^M$ is a constant coefficient subequation
for which $\mathbf{F}^M$ is a monotonicity cone.

When $M = \det$, we get the branches of the Monge-Ampère equation (Example B). However, this applies to many other interesting cases (such as Examples C, D, and K). Furthermore, for each subset $\Lambda \subset \mathbf{R}^n$ as in 14.1 below, one can construct a subequation $\mathbf{F}^{M,\Lambda}$ using the λ_k^M (see [HL$_7$]).

If the polynomial M is invariant under a subgroup $G \subset \mathrm{O}_n$, then these subequations carry over to any manifold with a topological G-structure.

Parabolic Subequations. The methods and results of this paper carry over effectively to parabolic equations. Suppose X is a riemannian manifold equipped with a riemannian G-subequation F for $G \subset \mathrm{O}_n$. We assume F is induced from a universal model

$$\mathbf{F} \;=\; \{J \in \mathbf{J}^2 : f(J) \geq 0\}$$

where $f : J^2(X) \to \mathbf{R}$ is G-invariant, $\mathcal{P}$- and $\mathcal{N}$-monotone, and Lipschitz in the reduced variables (p, A). Then on the riemannian product $X \times \mathbf{R}$ we have the associated G-universal parabolic subequation H defined by

$$f(J) - p_0 \;\geq\; 0$$

where p_0 denotes the u_t component of the 2-jet of u. The H-harmonic functions are solutions of the equation $u_t = f(u, Du, D^2u)$. Interesting examples which can be treated include:

(i) $f = \mathrm{tr}A$, the standard heat equation $u_t = \Delta u$ for the Laplace-Beltrami operator on X.

(ii) $f = \lambda_q(A)$, the qth ordered eigenvalue of A. This is the natural parabolic equation associated to the qth branch of the Monge-Ampère equation.

(iii) $f = \mathrm{tr}A + \frac{k}{|p|^2+\epsilon^2} p^tAp$ for $k \geq -1$ and $\epsilon > 0$. When $X = \mathbf{R}^n$ and $k = -1$, the solutions $u(x,t)$ of the associated parabolic equation, in the limit as $\epsilon \to 0$, have the property that the associated level sets $\Sigma_t \;\equiv\; \{x \in \mathbf{R}^n : u(x,t) = 0\}$ are evolving by mean curvature flow. (See the classical papers of Evans-Spruck [ES$_*$] and Chen-Giga-Goto [CGG$_*$], and the very nice accounts in [E] and [Gi].)

(iv) $f = \mathrm{tr}\{\arctan A\}$. When $X = \mathbf{R}^n$, solutions $u(x,t)$ have the property that the graphs of the gradients $\Gamma_t \;\equiv\; \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n = \mathbf{C}^n : y = D_x u(x,t)\}$ are Lagrangian submanifolds which evolve the initial data by mean curvature flow. (See [CCH] and references therein.)

Straightforward application of the techniques in this paper shows that:

Comparison holds for all riemannian G-subequations H on $X \times \mathbf{R}$.

By standard techniques one can prove more. Consider a compact subset $K \subset \{t \leq T\} \subset X \times \mathbf{R}$ and let $K_T \equiv K \cap \{t = T\}$ denote the terminal time slice of K. Let $\partial_0 K \;\equiv\; \partial K - \mathrm{Int}K_T$ denote the *parabolic boundary of* K. Here $\mathrm{Int}K$ denotes the relative interior in $\{t = T\} \subset X \times \mathbf{R}$. We say that *parabolic comparison holds for* H if for all such K (and T)

$$u + v \;\leq\; p \quad \text{on} \;\; \partial_0 K \qquad \Rightarrow \qquad u + v \;\leq\; p \quad \text{on} \;\; \mathrm{Int}K$$

for all $u \in H(K)$ and $v \in \widetilde{H}(K)$. Then one has that:

Parabolic comparison holds for all riemannian

G-subequations H on $X \times \mathbf{R}$.

Under further mild assumptions on f which are satisfied in the examples above, one also has existence results. Consider a domain $\Omega \subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex. Set $K = \overline{\Omega} \times [0, T]$. Then

$$\text{For each } \varphi \in C(\partial_0 K) \text{ there exists a unique function } u \in C(K) \text{ such that } u\big|_{\mathrm{Int} K} \text{ is } H\text{-harmonic and } u\big|_{\partial_0 K} = \varphi.$$

One also obtains corresponding long-time existence results. Details will appear elsewhere.

A brief outline of the paper. Section 2 along with Appendices A and B provide a self-contained treatment of general F-subharmonic functions and their properties. Section 3 introduces the concept of a *subequation* and discusses the natural duality among subequations. Riemannian manifolds are considered in Section 4 where it is shown that there is a natural splitting of the 2-jet bundle induced by the riemannian hessian. This splitting gives many geometric examples of subequations which are purely second-order. In Section 5 "universal" subequations are constructed. Suppose $\mathbf{F}$ is a constant coefficient (euclidean) subequation with compact invariance group $G \subset \mathrm{O}_n$, and X is a riemannian manifold with a topological G-structure. Then it is shown that the euclidean model $\mathbf{F}$ induces a *riemannian G-subequation* F on X with the property that F is locally jet-equivalent to $\mathbf{F}$. Section 6 discusses automorphisms of the 2-jet bundle and the general notion of jet-equivalence for subequations. In Section 7 a notion of *strictly* F-subharmonic functions is introduced for upper semicontinuous functions. In Section 8 a weak form of comparison is defined using this notion of strictness. It is shown that if weak comparison holds locally, then it is true globally. Section 9 addresses the question of when weak comparison implies comparison. We first note that this holds whenever approximation by strictly F-subharmonic functions is possible. The main discussion concerns how this "strict approximation" can be deduced from a form of monotonicity. This yields many geometric examples. In Section 10 we prove that weak comparison holds for any subequation which is locally (weakly) jet-equivalent to a constant coefficient subequation, and in particular for the riemannian G-subequations constructed in Section 5. The proof relies on the Theorem on Sums stated in Appendix C. In Section 11 we introduce the notion of the *asymptotic interior* $\overrightarrow{F}$ of a reduced subequation F. This leads to the notion of strict F-boundary-convexity which implies the existence of barriers essential to existence proofs. For riemannian G-subequations the existence of these barriers is actually equivalent to strict F-convexity. Section 12 addresses the existence question for the Dirichlet problem. Assuming strict boundary convexity for both the subequation F and its dual $\widetilde{F}$, several existence theorems are proved. Section 13 compiles and summarizes the results established for the Dirichlet problem.

The remaining sections are devoted to examples. Section 14 examines O_n-universal subequations. These are subequations that make sense on any riemannian manifold. Particular attention is paid to subequations which are purely second-order. Analogous results in the complex and quaternionic case are examined in Section 15. Section 16 discusses the subequations $F(\mathbb{G})$ geometrically defined by a closed subset $\mathbb{G}$ of the Grassmannian. Section 17 applies the theory to equations defined in terms of the principal curvatures of the graph of u. Section 18 applies affine jet-equivalence to give results for inhomogeneous and other equations. Section 19 treats the Calabi-Yau–type equations on almost complex hermitian manifolds.

We note that in [AFS] and [PZ], standard viscosity theory has been retrofitted to riemannian manifolds by using the distance function, parallel translation, Jacobi fields, etc,. In our approach this machinery is not necessary. We get by with the standard viscosity techniques (cf. [CIL], [C]).

Acknowledgments. H.B. Lawson was partially supported by the N.S.F.

2. F-Subharmonic Functions

Let X be a smooth n-dimensional manifold. Denote by $J^2(X) \to X$ the bundle of 2-jets whose fiber at a point x is the quotient

$$J^2_x(X) \;=\; C^\infty_x / C^\infty_{x,3},$$

where C^∞_x denotes the germs of smooth functions at x and $C^\infty_{x,3}$ the subspace of germs which vanish to order three at x. Given a smooth function u on X, let $J^2_x u \in J^2_x(X)$ denote its 2-jet at x, and note that $J^2 u$ is a smooth section of the bundle $J^2(X)$.

The bundle on 1-jets $J^1(X)$ is defined similarly and has a natural splitting $J^1(X) = \mathbf{R} \oplus T^*X$ with $J^1_x u = (u(x), (du)_x)$. There is a short exact sequence of bundles

$$0 \to \mathrm{Sym}^2(T^*X) \to J^2(X) \to J^1(X) \to 0. \tag{2.1}$$

Here the space $\mathrm{Sym}^2(T^*_x X)$ of symmetric bilinear forms on $T_x X$ is embedded as the space of 2-jets of functions with critical value zero at the point x, i.e.,

$$\mathrm{Sym}^2(T^*_x X) \cong \{J^2_x u : u(x) = 0, (du)_x = 0\}.$$

Note that if u is such a function, and V, W are vector fields near x, then $(\mathrm{Hess}_x u)(V, W) = V \cdot W \cdot u = W \cdot V \cdot u + [V, W] \cdot u = W \cdot V \cdot u = (\mathrm{Hess}_x u)(W, V)$ is a well-defined symmetric form on $T_x X$. However, for functions u with $(du)_x \neq 0$, there is no natural definition of $\mathrm{Hess}_x u$, i.e., the sequence (2.1) has no natural splitting. Choices of splittings correspond to definitions of a hessian, and there is a canonical one for each riemannian metric, as we shall see in Section 4.

At a minimum point x for a smooth function u, we have $(du)_x = 0$ (so that $\mathrm{Hess}_x u \in \mathrm{Sym}^2(T^*X)$ is well defined), and $\mathrm{Hess}_x u \geq 0$. The isomorphism

(2.2)
$$\{H \in \mathrm{Sym}^2(T^*X) : H \geq 0\} \;\cong\; \{J^2_x u : u \geq 0 \;\text{ near } x \text{ and } u(x) = 0\}$$

defines a cone bundle

$$\mathcal{P} \;\subset\; \mathrm{Sym}^2(T^*X) \;\subset\; J^2(X)$$

with $\mathcal{P}_x$ defined by (2.2).

Given an arbitrary subset $F \subset J^2(X)$, a function $u \in C^2(X)$ will be called **F-subharmonic** if its 2-jet satisfies

$$J^2_x u \;\in\; F_x \quad \text{for all } x \in X,$$

and **strictly F-subharmonic** if its 2-jet satisfies

$$J^2_x u \;\in\; (\mathrm{Int} F)_x \quad \text{for all } x \in X.$$

These notions are of limited interest for general sets F.

Definition 2.1. A subset $F \subset J^2(X)$ satisfies the **positivity condition** if

$$F + \mathcal{P} \;\subseteq\; F. \tag{P}$$

A subset $F \subset J^2(X)$ which satisfies (P) will be called **$\mathcal{P}$-monotone**.

Note that condition (P) implies that

$$\mathrm{Int} F + \mathcal{P} \;\subset\; \mathrm{Int} F.$$

Monotonicity is a key concept in this paper. For arbitrary subsets $M, F \subset J^2(X)$ we say that F is **M-monotone** or that M is a **montonicity set for** F if

$$F + M \subset F.$$

It is necessary and quite useful to extend the definition of F-subharmonic to non-differentiable functions u. Let $\mathrm{USC}(X)$ denote the set of $[-\infty, \infty)$-valued, upper semicontinuous functions on X.

Definition 2.2. A function $u \in \mathrm{USC}(X)$ is said to be **F-subharmonic** if for each $x \in X$ and each function φ which is C^2 near x, one has that

$$\left\{ \begin{array}{lll} u - \varphi & \leq 0 & \text{near } x_0 \text{ and} \\ & = 0 & \text{at } x_0 \end{array} \right\} \quad \Rightarrow \quad J^2_x \varphi \;\in\; F_x. \tag{2.3}$$

We denote by $F(X)$ the set of all such functions.

Note that if $u \in C^2(X)$, then

$$u \in F(X) \quad \Rightarrow \quad J^2_x u \in F_x \qquad \forall\, x \in X \tag{2.4}$$

since the test function φ may be chosen equal to u in (2.3). The converse is not true for general subsets F. However, we have the following.

Proposition 2.3. *Suppose F satisfies the positivity condition (P) and $u \in C^2(X)$. Then*

$$J^2_x u \ \in\ F_x \ \text{ for all } x \in X \qquad \Rightarrow \qquad u \ \in\ F(X). \tag{2.5}$$

Proof. Assume $J^2_{x_0} u \in F_{x_0}$ and φ is a C^2-function such that

$$\left\{ \begin{matrix} u - \varphi & \leq & 0 & \text{near } x_0 \\ & = & 0 & \text{at } x_0 \end{matrix} \right\}.$$

Since $(\varphi - u)(x_0) = 0$, $d(\varphi - u)_{x_0} = 0$, and $\varphi - u \geq 0$ near x_0, we have $J^2_{x_0}(\varphi - u) \in \mathcal{P}_{x_0}$ by definition. Now the positivity condition implies that $J^2_{x_0}\varphi \in J^2_{x_0} u + \mathcal{P}_{x_0} \subset F_{x_0}$. This proves that $u \in F(X)$. ∎

The proof above shows that *we must assume F satisfies the positivity condition.* Otherwise the definition of F-subharmonicity would not extend the natural one for smooth functions.

There is an equivalent definition of F-subharmonic functions which is quite useful. We record it here and prove it in Appendix A (Proposition A.1 (IV)).

Lemma 2.4. *Fix $u \in \mathrm{USC}(X)$, and suppose that $F \subset J^2(X)$ is closed. Then $u \notin F(X)$ if and only if there exists a point $x_0 \in X$, local coordinates x at x_0, $\alpha > 0$, and a quadratic function*

$$q(x) \ = \ r + \langle p, x - x_0 \rangle + \tfrac{1}{2}\langle A(x - x_0), x - x_0 \rangle$$

with $J^2_{x_0}(q) \notin F_{x_0}$ so that

$$\begin{matrix} u(x) - q(x) & \leq & -\alpha|x - x_0|^2 & \quad \textit{near } x_0 & \quad \textit{and} \\ & = & 0 & \quad \textit{at } \ x_0 & \end{matrix}$$

Remark 2.5. (a). The hypothesis that F is closed in Lemma 2.4 can be substantially weakened. It suffices to assume only that the fibers of F under the map $J^2(X) \to J^1(X)$ are closed. In fact, Proposition A.1 is proved under this weaker hypothesis.

(b). The positivity condition is rarely used in proofs. This is because without it $F(X)$ is empty and the results are trivial. For example, the positivity condition is not required in the following theorem. (F need only be closed.)

It is remarkable, at this level of generality, that F-subharmonic functions share many of the important properties of classical subharmonic functions.

Theorem 2.6. Elementary Properties of F-Subharmonic Functions. *Let F be an arbitrary closed subset of $J^2(X)$.*

(A) (Maximum Property) *If* $u, v \in F(X)$, *then* $w = \max\{u, v\} \in F(X)$.
(B) (Coherence Property) *If* $u \in F(X)$ *is twice differentiable at* $x \in X$, *then* $J^2_x u \in F_x$.
(C) (Decreasing Sequence Property) *If* $\{u_j\}$ *is a decreasing* ($u_j \geq u_{j+1}$) *sequence of functions with all* $u_j \in F(X)$, *then the limit* $u = \lim_{j\to\infty} u_j \in F(X)$.
(D) (Uniform Limit Property) *Suppose* $\{u_j\} \subset F(X)$ *is a sequence which converges to* u *uniformly on compact subsets to* X; *then* $u \in F(X)$.
(E) (Families Locally Bounded Above) *Suppose* $\mathcal{F} \subset F(X)$ *is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization* v^* *of the upper envelope*

$$v(x) \;=\; \sup_{f\in\mathcal{F}} f(x)$$

belongs to $F(X)$.

Proof. See Appendix B.

Cautionary Note 2.7. Despite the elementary proofs of the properties in Theorem 2.6, illustrating how well adapted Definition 2.2 is to nonlinear theory, there are difficulties with the linear theory. If F_1 and F_2 are $\mathcal{P}$-monotone subsets, then the (fiberwise) sum $F_1 + F_2$ is also obviously a $\mathcal{P}$-monotone subset. However, the property

$$u \in F_1(X), v \in F_2(X) \quad \Rightarrow \quad u + v \in (F_1 + F_2)(X)$$

is difficult to deduce from Definition 2.2 even in the basic case where $F_1 = F_2 = F_1 + F_2$ is the linear "subequation" on $\mathbf{R}^n$ defined by $\Delta u \geq 0$.

3. Dirichlet Duality and the Notion of a Subequation

The following concept is the lynchpin for the Dirichlet problem.

Definition 3.1. Given a subset $F \subset J^2(X)$, the **Dirichlet dual** $\widetilde{F}$ of F is defined by

$$\widetilde{F} \;=\, \sim(-\mathrm{Int}F) \;=\; -(\sim \mathrm{Int}F). \tag{3.1}$$

Note the obvious properties:

(1) $F_1 \subset F_2 \quad \Rightarrow \quad \widetilde{F}_2 \subset \widetilde{F}_1$.

(2) $\widetilde{F_1 \cap F_2} \;=\; \widetilde{F}_1 \cup \widetilde{F}_2$

(3) $\widetilde{\widetilde{F}} \;=\; F$ if and only if $F \;=\; \overline{\mathrm{Int}F}$.

Thus to have a true duality with $\widetilde{\widetilde{F}} = F$ we must assume that $F = \overline{\mathrm{Int}F}$. For simplicity we also want to compute the dual fiberwise in the jet bundle. Consequently we will assume the following three topological conditions on F, combined as condition (T).

(T) (*i*) $F = \overline{\mathrm{Int}F}$, (*ii*) $F_x = \overline{\mathrm{Int}F_x}$, (*iii*) $\mathrm{Int}F_x = (\mathrm{Int}F)_x$

(where $\mathrm{Int}F_x$ denotes interior with respect to the fiber). It is then easy to see that the fiber of $\widetilde{F}$ at x is given by $-(\sim \mathrm{Int}F_x) = \sim(-\mathrm{Int}F_x) = \widetilde{(F_x)}$, so there is no ambiguity in the notation $\widetilde{F}_x$.

Definition 3.2. A subset $F \subset J^2(X)$ satisfying (T) will be called a **T-subset**.

Lemma 3.3. *Suppose $F \subset J^2(X)$ has the property that $F = \overline{\mathrm{Int}F}$. Then*

(a) *F satisfies condition (P) $\iff$ $\widetilde{F}$ satisfies condition (P).*

(b) *F satisfies condition (T) $\iff$ $\widetilde{F}$ satisfies condition (T).*

Proof. Assertion (b) is straightforward. To prove (a) we use another property.

Lemma 3.4. *Suppose that F is a T-subset. Then*

$$(4) \quad \widetilde{F_x + J} = \widetilde{F}_x - J \qquad \text{for all } J \in J^2_x(X).$$

Proof. Fix $J \in J^2_x(X)$. Then $J' \in \widetilde{F_x + J} \iff -J' \notin \mathrm{Int}(F_x + J) = \mathrm{Int}F_x + J \iff -(J' + J) \notin \mathrm{Int}F_x \iff J' + J \in \widetilde{F}_x \iff J' \in \widetilde{F}_x - J$. ∎

Corollary 3.5. *Suppose F is a T-subset of $J^2(X)$ and M is an arbitrary subset of $J^2(X)$. Then*

$$F \text{ is } M\text{-monotone} \iff \widetilde{F} \text{ is } M\text{-monotone.}$$

Proof. Fix $J \in M_x$ and assume $F_x + J \subset F_x$, or equivalently $F_x \subset F_x - J$. By (1) this implies that $\widetilde{F_x - J} \subset \widetilde{F}_x$. By (4) we have $\widetilde{F_x - J} = \widetilde{F}_x + J$ so that $\widetilde{F}_x + J \subset \widetilde{F}_x$. The converse follows from (3) and (b). ∎

Proof of (a). Take $M = \mathcal{P}$ in Corollary 3.5. ∎

Given a closed subset $F \subset J^2(X)$, note that

$$(3.2) \qquad \partial F = F \cap (\sim \mathrm{Int}F) = F \cap (-\widetilde{F}).$$

Definition 3.6. A function u is said to be **F-harmonic** if

$$u \in F(X) \quad \text{and} \quad -u \in \widetilde{F}(X).$$

Thus, under condition (T) a function $u \in C^2(X)$ is F-harmonic if and only if

$$(3.3) \qquad J^2_x u \in \partial F_x \quad \text{for all } x \in X.$$

Note also that an F-harmonic function is automatically continuous (since both u and $-u$ are upper semicontinuous). The topological condition (T) implies that $\widetilde{\widetilde{F}} = F$ and hence that

$$u \text{ is } F\text{-harmonic} \quad \iff \quad -u \text{ is } \widetilde{F}\text{-harmonic}.$$

The focal point of this paper is the **Dirichlet problem**, abbreviated (DP). Given a compact subset $K \subset X$ and a function $\varphi \in C(\partial K)$, a function $u \in C(K)$ is a solution to (DP) if

$$(1) \quad u \text{ is } F\text{-harmonic on Int}K \quad \text{and} \quad (2) \quad u\big|_{\partial K} \;=\; \varphi.$$

Example 3.7. Consider the one variable first-order subset F, defined by $|p| \leq 2|x|$, and its dual $\widetilde{F}$ defined by $|p| \geq 2|x|$. Of the conditions (P) and (T), the subset F satisfies all but condition (ii) of (T) and its dual $\widetilde{F}$ satisfies all but (iii) of (T). Both functions $x^2 - 1$ and $-x^2 + 1$ are F-harmonic and have the same boundary values on $[-1, 1]$. Thus without the full condition (T), solutions of the Dirichlet problem may not be unique.

Subequations. Uniqueness for the Dirichlet problem requires a third hypothesis in addition to (P) and (T). Consider the elementary canonical splitting

$$J^2(X) \;=\; \mathbf{R} \oplus J^2_{\text{red}}(X)$$

where $\mathbf{R}$ denotes the 2-jets of (locally) constant functions and $J^2_{\text{red}}(X)_x \equiv \{J^2_x u : u(x) = 0\}$ is the space of *reduced* 2-jets at x, with the short exact sequence of bundles

$$0 \;\to\; \text{Sym}^2(T^*X) \;\to\; J^2_{\text{red}}(X) \;\to\; T^*X \;\longrightarrow\; 0$$

which is the same as (2.1) except for the trivial factor $\mathbf{R}$.

We define

$$\mathcal{N} \;\subset\; \mathbf{R} \;\subset\; J^2(X)$$

to have fibers $\mathcal{N}_x = \mathbf{R}^- = \{c \in \mathbf{R} : c \leq 0\}$.

Definition 3.8. A subset $F \subset J^2(X)$ satisfies the **negativity condition** if

$$\text{(N)} \qquad F + \mathcal{N} \;\subseteq\; F.$$

A subset $F \subset J^2(X)$ satisfying (N) will also be called **$\mathcal{N}$-monotone**.

Now we can add to Lemma 3.2 a third conclusion (assuming F satisfies (T)):

$$\text{(c)} \quad F \text{ satisfies condition (N)} \quad \iff \quad \widetilde{F} \text{ satisfies condition (N)}$$

by taking $M = \mathcal{N}$ in Corollary 3.5.

In this paper the main results concerning existence and uniqueness for the Dirichlet problem assume that F satisfies (P), (T), and (N). This is formalized as follows.

Definition 3.9. By a **subequation** F on a manifold X we mean a subset

$$F \ \subset\ J^2(X)$$

which satisfies the three conditions (P), (T), and (N).

Proposition 3.10.

$$F \ \text{ is a subequation} \qquad \Longleftrightarrow \qquad \widetilde{F} \ \text{ is a subequation}$$

Proof. See (a), (b), and (c) above. ■

Our investigation of the (DP) for a subequation involves two additional subequations, which are constructed from the original and have two additional properties. One is the *cone property:*

$$J \in F \quad \Rightarrow \quad tJ \in F \ \text{ for all } t > 0,$$

i.e., Each fiber F_x is a cone with vertex at the origin. Stronger yet is the *convex cone property*:

Each fiber F_x is a convex cone with vertex at the origin.

Definition 3.11. A closed subset $F \subset J^2(X)$ having properties (P), (T), (N), and the cone property will be called a *cone subequation.* If it also has the convex cone property it will be called a *convex cone subequation.*

Euclidean (or Constant Coefficient) Subequations. The 2-jet bundle on $X = \mathbf{R}^n$ is canonically trivialized by

$$J^2_x u \ = \ \big(u(x), D_x u, D^2_x u\big) \tag{3.4}$$

where

$$D_x u \ = \ \left(\tfrac{\partial u}{\partial x_1}(x), \ldots, \tfrac{\partial u}{\partial x_n}(x)\right) \quad \text{ and } \quad D^2_x u \ = \ \left(\left(\tfrac{\partial^2 u}{\partial x_i \partial x_j}(x)\right)\right)$$

are the first and second derivatives of u at x. That is, for any open subset $X \subset \mathbf{R}^n$ there is a canonical trivialization

$$J^2(X) \ \cong\ X \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n) \tag{3.4'}$$

with fiber

$$\mathbf{J}^2 \ = \ \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n).$$

The standard notation $J = (r, p, A) \in \mathbf{J}^2$ will be used for the coordinates on $\mathbf{J}^2$.

Definition 3.12. Any subset $\mathbf{F} \subset \mathbf{J}^2$ which satisfies (P), (N), and for which $\mathbf{F} = \overline{\mathrm{Int}\mathbf{F}}$ determines a **euclidean** or **constant coefficient subequation** on any open subset $X \subset \mathbf{R}^n$ by setting $F = X \times \mathbf{F} \subset J^2(X)$ (with constant fiber $F_x \equiv \mathbf{F}$). For simplicity we shall often denote this subequation simply by $\mathbf{F}$ (with the euclidean coordinates implied), and refer to the F-subharmonic functions as $\mathbf{F}$-subharmonic functions.

Note that our assumptions on $\mathbf{F}$ imply the full condition (T) for F.

In this paper our first concern will be to start with an arbitrary euclidean subequation $\mathbf{F}$ and then to describe those riemannian manifolds which have a variable coefficient subequation F modeled on $\mathbf{F}$. We will give an explicit construction of this F. When, for example, $\mathbf{F}$ is the euclidean Laplace subequation, this analogue can be constructed on any riemannian manifold X and is simply the Laplace-Beltrami subequation on X.

The general linear group $\mathrm{GL}_n(\mathbf{R})$ has a natural action on $\mathbf{J}^2$ given by

$$h(r,p,A) \;=\; (r, hp, hAh^t) \qquad \text{for } h \in \mathrm{GL}_n(\mathbf{R}). \tag{3.5}$$

Each euclidean subequation $\mathbf{F} \subset \mathbf{J}$ has a **compact invariance group**

$$G(\mathbf{F}) \;=\; \{h \in \mathrm{O}_n : h(\mathbf{F}) = \mathbf{F}\} \tag{3.6}$$

which will be instrumental in determining which riemannian manifolds can be fitted with an analogous or companion subequation F. When $G(\mathbf{F}) = \mathrm{O}_n$, no restriction on X will be required. However, even the extreme case $G(\mathbf{F}) = \{I\}$ is not vacuous and will prove interesting.

4. The Riemannian Hessian—A Canonical Splitting of $J^2(X)$

4.1. The riemannian hessian. Assume now that X is equipped with a riemannian metric. Then, using the riemannian connection, any C^2-function u on X has a canonically defined *riemannian hessian* at every point given as follows. Fix $x \in X$ and vector fields V and W defined near x. Define

$$(\mathrm{Hess}_x u)(V,W) \;\equiv\; V \cdot W \cdot u - (\nabla_V W) \cdot u \tag{4.1}$$

where the RHS is evaluated at x. Since $(\nabla_V W) \cdot u - (\nabla_W V) \cdot u = [V,W] \cdot u$, we see that $(\mathrm{Hess}_x u)(V,W)$ is symmetric and depends only on the values of V and W at the point x, i.e., $\mathrm{Hess}\, u$ is a section of $\mathrm{Sym}^2(T^*X)$. Of course, at a critical point the riemannian hessian always agrees with the hessian defined at the beginning of Section 2.

The subequations we shall consider will be obtained by putting constraints on the *riemannian 2-jet*

$$(u, du, \mathrm{Hess}\, u) \tag{4.2}$$

which combines the riemannian hessian with the exterior derivative.

4.2. The canonical splitting. The riemannian hessian provides a bundle isomorphism

$$J^2(X) \;\xrightarrow{\cong}\; \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \tag{4.3}$$

by mapping

$$J^2_x u \;\mapsto\; (u(x), (du)_x, \mathrm{Hess}_x u)$$

for a C^2-function u at x. Note that the right hand side depends only on the 2-jet of u at x, and hence this is a well defined bundle map. It provides a canonical splitting of the short exact sequence (2.1).

4.3. Local trivializations associated to framings. The picture can be trivialized by a choice of local framing $e = (e_1, \dots, e_n)$ of the tangent bundle TX on some neighborhood U. (This framing is *not* required to be orthonormal.) The canonical splitting (4.3) determines a trivialization of $J^2(U)$ given at $x \in U$ by

$$\Phi^e : J^2_x(U) \longrightarrow \mathbf{R} \oplus \mathbf{R}^n \oplus \mathrm{Sym}^2(\mathbf{R}^n) \qquad \text{defined by} \tag{4.4a}$$

$$\Phi^e\left(J^2_x u\right) \equiv (u, e(u), (\mathrm{Hess}\, u)(e,e)) \tag{4.4b}$$

where $e(u) = (e_1 u, \dots, e_n u)$ and $(\mathrm{Hess}\, u)(e,e)$ is the $n \times n$-matrix with entries $(\mathrm{Hess}\, u)\,(e_i, e_j)$.

Lemma 4.1 (Change of Frame). *Under a change of frame $e' = he$ (that is, $e'_i = \sum_j h_{ij} e_j$) where h is a smooth $\mathrm{GL}_n(\mathbf{R})$-valued function, one has*

$$\left(u, e'(u), (\mathrm{Hess}\, u)(e', e')\right) = \left(u, he(u),\ h(\mathrm{Hess}\, u)(e,e)h^t\right). \tag{4.5}$$

Said differently, for a 2-jet $J \in J^2_x X$,

$$\Phi^e(J) = (r, p, A) \qquad \Rightarrow \qquad \Phi^{e'}(J) = (r, hp, hAh^t). \tag{4.6}$$

The elementary proof is omitted.

The rest of this section presents some general constructions of subequations using the riemannian hessian. For the sake of continuity the reader may want to pass directly to Section 5.

Some Classes of Subequations Based on the Riemannian Hessian

4.4. Pure Second-Order Subequations. These are the subequations which involve only the riemannian hessian and not the value of the function or its gradient. That is, with respect to the riemannian splitting (4.3), a subequation of the form

$$F = \mathbf{R} \oplus T^*X \oplus F'$$

for a closed subset $F' \subset \mathrm{Sym}^2(T^*X)$ will be called *pure (riemannian-) second-order*. In other words, $J^2_x u \in F$ if and only if $\mathrm{Hess}_x u \in F'$.

We now observe that a closed subset $F' \subset \mathrm{Sym}^2(T^*X)$ determines a pure second-order subequation if and only if positivity $F' + \mathcal{P} \subset F'$ holds and

$$\mathrm{Int} F'_x = (\mathrm{Int} F')_x \ \text{ for all } x, \tag{4.7}$$

where $\mathrm{Int} F'_x$ denotes interior with respect to the fiber $J^2_x(X)$. Condition (N) is obvious. It is clear that condition (P) for F is equivalent to $F' + \mathcal{P} \subset F'$, which implies that $\mathrm{Int} F' + \mathrm{Int} \mathcal{P} \subset \mathrm{Int} F'$, because any

$A \in \text{Int}F'_x$ can be extended to a local section of $\text{Int}F'$. Since $A + \epsilon I$ approximates any $A \in F'$, it follows that condition (P) implies

$$(i)\ \ F \ = \ \overline{\text{Int}F} \quad \text{and} \quad (ii)\ \ F_x \ = \ \overline{\text{Int}F_x}.$$

Thus with (4.7) condition (T) holds.

4.5. The complex and quaternionic hessians. Let X be a riemannian manifold equipped with a pointwise orthogonal almost complex structure $J : TX \to TX$. On this hermitian almost complex manifold, one can define the *complex hessian* of a C^2-function u by

$$\text{Hess}^{\mathbf{C}} u \ \equiv \ \tfrac{1}{2} \left\{ \text{Hess}\, u - J(\text{Hess}\, u) J \right\}. \tag{4.8}$$

This is a hermitian symmetric quadratic form on the complex tangent spaces of X. In particular its eigenvalues are real with even multiplicity and its eigenspaces are J-invariant. (See Section 15 for more details.)

Analogously, suppose X is equipped with a hermitian almost quaternionic structure, i.e., orthogonal bundle maps $I, J, K : TX \to TX$ satisfying the standard quaternionic identities: $I^2 = J^2 = K^2 = -1,\ IJ = -JI = K$, etc. Then one can define the *quaternionic hessian*

(4.9)

$$\text{Hess}^{\mathbf{H}} u \ \equiv \ \tfrac{1}{4} \left\{ \text{Hess}\, u - I(\text{Hess}\, u) I - J(\text{Hess}\, u) J - K(\text{Hess}\, u) K \right\}.$$

A number of basic subequations are defined in terms of these hessians. Primary among them are the following.

4.6. The Monge-Ampère subequations. A classical subequation, which is defined on any riemannian manifold X and will play an important role in this paper, is the (real) *Monge-Ampère subequation*

$$P \ \equiv \ P^{\mathbf{R}} \ \equiv \ \{J^2(u) : \text{Hess}\, u \geq 0\}. \tag{4.10}$$

When X carries an almost complex or quaternionic structure as above, we also have the associated *complex Monge-Ampère subequation*

$$P^{\mathbf{C}} \ \equiv \ \{J^2(u) : \text{Hess}^{\mathbf{C}} u \geq 0\} \tag{4.11}$$

and *quaternionic Monge-Ampère subequation*

$$P^{\mathbf{H}} \ \equiv \ \{J^2(u) : \text{Hess}^{\mathbf{H}} u \geq 0\}. \tag{4.12}$$

One easily checks that these sets are in fact subequations and have the convex cone property. They provide important examples of *monotonicity cones* for many other subequations and play an important role in the study of those subequations (see Section 8). In addition, they typify an important general construction which we now present.

4.7. Geometrically defined subequations—Grassmann structures. Consider the Grassmann bundle $\pi : G(p, TX) \to X$ with fibres $G(p, T_x X)$, the set of unoriented p-planes ξ through the origin in $T_x X$. If X is a riemannian manifold, then we can identify $\xi \in G(p, T_x X)$ with orthogonal projection $P_\xi \in \text{Sym}^2(T_x^* X)$ onto ξ.

Given $A \in \mathrm{Sym}^2(T_x^*X)$ and $\xi \in G(p, T_xX)$, the *ξ-trace of A* is defined by

$$\mathrm{tr}_\xi A \;=\; \langle A, P_\xi \rangle \tag{4.13}$$

using the natural inner product on $\mathrm{Sym}^2(T_x^*X)$. Equivalently,

$$\mathrm{tr}_\xi A \;=\; \mathrm{trace}\left(A\big|_\xi \right) \tag{4.14}$$

where $A\big|_\xi \in \mathrm{Sym}^2(\xi)$ is the restriction of A to ξ.

Note that for any closed subset $\mathbb{G}_x \subset G(p, T_xX)$, the set

$$F(\mathbb{G}_x) \;=\; \{A \in \mathrm{Sym}^2(T_x^*X) : \mathrm{tr}_\xi A \geq 0 \;\;\forall\, \xi \in \mathbb{G}_x\}$$

is a closed convex cone with vertex at the origin in $\mathrm{Sym}^2(T_x^*X)$. Moreover, F_x automatically satisfies positivity since if $P \geq 0$, $\mathrm{tr}_\xi P \geq 0$ for all $\xi \in G(p, T_xX)$.

Given $\mathbb{G} \subset G(p, TX)$, define $F(\mathbb{G}) \subset \mathrm{Sym}^2(T^*X)$ to be the subset whose fibre at x is $F(\mathbb{G}_x)$. It is left to the reader to verify that: $F(\mathbb{G})$ is a closed subset if and only if $\pi : \mathbb{G} \to X$ is a local surjection. Furthermore, the topological condition (T)(iii), i.e., (4.7), is satisfied in this case. It now follows from the discussion in 4.4 (Pure Second Order Subequations) that the following holds.

Proposition 4.2. *Given a closed subset $\mathbb{G} \subset \mathrm{Sym}^2(T^*X)$ with $\pi : \mathbb{G} \to X$ a local surjection, the subset*

$$\mathbf{R} \oplus T^*X \oplus F(\mathbb{G})$$

is a pure second order convex cone subequation.

The set $F(\mathbb{G})$ is said to be *geometrically defined by* $\mathbb{G}$, and the $F(\mathbb{G})$-subharmonic functions will be referred to as $\mathbb{G}$-*plurisubharmonic functions*.

4.8. Gradient independent subequations. On a riemannian manifold X there are the subequations which only involve the riemannian hessian and the value of the function. That is, with respect to the riemannian splitting (4.3), a subset of the form

$$F \;=\; T^*X \oplus F'$$

for a closed subset $F' \subset \mathbf{R} \oplus \mathrm{Sym}^2(T^*X)$ is said to be *gradient-independent*. Note that $F' \equiv \mathcal{N} + \mathcal{P} \subset \mathbf{R} \oplus \mathrm{Sym}^2(T^*X)$ provides an example of a gradient-independent subequation. It is also a convex cone subequation.

Note that a closed subset $F = T^*X \oplus F'$ satisfies (P) and (N) if and only if

$$F' + \mathcal{N} + \mathcal{P} \;\subset\; F'.$$

The topological conditions (T)(i) and (T)(ii) then follow from (P) and (N) by using $(r, A) = \lim_{\epsilon \to 0}(r - \epsilon, A + \epsilon I)$. This proves that $F \subset J^2(X)$

is a gradient-independent subequation if and only if F is of the form $F = T^*X \oplus F'$ and F' satisfies:

$$F' \text{ is closed}, \quad F' + \mathcal{N} + \mathcal{P} \subset F', \quad \text{and} \quad \text{Int}F'_x = (\text{Int}F')_x \text{ for all } x.$$

Said differently, a closed subset $F' \subset \mathbf{R} \oplus \text{Sym}^2(T^*X)$ with $\text{Int}F'_x = (\text{Int}F')_x$ for all x is a (gradient-independent) subequation if and only if F' is $(\mathcal{N} + \mathcal{P})$-monotone.

4.9. Subequations of Reduced Type. These are the subequations $F \subset J^2(X)$ that do not involve the dependent variable r in the fibers of $J^2(X)$. That is, $F = \mathbf{R} \times F'$ where F' is a subset of the reduced 2-jet bundle $J^2_{\text{red}}(X)$, and we are using the natural bundle splitting $J^2(X) = \mathbf{R} \times J^2_{\text{red}}(X)$.

5. Universal Subequations on Manifolds with Topological G-Structure

In this section we construct the subequations of principal interest in the paper. The construction starts with a "universal model," which is a euclidean (constant coefficient) subequation $\mathbf{F} \subset \mathbf{J}^2$. The idea is to find subequations $F \subset J^2(X)$ which are locally modeled on $\mathbf{F}$. To accomplish this, we recall from (3.6) the compact invariance group

$$G \;=\; G(\mathbf{F}) \;\equiv\; \{g \in \text{O}_n : g(\mathbf{F}) = \mathbf{F}\} \tag{5.1}$$

where O_n acts on $\mathbf{J}^2$ by

$$g(r, p, A) \;\equiv\; (r, gp, gAg^t). \tag{5.2}$$

The main point is that whenever a riemannian manifold X is given a topological G-structure, we can construct the desired subequation by using this structure and the canonical splitting of $J^2(X)$ in the previous section. Subequations F constructed from an $\mathbf{F}$ in this way will be called *riemannian G-subequations.*

5.1. Riemannian G-manifolds. Recall that for a fixed subgroup

$$G \subseteq \text{O}_n,$$

a **topological G-structure on** X is a family of C^∞ local trivializations of TX over open sets in a covering $\{U_\alpha\}$ of X whose transition functions have values in G, i.e., the transition function from the α-trivialization to the β-trivialization is just a map $g^{\beta,\alpha} : U_\alpha \cap U_\beta \longrightarrow G$. A local trivialization on U_α is simply a choice of framing $e^\alpha_1, \ldots, e^\alpha_n$ for TX over U_α, and one can think of these trivializations as a family of *admissible framings* or *admissible G-framings* for TX. At each point $x \in X$, the structure determines a family of admissible G-frames, in the subset of all tangent frames, on which G acts simply and transitively. This gives a principal G-bundle in the bundle of all tangent frames.

Definition 5.1. A **riemannian G-manifold** is a riemannian manifold equipped with a topological G-structure.

Important Notes. (1) A topological G-structure on X (for $G \subset \mathrm{O}_n$) determines a second riemannian metric on X by declaring the admissible G frames to be orthonormal. These two metrics coincide if the admissible frame fields are orthonormal in the original metric, and this will typically be the case. However, the results will hold without this assumption.

(2) Furthermore, one can as well consider topological G-structures on X where G is a subgroup of the *full invariance group*

$$\widetilde{G} \;=\; \widetilde{G}(\mathbf{F}) \;\equiv\; \{g \in \mathrm{GL}_n : g(\mathbf{F}) = \mathbf{F}\}.$$

The arguments presented here carry through in this more general case, as the reader can easily verify. This has genuine applicability since there are interesting subequations which are invariant under groups like the conformal group, or $\mathrm{GL}_n(\mathbf{R})$ or $\mathrm{GL}_n(\mathbf{C})$. In these cases one can do local calculations with more general frame fields. However, no new subequations arise from this generalization, and so our exposition is restricted to compact G.

Important Point: The compact invariance group G is not the riemannian holonomy group (for either metric). No integrability assumption is made. Topological G-structures are soft and easily constructed.

5.2. Riemannian G-subequations. The idea now is the following. Suppose X is a riemannian G-manifold with admissible framings $\{(U_\alpha, e^\alpha)\}$. Then each framing e^α determines a local trivialization of $J^2(U_\alpha)$ given by (4.4b). Now if G lies in the invariance group of a euclidean subequation $\mathbf{F}$, the local subequations $U_\alpha \times \mathbf{F}$ will be preserved under changes of framing and thereby determine a global subequation F on X. We formalize this in the following lemma.

Lemma 5.2. *Suppose $\mathbf{F}$ is a euclidean subequation with compact invariance group G and X is a riemannian G-manifold. For $x \in X$, the condition on a 2-jet $J \equiv J^2_x u$ that*

$$\Phi^e(J) \equiv (u(x), e_x(u), (\mathrm{Hess}_x u)(e,e)) \;\in\; \mathbf{F} \tag{5.3}$$

is independent of the choice of G-frame e at x. Hence there is a well defined subset $F \subset J^2(X)$ given by

$$J \in F_x \quad \Leftrightarrow \quad \Phi^e\,(J)\,(x) \in \mathbf{F}. \tag{5.4}$$

Proof. Note that (5.4) is independent of the G-frame e, since if $e' = he$ is another admissible G-frame on a neighborhood of x, then $\Phi^{e'}(J) = h\,(\Phi^e(J)) \in h(\mathbf{F}) = \mathbf{F}$ by (5.1), (5.2), and the change of frame formula (4.6). Hence, F is well defined by (5.4). ∎

It is easy to see that

$$F \text{ satisfies (P)} \iff \mathbf{F} \text{ satisfies (P)},$$
$$F \text{ satisfies (N)} \iff \mathbf{F} \text{ satisfies (N), and}$$
$$F \text{ satisfies (T)} \iff \mathbf{F} \text{ satisfies (T)},$$

so that F is a subequation on X.

Definition 5.3. The subequation F defined by (5.4) above will be called the *riemannian G-subequation on X with euclidean model* $\mathbf{F}$. For simplicity the F-subharmonic functions on X will be called $\mathbf{F}$-subharmonic.

Note that $\mathbf{F}$ is G-invariant if and only if $\widetilde{\mathbf{F}}$ is G-invariant. It is also easy to show:

Lemma 5.4. *The subequation $\widetilde{F}$ is the riemannian G-subequation on X with euclidean model $\widetilde{\mathbf{F}}$.*

The extreme cases are where $G = \{I\}$ or $G = \mathrm{O}_n$. Both are interesting.

Example $G = \{I\}$. Here $\mathbf{F}$ can be any euclidean subequation. A riemannian $\{I\}$-manifold is one which is equipped with n vector fields which are linearly independent at every point. For example, such fields can be found on any orientable riemannian 3-manifold. They also exist on any Lie group with a metric invariant under left translations by elements of the group. Thus, for an arbitrary euclidean subequation $\mathbf{F}$ there are plenty of riemannian manifolds which support a riemannian subequation F modeled on $\mathbf{F}$.

Example $G = \mathrm{O}_n$. Given a euclidean subequation $\mathbf{F}$ which is invariant under the full orthogonal group, there exists a companion riemannian O_n-subequation F on *every* riemannian manifold.

Example $G = \mathrm{U}_n$. If X is an almost complex manifold with a compatible (hermitian) metric, then one can consider $\mathbf{F} = \mathbf{R} \times \mathbf{R}^{2n} \times \mathbf{P_C}$ where $\mathbf{P_C} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ are the matrices whose hermitian symmetric part is ≥ 0 (cf. (4.8), (4.11), and Section 15). This leads to solving the Dirichlet problem for the homogeneous complex Monge-Ampere equation on domains in X.

Example $G = \mathrm{Sp}_1 \cdot \mathrm{Sp}_n$ or Sp_n. There are analogues of the preceding example on any almost quaternionic or almost hyperkahler manifold with a compatible metric.

5.3. Riemannian G-subequations in local coordinates. Finally we derive the key formula needed to prove a comparison result in Section 10. Recall that the riemannian hessian has a simple expression in terms of local coordinates $x = (x_1, \ldots, x_n)$ on X, namely,

$$(\mathrm{Hess} u)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k(x) \frac{\partial u}{\partial x_k}, \tag{5.5}$$

where Γ_{ij}^k denote the *Christoffel Symbols* of the metric connection defined by the relation $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}$. The equation (5.5) can be written more succinctly as

$$\text{(5.5')} \qquad (\mathrm{Hess}\, u)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = D^2 u - \Gamma_x(Du)$$

where Du and D^2u are the first and second derivatives of u in the coordinates x and

$$\Gamma_x : \mathbf{R}^n \to \mathrm{Sym}^2(\mathbf{R}^n)$$

denotes the linear Christoffel map defined above.

Proposition 5.5. *Let F be a riemannian G-subequation on X with euclidean model $\mathbf{F}$ on a riemannian G-manifold X. Suppose $x = (x_1, \ldots, x_n)$ is a local coordinate system on U and that $e_1, \ldots, e_n$ is an admissible G-frame on U. Let h denote the GL_n-valued function on U defined by $e = h\frac{\partial}{\partial x}$. Then a C^2-function u is F-subharmonic on U if and only if*

$$\text{(5.6)} \qquad \left(u,\ hDu, h\left(D^2u - \Gamma(Du)\right)h^t\right) \in \mathbf{F} \qquad \text{on } U.$$

Remark 5.6. Fix $J \in J_x^2$ with $x \in U$. Then by using the standard isomorphism $J_x^2 \cong \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$ induced by the coordinate system $x = (x_1, \ldots, x_n)$ to represent J as $J = (r, p, A) = (u, Du, D^2u)$, condition (5.4) can be restated as saying that

$$J \in F_x \qquad \iff \qquad (r, hp, h(A - \Gamma(p))h^t) \in \mathbf{F}.$$

Proof. By (5.3) we must show that

$$\text{(5.7)} \qquad (u, e(u), (\mathrm{Hess}\, u)(e.e)) = \left(u, hDu, h(D^2u - \Gamma(Du))h^t\right).$$

The fact that $e(u) = hDu$, i.e., that $e_i(u) = \sum_{j=1}^n h_{ij}\frac{\partial u}{\partial x_j}$, is obvious. For the proof that the matrix $(\mathrm{Hess}\, u)(e_i, e_j)$ equals $h(D^2u - \Gamma(Du))h^t$, note that

$$\begin{aligned}(\mathrm{Hess}\, u)(e_i, e_j) &= (\mathrm{Hess}\, u)\left(\sum_{k=1}^n h_{ik}\frac{\partial}{\partial x_k}, \sum_{\ell=1}^n h_{j\ell}\frac{\partial}{\partial x_\ell}\right) \\ &= \sum_{k,\ell}^n h_{ik}(\mathrm{Hess}\, u)\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right) h_{j\ell}\end{aligned}$$

and then apply (5.5). ∎

6. Jet-Equivalence of Subequations

In this section we introduce the important notion of jet-equivalence for subequations. This concept enables us to solve the Dirichlet problem for a very broad spectrum of subequations on manifolds—namely those which are locally jet-equivalent to a euclidean one. They include the subequations considered so far. In fact, Proposition 5.5 just states that F is jet-equivalent to $\mathbf{F}$. However, jet-equivalence goes far beyond this realm, to quite general variable coefficient inhomogeneous equations. These include the basic Calabi-Yau equation (see Example 6.15 below). It overcomes an important obstacle in understanding, on an almost complex manifold, the *intrinsic homogeneous complex Monge-Ampère subequation* (i.e., intrinsic plurisubharmonic functions) where no hermitian metric is used (cf. [HL$_8$]).[1]

6.1. Automorphisms. To define jet-equivalence, we first need to understand the bundle automorphisms of $J^2(X)$.

Definition 6.1. (a) An **automorphism** of the reduced jet bundle $J^2_{\rm red}(X)$ is a bundle isomorphism $\Phi : J^2_{\rm red}(X) \to J^2_{\rm red}(X)$ such that with respect to the short exact sequence

$$0 \longrightarrow \mathrm{Sym}^2(T^*X) \longrightarrow J^2_{\rm red}(X) \longrightarrow T^*X \longrightarrow 0 \tag{6.1}$$

we have

$$\Phi(\mathrm{Sym}^2(T^*X)) = \mathrm{Sym}^2(T^*X) \tag{6.2}$$

so there is an induced bundle automorphism

$$g = g_\Phi : T^*X \longrightarrow T^*X \tag{6.3}$$

and we further require that there exist a second bundle automorphism

$$h = h_\Phi : T^*X \longrightarrow T^*X \tag{6.3'}$$

such that on $\mathrm{Sym}^2(T^*X)$, Φ has the form $\Phi(A) = hAh^t$, i.e.,

$$\Phi(A)(v,w) = A(h^t v, h^t w) \qquad \text{for } v,w \in TX. \tag{6.4}$$

(b) An **automorphism** of the full jet bundle $J^2(X) = \mathbf{R} \oplus J^2_{\rm red}(X)$ is a bundle isomorphism $\Phi : J^2(X) \to J^2(X)$ which is the direct sum of the identity on $\mathbf{R}$ and an automorphism of $J^2_{\rm red}(X)$.

Lemma 6.2. *The automorphisms of $J^2(X)$ form a group. They are the sections of the bundle of groups whose fiber at $x \in X$ is the group of automorphisms of $J^2_x(X)$ defined by (6.2), (6.3), and (6.4) above.*

---

[1] Examples show that even on $\mathbf{C}^n$ there are hermitian metrics for which not all the classical psh-functions are hermitian psh (cf. [HL$_8$]).

Proof. We first show that the composition of two automorphisms of $J^2(X)$ is again an automorphism. Suppose Ψ and Φ are bundle automorphisms. Then $\Psi\circ\Phi$ clearly satisfies condition (6.2) and one sees easily that $g_{\Psi\circ\Phi} = g_\Psi \circ g_\Phi$. Finally, $(\Psi\circ\Phi)(A) = \Psi(\Phi(A)) = h_\Psi \Phi(A) h_\Psi^t = h_\Psi h_\Phi A h_\Phi^t h_\Psi^t = (h_\Psi \circ h_\Phi) A (h_\Psi \circ h_\Phi)^t = h_{\Psi\circ\Phi} A h_{\Psi\circ\Phi}^t$. The proof that the inverse of an automorphism is an automorphism is similar. ∎

Proposition 6.3. *With respect to any splitting*

$$J^2(X) \;=\; \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X)$$

of the short exact sequence (6.1), a bundle automorphism has the form

$$\Phi(r,p,A) \;=\; (r, gp, hAh^t + L(p)) \tag{6.5}$$

where $g, h : T^*X \to T^*X$ are bundle isomorphisms and L is a smooth section of the bundle $\mathrm{Hom}\,(T^*X, \mathrm{Sym}^2(T^*X))$.

Proof. With respect to a splitting $J^2(X) \cong T^*X \oplus \mathrm{Sym}^2(T^*X)$, any bundle isomorphism is of the form $\Phi(p,A) = (gp + M(A), H(A) + L(p))$. The requirement that Φ leaves $\mathrm{Sym}^2(T^*X)$ invariant implies that the point-wise linear map M is zero, and the property that $\Phi(0,A) = (0, hAh^t)$ implies that $H(A) = hAh^t$. ∎

Example 6.4. Given a local coordinate system $(x_1, \dots, x_n)$ on an open set $U \subset X$, the canonical trivialization

$$J^2(U) \;=\; U \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n) \tag{6.6}$$

is determined by $J_x^2 u = (u, Du, D^2u)$ evaluated at x, where $Du = (u_1, \dots, u_n)$ and $D^2u = ((u_{ij}))$ (cf. Remark 2.8). With respect to this splitting, every automorphism is of the form

$$\Phi(u, Du, D^2u) \;=\; (u,\; gDu,\; h\cdot D^2u\cdot h^t + L(Du)) \tag{6.7}$$

where $g_x, h_x \in \mathrm{GL}_n$ and $L_x : \mathbf{R}^n \to \mathrm{Sym}^2(\mathbf{R}^n)$ is linear for each point $x \in U$.

Example 6.5. The trivial 2-jet bundle on $\mathbf{R}^n$ has fiber

$$\mathbf{J}^2 = \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$$

with automorphism group

$$\mathrm{Aut}(\mathbf{J}^2) \;\equiv\; \mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{Hom}\,(\mathbf{R}^n, \mathrm{Sym}^2(\mathbf{R}^n))$$

where the action is given by

$$\Phi_{(g,h,L)}(r,p,A) \;=\; (r,\; gp,\; hAh^t + L(p)).$$

Note that the group law is

$$(\bar g, \bar h, \bar L)\cdot(g,h,L) \;=\; (\bar g g,\; \bar h L \bar h^t + \bar L \circ g).$$

Remark 6.6. Automorphisms at a point, with $g = h$, appear naturally when one considers the action of diffeomorphisms. Namely, if φ is a diffeomorphism fixing a point x_0, then in local coordinates (as in

Example 6.4 above) the right action on $J^2_{x_0}$, induced by the pull-back φ^* on 2-jets, is given by (6.7), where $g_{x_0} = h_{x_0}$ is the transpose on the Jacobian matrix $((\frac{\partial \varphi^i}{\partial x_j}))$ and $L_{x_0}(Du) = \sum_{k=1}^n u_k \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j}(x_0)$.

Cautionary Note 1. Despite the remark above, automorphisms of the 2-jet bundle $J^2(X)$, even those with $g = h$, have little to do with global diffeomorphisms or global changes of coordinates. In fact, an automorphism radically restructures $J^2(X)$ in that the image of an integrable section (one obtained by taking $J^2 u$ for a fixed smooth function u on X) is essentially never integrable.

6.2. Jet-Equivalence of subequations.

Definition 6.7. Two subequations $F, F' \subset J^2(X)$ are said to be **jet-equivalent** if there exists an automorphism $\Phi : J^2(X) \to J^2(X)$ with $\Phi(F) = F'$.

Cautionary Note 2. A jet-equivalence $\Phi : F \to F'$ does not take F-subharmonic functions to F'-subharmonic functions. In fact, as mentioned above, for $u \in C^2$, $\Phi(J^2 u)$ is almost never the 2-jet of a function. It happens if and only if $\Phi(J^2 u) = J^2 u$.

Nevertheless, it is easily checked that if Φ is an automorphism and F is a subequation, then $\Phi(F)$ is also a subequation.

Definition 6.8. We say that a subequation $F \subset J^2(X)$ **is locally jet-equivalent to a euclidean subequation** if each point x has a coordinate neighborhood U such that $F\big|_U$ is jet-equivalent to a euclidean (constant coefficient) subequation $U \times \mathbf{F}$ in those coordinates. Such a coordinate chart will be called **distinguished**.

Using Example 6.4, we see that Proposition 5.5 can be restated as follows. Let $\mathbf{F}$ be a euclidean subequation with compact invariance group $G = G(\mathbf{F})$ (see (3.6)).

Proposition 6.9. *Suppose that F is a riemannian G-subequation with euclidean model $\mathbf{F}$ on a riemannian G-manifold X. Then F is locally jet-equivalent to $\mathbf{F}$ on X.*

Lemma 6.10. *Suppose X is connected and $F \subset J^2(X)$ is locally jet-equivalent to a euclidean subequation on X. Then there is a euclidean subequation $\mathbf{F} \subset \mathbf{J}^2$, unique up to automorphisms, such that F is jet-equivalent to $U \times \mathbf{F}$ on every distinguished coordinate chart.*

Proof. In the overlap of any two distinguished charts $U_1 \cap U_2$, choose a point x. Then the local automorphisms Φ_1 and Φ_2, restricted to F_x, determine an automorphism taking $\mathbf{F}_1$ to $\mathbf{F}_2$. Thus the local euclidean subequations on these charts are all equivalent under automorphisms, and they can be made equal by applying the appropriate constant automorphism on each chart. ∎

6.3. Affine automorphisms and affine jet-equivalence. The automorphism group $\mathrm{Aut}(\mathbf{J}^2)$ can be naturally extended by the translations of $\mathbf{J}^2$. Recall that the group of affine transformations of a vector space V is the product $\mathrm{Aff}(V) = \mathrm{GL}(V) \times V$ acting on V by $(g, v)(u) = g(v) + u$. The group law is $(g, v) \cdot (h, w) = (gh, v + g(w))$. There is a short exact sequence $0 \to V \to \mathrm{Aff}(V) \xrightarrow{\pi} \mathrm{GL}(V) \to \{I\}$.

Definition 6.11. The **affine automorphism group** is the inverse image

$$\mathrm{Aut}_{\mathrm{aff}}(\mathbf{J}^2) = \pi^{-1}(\mathrm{Aut}(\mathbf{J}^2))$$

of $\mathrm{Aut}(\mathbf{J}^2) \subset \mathrm{GL}(\mathbf{J}^2)$ under the surjective group homomorphism $\pi : \mathrm{Aff}(\mathbf{J}^2) \to \mathrm{GL}(\mathbf{J}^2)$.

Note that any affine automorphism $\widetilde{\Phi}$ can be written in the form

$$\widetilde{\Phi} \;=\; \Phi + J \tag{6.8}$$

where Φ is a (linear) automorphism and J is a section of the bundle $J^2(X)$.

Using the affine automorphism group, we expand our notion of jet-equivalence in an important way. Let F and F' be subequations on a manifold X.

Definition 6.12. Two subequations $F, F' \subset J^2(X)$ are said to be **affinely jet-equivalent** if there exists an affine automorphism $\widetilde{\Phi} : J^2(X) \to J^2(X)$ with $\widetilde{\Phi}(F) = F'$.

Definition 6.13. We say that a subequation $F \subset J^2(X)$ **is locally affinely jet-equivalent to a euclidean subequation** if each point x has distinguished coordinate neighborhoods U such that $F\big|_U$ is affinely jet-equivalent to a euclidean (constant coefficient) subequation $U \times \mathbf{F}$ in those distinguished coordinates.

Lemma 6.14. *Suppose F is a subequation on a coordinate chart U with a given affine jet-equivalence to a euclidean subequation $U \times \mathbf{F}$. Write the affine automorphism $\widetilde{\Phi}$ as in (6.8) above so that*

$$J \in F_x \quad \iff \quad \Phi_x(J) + J_x \;\in\; \mathbf{F}$$

for $x \in U$. Then

$$J \in \widetilde{F}_x \quad \iff \quad \Phi_x(J) - J_x \;\in\; \widetilde{\mathbf{F}}.$$

Proof. $J \in \widetilde{F}_x \iff -J \notin \mathrm{Int}F_x \iff \Phi_x(-J) + J_x \notin \mathrm{Int}\mathbf{F} \iff -\{\Phi_x(J) - J_x\} \notin \mathrm{Int}\mathbf{F} \iff \Phi_x(J) - J_x \in \widetilde{\mathbf{F}}$. (Recall that $\mathrm{Int}F_x$ denotes interior with respect to the fiber.) ∎

Example 6.15. (The Calabi-Yau Equation). Let X be an almost complex hermitian manifold (a riemannian U_n-manifold), and consider the subequation $F \subset J^2(X)$ determined by the euclidean subequation:

$$A_{\mathbf{C}} + I \ \geq\ 0 \quad \text{and} \quad \det{}_{\mathbf{C}}\{A_{\mathbf{C}} + I\} \ \geq\ 1,$$

where $A_{\mathbf{C}} \equiv \frac{1}{2}(A - JAJ)$ is the hermitian symmetric part of A. Let $f > 0$ be a smooth positive function on X and write $f = h^{-2n}$. Consider the global affine jet-equivalence of $J^2(X)$ given by

(6.9)

$$\widetilde{\Phi}(r,p,A) \ =\ (r,p,(hI)A(hI)^t + (h^2-1)I) \ =\ (r,p,h^2A + (h^2-1)I)$$

and set $F_f = \widetilde{\Phi}^{-1}(F)$. Then

$$\begin{aligned} (r,p,A) \in F_f \iff &\quad h^2(A_{\mathbf{C}} + I) \geq 0 \ \text{ and } \ \det{}_{\mathbf{C}}\{h^2(A_{\mathbf{C}} + I)\} \geq 1 \\ \iff &\quad (A_{\mathbf{C}} + I) \geq 0 \ \text{ and } \ \det{}_{\mathbf{C}}\{(A_{\mathbf{C}} + I)\} \geq f \end{aligned}$$

so the F_f-harmonic functions are functions u with $\mathrm{Hess}_{\mathbf{C}} u + I \geq 0$ (quasi-plurisubharmonic) and $\det_{\mathbf{C}}\{\mathrm{Hess}_{\mathbf{C}} u + I\} = f$. If X is actually a complex manifold of dimension n with Kähler form ω, this last equation can be written in the more familiar form

$$\left(\frac{1}{i}\partial\overline{\partial}u + \omega\right)^n \ =\ f\omega^n.$$

One can similarly treat the equation

$$\left(\frac{1}{i}\partial\overline{\partial}u + \omega\right)^n \ =\ e^u f\omega^n,$$

or the same equation with e^u replaced by any non-decreasing positive function $F(u)$.

There are many further examples illustrating the flexibility and power of using local affine jet-equivalence. We present some of them in Section 18.

7. Strictly F-Subharmonic Functions

Consider a second-order subequation $F \subset J^2(X)$ on a manifold X. For the more general case of subsets which are just $\mathcal{P}$-monotone, see Remark 7.8. Strict subharmonicity for C^2-functions is unambiguous.

Definition 7.1. A function $u \in C^2(X)$ is said to be **strictly** F-**subharmonic on** X if

$$J^2_x u \ \in\ \mathrm{Int} F \quad \text{for all } x \in X. \tag{7.1}$$

There is more than one way to extend this notion to functions $u \in F(X)$ which are not C^2. The choice made in Definition 7.4 will be shown to be useful in Section 8 and in discussing the Dirichlet problem.

Fix a metric on the vector bundle $J^2(X)$. (If X is given a riemannian metric, this metric together with the canonical splitting (4.3) determines a metric on $J^2(X)$.)

Definition 7.2. For each $c > 0$ we define the c**-strict** subset $F^c \subset F$ by

$$F_x^c \;=\; \{J \in F_x : \mathrm{dist}(J, \sim F_x) \geq c\} \tag{7.2}$$

where dist denotes distance in the fiber $J_x^2(X)$.

Lemma 7.3. *Suppose that F is a subequation. Then the set F^c is closed and is both $\mathcal{P}$- and $\mathcal{N}$-monotone. However, F^c does not necessarily satisfy (T).*

Proof. Note that for $J \in J_x^2(X)$,

$$J \in F_x^c \qquad \iff \qquad B_x(J, c) \subseteq F_x \tag{7.3}$$

where $B_x(J, c) \subset J_x^2(X)$ is the closed metric ball of radius c about the point J in the fiber. Suppose now that $J_i \in F_{x_i}^c$ is a sequence converging to J at x. Then $B_{x_i}(J_i, c) \subseteq F_{x_i}$ for all i, and since F is closed we conclude that $B_x(J, c) \subseteq F_x$. Hence F^c is closed.

Suppose now that $P \in \mathcal{P}_x$. Then by condition (P) for F we see that $B_x(J, c) \subseteq F_x \;\Rightarrow\; B_x(J + P, c) = B_x(J, c) + P \subseteq F_x$, and so condition (P) holds for F^c. The proof that F^c satisfies condition (N) is the same.

Example. Let K denote the union of the unit disk $\{|z| \leq 1\}$ with the interval $[1, 2]$ on the x-axis in $\mathbf{R}^2$. Let $K_c = \{z \in \mathbf{R}^2 : \mathrm{dist}(x, K) \leq c\}$. Define F by requiring $p \in K_c$. Then F^c is easily seen to be equal to K. Now F is a subset which satisfies (P), (N), and (T), whereas F^c is a subset which satisfies (P) and (N) but not (T). ∎

Definition 7.4. A function u on X is **strictly F-subharmonic** if for each point $x \in X$ there is a neighborhood B of x and $c > 0$ such that u is F^c-subharmonic on B. Let $F_{\mathrm{strict}}(X)$ denote the space of such functions.

For a C^2 function it is easy to see that the two definitions of strictly F-subharmonicity, given in Definitions 7.1 and 7.4, agree (see Remark 7.8). Moreover, it is easy to see that Definition 7.4 is independent of the choice of metric on the bundle $J^2(X)$.

Strictly F-subharmonic functions are stable under smooth perturbations.

Lemma 7.5. (Stability). *Suppose $u \in F^c(X)$ and $\psi \in C^2(X)$. For each precompact open subset $Y \subset\subset X$*

$$u + \delta\psi \;\in\; F^{\frac{c}{2}}(Y) \qquad \text{if } \delta \text{ is sufficiently small.}$$

Proof. By Definition 2.2 it will suffice to show that for all $\delta > 0$ sufficiently small

$$F_x^c + \delta J_x^2 \psi \;\subset\; F_x^{\frac{c}{2}} \quad \text{for all } x \in Y. \tag{7.4}$$

Choose $\delta > 0$ so that

$$\delta \| J_x^2 \psi \| \ < \ \tfrac{c}{2} \quad \text{for all } x \in Y. \tag{7.5}$$

Fix $J \in F_x^c$ with $x \in Y$. By (7.3) we have $B_x(J, c) \subset F_x$. Note then that $B_x(J + \delta J_x^2 \psi, \frac{c}{2}) = B_x(J, \frac{c}{2}) + \delta J_x^2 \psi$ is contained in $B_x(J, c) \subset F_x$ by (7.5). Again using (7.3), this shows that $J + \delta J_x^2 \psi \in F_x^{\frac{c}{2}}$. ■

Corollary 7.6. *Suppose $u \in F_{\text{strict}}(X)$ and $\psi \in C^\infty_{\text{cpt}}(X)$. Then*

$$u + \delta \psi \ \in \ F_{\text{strict}}(X) \qquad \textit{if } \delta \textit{ is sufficiently small.}$$

We shall need the following two properties.

Lemma 7.7.

(i) $u, v \ \in \ F_{\text{strict}}(X) \quad \Rightarrow \quad \max\{u, v\} \ \in \ F_{\text{strict}}(X)$.

(ii) *If F satisfies the negativity condition* (N)*, then*

$$u \ \in \ F_{\text{strict}}(X) \ \text{ and } \ c > 0 \quad \Rightarrow \quad u - c \ \in \ F_{\text{strict}}(X).$$

The proof is straightforward and omitted.

Remark 7.8. ($\mathcal{P}$-monotone subsets). The results of this section remain true for $\mathcal{P}$-monotone subsets and for $(\mathcal{P} + \mathcal{N})$-monotone subsets which are not necessarily subequations (i.e., condition (T) may not be satisfied). This is important in Section 11, on boundary convexity, where the results are applied to a $\mathcal{P}$-monotone subset $\overrightarrow{F}$ which is open. Everything is straightforward except the proof of the assertion

$$\psi \in C^2(X) \cap F_{\text{strict}}(X) \qquad \Rightarrow \qquad J_x^2 \psi \in \text{Int} F \ \text{ for all } x \in X. \tag{7.6}$$

In general, (7.6) cannot be proved by establishing that $F^c \subset \text{Int} F$. For example, take $X = \mathbf{R}$ and define F by requiring $p \geq 0$ if $x \leq 0$ and $p \geq 1$ if $x > 0$. Then the point $x = 0, p = \frac{1}{2}$ belongs to $F^{\frac{1}{2}}$ but not $\text{Int} F$.

We prove (7.6) as follows. Fix $x_0 \in X$ and trivialize $J^2(X) = U \times \mathbf{J}^2$ on a neighborhood U of x_0 via a choice of orthonormal frame field. For such a choice, the fiber metric on $\mathbf{J}$ is constant. Now ψ is c-strict for some $c > 0$ in a smaller neighborhood U of x_0, i.e., $B(J_x^2 \psi, c) \ \subset \ F_x$ for $x \in U$. By continuity we have $|J_x^2 \psi - J_{x_0}^2 \psi| \leq \frac{c}{2}$ on a neighborhood V of x_0, and therefore $B(J_{x_0}^2 \psi, \frac{c}{2}) \ \subset \ F_x$ for $x \in W \equiv U \cap V$. Thus $W \times B(J_{x_0}^2 \psi, \frac{c}{2}) \subset F$ and in particular $J_{x_0}^2 \psi \in \text{Int} F$. ■

The following elementary example shows that $J_x^2 \psi \in \text{Int}(F_x)$ for x near x_0 (rather than $J_{x_0}^2 \psi \in (\text{Int} F)_{x_0}$ as in (7.1)) is not sufficient to guarantee that ψ is c-strict near x_0. Let F be the one variable subequation defined by $\{|p| \leq |x|\} \cup (\{0\} \times [-1, 1])$. Take $\psi \equiv 0$. Then:

1) ψ is not strictly F-subharmonic since $J_0^2 \psi \notin \text{Int} F$,
2) ψ is not c-strict near $x = 0$ since $F_x^c = \emptyset$ for $0 < |x| < c$, but
3) $J_x^2 \psi \in \text{Int}(F_x)$ for all x.

8. Comparison Theory—Local to Global

In this section we begin our analysis of the uniqueness/comparison question for F-harmonic functions with given boundary values on a compact subset K of X. Let us set the notation

$$F(K) \equiv \left\{u \in \mathrm{USC}(K) : u\big|_{\mathrm{Int}K} \in F(\mathrm{Int}K)\right\}.$$

Then the comparison principle can be stated as follows:

$$\text{(8.1)} \qquad \begin{array}{c} \text{For all} \ \ u \in F(K) \quad \text{and} \quad -w \in \widetilde{F}(K) \\ u \leq w \ \text{ on } \ \partial K \quad \Rightarrow \quad u \leq w \ \text{ on } \ K. \end{array}$$

However, we prefer to state it in the following form which invokes duality.

Definition 8.1. We say that **comparison holds for F on X** if for all compact sets $K \subset X$, whenever

$$u \in F(K) \quad \text{and} \quad v \in \widetilde{F}(K),$$

the **Zero Maximum Principle** holds for $u+v$ on K, that is,

$$\text{(ZMP)} \qquad u+v \leq 0 \quad \text{on} \quad \partial K \qquad \Rightarrow \qquad u+v \leq 0 \quad \text{on} \quad K.$$

If comparison holds, it is immediate that

Uniqueness for the Dirichlet problem holds, that is:

If two F-harmonic functions agree on ∂K, they must agree on K.

Local comparison does not imply global comparison. However, for a weakened form of comparison, local does imply global. This is the main result of this section.

Definition 8.2. We say that **weak comparison holds for F on X** if for all compact subsets $K \subset X$, and functions

$$u \in F^c(K), \quad v \in \widetilde{F}(K), \quad c > 0,$$

the Zero Maximum Principle (ZMP) holds for $u+v$ on K. We say that **local weak comparison holds for F on X** if for all $x \in X$, there exists a neighborhood U of x such that weak comparison holds for F on U.

Theorem 8.3. *Suppose that F is a subequation on a manifold X. If local weak comparison holds for F on X, then weak comparison holds for F on X.*

Proof. Suppose weak comparison fails for F on X. Then there exist $c > 0$, $u \in F^c(X)$, $v \in \widetilde{F}(X)$, and compact $K \subset X$ with

$$u+v \leq 0 \quad \text{on} \quad \partial K \quad \text{but} \quad \sup_K (u+v) > 0.$$

Choose a maximum point $x_0 \in \mathrm{Int}K$ and let $M = u(x_0) + v(x_0) > 0$ denote the maximum value. Fix local coordinates x on a neighborhood of x_0 containing $U = \{x : |x - x_0| < \rho\}$. By choosing ρ sufficiently small, we can assume that weak comparison holds for F on U.

By the Stability Lemma 7.5, we have $u' = u - \delta|x - x_0|^2 \in F^{\frac{c}{2}}(U)$ if δ is chosen small enough. Now x_0 is the unique maximum point for $u' + v$ on U, with maximum value M. Choose $K_0 = \{x : |x - x_0| \leq \rho/2\} \subset U$. Then $\sup_{\partial K_0}(u' + v) = M' < M$. Set $u'' = u' - \max\{0, M'\}$. Then by condition (N) for $F^{\frac{c}{2}}$ (see Lemma 7.3), we have $u'' \in F^{\frac{c}{2}}(U)$. Moreover,

$$\sup_{\partial K_0}(u'' + v) \ \leq\ 0 \quad \text{while} \quad u''(x_0) + v(x_0) \ =\ M - \max\{0, M'\} \ >\ 0.$$

Thus, weak comparison fails for F on U, contrary to assumption. ∎

Weak comparison can be strengthened as follows. Define

$$F_{\text{strict}}(K) \ =\ \left\{u \in \mathrm{USC}(K) : u\big|_{\mathrm{Int}K} \in F_{\text{strict}}(\mathrm{Int}K)\right\}. \tag{8.2}$$

Lemma 8.4. *Suppose weak comparison holds for F on X. For all compact sets $K \subset X$, if $u \in F_{\text{strict}}(K)$ and $v \in \widetilde{F}(K)$, then $u + v$ satisfies the (ZMP).*

Proof. We assume $u + v \leq 0$ on ∂K. Since $u \in \mathrm{USC}(K)$, for each $\delta > 0$ the set

$$U_\delta \ =\ \{x \in K : u(x) + v(x) < \delta\}$$

is an open neighborhood of ∂K in K. Exhaust $\mathrm{Int}K$ by compact sets K_ϵ with $\mathrm{Int}K = \bigcup_\epsilon \mathrm{Int}K_\epsilon$. Then $\partial K_\epsilon \subset U_\delta$ for $\epsilon > 0$ small. Now $u - \delta + v \leq 0$ on ∂K_ϵ. However, $u - \delta$ is F^c-subharmonic on $\mathrm{Int}K_\epsilon$ for some $c > 0$. Hence, (WC) for K_ϵ states that $u - \delta + v \leq 0$ on K_ϵ. Thus $u - \delta + v \leq 0$ on $\mathrm{Int}K$. Hence $u + v \leq 0$ on K. ∎

9. Strict Approximation and Monotonicity Subequations

In this section we discuss certain global approximation techniques which can be used to deduce comparison from weak comparison. Consider a general second-order subequation F on a manifold X.

Definition 9.1. We say that **strict approximation** holds for F on X if for each compact set $K \subset X$, each function $u \in F(X)$ can be uniformly approximated by functions in $F_{\text{strict}}(K)$.

When strict approximation is available, it is easy to show that weak comparison implies comparison.

Theorem 9.2. (Global Comparison). *Suppose F is a subequation on a manifold X. Assume that both local weak comparison and the strict approximation property hold for F on X. Then comparison holds for F on X.*

Proof. Suppose $u \in F(K)$, $v \in \widetilde{F}(K)$, and $u + v \leq 0$ on ∂K. By Theorem 8.3 we can assume that weak comparison holds. By strict approximation, for each $\epsilon > 0$, there exists $\overline{u} \in F_{\text{strict}}(K)$ with $u - \epsilon \leq \overline{u} \leq u + \epsilon$ on K. Now property (N) implies that $\overline{u} - \epsilon \in F_{\text{strict}}(K)$. Note that $\overline{u} - \epsilon + v \leq u + v \leq 0$ on ∂K. Lemma 8.4 states that $\overline{u} - \epsilon + v \leq 0$ on K. This proves that $u + v$ satisfies the (ZMP). ∎

Example 9.3. (The Eikonal Equation). This subequation $\mathbf{F}$ on $\mathbf{R}^n$ is defined by $|\nabla u| \leq 1$. Given $u \in \mathbf{F}(K)$, set $u_\epsilon = (1-\epsilon)u$. Then $u_\epsilon \in F^\epsilon(K)$ because if φ is a test function for $(1-\epsilon)u$ at x_0, then $\frac{1}{1-\epsilon}\varphi$ is a test function for u at x_0. Thus $|\nabla \varphi(x_0)| \leq 1 - \epsilon$.

In contrast to this example, for the geometric subequations F that are of primary interest in this paper, the approximations u_ϵ will be of the form

$$u_\epsilon \;=\; u + \epsilon\psi \qquad 0 < \epsilon \leq \epsilon_0 \tag{9.1}$$

where ψ is a C^2-function independent of u. The function ψ will be referred to as an *approximator* for F.

Suppose $M \subset J^2(X)$ is a subset such that the fiber-wise sum

$$F + M \;\subset\; F \quad \text{and} \quad \epsilon M \;\subset\; M \qquad \text{for } 0 < \epsilon \leq \epsilon_0. \tag{9.2}$$

If ψ is a C^2-function which is strictly M-subharmonic, then ψ is an approximator for F (see the proof of Theorem 9.5). However, condition (9.2) implies that at each point x

$$F_x + \alpha_1 J_1 + \alpha_2 J_2 \;\subset\; F_x \quad \text{for all } \alpha_1 > 0, \alpha_2 > 0 \text{ and } J_1, J_2 \in M_x. \tag{9.3}$$

Hence we might as well assume that M is a convex cone, but at this point it is convenient not to assume that $\overline{\mathbf{M}}$ is a subequation, i.e., that $\mathbf{M}$ satisfies (P) and (N). See Remark 9.11.

Definition 9.4. A subset $M \subset J^2(X)$ will be called **a convex monotonicity cone for** F if

(1) M is a convex cone with vertex at the origin, and

(2) $F + M \;\subset\; F$.

Theorem 9.5. *Suppose M is a convex monotonicity cone for F as above. If there exists $\psi \in C^2(X)$ which is strictly M-subharmonic, then strict approximation holds for F on X.*

Proof. It will suffice to establish the following.

Assertion 9.6. *For each compact subset $K \subset X$, there exists $\delta > 0$ such that*

$$u + \epsilon\psi \;\in\; F^{\epsilon\delta}(K) \quad \text{for all } u \in F(K) \quad \text{and for all } \epsilon > 0.$$

To begin, note that $u - \varphi$ has local maximum 0 at a point x if and only if $u + \epsilon\psi - (\varphi + \epsilon\psi)$ has local maximum 0 at x. Hence we must show that under the hypothesis $J^2_x\varphi \in F_x$ we have that $J^2_x(\varphi + \epsilon\psi) = J^2_x\varphi + \epsilon J^2_x\psi \in F^{\epsilon\delta}_x$; in other words, that $F_x + \epsilon J^2_x\psi \subset F^{\epsilon\delta}_x$.

Since $\psi \in C^2(X)$, $J^2_x\psi$ is a continuous function of x. Hence $\{J^2_x\psi : x \in K\}$ is compact. That ψ is strictly M-subharmonic implies that $\{J^2_x\psi : x \in K\}$ is a compact subset of Int$M$. Take $\delta =$ the distance from $\{J^2_x\psi : x \in K\}$ to $\sim$ Int$M$. Then $B(J^2_x\psi, \delta) \subset M_x$ for all $x \in K$ where B denotes the ball in the fiber. Suppose now that $J \in F_x$ and $x \in K$. Then

$$B(J + \epsilon J^2_x\psi, \epsilon\delta) \;=\; J + \epsilon B(J^2_x\psi, \delta) \;\subset\; F_x + M_x \;\subset\; F_x$$

as desired. ∎

Combining Theorem 9.2 with Theorem 9.5 yields the version of global comparison that will be used in this paper.

Theorem 9.7. *Suppose F is a subequation on a manifold X. Assume that X supports a C^2 function which is strictly M-subharmonic, where M is a monotonicity cone for F. Then local weak comparison for F implies global comparison for F on X.*

Remark 9.8. (Circular Monotonicity Cones). In a situation where the C^2-function ψ is given, the simplest monotonicity cone to consider is one whose fiber at each point x is a circular cone $C(J)$ about $J \equiv J^2_x\psi$. If $\delta = \text{dist}(J^2_x\psi, \sim \text{Int}M)$, as in the above proof, then the circular cone can be taken to be the cone $C^\delta(J)$ on the ball $B(J, \delta)$. The cross-section of this cone $C^\delta(J)$ by the hyperplane (through J) perpendicular to J, is a ball of radius R in this hyperplane, where, setting $\gamma = 1/R$, one calculates that

$$\delta \;=\; \frac{|J|}{\sqrt{1+\gamma^2|J|^2}}.$$

This same cone will be denoted by $C_\gamma(J)$ when γ is to be emphasized.

Lemma 9.9. *Suppose that F is a subequation with $F_x \neq \emptyset$ and $F_x \neq J^2_x(X)$, and fix $J \in J^2_x(X)$. The following are equivalent.*

(1) *F_x is $C_\gamma(J)$-monotone.*

(2) *The boundary ∂F_x can be graphed over the hyperplane $J^\perp$ with graphing function f which is γ-Lipschitz, i.e., for $J_0 \in J^\perp$,*

$$J_0 + tJ \in F_x \quad \iff \quad t \geq f(J_0),$$

and for all $\bar{J}_0, J_0 \in J^\perp$,

$$-\gamma|J_0| \;\leq\; f(\bar{J}_0 + J_0) \;\leq\; \gamma|J_0|.$$

The elementary proof is left to the reader.

Remark 9.10. Suppose $\mathbf{F}$ is a universal model (i.e., a constant coefficient subequation) which is G-invariant. If $\mathbf{F}$ has a convex monotonicity cone $\mathbf{M}$ which is G-invariant, then on a manifold X with topological

G-structure, the induced subequation F and the induced convex monotonicity cone M satisfy $F + M \subset F$, i.e., M is a convex monotonicity cone for F on X.

Remark 9.11. The global comparison Theorems 9.2 and 9.7 are useful even locally. In Section 10 we will prove that weak comparison holds for any constant coefficient subequation $\mathbf{F}$ on $\mathbf{R}^n$. Consequently, if $\mathbf{F}$ has a monotonicity cone $\mathbf{M}$ with non-empty interior, then local comparison holds for $\mathbf{F}$. To prove this, fix a point x_0 and pick a point $(r, p, A) \in \mathrm{Int}\mathbf{M}$ above x_0. Let ψ denote the quadratic function whose 2-jet at x_0 equals (r, p, A). Then ψ is strictly $\mathbf{M}$-subharmonic in a neighborhood of x_0.

Therefore, given a constant coefficient subequation $\mathbf{F}$, the key question is:

(9.4) When does $\mathbf{F}$ have a monotonicity cone $\mathbf{M}$ with interior?

We might as well assume that $\mathbf{M} = \overline{\mathrm{Int}\mathbf{M}}$, and (as noted prior to Definition 9.4) that $\mathbf{M}$ is convex.

Now $\mathbf{M}$ need not satisfy conditions (P) or (N), i.e., $\mathbf{M}$ need not be a subequation. (For a useful example, let $\mathbf{M} = C(J)$ in Remark 9.8.) However, such an $\mathbf{M}$ can always be enlarged to $\mathbf{M}' = \mathbf{M} + (\mathbf{R}_- \times \{0\} \times \mathcal{P})$ which satisfies:

(1) $\mathbf{M}' = \overline{\mathrm{Int}\mathbf{M}'}$ and $\mathbf{M}'$ is convex (as is true of $\mathbf{M}$),
(2) $\mathbf{M}'$ is both $\mathcal{N}$- and $\mathcal{P}$-monotone, i.e., $\mathbf{M}'$ is a subequation,
(3) $\mathbf{F} + \mathbf{M}' \subset \mathbf{F}$, i.e., $\mathbf{M}'$ is also a monotonicity cone for F.

This proves that the question (9.4) is equivalent to the question:
(9.5)
When does $\mathbf{F}$ have a convex conical monotonicity subequation $\mathbf{M}$?

Each such $\mathbf{M}$ is a convex cone with interior, and M contains $\mathbf{R}_- \times \{0\} \times \mathcal{P}$. Hence M can be thought of as a fattening of $\mathbf{R}_- \times \{0\} \times \mathcal{P}$ to a convex set with interior. The bigger $\mathbf{M}$ is, the more can be concluded about $\mathbf{F}$. However, as illustrated by the example $\mathbf{M} \equiv C(J)$, (9.4) may not be easier to answer since $\mathbf{M}'$ cannot be described directly without defining $\mathbf{M} = C(J)$ and then defining $\mathbf{M}'$ as the sum $\mathbf{M} + (\mathbf{R}_- \times \{0\} \times \mathcal{P})$.

In the next subsection we present a few of the basic examples of monotonicity subequations M on a manifold. They are all based on a universal euclidean model $\mathbf{M}$ satisfying Definition 9.4. We leave it to the reader to explicitly describe the model $\mathbf{M}$ in the examples.

Examples of Strict Approximation Using Monotonicity.

Most of the following examples are purely second-order. We use the canonical splitting from Section 4.2,

$$J^2(X) \ \cong\ \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X), \tag{9.6}$$

on the riemannian manifold X. They are the subequations $F \subset J^2(X)$ which are the pull-backs of subsets $F' \subset \mathrm{Sym}^2(T^*X)$ under the projection $J^2(X) \to \mathrm{Sym}^2(T^*X)$ induced by (9.6). (See Section 4.4.)

Example 9.12. (The Real Monge-Ampére Monotonicity Subequation). For all pure second-order subsets F the positivity condition (P) is equivalent to F being P-monotone where

$$P \;=\; \mathbf{R} \oplus T^*X \oplus \mathcal{P}$$

is the Monge-Ampére subequation discussed in Section 4.6. For a general riemannian manifold X, strict P-subharmonicity is the same as strict convexity.

Theorem 9.13. *Suppose X is a riemannian manifold which supports a strictly convex C^2 function. Then strict approximation holds for every pure second-order subequation F on X.*

For example, in $\mathbf{R}^n$ the function $|x|^2$ is strictly P-subharmonic. More generally, if X has sectional curvature ≤ 0, the function $\delta(x) = \mathrm{dist}(x, x_0)^2$ is strictly P-convex up to the first cut point of x_0. In particular, if X is complete and simply connected, then δ is globally strictly P-convex. Of course, on any riemannian manifold X the function δ is strictly P-subharmonic in a neighborhood of x_0 since $\mathrm{Hess}_{x_0}\delta = 2I$. Thus strict approximation holds for all pure second-order subequations in these cases.

Example 9.14. (The Complex Monge-Ampère Monotonicity Subequation). Suppose now that (X, J) is a hermitian almost complex manifold and consider the projection

$$J^2(X) \;\to\; \mathrm{Sym}^2_{\mathbf{C}}(T^*X)$$

given by projecting onto $\mathrm{Sym}^2(T^*X)$ and then taking the hermitian symmetric part (cf. 4.5). A subequation defined by pulling back a subset of $\mathrm{Sym}^2_{\mathbf{C}}(T^*X)$ will be called a *complex hessian* subequation. Each such F is $P^{\mathbf{C}}$-monotone where $P^{\mathbf{C}}$ is the complex Monge-Ampère subequation defined in 4.6.

The C^2 functions ψ on X which are strictly $P^{\mathbf{C}}$-subharmonic are just the classical $\mathbf{C}$-plurisubharmonic functions if X is a complex manifold. We will use this terminology even if J is not integrable.

Theorem 9.15. *Suppose F is a complex hessian subequation on a hermitian almost complex manifold X. If X supports a strictly* $\mathbf{C}$-*plurisubharmonic function of class C^2, then strict approximation holds for F on X.*

Example 9.16. (The Quaternionic Monge-Ampère Monotonicity Subequation). We leave it to the reader to formulate the analogous result on an almost quaternionic hermitian manifold which supports a strictly **H**-plurisubharmonic function.

Note that $P \subset P^{\mathbf{C}} \subset P^{\mathbf{H}}$ so the corresponding subharmonic functions are progressively easier to find.

Example 9.17. (Geometrically Defined Monotonicity Subequations). Suppose $M^{\mathbb{G}}$ is geometrically defined by a subset $\mathbb{G}$ of the Grassmann bundle $G(p, TX)$ as in Section 4.7. For example, the three Monge-Ampère cases over $K = \mathbf{R}, \mathbf{C}$, and $\mathbf{H}$ are geometrically defined by taking $\mathbb{G}$ to be the Grassmannian of K-lines $G(1, TX)$, $G_{\mathbf{C}}(1, TX) \subset G(2, TX)$, and $G_{\mathbf{H}}(1, TX) \subset G(4, TX)$, respectively. Other important examples are given by taking $\mathbb{G}$ to be all of $G(p, TX)$, $G_{\mathbf{C}}(p, TX)$, and $G_{\mathbf{H}}(p, TX)$ for general p.

Any calibration ϕ of degree p on a riemannian manifold X determines the subset

$$\mathbb{G}(\phi) \;=\; \{\xi \in G(p, TX) : \phi(\xi) = 1\}$$

of calibrated p-planes, and hence a convex cone subequation geometrically defined by $\mathbb{G}(\phi)$, as long as $\pi : \mathbb{G}(\phi) \to X$ is a local surjection (see Prop. 4.2). That is, a C^2 function u is **$\mathbb{G}(\phi)$-plurisubharmonic** if

$$\mathrm{tr}_\xi \mathrm{Hess}\, u \;\geq\; 0 \quad \text{for all } \xi \in \mathbb{G}(\phi).$$

Theorem 9.18. *Suppose $\mathbb{G}$ is a subset of the Grassmann bundle $G(p, TX)$ as in Section 4.7. If X supports a C^2 strictly $\mathbb{G}$-plurisubharmonic function, then strict approximation holds for any subequation on X which is $M^{\mathbb{G}}$ monotone.*

Of course, one such equation is $M^{\mathbb{G}}$ itself.

Example 9.19. (Gradient Independent Subequations). These are subsets $F \subset J^2(X)$ which are the pull-backs of subsets $F' \subset \mathbf{R} \oplus \mathrm{Sym}^2(T^*X)$ under the projection $J^2(X) \to \mathbf{R} \oplus \mathrm{Sym}^2(T^*X)$ induced by (9.4). See Section 4.8. All such sets are $\mathcal{M}^-$-monotone where

$$\mathcal{M}^- \;\equiv\; \mathbf{R}^- \oplus T^*X \oplus \mathcal{P} \tag{9.7}$$

if and only if they satisfy (P) and (N). Hence, gradient independent subequations are automatically $\mathcal{M}^-$-monotone.

More generally, if M' is any one of the pure second-order monotonicity subequations discussed above, then

$$M \;=\; \mathbf{R}^- \oplus T^*X \oplus M'$$

is a gradient independent monotonicity subequation. Each subequation F which is M monotone must be gradient independent. The same ψ's used in the previous examples will work for F. This is because on a compact subset K, $\psi - c$ is strictly negative for $c >> 0$. More precisely, Theorem 9.15 (and its complex and quaternionic versions) continues to hold for the more general gradient independent subequations.

10. A Comparison Theorem for G-Universal Subequations

The main result of this section can be stated as follows.

Theorem 10.1. *Suppose F is a subequation on a manifold X which is locally affinely jet-equivalent to a constant coefficient subequation $\mathbf{F}$. Then weak comparison holds for F on X.*

A case of particular interest in this paper is the following. Suppose X is a riemannian manifold equipped with a topological G-structure, for a closed subgroup $G \subset \mathrm{O}_n$. Suppose $\mathbf{F} \subset \mathbf{J}^2$ is a euclidean G-invariant subequation on $\mathbf{R}^n$. Let $F \subset J^2(X)$ denote a riemannian G-subequation with model fiber $\mathbf{F}$ on X given by Lemma 5.2.

Corollary 10.2. *Suppose F is a riemannian G-subequation on X. Then weak comparison holds for F-subharmonic functions on X.*

Proof. On any coordinate chart U with an admissible local framing, F is jet-equivalent to $\mathbf{F}$ (by Proposition 5.5). ∎

Remark. The following proof of local weak comparison uses the Theorem on Sums, which is discussed in Appendix C. The argument for equations which are honestly constant coefficient is particularly easy—see Corollary C.3.

Proof of Theorem 10.1. For clarity, we first present the proof in the case where F is locally (linearly) jet-equivalent to a constant coefficient equation as in Definition 6.7.

By Theorem 8.3 we need only prove weak comparison on a chart U where F is jet-equivalent to $\mathbf{F}$. Suppose weak comparison fails on U. We will derive a contradiction using the Theorem on Sums. Failure of comparison means that there exist $u \in F^c(U)$ and $v \in \widetilde{F}(U)$ such that $u + v$ does not satisfy the Zero Maximum Principle on some compact subset $K \subset U$. Let $h(x)$ and L_x determine the field of automorphisms taking F to $\mathbf{F}$. Theorem C.1 says that there exist a point $x_0 \in \mathrm{Int}K$, a sequence of numbers $\epsilon \searrow 0$ with associated points $z_\epsilon = (x_\epsilon, y_\epsilon) \to (x_0, x_0)$, and 2-jets:

$$\alpha_\epsilon \equiv (x_\epsilon, r_\epsilon, p_\epsilon, A_\epsilon) \in F^c_{x_\epsilon} \quad \text{and} \quad \beta_\epsilon \equiv (y_\epsilon, s_\epsilon, q_\epsilon, B_\epsilon) \in \widetilde{F}_{y_\epsilon}$$

with the following properties:

$$(10.1) \quad r_\epsilon = u(x_\epsilon), \quad s_\epsilon = v(y_\epsilon), \quad \text{and} \quad r_\epsilon + s_\epsilon = M_\epsilon \searrow M_0 > 0$$

$$(10.2) \qquad p_\epsilon = \frac{x_\epsilon - y_\epsilon}{\epsilon} = -q_\epsilon \quad \text{and} \quad \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \longrightarrow 0$$

$$(10.3) \qquad \begin{pmatrix} A_\epsilon & 0 \\ 0 & B_\epsilon \end{pmatrix} \leq \frac{3}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The jet-equivalence of F and $\mathbf{F}$ says that

$$(10.4)\qquad \alpha'_\epsilon \ \equiv\ (r'_\epsilon, p'_\epsilon, A'_\epsilon)\ \in\ \mathbf{F}^c \quad \text{and} \quad \beta'_\epsilon \ \equiv\ (s'_\epsilon, q'_\epsilon, B'_\epsilon)\ \in\ \widetilde{\mathbf{F}}$$

where

$$(10.5)\qquad r'_\epsilon \ =\ r_\epsilon, \quad p'_\epsilon \ =\ g(x_\epsilon)p_\epsilon, \quad A'_\epsilon \ =\ h(x_\epsilon)A_\epsilon h(x_\epsilon)^t + L_{x_\epsilon}(p_\epsilon)$$

$$(10.6)\qquad s'_\epsilon \ =\ s_\epsilon, \quad q'_\epsilon \ =\ g(y_\epsilon)q_\epsilon, \quad B'_\epsilon \ =\ h(y_\epsilon)B_\epsilon h(y_\epsilon)^t + L_{y_\epsilon}(q_\epsilon).$$

Since $\alpha'_\epsilon \in \mathbf{F}^c$, we also have that $\alpha''_\epsilon \equiv (r_\epsilon - M_\epsilon, p'_\epsilon, A'_\epsilon + P_\epsilon) \in \mathbf{F}^c$ for any $P_\epsilon \geq 0$.

By, (10.4) we have $-\beta'_\epsilon \notin \mathrm{Int}\mathbf{F}$. Hence

$$(10.7)\qquad 0\ <\ c\ \leq\ \mathrm{dist}(\alpha''_\epsilon, -\beta'_\epsilon)\ =\ |\alpha''_\epsilon + \beta'_\epsilon|.$$

To complete the proof, we show that $\alpha''_\epsilon + \beta'_\epsilon$ converges to zero. The first component of $\alpha''_\epsilon + \beta'_\epsilon$ is $r_\epsilon - M_\epsilon + s_\epsilon$, which tends to zero by (10.1).

The second component of $\alpha''_\epsilon + \beta'_\epsilon$ is

$$p'_\epsilon + q'_\epsilon \ =\ g(x_\epsilon)\frac{(x_\epsilon - y_\epsilon)}{\epsilon} - g(y_\epsilon)\frac{(x_\epsilon - y_\epsilon)}{\epsilon} \ =\ \Big(g(x_\epsilon) - g(y_\epsilon)\Big)\frac{(x_\epsilon - y_\epsilon)}{\epsilon}$$

which converges to zero as $\epsilon \to 0$ by (10.2).

It remains to find $P_\epsilon \geq 0$ so that the third component of $\alpha''_\epsilon + \beta'_\epsilon$, namely $A'_\epsilon + P_\epsilon + B'_\epsilon$, converges to zero. This will contradict (10.7).

Multiplying both sides in (10.3) by

$$\begin{pmatrix} h(x_\epsilon) & 0 \\ 0 & h(y_\epsilon) \end{pmatrix} \text{ on the left and } \begin{pmatrix} h(x_\epsilon)^t & 0 \\ 0 & h(y_\epsilon)^t \end{pmatrix} \text{ on the right}$$

gives

$$\begin{pmatrix} h(x_\epsilon)A_\epsilon h(x_\epsilon)^t & 0 \\ 0 & h(y_\epsilon)B_\epsilon h(y_\epsilon)^t \end{pmatrix} \ \leq\ \frac{3}{\epsilon}\begin{pmatrix} h(x_\epsilon)h(x_\epsilon)^t & -h(x_\epsilon)h(y_\epsilon)^t \\ -h(y_\epsilon)h(x_\epsilon)^t & h(y_\epsilon)h(y_\epsilon)^t \end{pmatrix}.$$

Restricting these two quadratic forms to diagonal elements (x, x) then yields

$$\begin{aligned} &h(x_\epsilon)A_\epsilon h(x_\epsilon)^t + h(y_\epsilon)B_\epsilon h(y_\epsilon)^t \\ &\qquad\leq\ \frac{3}{\epsilon}\left[h(x_\epsilon)(h(x_\epsilon)^t - h(y_\epsilon)^t) - h(y_\epsilon)(h(x_\epsilon)^t - h(y_\epsilon)^t)\right] \\ &\qquad=\ \frac{3}{\epsilon}(h(x_\epsilon) - h(y_\epsilon))(h(x_\epsilon)^t - h(y_\epsilon)^t) \\ &\qquad\leq\ \frac{\lambda}{\epsilon}\,|x_\epsilon - y_\epsilon|^2 \cdot I \qquad \text{for some } \ \lambda > 0. \end{aligned}$$

Thus there exists $P_\epsilon \in \mathbf{P}$ so that

$$h(x_\epsilon)A_\epsilon h(x_\epsilon)^t + h(y_\epsilon)B_\epsilon h(y_\epsilon)^t + P_\epsilon \ =\ \frac{\lambda}{\epsilon}\,|x_\epsilon - y_\epsilon|^2 \cdot I.$$

It now follows from the definitions in (10.5) and (10.6) that

$$(10.8)\qquad A'_\epsilon + B'_\epsilon + P_\epsilon \ =\ \frac{\lambda}{\epsilon}\,|x_\epsilon - y_\epsilon|^2 \cdot I + L_{x_\epsilon}(p_\epsilon) + L_{y_\epsilon}(q_\epsilon).$$

However,

$$\begin{aligned}|L_{x_\epsilon}(p_\epsilon) + L_{y_\epsilon}(q_\epsilon)| = & \left|(L_{x_\epsilon} - L_{y_\epsilon})\left(\frac{x_\epsilon - y_\epsilon}{\epsilon}\right)\right| \\ & \leq \|L_{x_\epsilon} - L_{y_\epsilon}\| \frac{|x_\epsilon - y_\epsilon|}{\epsilon} \\ = & \ O\left(\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}\right).\end{aligned}$$

Using (10.2), this shows that

$$A'_\epsilon + B'_\epsilon + P_\epsilon \ \cong \ \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \ \to \ 0 \quad \text{as } \epsilon \searrow 0. \tag{10.9}$$

This completes the proof in the case of linear jet-equivalence.

Suppose now that our local jet-equivalence is affine and can be written in the form $\widetilde{\Phi}_x = \Phi_x + J_0(x)$, where Φ is a linear jet-equivalence as above (cf. (6.8)). Then the proof above goes through essentially unchanged except that, in light of Lemma 6.9, we must replace (10.4) with

$$\begin{aligned} \alpha'_\epsilon + J_0(x_\epsilon) \ &\equiv \ (r'_\epsilon, p'_\epsilon, A'_\epsilon) + J_0(x_\epsilon) \ \in \ \mathbf{F}^c \\ \text{and} \quad \beta'_\epsilon - J_0(y_\epsilon) \ &\equiv \ (s'_\epsilon, q'_\epsilon, B'_\epsilon) - J_0(y_\epsilon) \ \in \ \widetilde{\mathbf{F}}, \end{aligned} \tag{10.4'}$$

We now observe that $J_0(x_\epsilon) - J_0(y_\epsilon) \to 0$, and the rest of the proof is exactly as written above. ∎

Combining Corollary 10.2 with Theorem 9.7 yields comparison for a wide class of riemannian G-subequations.

Theorem 10.3. *Suppose F is a riemannian G-subequation on a manifold X. If X supports a C^2 strictly M-subharmonic function, where M is a monotonicity cone for F, then comparison holds for F on X.*

11. Strictly F-Convex Boundaries and Barriers

In this section we introduce the notion of a strictly F-convex boundary for a general subequation F. This notion implies the existence of barriers, which are crucial for our main results. Strictly F-convex boundaries have a relatively simple geometric characterization in terms of the second fundamental form of $\partial\Omega$ (see Section 11.4). Thus even for very general equations, the geometry of the boundary enters explicity into the study of the Dirichlet problem.

The idea of an F-convex boundary is the following. Start with a general subequation F. For each $\lambda \in \mathbf{R}$, there is a reduced subequation F_λ defined by setting the r-variable equal to λ (e.g., $\Delta u \geq e^u$ becomes $\Delta u \geq e^\lambda$). To each F_λ we associate an *asymptotic interior* $\overrightarrow{F_\lambda}$. This is an open, point-wise conical set obtained by taking rays with conical neighborhoods that eventually lie in F. In many examples (such as $\Delta u \geq e^u$)

the asymptotic interiors $\overrightarrow{F_\lambda}$ all agree. Of course this includes the case where the subequation is independent of r to start with. Furthermore, in this case, if F is itself a cone, then $\overrightarrow{F} = \text{Int}F$. In general these asymptotic interiors are simpler than F, and as examples will show, the process of computing them is often quite straightforward.

Now the boundary of a domain $\partial\Omega$ is called *locally* $\overrightarrow{F_\lambda}$*-convex* if it has a local defining function which is strictly $\overrightarrow{F_\lambda}$-subharmonic, and it is said to be *locally* F*-convex* if it is locally $\overrightarrow{F_\lambda}$-convex for all λ. The main point of these definitions is that $\overrightarrow{F_\lambda}$-convexity at a point x implies the existence of a "λ-barrier" for F at x, and these barriers are exactly what is needed for existence of solutions to the Dirichlet problem.

One might ask about the simpler concept where one just takes the asymptotic interior of F directly. This can be done, but for many important equations involving the dependent variable (e.g. the Calabi-Yau–type equations in Example 6.15), this naïve F-convexity rules out all boundaries, whereas the refined one gives just the right condition (see Section 19). Moreover, if F is independent of the r-variable and has constant coefficients, one can show that the existence of barriers is actually equivalent to our notion of boundary convexity.

We note that for constant coefficient, pure second-order subequations on $\mathbf{R}^n$, convexity and the existence of barriers were analyzed in [HL$_4$].

We also note that under a mild hypothesis on F, the boundaries of small enough balls in any coordinate system are strictly F-convex (see Proposition 11.9 below).

11.1. Asymptotics. Consider the canonical decomposition $J^2(X) = \mathbf{R} \oplus J^2_{\text{red}}(X)$ with fiber coordinates $J \equiv (r, J_0)$, and refer to a subequation of the form $\mathbf{R} \oplus F$ with $F \subset J^2_{\text{red}}(X)$ as a subequation *of reduced type / independent of the r variable* (cf. Section 4.9). These subequations must be handled first.

Our first step is to replace F by an (asymptotically) smaller open set $\overrightarrow{F}$ which is a cone (i.e., each fiber is a cone with vertex the origin).

Definition 11.1. Suppose that $F \subset J^2_{\text{red}}(X)$ is a subequation independent of the r-variable. The **asymptotic interior** $\overrightarrow{F}$ of F is the set of all $J \in J^2_{\text{red}}(X)$ for which there exists a neighborhood $\mathcal{N}(J)$ in (the total space of) $J^2_{\text{red}}(X)$ and a number $t_0 > 0$ such that

$$t \cdot \mathcal{N}(J) \ \subset \ F \quad \text{for all } t \geq t_0. \tag{11.1}$$

Note that:

$$\text{If } F \text{ is a cone, then } \overrightarrow{F} \ = \ \text{Int}F, \tag{11.2}$$

but otherwise $\overrightarrow{F}$ is smaller than F asymptotically, and may be empty.

Proposition 11.2. *The asymptotic interior* $\overrightarrow{F}$ *is an open cone in* $J^2_{\text{red}}(X)$ *which satisfies condition (P).*

Proof. Obviously $\overrightarrow{F}$ is open and is a cone (i.e., $t\overrightarrow{F} = \overrightarrow{F}$ for all $t > 0$). To prove (P), suppose $J \in \overrightarrow{F}$ and $P_{x_0} \in \mathcal{P}_{x_0} \subset J^2_{x_0}(X)$. Extend P_{x_0} to a smooth section P of $\mathcal{P}$ near x_0. Note that the fiber-wise sum $\mathcal{N}(J) + P$ is a neighborhood $\mathcal{N}(J + P_{x_0})$ of $J + P_{x_0}$. (The fiber at x of the sum is defined to be empty if $\mathcal{N}(J)_x$ is empty.) Finally note that $t\mathcal{N}(J + P_{x_0})_x = t\mathcal{N}(J)_x + tP_x \subset F$ if $t \geq t_0$, since $t\mathcal{N}(J)_x \subset F$ if $t \geq t_0$. ∎

Lemma 11.3. *Suppose $F \subset J^2_{\text{red}}(X)$ is a reduced subequation. If F is defined by a G-invariant universal model $\mathbf{F}$ as in Lemma 5.2, then the asymptotic interior $\overrightarrow{F}$ of F is the bundle of open cones induced by the universal model $\overrightarrow{\mathbf{F}}$ via (5.3), where $\overrightarrow{\mathbf{F}}$ is the asymptotic interior of $\mathbf{F}$. More generally, jet-equivalence preserves asymptotic interiors.*

Proof. Exercise.

Definition 11.4. A C^2-function u with $J^2_x u \in \overrightarrow{F}$ for all x will be called *strictly $\overrightarrow{F}$ subharmonic.*

This is the notion required for boundary convexity and barriers for the Dirichlet problem. In particular, the question of when the closure of $\overrightarrow{F}$ is a subequation having $\overrightarrow{F}$ as its interior can be avoided.

We next observe that for many subequations F there are many local $\overrightarrow{F}$-subharmonic functions. Suppose we are in a local coordinate system for X with standard fiber coordinates (p, A) for $J^2_{\text{red}}(X)$.

Proposition 11.5. *If $\overrightarrow{F}_{x_0}$ is non-empty, then for all $\epsilon > 0$, $\overrightarrow{F}$ contains a subset of the form*

$$B(x_0, r_0) \times B(p_0, \delta_0) \times (\epsilon I + \text{Int}\mathcal{P}).$$

Moreover, if $\overrightarrow{F}_{x_0}$ contains a point of the form $(0, A)$, then the function $\rho \equiv |x - x_0|^2 - R^2$ is strictly $\overrightarrow{F}$-subharmonic on $B(x_0, R)$ for $R > 0$ sufficiently small.

Proof. If $(p, A) \in \overrightarrow{F}_{x_0}$, then by positivity we may replace A by λI (since $\lambda I - A > 0$ for large λ). By the openness of $\overrightarrow{F}$ there exist $r_0, \delta > 0$ such that $B(x_0, r_0) \times B(p, \delta) \times \{\lambda I\} \subset \overrightarrow{F}$, and therefore by positivity $B(x_0, r_0) \times B(p, \delta) \times (\lambda I + \mathcal{P}) \subset \overrightarrow{F}$. Applying the cone property proves the first assertion. For the second, note that if $p = 0$ then $B(x_0, r_0) \times B(0, \delta_0) \times (I + \mathcal{P}) \subset \overrightarrow{F}$. ∎

Without a point $(0, A_0) \in \overrightarrow{F}_{x_0}$, the second assertion can fail. Consider the subequation $A \geq p \geq 0$ on $\mathbf{R}$, or the subequation $A \geq p_0 I \geq 0$ on $\mathbf{R}^n$.

11.2. Boundary Convexity. Suppose Ω is a domain with smooth boundary $\partial\Omega$ in X (or some open subset which we also call X). By a *defining function* for $\partial\Omega$ we mean a smooth function ρ defined on a neighborhood of $\partial\Omega$ such that

$$\partial\Omega = \{x : \rho(x) = 0\}, \quad d\rho \neq 0 \text{ on } \partial\Omega, \quad \text{and} \quad \rho < 0 \text{ on } \Omega.$$

Note that for $x \in \partial\Omega$ we have $J^2_x\rho = \{0\} \times J^2_{\text{red},x}\rho$. For simplicity of notation, we set $J^2_x\rho = J^2_{\text{red},x}\rho$.

Proposition 11.6. *Suppose F is a reduced subequation on X with asymptotic interior $\overrightarrow{F}$. Let $\Omega \subset X$ be a domain with smooth boundary. Then for $x \in \partial\Omega$ the following are equivalent.*

(1) *There exists a local defining function ρ for $\partial\Omega$ near x such that*

$$J^2_x\rho \in \overrightarrow{F}_x \tag{11.3}$$

(so that ρ is strictly $\overrightarrow{F}$-subharmonic near x).

(2) *There exists a local defining function ρ for $\partial\Omega$ near x and $t_0 > 0$ such that*

$$J^2_x\rho + t(d\rho)_x \circ (d\rho)_x \in \overrightarrow{F}_x \quad \text{for all } t \geq t_0. \tag{11.4}$$

(3) *Given any defining function ρ for $\partial\Omega$ near x, there exists $t_0 > 0$ such that*

$$J^2_x\rho + t(d\rho)_x \circ (d\rho)_x \ \in \ \overrightarrow{F}_x \quad \text{for all } t \geq t_0. \tag{11.4}$$

Note that if (11.4) holds for $t = t_0$, then it holds for all $t \geq t_0$ by positivity for $\overrightarrow{F}$.

Definition 11.7. (Boundary Convexity). Let F be a reduced subequation on X and $\Omega \subset X$ a smoothly bounded domain. Then $\partial\Omega$ is **strictly F-convex at** $x \in \partial\Omega$ if the equivalent conditions (1), (2), and (3) hold at x. The boundary $\partial\Omega$ is said to be **strictly F-convex** if this holds at every point $x \in \partial\Omega$.

Proof of Proposition 11.6. (1) $\Rightarrow$ (3): Assume that (1) is true for the defining function ρ. Any other local defining function for $\partial\Omega$ is of the form $\widetilde{\rho} = u\rho$ for some smooth function $u > 0$. At $x \in \partial\Omega$ we have $d\widetilde{\rho} = ud\rho$ and $J^2\widetilde{\rho} = uJ^2\rho + du \circ d\rho$. Hence with $\epsilon > 0$ we have

$$\begin{aligned} J^2\widetilde{\rho} + td\widetilde{\rho} \circ d\widetilde{\rho} \ &= \ uJ^2\rho + du \circ d\rho + tu^2 d\rho \circ d\rho \\ &= \ u\left(J^2\rho - \epsilon \cdot I\right) + \left(tu^2 d\rho \circ d\rho + du \circ d\rho + u\epsilon \cdot I\right) \end{aligned}$$

at x. Now for $\epsilon > 0$ sufficiently small we have $u\left(J^2\rho - \epsilon \cdot I\right) \in \overrightarrow{F}$ by the assumption on ρ, the openness of $\overrightarrow{F}$ and the cone property for $\overrightarrow{F}$. For all t sufficiently large, the remaining term in the second line above lies in $\text{Int}\mathcal{P}_x \subset \text{Sym}^2(T^*_xX)$. Now apply the positivity condition (P) for $\overrightarrow{F}$.

(2) $\Rightarrow$ (1): Assume that (2) is true for the defining function ρ for $\partial\Omega$ and consider $\rho_t \equiv \rho + \frac{1}{2}t\rho^2$ for $t \geq 0$. Then ρ_t is also a defining function for $\partial\Omega$ and it has 2-jet

$$J^2\rho_t \;=\; J^2\rho + t d\rho \circ d\rho \qquad \text{on} \;\; \partial\Omega.$$

Hence ρ_t satisfies (1) if $t \geq t_0$.

Evidently (3) $\Rightarrow$ (2), and (1) $\Rightarrow$ (2) by positivity. ∎

From Proposition 11.6(3) we conclude the following.

Corollary 11.8. *Let $\Omega \subset\subset X$ be a pre-compact domain with a smooth strictly F-convex boundary, and ρ a global defining function for $\partial\Omega$. Then for all $t > 0$ sufficiently large, the defining function $\rho_t = \rho + \frac{1}{2}t\rho^2$ is strictly F-subharmonic in a neighborhood of $\partial\Omega$.*

Proof. Since $\rho + \frac{1}{2}t\rho^2$ is strictly $\overrightarrow{F}$-subharmonic in a neighborhood U_x of x in $\partial\Omega$ for $t \geq$ some t_x, we can pass to a finite covering of $\partial\Omega$ by such sets U_x and take t to be the largest t_x in the family. Then ρ_t is strictly F-subharmonic at all points of $\partial\Omega$ and hence in a neighborhood of $\partial\Omega$. ∎

The question of a global strictly F-subharmonic defining function on a neighborhood of $\overline{\Omega}$ will be discussed later.

A different question, namely, the local existence of strictly F-convex domains, has already been answered in Proposition 11.5. We restate it here in the language of F-convexity.

Proposition 11.9. *Let F be a reduced subequation on X and fix $q \in X$. Suppose the fiber F_q contains a critical jet $(0, A)$. Then in any local coordinate system, the balls $\{x : \|x - x(q)\| < R\}$ are all strictly F-convex for all $R > 0$ sufficiently small.*

Example. The strictness condition $J^2_x\rho \in \overrightarrow{F}$ in (1) is not independent of the defining function ρ. Suppose $F = \widetilde{\mathbf{P}}$, the constant coefficient subequation defined by requiring at least one eigenvalue of D^2u to be ≥ 0, and note that $F = \overrightarrow{F}$. The defining function $\rho(x) = 1 - |x|^2$ for the unit sphere S^{n-1} (with inwardly pointing gradient) is not $\widetilde{\mathbf{P}}$-subharmonic, whereas the defining function $\rho + \rho^2$ *is* strictly $\widetilde{\mathbf{P}}$-subharmonic near S^{n-1}.

We now complete Definition 11.7 by extending boundary convexity from reduced subequations to the general case.

Associated to a general subequation $F \subset J^2(X)$ is a family of reduced subequations $F_\lambda \subset J^2_{\text{red}}(X)$, $\lambda \in \mathbf{R}$, obtained by setting the r-variable equal to the constant λ. Said differently, F_λ is defined by

$$\{\lambda\} \times F_\lambda \;=\; F \cap \left\{ \{\lambda\} \times J^2_{\text{red}}(X) \right\}.$$

Each reduced subequation F_λ has an asymptotic interior $\overrightarrow{F_\lambda}$. (It is not uncommon for all the asymptotic interiors $\overrightarrow{F_\lambda}$ to agree even though F is not a reduced subequation.)

Definition 11.10.(Boundary Convexity Continued). Given a general subequation $F \subset J^2(X)$ and a domain $\Omega \subset X$ with smooth boundary, we say that $\partial\Omega$ is *strictly F-convex at a point x* if $\partial\Omega$ is strictly F_λ-convex at x for each $\lambda \in \mathbf{R}$ (cf. Definition 11.7). The boundary $\partial\Omega$ is called (globally) *F-convex* if it is F-convex at every $x \in \partial\Omega$.

11.3. Barriers. The existence of barriers at a boundary point $x_0 \in \partial\Omega$ of a domain Ω is needed for establishing boundary regularity for the Dirichlet problem.

Definition 11.11. Given $\lambda \in \mathbf{R}$, $x_0 \in \partial\Omega$, and a local defining function ρ for $\partial\Omega$ near x_0, we say that ρ *defines a λ-barrier for F at $x_0 \in \partial\Omega$* if there exist $C_0 > 0$, $\epsilon > 0$, and $r_0 > 0$ such that the function

$$\beta(x) \;=\; \lambda + C\left(\rho(x) - \epsilon\frac{|x-x_0|^2}{2}\right) \tag{11.5}$$

is strictly F-subharmonic on $B(x_0, r_0)$ for all $C \geq C_0$. If F is a reduced subequation, then we say ρ *defines a barrier for F at x_0*, since the same ρ works for all $\lambda \in \mathbf{R}$.

Theorem 11.12. (Existence of Barriers). *Suppose $\Omega \subset X$ is a domain with smooth boundary $\partial\Omega$ which is strictly F-convex at $x_0 \in \partial\Omega$. Then for each $\lambda \in \mathbf{R}$ there exists a local defining function ρ for $\partial\Omega$ near x_0 which defines a λ-barrier for F at x_0.*

Proof. Choose a local defining function ρ for $\partial\Omega$ near x_0 with $J \equiv (D_{x_0}\rho, D^2_{x_0}\rho) \in \overrightarrow{F}_{\lambda'}$ ($\lambda' > \lambda$). Then by the definition of $\overrightarrow{F}$ there exists a neighborhood $\mathcal{N}(J)$ in the total reduced 2-jet space $J^2_{\rm red}(X)$, and a number $C_0 > 0$ such that

$$C \cdot \mathcal{N}(J) \;\subset\; {\rm Int} F_{\lambda'} \quad \text{for all} \;\; C \geq C_0.$$

By shrinking we may assume that $\mathcal{N}(J) = B(x_0, r_0) \times B(D_{x_0}\rho, \delta_0) \times B(D^2_{x_0}\rho, \epsilon_0)$ is a product neighborhood in some local coordinate system.

Set $\alpha(x) \equiv \rho(x) - \epsilon\frac{|x-x_0|^2}{2}$ so our desired barrier will be $\beta(x) = \lambda + C\alpha(x)$. Then

$$J_{{\rm red},x}\alpha \;=\; \left(D_x\rho - \epsilon(x - x_0),\, D^2_x\rho - \epsilon I\right). \tag{11.6}$$

By choosing ϵ and r_0 sufficiently small we have

$$J_{{\rm red},x}\alpha \;\in\; \mathcal{N}(J) \qquad \text{if} \;\; x \in B(x_0, r_0).$$

This proves that

$$J_{{\rm red},x}\beta \;=\; C J_{{\rm red},x}\alpha \;\in\; {\rm Int} F_{\lambda'}$$

if $C \geq C_0$ and $x \in B(x_0, r_0)$.

Now by the negativity condition (N) we have that $F_\mu \supset F_{\lambda'}$ if $\mu < \lambda'$. Hence,

$$(-\infty, \lambda') \times \mathrm{Int} F_{\lambda'} \ \subset \ F$$

is an open subset in the full jet space $J^2(X)$, and it contains $J_x\beta$ for all $C \geq C_0$ and all $x \in B(x_0, r_0)$. ∎

Remark 11.13. (Existence of Barriers—Refinements). Actually, less than is stated in Definition 11.10 is required for the application to the Dirichlet problem. Namely, one only needs that for each $\lambda \in \mathbf{R}$, there is a continuous defining function ρ such that there exist $C_0 \geq 0$, $\epsilon > 0$, and $r_0 > 0$ so that

$$\beta(x) \ = \ \lambda + C\left(\rho(x) - \epsilon\frac{|x-x_0|^2}{2}\right) \ \in \ F_{\mathrm{strict}}\left(\overline{B(x_0,r_0)} \cap \overline{\Omega}\right) \tag{11.7}$$

is strictly F-subharmonic on $\overline{B(x_0,r_0)} \cap \overline{\Omega}$ for all $C \geq C_0$. (See Section 7 for the definition of $F_{\mathrm{strict}}(K)$.)

This applies to the intersection $\Omega = \Omega_1 \cap \Omega_2$ where $\Omega_1, \Omega_2 \subset X$ are two domains with smooth, strictly $\overrightarrow{F}$-convex boundaries. Fix $x_0 \in \partial\Omega_1 \cap \partial\Omega_2$ and $\lambda \in \mathbf{R}$. Assume that there exist local defining functions ρ_k for each $\partial\Omega_k$ near x_0 and constants $r_0 > 0$, $\epsilon > 0$, and $C_0 \geq 0$ such that $\beta_k(x) \equiv \lambda + C\left(\rho_k(x) - \epsilon|x-x_0|^2\right) \in \ F_{\mathrm{strict}}\left(\overline{B(x_0,r_0)} \cap \overline{\Omega}\right)$ for all $C \geq C_0$. Then

$$\max\{\beta_1, \beta_2\} \ \in \ \ F_{\mathrm{strict}}(\overline{B(x_0,r_0)} \cap \overline{\Omega}_1 \cap \overline{\Omega}_2)$$

for all $C \geq C_0$ by Lemma 7.7(i).

This remark is particularly useful in establishing existence for parabolic equations.

Our notion of strict boundary convexity for a subequation F is sufficient to ensure the existence of a family of F-barriers (Theorem 11.12). Might there be other different conditions which also ensure the existence of barriers? For reduced euclidean subequations, the answer is essentially no.

Proposition 11.14. *Let F be a reduced constant coefficient subequation on $\mathbf{R}^n$ and consider a domain $\Omega \subset\subset \mathbf{R}^n$. Suppose ρ is a defining function for $\partial\Omega$ at x_0 which defines a barrier for F at x_0. Then $J^2_{x_0}\rho \in \overrightarrow{F_{x_0}}$, i.e., $\partial\Omega$ is strictly F-convex at x_0.*

Proof (Outline). By hypothesis we have $C(D_x\rho - \epsilon(x-x_0),\, D^2_x\rho - \epsilon I) \in \mathrm{Int}F$ if $|x - x_0|$ and ϵ are small and C is large. One must show that $(D_x\rho - \epsilon(x-x_0),\, D^2_x\rho - \epsilon I)$ fills out a neighborhood $\mathcal{N}(J)$ of $J = (D_{x_0}\rho,\, D^2_{x_0}\rho)$ in the total space $J^2_{\mathrm{red}}(X)$. The only difficulty is in the p-variable. The map $x \mapsto D_x\rho - \epsilon(x - x_0)$ has derivative at x_0 equal to $D^2_{x_0} - \epsilon I$, which is invertible for almost all ϵ. Hence this map fills out a ball about $p_0 = D_{x_0}\rho$. ∎

Finally we point out that the asymptotic interior $\overrightarrow{F}$ of a reduced subequation $F \subset J^2_{\rm red}(X)$ could have been defined using another section J_1 of $J^2_{\rm red}(X)$ as the vertex rather than the zero section. Define the *asymptotic interior of F based at J_1*, denoted $\overrightarrow{F_{J_1}}$, to be the set of J for which there exists a neighborhood $\mathcal{N}(J)$ of J in $J^2_{\rm red}(X)$ and $t_0 > 0$ such that $J_1 + t\mathcal{N}(J) \ \subset \ F$ for all $t \geq t_0$. Then

$$\overrightarrow{F_{J_1}} \ = \ \overrightarrow{F_{J_2}} \quad \text{for any two sections} \ \ J_1 \ \text{ and } \ J_2. \tag{11.8}$$

The proof is left to the reader.

11.4. A Geometric Characterization of Boundary Convexity. In this subsection we see that on a riemannian manifold X, Proposition 11.6 enables us to characterize the $\overrightarrow{F}$-convexity of a boundary $\partial\Omega$ in terms of its second fundamental form $II_{\partial\Omega}$ with respect to the outward-pointing normal n. Recall the canonical decomposition of the 2-jet bundle given in Section 4.2:

$$J^2(X) = \mathbf{R} \oplus T^*X \oplus {\rm Sym}^2(T^*X). \tag{11.9}$$

Proposition 11.15. *The boundary $\partial\Omega$ is strictly F-convex at a point $x \in \partial\Omega$ if and only if*

$$(0, n, \, tP_n \oplus II_{\partial\Omega}) \ \in \ \overrightarrow{F_x} \qquad \text{for all } t \geq \text{some } t_0 \tag{11.10}$$

where P_n is orthogonal projection onto the normal line $\mathbf{R} \cdot n$ at x.

Note. Blocking with respect to the decomposition $T_xX = \mathbf{R} \cdot n \oplus T_x(\partial\Omega)$, (11.10) can be rewritten

$$\left(0, (1,0), \begin{pmatrix} t & 0 \\ 0 & II_{\partial\Omega} \end{pmatrix}\right) \ \in \ \overrightarrow{F_x} \qquad \text{for all } t \geq \text{some } t_0. \tag{11.10$'$}$$

Proof. Choose ρ to be the signed distance function on a neighborhood U of $\partial\Omega$. That is,

$$\rho(y) \ = \ \begin{cases} -{\rm dist}(y, \partial\Omega) & \text{if } \ y \in U^- \\ +{\rm dist}(y, \partial\Omega) & \text{if } \ y \in U^+ \end{cases}$$

where $U - \partial\Omega = U^+ \cup U^-$ and signs are chosen so that $\nabla\rho = n$ on the boundary $\partial\Omega$. Then it is a standard calculation (cf. [HL$_2$, (5.7)]) that

$${\rm Hess}_x\rho \ = \ \begin{pmatrix} 0 & 0 \\ 0 & II_{\partial\Omega} \end{pmatrix}$$

with respect to the splitting $T_xX = (\mathbf{R} \cdot \nabla\rho) \oplus T_x\partial\Omega$. Since $\nabla\rho = n$ at x, the assertion follows directly from Proposition 11.6 and Definition 11.7. ∎

11.5. Example: Equations Involving Curvatures of the Graph. The condition of $\overrightarrow{F}$-convexity is nicely illustrated by the following class of equations. Fix a closed subset $\mathbf{S} \subset \mathbf{R}^n$, invariant under permutation of coordinates and satisfying $S + (\mathbf{R}^+)^n \subset S$. Consider a function $u : \Omega \to \mathbf{R}$ on a smoothly bounded domain $\Omega \subset \mathbf{R}^n$. At each point $x \in \Omega$ let $\kappa(x) = (\kappa_1(x), \dots, \kappa_n(x))$ be the principal curvatures of the graph of u in $\mathbf{R}^{n+1}$. We define our subequation F by the condition that $\kappa(x) \in \mathbf{S}$ for all x. The simplest case, $\mathbf{S} = \{\sum_j \kappa_j \geq 0\}$, corresponds to the graph having non-negative mean curvature, and the resulting equation on u is the classical minimal surface equation.

This subequation determined by $\mathbf{S}$ involves only first and second derivatives $(Du, D^2u) = (p, A)$ and can be defined as follows. Consider $E \in \mathrm{Sym}^2(\mathbf{R}^n)$ given by

$$E \equiv I - \tfrac{1}{\nu(1+\nu)} p \circ p \qquad \text{where} \quad \nu = \sqrt{1 + |p|^2}.$$

Then the principal curvatures $\kappa_1, \dots, \kappa_n$ of the graph of u are the eigenvalues of the linear transformation $\frac{1}{\nu} EAE$. Thus we have that

$$F = \left\{(p, A) : \text{the eigenvalues of } \tfrac{1}{\nu} EAE \text{ lie in } \mathbf{S}\right\}.$$

Straightforward computation shows that

$$\overrightarrow{F} \supseteq \left\{(p, A) : \text{the eigenvalues of } P^\perp A P^\perp \text{ lie in } \operatorname{Int} \mathbf{S}\right\}.$$

where $P^\perp$ is orthogonal projection onto the hyperplane $p^\perp$. Thus we find the following.

Proposition 11.16. *The boundary $\partial\Omega$ is strictly $\overrightarrow{F}$-convex if its own principal curvatures $\kappa_1^{\partial\Omega}, \dots, \kappa_{n-1}^{\partial\Omega}$, satisfy*

$$(0, \kappa_1^{\partial\Omega}, \dots, \kappa_{n-1}^{\partial\Omega}) \in \operatorname{Int} \mathbf{S} \qquad \text{at each } x \in \partial\Omega.$$

Thus for the minimal surface equation we obtain the classical condition that the boundary has positive mean curvature at each point. More generally, let $\mathbf{S}_k$ be the closure of the component of $\{\sigma_1(\kappa) > 0, \dots, \sigma_k(\kappa) > 0\}$ containing $(1, \dots, 1)$, and let F_k be the corresponding subequation. Then we get the condition that $\sigma_1(\kappa) > 0, \dots, \sigma_k(\kappa) > 0$ for the principal curvatures of $\partial\Omega$. Similarly one could restrict further to the set $\sigma_k/\sigma_\ell > 1$ (or $\sigma_\ell/\sigma_k > 1$) as in [LE], and the corresponding condition comes out for the boundary. Of course Proposition 11.16 applies to very general sets $\mathbf{S}$.

Existence for the Dirichlet problem for these examples is the topic of Section 17.

12. The Dirichlet Problem—Existence

Throughout this section we assume that F is a subequation on a manifold X and that $\Omega \subset\subset X$ is a domain with smooth boundary $\partial\Omega$.

Furthermore, we assume that both $F_{\rm strict}(\overline{\Omega})$ and $\widetilde{F}_{\rm strict}(\overline{\Omega})$ contain at least one function bounded below. (See (8.2) for this notation.) This assumption is "minor;" for example, it is obvious locally for all subequations with $F_x \neq \emptyset$ and $\widetilde{F}_x \neq \emptyset$ for all x. Our key assumption in the existence theorems is that $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex.

Definition 12.1. Given a boundary function $\varphi \in C(\partial\Omega)$, consider the *Perron family*

$$\mathcal{F}(\varphi) \ \equiv\ \left\{ u \in {\rm USC}(\overline{\Omega}) : u\big|_{\Omega} \in F(\Omega) \ \text{ and } \ {\rm u}\big|_{\partial\Omega} \leq \varphi \ \right\}$$

and define the *Perron function*

$$U(x) \ \equiv\ \sup\{u(x) : u \in \mathcal{F}(\varphi)\}$$

to be the upper envelope of the Perron family.

We shall begin by isolating all the conclusions that hold only under the assumption of weak comparison. This is done in the next theorem and its corollary. In the two subsequent theorems the remaining gap in existence is filled in two different ways. In the first we see that, assuming comparison, the gap is easy to fill. In the second we assume constant coefficients and apply an argument of Walsh ([W]) to fill the gap.

The method of proof for the next theorem is the classical barrier argument. In the case where $F = \mathcal{P}_{\mathbf{C}}$ on $\mathbf{C}^n$, these arguments can be found as far back as Bremermann ([B]) except for the "bump argument" for part (3), given in the proof of Lemma $\widetilde{\mathbf{F}}$ below, which is due to Bedford and Taylor ([BT]). This argument was rediscovered by Ishii ([I]).

Theorem 12.2. *Suppose that $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex, and that weak comparison holds for F on X. Given $\varphi \in C(\partial\Omega)$, the Perron function U satisfies:*

(1) $U_* \ =\ U \ =\ U^* \ =\ \varphi$ *on* $\partial\Omega$,

(2) $U \ =\ U^*$ *is F-subharmonic on* Ω,

(3) $-U_*$ *is $\widetilde{F}$-subharmonic on* Ω.

Corollary 12.3. *If U is lower semicontinuous on Ω, i.e., if $U_* = U$, then:*

(a) $U \ \in\ C(\overline{\Omega})$,

(b) U *is F harmonic on* Ω,

(c) $U \ =\ \varphi$ *on* $\partial\Omega$,

i.e., U solves the Dirichlet problem on $\overline{\Omega}$ for boundary values φ.

Theorem 12.4. *Assume that comparison holds for F, and suppose that $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex. Then for each $\varphi \in C(\partial\Omega)$,* *the Perron function U solves the Dirichlet problem on Ω for boundary values φ.*

Theorem 12.5. *Suppose that F is a constant coefficient subequation on $X = \mathbf{R}^n$, or more generally suppose that $X = K/G$ is a riemannian homogeneous space and that F is a subequation which is invariant under the natural action of the Lie group K on $J^2(X)$.*

If $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex, then existence holds for the Dirichlet problem on Ω as above.

Remark 12.6. There is a hidden hypothesis in these existence results: the sets $\overrightarrow{F_\lambda}$ and $\overrightarrow{\widetilde{F}}_\lambda$ for $\lambda \in \mathbf{R}$ (see Section 11) must be non-empty in order for there to exist any domain Ω whose boundary is both F and $\widetilde{F}$ strictly convex. For example, for the Eikonal subequation $|p| \leq 1$ this fails and, as is well known, existence fails for general $\varphi \in C(\partial B)$ where B is a ball in $\mathbf{R}^n$. However, Theorem 12.5 does apply to the infinite Laplacian, defined by taking F to be the closure of the set $\{(r, p, A) \in J^2(\mathbf{R}^n) : \langle Ap, p\rangle > 0\}$. This example is self-dual, i.e., $\widetilde{F} = F$, and it is also a cone with $\overrightarrow{F} = \text{Int}F$. It is easy to check that $(r, p, A) \in \text{Int}F$ if and only if either $\langle Ap, p\rangle > 0$ or $A > 0$ and $p = 0$. Hence, strictly convex functions are strictly F-subharmonic and define domains for which existence holds.

Now for the proofs.

Lemma F.

$$U^*\big|_\Omega \ \in \ F(\Omega)$$

Lemma $\widetilde{\mathbf{F}}$.

$$-U_*\big|_\Omega \ \in \ \widetilde{F}(\Omega)$$

Proposition F. *Suppose $\partial\Omega$ is strictly F-convex at $x_0 \in \partial\Omega$. For each $\delta > 0$ small, there exists $\underline{u} \in \mathcal{F}(\varphi)$ with the additional properties:*

- $\underline{u}$ is continuous at x_0,
- $\underline{u}(x_0) = \varphi(x_0) - \delta$,
- $\underline{u} \in F_{\text{strict}}(\overline{\Omega})$.

Proposition $\widetilde{\mathbf{F}}$. *Suppose $\partial\Omega$ is strictly $\widetilde{F}$-convex at $x_0 \in \partial\Omega$. For each $\delta > 0$ small, there exists $\overline{u} \in \widetilde{\mathcal{F}}(-\varphi)$ with the additional properties:*

- $\overline{u}$ is continuous at x_0,
- $\overline{u}(x_0) = -\varphi(x_0) - \delta$,
- $\overline{u} \in \widetilde{F}_{\text{strict}}(\overline{\Omega})$.

Corollary F.

$$\varphi(x_0) \ \leq \ U_*(x_0)$$

Corollary $\widetilde{\mathbf{F}}$.

$$U^*(x_0) \ \leq\ \varphi(x_0)$$

Conclusion 1. (Boundary Continuity). *We have $U_* = U = U^* = \varphi$ on $\partial\Omega$. In particular, U is continuous at each point of $\partial\Omega$.*

Proof. By Corollaries F and $\widetilde{\mathrm{F}}$, we have $\varphi(x_0) \leq U_*(x_0) \leq U(x_0) \leq U^*(x_0) \leq \varphi(x_0)\ \forall\, x_0$. ∎

Conclusion 2. $U^* \ \in\ \mathcal{F}(\varphi)$.

Proof. Use Corollary $\widetilde{\mathrm{F}}$ together with Lemma F. ∎

Conclusion 3. $U = U^*$ on $\overline{\Omega}$.

Proof. We have $U^* \leq U$ on $\overline{\Omega}$ since $U^* \in \mathcal{F}(\varphi)$. ∎

Proof of Lemma F. Because of the families locally bounded above property, it suffices to show that $\mathcal{F}(\varphi)$ is uniformly bounded above. Pick $\psi \in \widetilde{F}_{\text{strict}}(\overline{\Omega})$ bounded below. Pick $c >> 0$ so that $\psi - c \leq -\varphi$ on $\partial\Omega$. By Lemma 7.7(ii,) we have $\psi - c \in \widetilde{F}_{\text{strict}}(\overline{\Omega})$. Given $u \in \mathcal{F}(\varphi)$, we have $u + (\psi - c) \leq 0$ on $\partial\Omega$. Weak comparison for $\widetilde{F}$ implies that $u + \psi - c \leq 0$ on $\overline{\Omega}$. Hence, $u \leq -\inf_{\overline{\Omega}} \psi + c$ for all $u \in \mathcal{F}(\varphi)$. ∎

Proof of Proposition F. Assume $\psi \in F_{\text{strict}}(\overline{\Omega})$ is bounded below on $\overline{\Omega}$. Since $\partial\Omega$ is strictly F-convex at x_0, Theorem 11.12 states that there exist a local defining function ρ for $\partial\Omega$ near x_0, and $r > 0, \epsilon_0 > 0$, and $C_0 > 0$ such that in some local coordinates

$$\begin{aligned} \beta(x) \equiv\ & \varphi(x_0) - \delta + C(\rho(x) - \epsilon|x - x_0|^2) \\ \in\ & F_{\text{strict}}(\overline{B(x_0, r)} \cap \overline{\Omega}) \quad \forall\, C \geq C_0 \ \text{ and } \ \forall \epsilon \leq \epsilon_0. \end{aligned} \tag{12.1}$$

Shrink $r > 0$ so that

$$\varphi(x_0) - \delta \ <\ \varphi(x) \quad \text{on } \partial\Omega \cap B(x_0, r). \tag{12.2}$$

Pick $N > \sup_{\partial\Omega} |\varphi| + \sup_{\overline{\Omega}} \psi$ so that

$$\psi - N \ <\ \varphi - \delta \quad \text{on } \partial\Omega. \tag{12.3}$$

Choose C so large that on $A \equiv (B(x_0, r) \sim B(x_0, r/2)) \cap \overline{\Omega}$ we have

$$\beta \ <\ \psi - N, \tag{12.4}$$

This is possible since $\rho(x) - \epsilon|x - x_0|^2$ is strictly negative on A and $\psi - N$ is bounded below on A.

By (12.4) we have that

$$\underline{u}(x) \ =\ \max\{\beta, \psi - N\}$$

is a well defined function on $\overline{\Omega}$ which is equal to $\psi - N$ outside $B(x_0, r/2)$.

Now we have $\psi - N \in\ F_{\text{strict}}(\overline{\Omega})$, because condition (N) is satisfied. Thus Lemma 7.7(i) and condition (12.1) imply that $\underline{u} \in\ F_{\text{strict}}(\overline{\Omega})$.

Outside the set $B(x_0, r/2) \cap \overline{\Omega}$, we have $\underline{u} = \psi - N$ which is $\leq \varphi$ on $\partial\Omega$ by (12.3), while on $B(x_0, r/2) \cap \partial\Omega$ we still have $\psi - N \leq \varphi$, but also

$$\begin{aligned}\beta = \ & \varphi(x_0) - \delta + C(\rho(x) - \epsilon|x - x_0|^2) \ = \ \varphi(x_0) - \delta - C\epsilon|x - x_0|^2 \\ & \leq \varphi(x_0) - \delta \ \leq \ \varphi(x)\end{aligned}$$

by (12.1). Thus $\underline{u}\big|_{\partial\Omega} \leq \varphi$. This proves that $\underline{u} \in \mathcal{F}(\varphi)$. Finally note that $\beta(x_0) = \varphi(x_0) - \delta$ which is $> \psi(x_0) - N$ by (12.3). Continuity of β and the upper semicontinuity of ψ implies that $\beta > \psi - N$ in a neighborhood of x_0. That is $\underline{u} = \beta$ in a neighborhood of x_0. Thus $\underline{u}$ is continuous at x_0, and $\underline{u}(x_0) = \varphi(x_0) - \delta$. ∎

Proof of Proposition $\widetilde{\mathbf{F}}$. This is merely Proposition F with an exchange of roles. ∎

Proof of Corollary F. Since $\underline{u} \in \mathcal{F}(\varphi)$, we have $\underline{u} \leq U$. Hence, $\underline{u}_* \leq U_*$. By the continuity of $\underline{u}$ at x_0 and the fact that $\underline{u}(x_0) = \varphi(x_0) - \delta$, we have $\varphi(x_0) - \delta \leq U_*(x_0)$ for all $\delta > 0$ small. ∎

Proof of Corollary $\widetilde{\mathbf{F}}$. Here we use weak comparison again. Choose $u \in \mathcal{F}(\varphi)$. Since $\overline{u} \leq -\varphi$ on $\partial\Omega$, we have $u + \overline{u} \leq 0$ on $\partial\Omega$. Since $\overline{u} \in \widetilde{F}_{\rm strict}(\overline{\Omega})$, weak comparison implies that $u + \overline{u} \leq 0$ on $\overline{\Omega}$. Therefore, $U + \overline{u} \leq 0$ on $\overline{\Omega}$, i.e., $U \leq -\overline{u}$ on $\overline{\Omega}$. Since $\overline{u}$ is continuous at x_0 and $\overline{u}(x_0) = -\varphi(x_0) - \delta$, this implies that $U^*(x_0) \leq \varphi(x_0) + \delta$ for all $\delta > 0$ small. ∎

Proof of Lemma $\widetilde{\mathbf{F}}$. Note that the Conclusions 1, 2, and 3 are now established. Suppose $-U_*\big|_{\Omega} \notin \widetilde{F}(\Omega)$. Then by Lemma 2.4 there exist $x_0 \in \Omega$, $\epsilon > 0$, and $\psi \in C^2$ near x_0 so that in local coordinates

$$\begin{aligned}&(1) \quad -U_* - \psi \ \leq \ -\epsilon|x - x_0|^2 \quad \text{near } x_0 \\ &(2) \quad -U_* - \psi \ = \ 0 \qquad\qquad\quad \text{at } x_0\end{aligned}$$

but

$$J^2_{x_0}\psi \ \notin \ \widetilde{F}_{x_0}, \qquad \text{i.e.,} \qquad -J^2_{x_0}\psi \ \in \ {\rm Int}F_{x_0}.$$

Then there exist $r > 0, \delta > 0$ so small that

$$u \ \equiv \ -\psi + \delta \ \text{ is } F\text{−subharmonic on } B(x_0, r)$$

Moreover, (1) implies that for $\delta > 0$ sufficiently small,

$$(1)' \quad u \ < \ U_* \text{ on a neighborhood of } \partial B(x_0, r).$$

Since $U_* \leq U$, statement $(1)'$ implies that the function

$$u' \ \equiv \ \begin{cases} U & \text{on } \ \overline{\Omega} - B(x_0, r) \\ \max\{U, u\} & \text{on } \ \overline{B(x_0, r)} \end{cases}$$

is F-subharmonic on Ω. Conclusion 1 says that $U = \varphi$ on $\partial\Omega$, which implies $u' \in \mathcal{F}(\varphi)$. Therefore, $u' \leq U$ on $\overline{\Omega}$, which implies in turn that $u \leq U$ on $B(x_0, r)$.

Now statement (2) above says $U_*(x_0) = u(x_0) - \delta$. Pick a sequence $x_k \to x_0$ with $\lim_{k\to\infty} U(x_k) = U_*(x_0)$. Then

$$(i) \quad \lim_{k\to\infty} U(x_k) \ = \ u(x_0) - \delta$$
$$(ii) \quad \lim_{k\to\infty} u(x_k) \ = \ u(x_0).$$

This implies that $u(x_k) > U(x_k)$ for all k large, contradicting the fact that $u \leq U$ on $B(x_0, r)$. ∎

This completes the proof of Theorem 12.2 and its Corollary 12.3. Interior continuity is all that remains in showing that $U\big|_\Omega$ is F-harmonic.

Proof of Theorem 12.4. (Assuming Comparison). By Corollary F and Lemma $\widetilde{\mathrm{F}}$

$$-U_* \ \in \ \widetilde{\mathcal{F}}(-\varphi).$$

In particular,

$$U - U_* \ \leq \ 0 \qquad \text{on } \partial\Omega.$$

Since $U\big|_\Omega \in F(\Omega)$ and $-U_*\big|_\Omega \in \widetilde{F}(\Omega)$, comparison implies

$$U - U_* \ \leq \ 0 \qquad \text{on } \overline{\Omega},$$

that is, $U \leq U_* \leq U$. Since $U = U^*$, we are done. ∎

Proof of Theorem 12.5. (Assuming Constant Coefficients): We suppose that $X = \mathbf{R}^n$ and that F has constant coefficients. Let $\Omega_\delta \equiv \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$ and $C_\delta \equiv \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) < \delta\}$. Suppose $\epsilon > 0$ is given. By the continuity of U at points of $\partial\Omega$ and the compactness of $\partial\Omega$, it follows easily that there exists a $\delta > 0$ such that

$$\text{(12.5)} \qquad \text{if } |y| \leq \delta, \ \ \text{then } U_y \leq U + \epsilon \ \ \text{on } C_{2\delta}$$

where $U_y(x) \equiv U(x+y)$ is the y-translate of U and where we define U to be $-\infty$ on $\mathbf{R}^n - \overline{\Omega}$. We claim that

$$\text{(12.6)} \qquad \text{if } |y| \leq \delta, \ \ \text{then } U_y \leq U + \epsilon \ \ \text{on } \overline{\Omega},$$

Setting $z = x + y$, this implies that

$$\text{if } z \in \overline{\Omega}, \ x \in \overline{\Omega}, \ \text{and } |z - x| \leq \delta, \ \ \text{then } U(z) \leq U(x) + \epsilon$$

and therefore by symmetry that $|U(z) - U(x)| \leq \epsilon$. Thus the proof is complete once (12.6) is established.

To prove (12.6), note first that $U_y - \epsilon \in F(\Omega_\delta)$ for each $|y| < \delta$ by the translation invariance of F and property (N). Since $U_y \leq U + \epsilon$ on the collar $C_{2\delta}$, one has

$$g_y \ \equiv \ \max\{U_y - \epsilon, U\} \ \in \ F(\Omega).$$

Now (12.5) implies that $g_y = U$ on $C_{2\delta}$. Therefore,

$$g_y \ \in\ \mathcal{F}(\varphi),$$

and hence $g_y \le U$ on $\overline{\Omega}$. This proves that

$$U_y - \epsilon \ \le g_y \ \le\ U \qquad \text{on} \ \ \Omega_\delta.$$

Combined with (12.5), this proves (12.6). ∎

This Walsh argument evidently applies to domains Ω in any riemannian homogeneous space $X = G/H$, provided that the equation F is invariant under the action of G on $J^2(X)$. ∎

Our existence result for general constant coefficient subequations can be restated as follows.

Theorem 12.7. *Suppose that $F \subset J^2(\mathbf{R}^n)$ is a constant coefficient subequation, and that Ω is a domain with smooth boundary $\partial\Omega$ which is both F and $\widetilde{F}$ strictly convex. Given $\varphi \in C(\partial\Omega)$, let U denote the F-Perron function on $\overline{\Omega}$ with boundary values φ, and let V denote the $\widetilde{F}$-Perron function on $\overline{\Omega}$ with boundary values $-\varphi$. Then both $-V$ and U are solutions to the Dirichlet problem for F with boundary values φ. Furthermore, any other such solution u satisfies*

$$-V \ \le\ u \ \le\ U \qquad \text{on} \ \ \overline{\Omega}.$$

Proof. This follows easily from Theorem 12.5. ∎

Example 12.8. (Existence without Uniqueness). Consider the dual subequations

$$\begin{aligned} F \ &\equiv\ \left\{(r,p,A) : A - \tfrac{1}{2}|p|^{\frac{1}{2}}\left(I + P_{[p]}\right) \ge 0\right\} \quad \text{and} \\ \widetilde{F} \ &\equiv\ \left\{(r,p,A) : A + \tfrac{1}{2}|p|^{\frac{1}{2}}\left(I + P_{[p]}\right) \ge 0\right\} \end{aligned} \tag{12.7}$$

(where $P_{[p]}$ is orthogonal projection onto the p-line). First note that $\mathrm{Int}\mathcal{P}$ is the asymptotic interior of both F and $\widetilde{F}$. Consequently, the boundaries $\partial\Omega$ which are strictly F- and $\widetilde{F}$-convex are precisely the boundaries which are classically strictly convex. Consider now the Dirichlet problem for the R-ball $\Omega = \{x : |x| < R\}$ with boundary-value function $\varphi = 0$ (hence $\widetilde{\varphi} = -\varphi = 0$ also). Then Theorem 12.7 applies. Moreover,

- The F-Perron function is $U \equiv 0$.
- The $\widetilde{F}$-Perron function is $V(x) \equiv \frac{1}{12}(R^3 - |x|^3)$.

Thus, in particular, while existence and weak comparison hold (cf. Theorem 10.1 or Corollary C.3), local comparison fails here.

Proof. Obviously $U \equiv 0$ is F-harmonic. Moreover,

$$-U(x) + \tfrac{\epsilon}{2}|x|^2 \ =\ \tfrac{\epsilon}{2}|x|^2 \ \ \text{is strictly } \widetilde{F}\text{−subharmonic.} \tag{12.8}$$

Assuming (12.8), weak comparison implies that on $\overline{\Omega}$:

$$u(x) + (-U(x) + \tfrac{\epsilon}{2}|x|^2) \leq \tfrac{\epsilon}{2}R^2 \quad \text{for all } u \in \mathcal{F}(\varphi).$$

Thus, $u \leq U$ for all $u \in \mathcal{F}(\varphi)$, i.e., U is the F-Perron function for $\varphi = 0$.

Similarly, calculation shows that $V(x) = \frac{1}{12}(R^3 - |x|^3)$ is $\widetilde{F}$-harmonic. Moreover,

$$-V(x) + \tfrac{\epsilon}{2}|x|^2 \quad \text{is strictly } F\text{-subharmonic.} \tag{12.9}$$

Assuming (12.9), weak comparison implies that on $\overline{\Omega}$:

$$(-V(x) + \tfrac{\epsilon}{2}|x|^2) + v(x) \ \leq \ \tfrac{\epsilon}{2}R^2 \quad \text{for all } v \in \widetilde{\mathcal{F}}(-\varphi).$$

Thus, $v \leq V$ for all $v \in \widetilde{\mathcal{F}}(\varphi)$, i.e., V is the $\widetilde{F}$-Perron function for $-\varphi = 0$.

To prove (12.8), simply note that the Hessian of $-U(x) + \frac{\epsilon}{2}|x|^2$ is $A = \epsilon \cdot I$ and apply the definition. To prove (12.9), note that for the function $-V(x) + \frac{\epsilon}{2}|x|^2$

$$p \ = \ \left(\tfrac{|x|}{4} + \epsilon\right) x \quad \text{and} \quad A \ = \ \left(\tfrac{|x|}{4} + \epsilon\right) \cdot I + \tfrac{|x|}{4} P_{[x]}.$$

The eigenspaces for $A - \frac{1}{2}|p|^{\frac{1}{2}}(I + P_{[p]})$ are:

$$[x] \text{ with eigenvalue } \tfrac{|x|}{2} + \epsilon - \left(\tfrac{|x|^2}{4} + \epsilon|x|\right)^{\frac{1}{2}} \text{ and}$$

$$[x]^{\perp} \text{ with eigenvalues } \tfrac{|x|}{4} + \epsilon - \tfrac{1}{2}\left(\tfrac{|x|^2}{4} + \epsilon|x|\right)^{\frac{1}{2}}.$$

It is easy to see that these eigenvalues are > 0 on $\overline{\Omega}$. ∎

Despite the above examples of strict approximation, it must fail in general for this equation since otherwise uniqueness would hold. More precisely, U has no F-strict approximation and V has no $\widetilde{F}$-strict approximation.

Remark 12.9. (Refinements).

1) Note that in the proof of Proposition F we only needed the barrier β defined in (12.1) to be strict on the Ω-portion of the ball, that is, we only really needed

$$\beta \in F_{\text{strict}}(\Omega \cap B(x_0, r)).$$

This can sometimes be established even though $\partial\Omega$ is not strictly F-convex. For example, suppose F is defined by

$$A \ \geq \ 0 \quad \text{and} \quad \det A \ \geq \ e^r \tag{12.10}$$

and note that $\overrightarrow{F}$ is defined by

$$A \ > \ 0 \quad \text{and} \quad r \ < \ 0.$$

Consider $\Omega = \{x : |x| < R\}$ and set $\rho(x) = \frac{1}{2}(|x|^2 - R^2)$. Note that with $\lambda = \varphi(x_0) - \delta$ the barrier satisfies $\beta(x) \leq \lambda$ on the Ω-portion of $B(x_0, r)$. Then it is easy to see that for $C > 0$ sufficiently large, we have $\beta \in F_{\text{strict}}(\Omega \cap B(x_0, r))$. This is because $J^2_x\beta = (\beta(x), C(x - 2\epsilon(x - x_0)), C(1-2\epsilon)I)$, so that $\det A - e^r \geq C^n(1-2\epsilon)^n - e^\lambda > 0$ for all $x \in \Omega \cap B(x_0, r)$. Thus Proposition F and Corollary F are valid for this F in spite of the fact that $\partial\Omega$ is not strictly $\overrightarrow{F}$-convex. In fact, Proposition F and Corollary F are false for the subequation $H \equiv \text{closure}\{\overrightarrow{F}\}$ unless $\varphi(x_0) \leq 0$. It is now easy to show that existence and uniqueness hold for the subequation F defined by (12.10), since $\psi(x) = \frac{1}{2}(|x|^2 - R^2)$ is a good approximator for F.

2) Notice that the weakened form of Proposition F, with $\underline{u} \in F(\overline{\Omega})$ but not necessarily strict, is all that is needed to prove Corollary F. However, to prove Corollary $\widetilde{\text{F}}$ from Proposition $\widetilde{\text{F}}$, weak comparison was used so that the strictness of $\overline{u} \in \widetilde{F}_{\text{strict}}(\overline{\Omega})$ cannot be dropped.

13. The Dirichlet Problem—Summary Results

We present here several summary results which follow from the work above. We make the following standing hypotheses in this section.

(i) F is a subequation on a riemannian manifold X.
(ii) $\Omega \subset\subset X$ is a domain with smooth boundary $\partial\Omega$.
(iii) Both $F_{\text{strict}}(\overline{\Omega})$ and $\widetilde{F}_{\text{strict}}(\overline{\Omega})$ have at least one function bounded below.

Consider the following

Dirichlet Problem for F:

Given $\varphi \in C(\partial\Omega)$, consider the Perron function

$$U \equiv \sup_{u \in \mathcal{F}(\varphi)} u \qquad \text{where } \mathcal{F}(\varphi) = \left\{ u \in F(\overline{\Omega}) : u\big|_{\partial\Omega} \leq \varphi \right\}.$$

Existence. For each $\varphi \in C(\partial\Omega)$, the Perron function satisfies

- U is F-harmonic,
- $U = \varphi$ on $\partial\Omega$,
- $U \in C(\overline{\Omega})$.

Uniqueness. U is the only function with these three properties.

In what follows, M_F denotes a monotonicity cone for F.

Theorem 13.1. *Suppose F is a riemannian G-subequation where X is provided with a topological G-structure. Suppose there exists a C^2 strictly M_F-subharmonic function on X.*

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem.

Theorem 13.1′. *Theorem 13.1 also holds for any subequation F which is locally affinely jet-equivalent to a riemannian G-subequation on X.*

Theorem 13.2. *Suppose F is a subequation for which weak comparison holds. Suppose there exists a C^2 strictly M_F-subharmonic function on X.*

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem.

Theorem 13.3. *Suppose comparison holds for a subequation F on X.*

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem.

Theorem 13.4. *Let $\mathbf{F}$ be a constant coefficient subequation on $\mathbf{R}^n$.*

(Existence.) *Then for every domain $\Omega \subset\subset X$ whose boundary is strictly $\mathbf{F}$- and $\widetilde{\mathbf{F}}$-convex, existence holds for the Dirichlet problem.*

(Comparison.) *If $\mathbf{F}$ is pure second-order, or more generally, independent of the gradient, then comparison holds for $\mathbf{F}$ on $\mathbf{R}^n$.*

This provides a new proof of existence and uniqueness for pure second-order subequations, established in [HL_4] using a result of Slodkowski [S].

Proof. The existence is just a restatement of Theorem 12.7. Since weak comparison holds for any constant coefficient subequation, comparison will follow from strict approximation. A closed subset $\mathbf{F} \subset \mathbf{J}^2$ is a pure second-order subequation if and only if $M = \mathbf{R} \times \mathbf{R}^n \times \mathcal{P}$ is a monotonicity set for $\mathbf{F}$. Since $|x|^2$ is strictly M-subharmonic, strict approximation holds for $\mathbf{F}$. More generally, recall that a closed subset $\mathbf{F} \subset \mathbf{J}^2$ is independent of the gradient if and only if $M \equiv \mathbf{R}_- \times \mathbf{R}^n \times \mathcal{P}$ is a monotonicity set for $\mathbf{F}$. In this case, the function $|x|^2 - R^2$ is strictly M-subharmonic on the ball of radius R about the origin. Since each compact subset $K \subset \mathbf{R}^n$ is contained in such a ball, strict approximation holds for $\mathbf{F}$. ∎

More generally we have the following (see Theorem 9.13).

Theorem 13.5. *Let $X = K/G$ be a riemannian homogeneous space and suppose F is a subequation which is invariant under the natural action of the Lie group K on $J^2(X)$.*

(Existence.) *Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, existence holds for the Dirichlet problem.*

(Comparison.) *If X supports a strictly convex C^2-function, then comparison holds for F on X.*

14. Universal Riemannian Subequations

In this section we consider subequations defined on any riemannian manifold by the requirement that $\mathrm{Hess}_x u \in \mathbf{F}$ (for $u \in C^2$), where $\mathbf{F}$ is a closed subset of $\mathrm{Sym}^2(\mathbf{R}^n)$ which is O_n-invariant. Recall from Section 4.4 that for such purely second-order closed subsets of $J^2(X)$, condition (N) is automatic and condition (P) implies condition (T).

Each O_n-invariant closed subset $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ determines a closed subset Λ, invariant under the permutation group π_n, and consists of all n-tuples of eigenvalues of A where $A \in \mathbf{F}$. Conversely, each closed subset $\Lambda \subset \mathbf{R}^n$ invariant under π_n determines a closed O_n-invariant subset $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$, namely, $\mathbf{F} \equiv \{A : \lambda(A) \in \Lambda\}$. Moreover,

$$\mathbf{F} \text{ satisfies (P)} \quad \iff \quad \Lambda + \mathbf{R}^n_+ \subset \Lambda \tag{14.1}$$

where $\mathbf{R}_+ = [0, \infty)$. The implication from left to right in (14.1) is obvious since Λ can be taken to be the subset of diagonal elements in $\mathbf{F}$, and $\mathbf{R}^n_+$ the set of diagonal elements in $\mathcal{P}$. To prove the reverse implication, consider the *ordered eigenvalues*

$$\lambda_1(A) \le \cdots \le \lambda_n(A) \tag{14.2}$$

of $A \in \mathrm{Sym}^2(\mathbf{R}^n)$. Here $\lambda_k(A)$, the kth smallest eigenvalue, is a well defined continuous function on $\mathrm{Sym}^2(\mathbf{R}^n)$. The standard fact needed is the *monotonicity of the ordered eigenvalues*:

$$A \le B \quad \Rightarrow \lambda_k(A) \le \lambda_k(B) \text{ for all } k \tag{14.3}$$

which follows from the minimax definition of $\lambda_k(A)$.

Set

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)).$$

Now assume $A \in \mathbf{F}$ and $P \ge 0$ so that $B = A + P \ge A$. Monotonicity (14.3) says that $\lambda(B)$ equals $\lambda(A)$ plus a vector in $\mathbf{R}^n_+$. Since $\lambda(A) \in \Lambda$, the assumption $\Lambda + \mathbf{R}^n_+ \subset \Lambda$ implies that $\lambda(B) \in \Lambda$. Hence $B \in \mathbf{F}$, that is, $\mathbf{F}$ satisfies (P), and (14.1) is proved.

Definition 14.1. Suppose Λ is a closed subset of $\mathbf{R}^n$ invariant under the permutation group π_n. If Λ satisfies

$$\Lambda + \mathbf{R}^n_+ \subset \Lambda, \tag{14.4}$$

then Λ will be referred to as a *positive* (or $\mathbf{R}^n_+$*-monotone*) set in $\mathbf{R}^n$.

In the final Remark 14.11 we give a canonical description of all possible $\mathbf{R}^n_+$-monotone sets. Otherwise, the remainder of this section is devoted to examples. Our discussion is confined to subsets $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$, but each $\mathbf{F}$ corresponds to a universal subequation in riemannian geometry, and our language reflects that fact.

Example 14.2. (Monge-Ampère—The Principal Branch). The Monge-Ampère equation $\det(A) = 0$ gives rise to several subequations or branches. The principal branch is just the subequation $\mathcal{P}$. In terms of the ordered eigenvalues, it is given by:

$$\lambda_1(A) \ge 0.$$

We call $\mathcal{P}$ a *branch* because $\partial\mathcal{P} \subset \{\det = 0\}$. However, there are further natural subequations with this property, as we shall see below.

One can show that on a connected open set $X \subset \mathbf{R}^n$ a function u is $\mathcal{P}$-subharmonic if and only if u is convex (or $\equiv -\infty$).

The dual subequation $\widetilde{\mathcal{P}}$ is defined by

$$\lambda_n(A) \ \geq \ 0.$$

In [HL$_4$] we proved that on open sets $X \subset \mathbf{R}^n$, a function u is $\widetilde{\mathcal{P}}$-subharmonic if and only if u is *subaffine*, that is,

$$u \leq a \ \text{ on } \partial K \quad \Rightarrow \quad u \leq a \ \text{ on } K$$

for each affine function a and each compact subset $K \subset X$. Thus u is $\mathcal{P}$-harmonic if and only if u is convex and $-u$ is subaffine. Since $\partial\widetilde{\mathcal{P}} \subset \{\det = 0\}$, the subequation $\widetilde{\mathcal{P}}$ is another branch of the Monge-Ampère equation.

Example 14.3. (The Other Branches of the Monge-Ampère Equation). Define the q^{th} branch $\mathcal{P}_q$ by the condition that

$$\lambda_q(A) \ \geq \ 0 \qquad (\text{at least } n-q+1 \text{ eigenvalues } \geq 0).$$

This gives n branches or subequations

$$\mathcal{P} \ = \ \mathcal{P}_1 \ \subset \ \cdots \ \subset \ \mathcal{P}_q \ \subset \ \cdots \ \subset \ \mathcal{P}_n \ = \ \widetilde{\mathcal{P}}$$

for the Monge-Ampère equation since, for each q,

$$\partial\mathcal{P}_q \ \subset \ \{\det = 0\}$$

and the positivity condition

$$\mathcal{P}_q + \mathcal{P} \ \subset \ \mathcal{P}_q$$

is satisfied by (14.3). Duality becomes

$$\widetilde{\mathcal{P}}_q \ = \ \mathcal{P}_{n-q+1}$$

since $\lambda_q(-A) \ = \ -\lambda_{n-q+1}(A)$. The principal branch $\mathcal{P}$ is a monotonicity subequation for each branch $\mathcal{P}_q$. Since each $\mathcal{P}_q$ is a cone, we have

$$\overrightarrow{\mathcal{P}}_q \ = \ \mathcal{P}_q.$$

Thus the boundary of a domain $\Omega \subset X$ is strictly $\mathcal{P}_q$-convex at a point x iff its second fundamental form $II_{\partial\Omega}$ (with respect to the interior-pointing normal) has at least $n-q$ principal curvatures > 0. Thus, $\partial\Omega$ is strictly $\widetilde{\mathcal{P}}_q = \mathcal{P}_{n-q+1}$-convex if $II_{\partial\Omega}$ has at least $q-1$ principal curvatures > 0.

Theorem 13.1 gives us the following result.

Theorem 14.4. *Let $\Omega \subset\subset X$ be a domain with smooth boundary in a riemannian manifold X. Suppose Ω admits a smooth strictly convex global defining function. Then the Dirichlet problem for every branch of the real Monge-Ampère equation is uniquely solvable for all continuous boundary functions.*

Furthermore, if X carries a strictly convex function defined on a neighborhood of $\overline{\Omega}$, then one can uniquely solve the Dirichlet problem for the branch $\mathcal{P}_q$ if the second fundamental form of $\partial\Omega$ has at least $\max\{q-1, n-q\}$ *principal curvatures* > 0 *at each point.*

Example 14.5. (Geometric p-Plurisubharmonicity). Let $G(p, \mathbf{R}^n)$ denote the Grassmannian of all p-dimensional subspaces of $\mathbf{R}^n$. Define $\mathbf{F}(G(p, \mathbf{R}^n))$ by requiring that

$$\text{(14.5)} \qquad \mathrm{tr}_W A \ \geq \ 0 \ \text{ for all } W \in G(p, \mathbf{R}^n),$$

As with all geometrically defined subequations, this is a convex cone subequation. Functions which are $F(G(p, \mathbf{R}^n))$-subharmonic are more appropriately called *geometrically p-plurisubharmonic* since one can prove that a function u is $\mathbf{F}(G(p, \mathbf{R}^n))$-subharmonic if and only if its restriction to every minimal p-dimensional submanifold of X is subharmonic with respect to the induced riemannian metric (or $\equiv -\infty$).

The subequation $\mathbf{F}(G(p, \mathbf{R}^n))$ can be written in terms of the ordered eigenvalues as

$$\lambda_1(A) + \cdots + \lambda_p(A) \ \geq \ 0,$$

i.e., all p-fold sums of the eigenvalues are ≥ 0.

The dual subequation $\widetilde{\mathbf{F}}(G(p, \mathbf{R}^n))$ can be described by either of the two equivalent conditions:

$$\mathrm{tr}_W A \ \geq \ 0 \quad \text{for some } \ W \in G(p, \mathbf{R}^n)$$

$$\lambda_{n-p+1}(A) + \cdots + \lambda_n(A) \ \geq \ 0.$$

Both the subequation $\mathbf{F}(G(p, \mathbf{R}^n))$ and its dual $\widetilde{\mathbf{F}}(G(p, \mathbf{R}^n))$ are branches of a polynomial equation. Define a polynomial M_p on $\mathrm{Sym}^2(\mathbf{R}^n)$ by

$$\text{(14.6)} \qquad M_p(A) \ = \ \prod_{|I|=p}{}' \left(\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A)\right)$$

where the prime indicates that the product is over all multi-indices $I = (i_1, \ldots, i_p)$ with $i_1 < i_2 < \cdots < i_p$. Then

$$A \in \partial\mathbf{F}(G(p, \mathbf{R}^n)) \ \text{ or } \ A \in \partial\widetilde{\mathbf{F}}(G(p, \mathbf{R}^n)) \qquad \Rightarrow \qquad M_p(A) \ = \ 0.$$

The equation $M_p(A) \ = \ 0$ has many branches, or subequations. Let

$$\lambda_I(A) \ \equiv \ \lambda_{i_1} + \cdots + \lambda_{i_p}$$

denote the Ith p-fold sum of ordered eigenvalues. These $N = \binom{n}{p}$ real numbers can be ordered as

$$\lambda_1(p, A) \ \leq \ \cdots \ \leq \ \lambda_N(p, A).$$

Define $\mathbf{F}_k, k = 1, \ldots, N$ by the condition that

$$\lambda_k(p, A) \ \geq \ 0.$$

That is, $A \in \mathbf{F}_k$ has at least $N-k+1$ p-fold sums of its eigenvalues ≥ 0. Note that if $B \in \mathbf{F}_1 = \mathbf{F}(G(p,\mathbf{R}^n))$ (the principal branch) and $A \in \mathbf{F}_k$, then $A+B \in \mathbf{F}_k$. That is,

$\mathbf{F}(G(p,\mathbf{R}^n))$ is a monotonicity subequation for each branch $\mathbf{F}_k$.

Since $\mathbf{F}_k$ is already a cone, we have that $\overrightarrow{\mathbf{F}}_k = \mathbf{F}_k$. The dual is given by

$$\widetilde{\mathbf{F}}_k \;=\; \mathbf{F}_{N-k+1}.$$

Note that by $(11.6)'$ strict boundary convexity of a domain Ω means that

$$\begin{pmatrix} t & 0 \\ 0 & II_{\partial\Omega} \end{pmatrix} \in \mathbf{F}_k \quad \text{for all } t > 0 \quad \text{sufficiently large.}$$

Note that the top $\binom{n-1}{p-1}$ p-fold sums of eigenvalues of this matrix are automatically positive for large t. Hence, if $\binom{n}{p} - k + 1 \leq \binom{n-1}{p-1}$ every boundary is automatically strictly $\mathbf{F}_k$-convex. Otherwise, $\mathbf{F}_k$-boundary convexity means that $II_{\partial\Omega}$ has at least $\binom{n}{p} - k + 1 - \binom{n-1}{p-1}$ p-fold sums of its principal curvatures ≥ 0.

We leave it to the reader to formulate corollaries of Theorem 13.1 for these equations (as we did for the Monge-Ampère equation above).

Remark 14.6. Each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ acts as a derivation D_A on $\Lambda^p\mathbf{R}^n$ and $D_A \in \mathrm{Sym}^2(\Lambda^p\mathbf{R}^n)$. The polynomial M_p is the restriction of the determinant on $\mathrm{Sym}^2(\Lambda^p\mathbf{R}^n)$ to $\mathrm{Image}(D_A)$

$$M_p(A) \;=\; \det(D_A)\,.$$

Example 14.7. (Elementary Symmetric Functions). Recall that the cone $\mathcal{P}$ can be defined by requiring that

$$\sigma_1(A) \;\geq\; 0,\; \sigma_2(A) \;\geq\; 0,\; \ldots\;,\; \sigma_n(A) \;\geq\; 0$$

where $\sigma_k(A)$ is the kth elementary symmetric function of the eigenvalues of A. For each k, $1 \leq k \leq n$, the condition

$$\sigma_1(A) \;\geq\; 0,\; \sigma_2(A) \;\geq\; 0,\; \ldots\;,\; \sigma_k(A) \;\geq\; 0$$

defines a convex cone subequation $\mathbf{F}(\sigma_k)$. One can show that this set $\mathbf{F}(\sigma_k)$ is exactly the closure of the connected component of the complement of $\{\sigma_k(A) = 0\}$ which contains the identity I.

The equation $\{\sigma_k(A) = 0\}$ has $k-1$ other branches

$$\mathbf{F}(\sigma_k) \;=\; \mathbf{F}_1(\sigma_k) \;\subset\; \mathbf{F}_2(\sigma_k) \;\subset\; \cdots \subset\; \mathbf{F}_k(\sigma_k), \tag{14.7}$$

each of which is $\mathbf{F}(\sigma_k)$-monotone, and for which $\widetilde{\mathbf{F}}_j(\sigma_k) = \mathbf{F}_{k-j+1}(\sigma_k)$. (See the general discussion in [HL_7].) The last branch $\mathbf{F}_k(\sigma_k)$, which is the dual to $\mathbf{F}_1(\sigma_k)$, is given by the condition

$$\sigma_1(A) \;\geq\; 0 \quad \text{or} \quad -\sigma_2(A) \;\geq\; 0 \quad \text{or} \quad \sigma_3(A) \;\geq\; 0 \quad \text{or} \quad \ldots \quad \text{or} \quad (-1)^{k-1}\sigma_k(A) \;\geq\; 0.$$

Theorem 14.8. *Let $\Omega \subset\subset X$ be a domain with smooth boundary in a riemannian manifold X. Suppose Ω is globally strictly $\mathbf{F}(\sigma_k)$-convex; that is, suppose there is a strictly $\mathbf{F}(\sigma_k)$-subharmonic defining function for Ω. Then for every branch of the equation $\sigma_k(\mathrm{Hess}\, u) = 0$, the Dirichlet problem is uniquely solvable for all continuous boundary data.*

Proof. The existence of a strictly $\mathbf{F}(\sigma_k)$-subharmonic defining function ρ implies that the boundary is strictly $\mathbf{F}(\sigma_k)$-convex, and from the inclusions (14.7) it is also $\mathbf{F}_j(\sigma_k)$-convex for each j. Since $\widetilde{\mathbf{F}}_j(\sigma_k) = \mathbf{F}_{k-j+1}(\sigma_k)$, the boundary convexity hypothesis of Theorem 13.1 is satisfied. Since $\mathbf{F}(\sigma_k)$ is a monotonicity subequation of $\mathbf{F}_j(\sigma_k)$ and ρ is strictly $\mathbf{F}(\sigma_k)$-subharmonic, the other hypothesis is satisfied and Theorem 13.1 applies. ∎

Example 14.9. (The Special Lagrangian Potential Equation). This is the subequation $\mathbf{F}_c$ defined by the condition

$$\mathrm{tr}\arctan A \ = \ \arctan(\lambda_1(A)) + \cdots + \arctan(\lambda_n(A)) \ \geq \ \frac{c\pi}{2}$$

for $c \in (-n, n)$. Since arctan is an odd function, the dual equation is again of this form. That is,

$$\widetilde{\mathbf{F}}_c \ = \ \mathbf{F}_{-c}.$$

For values of c where $\mathbf{F}_c$ is convex, the Dirichlet problem for this equation was studied in detail by Caffarelli, Nirenberg, and Spruck [CNS] who established existence, uniqueness, and regularity. Existence, uniqueness, and continuity for all other branches $\mathbf{F}_c$ were established in [HL_4].

One computes that the asymptotic interior of $\mathbf{F}_c$ is

$$\overrightarrow{\mathbf{F}_c} \ = \ \mathrm{Int}\mathcal{P}_q \ = \ \{A : \lambda_q(A) > 0\} \tag{14.8}$$

($\lambda_q(A)$ is the q^{th}, ordered eigenvalue) where q is the unique integer such that

$$\frac{n-c}{2} \ \leq \ q \ < \ \frac{n-c}{2} + 1. \tag{14.9}$$

Theorem 14.10. *Let X be a riemannian n-manifold X on which there exists some global strictly convex function. Let $\Omega \subset\subset X$ be a domain with smooth boundary and suppose $\partial\Omega$ is strictly $\mathcal{P}_q$-convex for an integer q satisfying*

$$1 \ \leq \ q \ < \ \frac{n}{2} + 1. \tag{14.10}$$

Then the Dirichlet problem for $\mathbf{F}_c$-harmonic functions is uniquely solvable on Ω for all continuous $\varphi \in \partial\Omega$ and for all c with

$$|c| \ < \ n - 2q + 2. \tag{14.11}$$

Proof. Recall that strict $\mathcal{P}_q$-convexity implies strict $\mathcal{P}_{q'}$-convexity for $q \le q' \le n$. Suppose $c \ge 0$ and (14.9) holds. Let $\widetilde{q}$ be the integer satisfying (14.9) with c replaced by $-c \le 0$. Then $\widetilde{q} \ge q$. Hence, by the first remark and (14.8) the hypothesis on the boundary is satisfied and Theorem 13.1 applies to $\mathbf{F}_c$. As c descends to zero, the q in (14.9) increases, so by the initial remark, the boundary hypothesis continues to be satisfied. ∎

The subequations $\mathbf{F}_c$ are related to the equation

$$\text{Im}\left\{e^{-i\theta}\text{det}_{\mathbf{C}}(I + i\text{Hess}\, u)\right\} \;=\; 0 \tag{14.12}$$

(for fixed θ) which arises in Special Lagrangian geometry [HL$_1$]. If u satisfies (14.12), then the graph $\{y = \nabla u\}$ in $\mathbf{R}^n \times \mathbf{R}^n$ is absolutely volume-minimizing. Note that $\text{Im}\left\{e^{-i\theta}\text{det}_{\mathbf{C}}(I + iA)\right\} = \text{Im}\left\{e^{-i\theta}\prod_k (I + i\lambda_k(A))\right\} = 0$ yields the congruence

$$\sum_k \arctan(\lambda_k(A)) \;\equiv\; \theta \pmod{\pi}.$$

Thus the equation (14.12) has many disjoint connected sheets corresponding to the subequations $\mathbf{F}_{2(\frac{\theta}{\pi}+k)}$ for either n or $n-1$ integer values of k. The maximal (and minimal) values were treated in [CNS] and the other values in the euclidean case in [HL$_4$].

An interesting case where all of the above applies is the cotangent bundle T^*K of a Lie group K. Fixing an orthonormal framing $e_1, \ldots, e_n$ with respect to a left-invariant metric gives a splitting

$$T^*K \;=\; K \times \mathbf{R}^n$$

which determines in an obvious way a hermitian almost complex structure and an invariant $(n, 0)$-form, i.e., an almost Calabi-Yau structure. By Theorem 14.10 we can solve the Dirichlet problem for the special Lagrangian potential equation on $\mathcal{P}_q$-convex domains $\Omega \subset K$ for q as above. For each solution u the graph of ∇u in T^*K will be a special Lagrangian submanifold.

Remark 14.11. (A canonical form for general $\mathbf{R}^n_+$-monotone sets Λ). Consider the hyperplane H in $\mathbf{R}^n$ perpendicular to $e = (1, \ldots, 1)$. The boundary of the positive orphant $\mathbf{R}^n_+$ is a graph over H. More precisely for $\mu = (\mu_1, \ldots, \mu_n) \in H$, set $\|\mu\|^+ \equiv -\mu_{\min}$ where $\mu_{\min} = \min\{\mu_1, \ldots, \mu_n\}$. Then

$$\partial\mathbf{R}^n_+ \;=\; \{\mu + \|\mu\|^+ e : \mu \in H\}.$$

Similarly,

$$\partial\mathbf{R}^n_- \;=\; \{\mu - \|\mu\|^- e : \mu \in H\}$$

where $\|\mu\|^- = \mu_{\max} = \max\{\mu_1, \ldots, \mu_n\}$.

One can characterize the positive sets Λ as follows. There exists a function $f : H \to \mathbf{R}$, invariant under the action of π_n on H, which is $\|\cdot\|^{\pm}$-Lipschitz, i.e.,

$$-\|\mu\|^- \ \le f(\lambda+\mu) - f(\lambda) \ \le \ \|\mu\|^+ \ \text{ for all } \lambda, \mu \in H \tag{14.13}$$

such that

$$\Lambda \ = \ \{\mu + te : \mu \in H \text{ and } t \ge f(\mu)\}. \tag{14.14}$$

Finally note that $\|\lambda+\mu\|^{\pm} \le \|\lambda\|^{\pm} + \|\mu\|^{\pm}$ and $\|\mu\|^- = \|-\mu\|^+$.

15. The Complex and Quaternionic Hessians

Virtually all the results of the previous section carry over directly to the complex and quaternionic hessians, that is, to the case of U_n and $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$ invariant equations.

The Complex Case. Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ where $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ denotes the standard complex structure. Then we have the set of *hermitian symmetric matrices*

$$\mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n) \ = \ \{A \in \mathrm{Sym}^2(\mathbf{R}^{2n}) : AJ = JA\}$$

and the natural projection

$$\pi : \mathrm{Sym}^2(\mathbf{R}^{2n}) \longrightarrow \mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n) \quad \text{given by } \pi(A) = \tfrac{1}{2}(A - JAJ).$$

If $A \in \mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n)$ and $A(e) = \lambda e$, then $A(Je) = \lambda Je$, and so $\mathbf{C}^n$ decomposes into a direct sum of n complex eigenlines with eigenvalues $\lambda_1(A) \le \cdots \le \lambda_n(A)$. Let $\mathcal{P}^{\mathbf{C}} = \pi(\mathcal{P}) = \{\lambda_1(A) \ge 0\}$ be the cone of positive hermitian symmetric matrices.

Note that closed subsets $\mathbf{F} \subset \mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n)$ invariant under U_n are uniquely determined by their eigenvalue set $\Lambda \subset \mathbf{R}^n$ (invariant under π_n). Moreover, one again has monotonicity of the ordered eigenvalues, so that positivity for invariant sets can be expressed as

$$\pi^{-1}(\mathbf{F}) + \mathcal{P} \subset \pi^{-1}(\mathbf{F}) \quad \iff \quad \mathbf{F} + \mathcal{P}^{\mathbf{C}} \subset \mathbf{F} \quad \iff \quad \Lambda + \mathbf{R}^n_+ \subset \Lambda.$$

Thus each positive π_n invariant set $\Lambda \subset \mathbf{R}^n$ determines a pure second-order U_n-invariant subequation $\pi^{-1}(\mathbf{F})$. All these equations carry over to any complex (or almost complex) manifold X with a hermitian -metric.

The complex Monge-Ampère equation

$$\det{}_{\mathbf{C}} A \ = \ \lambda_1(A) \cdots \lambda_n(A)$$

behaves exactly as in the real case with principal branch $\mathcal{P}^{\mathbf{C}}$ defined by $\{\lambda_1(A) \ge 0\}$ and the remaining branches defined by $\mathcal{P}^{\mathbf{C}}_q = \{\lambda_k(A) \ge 0\}$. The cone $\mathcal{P}^{\mathbf{C}}$ is a monotonicity subequation for each of the branches. The $\mathcal{P}^{\mathbf{C}}$-subharmonic functions are exactly the classical plurisubharmonic functions on X. The boundary of a domain is strictly $\mathcal{P}^{\mathbf{C}}$-convex if and only if it is classically strictly pseudo-convex. Theorem 13.1 now gives the following.

Theorem 15.1. *Let $\Omega \subset\subset X$ be a domain with smooth boundary in an almost complex hermitian manifold X. Suppose Ω admits a smooth strictly plurisubharmonic global defining function. Then the Dirichlet problem for every branch of the complex Monge-Ampère equation is uniquely solvable for all continuous boundary functions.*

Furthermore, if X carries some strictly plurisubharmonic function on a neighborhood of $\overline{\Omega}$, then one can uniquely solve the Dirichlet problem for the branch $\mathcal{P}_q^{\mathbf{C}}$ if the hermitian symmetric part of the second fundamental form of $\partial\Omega$ (i.e., the Levi form) has at least $\max\{q-1, n-q\}$ *strictly positive eigenvalues at each point.*

This generalizes a result of Hunt and Murray [HM] and Slodkowski [S] to the case of almost complex manifolds.

The discussion of geometric p-plurisubharmonicity, elementary symmetric functions, and the special Lagrangian potential equation all carry over, and analogues of Theorems 14.8 and 14.10 hold.

The Quaternionic Case. Consider $\mathbf{H}^n = (\mathbf{R}^{4n}, I, J, K)$ where $I, J, K : \mathbf{R}^{4n} \to \mathbf{R}^{4n}$ denote right scalar multiplication by the unit quaternions i, j, k. Then we have the set of *quaternionic hermitian symmetric matrices*

$$\mathrm{Sym}^2_{\mathbf{H}}(\mathbf{H}^n) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^{4n}) : AI = IA, AJ = JA \text{ and } AK = KA\}$$

and the natural projection

$$\pi : \mathrm{Sym}^2(\mathbf{R}^{2n}) \longrightarrow \mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n)$$
$$\text{given by } \pi(A) = \tfrac{1}{4}(A - IAI - JAJ - IKI).$$

If $A \in \mathrm{Sym}^2_{\mathbf{H}}(\mathbf{H}^n)$ and $A(e) = \lambda e$, then $A(Ie) = \lambda Ie$, $A(Je) = \lambda Je$, and $A(Ke) = \lambda Ke$, and so $\mathbf{H}^n$ decomposes into a direct sum of n quaternionic eigenlines with eigenvalues $\lambda_1(A) \le \cdots \le \lambda_n(A)$. Let $\mathcal{P}^{\mathbf{H}} = \pi(\mathcal{P}) = \{\lambda_1(A) \ge 0\}$ be the cone of positive quaternionic hermitian symmetric matrices. The discussion now completely parallels the one given for the complex case above. The discussion carries over with no change.

16. Geometrically Defined Subequations—Examples

The subequations $\mathbf{F}(\mathbb{G})$ which are geometrically defined by a subset $\mathbb{G}$ of the Grassmann bundle $G(p, TX)$ (as in Section 4.7) are always convex cone subequations. As we have seen, the universal subequations $\mathbf{F}(\mathbb{G})$ where $\mathbb{G}$ is one of the Grassmannians $G(p, \mathbf{R}^n), G_{\mathbf{C}}(p, \mathbf{C}^n)$, or $G_{\mathbf{H}}(p, \mathbf{H}^n)$ for $p = 1, \ldots, n$ are particularly interesting. These are principal branches of a polynomial equation $M_p = 0$. However, there are many additional interesting examples. For some of them there is no known polynomial operator.

We begin with the general result. Fix a closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$. Set $G = \{g \in \mathrm{O}_n : g(\mathbb{G}) = \mathbb{G}\}$ and let X be a riemannian manifold with topological G-structure so that the riemannian G-subequation $F(\mathbb{G})$ is defined on X. A domain $\Omega \subset\subset X$ is *strictly* $\mathbb{G}$*-convex* if it admits a strictly $\mathbb{G}$-plurisubharmonic defining function. The existence and topological structure of such domains has been studied in [$\mathrm{HL}_{2,5}$]. It is shown there that if $\partial\Omega$ is strictly $\mathbb{G}$-convex and if $\overline{\Omega}$ supports a strictly $\mathbb{G}$-plurisubharmonic function, then Ω is itself strictly $\mathbb{G}$-convex. From Theorem 13.1 we have the following.

Theorem 16.1. *Let X be a riemannian manifold with topological G-structure. Then for any strictly $\mathbb{G}$-convex domain $\Omega \subset\subset X$ the Dirichlet problem for $F(\mathbb{G})$-harmonic functions is uniquely solvable for all continuous boundary data.*

Example 16.2. (Calibrations). Fix a form $\phi \in \Lambda^p \mathbf{R}^n$ with comass 1, i.e.,

$$\|\phi\|_{\text{comass}} \;\equiv\; \sup\left\{ \left| \frac{\phi|_W}{\mathrm{vol}_W} \right| : W \in G(p, \mathbf{R}^n) \right\} \;\leq\; 1.$$

Given such a form, we define

$$\mathbb{G}(\phi) \;\equiv\; \left\{ W \in G(p, \mathbf{R}^n) : \left| \frac{\phi|_W}{\mathrm{vol}_W} \right| = 1 \right\}.$$

Let $G_\phi = \{g \in \ \mathbf{SO}_n : g^*\phi = \phi\}$ and note that G_ϕ preserves $\mathbb{G}(\phi)$. Therefore, when X has a topological G_ϕ-structure, Theorem 16.1 applies. In this case the $\mathbb{G}(\phi)$-plurisubharmonic and $F(\mathbb{G}(\phi))$-harmonic functions are simply called ϕ*-plurisubharmonic* and ϕ*-harmonic* functions. Specific examples are given at the end of the introduction.

Note that if X has a topological G_ϕ-structure, the form ϕ determines a global smooth p-form ϕ on all of X with $\|\phi\|_{\text{comass}} \equiv 1$. If $d\phi = 0$, then ϕ is a standard calibration [HL_1], and all ϕ-submanifolds are automatically minimal. For this case an analysis of ϕ-plurisubharmonic functions, ϕ-convexity, ϕ-positive currents, etc. is carried out in detail in [$\mathrm{HL}_{2,3}$].

Example 16.3. (Lagrangian Plurisubharmonicity). Take $\mathbb{G} \equiv \mathrm{LAG} \subset G_{\mathbf{R}}(n\,\mathbf{C}^n)$ to be the set of Lagrangian n-planes in $\mathbf{C}^n$. These planes have many equivalent descriptions. They occur, for example, as tangent planes to graphs of gradients over $\mathbf{R}^n \subset \mathbf{C}^n$ and also as rotations of $\mathbf{R}^n \subset \mathbf{C}^n$ by an element of the unitary group U_n. In particular, the invariance group which fixes LAG is U_n. There exists a polynomial M on $\mathrm{Sym}^2(\mathbf{R}^{2n})$ which is I-hyperbolic and of degree 2^n with the top branch of $\{M = 0\}$ equal to $\mathbf{F}(\mathrm{LAG})$. This is discussed in a separate paper [HL_6].

Example 16.4. $\mathbf{F}(\mathbb{G}(\phi))$ as a Monotonicity Subequation. To be specific, consider the associative calibration ϕ on $\mathbf{R}^7 = \mathrm{Im}\mathbf{O}$ and the set $\mathbb{G}(\phi)$ of associative 3-planes. The convex cone subequation $\mathbf{F}(\mathbb{G}(\phi))$ is not the principal branch of a polynomial equation $\{M = 0\}$. Nevertheless, one can construct subequations $\mathbf{F}$ which are $\mathbb{G}(\phi)$-monotone. For example, define $\mathbf{F}$ by the condition that $A \in \mathbf{F}$ if

$$\exists\, \xi, \eta \;\in\; \mathbb{G}(\phi) \;\text{ with } \xi \perp \eta \;\text{ and }\; \mathrm{tr}_\xi A \;\geq\; 0, \;\; \mathrm{tr}_\eta A \;\geq\; 0.$$

17. Equations Involving the Principal Curvatures of the Graph

Let $F_{\mathbf{S}}$ be one of the constant coefficient subequations in $\mathbf{R}^n$ discussed in Section 11.5. This is defined by the requirement that for C^2-functions u, the principal curvatures $(\kappa_1, \ldots, \kappa_n)$ of the graph of u lie in a given subset $\mathbf{S} \subset \mathbf{R}^n$ which is symmetric and satisfies positivity. Now one can check that

$$\widetilde{F_{\mathbf{S}}} \;=\; F_{\widetilde{\mathbf{S}}}$$

and in many interesting cases (such as those arising from Gårding hyperbolic polynomials) we have $\mathbf{S} \subset \widetilde{\mathbf{S}}$ (or the reverse), and so $F_{\mathbf{S}}$-convexity implies $F_{\widetilde{\mathbf{S}}}$-convexity. From Theorem 12.7 and Proposition 11.16 we have the following.

Theorem 17.1. *Let $F_{\mathbf{S}}$ be as above, and suppose $\Omega \subset\subset \mathbf{R}^n$ is a smoothly bounded domain which satisfies the geometric convexity condition:*

$$(0, \lambda_1(x), \ldots, \lambda_{n-1}(x)) \;\in\; \mathbf{S} \cap \widetilde{\mathbf{S}} \quad \text{at each} \quad x \in \partial\Omega,$$

$\lambda_1(x), \ldots, \lambda_{n-1}(x)$ *are the principal curvatures of the boundary at x. Then existence holds for the Dirichlet problem for $F_{\mathbf{S}}$ for all continuous boundary data. In fact, the further statements in Theorem 12.7 hold.*

Remark 17.2. For many interesting equations involving the elementary symmetric functions $\sigma_k(\kappa) = \sigma_k(\kappa_1, \ldots, \kappa_n)$, one also has uniqueness. See the very nice paper [LE] for example.

18. Applications of Jet-Equivalence— Inhomogeneous Equations

The notion of affine jet-equivalence is flexible and powerful. Given Theorem 10.1, it is possible to treat a vast array of equations on manifolds. One important case is the Calabi-Yau equation examined in 6.15. That example can be greatly generalized as follows.

Example 18.1. Let F be the riemannian G-subequation for $G = \mathrm{O}_n$ (or U_n or Sp_n) defined by

$$\sigma_1(A+I) \geq 0, \sigma_2(A+I) \geq 0, \ldots, \sigma_{k-1}(A+I) \geq 0 \quad \text{and} \quad \sigma_k(A+I) \geq 1$$

where $\sigma_\ell(A)$ is the ℓ^{th} elementary symmetric function in the eigenvalues of A (or the complex or quaternionic hermitian symmetric part of A). Let $\widetilde{\Phi}$ be the global affine jet-equivalence

$$\widetilde{\Phi}(r,p,A) \;=\; (r,p,(hI)A(hI)^t + (h^2-1)I) \;=\; (r,p,h^2A + (h^2-1)I)$$

defined in (6.9) and set $f = h^{-2k}$. Let $F_f = \widetilde{\Phi}^{-1}(F)$. Then a smooth F_f-harmonic function is one that satisfies

$$\sigma_1(\operatorname{Hess} u + I) \;\geq\; 0, \;\ldots\; , \sigma_k(\operatorname{Hess} u + I) \;\geq\; 0 \qquad \text{and}$$
$$\sigma_k(\operatorname{Hess} u + I) \;=\; f.$$

This example immediately generalizes by replacing $\sigma_k(A)$ with any homogeneous O_n-invariant polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$ which is hyperbolic with respect to the identity I (see 18.4 below).

Example 18.2. There is another (simpler) version of the examples above. Let F be the riemannian G-subequation for $G = \mathrm{O}_n$ (or U_n or Sp_n) defined by

$$\sigma_1(A) \;\geq\; 0, \;\ldots\; , \sigma_{k-1}(A) \;\geq\; 0 \quad \text{and} \quad \sigma_k(A+I) \geq 1$$

with $\sigma_\ell(A)$ as above. Let $\widetilde{\Phi}$ be the global affine jet-equivalence $\widetilde{\Phi}(r,p,A) = (r,p,(hI)A(hI)^t) = (r,p,h^2A)$ and set $f = h^{-2k}$. Then the smooth $F_f = \widetilde{\Phi}^{-1}(F)$ harmonic functions satisfy

$$\sigma_1(\operatorname{Hess} u) \;\geq\; 0, \;\ldots\; , \sigma_k(\operatorname{Hess} u) \;\geq\; 0 \qquad \text{and}$$
$$\sigma_k(\operatorname{Hess} u) \;=\; f.$$

Example 18.3. (Inhomogeneous Equations for $\lambda_q(\operatorname{Hess} u)$). Let F be a subequation on a manifold X and J any section of $J^2(X)$. Then $F_J \equiv F + J$ (fiber-wise translation) is a subequation affinely equivalent to F. These two subequations have the same asymptotic interior (cf. Section 11). Furthermore, any monotonicity cone for one is a monotonicity cone for the other (cf. Section 9).

A simple but illustrative example, for X riemannian, is given by using the canonical splitting (4.3) and taking $J_x = f(x)\mathrm{Id}$ where f is a smooth function on X and Id is the identity section of $\mathrm{Sym}^2(T^*X)$. Let $F = \mathcal{P}_q$, the q^{th} branch of the Monge-Ampère subequation (see Example 14.3). Then the F_J-harmonic functions are solutions of the equation

$$\lambda_q(\operatorname{Hess} u) \;=\; f(x)$$

where $\lambda_q(\operatorname{Hess} u)$ is the q^{th}-ordered eigenvalue of $\operatorname{Hess} u$.

Another example comes from $F = F(\mathbb{G})$, as in Section 4.7. Translating $F(\mathbb{G})$ by $J = \frac{1}{p} f \cdot \mathrm{Id}$ gives the subequation

$$\operatorname{tr}_\xi(\operatorname{Hess}_x u) \;\geq\; f(x) \quad \text{for all } \xi \in \mathbb{G} \text{ and all } x \in X.$$

The F_J-subharmonic functions are *quasi-$\mathbb{G}$-plurisubharmonic* with variable right hand side.

Further interesting examples are given by translating the subequations arising as branches of any Gårding I-hyperbolic polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$.

Example 18.4. (Gårding Hyperbolic Polynomials). The examples above can be vastly generalized by using Gårding's theory of hyperbolic polynomials [G]. (For details of what follows the reader is referred to [HL$_7$].) Let $M : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}$ be a homogeneous polynomial of degree ℓ which is hyperbolic with respect to the identity I. Then the Gårding cone $\mathbf{F}^M$ is defined to be the connected component of $\{M > 0\}$ containing I. It is a convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$ and is a monotonicity cone for the sets $\mathbf{F}^M(c) \equiv \{A \in \mathbf{F}^M : M(A) \geq c\}$ and also for the branches $\mathbf{F}^M = \mathbf{F}^M_1 \subset \mathbf{F}^M_2 \subset \mathbf{F}^M_3 \subset \cdots$ described in the introduction.

Suppose now that this Gårding polynomial M is invariant under a subgroup $G \subseteq \mathrm{O}_n$ and that X is a riemannian manifold with topological G-structure. Then each of the sets above determines a pure second-order subequation on X. For each of these, the basic one F^M, associated to $\mathbf{F}^M$, is a monotonicity cone. Suppose now that B is a smooth section of $F^M \subset \mathrm{Sym}^2(T^*X)$ and h is a positive smooth function. Consider the affine jet-equivalence Φ defined by $\Phi(r,p,A) = (r,p,hI(A+B(x))hI)$ using the canonical splitting of $J^2(X)$. Set $F_\Phi \equiv \Phi^{-1}(F^M(1))$ and let $f = h^{-2\ell}$. Then one finds as above that at any point $x \in X$

$$A \in F_\Phi \qquad \Longleftrightarrow \qquad A + B(x) \in F^M \ \text{ and } \ M(A+B(x)) \geq f(x).$$

Then a smooth F_Φ-harmonic function u is one that satisfies

$$\mathrm{Hess}_x u + B(x) \in F^M \qquad \text{and} \qquad M(\mathrm{Hess}_x u + B(x)) = f(x)$$

One can also translate the various branches F^M_k of the main subequation by the section B.

19. Equations of Calabi-Yau Type in the Almost Complex Case

Let X be an almost complex hermitian manifold and consider the subequation

$$\text{(19.1)} \qquad \mathrm{Hess}_{\mathbf{C}} u + I \ \geq\ 0 \quad \text{and} \quad \det{}_{\mathbf{C}}(\mathrm{Hess}_{\mathbf{C}} u + I) \ \geq\ F(u)f(x)$$

with $F, f > 0$ and F non-decreasing (e.g. $F(u) = e^u$) (cf. Example 6.15 and Section 15). Let $\Omega \subset\subset X$ be a domain with smooth boundary $\partial\Omega$. To address the Dirichlet problem for (19.1) on Ω we need to analyze the condition of F-convexity for $\partial\Omega$. For this we fix $\lambda \in \mathbf{R}$ and consider the subequation F_λ (independent of the r-variable) given in (p, A)-coordinates by

$$\text{(19.2)} \qquad A_{\mathbf{C}} + I \ \geq\ 0 \quad \text{and} \quad \det{}_{\mathbf{C}}(A_{\mathbf{C}} + I) \ \geq\ F(\lambda)f(x).$$

Recall the complex positivity cone $\mathcal{P}_{\mathbf{C}} = \{A : A_{\mathbf{C}} > 0\}$.

Definition 19.1. The domain $\Omega \subset X$ is *strictly pseudo-convex* if there exists a defining function ρ for Ω which is strictly $\mathcal{P}_{\mathbf{C}}$-subharmonic on $\overline{\Omega}$ (i.e., $\mathrm{Hess}_{\mathbf{C}}\rho > 0$ on $\overline{\Omega}$).

Lemma 19.2. *The boundary of a strictly pseudo-convex domain is strictly F and $\widetilde{F}$ convex.*

Proof. One computes directly that the open cone $\mathrm{Int}\mathcal{P}_{\mathbf{C}}$ lies in the asymptotic interior $\overrightarrow{F_\lambda}$ for any value of λ. One also sees directly that $\mathrm{Int}\mathcal{P}_{\mathbf{C}}$ lies in the asymptotic interior $\overrightarrow{\widetilde{F}}_\lambda$ for any value of λ. (Details are left as an exercise.) The assertion now follows from Definition 11.10. ∎

Theorem 19.3. *Let Ω be a strictly pseudo-convex domain in an almost complex hermitian manifold X. Then the Dirichlet problem for the Calabi-Yau equation (19.1) is uniquely solvable for any continuous function on the boundary $\partial\Omega$.*

Proof. First recall from Example 6.15 that F is locally affinely jet-equivalent to a constant coefficient subequation on X. It is straightforward to see that $\mathcal{P}_{\mathbf{C}}$ is a monotonicity cone for F, and by hypothesis there exists a global defining function for Ω which is strictly $\mathcal{P}_{\mathbf{C}}$-subharmonic. We have already seen that $\partial\Omega$ is strictly F and $\widetilde{F}$ convex. Hence Theorem 13.1′ applies. ∎

Admittedly the Calabi-Yau equation holds more interest in the (integrable) complex manifold case. However, it is surprising that the Dirichlet problem is uniquely solvable in this very general setting.

Example 19.4. Many examples can be obtained by starting with a strongly pseudo-convex domain Ω_0 in a complex manifold X_0 and then deforming the complex structure J_0 to a nearby almost complex structure which is not integrable. Any given hermitian metric on X_0 can be continuously deformed by averaging over the nearby J's. For small enough deformations any given strongly plurisubharmonic defining function for Ω_0 will remain $\mathcal{P}_{\mathbf{C}}$-subharmonic. Thus, the Dirichlet problem for (19.1) can be solved for all sufficiently nearby structures.

Appendix A. Equivalent Definitions F-Subharmonic

In this appendix we assume the following for each fiber $F_x \subset \mathbf{J}^2 = \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$. The positivity condition is not required, nor is F required to be closed. Our assumption is that for some $P > 0$:

(A.1) If $(r, p, A + \alpha P) \in F_x$ for all $\alpha > 0$ small, then $(r, p, A) \in F_x$.

Remark. Assuming the positivity condition (P), condition (A.1) is equivalent to requiring that *the J^1-fibers of F, that is, the fibers of F*

under the natural map $J^2(X) \to J^1(X)$, are closed. In terms of the standard coordinates, this means that

(A.1') Each $F_{x,r,p} = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : (x,r,p,A) \in F\}$ is closed.

For the proof, note that if $A \in \overline{F}_{x,r,p}$, then for each $\epsilon > 0$ there exists $A_\epsilon \in F_{x,r,p}$ with $A_\epsilon - A \le \epsilon P$. By (P) this implies $A + \epsilon P \in F_{x,r,p}$. Assuming (A.1), this proves that $A \in F_{x,r,p}$, i.e., this proves (A.1)$'$. The converse obviously holds for all $P > 0$. Thus, assuming positivity, (A.1) holds for one $P > 0$ if and only if it holds for all $P > 0$.

Proposition A.1. *Suppose that $u \in \mathrm{USC}(X)$ and $x_0 \in X$. Let $x = (x_1, \ldots, x_n)$ be local coordinates on a neighborhood of x_0. Then the following conditions I, II, III, and IV are equivalent.*

I. For all $\varphi \in C^2$ near x_0,

$$(1) \quad \left\{ \begin{array}{ll} u - \varphi & \le 0 \quad \text{near } x_0 \\ & = 0 \quad \text{at } x_0 \end{array} \right\} \quad \Rightarrow \quad J^2_{x_0}\varphi \in F_{x_0}$$

II. For all $(r,p,A) \in \mathbf{J}^2$,

$$(2) \quad \left\{ \begin{array}{ll} u(x) - \left[r + \langle p, x - x_0\rangle + \tfrac{1}{2}\langle A(x-x_0), x - x_0\rangle\right] & \le 0 \quad \text{near } x_0 \\ & = 0 \quad \text{at } x_0 \end{array} \right\}$$

$$\Rightarrow \quad (r,p,A) \in F_{x_0}.$$

III. For all $(r,p,A) \in \mathbf{J}^2$,

$$(3) \quad \left\{ \begin{array}{ll} u(x) - \left[r + \langle p, x - x_0\rangle + \tfrac{1}{2}\langle A(x-x_0), x - x_0\rangle \right. & \\ \qquad \left. + o(|x-x_0|^2)\right] & \le 0 \quad \text{near } x_0 \\ & = 0 \quad \text{at } x_0 \end{array} \right\}$$

$$\Rightarrow \quad (r,p,A) \in F_{x_0}.$$

IV. For all $(r,p,A) \in \mathbf{J}^2$ and $\alpha > 0$,

$$(4) \quad \left\{ \begin{array}{ll} u(x) - \left[r + \langle p, x - x_0\rangle \right. & \\ \qquad \left. + \tfrac{1}{2}\langle A(x-x_0), x - x_0\rangle\right] & \le -\alpha|x-x_0|^2 \quad \text{near } x_0 \\ & = 0 \quad \text{at } x_0 \end{array} \right\}$$

$$\Rightarrow \quad (r,p,A) \in F_{x_0}.$$

Proof. (I $\Rightarrow$ II): Given $(r,p,A) \in \mathbf{J}^2$ satisfying (2), set

$$\varphi = r + \langle p, x - x_0\rangle + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle.$$

Since the quadratic function φ satisfies (1), the condition I implies $(r,p,A) = J^2_{x_0}\varphi \in F_{x_0}$.

(II $\Rightarrow$ III): Given $(r,p,A) \in \mathbf{J}^2$ satisfying (3), it follows that $\forall\, \alpha > 0$

$$u(x) - \left[r + \langle p, x - x_0\rangle + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle\right] \le \alpha|x-x_0|^2 \quad \text{near } x_0$$
$$= 0 \quad \text{at } x_0$$

or equivalently

$$u(x) - \big[r + \langle p, x - x_0\rangle + \tfrac{1}{2}\langle (A + 2\alpha I)(x - x_0), x - x_0\rangle\big] \le 0 \quad \text{near } x_0$$
$$= 0 \quad \text{at } x_0.$$

However, this is just (2) for $(r, p, A+2\alpha I)$. Hence by II we have $(r, p, A+2\alpha I) \in F_{x_0}$ for all $\alpha > 0$, proving that $(r, p, A) \in F_{x_0}$ because of the assumption (A.1).

(III $\Rightarrow$ I): Given φ satisfying (1), the Taylor series for φ satisfies (3).

(II $\Rightarrow$ IV): This holds since (4) $\Rightarrow$ (2).

(IV $\Rightarrow$ II): Suppose that $(r, p, A) \in \mathbf{J}^2$ satisfies (2). Equivalently,

$$\begin{aligned} u(x) - \big[r + \langle p, x - x_0\rangle &+ \tfrac{1}{2}\langle (A + 2\alpha I)(x - x_0), x - x_0\rangle\big] \\ &\le -\alpha|x - x_0|^2 \quad \text{near } x_0 \\ &= 0 \qquad\qquad\quad \text{at } x_0. \end{aligned}$$

That is, $(r, p, A + 2\alpha I)$ satisfies (4). Therefore, by IV, $(r, p, A + 2\alpha I) \in F_{x_0}$ for all $\alpha > 0$. Finally (A.1) implies that $(r, p, A) \in F_{x_0}$. ■

Appendix B. Elementary Properties of F-Subharmonic Functions

The proof of Theorem 2.6, which lists the elementary properties of F-subharmonic functions, is given in this appendix. As explained in Remark 2.6, the positivity condition is not needed in these proofs. For Property (A), only condition A.1 is required. For Property (B), we only require that the J^1-fibers of F be closed (cf. (A.1)$'$). Not surprisingly, for (C), (D), and (E) the full hypothesis that F be closed is used.

Proofs.

(A) The condition that $\max\{u, v\} - \varphi \le 0$ near x_0 with equality at x_0 implies that for one of the functions u, v, say u, we have $u(x_0) = \varphi(x_0)$. In this case, $u - \varphi \le 0$ near x_0 with equality at x_0. Hence, $J^2_{x_0}\varphi \in F_{x_0}$.

(B) This follows from condition III in Proposition A.1.

The remaining properties are proved by a common method which uses Lemma 2.4.

(C) Recall the basic fact that if $\{v_j\} \subset \mathrm{USC}(X)$ is a decreasing sequence with limit v, then

$$\lim_{j\to\infty} \left\{ \sup_K v_j \right\} = \sup_K v \tag{B.1}$$

This is proven as follows. Given $\delta > 0$, the upper semicontinuity of v_j implies that the set $K_j = \{x \in K : v_j(x) \ge \sup_K v + \delta\}$ is compact. The sets K_j are decreasing since $\{v_j\}$ is decreasing. The pointwise convergence of $\{v_j\}$ to v implies that $\bigcap_j K_j = \emptyset$. Hence, there exists j_0 with $K_j = \emptyset$ for all $j \ge j_0$. That is, $v_j(x) < \sup_K v + \delta$ for all $j \ge j_0$ and $x \in K$.

Suppose now that $u \notin F(X)$. Then by Lemma 2.4 there exists $x_0 \in X$, local coordinates x about x_0, $\alpha > 0$, and a quadratic function $\varphi(x) = r + \langle p, x - x_0\rangle + \frac{1}{2}\langle A(x-x_0), x - x_0\rangle$ such that

$$
\text{(B.2)} \qquad \begin{aligned} u(x) - \varphi(x) \ &\le\ -\alpha|x - x_0|^2 \qquad \text{near } x_0 \qquad \text{and} \\ &=\ 0 \qquad\qquad\qquad\quad \text{at } \ x_0 \end{aligned}
$$

but $J^2_{x_0}\varphi = (r, p, A) \notin F_{x_0}$.

Pick a small closed ball B about x_0 so that $u - \varphi$ has a strict maximum over B at x_0. Choose a maximum point $x_j \in B$ for the function $u_j - \varphi$ over B. Fix a smaller open ball B' about x_0 and let $K = B - B'$ denote the corresponding compact annulus. Apply (B.1) to the decreasing sequence $v_j \equiv u_j - \varphi$. Since $\sup_K(u - \varphi) < 0$, this proves that $\sup_K(u_j - \varphi) < 0$ for all j sufficiently large.

Since the maximum value of $u_j - \varphi \ge u - \varphi$ over B is ≥ 0, this proves that $x_j \notin K$ i.e., $x_j \in B'$ for j large. Since B' was arbitrary, we have

$$
\text{(B.3)} \qquad \lim_{j\to\infty} x_j \ =\ x_0.
$$

In particular, $x_j \in \mathrm{Int}B$ is an interior maximum point for $u_j - \varphi$. Set

$$
r_j = u_j(x_j), \quad p_j = D_{x_j}\varphi = p + A(x_j - x_0), \quad A_j = D^2_{x_j}\varphi = A.
$$

This proves that

$$
(r_j, p_j, A_j) \in \ F_{x_j}
$$

since $u_j \in F(X)$.

Applying (B.1) to $K = B$ yields that $r_j = \sup_B u_j \searrow \sup_B u = r$. Hence, $\lim_{j\to\infty} r_j = r$. This proves that

$$
\text{(B.4)} \qquad \lim_{j\to\infty}(x_j, r_j, p_j, A_j) \ =\ (x_0, r, p, A).
$$

Since F is closed, this proves that $(x_0, r, p, A) \in F_{x_0}$, contrary to hypothesis.

(D) Since (B.1) is valid if $\{v_j\}$ converges uniformly to v, the proof of (D) is essentially the same as the proof of (C), except easier.

(E) Suppose $u \notin F(X)$. Then exactly as in the previous proofs we have (B.2) for v^*. Since

$$
r \ =\ v^*(x_0) \ =\ \lim_{k\to\infty} \sup_{|y-x_0|\le\frac{1}{k}} \left\{ \sup_{f\in\mathcal{F}} f(y) \right\},
$$

it follows easily that there exists a sequence $y_k \to x_0$ in $\mathbf{R}^n$ and a sequence $f_k \in \mathcal{F}$ such that

$$
\lim_{k\to\infty} f_k(y_k) \ =\ r.
$$

Choose a maximum point x_k for $f_k - \varphi$ over B, a small closed ball about x_0 on which the condition (B.2) holds. By taking a subsequence, we may assume that $x_k \to \bar{x} \in B$. Now

$$f_k(y_k) - \varphi(y_k) \ \le\ f_k(x_k) - \varphi(x_k). \tag{B.5}$$

The left hand side has limit zero. Hence,

$$\begin{aligned} 0\ &\le \liminf_{k\to\infty} f_k(x_k) - \varphi(\bar{x}) \\ &\le \limsup_{k\to\infty} v(x_k) - \varphi(\bar{x}) \ \le\ v^*(\bar{x}) - \varphi(\bar{x}), \end{aligned} \tag{B.6}$$

since by definition $f_k \le v$. This proves that $\bar{x} = x_0$ by (B.2). In particular, each x_k is interior to B for k large. Therefore, since each f_k is F-subharmonic, we find that

$$\left(f_k(x_k),\ D_{x_k}\varphi,\ D^2_{x_k}\varphi\right) \ \in\ F_{x_k}.$$

Note that (B.6) implies $\lim_{k\to\infty} f_k(x_k) = v^*(x_0) = \varphi(x_0)$. Since F is closed, we conclude that

$$J^2_{x_0}\varphi = \lim_{k\to\infty} \left(f_k(x_k),\ D_{x_k}\varphi,\ D^2_{x_k}\varphi\right) \ \in\ F_{x_0},$$

which is a contradiction. ∎

Appendix C. The Theorem on Sums

In this appendix we recall the fundamental Theorem on Sums, which plays a key role in viscosity theory ([CIL], [C]). We restate the result in a form which is particularly suited to our use.

Fix an open subset $X \subset \mathbf{R}^n$, and let $F, G \subset J^2(X)$ be two second-order partial differential subequations. Recall that a function $w \in \mathrm{USC}(X)$ satisfies the *Zero Maximum Principle* on a compact subset $K \subset X$ if

$$w \ \le\ 0 \ \text{ on } \partial K \qquad \Rightarrow \qquad w \ \le\ 0 \ \text{ on } K. \tag{ZMP}$$

Theorem C.1. *Suppose $u \in F(X)$ and $v \in G(X)$, but $u + v$ does not satisfy the Zero Maximum Principle (ZMP) on a compact set $K \subset X$. Then there exist a point $x_0 \in \mathrm{Int}K$ and a sequence of numbers $\epsilon \searrow 0$ with associated points $z_\epsilon = (x_\epsilon, y_\epsilon) \to (x_0, x_0)$ in $X \times X$, and there exist*

$$J_{x_\epsilon} \ \equiv\ (r_\epsilon, p_\epsilon, A_\epsilon) \ \in\ F_{x_\epsilon} \quad \text{and} \quad J_{y_\epsilon} \ \equiv\ (s_\epsilon, q_\epsilon, B_\epsilon) \ \in\ G_{y_\epsilon}$$

such that

(1) $r_\epsilon \ =\ u(x_\epsilon), \quad s_\epsilon \ =\ v(y_\epsilon), \quad \text{and} \quad r_\epsilon + s_\epsilon \ =\ M_\epsilon \ \searrow\ M_0 > 0,$

(2) $p_\epsilon \ =\ \dfrac{x_\epsilon - y_\epsilon}{\epsilon}, \quad q_\epsilon \ =\ \dfrac{y_\epsilon - x_\epsilon}{\epsilon} \ =\ -p_\epsilon, \quad \text{and} \quad \dfrac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \ \to\ 0,$

(3)
$$-\frac{3}{\epsilon}I \ \le\ \begin{pmatrix} A_\epsilon & 0 \\ 0 & B_\epsilon \end{pmatrix} \ \le\ \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Remark C.2. In fact, we have

$$J_{x_\epsilon} \ \in \ \overline{J^{2,+}_{x_\epsilon} u} \quad \text{and} \quad J_{y_\epsilon} \ \in \ \overline{J^{2,+}_{y_\epsilon} v},$$

where $J^{2,+}_x u$ denotes the upper 2-jet of u at x.

Restricting the right hand inequality in (3) to the diagonal yields

$$\text{(3')} \qquad A_\epsilon + B_\epsilon \ = \ -P_\epsilon \qquad \text{where} \ \ P_\epsilon \ \geq \ 0.$$

This is enough to prove a result which lies between weak comparison and comparison for a constant coefficient subequation $\mathbf{F}$ on $\mathbf{R}^n$.

Corollary C.3. *Suppose* $\mathbf{H}$ *and* $\mathbf{F}$ *are constant coefficient subequations with* $\mathbf{H} \subset \mathrm{Int}\mathbf{F}$. *Suppose* K *is a compact subset of* $\mathbf{R}^n$. *If* $u \in \mathbf{H}(K)$ *and* $v \in \widetilde{\mathbf{F}}(K)$, *then*

$$u + v \ \leq \ 0 \quad \text{on } \partial K \qquad \Rightarrow \qquad u + v \ \leq \ 0 \quad \text{on } K.$$

Proof. By (1), (2), and $(3)'$ we have

$$(-s_\epsilon, -q_\epsilon, -B_\epsilon) \ = \ (r_\epsilon - M_\epsilon, p_\epsilon, A_\epsilon + P_\epsilon).$$

Now $(r_\epsilon, p_\epsilon, A_\epsilon) \in \mathbf{H}$ implies $(r_\epsilon - M_\epsilon, p_\epsilon, A_\epsilon + P_\epsilon) \in \mathrm{Int}\mathbf{F}$.
However, $(s_\epsilon, q_\epsilon, B_\epsilon) \in \widetilde{\mathbf{F}}$ says that $(-s_\epsilon, -q_\epsilon, -B_\epsilon) \notin \mathrm{Int}\mathbf{F}$. ∎

Appendix D. Some Important Counterexamples

One might wonder whether comparison, or at least uniqueness for the Dirichlet problem, can be established if one weakens the assumption that there exists a strictly M-subharmonic function where $M + F \subset F$. Consider, for example, the case of domains which are strictly F-convex (i.e., for which there exists a globally strictly F-subharmonic defining function) but the condition $F + F \subset F$ is not satisfied. We give here an example of a strongly P_3-convex domain in a non-negatively curved space where comparison and, in fact, uniqueness for the Dirichlet problem fail.

Consider the standard riemannian 3-sphere $S^3 = \{x \in \mathbf{R}^4 : \|x\| = 1\}$ and the great circle $\gamma = \{(x_1, x_2, 0, 0) : x_1^2 + x_2^2 = 1\} \subset S^3$. Define

$$\delta : S^3 \ \to \mathbf{R} \quad \text{by} \quad \delta(x) \ \equiv \ \mathrm{dist}(x, \gamma)$$

where distance is taken in the 3-sphere metric. The level sets $T_s \equiv \delta^{-1}(s)$ for $0 < s < \pi/2$ are flat tori, which are orbits of the obvious T^2 torus action on $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$. The eigenvalues $\lambda_1(s), \lambda_2(s)$ of the second fundamental form II_s of T_s are constant on T_s and, by the Gauss curvature equation, satisfy

$$\lambda_1(s) \ = \ -\frac{1}{\lambda_2(s)}.$$

Straightforward calculation (cf. [HL_2, (5.7)]) shows that the riemannian hessian

$$\mathrm{Hess}\,\delta \ = \ \begin{pmatrix} 0 & 0 \\ 0 & II_s \end{pmatrix} \qquad \text{where} \ \ \delta = s \in (0, \pi/2),$$

from which it follows easily that

$$\mathrm{Hess}\,\left(\tfrac{1}{2}\delta^2\right) \ = \ \begin{pmatrix} 1 & 0 \\ 0 & \delta \cdot II_s \end{pmatrix} \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & s\lambda(s) & 0 \\ 0 & 0 & -\frac{s}{\lambda(s)} \end{pmatrix}$$

where $\delta = s \in (0, \pi/2)$. As $s \searrow 0$, $\lambda(s) = \frac{1}{s} + O(1)$, and so

$$\mathrm{Hess}\,\left(\tfrac{1}{2}\delta^2\right)\Big|_{s=0} \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, we see the following. Let

$$U \equiv S^3 - \widetilde{\gamma}$$

where $\widetilde{\gamma} = \{(0, 0, x_3, x_4) : x_3^2 + x_4^2 = 1\}$ is the "opposite" or "focal" geodesic to γ. Then

(D.1)
$$\mathrm{Hess}\,\left(\tfrac{1}{2}\delta^2\right) \ \text{ has 2 strictly positive eigenvalues everywhere on } U, \text{ and}$$

(D.2) $$\mathrm{Hess}\,\left(-\tfrac{1}{2}\delta^2\right) \ \text{ has 1 eigenvalue } \geq 0 \text{ everywhere on } U.$$

Conclusion D.1. (Co-convex $\not\Rightarrow$ Maximum Principle). Example (D.2) shows that on spherical domains the Maximum Principle fails for co-convex, i.e., $\widetilde{\mathcal{P}}$-subharmonic functions (where $\widetilde{\mathcal{P}}$ means at least one eigenvalue ≥ 0). In euclidean space, co-convex functions do satisfy the maximum principle. They are called *subaffine functions* and play an important role in [HL_4].

Consider now the product

$$U \times U \ \subset \ S^3 \times S^3$$

and define $\delta_k = \delta \circ \pi_k$ where $\pi_k : U \times U \to U$ denotes projection onto the kth factor. Set

$$\rho \equiv \tfrac{1}{2}\delta_1^2 + \tfrac{1}{2}\delta_2^2$$

and note that

$$\mathrm{Hess}\,\rho \ = \ \mathrm{Hess}\,\left(\tfrac{1}{2}\delta_1^2\right) \oplus \mathrm{Hess}\,\left(\tfrac{1}{2}\delta_2^2\right).$$

In particular, ρ has four strictly positive eigenvalues everywhere on $U \times U$. In the terminology of Section 7 this means that ρ *is strictly* P_2-*subharmonic on* $U \times U$. (Recall that P_q-subharmonic means that $\mathrm{Hess}\,u$ has at least $n - q = 6 - q$ eigenvalues ≥ 0.) Therefore the domain

(D.3) $$\Omega_c \ \equiv \ \{(x, y) \in U \times U : \rho(x, y) \leq c\} \ \text{ is strictly } P_2 \text{ convex}$$

for $0 < c < \frac{\pi^2}{8}$.

Consider now the functions

$$u_1 = -\tfrac{1}{2}\delta_1^2 \quad \text{and} \quad u_2 = -\tfrac{1}{2}\delta_2^2.$$

By (D.2) have the following. Recall that u is P_q-harmonic if $\lambda_{q+1} \equiv 0$ where $\lambda_1 \leq \cdots \leq \lambda_6$ are the eigenvalues of $\operatorname{Hess} u$.

Lemma D.2. *Each* $\operatorname{Hess} u_k$, $k = 1, 2$ *has two negative eigenvalues, three zero eigenvalues, and one non-negative eigenvalue at every point of* $U \times U$. *In particular,*

$$u_1, u_2 \ \in \ P_2(U \times U) \quad \text{and} \quad -u_1, -u_2 \ \in \ P_1(U \times U).$$

Furthermore, each u_k *is* P_2, P_3, *and* P_4*-harmonic, whereas each* $-u_k$ *is* P_1, P_2, *and* P_3*-harmonic on* $U \times U$.

Conclusion D.3. (Comparison Fails). Each domain Ω_c is strictly P_2-convex. Furthermore, there are smooth functions

$$u_1 \ \in \ P_2(\Omega_c) \quad \text{and} \quad u_2 \ \in \ \widetilde{P}_2(\Omega_c)$$

(since $P_2(\Omega_c) \subset P_3(\Omega_c) = \widetilde{P}_2(\Omega_c)$) such that

$$u_1 + u_2 \ \equiv \ -c \ < \ 0 \ \text{ on } \partial\Omega_c \quad \text{and} \quad \sup_{\Omega_c}(u_1 + u_2) \ = \ 0.$$

Hence, the Comparison Principle fails for P_2 on each Ω_c.

Conclusion D.4. (Uniqueness fails for the Dirichlet problem). Each domain Ω_c is strictly P_3-convex (since $P_2 \subset P_3$, and so P_2-subharmonic $\Rightarrow$ P_3-subharmonic). By the lemma we have that

$$\text{Both functions } \ u_1 \ \text{ and } \ (c - u_2) \ \text{ are } P_3\text{-- harmonic on } \Omega_c,$$

and

$$u_1 \ \equiv \ c - u_2 \ \text{ on } \ \partial\Omega_c.$$

Remark D.5. (Existence without uniqueness (again)). Note that the domains Ω_c are strictly $\overrightarrow{P}_2$ and $\overrightarrow{\overrightarrow{P}}_2$-convex (since $P_k = \overrightarrow{P}_k$ and $P_2 \subset P_3$). Since the isometry group of $S^3 \times S^3$ is transitive and preserves these subequations, Theorem 12.5 guarantees the existence of solutions to the Dirichlet problem for all continuous boundary data on each Ω_c. However, as seen above, uniqueness fails in general.

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