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WEIGHTED PROJECTIVE EMBEDDINGS, STABILITY OF ORBIFOLDS, AND CONSTANT SCALAR CURVATURE KÄHLER METRICS

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Abstract

We embed polarised orbifolds with cyclic stabiliser groups into weighted projective space via a weighted form of Kodaira embedding. Dividing by the (non-reductive) automorphisms of weighted projective space then formally gives a moduli space of orbifolds. We show how to express this as a *reductive* quotient and so a GIT problem, thus defining a notion of stability for orbifolds.

We then prove an orbifold version of Donaldson's theorem: the existence of an orbifold Kähler metric of constant scalar curvature implies K-semistability.

By extending the notion of slope stability to orbifolds, we therefore get an explicit obstruction to the existence of constant scalar curvature orbifold Kähler metrics. We describe the manifold applications of this orbifold result, and show how many previously known results (Troyanov, Ghigi-Kollár, Rollin-Singer, the AdS-CFT Sasaki-Einstein obstructions of Gauntlett-Martelli-Sparks-Yau) fit into this framework.

1. Introduction

The problem of finding canonical Kähler metrics on complex manifolds is central in Kähler geometry. Much of the recent work in this area centres around the conjecture of Yau, Tian, and Donaldson that the existence of a constant scalar curvature Kähler (cscK) metric should be equivalent to an algebro-geometric notion of stability. This notion, called "K-stability", should be understood roughly as follows. Suppose we are looking for such a metric on X whose Kähler form lies in the first Chern class of an ample line bundle L. Then, using sections of L^k, one can embed X in a large projective space $\mathbb{P}^{N_k}$ for $k \gg 0$, and stability is taken in a Geometric Invariant Theory (GIT) sense with respect to the automorphisms of these projective spaces as $k \to \infty$. By the Hilbert-Mumford criterion, this in turn can be viewed as a statement about numerical invariants coming from one-parameter degenerations

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of X. The connection with metrics is through the Kempf-Ness theorem, that a stable orbit contains a zero of the moment map. Here this says that a (Chow) stable X can be moved by an automorphism of $\mathbb{P}^{N_k}$ to be *balanced*, and then the restriction of the Fubini-Study metric on $\mathbb{P}^{N_k}$ approximates a cscK metric for $k \gg 0$.

In this paper we formulate and study a Yau-Tian-Donaldson correspondence for orbifolds. On the algebro-geometric side this involves orbifold line bundles, embeddings in weighted projective space, and a notion of stability for orbifolds. This is related in differential geometry to orbifold Kähler metrics (those which pull back to a genuine Kähler metric upstairs in an orbifold chart; downstairs these are Kähler metrics with cone angles $2\pi/m$ about divisors with stabiliser group $\mathbb{Z}/m := \mathbb{Z}/m\mathbb{Z}$) and their scalar curvature. So we restrict to the case of orbifolds with cyclic quotient singularities, but importantly we do allow the possibility of orbifold structure in codimension one.

Our motivation is not the study of orbifolds per se, but their applications to manifolds. Orbifold metrics are often the starting point for constructions of metrics on manifolds (see for instance [**GK07**], and the gluing construction of [**RS05**]) or arise naturally as quotients of manifolds (for instance, quasi-regular Sasaki-Einstein metrics on odd dimensional manifolds correspond to orbifold Kähler-Einstein metrics on the leaf space of their Reeb vector fields). What first interested us in this subject was the remarkable work of [**GMSY07**] finding new obstructions to the existence of Ricci-flat cone metrics on cones over singularities, Sasaki-Einstein metrics on the links of the singularities, and orbifold Kähler-Einstein metrics on the quotient. We wanted to understand their results in terms of stability. In fact, we found that most known results concerning orbifold cscK metrics could be understood through an extension of the "slope stability" of [**RT06**, **RT07**] to orbifolds.

The end product is a theory very similar to that of manifolds, but with a few notable differences requiring new ideas:

- Embedding an orbifold into projective space loses the information of the stabilisers, so instead we show how to embed them faithfully into weighted projective space. This requires the correct notion of ampleness for an orbi-line bundle L, and we are forced to use sections of *more than one* power L^k—in fact, at least as many as the order of the orbifold (defined in Section 2.1). Then the relevant stability problem is taken not with respect to the full automorphism group of weighted projective space (which is not reductive) but with respect to its reductive part (a product of general linear groups). This later quotient exactly reflects the ambiguity given by the choice of sections used in the embedding and, it turns out, gives the same moduli problem.

- By considering the relevant moment maps, we define the Fubini-Study Kähler metrics on weighted projective space required for stability. A difference between this and the smooth case is that the curvature of the natural hermitian metric on the hyperplane line bundle is not the Fubini-Study Kähler metric, though we prove that the difference becomes negligible asymptotically.
- A key tool connecting metrics of constant scalar curvature to stability is the asymptotic expansion of the Bergman kernel. To ensure an expansion on orbifolds similar to that on manifolds, we consider not just the sections of L^k but sections of L^{k+i} as i ranges over one or more periods. Moreover, these sections must be taken with appropriate weights to ensure contributions from the orbifold locus add up to give a global expansion. This is the topic of the companion paper [**RT11**], which also contains a discussion of the exact weights needed.

  This choice of weights can also be seen from the moment map framework. The stability we consider is with respect to a product of unitary groups acting on a weighted projective space, and since the centraliser of this group is large, the moment map is only defined up to some arbitrary constants. These correspond exactly to the weights required for the Bergman kernel expansion, and the main result of [**RT11**] is that there is a choice of weights (and thus a choice of stability notion) that connects with scalar curvature.
- The numerical invariants associated to orbifolds and their 1-parameter degenerations are not polynomial but instead consist of a polynomial "Riemann-Roch" term plus periodic terms coming from the orbifold strata. The definition of the numerical invariants needed for stability (such as the Futaki invariant) will be made by normalising these periodic terms so they have average zero, and then only using the Riemann-Roch part. Then calculations involving stability become identical to the manifold case, only with the canonical divisor replaced with the orbifold canonical divisor.

After setting up this general framework, our main result is one direction of the Yau-Tian-Donaldson conjecture for orbifolds.

Theorem 1.1. *Let (X, L) be a polarised orbifold with cyclic quotient singularities. If $c_1(L)$ admits an orbifold Kähler metric of constant scalar curvature, then (X, L) is K-semistable.*

Our approach follows the proof given for manifolds by Donaldson in [**Don05**]. An improvement by Stoppa [**Sto09**] says that, as long as one assumes a discrete automorphism group, the existence of a cscK metric actually implies K-stability—it is natural to ask if this too can be extended to orbifolds.

Finally, we give an orbifold version of the slope semistability of [**RT06**, **RT07**], which we show is implied by orbifold K-semistability. Together with Theorem 1.1, it gives an obstruction to the existence of orbifold cscK metrics. We use this to interpret some of the known obstructions in terms of stability, for instance the work of Troyanov on orbifold Riemann surfaces, Ghigi-Kollár on orbifold projective spaces, and Rollin-Singer on projectivisations of parabolic bundles. A particularly important class for this theory is Fano orbifolds, where cscK metrics are Kähler-Einstein and equivalent to certain quasi-regular Sasaki-Einstein metrics on odd dimensional manifolds. In this vein, we interpret the Lichnerowicz obstruction of Gauntlett-Martelli-Sparks-Yau in terms of stability.

1.1. Extensions. Non-cyclic orbifolds. We have restricted our attention purely to orbifolds with cyclic quotient singularities. It should extend easily to orbifolds whose stabilisers are products of cyclic groups by using several ample (in the sense of Section 2.5) line bundles to embed in a product of weighted projective spaces. To encompass also non-abelian orbifolds, one should replace the line bundle with a bundle of higher rank so that the local stabiliser groups can act effectively on the fibre over a fixed point, to give a definition of local ampleness mirroring 2.7 in the cyclic case. Then one would hope to embed into weighted Grassmannians. We thank Dror Varolin for this suggestion.

More general cone angles and ramifolds. It would be nice to extend our results from orbifold Kähler metrics—which have cone angles of the form $2\pi/p,\ p \in \mathbb{N}$, along divisors D—to metrics with cone angles which are any positive rational multiple of 2π. It should be possible to study these within the framework of algebro-geometric stability as well.

The one dimensional local model transverse to D is as follows. In this paper, to get cone angle $2\pi/m$ along $x = 0$ we introduce extra local functions $x^{\frac{k}{m}}$ (by passing the local m-fold cover and working with orbifolds). Therefore, to produce cone angles $2\pi p$ it makes sense to discard the local functions $x, x^2, \ldots, x^{p-1}$ and use only $1, x^p, x^{p+1}, \ldots$. (We could use $1, x^p, x^{2p}, \ldots$, i.e., pass to a p-fold quotient instead of an m-fold cover, but this would be less general, producing metrics invariant under $\mathbb{Z}/p$ rather than those with this invariance on only the tangent space at $x = 0$.)

The map (x^p, x^{p+1}) from $\mathbb{C}$ to $\mathbb{C}^2$ is a set-theoretic injection with image $\{v^p = u^{p+1}\} \subset \mathbb{C}^2$. For very small x (so that x^{p+1} is negligible compared to x^p) it is very close to the p-fold cover $x \mapsto x^p$. More precisely, $\{v^p = u^{p+1}\}$ has p local branches (interchanged by monodromy) all tangent to the u-axis. Going once round $x = 0$ through angle 2π, we go p times round $u = 0$ through angle $2\pi p$. Therefore, if we restrict a Kähler metric from $\mathbb{C}^2$ to $\{v^p = u^{p+1}\}$ and pullback to $\mathbb{C}$, we get a smooth Kähler metric away from $x = 0$ which has cone angle $2\pi p$ at the origin. Similarly, the map $(x^p, x^{p+1}, \ldots, x^{p+k})$ to $\mathbb{C}^{k+1}$ has the same property.

To work globally, one has to pick a splitting $H^0(X, L^k) \cong H^0(D, L^k) \oplus H^0(X, L^k(-D))$ and discard those functions in the second summand which do not vanish to at least order p along D. That is, we take the obvious map

$$X \to \mathbb{P}\big(H^0(D, L^k)^* \oplus H^0(L^k(-pD))^*\big). \tag{1.2}$$

So instead of Kodaira embedding, we take an injection which fails to be an embedding in the normal directions to D just as in the local model above. (More generally, to get cone angles $2\pi p/q$ one should apply the above description to an orbifold with $\mathbb{Z}/q$ stabilisers along D and injections instead into weighted projective spaces.) One might hope for a relation between balanced injections of X (1.2) and cscK metrics with prescribed cone angles along D. We thank Dmitri Panov for discussions about these "ramifolds". He has also pointed out that it is too ambitious to expect the full theory for manifolds and orbifolds to carry over verbatim to this setting since cscK metrics with cone angles greater than 2π can be non-unique. We hope to return to this in future work.

Zero cone angles, cuspidal metrics, and stability of pairs. It would be fruitful to consider the limit of large orbifold order. By this we mean fixing the underlying space X and a divisor D, then putting $\mathbb{Z}/m$-stabilisers along D (as in Section 2.2) and considering $m \gg 0$. Then, formally at least, stability in the limit $m \to \infty$ is the same as stability of the underlying space where the numerical invariants are calculated with K_X replaced with $K_X + D$. This has been studied by Székelyhidi [**Szé07**] under the name of "relative stability" of the pair (X, D), which he conjectures to be linked via a Yau-Tian-Donaldson conjecture to the existence of complete "cuspidal" cscK metrics on $X \backslash D$. And indeed one can think of orbifold metrics with cone angle $2\pi/m$ along a divisor D as tending (as $m \to \infty$) to a complete metric on $X \backslash D$ (thanks to Simon Donaldson and Dmitri Panov for explaining this to us).

Pairs. In principle, this paper gives many other ways of forming moduli spaces of pairs (X, D). Initially, one should take X smooth projective and D a simple normal crossings divisor which is a union of smooth divisors D_i. Labelling the D_i by integers $m_i > 0$ satisfying the conditions of Section 2.2, we get a natural orbifold structure on X from which we recover D as the locus with nontrivial stabiliser group. Taking (for instance) the orbifold line bundle produced by tensoring a polarisation on X by $\mathcal{O}(\sum_i D_i/m_i)$ gives an orbifold line bundle which is ample in the sense of Section 2.5. Embedding in weighted projective space as in Section 2.6 and dividing the resulting Hilbert scheme by the reductive group described in Section 2.10 gives a natural GIT problem and notion of stability.

One should then analyse which orbischemes appear in the compactification that this produces (in this paper we mainly study only smooth orbifolds and their cscK metrics). It is quite possible that the resulting stable pairs will form a new interesting class. Studying moduli and stability of varieties using GIT fell out of favour, not least because the singularities it allows are not those that arise naturally in birational geometry, but interesting recent work of Odaka [**Oda**] suggests a relationship between the newer notion of K-stability (rather than Chow stability) and semi-log-canonical singularities. It is therefore natural to wonder if orbifold K-stability of (X, D) is related to some special types of singularity of pairs (perhaps this is most likely in the $m \to \infty$ limit of the last section). In fact, the recent work of Abramovich-Hassett [**AH09**] precisely studies moduli of varieties and pairs using orbischemes, birational geometry, and the minimal model programme (but not GIT).

An obvious special case is curves with weighted marked points, as studied by Hassett [**Has03**] and constructed using GIT by Swinarski [**Swi**]. It is possible that Swinarski's construction can be simplified by using embeddings in weighted projective space instead of projective space, and even that his (difficult) stability argument might follow from the existence of an orbifold cscK metric.

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2. Orbifold embeddings in weighted projective space

The proper way to write this paper would be using Deligne-Mumford stacks, but this would alienate much of its potential readership (as well as the two authors). Most of our DM stacks are smooth, so there is an elementary description in terms of orbifolds, and it therefore makes sense to use it. However, at points (such as when we consider the central fibre of a degeneration of orbifolds) DM stacks, or orbischemes, are unavoidable. At this point most of the results we need (such as the appropriate version of Riemann-Roch) are only available in the DM stacks literature. So we adopt the following policy. Where possible, we phrase things in elementary terms using only orbifolds. We state the results we need in this language, even when the only proofs available are in the DM stacks literature. Where we do something genuinely new, we give proofs using the orbifold language, even though they of course apply more generally to orbischemes or DM stacks.

2.1. Orbibasics. We sketch some of the basics of the theory of orbifolds and refer the reader to [**BG08**, **GK07**] for more details. An orbifold consists of a variety X (either an algebraic variety or, for us, an analytic space), with only finite quotient singularities, that is covered by orbifold charts of the form $U \to U/G \cong V \subset X$, where V is an open set in X, U is an open set in $\mathbb{C}^n$, and G is a finite group acting effectively on U. We also insist on a minimality condition, that the subgroups of G given by the stabilisers of points of U generate G (otherwise one should make both U and G smaller—it is important that we are using the analytic topology here).

The gluing condition on charts is the following. If $V' \subset V$ are open sets in X with charts $U'/G' \cong V'$ and $U/G \cong V$, then there should exist a monomorphism $G' \hookrightarrow G$ and an injection $U' \hookrightarrow U$ commuting with the given G'-action on U' and its action through $G' \hookrightarrow G$ on U.

Notice that these injections are *not* in general unique, so the charts do not have to satisfy a cocycle condition upstairs, though of course they do downstairs where the open sets V glue to give the variety X. That is, the orbifold charts need not glue since an orbifold need not be a global quotient by a finite group, though we will see in Remark 2.16 that they *are* global $\mathbb{C}^*$-quotients under a mild condition.

It follows from the gluing condition that the *order* of a point $x \in X$—the size of the stabiliser of any lift of x is any orbifold chart—is well defined. The *order of* X is defined to be the least common multiple of the order of its points (which is finite if X is compact). The *orbifold locus* is the set of points with nontrivial stabiliser group.

In this paper we will mostly consider only compact orbifolds with cyclic stabiliser groups, so that each G is always cyclic.

By an *embedding* $f \colon X \to Y$ of orbifolds we shall mean an embedding of the underlying spaces of X and Y such that for every $x \in X$ there exist orbifold charts $U' \to U'/G \ni x$ and $U \to U/G \ni f(x)$ such that f lifts to an equivariant embedding $U' \hookrightarrow U$. We say that the orbifold structure on X is pulled back from that on Y. Similarly, we get a notion of isomorphism of orbifolds.

Given a point in the orbifold locus with stabiliser group $\mathbb{Z}/m$, call its preimage in a chart p, with maximal ideal $\mathfrak{m}_p$. Split its cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ into weight spaces under the group action (and use the fact that the ring of formal power series about that point is $\oplus_i S^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$) to see that locally analytically there is a chart $U \to U/(\mathbb{Z}/m)$ of the form

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1^{a_1}, z_2^{a_2}, \ldots, z_k^{a_k}, z_{k+1}, \ldots, z_n), \tag{2.1}$$

for some integers a_i which divide m. We call this an *orbifold point* of type $\frac{1}{m}(\lambda_1, \ldots, \lambda_k)$ if $\zeta \in \mathbb{Z}/m$ acts[1] as

$$\zeta \cdot (z_1, \ldots, z_k) = (\zeta^{\lambda_1} z_1, \ldots, \zeta^{\lambda_k} z_k).$$

The general principle is that any local object (e.g. a tensor) on an orbifold is defined to be an invariant object on a local chart (rather than an object downstairs on the underlying space). So an *orbifold Kähler metric* is an invariant Kähler metric on U for each orbifold chart $U \to U/G$ which glues: its pullback under an injection $U' \hookrightarrow U$ of charts above is the corresponding metric on U'. Such a metric descends to give a Kähler metric on the underlying space X, but with possible singularities along the orbifold locus.

For instance, the standard orbifold Kähler metric on $\mathbb{C}/(\mathbb{Z}/m)$ is given by $\frac{i}{2} dz\, d\bar{z}$, where z is the coordinate on $\mathbb{C}$ upstairs and $x = z^m$ is the coordinate on the scheme theoretic quotient $\mathbb{C}$. Downstairs this takes the form $\frac{i}{2} m^{-2} |x|^{\frac{2}{m}-2} dx\, d\bar{x}$, which is a singular Kähler metric on $\mathbb{C}$. The circumference of the circle of radius r about the origin is easily calculated to be $2\pi r/m$, so the metric has cone angle $2\pi/m$ at the origin, whereas usual Kähler metrics have cone angle 2π. More generally, for any divisor D in the orbifold locus with stabiliser group $\mathbb{Z}/m$, orbifold Kähler metrics on X have cone angle $2\pi/m$ along D. So it is important for us to think of $\mathbb{C}/(\mathbb{Z}/m)$ as an orbifold, and not as its scheme theoretic quotient $\mathbb{C}$.

Even when the stabilisers have codimension two (so that the orbifold is determined by the underlying variety with quotient singularities, and one "can forget" the orbifold structure if only interested in the algebraic or analytic structure), an orbifold metric is very different from the usual notion of a Kähler metric over the singularities (i.e. one which is locally the restriction of a Kähler metric from an embedding in a smooth ambient space).

2.2. Codimension one stabilisers. The cyclic orbifolds which will most interest us will be those for which the orbifold locus has codimension one. These are the orbifolds whose local model (2.1) has coprime weights a_i.

Therefore, globally the orbifold is described by the pair (X, Δ), where

- X is a *smooth* variety,
- Δ is a $\mathbb{Q}$-divisor of the form $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i$,
- the D_i are distinct smooth irreducible effective divisors,

[1]Here λ_i is a multiple of m/a_i, of course. We are disobeying Miles Reid and picking the usual identification of $\mathbb{Z}/m$ with the mth roots of unity.

- $D = \sum D_i$ has normal crossings, and
- the m_i are positive integers such that m_i and m_j are coprime if D_i and D_j intersect.

Then the stabiliser group of points in the intersection of several components D_i will be the product of groups $\mathbb{Z}/m_i$, and this is cyclic by the coprimality assumption.

Here Δ is the ramification divisor of the orbifold charts; see Example 2.8 for the expression of this in terms of the orbifold canonical bundle.

Notice that above we are also claiming the converse: that given such a pair (X, Δ), it is an easy exercise to construct an orbifold with stabiliser groups $\mathbb{Z}/m_i$ along the D_i, and this is unique. This can be generalised to Deligne-Mumford stacks [**Cad07**]; we give a global construction in (2.15).

Orbifolds with codimension one stabilisers were called "not well-formed" in the days when "we were doing the wrong thing" (Miles Reid, Alghero 2006). Then orbifolds were studied as a means to produce schemes, so only the quotient was relevant. The orbifold locus could be removed, since the quotient is smooth. Hence in much of the literature (e.g. [**Dol82**]), the not well-formed case is unfortunately ignored.

More generally, *any* orbifold can be dealt with in much the same way: it can be described by a pair (X, Δ) just as above, but where X has at worst finite cyclic quotient singularities. This is the point of view taken by [**GK07**].

2.3. Weighted projective spaces. The standard source of examples of orbifolds is weighted projective spaces. A graded vector space $V = \oplus_i V^i$ is equivalent to a vector space V with a $\mathbb{C}^*$-action, acting on V^i with weight i. Throughout this paper, V will always be finite dimensional, with all weights strictly positive. We can therefore form the associated weighted projective space $\mathbb{P}(V) := (V \backslash \{0\})/\mathbb{C}^*$. This is sometimes denoted $\mathbb{P}(\lambda_1, \ldots, \lambda_n)$, where $n = \dim V$ and the λ_j are the weights (so the number of λ_j that equal i is $\dim V^i$).

Let x_j, $j = 1, \ldots, n$, be coordinates on V such that x_j has weight $-\lambda_j$. Then $\mathbb{P}(V)$ is covered by the orbifold charts

$$\begin{array}{c} \{x_j = 1\} \cong \mathbb{C}^{n-1} \\ \downarrow \\ \mathbb{P}(V). \end{array} \tag{2.2}$$

The λ_jth roots of unity $\mathbb{Z}/\lambda_j \subset \mathbb{C}^*$ act trivially on the x_j coordinate, preserving the above $\mathbb{C}^{n-1}$ slice. The vertical arrow is the quotient by this $\mathbb{Z}/\lambda_j$; the generator $\exp(2\pi i/\lambda_j) \in \mathbb{C}^*$ acting by

$$(x_i) \mapsto (\exp(2\pi i \lambda_i/\lambda_j) x_i). \tag{2.3}$$

The order of $\mathbb{P}(V)$ is the least common multiple of the weights λ_j. If the λ_j have highest common factor $\lambda > 1$, then $\mathbb{P}(V)$ has generic stabilisers:

every point is stabilised by the λth roots of unity, and we will usually assume that this is not the case, so $\mathbb{P}(V)$ inherits the structure of an orbifold with cyclic stabiliser groups.

The orbifold points of $\mathbb{P}(V)$ are as follows. Each vertex

$$P_i := [0, \dots, 1, \dots, 0]$$

is of type $\frac{1}{\lambda_i}(\lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_N)$. The general points along the line P_iP_j are orbifold points of type $\frac{1}{\mathrm{hcf}(\lambda_i,\lambda_j)}(\lambda_1, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \lambda_N)$, with similar orbifold types along higher dimensional strata.

Thus if for some j the λ_i, $i \neq j$, have highest common factor $\lambda > 1$, then $\mathbb{P}(V)$ is not well formed: it has a divisor of orbifold points with stabiliser group containing $\mathbb{Z}/\lambda$ along $x_j = 0$. Replacing the λ_i, $i \neq j$, by λ_i/λ gives a well formed weighted projective space [**Dol82**, **Fle00**] which is just the underlying variety without the divisor of orbifold points. As discussed in the last section, it is important for us *not* to mess with the orbifold structure in this way.

Similarly, the map $\mathbb{P}^{n-1} \to \mathbb{P}(\lambda_1, \dots, \lambda_n)$, $[x_1, \dots, x_n] \mapsto [x_1^{\lambda_1}, \dots, x_n^{\lambda_n}]$ exhibits the underlying variety of weighted projective space as a global finite quotient of ordinary projective space. Again this does *not* give the right orbifold structure of (2.2), so we do not use it.

2.4. Orbifold line bundles and $\mathbb{Q}$-divisors. Locally, an orbifold line bundle is simply an equivariant line bundle on an orbifold chart. This differs from an ordinary line bundle pulled back from downstairs which satisfies the property that the G-action on the line over any fixed point is trivial. In other words, (the pull back to an orbifold chart of) an ordinary line bundle has a local invariant trivialisation, which an orbifold line bundle may not. So in general orbifold line bundles are *not* locally trivial.

To define them globally, we need some notation. Suppose that V_i, V_j, V_k are open sets in X with charts $U_i/G_i \cong V_i$, etc. Then by the definition of an orbifold, the overlaps $V_{ij} := V_i \cap V_j$, etc. also have charts $U_{ij}/G_{ij} \cong V_{ij}$ and inclusions $U_{ij} \hookrightarrow U_i$, $G_{ij} \hookrightarrow G_i$, etc.

Given local equivariant line bundles L_i over each U_i, the gluing (or cocycle) condition to define a global orbifold line bundle is the following. Pulling back L_j and L_i to U_{ij} (via its inclusions in U_j, U_i respectively), there should be isomorphisms ϕ_{ij} from the former to the latter, intertwining the actions of G_{ij}. Pulling back further to U_{ijk}, we call this isomorphism $\phi_{ij} \in L_i \otimes L_j^*$ (suppressing the pullback maps for clarity). The cocycle condition is that over U_{ijk},

$$\phi_{ij}\phi_{jk}\phi_{ki} \ \in \ L_i \otimes L_j^* \otimes L_j \otimes L_k^* \otimes L_k \otimes L_i^*$$

should be precisely the identity element 1.

The standard example is the orbifold canonical bundle K_{orb}, which is defined to be K_U on the chart U (with the obvious G-action induced from that on U) and which glues automatically.

Example 2.4. Take X a smooth space with a smooth divisor D along which we put $\mathbb{Z}/m$ stabiliser group to form the orbifold $(X,(1-1/m)D)$. Then the orbifold line bundle $\mathcal{O}\big(-\frac{1}{m}D\big)$ is easily defined as the ideal sheaf of the reduced pullback of D to any chart. In this way it glues automatically.

Locally it has generator z, a local coordinate upstairs cutting out the reduced pullback of D. But this has weight one under the $\mathbb{Z}/m$-action; it is not an invariant section, so does *not* define a section of the orbifold line bundle downstairs (z^{km-1} times this generator does, for all $k \geq 0$). Therefore, this orbifold bundle is *not* locally trivial: it is locally the trivial line bundle with the weight one *nontrivial* $\mathbb{Z}/m$-action.

Away from D, the section which is z^{-1} times by this weight one generator is both regular and invariant, so can be glued to the trivial line bundle. In this way one can give an equivalent definition of $\mathcal{O}\big(-\frac{1}{m}D\big)$ via transition functions, much as in the manifold case.

Taking tensor powers, we can form $\mathcal{O}\big(\frac{n}{m}D\big)$ for any integer n. This is an ordinary line bundle only for n/m an integer. The inclusion $\mathcal{O}(-\frac{1}{m}D) \hookrightarrow \mathcal{O}_X$ defines a canonical section $s_{D/m}$ of $\mathcal{O}(\frac{1}{m}D)$ which in the orbifold chart above looks like z vanishing on D.

The pushdown to the underlying manifold X of $\mathcal{O}\big(\frac{n}{m}D\big)$ is the ordinary line bundle given by the round down

$$\mathcal{O}\left(\left\lfloor\frac{n}{m}\right\rfloor D\right). \tag{2.5}$$

That is to say that the (invariant) sections of $\mathcal{O}\big(\frac{n}{m}D\big)$ are of the form $s_{D/m}^{\frac{n}{m}-\lfloor\frac{n}{m}\rfloor}t$, where t is any section of the ordinary line bundle $\mathcal{O}\big(\lfloor\frac{n}{m}\rfloor\big)$ on X.

Since tensor product does not commute with round down, we lose information by pushing down to X: the natural consequence of orbifold line bundles not being locally trivial.

More generally, on any orbifold given by a pair (X,Δ) as in Section 2.2, orbifold line bundles and their sections correspond to $\mathbb{Q}$-divisors such that the denominator of the coefficient of D_i must divide m_i, and any irreducible divisor D not in the list of D_i must have integral coefficients. The space of global sections of the orbifold line bundle is the space of sections of the round down. Care must be taken, however; for instance, if D_1 and D_2 have $\mathbb{Z}/m$-stabilisers along them and $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$, this certainly does *not* imply that $\mathcal{O}(D_1/m) \cong \mathcal{O}(D_2/m)$.

The *tautological line bundle* $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over the weighted projective space $\mathbb{P}(V)$ is the orbi-line bundle over $\mathbb{P}(V)$ with fibre over $[v]$ the union of the orbit $\mathbb{C}^*.v \subset V$ and $0 \in V$. (Any two elements in a fibre can be written $w_i = t_i.v$ for $t_i \in \mathbb{C},\ i = 1, 2$, so we can define the linear structure by $aw_1 + bw_2 := (at_1 + bt_2).v$. Ordinarily, this is not the linear structure on V and the fibre $\mathcal{O}_{[v]}(-1) \subset V$ is not a linear subspace.) Over the orbi-chart (2.2), this is the trivial line bundle $\mathbb{C}^{n-1} \times \mathbb{C}$ with the weight one $\mathbb{Z}/\lambda_j$-action on the line $\mathbb{C}$ times by its action (2.3) on $\mathbb{C}^{n-1}$. In other words, the map

$$\begin{aligned} \mathbb{C}^{n-1} \times \mathbb{C} &\to \mathbb{C}^n \\ (x_1, \dots, \widehat{x}_j, \dots, x_n, t) &\mapsto (t^{\lambda_1} x_1, \dots, t^{\lambda_j}, \dots, t^{\lambda_n} x_n) \end{aligned} \tag{2.6}$$

becomes $(\mathbb{Z}/\lambda_j)$-equivariant when we use the action (2.3) on $\mathbb{C}^{n-1}$, the standard weight-one action on $\mathbb{C}$, and the original weighted $\mathbb{C}^*$-action on $\mathbb{C}^n$. The map (2.6) is defined in order to take the trivialisation 1 of $\mathbb{C}$ to the tautological trivialisation of the pullback of the orbit to the chart (2.2) (a point of the chart (2.2) is a point of its own orbit and so trivialises it).

Note that Dolgachev [**Dol82**] uses the same notation $\mathcal{O}_{\mathbb{P}(V)}(-1)$ to denote the push forward of our $\mathcal{O}_{\mathbb{P}(V)}(-1)$ to the underlying space, thus rounding down fractional divisors. Therefore, $\mathcal{O}_{\mathbb{P}(V)}(a+b) = \mathcal{O}_{\mathbb{P}(V)}(a) \otimes \mathcal{O}_{\mathbb{P}(V)}(b)$ does not hold for his sheaves, but is true almost by definition for our orbifold line bundles.

As a trivial example, consider $\mathcal{O}(k)$ over the weighted projective line $\mathbb{P}(1, m)$. The first coordinate x on $\mathbb{C}^2$ has weight one, so restricts to a linear functional on orbits (the fibres of $\mathcal{O}(-1)$). It therefore defines a section of $\mathcal{O}(1)$ which vanishes at the orbifold point $x = 0$. Since x is the coordinate upstairs in the chart (2.2) and x^m the coordinate downstairs, this is $\frac{1}{m}$ times by a real manifold point. The coordinate y has weight m on the fibres of $\mathcal{O}(-1)$, so defines a section of $\mathcal{O}(m)$ which vanishes at the manifold point $y = 0$.

The underlying variety is the projective space on the degree m variables x^m, y, i.e. it is $\mathbb{P}^1$ with reduced points 0 and ∞ where these two variables vanish. Thus

$$\mathcal{O}_{\mathbb{P}(1,m)}(k) = \mathcal{O}\left(\frac{k}{m}(0)\right) = \mathcal{O}\left(\left\lfloor \frac{k}{m} \right\rfloor (\infty) + \left(\frac{k}{m} - \left\lfloor \frac{k}{m} \right\rfloor\right)(0)\right).$$

Similarly, on $\mathbb{P}(a, b)$ with $pa + qb = 1$, the underlying variety is the usual Proj of the graded ring on the degree ab generators x^b and y^a. Denote by 0 and ∞ the zeros of x^b and y^a, respectively. Then it is a nice exercise to check that the orbifold line bundle $\mathcal{O}_{\mathbb{P}(a,b)}(1)$ is isomorphic to

$$\mathcal{O}\left(\frac{p}{b}(0) + \frac{q}{a}(\infty)\right),$$

of degree $\frac{1}{ab}$.

2.5. Orbifold polarisations. To define orbifold polarisations, we need the right notion of ampleness or positivity. For manifolds (or schemes), this is engineered to ensure that the global sections of L generate the local ring of functions at each point. For orbifolds, this requires also a *local* condition on an orbifold line bundle L, as we explain using the simplest example. Consider the orbifold $\mathbb{C}/(\mathbb{Z}/2)$ with local coordinate z on $\mathbb{C}$ acted on by $\mathbb{Z}/2$ via $z \mapsto -z$. Then $x = z^2$ is a local coordinate on the quotient thought of as a manifold. Any line bundle pulled back from the quotient (i.e. which has trivial $\mathbb{Z}/2$-action upstairs when considered as a trivial line bundle there) has invariant sections $\mathbb{C}[x] = \mathbb{C}[z^2]$. Therefore, it sees the quotient only as a manifold, missing the extra functions of $\sqrt{x} = z$ that the orbifold sees. So we do not think of it as locally ample: if we tried to embed using its sectionsm we would "contract" the stabilisers, leaving us with the underlying manifold.

Conversely, the trivial line bundle upstairs with nontrivial $\mathbb{Z}/2$-action (acting as -1 on the trivialisation) has invariant sections $\sqrt{x}\,\mathbb{C}[x] = z\mathbb{C}[z^2]$. Its square has trivial $\mathbb{Z}/2$-action and has sections $\mathbb{C}[x] = \mathbb{C}[z^2]$ as above. Therefore, its sections and those of its powers generate the entire ring of functions $\mathbb{C}[\sqrt{x}] = \mathbb{C}[z]$ upstairs, and see the full orbifold structure.

Definition 2.7. An orbifold line bundle L over a cyclic orbifold X is *locally ample* if in an orbifold chart around $x \in X$, the stabiliser group acts faithfully on the line L_x. We say L is *orbi-ample* if it is both locally ample and globally positive. (By globally positive here we mean $L^{\mathrm{ord}(X)}$ is ample in the usual sense when thought of as a line bundle on the underlying space of X; from the Kodaira-Baily embedding theorem [**Bai57**] one can equivalently ask that L admits a hermitian metric with positive curvature.)

By a *polarised* orbifold we mean a pair (X, L) where L is an orbi-ample line bundle on X.

Note that ordinary line bundles on the underlying space are never ample on genuine orbifolds. Some care needs to be taken when applying the usual theory to orbi-ample line bundles. For instance, it is not necessarily the case that the tensor product of locally ample line bundles remain locally ample, but if L is locally ample, then so is L^{-1}. One can easily check that L is orbi-ample if and only if L^k is ample for one (or all) $k > 0$ coprime to $\mathrm{ord}(X)$.

Example 2.8. The orbifold canonical bundle K_{orb} is locally ample along divisors of orbifold points, but not necessarily at codimension two orbifold points. For instance, the quotient of $\mathbb{C}^2$ by the scalar action of ± 1 has trivial canonical bundle, so local ampleness is not determined in codimension one.

Suppose that X is smooth but with a divisor D with stabiliser group $\mathbb{Z}/m$. Locally write D as $x = 0$ and pick a chart with coordinate z such that $z^m = x$. Then the identity $dx = mz^{m-1}dz = mx^{1-\frac{1}{m}}dz$ shows that X has orbifold canonical bundle

$$K_{\mathrm{orb}} \;=\; K_X + \left(1 - \frac{1}{m}\right) D \;=\; K_X + \Delta,$$

where K_X is the canonical divisor of the variety underlying X. More generally, if the orbifold locus is a union of divisors D_i with stabiliser groups $\mathbb{Z}/m_i$, then $K_{\mathrm{orb}} = K_X + \Delta$, where $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i$ as in Section 2.2.

Example 2.9. The hyperplane bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on any weighted projective space $\mathbb{P}(V)$ is locally ample, and it is actually orbi-ample since some power is ample [**Dol82**, proposition 1.3.3] (we shall also show below that it admits a hermitian metric with positive curvature). The pullback of an orbi-ample bundle along an orbifold embedding is also orbi-ample, and thus any orbifold embedded in weighted projective space admits an orbi-ample line bundle. If (X, Δ) is an orbifold, X is smooth, and H is an ample divisor on X, then the orbifold bundle $H + \Delta$ of Section 2.2 is orbi-ample if and only if $H + \Delta$ is an ample $\mathbb{Q}$-divisor on X.

2.6. Orbifold Kodaira embedding. Fix a polarised orbifold (X, L) and $k \gg 0$. Let i run throughout a fixed indexing set $0, 1, \ldots, M$, where $M \geq \mathrm{ord}(X)$, and let V be the graded vector space

$$V = \bigoplus_i V^{k+i} := \bigoplus_i H^0(L^{k+i})^*.$$

We give the ith summand weight $k+i$. Map X to the weighted projective space $\mathbb{P}(V)$ by

$$\phi_k(x) := \left[\, \oplus_i \mathrm{ev}_x^{k+i} \,\right]. \tag{2.10}$$

Here we fix a trivialisation of L_x on an orbifold chart, inducing trivialisations of all powers L_x^{k+i}, and then ev_x^{k+i} is the element of $H^0(L^{k+i})^*$ which takes a section $s \in H^0(L^{k+i})$ to $s(x) \in L_x^{k+i} \cong \mathbb{C}$. The weights are chosen so that a change in trivialisation induces a change in $\oplus_i \mathrm{ev}_x^{k+i}$ that differs only by the action of $\mathbb{C}^*$ on V.

Picking a basis s_j^{k+i} for $H^0(L^{k+i})$, then, the map can be described by

$$\phi_k(x) = \left[(s_j^{k+i}(x))_{i,j}\right].$$

This map is well defined at all points x for which there exists a global section of some L^{k+i} not vanishing at x.

Proposition 2.11. *If (X, L) is a polarised orbifold, then for $k \gg 0$ the map* (2.10) *is an embedding of orbifolds (i.e. the orbifold structure* *on X is pulled back from that on the weighted projective space $\mathbb{P}(V)$) and*

$$\phi_k^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong L.$$

Proof. Fix $x \in X$. It has stabiliser group $\mathbb{Z}/m$ for some $m \geq 1$, and a local orbifold chart $U/(\mathbb{Z}/m)$. Let $y \in U$ (with maximal ideal $\mathfrak{m}_y$) map to x, and decompose $\mathfrak{m}_y/\mathfrak{m}_y^2 = \oplus_l V^l$ into weight spaces. Since we have chosen the indexing set for i to range over at least a full period of length m, at least one of the L_y^{k+i} has weight 0 and, for each l, there is at least one i_l in the indexing set such that $L_y^{k+i} \otimes V^{i_l}$ has weight 0.

Therefore, each of these $\mathbb{Z}/m$-modules has invariant local generators, defining local sections of the appropriate power of L on X. For $k \gg 0$ these extend to global sections, by ampleness. (The pushdowns of the powers of L from the orbifold to the underlying scheme give sheaves which all come from a finite collection of sheaves tensored by a line bundle. For $k \gg 0$ this line bundle becomes very positive, and so eventually has no cohomology. This value of k can be chosen uniformly for all y by cohomology vanishing for a *bounded* family of sheaves on a scheme.)

Therefore, trivialising L locally, the sections generate $\mathcal{O}_y$ and $\mathfrak{m}_y/\mathfrak{m}_y^2$, so the pullback of the local functions on $\mathbb{P}(V)$ (the polynomials in $(x_i)_{i\neq j}$ on the orbifold chart (2.2)) generate the local functions on U. It follows that the map is an embedding for large k.

Invariantly, the map (2.10) can be described as follows. Any lift $\tilde{x} \in L_x^{-1}$ of x is a linear functional on L_x. Similarly, $\tilde{x}^{\otimes(k+1)}$ is a linear functional on L_x^{k+i}. Composed with the evaluation map, $\mathrm{ev}_x^{k+i}\colon H^0(L^{k+i}) \to L_x^{k+i}$ gives

$$\tilde{x}^{\otimes(k+1)} \circ \mathrm{ev}_x^{k+i} : \ H^0(L^{k+i}) \to \mathbb{C}.$$

Therefore,

$$\oplus_i\big(\tilde{x}^{\otimes(k+1)} \circ \mathrm{ev}_x^{k+i}\big) \ \in \ \bigoplus_i H^0(L^{k+i})^* = V$$

is a well defined point, with no $\mathbb{C}^*$-scaling ambiguities or choices. In other words, (2.10) lifts to a natural $\mathbb{C}^*$-equivariant embedding of the orbi-line

$$L_x^{-1} \ \hookrightarrow \ \bigoplus_i H^0(L^{k+i})^* \tag{2.12}$$

onto the $\mathbb{C}^*$-orbit over the point (2.10). This makes it clear that under this weighted Kodaira embedding, the pullback of the $\mathcal{O}_{\mathbb{P}(V)}(-1)$ orbifold line bundle over $\mathbb{P}(V)$ is L^{-1}. q.e.d.

Remark 2.13. That $\phi_k^*\mathcal{O}_{\mathbb{P}(V)}(-1) = L^{-1}$, even though the embedding uses the sections of $L^k, \dots, L^{k+M}$ and not those of L, follows from the fact that we give $H^0(L^{k+i})^*$ weight $k+i$. This might come as a surprise and appear to contradict what we know about Kodaira embedding for manifolds. For instance, suppose we embed the *manifold* $\mathbb{P}^1$ using $\mathcal{O}(2)$. Under the normal Kodaira embedding, we get a conic in $\mathbb{P}^2 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(2)^*)$ such that the pullback of $\mathcal{O}_{\mathbb{P}^2}(-1)$ is $\mathcal{O}_{\mathbb{P}^1}(-2)$.

However, from the above orbifold perspective, this is *not* an embedding of $\mathbb{P}^1$, but of the orbifold $\mathbb{P}^1/(\mathbb{Z}/2)$, where the $\mathbb{Z}/2$-action is trivial. We see this as follows. At the level of line bundles (2.12), it is an embedding of $\mathcal{O}_{\mathbb{P}^1}(-1)\big/(\mathbb{Z}/2)$ into $\mathcal{O}_{\mathbb{P}^2}(-2)$, where the $\mathbb{Z}/2$-action is by -1 on each fibre. As a manifold this quotient is indeed $\mathcal{O}_{\mathbb{P}^1}(-2)$, but as an orbifold it is instead an orbifold line bundle over the orbifold $\mathbb{P}^1/(\mathbb{Z}/2)$, where the $\mathbb{Z}/2$-action is trivial.

Remark 2.14. When we began this project we considered a different, perhaps more natural, weighted projective embedding. We embedded in the same way in

$$\mathbb{P}\Big(\bigoplus_i H^0(L^{ik})^*\Big),$$

where we give $H^0(L^{ik})^*$ weight i (*not* ik). (Notice how this cures the problem with Veronese embeddings described in Remark 2.13 above.) This can also be shown to pull back the orbifold structure of weighted projective space to that of X when L is ample, and to pull $\mathcal{O}(1)$ back to L^k. However, the corresponding Bergman kernel turns out not to be relevant to constant scalar curvature orbifold Kähler metrics. We learnt about the related alternative embedding (2.10) from Dan Abramovich; see [**AH09**]. The idea of using weighted projective embeddings certainly goes back further to Miles Reid; see for instance [**Rei**].

2.7. OrbiProj. It is similarly simple to write down an orbifold version of the Proj construction, using the whole graded ring $\oplus_k H^0(L^k)$ at once. Given a finitely generated graded ring $R = \oplus_{k\geq 0} R_k$ (not necessarily generated in degree 1!), we can form the scheme Proj R in the usual way [**Har77**, proposition II.2.5]. However, this loses information (for instance, we could throw away all the graded pieces except the R_{nk}, $k \gg 0$, and get the same result).

We endow Proj R with an orbischeme structure by describing the orbischeme charts. Fix a homogeneous element $r \in R_+$ and consider the Zariski-open subset Spec $R_{(r)} = (\text{Proj } R)\backslash\{r = 0\}$. (As usual, $R_{(r)}$ is the degree zero part of the localised ring $r^{-1}R$.) Then

$$\text{Spec } \frac{R}{(r-1)} \longrightarrow \text{Spec } R_{(r)}$$

is our orbi-chart. Here $R/(r-1)$ is the quotient of R (thought of as a ring and forgetting the grading) by the ideal $(r-1)$. The map from $R_{(r)}$ sets r to 1.

More simply but less invariantly, pick homogeneous generators and relations for the graded ring R. Then Proj R is embedded in the weighted projective space on the generators, cut out by the equations defined by the relations.

Given a projective scheme (X, L) and a Cartier divisor $D \subset X$, this gives a very direct way to produce Cadman's rth root orbischeme $\big(X, \big(1-\frac{1}{r}\big)D\big)$ [**Cad07**]. This has underlying scheme X but with stabilisers $\mathbb{Z}/r$ along D, and in the above notation it is simply

$$\left(X, \left(1-\frac{1}{r}\right)D\right) \;=\; \mathrm{Proj}\;\bigoplus_{k\geq 0} H^0\Big(X, \mathcal{O}\left(\left\lfloor\frac{k}{r}\right\rfloor D\right)\otimes L^k\Big). \tag{2.15}$$

The hyperplane line bundle $\mathcal{O}(1)\otimes L^{-1}$ on this Proj is $\mathcal{O}\big(\frac{1}{r}D\big)$. Picking generators and relations for the above graded ring, we see the rth root orbischeme very concretely, cut out by equations in weighted projective space.

Remark 2.16. Although orbifolds need not be global quotients by *finite* groups, we see that polarised orbifolds *are* global quotients of varieties by $\mathbb{C}^*$-actions. In terms of the weighted Kodaira embedding of Proposition 2.11, we take the total space of L^{-1} over X, minus the zero section, and divide by the natural $\mathbb{C}^*$-action on the fibres. Equivalently, we express the orbifold Proj of the graded ring R as the quotient of $\mathrm{Spec}\,(R)\backslash\{0\}$ by the action of $\mathbb{C}^*$ induced by the grading.

2.8. Orbifold Riemann-Roch. Suppose that L is an orbifold polarisation on X. We will need the asymptotics of $h^0(L^k)$ for $k \gg 0$. These follow from Kawazaki's orbifold Riemann-Roch theorem [**Kaw79**], or Toën's for Deligne-Mumford stacks [**Toë99**], and some elementary algebra (see for example [**Rei87**] in the well-formed case). Alternatively, they follow from the weighted Bergman kernel expansion (see [**RT11**, corollary 1.12]), or by embedding in weighted projective space and taking hyperplane sections in the usual way. The result is that

$$h^0(L^k) \;=\; \frac{\int_X c_1(L)^n}{n!}\,k^n \;-\; \frac{\int_X c_1(L)^{n-1}.c_1(K_{\mathrm{orb}})}{2(n-1)!}\,k^{n-1} \;+\; \tilde{o}(k^{n-1}). \tag{2.17}$$

Here and in what follows, we define $\tilde{o}(k^{n-1})$ to mean a sum of functions of k that can be written as $r(k)\delta(k) + O(k^{n-2})$, where $r(k)$ is a polynomial of degree $n-1$ and $\delta(k)$ is periodic in k with period $m = \mathrm{ord}(X)$ and *average zero*:

$$\delta(k) = \delta(k+m), \qquad \sum_{u=1}^{m}\delta(u) = 0.$$

Therefore, the average of $\tilde{o}(k^{n-1})$ over a period is in fact $O(k^{n-2})$, and we think of it as being a lower order term than the two leading ones of (2.17).

Here we are also using integration of Chern-Weil forms on orbifolds (or intersection theory on DM stacks). Of course, integration works for orbifolds just as it does for manifolds; it is defined in local charts, but then the local integral is divided by the size of the group. It also extends easily to orbischemes, just as usual integration works on schemes once we weight by local multiplicities.

We give a simple example which nonetheless illustrates a number of the issues we have been considering.

Example 2.18. Let $\mathbb{Z}/m$ act on ordinary $\mathbb{P}^1$ and the tautological line bundle over it by $\lambda\cdot[x,y]=[\lambda x,y]$. Then the quotient X is naturally an orbifold with $\mathbb{Z}/m$ stabilisers at the two points $x=0$ and $y=0$. And the quotient of $\mathcal{O}(-1)$ is naturally an orbifold line bundle L_X^{-1} over X.

However, L_X is not locally ample at $x=0$, since the above action is trivial on the fibre over $x=0$. So we "contract" the orbifold structure of X at this point to produce another orbifold Y by ignoring the stabiliser group at $x=0$ and thinking of it locally as a manifold. Only the orbifold point $y=0$ survives, and L_X automatically descends to an ample orbifold line bundle L_Y on Y, to which orbifold Riemann-Roch (2.17) should therefore apply.

The sections of L_Y^k (or those of L_X^k; they are the same) are the invariant sections of $\mathcal{O}_{\mathbb{P}^1}(k)$, which has basis $y^k,\, y^{k-m}x^m,\dots,\, y^{k-m\lfloor\frac{k}{m}\rfloor}x^{m\lfloor\frac{k}{m}\rfloor}$. In particular, $Y=\mathbb{P}\langle x^m,y\rangle=\mathbb{P}(m,1)$ and $h^0(L^k)=\lfloor\frac{k}{m}\rfloor+1$.

Writing this as $\frac{k}{m}+1-\frac{m-1}{2m}+\delta(k)$, where δ is periodic with average zero, we find

$$h^0(L^k)=\frac{k}{m}-\frac{1}{2}\left(-2+\left(1-\frac{1}{m}\right)\right)+\delta(k)=k\deg L-\frac{1}{2}\deg K_{\mathrm{orb}}+\delta(k).$$

Hence, as expected, the single $\mathbb{Z}/m$-orbifold point of Y adds $1-1/m$ to the degree of K_{orb}, and the other orbifold point of X does not show up.

2.9. Equivariant case. Fix a polarised orbifold (X,L) as above, but now with a $\mathbb{C}^*$-action on L linearising one on X. We need a similar expansion for the weight of a $\mathbb{C}^*$-action on $H^0(L^k)$. Instead of using the full equivariant Riemann-Roch theorem, we follow Donaldson in deducing what we need by using $\mathbb{P}^1$ to approximate $B\mathbb{C}^*=\mathbb{P}^\infty$ and applying the above orbifold Riemann-Roch asymptotics to the total space of the associated bundle over $\mathbb{P}^1$.

So let $\mathcal{O}_{\mathbb{P}^1}(1)^*$ denote the principal $\mathbb{C}^*$-bundle over $\mathbb{P}^1$ given by the complement of the zero-section in $\mathcal{O}(1)$. Form the associated (X,L)-bundle

$$(\mathcal{X},\mathcal{L}):=\mathcal{O}_{\mathbb{P}^1}(1)^*\times_{\mathbb{C}^*}(X,L).$$

Let $\pi\colon\mathcal{X}\to\mathbb{P}^1$ denote the projection. Then it is clear that $\pi_*\mathcal{L}^k$ is the associated bundle of the $\mathbb{C}^*$-representation $H^0(X,L^k)$. Splitting the latter into one dimensional weight spaces splits the former into line bundles. A line with weight i becomes the line bundle $\mathcal{O}(i)$. It follows that the *total weight* (i.e. the weight of the induced action on the determinant) of the $\mathbb{C}^*$-action on $H^0(X, L^k)$ is the first Chern class of $\pi_*\mathcal{L}^k$. Therefore,

$$w(H^0(X, L^k)) = \chi(\mathbb{P}^1, \pi_*\mathcal{L}^k) - \operatorname{rank}(\pi_*\mathcal{L}^k) = \chi(\mathcal{X}, \mathcal{L}^k) - \chi(X, L^k).$$

In particular, orbifold Riemann-Roch on $\mathcal{X}$ and X show that this has an expansion $b_0 k^{n+1} + b_1 k^n + \tilde{o}(k^n)$, where $b_0 = \int_{\mathcal{X}} \frac{c_1(\mathcal{L})^n}{(n+1)!}$.

We can express b_0 as an integral over X as follows. Take a hermitian metric h on L which is invariant under the action of $S^1 \subset \mathbb{C}^*$ and which has positive curvature $2\pi\omega$. Let σ denote the resulting connection 1-form on the principal S^1-bundle given by the unit sphere bundle $S(L)$ of L.

Differentiating the S^1-action gives a vector field v on $S(L)$. Then $\sigma(v)$ is the pullback of a function H on X. With respect to the symplectic form ω, this H is a hamiltonian for the S^1-action on X.

Write $(\mathcal{X}, \mathcal{L})$ as the associated bundle to the S^1-principal bundle $S(\mathcal{O}_{\mathbb{P}^1}(1))$ as follows;

$$(\mathcal{X}, \mathcal{L}) = S(\mathcal{O}_{\mathbb{P}^1}(1)) \times_{S^1} (X, L).$$

The Fubini-Study connection on $\mathcal{O}_{\mathbb{P}^1}(1)$ and the connection σ on L induce natural connections on $\mathcal{X} \to \mathbb{P}^1$ and on $\mathcal{L} \to \mathcal{X}$. In [**Don05**, section 5.1], Donaldson shows that the latter has curvature $H\omega_{FS} + \omega$. (Here ω_{FS} is pulled back from $\mathbb{P}^1$, and we think of ω as a form on $\mathcal{X}$ by using its natural connection over $\mathbb{P}^1$ to split its tangent bundle as $T\mathcal{X} = T\mathbb{P}^1 \oplus TX$.) Therefore, b_0 equals

$$\frac{1}{(n+1)!}\int_{\mathcal{X}} (H\omega_{FS} + \omega)^{n+1} = \frac{n+1}{(n+1)!}\int_{\mathbb{P}^1} \omega_{FS} \int_X H\omega^n = \int_X H\frac{\omega^n}{n!}.$$

This proves

Proposition 2.19. *The total weight of the $\mathbb{C}^*$-action on $H^0(L^k)$ is*

$$w(H^0(X, L^k)) = b_0 k^{n+1} + b_1 k^n + \tilde{o}(k^n), \quad \textit{where} \;\; b_0 = \int_X H\,\frac{\omega^n}{n!}.$$

We will apply this to weighted projective space $X = \mathbb{P}(V)$ and also to its sub-orbischemes, where the integral on the right must then take into account scheme-theoretic multiplicities and the possibility of generic stabiliser (so if an irreducible component of X has generic stabiliser $\mathbb{Z}/m$, then the integral over it is $\frac{1}{m}$ times the integral over the underlying scheme).

Finally, we remark that working with $\mathcal{O}_{\mathbb{P}^2}(1)$ in place of $\mathcal{O}_{\mathbb{P}^1}(1)$ replaces the trace of the infinitesimal action on $H^0(X, L^k)$ (i.e. the total weight) by the trace of the *square* of the infinitesimal action on $H^0(X, L^k)$, proving it equals

$$c_0 k^{n+2} + O(k^{n+1}) \quad \text{where } c_0 = \int_X H^2 \frac{\omega^n}{n!}\,. \tag{2.20}$$

2.10. Reducing to the reductive quotient. To form a moduli space of polarised varieties (X, L), one first embeds X in projective space $\mathbb{P}$ with a high power of L, thus identifying X with a point of the relevant Hilbert scheme of subvarieties of $\mathbb{P}$. It is easy to see that two points of the Hilbert scheme correspond to abstractly isomorphic polarised varieties if and only if they differ by an automorphism of $\mathbb{P}$. Therefore, a moduli space of varieties can be formed by taking the GIT quotient of the Hilbert scheme by the special linear group. (Different choices of linearisations of the action give different notions of stability of varieties.)

By Proposition 2.11 we can now mimic this for polarised orbifolds, first embedding in a *weighted* projective space $\mathbb{P}$. The Hilbert scheme of sub-orbischemes of $\mathbb{P}$ has been constructed in [**OS03**]. Therefore, we are left with the problem of quotienting this by the action of $\mathrm{Aut}(\mathbb{P})$.

At first sight, this seems difficult because $\mathrm{Aut}(\mathbb{P})$ is not reductive. Classical GIT works only for reductive groups (though a remarkable amount of the theory has now been pushed through in the nonreductive case [**DK07**]).

As a trivial example, consider $\mathbb{P}(1, 2)$ embedded by the identity map in itself. The automorphisms contain a nonreductive piece $\mathbb{C}$ in which $t \in \mathbb{C}$ acts by

$$[x, y] \ \mapsto \ [x, y + tx^2]. \tag{2.21}$$

However, this arises because $\mathbb{P}(1, 2)$ has not been Kodaira embedded as described in Section 2.6. Using *all* sections of $H^0(\mathcal{O}(1)) = \langle x\rangle$ and $H^0(\mathcal{O}(2)) = \langle x^2, y\rangle$ (not just x and y), we embed instead via

$$\mathbb{P}(1, 2) \hookrightarrow \mathbb{P}(1, 2, 2), \qquad [x, y] \mapsto [x, x^2, y].$$

Then the nonreductive $\mathbb{C}$ lies in a reductive subgroup of $\mathrm{Aut}(\mathbb{P}(1, 2, 2))$. Namely, (2.21) can be realised as the restriction to $\mathbb{P}(1, 2)$ of the automorphism

$$[A, B, C] \mapsto [A, B, C + tB]$$

lying in the reductive subgroup $SL(H^0(\mathcal{O}(2))) \subset \mathrm{Aut}\,\mathbb{P}(1, 2, 2)$. Of course, it can also be seen as the restriction of $[A, B, C] \mapsto [A, B, C + tA^2]$, another nonreductive $\mathbb{C}$ subgroup, but the point is that our embedding has a stabiliser in $\mathrm{Aut}(\mathbb{P}(1, 2, 2))$, and this causes the two copies of $\mathbb{C}$ restrict to the same action.

Having seen an example, the general case is actually simpler. Given a polarised variety (X, L), pick an isomorphism from $H^0(X, L^{k+i})$ to a fixed vector space V^{k+i}. Then from Section 2.6 we get an embedding of X into $\mathbb{P}(\oplus_i (V^{i+k})^*)$. This embedding is *normal*—the restriction map

$H^0(\mathcal{O}_{\mathbb{P}}(k+i)) \to H^0(\mathcal{O}_X(k+i))$ is an isomorphism by construction. The next result says that the resulting point of the Hilbert scheme of $\mathbb{P}$ is unique up to the action of the *reductive* group $\prod_i GL(V^{k+i})$.

Proposition 2.22. *Two normally embedded orbifolds X_1, X_2 sitting in $\mathbb{P}(\oplus_i (V^{i+k})^*)$ are abstractly isomorphic polarised varieties if and only if there is $g \in \prod_i GL(V^{k+i})$ such that $g.X_1 = X_2$.*

Proof. If the $(X_j, \mathcal{O}_{X_j}(1)))$ are abstractly isomorphic, then their spaces of sections $H^0(\mathcal{O}_{X_j}(k+i))$ are isomorphic vector spaces. Under this isomorphism, the two identifications $H^0(\mathcal{O}_{X_j}(k+i)) \cong V^{i+k},\ j=1,2,$ therefore differ by an element $g_{k+i} \in GL(V^{k+i})$. Then $g := \oplus_i g_{k+i}$ takes $X_1 \subset \mathbb{P}$ to X_2.

The converse is of course trivial, needing only the fact that the action of $\prod_i GL(V^{k+i})$ preserves the polarisation $\mathcal{O}_{\mathbb{P}}(1)$. q.e.d.

Therefore, one can set up a GIT problem to form moduli of orbifolds, just as Mumford did for varieties.

Firstly, one needs Matsusaka's big theorem for orbifolds, to ensure that for a fixed $k \gg 0$, uniform over all smooth polarised orbifolds of the same topological type, the orbifold line bundles L^{k+i} have the number of sections predicted by orbifold Riemann-Roch. This follows by pushing down to the underlying variety, which has only quotient, and so rational, singularities, to which [**Mat86**, theorem 2.4] applies.

We can thus embed them all in the same weighted projective space. Then one can remove those suborbifolds of weighted projective space whose embedding is non-normal, since they are easily seen to be unstable for the action of $\prod_i GL(V^{k+i})$. Thus by the above result, orbits on the Hilbert scheme really correspond to isomorphism classes of polarised orbifolds. Finally, one should compactify with orbischemes (or Deligne-Mumford stacks) to get proper moduli spaces of stable objects. We do not pursue this here, as only smooth orbifolds and their stability are relevant to cscK metrics, but many of the foundations are worked out in [**AH09**]. (Their point of view is slightly different from ours—their notion of stability is related to the minimal model programme rather than GIT, and they form moduli using the machinery of stacks.)

3. Metrics and balanced orbifolds

Our next point of business is to generalise the Fubini-Study metric to weighted projective space. Anticipating the application we have in mind, fix some $k \geq 0$ and let $V = \oplus_{i=1}^M V^{k+i}$ be a finite dimensional graded vector space. By a metric $|\cdot|_V$ on a V we will mean a hermitian metric which makes the vector spaces V^p and V^q orthogonal for $p \neq q$. Thus a metric on V is simply given by a hermitian metric $|\cdot|_{V^p}$ on each

V^p. By a graded orthonormal basis $\{t^p_\alpha\}$ for V we mean an orthonormal basis $\{t^p_1, \dots, t^p_{\dim V^p}\}$ for V^p for each $p = k+1, \dots, k+M$.

As usual, let $\mathbb{P}(V)$ be the weighted projective space obtained by declaring that V^{k+i} has weight $k+i$. The unitary group $U := \prod_i U(V^{k+i})$ acts on V with moment map

$$\mu_U(v) = \frac{1}{2}\left(v \otimes v^* - \bigoplus_i c_i \operatorname{Id}_{V^{k+i}}\right) \ \in \ \oplus_i \mathfrak{u}(V^{k+i})^*. \tag{3.1}$$

Here the c_i are arbitrary real constants, which we will take to be positive, and $v^* \in V^*$ is the linear functional corresponding to v under the hermitian inner product. Therefore, the $U(1)$ action on V which acts on V^{k+i} with weight $k+i$ has moment map $\mu_{U(1)} = \operatorname{Tr}_w \circ \mu_U$, where $\operatorname{Tr}_w \colon \mathfrak{u}^* \to \mathfrak{u}(1)^*$ is the projection $\operatorname{Tr}_w(\oplus_i A^i) = \sum_i (k+i)\operatorname{tr}(A^i)$. Thus if $v = \oplus_i v_{k+i}$, then

(3.2)

$$\mu_{U(1)}(v) = \frac{1}{2}\left(\sum_i (k+i)|v_{k+i}|^2 - c\right), \text{ where } \ c := \sum_i (k+i) c_i \dim V^{k+i}.$$

Definition 3.3. The *Fubini-Study* orbifold Kähler metric ω_{FS} associated to $|\cdot|_V$ is $\frac{1}{c}$ times the metric on $\mathbb{P}(V)$ which results from viewing it as the symplectic quotient $\mu^{-1}_{U(1)}(0)/U(1)$ and taking the Kähler reduction of the metric $|\cdot|_V$ under the isometric action of $U(1)$.

This is an orbifold Kähler metric: on the orbifold chart (2.2), it pulls back to a genuine Kähler metric on $\mathbb{C}^{n-1}$. In fact, it follows from Lemma 3.6 below that it is the curvature of a hermitian metric h_1 on the orbifold line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$. The dual of this hermitian metric is one of three natural candidates for the name of Fubini-Study metric on $\mathcal{O}_{\mathbb{P}(V)}(-1)$. A second natural choice h_2 is given by $|v|^2_{h_2} = \sum_i |v_{k+i}|^{\frac{2}{k+i}}$ (note that $|v|^2 = \sum_i |v_{k+i}|^2$ does not scale correctly under the action of $\mathbb{C}^*$ to define a hermitian metric). However, it is the third candidate $h_3 = h_{FS}$ below that we choose. It should be noted that only on an unweighted projective space do all three agree and metrics. It seems that h_{FS} is a special case of the more general metrics on line bundles over toric varieties constructed by Batyrev-Tschinkel [**BT95**, section 2.1].

Definition 3.4. The *Fubini-Study metric* h_{FS} on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is the hermitian metric defined by setting the points of $\mu^{-1}_{U(1)}(0)$ to have norm 1. Therefore,

$$|v|_{h_{FS}} := \frac{1}{\lambda(v)}\,,$$

where $\lambda(v).v$ is the unique point of $\mu_{U(1)}^{-1}(0)$ in the orbit $(0,\infty).v$. That is, by (3.2), $\lambda(v)$ is the unique positive real solution to

$$\sum_i (k+i)\lambda(v)^{2(k+i)}|v_{k+i}|^2 = c. \tag{3.5}$$

We also use h_{FS} to denote the induced metrics on $\mathcal{O}_{\mathbb{P}(V)}(i)$.

The discrepancy between ω_{FS} and the curvature form $2\pi\omega_{h_{FS}} := i\partial\overline{\partial}\log h_{FS}$ of the metric h_{FS} on $\mathcal{O}_{\mathbb{P}(V)}(1)$ can be deduced from a result in [**BG97**].

Lemma 3.6. *We have*

$$\omega_{FS} = \omega_{h_{FS}} + \frac{i}{2c}\partial\overline{\partial} f, \tag{3.7}$$

where $f\colon \mathbb{P}(V) \to \mathbb{R}$ is the function

$$f := \sum_i \sum_\alpha |t^i_\alpha|^2_{h_{FS}}. \tag{3.8}$$

Here $\{t^i_\alpha\}$ is a $|\cdot|_V$-orthonormal basis of V^*, so each t^i_α defines a section of $\mathcal{O}_{\mathbb{P}(V)}(i)$, whose pointwise h_{FS}-norm is what appears in (3.8).

Proof. Let $p\colon V\backslash\{0\} \to \mathbb{P}(V)$ be projection to the quotient. We use [**BG97**, 3.1]; in their notation we set χ to be the cth power homomorphism from S^1 to itself and shift our moment map by $\frac{c}{2}$ to agree with theirs. The result is that the pullback of the Kähler form produced by symplectic reduction is

$$p^*(c\,\omega_{FS}) = \frac{i}{2}\partial\overline{\partial}\,|\lambda(v).v|^2_V + \frac{i}{2\pi}\partial\overline{\partial}\log\lambda(v)^c, \tag{3.9}$$

where $\lambda(v) \in (0,\infty)$ is defined as in (3.5) so that $\lambda(v).v \in \mu_{U(1)}^{-1}(0)$.

Over an open set of $\mathbb{P}(V)$ pick a holomorphic section, or multisection, of p, lifting x to $v = v(x)$. Then the curvature of h_{FS} on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is $i\partial\overline{\partial}\log|v|_{h_{FS}}$, which by Definition 3.4 is $i\partial\overline{\partial}\log\lambda(v)^{-1}$. Therefore, the curvature of $\mathcal{O}_{\mathbb{P}(V)}(1)$ is $i\partial\overline{\partial}\log\lambda(v)$ and we can rewrite (3.9) (divided through by c) as

$$p^*(\omega_{FS}) = \frac{i}{2c}\partial\overline{\partial}\,|\lambda(v).v|^2_V + p^*\omega_{h_{FS}}.$$

Then at $v \in V\backslash\{0\}$ lying over a point $x \in \mathbb{P}(V)$ we calculate $|\lambda(v).v|^2_V$ as

$$\sum_i |\lambda(v).v|^2_{V_i} = \sum_i\sum_\alpha |t^i_\alpha(\lambda(v).v)|^2 = \sum_i\sum_\alpha |t^i_\alpha|^2_{h_{FS},x}\,.$$

The last equality follows from the definition of h_{FS} (3.4), since $\lambda(v).v$ lies in $\mu_{U(1)}^{-1}(0)$. q.e.d.

The restriction of μ_U (3.1) to $\mu_{U(1)}^{-1}(0)$ descends to $\mathbb{P}(V)$ as the moment map m for the induced action of $U/U(1)$ on $\mathbb{P}(V)$:

$$m([v]) = \frac{1}{2}\bigoplus_i \left(\lambda^{2(k+i)}(v)\, v_{k+i} \otimes v_{k+i}^* - c_i \operatorname{Id}_{V^{k+i}}\right), \tag{3.10}$$

with $\lambda(v)$ as defined in (3.5). Integrating this allows us to define a notion of balanced orbifolds.

Definition 3.11. Given an orbifold embedding $X \subset \mathbb{P}(V)$, define

$$M(X) = \int_X m\,\frac{\omega_{FS}^n}{n!}\,,$$

where m is the moment map from (3.10). We say that an orbifold $X \subset \mathbb{P}(V)$ is *balanced* if $M(X) = 0$.

Remark 3.12. The balanced condition depends on $|\cdot|_V$ and on the choice of constants c_i. Later we will choose specific constants to ensure a connection with scalar curvature.

Just as in the manifold situation [**Don01**, **Wan04**], M is the moment map for the action of $U/U(1)$ on Olsson and Starr's Hilbert scheme [**OS03**] of sub-orbischemes of $\mathbb{P}(V)$ endowed with its natural L^2-symplectic form. To make sense of this statement, one can either work purely formally, make a precise statement at smooth points, or restrict attention to a single orbit of $\operatorname{Aut}(\mathbb{P}(V))$; the latter is smooth and all we will need in the application to constant scalar curvature. For $X \subset \mathbb{P}(V)$ and v, w sections of $T\mathbb{P}(V)|_X$, their pairing with the symplectic form is defined to be

$$\Omega(v,w) := \int_X v \lrcorner \left(w \lrcorner \frac{\omega^{n+1}}{(n+1)!}\right).$$

The moment map calculation is the following. We let $A = \oplus_i A^{k+i}$ be a graded hermitian matrix generating the 1-parameter subgroup $\exp(tA)$ of automorphisms of $\mathbb{P}(V)$, inducing the vector field v_A on $\mathbb{P}(V)$. Since $m_A := \operatorname{tr}(mA)$ is a hamiltonian for v_A, we have $v_A \lrcorner\, \omega = dm_A$. Moving in the Hilbert scheme down a vector field v on $\mathbb{P}(V)$, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \operatorname{tr}(M(X)A) &= \int_X \mathcal{L}_v \left(m_A \frac{\omega^n}{n!}\right) \\
&= \int_X v(m_A)\frac{\omega^n}{n!} + \int_X m_A d\left(v \lrcorner \frac{\omega^n}{n!}\right) \\
&= \int_X \omega(v, v_A)\frac{\omega^n}{n!} - \int_X d(m_A) \wedge \left(v \lrcorner \frac{\omega^n}{n!}\right) \\
&= \int_X \omega(v, v_A)\frac{\omega^n}{n!} - \int_X (v_A \lrcorner\, \omega) \wedge \left(v \lrcorner \frac{\omega^n}{n!}\right) \\
&= \int_X v \lrcorner \left(v_A \lrcorner \left(\frac{\omega^{n+1}}{(n+1)!}\right)\right) = \Omega(v, v_A).
\end{aligned} \tag{3.13}$$

To express the balanced condition in terms of sections of line bundles, fix a polarised orbifold (X, L) with cyclic stabiliser groups. Embed X in weighted projective space with $k \gg 0$ as in Section 2.6:
(3.14)

$$\phi_k \colon X \hookrightarrow \mathbb{P}(V) \quad \text{where} \quad V = \bigoplus_{i=1}^{M} H^0(L^{k+i})^* \text{ and } L = \phi_k^* \mathcal{O}_{\mathbb{P}(V)}(1).$$

A metric $|\cdot|_V$ on V induces by Definition 3.4 a Fubini-Study metric on $\mathcal{O}(1)$, and so one on L which we also denote by h_{FS}. The next lemma expresses the balanced condition in terms of coordinates on V given by a graded $|\cdot|_V$-orthonormal basis $\{t^i_\alpha\}$, where $t^i_\alpha \in H^0(L^{k+i})$. To ease notation, we write

$$\mathrm{vol} := \int_X \frac{c_1(L)^n}{n!}.$$

Lemma 3.15. *With respect to these coordinates, the matrix $M(X) = \oplus_i M^i(X)$ has entries*

$$(M^i(X))_{\alpha\beta} = \frac{1}{2}\left(\int_X (t^i_\alpha, t^i_\beta)_{h_{FS}} \frac{\omega_{FS}^n}{n!} - c_i \,\mathrm{vol}\, \delta_{\alpha\beta}\right).$$

Proof. Given a point x in X, let $\tilde{x} \in L_x^{-1}$ be any non-zero lift, and write $t^i_\alpha(\tilde{x})$ for the complex number $(\tilde{x}^{\otimes(k+i)}, t^i_\alpha(x))$. Then

$$(t_\alpha, t_\beta)_{h_{FS}} = \lambda(\tilde{x})^{2(k+i)} t^i_\alpha(\tilde{x}) \overline{t^i_\beta(\tilde{x})}$$

where $\lambda(\tilde{x})$ is the positive solution to $\sum_i (k+i)\lambda(\tilde{x})^{2(k+i)} \sum_\alpha |t^i_\alpha(\tilde{x})|^2 = c$. Now the embedding of X maps x to the point with coordinates $[t^i_\alpha(x)]$, so in these coordinates $m(x) = \oplus_i m^i(x)$ where

$$(m^i(x))_{\alpha\beta} = \frac{1}{2}\left(\lambda(\tilde{x})^{2(k+i)} t^i_\alpha(\tilde{x})\overline{t^i_\beta(\tilde{x})} - c_i \delta_{\alpha\beta}\right) = \frac{1}{2}\left((t^i_\alpha, t^i_\beta)_{h_{FS}} - c_i \delta_{\alpha\beta}\right)$$

and the result follows by integrating over X. q.e.d.

The balanced condition can also be expressed in terms of hermitian metrics on L. Let $\mathcal{K}(c_1(L))$ denote the orbifold Kähler metrics on X which are $(2\pi)^{-1}$ times the curvature of an orbifold hermitian metric on L. Define maps

$$\{\text{hermitian metrics on } L\} \times \mathcal{K}(c_1(L)) \underset{\text{FS}}{\overset{\text{Hilb}}{\rightleftarrows}} \{\text{metrics on } V\}$$

as follows:

- If $|\cdot|_V$ is a metric on $V := \oplus_i H^0(L^{k+i})^*$, then

  $$FS(|\cdot|_V) = (\phi_k^* h_{FS}, \phi_k^* \omega_{FS}),$$

  where h_{FS} and ω_{FS} are Fubini-Study metrics associated to $|\cdot|_V$.

- If h is a hermitian metric on L and ω a Kähler metric in $\mathcal{K}(c_1(L))$ the metric $\mathrm{Hilb}(h,\omega)$ on V is defined by requiring that for $s \in H^0(L^{k+i})$,

$$|s|^2_{\mathrm{Hilb}(h,\omega)} = \frac{1}{c_i \operatorname{vol}} \int_X |s|^2_h \frac{\omega^n}{n!} \,. \tag{3.16}$$

  Notice this differs from the usual L^2-metric by the $c_i \operatorname{vol}$ factors. Obviously, V and the maps Hilb and FS depend on k, and this will always be clear from the context.

Definition 3.17. We say that the pair (h,ω) is *balanced* at level k if it is a fixed point of $\mathrm{FS} \circ \mathrm{Hilb}$. A metric $|\cdot|_V$ is said to be *balanced* if it is a fixed point of $\mathrm{Hilb} \circ \mathrm{FS}$.

Proposition 3.18. *A metric $|\cdot|_V$ on V is balanced if and only if $\phi_k \colon X \subset \mathbb{P}(V)$ is a balanced orbifold.*

Proof. Given a metric $|\cdot|_V$, let $(h_{FS},\omega_{FS}) = FS(|\cdot|_V)$ and $\{t^i_\alpha\}$ be a graded $|\cdot|_V$-orthonormal basis for V. Then by Lemma 3.15, $M(X) = 0$ if and only if

$$\frac{1}{c_i \operatorname{vol}} \int_X (t^i_\alpha, t^i_\beta)_{h_{FS}} \frac{\omega^n_{FS}}{n!} = \delta_{\alpha\beta} \quad \text{for all } i, \alpha, \beta,$$

if and only if $\{t^i_\alpha\}$ is orthonormal with respect to the $\mathrm{Hilb}(h_{FS},\omega_{FS})$ metric, if and only if it is the same metric as $|\cdot|_V$. q.e.d.

Another way to express the balanced condition is through Bergman kernels.

Definition 3.19. Let h be a hermitian metric on L and ω be a Kähler metric on X. The *weighted Bergman kernel* is the function

$$B_k = B_k(h,\omega) := \operatorname{vol} \sum_i c_i(k+i) \sum_\alpha |s^i_\alpha|^2_h$$

where $\{s^i_\alpha\}$ is a graded basis of $\oplus_i H^0(L^{k+i})$ that is orthonormal with respect to the L^2-metric defined by (h,ω). Equivalently,

$$B_k = \sum_i (k+i) \sum_\alpha |t^i_\alpha|^2_h$$

where $\{t^i_\alpha\}$ is orthonormal with respect to the $\mathrm{Hilb}(h,\omega)$ metric. Of course, B_k is independent of these choices of basis.

If B_k is constant over X, then we see by integrating over X that this constant is necessarily $c = \sum_i c_i(k+i)h^0(L^{k+i})$. In the unweighted case, B_k can be written invariantly in terms of the ratio of the hermitian metrics h and h_{FS} on L. We have the following analogue here.

Proposition 3.20. *Fix a hermitian metric h on L and a Kähler metric $\omega \in \mathcal{K}(c_1(L))$ and let $(h_{FS},\omega_{FS}) = FS \circ \mathrm{Hilb}(h,\omega)$. Then $h = h_{FS}$ if and only if $B_k(h,\omega) \equiv c$ is constant on X.*

Proof. Let $\{t^i_\alpha\}$ be a graded basis for $\oplus_i H^0(L^{k+i})$ that is orthonormal with respect to the $\mathrm{Hilb}(h,\omega)$-metric. For $x \in X$, let $\tilde{x}$ be any non-zero lift in $L^{-1}|_x$. Then

$$\begin{aligned}
\sum_i (k+i) \sum_\alpha |t^i_\alpha(x)|^2_{h_{FS}} &= \sum_i (k+i) \sum_\alpha |(\tilde{x}, t^i_\alpha(x))|^2 |\tilde{x}|^{-2(k+i)}_{h_{FS}} \\
&= \sum_i (k+i) \sum_\alpha \lambda(\tilde{x})^{2(k+i)} |(\mathrm{ev}^i_{\tilde{x}}, t^i_\alpha)|^2 \\
&= \sum_i (k+i) \lambda(\tilde{x})^{2i} |\,\mathrm{ev}^i_{\tilde{x}}\,|^2 = c, \qquad (3.21)
\end{aligned}$$

from the definition of dual norms, the fact that $\lambda(\tilde{x}) = |\tilde{x}|^{-1}_{h_{FS}}$, and the defining equation for $\lambda(\tilde{x})$ (3.5). Thus if $h = h_{FS}$, then B_k is constant. Conversely, if $\beta := h_{FS}/h$ we have

$$c = \sum_i (k+i) \sum_\alpha |t^i_\alpha(x)|^2_{h_{FS}} = \sum_i^m (k+i) \sum_\alpha \beta(x)^{(k+i)} |t^i_\alpha(x)|^2_h. \qquad (3.22)$$

Now note that for fixed x, the quantity $u_i = (k+i)\sum_\alpha |t^i_\alpha(x)|^2_h$ is nonnegative for each i, so there is a unique positive real solution to the equation $\sum_{i=1} \beta^{2(k+i)}(x) u_i = c$. If $B_k \equiv c$ is constant, then $\beta(x) = 1$ is one solution, and thus the unique solution, so $h = h_{FS}$. q.e.d.

4. Limits of Fubini-Study metrics

The connection between constant scalar curvature metrics and stability comes through the asymptotics of Fubini-Study metrics. The crucial ingredient is the asymptotics, as $k \to \infty$, of the weighted Bergman kernel of Definition 3.19:

$$B_k = \mathrm{vol} \sum_i c_i (k+i) \sum_\alpha |s^i_\alpha|^2_h.$$

Here $\{s^i_\alpha\}$ is a basis of $H^0(L^{k+i})$ that is orthonormal with respect to the L^2-metric induced by h and ω. Ensuring that this is related to scalar curvature requires a particular choice of c_i, so for concreteness assume from now on they are chosen by requiring

$$\sum_i c_i t^i := (t^{\mathrm{ord}(X)-1} + t^{\mathrm{ord}(X)-2} + \cdots + 1)^{p+1} \qquad (4.1)$$

for some sufficiently large integer p. We prove in [**RT11**, 1.7 and 4.13] that with this choice of c_i there is an asymptotic expansion

$$B_k = b_0 k^{n+1} + b_1 k^n + \cdots \quad \text{as } k \to \infty \qquad (4.2)$$

for some smooth functions b_i. Taking larger values of p yields a stronger expansion: in fact, if $p \geq r + q$ for integers $r, q \geq 0$, then (4.2) holds up to terms of order $O(k^{n+1-r})$ in the C^q-norm. By this we mean that there is a constant C such that for all k,

$$\left\| B_k - b_0k^{n+1} - b_1k^n - \cdots - b_{r-1}k^{n+1-(r-1)} \right\| \leq Ck^{n+1-r},$$

where the norm is the C^q-norm taken over X in the orbifold sense, with the pointwise norm of the derivatives measured with respect to the metric defined by ω. Moreover, the constant C can be taken to be uniform for (h,ω) in a compact set.

To achieve what we need in this paper, it is sufficient to select $p=5$, so in particular there is a C^2-expansion involving the top two terms b_0 and b_1; however, nothing is lost if the reader prefers to take a larger p for simplicity. Moreover, if $2\pi\omega_h$ denotes the curvature $i\partial\overline{\partial}\log h$ of h, the top two coefficients are given by [**RT11**, 1.11]

$$\begin{aligned} b_0 &= \operatorname{vol} \frac{\omega_h^n}{\omega^n}\sum_i c_i, \\ b_1 &= \operatorname{vol} \frac{\omega_h^n}{\omega^n}\sum_i c_i\left((n+1)i + \operatorname{tr}_{\omega_h}(\operatorname{Ric}(\omega)) - \frac{1}{2}\operatorname{Scal}(\omega_h)\right). \end{aligned}$$

In particular, if $2\pi\omega$ is in fact the curvature of h, this simplifies to

$$b_0 = \operatorname{vol}\sum_i c_i, \qquad b_1 = \operatorname{vol}\sum_i c_i\left((n+1)i + \frac{1}{2}\operatorname{Scal}(\omega)\right). \tag{4.3}$$

Observe that in this case the top order term, b_0, is constant over X.

Integrating the expansion over X shows $c = \sum_i c_i(k+i)h^0(L^{k+i})$ is polynomial modulo small terms (this is shown directly in Lemma 6.5). In fact,

$$c = \operatorname{vol}\sum_i c_i\left[k^{n+1} + \left((n+1)i + \frac{\overline{S}}{2}\right)k^n\right] + O(k^{n-1}), \tag{4.4}$$

where $\overline{S}$ denotes the average of the scalar curvature of any Kähler metric in $\mathcal{K}(c_1(L))$.

Similarly [**RT11**, remark 4.13], there is also an asymptotic expansion

$$\operatorname{vol}\sum_i c_i \sum_\alpha |s_\alpha^i|_h^2 \;=\; b_0k^n + b_1'k^{n-1} + \cdots \tag{4.5}$$

for some function b_1', and where b_0 is as above. Here the choice of c_i is as above (4.1), and if $p \geq r+q$, the expansion is in the C^q-norm up to terms of order $O(k^{n-r})$.

In what follows, fix a hermitian metric h on L and Kähler metric $\omega \in \mathcal{K}(c_1(L))$, and let $(h_{FS,k}, \omega_{FS,k})$ be the pair $FS \circ \operatorname{Hilb}(h,\omega)$ coming from the embedding $X \subset \mathbb{P}(\oplus_i H^0(L^{k+i})^*)$. For embeddings of manifolds in ordinary projective space, the asymptotics of $h/h_{FS,k}$ are those of the Bergman kernel. For orbifolds, the fact that the Fubini-Study fibre metric is defined implicitly in Definition 3.4 means that we have to work harder.

Theorem 4.6. *Suppose that $2\pi\omega$ is the curvature of h. Then the pair $(h_{FS,k}, \omega_{FS,k})$ converges to (h,ω) as k tends to infinity. In fact, if $\overline{S}$ denotes the average of the scalar curvature, then*

$$\frac{h_{FS,k}}{h} = 1 + \frac{\overline{S} - \mathrm{Scal}(\omega)}{2} k^{-2} + O(k^{-3}) \tag{4.7}$$

in the C^2-norm, and

$$\omega = \omega_{FS,k} + O(k^{-2}) \tag{4.8}$$

in C^0. In particular, the set of Fubini-Study Kähler metrics is dense in $\mathcal{K}(c_1(L))$.

Remark 4.9. The theorem can be generalised to the case that ω is not the curvature of h, in which case there will be an additional $O(k^{-1})$ term appearing in the expansion of $h_{FS,k}/h$.

Proof of (4.7). The aim is to find an asymptotic expansion of

$$\alpha_k := \frac{h_{FS,k}}{h}.$$

Set $\mathcal{B}_r := \sum_\alpha |t_\alpha^r|_h^2$ where $\{t_\alpha^r\}$ is a basis of $H^0(L^r)$ that is orthonormal with respect to the Hilb(h,ω)-norm from (3.16), so that $B_k = \sum_i (k+i)\mathcal{B}_{k+i}$. Then if $0 \neq \tilde{x} \in L_x^{-1}$,

$$\begin{aligned} \sum_i (k+i)\alpha_k^{k+i}\mathcal{B}_{k+i} &= \sum_i (k+i)\|\tilde{x}^{k+i}\|_{h_{FS,k}}^{-2} \sum_\alpha |t_\alpha^{k+i}(\tilde{x})|_h^2 \\ &= \sum_i (k+i)\|\tilde{x}^{k+i}\|_{h_{FS,k}}^{-2} \sum_\alpha \|\tilde{x}\|_{\mathrm{Hilb}(h,\omega)}^2 \\ &= c, \end{aligned} \tag{4.10}$$

where the second equality uses the fact that the t_α^{k+i} are orthonormal, the third inequality comes from the definition of the FS-norm (3.5), and as in (3.2), $c = \sum_i c_i(k+i)h^0(L^{k+i})$ is constant over X.

We aim first for an asymptotic expansion of α_k that holds in C^0. Say a sequence a_k of real numbers is of order $\Omega(k^p)$ if there is a $\delta > 0$ such that $a_k \geq \delta k^p$ for $p \gg 0$. A sequence of real-valued functions f_k on X is of order $\Omega(k^p)$ if there is a $\delta > 0$ with $f_k \geq \delta k^p$ uniformly on X for all $p \gg 0$.

Step 1: We show $\alpha_k = 1 + O(k^{-1})$ in C^0. Observe that from (4.3) and (4.4),

$$B_k = \sum_i (k+i)\mathcal{B}_{k+i} = \mathrm{vol} \sum_i c_i k^{n+1} + O(k^n) = c + O(k^n).$$

Taking the difference with (4.10) gives

$$\sum_i (k+i)\Big(\alpha_k^{k+i}-1\Big)\mathcal{B}_{k+i} = O(k^n),$$

and so

$$(\alpha_k - 1)\sum_i (k+i)\Big[1+\alpha_k+\alpha_k^2+\cdots+\alpha_k^{k+i-1}\Big]\mathcal{B}_{k+i} = O(k^n). \tag{4.11}$$

Now α_k is pointwise positive, so the term in square brackets is at least 1, and $\sum_i (k+i)\mathcal{B}_{k+i} = \Omega(k^{n+1})$, so the sum on the left hand side is $\Omega(k^{n+1})$. Thus $\alpha_k - 1 = O(k^{-1})$ as claimed.

Step 2: There are positive constants C_1, C_2 such that

$$\begin{array}{rcll} C_1 & \leq & \alpha_k^j & \text{for all } \dfrac{k}{2} \leq j \leq k, \\ \alpha_k^j & \leq & C_2 & \text{for all } 0 \leq j \leq k. \end{array} \tag{4.12}$$

Proof. As $\alpha_k = 1 + O(k^{-1})$ we have $C_1^2 \leq \alpha_k^k \leq C_2$ for some $C_1 \in (0,1)$, $C_2 > 1$ and all $k \gg 0$. Thus for $j \geq \frac{k}{2}$ we have $\alpha_k^j \geq C_1$ and for $j \leq k$ we have $\alpha_k^j \leq C_2$. q.e.d.

Using this, we can improve on Step 1 by observing that the term in square brackets in (4.11) is of order $\Omega(k)$ since each power of α_k is nonnegative, and there are at least $k/2$ terms bounded from below by C_1. Hence

$$\alpha_k - 1 = O(k^{-2}) \quad \text{in } C^0.$$

Step 3: Next define

$$\beta_k = 1 + \frac{\overline{S} - \mathrm{Scal}(\omega)}{2} k^{-2}. \tag{4.13}$$

We claim that

$$\sum_i (k+i)\beta_k^{k+i}\mathcal{B}_{k+i} = c + O(k^{n-1}) \quad \text{in } C^0. \tag{4.14}$$

That is, the β_k satisfy an implicit equation very close to the one (4.10) satisfied by the α_k, which we shall use to deduce that they are approximately equal.

Proof. Note

$$\beta_k^{k+i} = 1 + \frac{\overline{S} - \mathrm{Scal}(\omega)}{2} k^{-1} + O(k^{-2}).$$

So using the asymptotic expansion (4.2, 4.3) of the weighted Bergman kernel $B_k = \sum(k+i)\mathcal{B}_{k+i}$,

$$\begin{aligned}\sum_i (k+i)\beta_k^{k+i}\mathcal{B}_{k+i} &= \sum_i (k+i)\left(1+\frac{\overline{S}-\mathrm{Scal}(\omega)}{2k}+O(k^{-2})\right)\mathcal{B}_{k+i} \\ &= \mathrm{vol}\sum_i c_i\left[k^{n+1}+\left(\frac{\overline{S}-\mathrm{Scal}(\omega)}{2}+(n+1)i+\frac{\mathrm{Scal}(\omega)}{2}\right)k^n\right] \\ &\quad + O(k^{n-1}) \\ &= \mathrm{vol}\sum_i c_i\left[k^{n+1}+\left((n+1)i+\frac{\overline{S}}{2}\right)k^n\right]+O(k^{n-1}) \quad \text{in } C^0, \end{aligned} \tag{4.15}$$

since $\mathcal{B}_{k+i} = O(k^n)$ in C^0. Comparing with (4.4) proves the claim. q.e.d.

Step 4: To simplify notation, set

$$\gamma_k := \alpha_k^{k+i-1} + \alpha_k^{k+i-2}\beta_k + \cdots + \beta_k^{k+i-1}.$$

Taking the difference between the implicit equations (4.10) and (4.14) for α_k and β_k yields

$$(\alpha_k - \beta_k)\sum_i (k+i)\gamma_k\mathcal{B}_{k+i} = O(k^{n-1}) \quad \text{in } C^0. \tag{4.16}$$

From (4.12) and the definition (4.13) of β_k we see that $\gamma_k = \Omega(k)$. Therefore, by (4.16),

$$\alpha_k = \beta_k + O(k^{-3}) = 1 + \frac{\overline{S}-\mathrm{Scal}(\omega)}{2}k^{-2} + O(k^{-3}), \tag{4.17}$$

which is the expansion we wanted at the level of C^0-norms.

Step 5: To extend this to the C^2-norm, we actually require an expansion in the C^0-norm to higher order (this is because, although the pieces of the Bergman kernel $\mathcal{B}_{k+i}$ are of order $O(k^n)$, their derivatives $D^p\mathcal{B}_{k+i}$ are of order $O(k^{n+p})$ [**RT11**, corollary 4.10], resulting in a loss of a factor of k for each derivative we take). To achieve this, replace β_k with

$$\beta_k = 1 + \frac{\overline{S}-\mathrm{Scal}(\omega)}{2}k^{-2} + \tau_1 k^{-3} + \tau_2 k^{-4},$$

where the τ_i are smooth functions independent of k. Then the coefficient of k^{n-1} in (4.15) is $b_0\tau_1 + f$, where f is independent of k and the τ_i. Similarly, the coefficient of k^{n-2} is $b_0\tau_2 + g$, where g is independent of k and τ_2.

So setting $\tau_1 = -f/b_0$ and $\tau_2 = -g/b_0$, we may assume that the k^{n-1} and k^{n-2} terms in (4.15) vanish. Therefore,

$$\sum_i (k+i)\beta_k^{k+i}\mathcal{B}_{k+i} = c + O(k^{n-3+p}) \text{ in } C^p \text{ for } p = 0,1,2, \tag{4.18}$$

where we have used $\mathcal{B}_{k+i} = O(k^{n+p})$ in C^p in place of the original argument using $\mathcal{B}_{k+i} = O(k^n)$ in C^0. Thus

$$(\alpha_k - \beta_k)\sum_i (k+i)\gamma_k \mathcal{B}_{k+i} = O(k^{n-3+p}) \text{ in } C^p,\ p = 0, 1, 2. \tag{4.19}$$

In particular, $\alpha_k = \beta_k + O(k^{-5})$ in C^0.

Now to bound $D\alpha_k$, differentiate (4.10) to get

$$D\alpha_k \sum_i (k+i)^2 \alpha_k^{k+i-1}\mathcal{B}_{k+i} = -\sum_i (k+i)\alpha_k^{k+i} D\mathcal{B}_{k+i}. \tag{4.20}$$

Since powers of α_k are bounded above uniformly (4.12) and $D\mathcal{B}_{k+i} = O(k^{n+1})$, the sum on the right hand side is $O(k^{n+2})$. On the other hand, using the lower bound in (4.12), the sum on the left hand side is of order $\Omega(k^{n+2})$, and hence $D\alpha_k = O(1)$. We claim that $D\gamma_k = O(k^2)$. In fact, both α_k^j and β_k^j are uniformly bounded from above for all k and all $j \leq k+i$. Thus if $u + v \leq k + i$,

$$D(\alpha_k^u \beta_k^v) = u\alpha_k^{u-1}\beta_k^v D\alpha_k + v\alpha_k^u\beta_k^{v-1}D\beta_k = O(k), \tag{4.21}$$

since $D\alpha_k = O(1)$ and $D\beta_k = O(k^{-2})$. Thus $D\gamma_k$ is a sum of $O(k)$ terms, each of order $O(k)$, and so $D\gamma_k = O(k^2)$ as claimed.

So we know $\gamma_k\mathcal{B}_{k+i} = O(k^{n+1})$ and $D(\gamma_k\mathcal{B}_{k+i}) = O(k^{n+2})$. Differentiating the $p = 1$ statement of (4.19) and using $\gamma_k = \Omega(k)$ yields

$$\begin{aligned}(D\alpha_k - D\beta_k)\Omega(k^{n+2}) &= -(\alpha_k - \beta_k)\sum_i (k+i)D(\gamma_k\mathcal{B}_{k+i}) + O(k^{n-2}) \\ &= O(k^{n-2})\end{aligned}$$

as $\alpha_k - \beta_k = O(k^{-5})$. Hence $D\alpha_k = D\beta_k + O(k^{-4})$, and thus we have $\alpha_k - \beta_k = O(k^{-4})$ in C^1. In particular, $D\alpha_k = O(k^{-2})$.

A similar argument applies to the second derivative. Differentiating (4.20) yields

$$\begin{aligned}(D^2\alpha_k)\Omega(k^{n+2}) = &-2\sum_i (k+i)^2\alpha_k^{k+i-1}D\alpha_k D\mathcal{B}_{k+i} \\ &-\sum_i (k+i)\alpha_k^{k+i}D^2\mathcal{B}_{k+i} \\ &-\sum_i (k+i)^2(k+i-1)\alpha_k^{k+i-2}(D\alpha_k)^2\mathcal{B}_{k+i}\end{aligned}$$

which is $O(k^{n+3})$. Thus $D^2\alpha_k = O(k)$. If $u + v \leq k + i$, then

$$D^2(\alpha_k^u\beta_k^v) = u\alpha_k^{u-1}\beta_k^v D^2\alpha_k + v\alpha_k^u\beta_k^{v-1}D^2\beta_k + O(k^{-2})$$

since $D\alpha_k$ and $D\beta_k$ are both $O(k^{-2})$. Therefore, $D^2(\alpha_k^u\beta_k^v) = O(k^2)$ which implies that $D^2\gamma_k = O(k^3)$ and hence $D^2(\gamma_k\mathcal{B}_{k+i}) = O(k^{n+3})$.

Now taking the second derivative of the $p = 2$ statement in (4.19),

$$\begin{aligned}(D^2\alpha_k - D^2\beta_k)\Omega(k^{n+2}) &= -(\alpha_k - \beta_k)\sum_i (k+i)D^2(\gamma_k\mathcal{B}_{k+i}) \\ &\quad - 2(D\alpha_k - D\beta_k)\sum_i (k+i)D(\gamma_k\mathcal{B}_{k+i}) + O(k^{n-1}).\end{aligned}$$

Since $\alpha_k - \beta_k = O(k^{-5})$, $D\alpha_k - D\beta_k = O(k^{-4})$, and $D(\gamma_k\mathcal{B}_{k+i}) = O(k^{n+2})$, this is $O(k^{n-1})$. Hence $D^2\alpha_k = D^2\beta_k + O(k^{-3})$ as required. q.e.d.

Proof of (4.8). From Lemma 3.6 we have $\omega_{FS,k} = \omega_{h_{FS,k}} + \frac{i}{2c}\partial\overline{\partial} f_k$, where

$$f_k = \operatorname{vol}\sum_i c_i \sum_\alpha |s^i_\alpha|^2_{h_{FS,k}}$$

and the $\{s^i_\alpha\}$ is a graded basis of $\oplus_i H^0(L^{k+i})$ that is orthonormal with respect to the L^2-norm defined by (h,ω). (So $t^i_\alpha := \sqrt{c_i \operatorname{vol}}\, s^i_\alpha$ is an orthonormal basis with respect to the $\operatorname{Hilb}(h,\omega)$ metric.)

Applying $\partial\overline{\partial}\log$ to (4.7) shows that $\omega_{h_{FS,k}} = \omega_h + O(k^{-2}) = \omega + O(k^{-2})$ in C^0, so $\omega_{FS,k} = \omega + \frac{i}{2c}\partial\overline{\partial} f_k + O(k^{-2})$. So since c is of order $\Omega(k^{n+1})$, to prove (4.8) it will be sufficient to show that f_k is constant on X to $O(k^{n-1})$ in C^2-norm. Applying the expansion (4.7),

$$\begin{aligned} f_k(x) &= \operatorname{vol}\sum_i c_i \frac{h^{k+i}_{FS,k}}{h^{k+i}} \sum_\alpha |s_\alpha(x)|^2_h \\ &= \operatorname{vol}\sum_i c_i\Big(1 + \frac{\operatorname{Scal}(\omega) - \overline{S}}{2k} + O(k^{-2})\Big)\sum_\alpha |s_\alpha(x)|^2_h \\ &= b_0 k^n + O(k^{n-1}), \end{aligned}$$

by (4.5), where b_0 is constant. q.e.d.

5. Limits of balanced metrics

We digress in this section from our proof of Donaldson's theorem to give another application of the weighted Bergman kernel that illustrates the connection between balanced metrics and metrics of constant scalar curvature.

Theorem 5.1. *Let (h_k, ω_k) be a pair that is balanced for the embedding $X \subset \mathbb{P}(\oplus_i H^0(L^{k+i})^*)$, and suppose this sequence converges in C^2 to a limit (h,ω). Then $2\pi\omega$ is the curvature of h and* $\operatorname{Scal}(\omega)$ *is constant.*

Proof. Letting $2\pi\omega_{h_k}$ denote the curvature of h_k, by Lemma 3.6 we have

$$\omega_k = \omega_{h_k} + \frac{i}{2c}\partial\overline{\partial} f_k, \tag{5.2}$$

where

$$f_k(x) = \mathrm{vol} \sum_i c_i \sum_\alpha |s^i_\alpha(x)|^2_{h_k}$$

and $\{s^i_\alpha\}$ is a graded orthonormal basis of $\oplus_i H^0(L^{k+i})$ with respect to the L^2-metric defined by (h_k, ω_k). (Here we are using the balanced condition: that (h_k, ω_k) is the Fubini-Study metric induced from this L^2-metric.)

By (4.5) we have the C^4-estimate

$$f_k = \mathrm{vol}\, \frac{\omega^n_{h_k}}{\omega^n_k} \sum_i c_i k^n + O(k^{n-1}). \tag{5.3}$$

(The estimate is in C^4 rather than C^2 since we only require it to top order. Moreover, we have used here that the sequence (h_k, ω_k) converges so lies in a compact set, and thus the $O(k^{n-1})$ can be taken uniformly.) Since $c = \sum_i c_i(k+i)h^0(L^{k+i})$ is of order $\Omega(k^{n+1})$, we deduce from (5.2) that $\omega_k = \omega_{h_k} + O(k^{-1})$ in C^2.

In turn, this implies that $\omega^n_{h_k}/\omega^n_k = 1 + O(k^{-1})$, which we can feed back into (5.3) to give $\partial\overline{\partial} f_k = O(k^{n-1})$. Hence in fact

$$\omega_k = \omega_{h_k} + O(k^{-2}).$$

In particular, taking the limit as $k \to \infty$ implies that $\omega = \omega_h$, i.e. that $2\pi\omega$ is the curvature of h.

Therefore, $\omega^n_{h_k}/\omega^n_k = 1 + O(k^{-2})$ and

$$\mathrm{tr}_{\omega_{h_k}}(\mathrm{Ric}(\omega_k)) = \mathrm{tr}_{\omega_k}(\mathrm{Ric}(\omega_k)) + O(k^{-2}) = \mathrm{Scal}(\omega_k) + O(k^{-2}).$$

Thus the asymptotic expansion (4.2) for the weighted Bergman kernel becomes

$$B_k = \mathrm{vol} \sum_i (k+i) c_i \sum_\alpha |s^i_\alpha|^2 = \mathrm{vol} \sum_i c_i k^{n+1} + b_1 k^n + O(k^{n-1}), \tag{5.4}$$

where $b_1 = \mathrm{vol} \sum_i c_i \left((n+1)i + \frac{1}{2}\mathrm{Scal}(\omega_k)\right)$. But by Proposition 3.20 the balanced condition implies that this weighted Bergman kernel is the constant

$$c = \mathrm{vol} \sum_i c_i k^{n+1} + O(k^n).$$

So the coefficient of k^{n+1} agrees with that of (5.4). Taking coefficients of k^n gives, after some rearranging, a constant $\overline{S}$ independent of k such that

$$\mathrm{Scal}(\omega_k) - \overline{S} = O(k^{-1}).$$

Taking k to infinity yields $\mathrm{Scal}(\omega) = \overline{S}$ as required. q.e.d.

Remark 5.5. The previous theorem was first observed by Donaldson [**Don01**] in the case of manifolds embedded in projective space. In the same paper, Donaldson also proves a much harder converse: a cscK metric implies the existence of balanced metrics for large k. We expect that this converse can also be generalised to orbifold embeddings in weighted projective space, but have not attempted to prove it.

6. K-stability as an obstruction to orbifold cscK metrics

We now have the tools required to prove the orbifold version of Donaldson's theorem, and start with the precise definition of stability.

6.1. Definition of orbifold K-stability. Fix a compact dimensional polarised orbifold (X, L) of dimension n with cyclic quotient singularities.

Definition 6.1. A test configuration for (X, L) consists of a pair $(\pi\colon \mathcal{X} \to \mathbb{C}, \mathcal{L})$ where $\mathcal{X}$ is an orbischeme, π is flat, and $\mathcal{L}$ is an ample orbi-line bundle along with a $\mathbb{C}^*$-action such that (1) the action is linear and covers the usual action on $\mathbb{C}$ and (2) the general fibre $\pi^{-1}(t)$ of the test configuration is (X, L).

Test configurations arise from the action of a one-parameter $\mathbb{C}^*$-subgroup of the automorphisms of weighted projective space $\mathbb{P}$ on an orbifold embedded in $\mathbb{P}$. In general, the limit $\mathcal{X}_0 = \pi^{-1}(0)$ will not itself be an orbifold, as it may have scheme structure or entire components consisting of points with nontrivial stabilisers. In general, one should allow $\mathcal{X}$ to be a Deligne-Mumford stack, but for most of the applications in this paper $\mathcal{X}$ will itself be an orbifold.

Conversely, we can realise an abstract test configuration via a $\mathbb{C}^*$-action on weighted projective space, just as in the manifold case [**RT07**, proposition 3.7]. Using the orbi-ampleness of $\mathcal{L}$, we can embed $\mathcal{X}$ into the weighted projective bundle $\mathbb{P}(\oplus_i(\pi_*\mathcal{L}^{k+i})^*)$ over the base curve $\mathbb{C}$ for $k \gg 0$, such that the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ is $\mathcal{L}$. Pick a trivialisation of the bundle, making it isomorphic to $\mathbb{P}(V) \times \mathbb{C}$, where $V = \oplus_i H^0(\mathcal{X}_0, \mathcal{L}^{k+i}|_{\mathcal{X}_0})^*$. Thus the $\mathbb{C}^*$-action on V arising from the one on the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$ induces a diagonal $\mathbb{C}^*$-action on $\mathbb{P}(V) \times \mathbb{C} \supset \mathcal{X}$, giving the original test configuration.

By Proposition 2.19 we can write the total weight of the $\mathbb{C}^*$-action on $H^0(L^k)$ as

$$w(H^0(L^k)) = w(k) + \tilde{o}(k^n), \tag{6.2}$$

where $w(k)$ is a polynomial $b_0 k^{n+1} + b_1 k^n$ of degree $n+1$. Similarly,

$$h^0(L^k) = h(k) + \tilde{o}(k^{n-1}), \tag{6.3}$$

where $h(k) = a_0 k^n + a_1 k^{n-1}$.

Definition 6.4. The *Futaki invariant* of the test configuration $(\mathcal{X}, \mathcal{L})$ is the $F_1 = \frac{a_0 b_1 - a_1 b_0}{a_0^2}$ term in the expansion

$$\frac{w(k)}{kh(k)} = F_0 + \frac{F_1}{k} + O\left(\frac{1}{k^2}\right).$$

We say (X, L) is *K-semistable* if $F_1 \geq 0$ for any test configuration with general fibre (X, L). We say it is *K-polystable* if in addition $F_1 = 0$ only if the test configuration is a product $\mathcal{X} = X \times \mathbb{C}$, i.e. it arises from a $\mathbb{C}^*$-action on X.

In other words, we are simply ignoring the non-polynomial terms in the Hilbert and weight functions, and then defining stability exactly as for manifolds.

One reason this is a sensible stability notion related to scalar curvature is given by our next result. This shows that taking a weighted sum with our choice of c_i kills the periodic terms, a result we will apply later to both w (6.2) and h (6.3).

Lemma 6.5. *Let H be a function of the form*

$$H(k) = h(k) + \epsilon_h(k),$$

where h is a polynomial of degree n and ϵ_h is a sum of terms of the form $r(k)\delta(k)$ where r is a polynomial of degree $n-1$ and $\delta(k)$ is periodic with period m and average zero. Then

$$\sum_i c_i H(k+i) = \sum_i c_i h(k+i) + O(k^{n-4}).$$

Proof. First we claim that if $0 \leq p \leq 3$, then $\sum_{i \equiv u} c_i i^p$ is independent of u. To see this, let $m = \operatorname{ord}(X)$ and observe that by (4.1), $\sum_i c_i t^i$ has a root of order at least 4 at every nontrivial mth root of unity. Thus if $\sigma^m = 1$ with $\sigma \neq 1$, then $\sum_i c_i i^p \sigma^{ri} = 0$ for $1 \leq r \leq m-1$. So given any u,

$$\sum_i i^p c_i = \sum_{r=0}^{m-1} \sigma^{-ru} \sum_i i^p c_i \sigma^{ri} = \sum_i i^p c_i \left(\sum_{r=0}^{m-1} \sigma^{(i-u)r} \right) = m \sum_{i \equiv u} i^p c_i,$$

which proves the claim.

We have to show that $\sum_i c_i r(k+i)\delta(k+i) = O(k^{n-4})$. By the claim,

$$\begin{aligned} \sum_i c_i i^p \delta(k+i) &= \sum_{u=1}^{m} \sum_{i \equiv u-k \bmod m} c_i i^p \delta(u) \\ &= \frac{1}{m} \sum_{u=1}^{m} \delta(u) \sum_i c_i i^p \ = 0 \end{aligned}$$

for $0 \leq p \leq 3$. Hence the $k^d, \ldots, k^{d-3}$ terms in $\sum_i c_i (k+i)^d \delta(k+i)$ vanish, and the sum is $O(k^{n-4})$ if $d \leq n$. The result for general polynomials r follows by linearity. q.e.d.

6.2. Orbifold version of Donaldson's theorem. To recall the general setup, let h be a hermitian metric on L with positive curvature $2\pi\omega$, and for $k \gg 0$ consider the $\mathrm{Hilb}(h,\omega)$ metric on $\oplus_i H^0(L^{k+i})$ from (3.16). From the embedding $X \subset \mathbb{P}(\oplus_i H^0(L^{k+i})^*)$, we produced in Definition 3.11 a hermitian matrix $M(X) = M_k(X)$ (and defined the embedding to be balanced at level k when $M_k(X)$ vanishes). Using the norm $\|A\|^2 = \mathrm{tr}(AA^*)$ on hermitian matrices, the following is the key estimate.

Theorem 6.6. *There is a constant C such that*

$$\|M_k(X)\| \leq Ck^{\frac{n-2}{2}}\|\mathrm{Scal}(\omega) - \overline{S}\|_{L^2} + O(k^{\frac{n-4}{2}}),$$

where the L^2-norm is taken with respect to the volume form determined by ω.

Proof. To ease notation, we write $M = M_k(X)$. Since $\|M\|$ is unchanged by a unitary transformation, we may pick $\mathrm{Hilb}(h,\omega)$-orthonormal coordinates $\{t^i_\alpha\}$ such that $M = \oplus M^i$ with each M^i diagonal. Thus M^i has entries (3.15)

$$\begin{aligned} M^i_{\alpha\alpha} &= \frac{1}{2}\left(\int_X |t^i_\alpha|^2_{h_{FS,k}} \frac{\omega^n_{FS,k}}{n!} - c_i \,\mathrm{vol}\right) \\ &= \frac{1}{2}\left(\int_X |t^i_\alpha|^2_h \frac{h^{k+i}_{FS,k}}{h^{k+i}} \frac{\omega^n_{FS,k}}{n!} - c_i \,\mathrm{vol}\right), \end{aligned}$$

where $h_{FS,k}$ and $\omega_{FS,k}$ are the induced Fubini-Study metrics. Using the expansion of $h_{FS,k}/h = 1 + (\overline{S} - S)/2k^2 + O(k^{-3})$ of Theorem 4.6, we can write $M = A + B$ where $B^i_{\alpha\alpha} = O(k^{-2})$ and

$$A^i_{\alpha\alpha} = \frac{1}{2}\left(\int_X |t^i_\alpha|^2_h \left(1 + \frac{\overline{S} - S}{2k}\right)\frac{\omega^n}{n!} - c_i \,\mathrm{vol}\right) = \frac{1}{4k}\int_X |t^i_\alpha|^2_h (\overline{S} - S)\frac{\omega^n}{n!}.$$

Here we have used $\|t^i_\alpha\|^2_{L^2} = c_i \,\mathrm{vol}$ from the definition of the $\mathrm{Hilb}(h,\omega)$ norm. Using the Cauchy-Schwarz inequality,

$$\begin{aligned} |A^i_{\alpha\alpha}|^2 &\leq \frac{1}{16k^2}\int_X |t^i_\alpha|^2_h \frac{\omega^n}{n!}\int_X |t^i_\alpha|^2_h |\overline{S} - S|^2 \frac{\omega^n}{n!} \\ &\leq \frac{C'}{k^2}\int_X |t^i_\alpha|^2_h |\overline{S} - S|^2 \frac{\omega^n}{n!}, \end{aligned}$$

for some constant C'. Thus from the weak form of the expansion $\sum_i \sum_\alpha |t^i_\alpha|^2_h = O(k^n)$,

$$\begin{aligned} \|A\|^2 &\leq \frac{C'}{k^2}\int_X \sum_{i,\alpha} |t^i_\alpha|^2_h |\overline{S} - S|^2 \frac{\omega^n}{n!} \\ &\leq C'' k^{n-2}\|S - \overline{S}\|^2_{L^2}. \end{aligned}$$

Therefore, $\|M\| \leq \|A\| + \|B\| \leq Ck^{\frac{n-2}{2}}\|S - \overline{S}\|_{L^2} + \|B\|$, where $C = \sqrt{C''}$. But B is diagonal with $O(k^n)$ entries of size $O(k^{-2})$, so $\|B\|^2 = O(k^{n-4})$. q.e.d.

Now let $(\mathcal{X}, \mathcal{L})$ be a nontrivial test configuration for (X, L), embedded in $\mathbb{P}(V) \times \mathbb{C}$ (with $V = \oplus_i H^0(X_0, L_0^{k+i})^*$) as before, induced by a $\mathbb{C}^*$-action on $\mathbb{P}(V)$ that takes X to the limit X_0. Suppose that L has a metric h with positive curvature $2\pi\omega$, inducing the Hilb(h, w)-metric on $\oplus_i H^0(X, L^{k+i})$. Applying [**Don05**, lemma 2] to each of the spaces $H^0(X, L^{k+i})$, we get a metric on V such that the induced S^1-action is unitary. Therefore, the infinitesimal generator A^{k+i} of the induced action on $H^0(L_0^{k+i})$ is hermitian. We set

$$A := \bigoplus_i A^{k+i}. \tag{6.7}$$

As in Section 2.9, we get a hamiltonian H_A for the S^1-action on $\mathbb{P}(V)$ by contracting its vector field on the circle bundle $S(\mathcal{O}_{\mathbb{P}(V)}(1))$ with the connection 1-form whose curvature is ω_{FS}. It is

$$H_A \colon \mathbb{P}(V) \to \mathbb{R}, \qquad H_A([v]) = \frac{1}{c}\sum_i \lambda^{2(k+i)}(v)\langle A^{k+i}v_{k+i}, v_{k+i}\rangle,$$

using the inner product $\langle \cdot, \cdot \rangle$ on V and λ as defined in (3.5). This differs from our usual hamiltonian $m_A = \operatorname{tr}(mA)$ of (3.10) by the additive constant $\sum_i c_i \dim V^{k+i}$, and by the multiplicative factor $\frac{1}{c}$. (The latter scaling compensates for the fact that m_A is the hamiltonian for $c\omega_{FS}$; see Definition 3.3.)

By Proposition 2.19, then, the polynomial part of the total weight of the $\mathbb{C}^*$-action on V^* is $w(k) = b_0 k^{n+1} + b_1 k^n$, where $b_0 = \int_{X_0} H_A \frac{\omega_{FS}^n}{n!}$. From this we can define the Futaki invariant $F_1(\mathcal{X}, \mathcal{L})$ of the test configuration $(\mathcal{X}, \mathcal{L})$ as in Definition 6.4.

Theorem 6.8. *In the set-up as above, suppose that ω has constant scalar curvature. Then $F_1(\mathcal{X}, \mathcal{L}) \geq 0$.*

Proof. In the notation above, set $s = \log t$ and let $X_t = \exp(sA).X$ denote the fibre of the given test configuration over $t \in \mathbb{C}$, with central fibre X_0 the limit of $\exp(sA).X$ as $s \to -\infty$.

For fixed k, $\operatorname{tr}(M_k(X_s)A)$ is an increasing function of $s \in \mathbb{R}$, because $\operatorname{tr}(M(X)A)$ is a hamiltonian for the action of $\exp(sA)$ on the space of sub-orbifolds of $\mathbb{P}(V)$. Explicitly, substituting $v = Jv_A$ into (3.13) shows that the derivative of $\operatorname{tr}(M_k(X_s)A)$ is $\Omega(Jv_A, v_A) > 0$. Therefore,

$$\operatorname{tr}(AM_k(X)) = \operatorname{tr}(AM_k(X_1)) \geq \lim_{s\to-\infty} \operatorname{tr}(AM_k(X_t)) = \operatorname{tr}(AM_k(X_0)).$$

Recalling the definition of $M_k(X)$ (3.11), this gives

$$\begin{aligned}\|A\|\|M_k(X)\| \;\geq\; \int_{X_0} m_A \frac{\omega_{FS}^n}{n!} &= \int_{X_0} cH_A \frac{\omega_{FS}^n}{n!} - \mathrm{vol}\sum_i c_i \operatorname{tr}(A^{k+i}) \\ &= cb_0 - a_0 \sum_i c_i w(H^0(L^{k+i})), \qquad (6.9)\end{aligned}$$

by Proposition 2.19. Here we are writing $h^0(L^k) = a_0k^n + a_1k^{n-1} + \widetilde{o}(k^{n-1})$ and $w(H^0(L^k)) = b_0k^{n+1} + b_1k^n + \widetilde{o}(k^n)$. Lemma 6.5 then gives

$$\begin{aligned} c = \sum_i c_i(k+i)h^0(L^{k+i}) &= \tilde{a}_0k^{n+1} + \tilde{a}_1k^n + O(k^{n-1}), \\ \text{and} \quad \sum_i c_i w(H^0(L^{k+i})) &= \tilde{b}_0k^{n+1} + \tilde{b}_1k^n + O(k^{n-1}),\end{aligned}$$

where

$$\begin{aligned} \tilde{a}_0 = a_0\sum_i c_i \quad &\text{and} \quad \tilde{a}_1 = \sum_i c_i(a_0 i(n+1) + a_1), \\ \tilde{b}_0 = b_0\sum_i c_i \quad &\text{and} \quad \tilde{b}_1 = \sum_i c_i(b_0 i(n+1) + b_1).\end{aligned}$$

Therefore, (6.9) becomes

$$\begin{aligned}\|A\|\|M_k(X)\| &\geq c\left(b_0 - a_0\frac{\tilde{b}_0k^{n+1} + \tilde{b}_1k^n + O(k^{n-1})}{\tilde{a}_0k^{n+1} + \tilde{a}_1k^n + O(k^{n-1})}\right) \\ &= ca_0\left(k^{-1}\frac{\tilde{b}_0\tilde{a}_1 - \tilde{b}_1\tilde{a}_0}{\tilde{a}_0^2} + O(k^{-2})\right) \\ &= ca_0\left(k^{-1}\frac{b_0a_1 - b_1a_0}{a_0^2} + O(k^{-2})\right) \\ &= ca_0\left(-k^{-1}F_1 + O(k^{-2})\right).\end{aligned}$$

Now $\|A\|^2 = |\operatorname{tr} A^2| = O(k^{n+2})$ by (2.20), and c is strictly of order $O(k^{n+1})$. So Theorem 6.6 now gives

$$k^{\frac{n-2}{2}}\|\mathrm{Scal}(\omega) - \overline{S}\|_{L^2} + O(k^{\frac{n-4}{2}}) \geq Ck^{\frac{n}{2}}\left(-k^{-1}F_1 + O(k^{-2})\right)$$

for some constant $C > 0$. Hence when $\mathrm{Scal}(\omega)$ is constant (and therefore equal to $\overline{S}$), we see that $F_1 \geq 0$. q.e.d.

Corollary 6.10. *Let (X, L) be a polarised orbifold with cyclic stabiliser groups. If X admits an orbifold Kähler metric $\omega \in \mathcal{K}(c_1(L))$ with constant scalar curvature, then (X, L) is K-semistable.*

7. Slope stability of orbifolds

To get examples where K-stability obstructs the existence of constant scalar curvature metrics, we need a supply of test configurations for which we can calculate the Futaki invariant. To this end, we briefly describe the notion of slope stability. The detailed descriptions in [**RT06**, **RT07**] extend easily from manifolds to orbifolds with a few minor changes.

Fix an n-dimensional polarised orbifold (X, L) and a sub-orbischeme (or substack) $Z \subset X$: an invariant subscheme Z_U in each orbifold chart $U \to U/G \subset X$ such that for each injection of charts $U' \hookrightarrow U$, the subscheme $Z_{U'}$ is the scheme-theoretic intersection $Z_U \cap U'$. In most of our examples, Z will be smooth but with generic stabilisers. Working equivariantly in charts, one can produce a new orbischeme, the blowup $\pi\colon \hat{X} \to X$ of X along Z. Locally this is the blowup of U in Z_U divided by the induced action of the Galois group on this blowup. The exceptional divisors glue to give an orbifold exceptional divisor $E \subset \hat{X}$.

For large N, $\pi^* L^N(-E)$ is positive. (From now on we will suppress π^*.) Thus we can define the *Seshadri constant* by

$$\epsilon_{\rm orb}(Z) = \sup\left\{x \in \mathbb{Q}_+ \colon (L(-xE))^M \text{ is ample for some } M \in \mathbb{N}\right\}.$$

For example, if we put $\mathbb{Z}/m$ stabilisers along a smooth divisor $D \subset X$, then as in Section 2.4 there is a well defined orbi-divisor D/m whose Seshadri constant $\epsilon_{\rm orb}(D/m) = m\epsilon(D)$ is m-times the usual Seshadri constant of D in the underlying space of X.

To get a test configuration from Z, consider the suborbifold $Z \times \{0\} \subset X \times \mathbb{C}$. Blowing this up gives the degeneration $\mathcal{X} \to X \times \mathbb{C} \to \mathbb{C}$ to the normal cone of Z with exceptional divisor P. As shown in [**RT07**, proposition 4.1] for schemes (and the same results go through easily for orbifolds), $\epsilon_{\rm orb}(Z \times \{0\}) = \epsilon_{\rm orb}(Z)$. Let $p\colon \mathcal{X} \to X$ be the projection. Then for generic $c \in (0, \epsilon_{\rm orb}(Z)) \cap \mathbb{Q}$, general integer powers of $\mathcal{L}_c := p^*L(-cP)$ define a polarisation of $\mathcal{X}$. The natural action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ (trivial on (X, L), weight one on $\mathbb{C}$) lifts naturally to a linearised action on $(\mathcal{X}, \mathcal{L})$, and thus for such c we have a test configuration $(\mathcal{X}, \mathcal{L}_c)$ with general fibre (X, L). The central fibre is $\mathcal{X}_0 = \hat{X} \cup_E P$ consisting of the blowup $\hat{X} \to X$ along Z glued to P along E, and the induced $\mathbb{C}^*$ action is trivial on $\hat{X}$ and acts by scaling P along the normal to E.

As usual, we write

$$h^0(L^k) = a_0 k^n + a_1 k^{n-1} + \tilde{o}(k^{n-1}), \tag{7.1}$$

and then define the *slope* of (X, L) to be

$$\mu(X, L) = \frac{a_1}{a_0} = -\frac{n\int_X c_1(K_{\rm orb}).c_1(L)^{n-1}}{2\int_X c_1(L)^n},$$

by (2.17). To define the slope of $Z \subset X$, we work on the orbifold blowup $\pi\colon \hat{X} \to X$ along Z with exceptional divisor E. Then orbifold Riemann-Roch to $L^k(-\frac{j}{k}E)$ for fixed j (and $k = jK$ for some integer K) takes the form

$$h^0(L^k(-jE)) = p(k,j) + \epsilon_p(k,j). \tag{7.2}$$

Here p is a polynomial of two variables of total degree n and ϵ_p is a sum of terms of the form $r_p\delta'$ where r_p is a polynomial of two variables of total degree $n-1$ and $\delta' = \delta'(k,j)$ is periodic in each variable with average $\sum_{k,j=1}^{M} \delta'(k,j) = 0$. Define polynomials $a_i(x)$ by

$$p(k,xk) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}) \quad \text{for } kx \in \mathbb{N}.$$

Then the *slope* of Z (with respect to c) is

$$\mu_c(\mathscr{I}_Z) := \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2}dx}{\int_0^c a_0(x)dx}\,. \tag{7.3}$$

The only difference from the manifold case is that we ignored the periodic terms in the relevant Hilbert functions. This amounts to replacing K_X by K_{orb}.

Definition 7.4. We say that (X,L) is *slope semistable with respect to Z* if

$$\mu_c(\mathscr{I}_Z) \le \mu(X) \quad \text{ for all } 0 < c < \epsilon_{\mathrm{orb}}(Z).$$

We say that X is *slope semistable* if it is slope semistable with respect to all sub-orbischemes $Z \subset X$.

Alternatively, just as in the manifold case [**RT06**, definition 3.13], we can put $\tilde{a}_i(x) := a_i - a_i(x)$ and define the *quotient slope* of Z as

$$\mu_c(\mathcal{O}_Z) := \frac{\int_0^c \tilde{a}_1(x) + \frac{\tilde{a}_0'(x)}{2}dx}{\int_0^c \tilde{a}_0(x)dx}\,, \tag{7.5}$$

and (X,L) is slope semistable with respect to Z if and only if $\mu(X) \le \mu_c(\mathcal{O}_Z)$ for all $0 < c < \epsilon_{\mathrm{orb}}(Z)$.

One can check easily that slope semistability is invariant upon replacing L by a positive power. The point of these definitions is that the sign of the Futaki invariant of the test configuration given by deformation to the normal cone of Z is the same as the sign of $\mu(X) - \mu_c(\mathscr{I}_Z)$, resulting in the following slope obstruction to stability.

Theorem 7.6. *If (X,L) is K-semistable, then it is slope semistable.*

Proof. The argument is essentially the same as that in the smooth case [**RT07**, section 4]; only the Riemann-Roch formula changes. Since being not slope semistable is an open condition, we may without loss of generality assume that $c < \epsilon_{\mathrm{orb}}(Z)$ is general, and so by rescaling L we may assume that c is integral and coprime to m, making $(\mathcal{X}, \mathcal{L}_c)$ a test configuration. The space of sections on the central fibre of the test configuration splits as

$$H^0_{\mathcal{X}_0}(\mathcal{L}_c^k) = H^0_X(L^k \otimes \mathcal{I}_Z^{ck}) \ \oplus \bigoplus_{j=1}^{ck} t^j \, \frac{H^0_X(L^k \otimes \mathcal{I}_Z^{ck-j})}{H^0_X(L^k \otimes \mathcal{I}_Z^{ck-j+1})} \, . \tag{7.7}$$

Here $H^0_X(L^k \otimes \mathcal{I}_Z^j)$ is the space of sections of L^k which vanish to order j on Z (in an orbi-chart). The coordinate t is pulled back from the base $\mathbb{C}$, and is acted on by $\mathbb{C}^*$ with weight -1. Therefore (7.7) is the weight space decomposition of $H^0_{\mathcal{X}_0}(\mathcal{L}_c^k)$, with total weight

$$w(H^0_{\mathcal{X}_0}(\mathcal{L}_c^k)) \ \ = \ \ -\sum_{j=1}^{ck} j\Big(h^0_X(L^k \otimes \mathcal{I}_Z^{ck-j}) - h^0_X(L^k \otimes \mathcal{I}_Z^{ck-j+1})\Big).$$

Some manipulation, and the vanishing of higher cohomology of the push-downs of these sheaves to the underlying scheme, give

$$\begin{aligned}\sum_{j=1}^{ck} h^0_X(L^k \otimes \mathcal{I}_Z^j) - ckh^0_X(L^k) &= \sum_{j=1}^{ck} h^0_{\hat{X}}(L^k(-jE)) - ckh^0(L^k) \\ &= \sum_{j=1}^{ck} p(k,j) + \epsilon_p(k,j) - ck[h(k) + \tilde{o}(k^{n-1})].\end{aligned}$$

By Lemma 7.8 below, the periodic terms do not contribute to the top two order parts of this sum, so the leading order polynomial parts of the weight are

$$w(k) = \sum_{j=1}^{ck} p(j,k) - ckh(k) + \tilde{o}(k^n).$$

The calculation of the Futaki invariant is now exactly as in the smooth case [**RT07**, proposition 4.14 and equation 4.19], yielding

$$F_1(\mathcal{X}, \mathcal{L}_c) = (\mu(X) - \mu_c(\mathcal{I}_Z)) \frac{\int_0^c a_0(x)dx}{a_0} \, .$$

This is nonnegative if and only if X is slope semistable with respect to Z. q.e.d.

Lemma 7.8. *Suppose $\delta(k,j)$ is periodic in each variable with period m and average $\sum_{k,j=1}^m \delta(k,j) = 0$. Suppose also that $r(k,j)$ is a polynomial of two variables of total degree $n-1$ and c is a fixed integer. Then*

$$\sum_{j=1}^{ck} r(k,j)\delta(k,j) = \epsilon(k) + O(k^{n-1}),$$

where $\epsilon(k)$ is a sum of terms of the form $r(k)\delta'(k)$, with r a polynomial of degree n and δ' periodic of average zero.

Proof. By linearity it is sufficient to consider the case where $r(k,j) = j^{n-1}$. Set $j_0 = \left\lfloor \frac{ck}{m} \right\rfloor m$ and split the sum into two pieces depending on whether $j \leq j_0$ or $j \geq j_0 + 1$. In the first case, writing $j = mu + v$,

$$\sum_{j=0}^{j_0} j^{n-1}\delta(k,j) = \sum_{v=1}^{m} \delta(k,v) \sum_{u} (um+v)^{n-1}.$$

This splits into pieces of the form $P(k)\epsilon(k)$ where $\epsilon(k)$ is periodic and P is a polynomial of degree at most n, with the degree being equal to n only when $\epsilon(k) = \sum_{v=1}^{m} \delta(v,k)$, in which case ϵ has average zero. Then note that if P has degree $n-1$, there is a constant a so that $\epsilon - a$ has average zero, and $P\epsilon = P(\epsilon - a) + aP = P(\epsilon - a) + O(k^{n-1})$. Thus, after some rearrangement, this part of the sum is of the form claimed. The sum for $j \geq j_0 + 1$ immediately splits into terms of the form $P(k)\epsilon(k)$ where P has degree at most $n-1$ and ϵ is periodic, so by the same argument these terms are also of the required form. q.e.d.

We can calculate the slope of (sufficiently nice) suborbifolds much as in the manifold case. For instance, let $\hat{X} \xrightarrow{\pi} X$ be the orbifold blowup along a smooth Z of codimension $r \geq 2$, with orbifold exceptional divisor E. Then $K_{\mathrm{orb},\hat{X}} = \pi^* K_{\mathrm{orb},X} + (r-1)E$, and

$$\begin{aligned} a_0(x) &= -\frac{\int_{\hat{X}} c_1(L(-xE))^n}{n!}, \\ a_1(x) &= -\frac{\int_{\hat{X}} c_1(K_{\mathrm{orb},\hat{X}}) c_1(L(-xE))^{n-1}}{2(n-1)!}. \end{aligned} \tag{7.9}$$

So the formulae only differ from those in [**RT06**] in replacing K_X by K_{orb}. For example, if Z is as small as possible—the invariant subvariety defined by a reduced fixed point upstairs in an orbifold chart—then these quickly imply

$$\mu_c(\mathcal{O}_Z) = \frac{n(n+1)}{2c}. \tag{7.10}$$

Notice the order of the stabiliser group at this point does not feature; however, it enters into the Seshadri constant of Z and so does affect slope stability.

Similarly, if Z is an orbifold divisor in an orbifold surface, then

$$\mu_c(\mathcal{O}_Z) = \frac{3(2L.Z - c(K_{\mathrm{orb}}.Z + Z^2)}{2c(3L.Z - cZ^2)}.$$

8. Applications and further examples

8.1. Orbifold Riemann surfaces. By an orbifold Riemann surface we mean an orbifold of complex dimension one. This is equivalent to the data of a Riemann surface of genus $g \geq 0$ and r points $p_1, \ldots, p_r \in X$ marked by orders of stabiliser groups $m_1, \ldots, m_r \geq 2$. We assume $r \geq 1$.

Theorem 8.1. *A polarised Riemann surface (X, L) is slope semistable if and only if*

$$2g + \sum_{i=1}^{r} \left(\frac{m_i - 1}{m_i} \right) \geq 2 \max_{i=1,\dots,r} \left\{ \frac{m_i - 1}{m_i} \right\} \tag{8.2}$$

Proof. The orbifold canonical bundle of X is

$$K_{\mathrm{orb}} = K_X + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) p_i$$

so the slope of X is

$$\mu(X, L) = -\frac{\deg K_{\mathrm{orb}}}{2 \deg L} = \frac{1 - g - \frac{1}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right)}{\deg L}. \tag{8.3}$$

Without loss of generality, assume $m_1 \geq m_2 \geq \cdots \geq m_r$. Let $Z = \{p_1/m\}$ be the orbifold point of order m_1 with a reduced lift upstairs. Then

$$\mu_c(\mathcal{O}_Z) = c^{-1} \quad \text{and} \quad \epsilon_{\mathrm{orb}}(Z, X, L) = m_1 \deg L.$$

Now if (X, L) is semistable, then $\mu_\epsilon(\mathcal{O}_Z) \geq \mu(X)$, so

$$\frac{1}{m_1} \geq 1 - g - \frac{1}{2} \sum_{i=1}^{r} \frac{m_i - 1}{m_i}$$

which rearranges to give the inequality (8.2).

For the converse suppose that (8.2) holds and consider an orbifold subspace $Z \subset X$. This Z is an orbifold divisor whose degree is a rational number $q \geq \frac{1}{m_1}$. Thus $\epsilon := \epsilon_{\mathrm{orb}}(Z, L) = \frac{1}{q} \deg L \leq m_1 \deg L$. As $\mu_c(\mathcal{O}_Z) = c^{-1}$ is decreasing with respect to c, we get $\mu_c(\mathcal{O}_Z) \geq \epsilon^{-1} \geq (m_1 \deg L)^{-1}$. But (8.2) implies that this is greater than or equal to $\mu(X, L)$ (8.3), so X is slope semistable. q.e.d.

Remark 8.4. 1) It is hard to either violate or achieve equality in the inequality (8.2). Either would imply that $2g + 0 \leq 2 \max(1 - \frac{1}{m_i}) < 2$ and so $g = 0$. Then since each integer $m_i \geq 2$, we find there are only three cases in which an orbifold Riemann surface is not strictly slope stable:

a) $g = 0,\ r = 1$ (this gives $\mathbb{P}(1, m)$),

b) $g = 0,\ r = 2,\ m_1 \neq m_2$ (giving $\mathbb{P}(m_1, m_2)$ if $\mathrm{hcf}(m_1, m_2) = 1$), and

c) $g = 0,\ r = 2,\ m_1 = m_2$.

In the first two cases, (X, L) is not slope semistable and so not cscK. In the third case, (X, L) is actually slope polystable, as we now describe. The only way in which $\mu_c(\mathcal{O}_Z) = \mu(X)$ can occur in the proof of (8.1) is if $Z = \{p_1/m_1\}$ or $Z = \{p_2/m_2\}$ and $c = \epsilon(Z) = 2$. In this case, deformation to the normal cone $\mathcal{X}, \mathcal{L}_c)$ has $\mathcal{L}_c$ only semi-ample, pulled back from the contraction of the proper transform of the central fibre $X \times \{0\}$. This contraction is in fact a product configuration $X \times \mathbb{C}$ (with a nontrivial $\mathbb{C}^*$-action).

2) The stability condition (8.2) is actually a special case of (8.10) and thus an manifestation of the index obstruction which we discuss below.

Slope stability of orbifold Riemann surfaces fits perfectly into the known theory of orbifold cscK metrics which has been studied by several authors including Picard [**Pic05**], McOwen [**McO88**], and Troyanov [**Tro89**, **Tro91**]. In the terminology of this paper, Troyanov's results can be paraphrased as follows:

Corollary 8.5 (Troyanov)**.** *Let (X, L) be a polarised orbifold Riemann surface. Then $c_1(L)$ admits an orbifold cscK metric if and only if it is slope polystable.*

Proof. The main theorems in [**Tro91**] imply that X admits a cscK metric when strict inequality holds in (8.2) i.e. as long as (X, L) is slope stable. (To compare our notation with Troyanov's, set $\theta_i = \frac{2\pi}{m_i}$, and $\chi_{\text{orb}} = -\deg K_{\text{orb}}$. Then this is Theorem A in [**Tro91**] when $\chi_{\text{orb}} < 0$, Proposition 2 when $\chi_{\text{orb}} = 0$, and Theorem C when $\chi_{\text{orb}} > 0$.) The only way that (X, L) can be slope polystable and not slope stable is case c): if $g = 0$, $r = 2$ and $m_1 = m_2$. In this case, X is the global quotient $\mathbb{P}^1/(\mathbb{Z}/m)$ with orbifold cscK metric descended from the Fubini-Study metric on $\mathbb{P}^1$.

For the converse, if (X, L) is not slope polystable, then $g = 0$ and either a) $r = 1$ or b) $r = 2$ and $m_1 \neq m_2$. In these two cases, (X, L) is not slope semistable, which by Corollary 6.10 and Theorem 7.6 implies that X does not admit an orbifold cscK metric. q.e.d.

Remark 8.6. The statement that if $g = 0$ and a) $r = 1$ or b) $r = 2$ and $m_1 \neq m_2$ then X does not admit a cscK metric has also been proved by Troyanov [**Tro89**, theorem I]. Troyanov's work applies much more generally to cone angles not necessarily of the form $2\pi/m$, and this is also studied further in [**Che98**, **CL95**, **LT92**]. We hope to return to cone angles in $2\pi\mathbb{Q}_+$ using the method described in the Introduction.

8.2. Index obstruction to stability. Recall that the index $\text{ind}(X)$ of a Fano manifold X is defined to be the largest integer r such that K_X^{-1} is linearly equivalent to rD for some Cartier divisor $D \subset X$, and it is well known that if X is smooth, then $\text{ind}(X) \leq n + 1$ with equality if and only if $X = \mathbb{P}^n$. By contrast, for an Fano orbifold (X, Δ) it is possible that $K_{\text{orb}}^{-1} \cong \mathcal{O}(rD)$ where D is an orbi divisor and $r \in \mathbb{N}$ is larger than $n+1$. We will show that this prevents (X, Δ) from being K-stable. In fact, the same is true under the weaker condition that $K_{\text{orb}}^k \cong \mathcal{O}(krD)$ for some $k \in \mathbb{N}$ and $n + 1 \leq r \in \mathbb{Q}$.

Theorem 8.7. *(Index Obstruction) Let $(X, K_{orb}^{-1}) \supset D$ be a Fano orbifold and an orbi divisor. Suppose that $K_{orb}^{-k} \cong \mathcal{O}(krD)$ for some $k > 0$ and $r \in \mathbb{Q}_+$. If $r > n+1$, then (X, K_{orb}^{-1}) is slope unstable, and thus does not admit an orbifold Kähler-Einstein metric.*

Proof. Set $L = K_{\mathrm{orb}}^{-1}$. Using (7.9) to calculate the slope,

$$\begin{aligned} a_0(x) &= \frac{1}{n!}\int_X (c_1(L) - xc_1(D))^n = \frac{1}{n!}(r-x)^n \int_X c_1(D)^n, \\ a_1(x) + \frac{a_0'(x)}{2} &= -\frac{1}{2(n-1)!}(r-1)(r-x)^{n-1}\int_X c_1(D)^n. \end{aligned}$$

Now $a_0 = \frac{r^n}{n!}\int_X c_1(D)^n$ and $a_1 = \frac{r^n}{2(n-1)!}\int_X c_1(D)^n$, so $\mu(X,L) = \frac{n}{2}$. The Seshadri constant of D is r. Using the definition of the slope (7.5),

$$\mu_r(\mathcal{O}_D) = \frac{(n+1)((n-1)r+1)}{2nr},$$

which is less than $\mu(X) = n/2$ if and only if $r > n+1$. q.e.d.

Remark 8.8. At the level of Kähler-Einstein metrics, the analogous result has already been proved by Gauntlett-Martelli-Sparks-Yau using the "Lichnerowicz obstruction" to the existence of Sasaki-Einstein metrics with non-regular Reeb vector fields [**GMSY07**, section 2.2]. In fact it was their work that originally motivated this paper. In the same paper the authors discuss the "Bishop obstruction" which we have been unable to interpret in terms of stability.

Example 8.9. (Weighted projective space) Consider weighted projective space $\mathbb{WP} = \mathbb{P}(\lambda_0, \dots, \lambda_n)$, with $\lambda_0 \le \lambda_1 \le \dots \le \lambda_n$ *not all equal*. Then $\{x_0 = 0\}$ defines an effective divisor in $\mathcal{O}(\lambda_0)$, while $K_{\mathrm{orb}}^{-1} \cong \mathcal{O}(\sum_i \lambda_i)$. Since $\sum_i \lambda_i > (n+1)\lambda_0$, the index obstruction shows that $\mathbb{WP}$ is unstable, recovering the well known fact that it does not admit an orbifold cscK metric.

Example 8.10. (Orbifold projective space) Let $X = \mathbb{P}^n$ and take $n+2$ hyperplanes $H_1, \dots, H_{n+2}$ in general position, and integers $m_i \ge 2$. Setting

$$\Delta = \sum_{i=1}^{n+2}\left(1 - \frac{1}{m_i}\right) H_i,$$

we consider the orbifold $(\mathbb{P}^n, \Delta)$. Then $K_{\mathrm{orb}}^{-1} = K_{\mathbb{P}^n}^{-1}(-\Delta)$ becomes equivalent after passing to powers to

$$\mathcal{O}\left(n+1-\sum_{i=0}^{n+2}\left(1-\frac{1}{m_i}\right)\right) = \mathcal{O}\left(-1+\sum_{i=1}^{n+2}\frac{1}{m_i}\right). \tag{8.11}$$

Thus $(\mathbb{P}^n, \Delta)$ is a Fano orbifold as long as $\sum_{i=1}^{n+2}\frac{1}{m_i} > 1$.

The right hand side of (8.11) can be written

$$\mathcal{O}\left(m_j\left(-1+\sum_{i=1}^{n+2}\frac{1}{m_i}\right)D_j\right),$$

where $D_j = \frac{1}{m_j}H_j$ is an orbi divisor. Thus by the index obstruction, if (X, Δ) is a semistable Fano orbifold, then

$$\sum_{i=1}^{n+2}\frac{1}{m_i} \leq 1+(n+1)\min_{1\leq i\leq n+2}\left(\frac{1}{m_i}\right). \tag{8.12}$$

Remark 8.13. The previous example is considered by Ghigi-Kollár [**GK07**, example 43]. They show that as long as

$$1 < \sum_{i=1}^{n+2}\frac{1}{m_i} < 1+(n+1)\min_{1\leq i\leq n+2}\left(\frac{1}{m_i}\right)$$

then (X, Δ) admits a Kähler-Einstein metric. Thus the previous example suggests this condition is strict (our slightly weaker inequality comes from only having a proof that a cscK metric implies semistability rather than polystability). We remark that Ghigi-Kollár also prove a much more general condition under which a Kähler-Einstein Fano manifold with boundary divisor Δ yields a Kähler-Einstein orbifold (X, Δ) [**GK07**, theorem 41]. It is not the case that this condition is simply the index obstruction, and we have not been able to determine if this condition is related to slope stability or if it is also strict.

8.3. Orbifold ruled surfaces. Let (Σ, L) be a polarised orbifold Riemann surface and $\pi\colon E \to \Sigma$ be an orbifold vector bundle of rank r. Then $\mathbb{P}(E)$ is itself naturally an orbifold: on a chart $U \to U/G$ of Σ, the G action on $E|_U$ induces an action on $\mathbb{P}(E|_U)$ (which is effective as the action on U is) and these give orbifold charts on $\mathbb{P}(E)$. Suppose that the G-action on the fibres E over points of Σ with stabiliser group G has distinct eigenvalues, so that $\mathbb{P}(E)$ has codimension two orbifold locus and all fibres are finite quotients of $\mathbb{P}^{r-1}$. The hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is both locally ample and relatively ample, so $L_m := \mathcal{O}_{\mathbb{P}(E)}(1)\otimes\pi^*L^m$ is ample for m sufficiently large.

We claim that stability of $\mathbb{P}(E)$ is connected to stability of the underlying bundle E. Here stability of a bundle is to be taken in the sense of Mumford, so define

$$\mu_E := \frac{\deg E}{\operatorname{rank} E}$$

where the degree is taken in the orbifold sense. Then E is defined to be stable if for all orbifold bundles F with a proper injection $F \subset E$ we have $\mu_F < \mu_E$.

Now if $F \subset E$, then $\mathbb{P}(F)$ is a suborbifold of $\mathbb{P}(E)$. Using $\pi_*\mathcal{O}_{\mathbb{P}(E)}(k) = S^kE^*$, one can use orbifold Riemann-Roch to compute the slope of each in exactly the same way as in the manifold case [**RT06**, section 5.4]. The upshot is that the Seshadri constant of $\mathbb{P}(F)$ is $\epsilon_{\text{orb}}(\mathbb{P}(F)) = 1$ and

$$\mu_1(\mathcal{O}_{\mathbb{P}(F)}) - \mu(\mathbb{P}(E)) = C(\mu_E - \mu_F)\Big(rm + (r-1)\mu(\Sigma) - r\mu_E\Big)$$

for some $C > 0$, where $\mu(\Sigma) = -\deg K_{\text{orb}}/2\deg L$ is the orbifold slope of (Σ, L). The term inside the last set of brackets is positive for any m sufficiently positive that L_m is ample (it is essentially the volume of $(\mathbb{P}(E), L_m)$). Therefore, if E is unstable as an orbifold vector bundle, then $(\mathbb{P}(E), L_m)$ is slope unstable as an orbifold. This result also generalises to higher dimensional base as long as one works near the adiabatic limit of sufficiently large m, just as in the manifold case.

If E is polystable, then $\mathbb{P}(E)$ carries an orbifold cscK metric; see for example [**RS05**]. We therefore get a (partial) converse—for strictly unstable bundles, $(\mathbb{P}(E), L_m)$ does *not* carry an orbifold cscK metric for any m. (The discrepancy lies in strictly semistable, but not polystable, bundles.)

In fact, Rollin and Singer phrase their results in terms of parabolic bundles, but there is a complete correspondence between orbifold bundles E on Σ and parabolic vector bundles E' on the underlying space of Σ. In the notation of Section 2.4, the bundle E' is the pushdown of E from the orbifold to its underlying space; this is therefore the vector bundle analogue of rounding down of $\mathbb{Q}$-divisors in the line bundle case. The information lost is then encoded via the parabolic structure on E' at each of the orbifold points x, with rational weights of the form $p_j/\text{ord}(x)$ for $p_j < \text{ord}(x)$ corresponding to the weights of the action on E_x. See for example [**FS92**, section 5]. Moreover, this correspondence preserves subobjects and their degrees, where the parabolic degree of E' is defined as

$$\operatorname{pardeg} E' = \deg E' + \sum_{x,j} m_{x,j} \frac{p_j}{\text{ord}(x)}\,.$$

Here the sum is over all orbifold points x, and if the parabolic structure over x is given by the flag $F_0 \subset F_1 \subset \ldots \subset E'_x$ then, $m_{x,j} = \dim F_j/F_{j+1}$. Thus orbifold stability of E corresponds precisely to the parabolic stability of E'.

Rollin and Singer [**RS05**] use such orbifold cscK metrics as a starting point to produce ordinary cscK metrics (with zero scalar curvature, in fact) on small blowups of the orbifolds $\mathbb{P}(E)$, using a gluing method. Our results suggest that if E is unstable, destabilised by F, then one should be able to slope destabilise such blowups using the pullback (or proper transform) of $\mathbb{P}(F)$.

8.4. Slope stability of canonically polarised orbifolds. By the orbifold version of the Aubin-Yau theorems, orbifolds which have positive or trivial canonical bundle admit orbifold Kähler-Einstein metrics.

Therefore by Corollary 6.10 they are K-semistable, and so by Theorem 7.6 are also slope semistable. In fact, this can be proved directly. That is, suppose that (X, L) is a polarised orbifold and either

1) K_{orb} is numerically trivial and L is arbitrary *or*
2) $L = K_{\text{orb}}$.

Then (X, L) is slope stable. The proof is the same as the manifold case (see [**RT06**, Theorem 5.4]) or [**RT07**, theorem 8.4], with K_X replaced by K_{orb}, so we do not repeat it here.

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