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LOCALLY STRONGLY CONVEX AFFINE HYPERSURFACES WITH PARALLEL CUBIC FORM

Zejun Hu, Haizhong Li & Luc Vrancken

Abstract

We give a complete classification of locally strongly convex affine hypersurfaces of $\mathbb{R}^{n+1}$ with parallel cubic form with respect to the Levi-Civita connection of the affine Berwald-Blaschke metric. It turns out that all such affine hypersurfaces are quadrics or can be obtained by applying repeatedly the Calabi product construction of hyperbolic affine hyperspheres, using as building blocks either the hyperboloid, or the standard immersion of one of the symmetric spaces $\mathbf{SL}(m,\mathbb{R})/\mathbf{SO}(m)$, $\mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m)$, $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$, or $\mathbf{E}_{6(-26)}/\mathbf{F}_4$.

1. Introduction

In this paper, we study affine hypersurfaces of $\mathbb{R}^{n+1}$. The study of affine differential geometry originates with the work of Blaschke and his coworkers at the beginning of the twentieth century [**Bl**]. A more modern structural approach to this field was given by Nomizu at the 1984 conference Differential Geometry Meeting in Münster [**N**].

In the case that the hypersurface is nondegenerate, it is well known how to induce an affine connection ∇ and a symmetric bilinear form h, called the affine metric, on M. This is done by constructing a canonical transversal vector field to the immersion, called the affine normal. The classical Pick-Berwald theorem states that the induced affine connection coincides with the Levi-Civita connection of the affine metric if and only if the hypersurface is a quadric. For that reason, the difference tensor

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,$$

where $\hat{\nabla}$ is the Levi-Civita connection of the affine metric, plays a fundamental role in affine differential geometry.

Here in this paper, we will always assume that the hypersurface is locally strongly convex, i.e., the affine metric is definite. In this case, if necessary by changing the sign of the affine normal, we may always assume that the affine metric is positive definite. In particular, we will

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be interested in the question when the difference tensor K is parallel with respect to the Levi-Civita connection of the affine metric. As

$$h(K(X,Y),Z) = -\tfrac{1}{2}(\nabla h)(X,Y,Z) = -\tfrac{1}{2}C(X,Y,Z), \tag{1.1}$$

this is equivalent with demanding that the cubic form C is parallel with respect to the Levi-Civita connection $\hat{\nabla}$. The above question was initiated by the work of Bokan, Nomizu, and Simon [**BNS**], who showed that such hypersurfaces are necessarily affine hyperspheres. Actually, using [**DVY**] and [**HLSV**] we can see that such hypersurfaces, provided the difference tensor does not vanish identically, are necessarily hyperbolic homogeneous affine hyperspheres. Following the work of [**BNS**], further results in low dimensions were obtained in [**MN**], [**V1**], [**DV1**], [**DVY**], and [**HLSV**]. For all dimensions, a similar problem was studied in [**LW**] for centroaffine hypersurfaces of $\mathbb{R}^{n+1}$ under the additional condition that the Blaschke metric is flat.

Note that from a global point of view, locally strongly convex hyperbolic affine hyperspheres have been widely studied; see amongst others the works of [**C**], [**CY**], and the recent survey paper [**Lo**] (see references there). Even assuming such global conditions, the class of hyperbolic affine hyperspheres is surprisingly large. Even more, locally, in arbitrary dimensions one is still far away from a complete understanding of such hypersurfaces. Worthwhile to mention from a local point of view are the classification of the affine hyperspheres with constant sectional curvature (see [**VLS**] for the locally strongly convex case, or [**V2**] for the general case with non-vanishing Pick invariant) and the Calabi construction ([**C**], [**DV2**]) of hyperbolic affine hyperspheres which allows to associate with two hyperbolic affine hyperspheres $\psi_1 : M_1 \to \mathbb{R}^{n_1+1}$ and $\psi_2 : M_2 \to \mathbb{R}^{n_2+1}$ two new immersions φ and $\tilde{\varphi}$: for $p \in M_1, t \in \mathbb{R}$,

$$\varphi(p,t) = (c_1 e^{\frac{t}{\sqrt{n_1+1}}}\psi_1(p), c_2 e^{-\sqrt{n_1+1}t}) \in \mathbb{R}^{n_1+2},$$

and, for $p \in M_1, q \in M_2, t \in \mathbb{R}$,

$$\tilde{\varphi}(p,q,t) = (c_1 e^{\sqrt{\frac{n_2+1}{n_1+1}}t}\psi_1(p), c_2 e^{-\sqrt{\frac{n_1+1}{n_2+1}}t}\psi_2(q)) \in \mathbb{R}^{n_1+n_2+2},$$

which are both again hyperbolic affine hyperspheres. Here, φ and $\tilde{\varphi}$ are respectively called the Calabi product of an affine hypersphere and a point, and the Calabi product of two hyperbolic affine hyperspheres. Note that a straightforward calculation (see [**HLV**]) shows that the Calabi product of hyperbolic affine hyperspheres with parallel difference tensor again has parallel difference tensor.

A decomposition theorem, which can be seen as a converse of the previous statement, was obtained in [**HLV**]. In this paper we further develop the techniques started in [**HLSV**] in order to obtain the following complete classification.

Classification Theorem. *Let M^n be an n-dimensional ($n \geq 2$) locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{\nabla} C = 0$. Then M is a quadric (i.e., $C = 0$) or a hyperbolic affine hypersphere with $C \neq 0$; in the latter case either*

(i) *M^n is obtained as the Calabi product of a lower dimensional hyperbolic affine hypersphere with parallel cubic form and a point, or*
(ii) *M^n is obtained as the Calabi product of two lower dimensional hyperbolic affine hyperspheres with parallel cubic form, or*
(iii) *$n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, and (M^n, h) is isometric with $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m)$, and the immersion is affinely equivalent to the standard embedding of $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m) \hookrightarrow \mathbb{R}^{n+1}$, or*
(iv) *$n = m^2 - 1$, $m \geq 3$, and (M^n, h) is isometric with $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m)$, and the immersion is affinely equivalent to the standard embedding of $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m) \hookrightarrow \mathbb{R}^{m^2}$, or*
(v) *$n = 2m^2 - m - 1$, $m \geq 3$, and (M^n, h) is isometric with $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$, and the immersion is affinely equivalent to the standard embedding of $\mathbf{SU}^*(2m)/\mathbf{Sp}(m) \hookrightarrow \mathbb{R}^{n+1}$, or*
(vi) *$n = 26$ and (M^{26}, h) is isometric with $\mathbf{E}_{6(-26)}/\mathbf{F}_4$, and the immersion is affinely equivalent to the standard embedding of $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$.*

Note that the above theorem implies that all hyperbolic affine hyperspheres with parallel cubic form can be obtained by applying repeatedly the Calabi product construction of hyperbolic affine hyperspheres, using either the hyperboloid, or the standard immersion of one of the symmetric spaces $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m)$, $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m)$, $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$, or $\mathbf{E}_{6(-26)}/\mathbf{F}_4$ as building blocks.

The paper is organized as follows. In section 2, we review relevant materials and some lemmas, which include the decomposition of the tangent space into 3 orthogonal distributions $\mathcal{D}_1$ (which is 1 dimensional), $\mathcal{D}_2$, and $\mathcal{D}_3$, and the definition and properties of a bilinear map L from $\mathcal{D}_2 \times \mathcal{D}_2$ to $\mathcal{D}_3$. In section 3, we introduce, for any unit vector $v \in \mathcal{D}_2$, a linear map $P(v) : \mathcal{D}_2 \to \mathcal{D}_2$ and study its properties. In section 4, we use the previous results to obtain a direct sum decomposition for $\mathcal{D}_2$. We prove that there exist an integer k_0 and unit vectors $v_1, \ldots, v_{k_0} \in \mathcal{D}_2$ such that $\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \cdots \oplus \{v_{k_0}\} \oplus V_{v_{k_0}}(0)$. Here $V_{v_j}(0)$ is the eigenspace of $P(v_j)$ with eigenvalue 0. We also find that $\dim V_{v_1}(0) = \cdots = \dim V_{v_{k_0}}(0)$, which we denote by $\mathfrak{p}$, can only be equal to $0, 1, 3$, or 7. In the final four sections, we consider each of the four cases separately and in each case we obtain a complete classification of the affine hypersurfaces with parallel cubic form.

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2. Preliminaries and Lemmas

For a locally strongly convex hypersurface $F : M^n \hookrightarrow \mathbb{R}^{n+1}$, denote by h the affine metric and let S be the affine shape operator. M^n is called an affine hypersphere if $S = \lambda\, id$, where $\lambda = H := \frac{1}{n}\operatorname{tr} S$ is the affine mean curvature. Assuming that the affine normal is chosen such that h is positive definite, F is called a proper affine hypersphere if $H \neq 0$; if $H > 0$, the proper affine hypersphere is called elliptic, for $H < 0$ hyperbolic. If $H = 0$, the affine hypersphere is called improper or parabolic.

The curvature tensor $\hat{R}$ of $\hat{\nabla}$ is related to S and K by an equation of Gauß type

$$\begin{aligned}\hat{R}(X,Y)Z =& \tfrac{1}{2}\{h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\}\\ &- [K_X, K_Y]Z.\end{aligned}$$

In particular, for affine hyperspheres we have $S = H\, id$ and thus

$$\hat{R}(X,Y)Z \;=\; H\cdot\big(h(Y,Z)X - h(X,Z)Y\big) - [K_X, K_Y]Z. \tag{2.1}$$

We also recall the definition of the curvature tensor acting as derivation:

$$\begin{aligned}\big(\hat{R}(X,Y)K\big)(Z,W) =& \hat{R}(X,Y)K(Z,W)\\ &- K(\hat{R}(X,Y)Z,W) - K(Z,\hat{R}(X,Y)W).\end{aligned} \tag{2.2}$$

Moreover, K satisfies the *apolarity condition*, namely $\operatorname{tr} K_X = 0$ for all X, and has the property that $h(K(X,Y),Z)$ is totally symmetric in X, Y and Z.

Now we assume that M^n is an n-dimensional, locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ which has parallel cubic form, i.e., $\hat{\nabla} C = 0$, or equivalently $\hat{\nabla} K = 0$. Thus, according to [**BNS**], M^n is an affine hypersphere.

Since $\hat{\nabla} C = 0$ implies that $h(C,C)$ is constant, there are two cases. If $h(C,C) = 0$, then $C = 0$ and M^n is an open part of a quadric. Otherwise, C never vanishes, and we assume this from now on. Then, according to [**DVY**], M^n is a locally homogeneous hyperbolic affine hypersphere.

Note that the vanishing of $\hat{\nabla} K$ implies by the Ricci identity that

$$(\hat{R}(X,Y)K)(Z,W) = 0.$$

Using this property, following an idea first introduced by Ejiri [**E**], and since then widely applied and very useful for solving various problems (see e.g. [**DVY**] and [**VLS**]), a special orthonormal basis with respect to the affine metric can be constructed.

Lemma 2.1 (see [**DVY**], [**HLSV**]). *Let M^n be a hyperbolic affine hypersphere such that $S = H \cdot id$ with $H < 0$, and with parallel cubic form. Then, for every $x_0 \in M^n$, there exists an orthonormal basis $\{e_j\}_{1\le j\le n}$, satisfying $K_{e_1}e_j = \lambda_j e_j$, and there exists a number r, $1 \le r < n$, such that*

$$\begin{aligned} \lambda_2 = \lambda_3 = \cdots = \lambda_r = \tfrac{1}{2}\lambda_1, \\ \lambda_{r+1} = \cdots = \lambda_n = -\tfrac{r+1}{2(n-r)}\,\lambda_1 \;=:\; \mu \end{aligned} \tag{2.3}$$

and

$$-H \;=\; \lambda_1^2\, \tfrac{(r+1)^2 + 2(r+1)(n-r)}{4(n-r)^2}. \tag{2.4}$$

Therefore, for a locally strongly convex affine hypersurface with parallel cubic form, considering that it is affine homogeneous, we will restrict our discussions to a fixed point $x_0 \in M^n$, and deal with $(n-1)$ cases $\{\mathfrak{C}_r\}_{1\le r\le n-1}$ as follows:

Case $\mathfrak{C}_1$: $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -\frac{\lambda_1}{n-1}$.

Case $\mathfrak{C}_r$: $\lambda_2 = \cdots = \lambda_r = \frac{1}{2}\lambda_1$ and $\lambda_{r+1} = \cdots = \lambda_n = -\frac{r+1}{2(n-r)}\lambda_1$ for $2 \le r \le n-1$.

To discuss these cases, we first recall the following observations:

Lemma 2.2 (see [**HLSV**]). *If $r > \frac{1}{3}(2n-1)$, then the Case $\mathfrak{C}_r$ does not occur.*

From Lemma 2.2 we see that only the cases $\{\mathfrak{C}_r\}_{1\le r\le \bar{n}}$ are left to be studied, where $\bar{n}$ denotes the largest integer less than or equal to $(2n-1)/3$.

Lemma 2.3 (see [**HLSV**]). *If $2n \equiv 1 \mod 3$, then for the Case $\mathfrak{C}_{\bar{n}}$, we have:*

$$h(K_{e_j}e_k, e_l) = 0, \quad \text{for all } \; j,k,l \ge \bar{n}+1.$$

For case $\mathfrak{C}_r$ with $r \le \bar{n}$, we define $\mathcal{D}_2 := \operatorname{span}\{e_2,\dots,e_r\}$ and $\mathcal{D}_3 := \operatorname{span}\{e_{r+1},\dots,e_n\}$. Then we have

Lemma 2.4 (see [**HLSV**]). (1) *For the Case $\mathfrak{C}_r$, if $v \in \mathcal{D}_2$ and $w \in \mathcal{D}_3$, then $K_v w \in \mathcal{D}_2$.* (2) *If $\dim \mathcal{D}_2 \ge 1$, then for any $v_1, v_2, v_3 \in \mathcal{D}_2$, $h(K_{v_1}v_2, v_3) = 0$.*

For the general case $\mathfrak{C}_r$ with $\dim \mathcal{D}_2 \geq 1$, we recall from [**HLSV**] the definition of the bilinear map L on $\mathcal{D}_2 \times \mathcal{D}_2$ with image in $\mathcal{D}_3$ by

$$L(v_1, v_2) := K(v_1, v_2) - \tfrac{1}{2}\lambda_1 h(v_1, v_2)e_1, \qquad v_1, v_2 \in \mathcal{D}_2. \tag{2.5}$$

Then we have

Lemma 2.5 (see [**HLSV**]). *Assume that* $\dim \mathcal{D}_2 \geq 1$. *Then we have*

(1) *L is isotropic in the sense that (see* [**O**], [**V3**]*)*

$$h(L(v,v), L(v,v)) = \tfrac{n+1}{4(n-r)}\, \lambda_1^2\, (h(v,v))^2, \quad v \in \mathcal{D}_2. \tag{2.6}$$

Moreover, linearizing the above expression, it follows for orthonormal vectors X, Y, Z, and W in $\mathcal{D}_2$ that

$$h(L(X,X), L(X,Y)) = 0, \tag{2.7}$$

$$h(L(X,X), L(Y,Y)) + 2h(L(X,Y), L(X,Y)) = \tfrac{n+1}{4(n-r)}\lambda_1^2, \tag{2.8}$$

$$h(L(X,X), L(Y,Z)) + 2h(L(X,Y), L(X,Z)) = 0, \tag{2.9}$$

$$\begin{aligned} &h(L(X,Y), L(Z,W)) + h(L(X,Z), L(W,Y)) \\ &\qquad + h(L(X,W), L(Y,Z)) = 0. \end{aligned} \tag{2.10}$$

(2) *For Case $\mathfrak{C}_r$ with $\mathrm{Im}\,(L) \neq \mathcal{D}_3$, we have*

$$K(L(v_1, v_2), w) = -\tfrac{(n+1)(r+1)}{4(n-r)^2}\, h(v_1, v_2)\, \lambda_1^2\, w, \tag{2.11}$$

where $v_1, v_2 \in \mathcal{D}_2$ and $w \in \mathcal{D}_3$ such that $w \perp \mathrm{Im}\,(L)$.

We remark that the operator L and its properties will play a crucial role in our investigations. Besides the many properties it possesses which we will derive in the next two sections, we will see from the fifth and later sections that $\mathrm{Tr}\, L = 0$ if and only if the dimension n is completely determined, and moreover in that case it is surprisingly related to the normed division algebras; the latter appears only in dimension $1, 2, 4$, and 8: the real numbers, complex numbers, quaternions, and octonions.

The first property of L allows us to calculate the difference tensor on the image of L. It states;

Lemma 2.6. *For Case $\mathfrak{C}_r$ with $r \geq 2$, let $\{v_1, \ldots, v_{r-1}\}$ be an orthonormal basis of $\mathcal{D}_2$; then we have*

$$\begin{aligned} &K(L(v_p, v_j), L(v_k, v_\ell)) \\ &= -\tfrac{r+1}{2(n-r)}\lambda_1 h(L(v_p, v_j), L(v_k, v_\ell))e_1 \\ &\quad - \tfrac{(n+1)(r+1)\lambda_1^2}{4(n-r)^2} h(v_p, v_j)L(v_k, v_\ell) \\ &\quad + \sum_{m=1}^{r-1} h(L(v_p, v_m), L(v_k, v_\ell))L(v_j, v_m) \\ &\quad + \sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_k, v_\ell))L(v_p, v_m). \end{aligned} \tag{2.12}$$

Proof. From the computation

$$\begin{aligned} \hat{R}(e_1, v_1)v_2 =& \; Hh(v_1, v_2)e_1 - K_{e_1}K_{v_1}v_2 + K_{v_1}K_{e_1}v_2 \\ =& \; Hh(v_1, v_2)e_1 - K_{e_1}\left[\tfrac{\lambda_1}{2}h(v_1, v_2)\, e_1 + L(v_1, v_2)\right] + \tfrac{\lambda_1}{2}K_{v_1}v_2 \\ =& \left(H - \tfrac{\lambda_1^2}{2}\right) h(v_1, v_2)\, e_1 - K_{e_1}L(v_1, v_2) \\ & + \tfrac{\lambda_1}{2}\left[\tfrac{\lambda_1}{2}h(v_1, v_2)e_1 + L(v_1, v_2)\right] \\ =& \left(H - \tfrac{\lambda_1^2}{4}\right) h(v_1, v_2)\, e_1 + \tfrac{n+1}{2(n-r)}\lambda_1 L(v_1, v_2), \end{aligned} \tag{2.13}$$

we have for $v, \tilde{v} \in \mathcal{D}_2$ that

$$\hat{R}(e_1, v)\tilde{v} = -\tfrac{(n+1)^2\lambda_1^2}{4(n-r)^2}h(v, \tilde{v})e_1 + \tfrac{(n+1)}{2(n-r)}\lambda_1 L(v, \tilde{v}).$$

As our hypersurface has parallel cubic form, we have that

$$\begin{aligned} &\hat{R}(e_1, v_p)K(v_j, L(v_k, v_\ell)) \\ &= K(\hat{R}(e_1, v_p)v_j, L(v_k, v_\ell)) + K(v_j, \hat{R}(e_1, v_p)L(v_k, v_\ell)). \end{aligned} \tag{2.14}$$

As $K(v_j, L(v_k, v_\ell)) \in \mathcal{D}_2$ (Lemma 2.4 (1)), we can write:

$$\begin{aligned} K(v_j, L(v_k, v_\ell)) &= \sum_{m=1}^{r-1} h(K(v_j, L(v_k, v_\ell)), v_m)v_m \\ &= \sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_k, v_\ell))v_m. \end{aligned}$$

Using the above formulas, we find that

$$\begin{aligned}
&\hat{R}(e_1,v_p)K(v_j,L(v_k,v_\ell))\\
&=\sum_{m=1}^{r-1} h\big(L(v_j,v_m),L(v_k,v_\ell)\big)\hat{R}(e_1,v_p)v_m\\
&=\sum_{m=1}^{r-1} h\big(L(v_j,v_m),L(v_k,v_\ell)\big)\Big(-\tfrac{(n+1)^2\lambda_1^2}{4(n-r)^2}h(v_p,v_m)e_1\\
&\qquad+\tfrac{(n+1)}{2(n-r)}\lambda_1 L(v_p,v_m)\Big)\\
&=-\tfrac{(n+1)^2\lambda_1^2}{4(n-r)^2}h\big(L(v_j,v_p),L(v_k,v_\ell)\big)e_1\\
&\qquad+\tfrac{(n+1)}{2(n-r)}\lambda_1\sum_{m=1}^{r-1} h(L(v_j,v_m),L(v_k,v_\ell))L(v_p,v_m).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\hat{R}(e_1,v_p)L(v_k,v_\ell)&=-K(e_1,K(v_p,L(v_k,v_\ell)))+K(v_p,K(e_1,L(v_k,v_\ell)))\\
&=-\tfrac{1}{2}\lambda_1 K(v_p,L(v_k,v_\ell))-\tfrac{(r+1)}{2(n-r)}\lambda_1 K(v_p,L(v_k,v_\ell))\\
&=-\tfrac{(n+1)}{2(n-r)}\lambda_1 K(v_p,L(v_k,v_\ell))\\
&=-\tfrac{(n+1)}{2(n-r)}\lambda_1\sum_{m=1}^{r-1} h(L(v_p,v_m),L(v_k,v_\ell))v_m.
\end{aligned}$$

It follows that

$$\begin{aligned}
&K\left(v_j,\hat{R}(e_1,v_p)L(v_k,v_\ell)\right)\\
&=-\tfrac{(n+1)}{2(n-r)}\lambda_1\sum_{m=1}^{r-1} h(L(v_p,v_m),L(v_k,v_\ell))K(v_j,v_m)\\
&=-\tfrac{(n+1)}{4(n-r)}\lambda_1^2\sum_{m=1}^{r-1} h(L(v_p,v_m),L(v_k,v_\ell))h(v_j,v_m)e_1\\
&\quad-\tfrac{(n+1)}{2(n-r)}\lambda_1\sum_{m=1}^{r-1} h(L(v_p,v_m),L(v_k,v_\ell))L(v_j,v_m)\\
&=-\tfrac{(n+1)}{4(n-r)}\lambda_1^2 h(L(v_p,v_j),L(v_k,v_\ell))e_1\\
&\quad-\tfrac{(n+1)}{2(n-r)}\lambda_1\sum_{m=1}^{r-1} h(L(v_p,v_m),L(v_k,v_\ell))L(v_j,v_m).
\end{aligned}$$

Finally, we obtain that

$$\begin{aligned}
&K(\hat{R}(e_1, v_p)v_j, L(v_k, v_\ell)) \\
&= K(-\frac{(n+1)^2\lambda_1^2}{4(n-r)^2} h(v_p, v_j)e_1 + \tfrac{(n+1)}{2(n-r)}\lambda_1 L(v_p, v_j), L(v_k, v_\ell)) \\
&= \tfrac{(n+1)^2(r+1)\lambda_1^3}{8(n-r)^3} h(v_p, v_j)L(v_k, v_\ell) + \tfrac{(n+1)}{2(n-r)}\lambda_1 K(L(v_p, v_j), L(v_k, v_\ell)).
\end{aligned}$$

Therefore, combining all the three terms, we find that

$$\begin{aligned}
&\tfrac{(n+1)}{2(n-r)}\lambda_1 K(L(v_p, v_j), L(v_k, v_\ell)) \\
&= -\tfrac{(n+1)^2(r+1)\lambda_1^3}{8(n-r)^3} h(v_p, v_j)L(v_k, v_\ell) + \tfrac{(n+1)}{4(n-r)}\lambda_1^2 h(L(v_p, v_j), L(v_k, v_\ell))e_1 \\
&\quad + \tfrac{(n+1)}{2(n-r)}\lambda_1 \sum_{m=1}^{r-1} h(L(v_p, v_m), L(v_k, v_\ell))L(v_j, v_m) \\
&\quad - \tfrac{(n+1)^2\lambda_1^2}{4(n-r)^2} h(L(v_j, v_p), L(v_k, v_\ell))e_1 \\
&\quad + \tfrac{(n+1)}{2(n-r)}\lambda_1 \sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_k, v_\ell))L(v_p, v_m) \\
&= -\tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 h(L(v_j, v_p), L(v_k, v_\ell))e_1 \\
&\quad - \tfrac{(n+1)^2(r+1)\lambda_1^3}{8(n-r)^3} h(v_p, v_j)L(v_k, v_\ell) \\
&\quad + \tfrac{(n+1)}{2(n-r)}\lambda_1 \sum_{m=1}^{r-1} \Big(h(L(v_p, v_m), L(v_k, v_\ell))L(v_j, v_m) \\
&\qquad\qquad + h(L(v_j, v_m), L(v_k, v_\ell))L(v_p, v_m)\Big).
\end{aligned}$$

Simplifying the above expression, we get (2.12). q.e.d.

We note that equation (2.12) has very important consequences which will be used in sequel sections. For example, we have

Lemma 2.7. *For Case $\mathfrak{C}_r$ with $r \geq 3$, let $\{v_1, \ldots, v_{r-1}\}$ be an orthonormal basis of $\mathcal{D}_2$; then for $p \neq j$, we have*

$$\begin{aligned}
0 &= \Big(\tfrac{(n+1)(2n-r+1)\lambda_1^2}{4(n-r)^2} - 4h(L(v_j, v_p), L(v_j, v_p))\Big) L(v_p, v_j) \\
&\quad + \sum_{m\neq p} \Big[h(L(v_p, v_m), L(v_j, v_j)) \\
&\qquad - 2h(L(v_j, v_m), L(v_p, v_j))\Big] L(v_j, v_m).
\end{aligned} \tag{2.15}$$

In particular, if $L(v_1, v_2) \neq 0$ and $L(v_1, v_m)$ is orthogonal to $L(v_1, v_2)$ for all $m \neq 2$, then

$$h(L(v_1, v_2), L(v_1, v_2)) = \tfrac{(n+1)(2n-r+1)\lambda_1^2}{16(n-r)^2}. \tag{2.16}$$

Proof. These are direct consequences of (2.12). We interchange the couples of indices $\{p, j\}$ and $\{k, \ell\}$ to find the following condition:

$$\begin{aligned} 0 = &-\tfrac{(n+1)(r+1)\lambda_1^2}{4(n-r)^2}\Big(h(v_p, v_j)L(v_k, v_\ell) - h(v_k, v_\ell)L(v_p, v_j)\Big) \\ &+ \sum_{m=1}^{r-1} h(L(v_p, v_m), L(v_k, v_\ell))L(v_j, v_m) \\ &+ \sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_k, v_\ell))L(v_p, v_m) \\ &- \sum_{m=1}^{r-1} h(L(v_k, v_m), L(v_p, v_j))L(v_\ell, v_m) \\ &- \sum_{m=1}^{r-1} h(L(v_\ell, v_m), L(v_p, v_j))L(v_k, v_m). \end{aligned} \tag{2.17}$$

Now we take $j = k = \ell \neq p$; then equation (2.17) reduces to

$$\begin{aligned} 0 = &\tfrac{(n+1)(r+1)\lambda_1^2}{4(n-r)^2}L(v_p, v_j) + \sum_{m=1}^{r-1} h(L(v_p, v_m), L(v_j, v_j))L(v_j, v_m) \\ &+ \sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_j, v_j))L(v_p, v_m) \\ &- 2\sum_{m=1}^{r-1} h(L(v_j, v_m), L(v_p, v_j))L(v_j, v_m) \\ = &\Big[\tfrac{(n+1)(r+1)\lambda_1^2}{4(n-r)^2} + h(L(v_p, v_p), L(v_j, v_j)) + h(L(v_j, v_j), L(v_j, v_j)) \\ &\qquad - 2h(L(v_j, v_p), L(v_p, v_j))\Big]L(v_p, v_j) \\ &+ \sum_{m\neq p}\Big(h(L(v_p, v_m), L(v_j, v_j)) - 2h(L(v_j, v_m), L(v_p, v_j))\Big)L(v_j, v_m). \end{aligned}$$

Using the isotropy condition, this immediately gives (2.15). Then (2.16) follows by taking $j = 1$ and $p = 2$ in the equation (2.15), and by using (2.9). q.e.d.

If $\dim \mathcal{D}_2 \leq 2$ or $\dim(\mathrm{Im}\,(L)) = 1$, then by theorems 4.1, 5.1, 6.1, 6.2 of [**HLSV**], we have the conclusions in the Classification Theorem. Recall also from lemma 7.1 of [**HLSV**] that if $\dim \mathcal{D}_2 \geq 3$ and $\dim(\mathrm{Im}\,(L)) \geq 2$, then $\dim(\mathrm{Im}\,(L)) \geq 3$. Hence from now on, even if sometimes not necessary, we can assume that $\dim \mathcal{D}_2 = r - 1 \geq 3$ and $\dim(\mathrm{Im}\,(L)) \geq 3$.

Now for a pair $v_1, v_2 \in \mathcal{D}_2$ of orthonormal vectors, we define a function g by

$$g(v_1, v_2) = h(L(v_1, v_2), L(v_1, v_2)). \tag{2.18}$$

Note that the set of such vectors can be identified with the $(r-1)$-dimensional unit hypersphere, and as such we can choose (v_1, v_2) such that the absolute maximum for g is attained. We extend v_1, v_2 to get an orthonormal basis $\{v_1, v_2, \dots, v_{r-1}\}$ of $\mathcal{D}_2$. Observe that, for all $k \geq 3$, we have

$$\tfrac{d}{dt}\big|_{t=0} g(v_1, \cos t\, v_2 + \sin t\, v_k) = 0, \quad \tfrac{d}{dt}\big|_{t=0} g(\cos t\, v_1 + \sin t\, v_k, v_2) = 0.$$

This implies (see (7.2) of [**HLSV**]):

$$h(L(v_1, v_2), L(v_1, v_k)) = 0 = h(L(v_1, v_2), L(v_2, v_k)), \ \forall\ k \geq 3. \tag{2.19}$$

Then by (2.7), Lemma 2.7, and noting that $L(v_1, v_2) \neq 0$, we find that (2.16) holds. We have proved

Lemma 2.8. *For Case $\mathfrak{C}_r$ with $r \geq 4$ and $\dim(\mathrm{Im}\,(L)) \geq 2$, if $v_1, v_2 \in \mathcal{D}_2$ are orthogonal unit vectors, then the function g attains maximum in (v_1, v_2) if and only if*

$$h(L(v_1, v_2), L(v_1, v_2)) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2. \tag{2.20}$$

Moreover, if we further take v_3 orthogonal to the couple v_1, v_2 such that $h(L(v_1, v_3), L(v_1, v_3))$ is the maximum of g over the complement of $\mathrm{span}\{v_1, v_2\}$ in $\mathcal{D}_2$, then from

$$\tfrac{d}{dt}\big|_{t=0} g(v_1, \cos t\, v_3 + \sin t\, v_k) = 0,$$

we further have

$$h(L(v_1, v_3), L(v_1, v_k)) = 0$$

for all $k \geq 4$. Therefore, taking $j = 1$ and $p = 3$ in (2.15), we see that

$$\Big(h(L(v_1, v_3), L(v_1, v_3)) - \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\Big)\, L(v_1, v_3) = 0. \tag{2.21}$$

3. A map $P(v) : \mathcal{D}_2 \to \mathcal{D}_2$ for unit vector $v \in \mathcal{D}_2$

In this section, we define for any given unit vector $v \in \mathcal{D}_2$ a linear map $P(v) : \mathcal{D}_2 \to \mathcal{D}_2$ by

$$h(v^*, P(v)\tilde{v}^*) = h(L(v, v^*), L(v, \tilde{v}^*)), \quad v^*,\ \tilde{v}^* \in \mathcal{D}_2. \tag{3.1}$$

It is easily seen that $P(v)$ is well defined and it is a symmetric operator with respect to h. In fact, we can write

$$P(v)\tilde{v}^* = \sum_{k=2}^{r} h(L(v, e_k), L(v, \tilde{v}^*))e_k.$$

Moreover, we have

Lemma 3.1. *For any unit vector $v \in \mathcal{D}_2$, the operator $P(v) : \mathcal{D}_2 \to \mathcal{D}_2$ has $\sigma = \frac{n+1}{4(n-r)}\lambda_1^2$ as an eigenvalue with multiplicity one, and the corresponding eigenspace is spanned by v. The remaining eigenvalues of $P(v)$ are 0 and $\frac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$.*

Proof. According to (2.7), if $v^* \perp v$, then

$$h(v^*, P(v)v) = h(L(v, v^*), L(v, v)) = 0.$$

This implies that $P(v)v = \sigma v$. By definition and (2.6) we have

$$\sigma = h(v, P(v)v) = h(L(v, v), L(v, v)) = \tfrac{n+1}{4(n-r)}\lambda_1^2.$$

Next, we set $v = v_1$ and $\sigma = \sigma_1$. We take an orthonormal basis $\{v_m\}$ of $\mathcal{D}_2$ consisting of eigenvectors of $P(v)$ such that $P(v)v_m = \sigma_m v_m$, $1 \le m \le r-1$. We take the product of formula (2.15) for $j = 1$ and any $p \ge 2$ with $L(v_1, v_p)$. We have

$$\begin{aligned} h(L(v_1, v_p), L(v_1, v_p))\Big(&\tfrac{(n+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 \\ &- 4h(L(v_1, v_p), L(v_1, v_p))\Big) = 0. \end{aligned} \tag{3.2}$$

Here we have used the fact that, for all $m \neq p$,

$$h(L(v_1, v_p), L(v_1, v_m)) = h(v_p, P(v)v_m) = h(v_p, \sigma_m v_m) = 0.$$

By (3.2), we get either $h(L(v_1, v_p), L(v_1, v_p)) = 0$, or

$$h(L(v_1, v_p), L(v_1, v_p)) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2.$$

On the other hand, we have $\sigma_p = h(v_p, P(v)v_p) = h(L(v_1, v_p), L(v_1, v_p))$. This completes the proof. q.e.d.

In the following we denote by $V_v\left(\frac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$ and $V_v(0)$ the eigenspaces of $P(v)$ with respect to the eigenvalues $\frac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$ and 0, respectively.

Lemma 3.2. *Let $u, v \in \mathcal{D}_2$ be two unit orthogonal vectors. Suppose that $u \in V_v(0)$. Then we have* (i) *$L(u, v) = 0$,* (ii) *$L(u, u) = L(v, v)$,* (iii) *$v \in V_u(0)$,* (iv) *$P(u) = P(v)$ on $\{u, v\}^\perp$.*

Proof. As $u \in V_v(0)$, we have $h(L(u, v), L(u, v)) = h(u, P(v)u) = 0$, i.e., $L(u, v) = 0$. By (2.6) and (2.8) we have

$$h(L(u, u), L(u, u)) = h(L(v, v), L(v, v)) = h(L(u, u), L(v, v)) = \sigma.$$

Applying the Cauchy-Schwarz inequality, we obtain $L(u, u) = L(v, v)$. On the other hand, for any $w \perp v$, the fact that $L(u, v) = 0$ implies that

$$h(w, P(u)v) = h(L(u, v), L(u, w)) = 0.$$

It follows that $P(u)v = \beta v$ and $\beta = h(v, P(u)v) = h(L(u, v), L(u, v)) = 0$. Hence $v \in V_u(0)$.

To prove the final assertion, we take any unit vector $u_1 \in \{u, v\}^{\perp}$ and use (2.8) to see that

$$\begin{aligned} h(u_1, P(v)u_1) &= h(L(u_1, v), L(u_1, v)) \\ &= \tfrac{n+1}{8(n-r)^2}\lambda_1^2 - \tfrac{1}{2}h(L(u_1, u_1), L(v, v)) \\ &= \tfrac{n+1}{8(n-r)^2}\lambda_1^2 - \tfrac{1}{2}h(L(u_1, u_1), L(u, u)) \\ &= h(L(u_1, u), L(u_1, u)) = h(u_1, P(u)u_1). \end{aligned}$$

Similarly, for orthonormal vectors $u_1, u_2 \in \{u, v\}^{\perp}$, by (2.9) we have

$$\begin{aligned} h(u_1, P(v)u_2) &= h(L(u_1, v), L(u_2, v)) \\ &= -\tfrac{1}{2}h(L(u_1, u_2), L(v, v)) \\ &= -\tfrac{1}{2}h(L(u_1, u_2), L(u, u)) \\ &= h(L(u_1, u), L(u_2, u)) = h(u_1, P(u)u_2). \end{aligned}$$

Note also that

$$\begin{aligned} h(u, P(v)u_1) &= h(L(u, v), L(u_1, v)) \\ &= 0 = h(L(u, u), L(u_1, u)) = h(u, P(u)u_1), \\ h(v, P(v)u_1) &= h(L(v, v), L(u_1, v)) \\ &= 0 = h(L(u, v), L(u_1, u)) = h(v, P(u)u_1). \end{aligned}$$

In summary, we have proved $P(u) = P(v)$ on $\{u, v\}^{\perp}$. q.e.d.

Lemma 3.3. *Let $v, \tilde{v} \in \mathcal{D}_2$ be two unit orthogonal vectors. Then the equality*

$$h(L(v, \tilde{v}), L(v, \tilde{v})) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2 \tag{3.3}$$

holds if and only if $\tilde{v} \in V_v\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$. Moreover, if we assume $u \in V_v(0)$ and (3.3) holds, then $u \in V_{\tilde{v}}\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$.

Proof. If $\tilde{v} \in V_v\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$, then

$$h(L(v, \tilde{v}), L(v, \tilde{v})) = h(\tilde{v}, P(v)\tilde{v}) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2.$$

Conversely, if $h(L(v, \tilde{v}), L(v, \tilde{v})) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$, then the function g attains maximum at $(v, \tilde{v})$. It follows that, for any $w \perp \tilde{v}$,

$$h(w, P(v)\tilde{v}) = h(L(v, \tilde{v}), L(v, w)) = 0.$$

Therefore, we have $P(v)\tilde{v} = \tau\tilde{v}$ and $\tau = h(\tilde{v}, P(v)\tilde{v}) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$. This proves the first claim.

Now we assume (3.3). If $u \in V_v(0)$, then by the facts

$$v \in V_v(\sigma), \quad \tilde{v} \in V_v\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right),$$

we see that $u, v, \tilde{v}$ are orthonormal vectors and therefore, by Lemma 3.2, $P(u)\tilde{v} = P(v)\tilde{v}$. It follows that

$$h(L(u,\tilde{v}), L(u,\tilde{v})) = h(\tilde{v}, P(u)\tilde{v}) = h(\tilde{v}, P(v)\tilde{v}) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$$

and thus $u \in V_{\tilde{v}}\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$. q.e.d.

Lemma 3.4. *Let $u_1, u_2, u_3 \in \mathcal{D}_2$ be orthonormal vectors satisfying the condition*

$$h(L(u_1,u_3), L(u_1,u_3)) = h(L(u_2,u_3), L(u_2,u_3)) = \tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2.$$

Then for any vector $v \in \mathcal{D}_2$, we have $h(L(u_1,u_2), L(u_3,v)) = 0$.

Proof. By Lemma 3.3, we see that $u_1, u_2 \in V_{u_3}\left(\tfrac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2\right)$. Using the linearity of the assertion, we may assume that v is an eigenvector of $P(u_3)$.

We choose an orthonormal basis $\{v_m\}$ of $\mathcal{D}_2$ consisting of eigenvectors of $P(u_3)$ such that $v_1 = u_1, v_2 = u_2, v_3 = u_3$. We now use (2.17) for $p = 1, j = 2, k = \ell = 3$ to obtain

$$\begin{aligned} 0 = {} & \tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 L(v_1,v_2) + \sum_{m=1}^{r-1} h(L(v_1,v_m), L(v_3,v_3))L(v_2,v_m) \\ & + \sum_{m=1}^{r-1} h(L(v_2,v_m), L(v_3,v_3))L(v_1,v_m) \\ & - 2\sum_{m=1}^{r-1} h(L(v_3,v_m), L(v_1,v_2))L(v_3,v_m). \end{aligned} \tag{3.4}$$

As $m = 3$ and $k = 1, 2$, it follows that

$$h(L(v_k,v_m), L(v_3,v_3)) = 0,$$

and if $k = 1, 2$ and $m \neq k, 3$, we have that

$$h(L(v_k,v_m), L(v_3,v_3)) = -2h(v_k, P(v_3)v_m) = 0;$$

we see that (3.4) reduces to

$$\begin{aligned} 0 = {} & \tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 L(v_1,v_2) - 2\sum_{m=1}^{r-1} h(L(v_3,v_m), L(v_1,v_2))L(v_3,v_m) \\ & + h(L(v_1,v_1), L(v_3,v_3))L(v_2,v_1) + h(L(v_2,v_2), L(v_3,v_3))L(v_1,v_2). \end{aligned}$$

Furthermore, from (2.8) and the assumption, we find that

$$\begin{aligned} & \tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 + h(L(v_1,v_1), L(v_3,v_3)) + h(L(v_2,v_2), L(v_3,v_3)) \\ & = \tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 + \tfrac{n+1}{2(n-r)}\lambda_1^2 - 2h(L(u_1,u_3), L(u_1,u_3)) \\ & \quad - 2h(L(u_2,u_3), L(u_2,u_3)) = 0. \end{aligned}$$

Therefore, from (3.4) we get $\sum_{m=1}^{r-1} h(L(v_3,v_m),L(v_1,v_2))L(v_3,v_m)=0$, or equivalently, by Lemma 3.2,

$$\sum_{v_m\notin V_{u_3}(0)} h(L(v_3,v_m),L(v_1,v_2))L(v_3,v_m)=0. \tag{3.5}$$

Now note that for $v_p, v_q \notin V_{u_3}(0)$, we have

$$h(L(v_3,v_p),L(v_3,v_q))=h(v_p,P(v_3)v_q)=0, \quad \text{if} \quad p\neq q.$$

Then (3.5) implies that if $v_m \notin V_{u_3}(0)$, then $h(L(u_1,u_2),L(u_3,v_m))=0$.

On the other hand, if $v_m\in V_{u_3}(0)$, then $h(L(u_3,v_m),L(u_3,v_m)) = h(v_m,P(u_3)v_m)=0$ and hence $L(u_3,v_m)=0$. Therefore we also have $h(L(u_1,u_2),L(u_3,v_m))=0$.

We have now completed the proof of Lemma 3.4. q.e.d.

4. A decomposition of $\mathcal{D}_2$

In this section, we introduce a direct sum decomposition for $\mathcal{D}_2$, which turns out crucial for our purpose.

Pick any unit vector $v_1\in\mathcal{D}_2$ and write $\tau=\frac{(n+1)(2n-r+1)}{16(n-r)^2}\lambda_1^2$; then by Lemma 3.1, we have a direct sum decomposition for $\mathcal{D}_2$

$$\mathcal{D}_2=\{v_1\}\oplus V_{v_1}(0)\oplus V_{v_1}(\tau),$$

where here and later on, we denote also by $\{\cdot\}$ the vector space spanned by its elements. If $V_{v_1}(\tau)\neq\emptyset$, we take an arbitrary unit vector $v_2\in V_{v_1}(\tau)$. Then by Lemma 3.3 we have:

$$v_1\in V_{v_2}(\tau), \quad V_{v_1}(0)\subset V_{v_2}(\tau), \quad V_{v_2}(0)\subset V_{v_1}(\tau).$$

From this we deduce that

$$\mathcal{D}_2=\{v_1\}\oplus V_{v_1}(0)\oplus\{v_2\}\oplus V_{v_2}(0)\oplus\Big(V_{v_1}(\tau)\cap V_{v_2}(\tau)\Big).$$

If $V_{v_1}(\tau)\cap V_{v_2}(\tau)\neq\emptyset$, we further pick a unit vector $v_3\in V_{v_1}(\tau)\cap V_{v_2}(\tau)$. Then

$$\mathcal{D}_2=\{v_3\}\oplus V_{v_3}(0)\oplus V_{v_3}(\tau)$$

and by Lemma 3.3 we have

$$v_1,v_2\in V_{v_3}(\tau), \quad V_{v_1}(0),V_{v_2}(0)\subset V_{v_3}(\tau).$$

It follows that

$$\begin{aligned}\mathcal{D}_2=\{v_1\}\oplus V_{v_1}(0)\oplus\{v_2\}\oplus V_{v_2}(0)\oplus\{v_3\}\oplus V_{v_3}(0)\\ \oplus\Big(V_{v_1}(\tau)\cap V_{v_2}(\tau)\cap V_{v_3}(\tau)\Big).\end{aligned}$$

Considering that $\dim\mathcal{D}_2=r-1$ is finite, by induction, we get

Proposition 4.1. *There exist an integer k_0 and unit vectors $v_1,\dots,v_{k_0}$ of $\mathcal{D}_2$ such that*

$$\mathcal{D}_2=\{v_1\}\oplus V_{v_1}(0)\oplus\cdots\oplus\{v_{k_0}\}\oplus V_{v_{k_0}}(0). \tag{4.1}$$

In what follows, we will study the decomposition (4.1) in more detail.

Lemma 4.1. (i) *With the assumption* $\dim(\mathrm{Im}\,(L)) \geq 2$*, we have* $k_0 \geq 2$*.*

(ii) *For any unit vector* $u_1 \in V_{v_1}(0)$*, we have* $\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0)$*.*

(iii) *For any unit vectors* $u_1, \tilde{u}_1 \in \{v_1\} \oplus V_{v_1}(0)$ *and* $u_1 \perp \tilde{u}_1$*, we have* $L(u_1, \tilde{u}_1) = 0$*.*

Proof. (i) If $k_0 = 1$ and $\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0)$, then for any unit vector $v \in V_{v_1}(0)$ we have $L(v_1, v) = 0$ and thus $L(v, v) = L(v_1, v_1)$. This implies that $\dim(\mathrm{Im}\,(L)) = 1$, contradicting the assumption $\dim(\mathrm{Im}\,(L)) \geq 2$.

(ii) If we have $w \in V_{v_1}(0)$ such that $w \perp u_1$, then by Lemma 3.2, $P(u_1)w = P(v_1)w = 0$ and therefore $w \in V_{u_1}(0)$. Similarly, if $w \in V_{u_1}(0)$ such that $w \perp v_1$, then $w \in V_{v_1}(0)$. This proves the assertion.

(iii) Let $\{u_0^1 = v_1, u_1^1, \ldots, u_p^1\}$ be an orthonormal basis of $\{v_1\} \oplus V_{v_1}(0)$. By (ii) we see that

$$\{v_1\} \oplus V_{v_1}(0) = \{u_j^1\} \oplus V_{u_j^1}(0), \quad 1 \leq j \leq p.$$

This shows that $u_j^1 \in V_{u_\ell^1}(0)$ for $j \neq \ell$. Then we have $L(u_j^1, u_\ell^1) = 0$ and $L(u_j^1, u_j^1) = L(u_\ell^1, u_\ell^1)$ for all $j \neq \ell$. In particular, we have $L(u_1, \tilde{u}_1) = 0$ and $L(u_1, u_1) = L(v_1, v_1)$. q.e.d.

Lemma 4.2. *In the decomposition* (4.1)*, if we pick a unit vector* $u_2 \in V_{v_2}(0)$*, then there exists a unique unit vector* $u_1 \in V_{v_1}(0)$ *such that* $L(v_1, u_2) = L(v_2, u_1)$*.*

Proof. Let $\{u_1^\ell, \ldots, u_{p_\ell}^\ell\}$ be an orthonormal basis of $V_{v_\ell}(0),\ 1 \leq \ell \leq k_0$ such that $u_1^2 = u_2$. Then

$$\{v_1, \ldots, v_{k_0}; u_1^1, \ldots, u_{p_1}^1; \ldots; u_1^{k_0}, \ldots, u_{p_{k_0}}^{k_0}\} := \{\tilde{u}_m\}_{1 \leq m \leq r-1}$$

forms an orthonormal basis of $\mathcal{D}_2$. Now we use (2.12) with $p = k = 2, \ell = 1$ and $v_j = u_1^2$, together with the facts that $L(v_2, u_2) = 0$ and $h(L(v_1, v_2), L(v_1, v_2)) = \tau$ being a maximum, and we obtain

$$\begin{aligned} 0 &= K(L(v_2, u_2), L(v_1, v_2)) \\ &= -\tfrac{(r+1)}{2(n-r)}\lambda_1 h(L(u_2, v_2), L(v_1, v_2))e_1 \\ &\quad + \sum_{m=1}^{r-1} h(L(v_2, \tilde{u}_m), L(v_1, v_2))L(u_2, \tilde{u}_m) \\ &\quad + \sum_{m=1}^{r-1} h(L(u_2, \tilde{u}_m), L(v_1, v_2))L(v_2, \tilde{u}_m) \\ &= h(L(v_2, v_1), L(v_1, v_2))L(u_2, v_1) \\ &\quad + \sum_{m=1}^{r-1} h(L(u_2, \tilde{u}_m), L(v_1, v_2))L(v_2, \tilde{u}_m). \end{aligned} \tag{4.2}$$

To deal with the last summation, we first use Lemma 3.4 and that $h(L(v_1, v_2), L(v_1, v_2)) = \tau$ is a maximum to see that

$$h(L(u_2, v_\ell), L(v_1, v_2)) = 0, \quad 1 \le \ell \le k_0.$$

For $\ell \ge 3$ and $1 \le q \le p_\ell$, we have

$$\begin{aligned} h(L(v_1, u^\ell_q), L(v_1, u^\ell_q)) &= h(u^\ell_q, P(v_1)u^\ell_q) = \tau \\ &= h(u^\ell_q, P(v_2)u^\ell_q) = h(L(v_2, u^\ell_q), L(v_2, u^\ell_q)). \end{aligned}$$

Then by Lemma 3.4 we obtain

$$h(L(u_2, u^\ell_q), L(v_1, v_2)) = 0, \quad 3 \le \ell \le k_0,\ 1 \le q \le p_\ell.$$

By Lemma 4.1 (iii), we see that if $2 \le j \le p_2$, then $L(u_2, u^2_j) = 0$. If $j = 1$, then by $u^2_1 = u_2$ and (2.9) we have

$$h(L(u_2, u^2_1), L(v_1, v_2)) = -2h(L(u_2, v_1), L(u_2, v_2)) = 0.$$

Putting the above results into (4.2), we get

$$\tau L(v_1, u_2) = -\sum_{m=1}^{p_1} h(L(u_2, u^1_m), L(v_1, v_2))L(v_2, u^1_m). \tag{4.3}$$

Let us choose $u_1 = -\frac{1}{\tau}\sum_{m=1}^{p_1} h(L(u_2, u^1_m), L(v_1, v_2))u^1_m$; then clearly we have $u_1 \in V_{v_1}(0)$ and $L(v_1, u_2) = L(v_2, u_1)$.

Suppose $\tilde{u}_1 = \sum_{m=1}^{p_1} a_m u^1_m \in V_{v_1}(0)$ such that $L(v_1, u_2) = L(v_2, \tilde{u}_1)$; then by (4.3) we get

$$\sum_{m=1}^{p_1} \left(a_m + \tfrac{1}{\tau}h(L(u_2, u^1_m), L(v_1, v_2))\right) L(v_2, u^1_m) = 0. \tag{4.4}$$

Note that as $u^1_m \in V_{v_1}(0) \subset V_{v_2}(\tau)$, it implies that

$$h(L(v_2, u^1_j), L(v_2, u^1_\ell)) = h(u^1_j, P(v_2)u^1_\ell) = \tau\delta_{j\ell},$$

i.e., $\{\frac{1}{\sqrt{\tau}}L(v_2, u^1_m)\}_{1\le m\le p_1}$ consists of orthonormal vectors. Then (4.4) shows that

$$a_m = -\tfrac{1}{\tau}h(L(u_2, u^1_m), L(v_1, v_2)), \quad 1 \le m \le p_1.$$

This clearly proves the uniqueness of u_1.

To show that vector $u_1 \in V_{v_1}(0)$ satisfying $L(v_1, u_2) = L(v_2, u_1)$ must be of unit length, we write $u_1 = \alpha\tilde{u}_1$, where $\tilde{u}_1 \in V_{v_1}(0) \subset V_{v_2}(\tau)$ is unit. Then, at one side as $u_2 \in V_{v_2}(0) \subset V_{v_1}(\tau)$,

$$h(L(v_2, u_1), L(v_2, u_1)) = h(L(v_1, u_2), L(v_1, u_2)) = h(u_2, P(v_1)u_2) = \tau.$$

On the other side, we have

$$\begin{aligned} h(L(v_2, u_1), L(v_2, u_1)) &= \alpha^2 h(L(v_2, \tilde{u}_1), L(v_2, \tilde{u}_1)) \\ &= \alpha^2 h(\tilde{u}_1, P(v_2)\tilde{u}_1) = \alpha^2\tau. \end{aligned}$$

Hence $\alpha^2 = 1$ and u_1 is a unit vector. q.e.d.

To generalize Lemma 4.1 (ii), we can show the following

Lemma 4.3. *If* $u_1 \in \{v_1\} \oplus V_{v_1}(0)$ *is a unit vector, then*

$$\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0).$$

Proof. According to Lemma 4.1 (ii), we need only to consider the case that $u_1 \notin V_{v_1}(0)$ and $u_1 \neq v_1$.

If $\dim V_{v_1}(0) = 0$, there is nothing to prove. We now assume $p_1 = \dim V_{v_1}(0) \geq 1$ and let $\{u_1^1, \ldots, u_{p_1}^1\}$ be an orthonormal basis of $V_{v_1}(0)$ such that $u_1 = \cos\theta v_1 + \sin\theta u_1^1$.

If $p_1 \geq 2$, then by Lemma 4.1 (ii) we have $\{v_1\} \oplus V_{v_1}(0) = \{u_2^1\} \oplus V_{u_2^1}(0)$. It follows that $u_1 \in \{u_2^1\} \oplus V_{u_2^1}(0)$. However, we have $u_1 \perp u_2^1$; this implies that $u_1 \in V_{u_2^1}(0)$. By Lemma 4.1 (ii) again we get $\{u_2^1\} \oplus V_{u_2^1}(0) = \{u_1\} \oplus V_{u_1}(0)$. Hence $\{u_1\} \oplus V_{u_1}(0) = \{v_1\} \oplus V_{v_1}(0)$ as claimed.

Finally, if $p_1 = 1$: denote $\tilde{u}_1 = -\sin\theta v_1 + \cos\theta u_1^1$. Then $\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus \{\tilde{u}_1\}$. Therefore, to show that $\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0)$, it suffices to show that $\{\tilde{u}_1\} = V_{u_1}(0)$.

For that purpose, we take at this point an orthonormal basis $\{v_1, v_2 = u_1^1, v_3, \ldots, v_{r-1}\}$ of $\mathcal{D}_2$. Then by (2.6), (2.8), and the fact $L(v_1, u_1^1) = 0$ we have the following calculation:

$$\begin{aligned} &h(v_1, P(u_1)\tilde{u}_1) = h(L(v_1, u_1), L(\tilde{u}_1, u_1)) \\ &\quad = \sin\theta\cos^2\theta\Big(h(L(v_1, v_1), L(u_1^1, u_1^1)) - h(L(v_1, v_1), L(v_1, v_1))\Big) = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &h(v_2, P(u_1)\tilde{u}_1) = h(L(u_1^1, u_1), L(\tilde{u}_1, u_1)) \\ &\quad = \sin^2\theta\cos\theta\Big(h(L(u_1^1, u_1^1), L(u_1^1, u_1^1)) - h(L(u_1^1, u_1^1), L(v_1, v_1))\Big) = 0. \end{aligned}$$

For each $v = v_3, \ldots, v_{r-1} \in V_{v_1}(\tau)$, as Lemma 3.3 gives that $u_1^1 \in V_v(\tau)$, we easily obtain

$$\begin{aligned} &h(v, P(u_1)\tilde{u}_1) = h(L(v, u_1), L(\tilde{u}_1, u_1)) \\ &\quad = h\Big(\cos\theta L(v, v_1) + \sin\theta L(v, u_1^1), \sin\theta\cos\theta(L(u_1^1, u_1^1) - L(v_1, v_1))\Big) \\ &\quad = \sin\theta\cos^2\theta\, h(L(v, v_1), L(u_1^1, u_1^1)) - \sin^2\theta\cos\theta\, h(L(v, u_1^1), L(v_1, v_1)) \\ &\quad = -2\sin\theta\cos^2\theta\, h(v_1, P(u_1^1)v) + 2\sin^2\theta\cos\theta\, h(u_1^1, P(v_1)v) = 0. \end{aligned}$$

From the above calculations, we have proved $P(u_1)\tilde{u}_1 = 0$, i.e., $\tilde{u}_1 \in V_{u_1}(0)$. We only need to prove $\dim V_{u_1}(0) = 1$. Assume $\dim V_{u_1}(0) \geq 2$; pick $\hat{u}_1 \in V_{u_1}(0)$, such that $\langle \hat{u}_1, \tilde{u}_1 \rangle = 0$. By Lemma 4.1 (iii), $L(\tilde{u}_1, \hat{u}_1) = 0$. By Lemma 3.2 (i), $L(u_1, \tilde{u}_1) = 0$, $L(u_1, \hat{u}_1) = 0$. Noting $v_1 = \cos\theta u_1 - \sin\theta\tilde{u}_1$, $u_1^1 = \sin\theta u_1 + \cos\theta\tilde{u}_1$, we have $L(v_1, \tilde{u}_1) = L(\cos\theta u_1 - \sin\theta\tilde{u}_1, \tilde{u}_1) = 0$, thus $\tilde{u}_1 \in V_{v_1}(0)$, but $u_1^1 \in V_{v_1}(0)$. Noting $\langle \hat{u}_1, u_1^1 \rangle = 0$, we have $\dim V_{v_1}(0) \geq 2$, a contradiction with $p_1 = 1$. We have completed the proof of the lemma. q.e.d.

Lemma 4.4. *In the decomposition* (4.1)*, if for* $1 \leq \ell \leq k_0$ *we write* $V_\ell = \{v_\ell\} \oplus V_{v_\ell}(0)$*, then we have:*

(1) *For any unit vector* $a \in V_j$*,*

$$K(L(a,a), L(a,a)) = -\tfrac{(n+1)(r+1)}{8(n-r)^2}\lambda_1^3 e_1 + \tfrac{(n+1)(2n-3r-1)}{4(n-r)^2}\lambda_1^2 L(a,a). \tag{4.5}$$

(2) *For any unit vectors* $a \in V_j,\ b \in V_\ell,\ j \neq \ell$*,*

$$K(L(a,a), L(a,b)) = \tfrac{(n+1)(2n-3r-1)}{8(n-r)^2}\lambda_1^2 L(a,b), \tag{4.6}$$

$$\begin{aligned} K(L(a,a), L(b,b)) =& \tfrac{(n+1)(r+1)^2}{16(n-r)^3}\lambda_1^3 e_1 \\ &- \tfrac{(n+1)(r+1)\lambda_1^2}{4(n-r)^2}(L(a,a) + L(b,b)), \end{aligned} \tag{4.7}$$

$$K(L(a,b), L(a,b)) = -\tfrac{r+1}{2(n-r)}\lambda_1 \tau e_1 + \tau(L(a,a) + L(b,b)). \tag{4.8}$$

(3) *For unit vectors* $a \in V_j$*,* $b \in V_\ell$*,* $c \in V_q$ *and* j*,* ℓ*,* q *being distinct,*

$$K(L(a,b), L(a,c)) = \tau L(b,c), \tag{4.9}$$

$$K(L(a,a), L(b,c)) = -\tfrac{(n+1)(r+1)}{4(n-r)^2}\lambda_1^2 L(b,c). \tag{4.10}$$

(4) *For orthogonal unit vectors* a_1*,* $a_2 \in V_j$ *and unit vectors* $b \in V_\ell$*,* $c \in V_q$ *with* j*,* ℓ*,* q *being distinct, we have that*

$$K(L(a_1,b), L(a_2,c)) = \tau L(b,c'), \tag{4.11}$$

where $c' \in V_q$ *is the unique unit vector satisfying* $L(a_2,c) = L(a_1,c')$*.*

Proof. We take an orthonormal basis of $\mathcal{D}_2$ in such a way that it consists of all the orthonormal basis of V_j, $1 \leq j \leq k_0$. Then (1)–(4) are direct consequences of Lemma 2.6. Take (4.9), for example: we combine Lemma 2.6 with the fact $h(L(a,b), L(a,c)) = h(b, P(a)c)) = \tau h(b,c) = 0$ and the isotropic properties of L. Then we get (4.9). From (4.9), Lemmas 4.2 and 4.3, we can get (4.11). q.e.d.

Proposition 4.2. *In the decomposition* (4.1)*, if* $k_0 \geq 2$*, then*

$$\dim V_{v_1}(0) = \cdots = \dim V_{v_{k_0}}(0).$$

Moreover, the dimension which we denoted by $\mathfrak{p}$ *can only be equal to* $0, 1, 3$*, or* 7*.*

Proof. As a direct consequence of Lemma 4.2, for any $j \neq \ell$, we can define a one-to-one linear map from $V_{v_j}(0)$ to $V_{v_\ell}(0)$, which preserves length of vectors. Hence $V_{v_j}(0)$ and $V_{v_\ell}(0)$ are isomorphic and have the same dimension which we denote by $\mathfrak{p}$. To make the following discussion meaningful, we now assume $\mathfrak{p} \geq 1$.

Set $V_\ell = \{v_\ell\} \oplus V_{v_\ell}(0)$, $1 \leq \ell \leq k_0$. Let $\{v_\ell, u_1^\ell, \ldots, u_{\mathfrak{p}}^\ell\}$ be an orthonormal basis of V_ℓ. For each $j = 1, \ldots, \mathfrak{p}$, Lemmas 4.2 and 4.3

show that we can define a linear map $\mathfrak{T}_j : V_1 \to V_1$ such that for any unit vector $v \in V_1$, $\mathfrak{T}_j(v)$ is the unique element of $V_v(0) \subset V_1$ satisfying

$$L(v, u_j^2) = L(v_2, \mathfrak{T}_j(v)). \tag{4.12}$$

We remark that we can uniquely define $\mathfrak{T}_j(v) \in \{v\} \oplus V_v(0) = V_1$ by (4.11), without having to assume that $\mathfrak{T}_j(v) \in V_v(0)$. In fact, if we have a unit vector $x \in V_v(0)$ such that $L(v_2, ax + bv) = L(v, u_j^2)$, then from the definition of $\mathfrak{T}_j(v)$ we have $L(v_2, ax - \mathfrak{T}_j(v)) = -bL(v_2, v)$. As the function g defined by (2.18) attains a maximum at (v_2, v), it follows that $L(v_2, w) \perp L(v_2, v)$ for all $w \perp v$. Hence we see that $b = 0$ and $ax = \mathfrak{T}_j(v)$.

The linear map $\mathfrak{T}_j : V_1 \to V_1$ has the fundamental properties:

(P1) $h(\mathfrak{T}_j(v), \mathfrak{T}_j(v)) = h(v, v)$, i.e., $\mathfrak{T}_j$ preserves the length of vectors.
(P2) For all $v \in V_1$, we have $\mathfrak{T}_j(v) \perp v$.
(P3) $\mathfrak{T}_j^2 = -id$.
(P4) For all $j \neq \ell$ and $v \in V_1$, we get that $h(\mathfrak{T}_j(v), \mathfrak{T}_\ell(v)) = 0$.

Since the properties (P1) and (P2) can be easily seen from Lemma 4.2 and the definition of $\mathfrak{T}_j$, we only need to verify explicitly (P3) and (P4).

For any unit vector $v \in V_1$, we have

$$L(v_2, \mathfrak{T}_j^2(v)) = L(\mathfrak{T}_j(v), u_j^2). \tag{4.13}$$

Using the fact $\{\mathfrak{T}_j(v)\} \oplus V_{\mathfrak{T}_j(v)}(0) = V_1$ and $u_j^2 \in V_{v_2}(0) \subset V_{\mathfrak{T}_j(v)}(\tau)$, we have

$$\begin{aligned} h\big(L(\mathfrak{T}_j(v), u_j^2), L(\mathfrak{T}_j(v), u_j^2)\big) &= h\big(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\big) \\ &= h(L(v, v_2), L(v, v_2)) = \tau. \end{aligned} \tag{4.14}$$

Since $v, \mathfrak{T}_j(v), v_2, u_j^2$ are orthonormal vectors, by $L(v_2, u_j^2) = 0$, (2.10), and (4.12) we see that

$$\begin{aligned} 0 &= h\big(L(v, v_2), L(\mathfrak{T}_j(v), u_j^2)\big) \\ &\quad + h\big(L(v, \mathfrak{T}_j(v)), L(v_2, u_j^2)\big) + h\big(L(v, u_j^2), L(v_2, \mathfrak{T}_j(v))\big) \\ &= h\big(L(v, v_2), L(\mathfrak{T}_j(v), u_j^2)\big) + h\big(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\big). \end{aligned}$$

Applying (4.14) and the Cauchy-Schwarz inequality, we deduce

$$L(\mathfrak{T}_j(v), u_j^2) = -L(v, v_2). \tag{4.15}$$

Combining (4.13) and (4.15), we get $L(\mathfrak{T}_j^2(v) + v, v_2) = 0$. As $\mathfrak{T}_j^2(v) + v \in V_1 \subset V_{v_2}(\tau)$, it follows that

$$\begin{aligned} 0 &= h\big(L(\mathfrak{T}_j^2(v) + v, v_2), L(\mathfrak{T}_j^2(v) + v, v_2)\big) \\ &= h\big(\mathfrak{T}_j^2(v) + v, P(v_2)(\mathfrak{T}_j^2(v) + v)\big) = \tau h\big(\mathfrak{T}_j^2(v) + v, \mathfrak{T}_j^2(v) + v\big). \end{aligned}$$

Hence $\mathfrak{T}_j^2(v) = -v$ for a unit vector v and then by linearity for all $v \in V_1$, as claimed by (P3).

To verify (P4), we note that, if $j \neq \ell$, by definition

$$\mathfrak{T}_j(v), \mathfrak{T}_\ell(v) \in V_v(0);\;\; L(v_2, \mathfrak{T}_j(v)) = L(v, u_j^2) \perp L(v, u_\ell^2) = L(v_2, \mathfrak{T}_\ell(v)).$$

If we assume $\mathfrak{T}_\ell(v) = a\mathfrak{T}_j(v) + x$, where $x \perp \mathfrak{T}_j(v)$ and $x \in V_v(0)$, then

$$\begin{aligned} 0 &= h\big(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_\ell(v))\big) \\ &= h\big(L(v_2, \mathfrak{T}_j(v)), aL(v_2, \mathfrak{T}_j(v)) + L(v_2, x)\big) \\ &= ah\big(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\big) = a\tau. \end{aligned}$$

Thus $a = 0$ and therefore $\mathfrak{T}_j(v) \perp \mathfrak{T}_\ell(v)$, as claimed.

We now look at the unit hypersphere $S^{\mathfrak{p}}(1) \subset V_1$; then the above properties (P1)–(P4) show that at $v \in S^{\mathfrak{p}}(1)$

$$T_v S^{\mathfrak{p}}(1) = \{\mathfrak{T}_1(v), \ldots, \mathfrak{T}_{\mathfrak{p}}(v)\}.$$

Hence, by the properties (P1)–(P4), $S^{\mathfrak{p}}(1)$ is parallelizable. Then, according to R. Bott and J. Milnor [**BM**] and M. Kervaire [**Ke**], the dimension $\mathfrak{p}$ can only be equal to $1, 3$, or 7. q.e.d.

From theorems 4.1, 5.1, 6.1, and 6.2 in [**HLSV**], and combining with the above Proposition 4.2, we see that, in order to complete the proof of the Classification Theorem, it is sufficient to deal with the four cases that $\mathfrak{p} = 0$, 1, 3, 7. These will be carried out in the remaining sections and the results are stated as Theorem 5.1, Theorem 6.1, Theorem 7.1, and Theorem 8.1, respectively.

5. Hypersurfaces in $\mathbb{R}^{n+1}$ with $\mathfrak{p} = 0$

In this section, we will prove the following theorem.

Theorem 5.1. *Let M^n be a locally strongly convex affine hypersurface of $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\dim \mathcal{D}_2 = r - 1 \geq 2$ and $\mathfrak{p}$ defined in the previous section satisfies $\mathfrak{p} = 0$, then $n \geq \frac{1}{2}r(r+1) - 1$. Moreover, either we have*

(i) *$n = \frac{1}{2}r(r+1)$, and M^n can be decomposed as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point, or*

(ii) *$n > \frac{1}{2}r(r+1)$, and M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form, or*

(iii) *$n = \frac{1}{2}r(r+1) - 1$, and up to a suitable homothetic transformation, M^n is affine equivalent to an open part of the standard embedding $\mathbf{SL}(r, \mathbb{R})/\mathbf{SO}(r) \hookrightarrow \mathbb{R}^{n+1}$, which is explicitly described at the end of this section.*

Proof. In the present situation, the decomposition (4.1) reduces to

$$\mathcal{D}_2 = \{v_1\} \oplus \cdots \oplus \{v_{k_0}\}.$$

Then $\dim \mathcal{D}_2 = k_0 = r-1$ and $\{v_1, \dots, v_{k_0}\}$ forms an orthonormal basis of $\mathcal{D}_2$. According to Lemma 3.4 and the fact that for $j \neq \ell$, $v_j \in V_{v_\ell}(\tau)$, we have

$$h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau, \quad j \neq \ell, \tag{5.1}$$

$$h(L(v_j, v_{\ell_1}), L(v_j, v_{\ell_2})) = 0, \quad j, \ell_1, \ell_2 \text{ distinct}, \tag{5.2}$$

$$h(L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})) = 0, \quad j_1, j_2, j_3, j_4 \text{ distinct}. \tag{5.3}$$

It follows that $\left\{\frac{1}{\sqrt{\tau}} L(v_j, v_\ell)\right\}_{1 \le j < \ell \le k_0}$ consists of $\frac{1}{2}(r-1)(r-2)$ orthonormal vectors. For $\{L(v_j, v_j)\}_{1 \le j \le k_0}$, we note that

$$h(L(v_j, v_j), L(v_j, v_j)) = \tfrac{n+1}{4(n-r)} \lambda_1^2, \quad 1 \le j \le k_0, \tag{5.4}$$

$$h(L(v_j, v_j), L(v_\ell, v_\ell)) = \tfrac{n+1}{4(n-r)} \lambda_1^2 - 2\tau = -\tfrac{(n+1)(r+1)}{8(n-r)^2} \lambda_1^2, \quad 1 \le j \neq \ell \le k_0, \tag{5.5}$$

$$h(L(v_j, v_j), L(v_j, v_\ell)) = 0, \quad 1 \le j \neq \ell \le k_0, \tag{5.6}$$

$$h(L(v_j, v_j), L(v_{\ell_1}, v_{\ell_2})) = 0, \quad 1 \le j, \ell_1, \ell_2 \;\text{ distinct and } \le k_0. \tag{5.7}$$

Then $\{L_j := L(v_1, v_1) + \cdots + L(v_j, v_j) - j L(v_{j+1}, v_{j+1})\}_{1 \le j \le r-2}$ are $r-2$ mutually orthogonal vectors which are all orthogonal to $L(v_j, v_\ell)$, $j \neq \ell$. Moreover, we easily have $h(L_j, L_j) = 2j(j+1)\tau \neq 0$. Hence

$$\begin{cases} w_{j\ell} = \frac{1}{\sqrt{\tau}} L(v_j, v_\ell), & 1 \le j < \ell \le r-1; \\ w_j = \frac{1}{\sqrt{2j(j+1)\tau}} L_j, & 1 \le j \le r-2 \end{cases} \tag{5.8}$$

are $\frac{1}{2}(r-1)(r-2) + (r-2)$ orthonormal vectors in $\mathrm{Im}\,(L) \subset \mathcal{D}_3$.

Finally, it is easily known that $\mathrm{Tr}\, L = L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})$ is orthogonal to the above $\frac{1}{2}(r-1)(r-2) + (r-2)$ vectors and satisfies

$$h(\mathrm{Tr}\, L, \mathrm{Tr}\, L) = \tfrac{(n+1)(r-1)(2n-r^2-r+2)}{8(n-r)^2} \lambda_1^2. \tag{5.9}$$

The above results imply that

$$\begin{aligned} n = 1 + \dim \mathcal{D}_2 + \dim \mathcal{D}_3 &\ge 1 + (r-1) + \tfrac{1}{2}(r-1)(r-2) + (r-2) \\ &= \tfrac{1}{2} r(r+1) - 1. \end{aligned}$$

Moreover, by (5.9) we see that $\mathrm{Tr}\, L = 0$ if and only if $n = \frac{1}{2} r(r+1) - 1$.

So, there are three cases to be considered: **(i)** $n = \frac{1}{2} r(r+1)$; **(ii)** $n > \frac{1}{2} r(r+1)$; **(iii)** $n = \frac{1}{2} r(r+1) - 1$.

For **Case (i)** and **Case (ii)**, we define a unit vector

$$t = \frac{4(n-r)}{\sqrt{2(n+1)(r-1)(2n-r^2-r+2)}\,\lambda_1} \mathrm{Tr}\, L.$$

In **Case (i)**, the previous results and particularly (5.9) show that $\{t;\; w_{j\ell} \,|\, 1 \le j < \ell \le r-1;\; w_j \,|1 \le j \le r-2\}$ is an orthonormal basis of $\mathrm{Im}\,(L) = \mathcal{D}_3$. By direct calculations with application of Lemma 2.4, Lemma 4.4, and (5.1)–(5.8), we have

Lemma 5.1. *From the above assumptions, we obtain that*

$$
\begin{cases}
K(t,e_1) = -\frac{r+1}{2(n-r)}\lambda_1 t,\\
K(t,t) = -\frac{r+1}{2(n-r)}\lambda_1 e_1 + \frac{2n-2r^2-r+3}{(n-r)(2n-r^2-r+2)}\sqrt{\frac{(n+1)(2n-r^2-r+2)}{2(r-1)}}\lambda_1 t,\\
K(t,v_j) = \frac{1}{2(n-r)}\sqrt{\frac{(n+1)(2n-r^2-r+2)}{2(r-1)}}\lambda_1 v_j, \quad 1 \le j \le r-1,\\
K(t,w_j) = \frac{1}{n-r}\sqrt{\frac{(n+1)(2n-r^2-r+2)}{2(r-1)}}\lambda_1 w_j, \quad 1 \le j \le r-2,\\
K(t,w_{j\ell}) = \frac{1}{n-r}\sqrt{\frac{(n+1)(2n-r^2-r+2)}{2(r-1)}}\lambda_1 w_{j\ell}, \quad 1 \le j < \ell \le r-1.
\end{cases}
\tag{5.10}
$$

Put $T = \alpha e_1 + \beta t,\ T^* = -\beta e_1 + \alpha t$, where

$$
\alpha = \sqrt{\frac{2n-r^2-r+2}{(2n-r+1)r}}, \quad \beta = \sqrt{\frac{2(n+1)(r-1)}{(2n-r+1)r}}.
\tag{5.11}
$$

Then $\{T,\, T^*;\ v_j\,|_{1\le j\le r-1};\ w_m\,|_{1\le m\le r-2};\ w_{k\ell}\,|_{1\le k<\ell\le r-1}\}$ is an orthonormal basis of $T_{x_0}M$. By Lemma 5.1 we easily obtain the following

Lemma 5.2. *Under the above assumptions, it holds*

$$
\begin{cases}
K(T,T) = \eta_1 T, \quad K(T,v_j) = \eta_2 v_j,\ 1 \le j \le r-1,\\
K(T,T^*) = \eta_2 T^*, \quad K(T,w_j) = \eta_2 w_j, \quad 1 \le j \le r-2,\\
K(T,w_{j\ell}) = \eta_2 w_{j\ell}, \quad 1 \le j < \ell \le r-1,
\end{cases}
\tag{5.12}
$$

where η_1 and η_2 are defined by

$$
\begin{cases}
\eta_1 = \frac{n-r^2-r+1}{(n-r)r}\sqrt{\frac{(2n-r+1)r}{2n-r^2-r+2}}\lambda_1,\\
\eta_2 = \frac{2n-r^2-r+2}{2(n-r)r}\sqrt{\frac{(2n-r+1)r}{2n-r^2-r+2}}\lambda_1,
\end{cases}
\tag{5.13}
$$

which satisfy the relation

$$
\eta_1\eta_2 - \eta_2^2 = -\frac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda.
\tag{5.14}
$$

Based on the conclusions of Lemma 5.2, we can apply theorem 4 of [**HLV**] to conclude that in Case (i), M^n can be decomposed as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point.

For **Case (ii)**, we see that

$$\{t;\ w_{j\ell}\,|\,1 \le j < \ell \le r-1;\ w_j\,|1 \le j \le r-2\}$$

is still an orthonormal basis of $\mathrm{Im}\,(L)$. But now we no longer have that $\mathrm{Im}\,(L)$ coincides with $\mathcal{D}_3$. Denote $\tilde{n} = n - \frac{1}{2}r(r+1)$ and choose $\tilde{w}_1, \ldots, \tilde{w}_{\tilde{n}}$ in the orthogonal complement of $\mathrm{Im}\,(L)$ in $\mathcal{D}_3$ such that

$$\{t;\ w_{j\ell}\,|_{1\le j<\ell\le r-1};\ w_j\,|_{1\le j\le r-2};\ \tilde{w}_m\,|_{1\le m\le\tilde{n}}\}$$

is an orthonormal basis of $\mathcal{D}_3$. Then, besides (5.10), we further use (2) of Lemma 2.5 to get, for $1 \le m \le \tilde{n}$,

$$K(t, \tilde{w}_m) = -\tfrac{r^2-1}{(n-r)(2n-r^2-r+2)}\sqrt{\tfrac{(n+1)(2n-r^2-r+2)}{2(r-1)}}\lambda_1 \tilde{w}_m. \tag{5.15}$$

Now we define T and T^* as in Case (i). Similar to Lemma 5.2, we can easily show the following

Lemma 5.3. *For Case (ii), it holds*

$$\begin{cases} K(T,T) = \eta_1 T, \quad K(T, v_j) = \eta_2 v_j, \quad 1 \le j \le r-1, \\ K(T,T^*) = \eta_2 T^*, \quad K(T, w_j) = \eta_2 w_j, \quad 1 \le j \le r-2, \\ K(T, w_{j\ell}) = \eta_2 w_{j\ell}, \quad 1 \le j < \ell \le r-1, \\ K(T, \tilde{w}_m) = \eta_3 \tilde{w}_m, \quad 1 \le m \le \tilde{n}, \end{cases} \tag{5.16}$$

where η_1, η_2, and η_3 are defined by (5.13) *and*

$$\eta_3 = -\tfrac{r+1}{2(n-r)}\sqrt{\tfrac{(2n-r+1)r}{2n-r^2-r+2}}\lambda_1, \tag{5.17}$$

which satisfy the relation $\eta_2 \ne \eta_3$, $2\eta_2 \ne \eta_1 \ne 2\eta_3$, and

$$\eta_1 = \eta_2 + \eta_3, \quad \eta_2\eta_3 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{5.18}$$

Based on the conclusions of Lemma 5.3, we can apply theorem 3 of [**HLV**] to conclude that in Case (ii), M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form.

Finally, for **Case (iii)**, we have that

$$\mathcal{D}_3 = \{w_{j\ell}\}_{1\le j<\ell\le r-1} \oplus \{w_j\}_{1\le j \le r-2}.$$

It follows that

$$\{e_1;\ v_j\,|_{1\le j\le r-1};\ w_{j\ell}\,|_{1\le j<\ell\le r-1};\ w_j\,|_{1\le j\le r-2}\} \tag{5.19}$$

is an orthonormal basis of $T_{x_0}M$. Now, applying Lemma 2.4, Lemma 4.4, and the previous (5.1)–(5.7), we can calculate all the components of the difference tensor with respect to the basis (5.19). Particularly, according to the definition of $\{w_j\}$, we easily derive the following formulas:

$$
(5.20)\qquad \begin{cases}
L(v_1,v_1)=\sum_{j=2}^{r-1}\sqrt{\frac{2\tau}{j(j-1)}}\,w_{j-1},\\
L(v_2,v_2)=\sum_{j=3}^{r-1}\sqrt{\frac{2\tau}{j(j-1)}}\,w_{j-1}-\sqrt{\tau}\,w_1,\\
L(v_3,v_3)=\sum_{j=4}^{r-1}\sqrt{\frac{2\tau}{j(j-1)}}\,w_{j-1}-\sqrt{\frac{2\times 2\tau}{3}}\,w_2,\quad\dots\\
L(v_{r-2},v_{r-2})=\sqrt{\frac{2\tau}{(r-1)(r-2)}}\,w_{r-2}-\sqrt{\frac{2(r-3)\tau}{r-2}}\,w_{r-3},\\
L(v_{r-1},v_{r-1})=-\sqrt{\frac{2(r-2)\tau}{r-1}}\,w_{r-2}.
\end{cases}
$$

To calculate $K(v_j,w_l)$ and $K(v_j,w_{j_1j_2})$, we use Lemma 2.4 to obtain

$$
\begin{aligned}
K(v_j,w_\ell)&=\sum_{m=1}^{r-1}h(K(v_j,w_\ell),v_m)v_m=\sum_{m=1}^{r-1}h(L(v_j,v_m),w_\ell)v_m\\
&=\tfrac{1}{\sqrt{2\ell(\ell+1)\tau}}\sum_{m=1}^{r-1}h\left(L(v_j,v_m),\sum_{\tilde m=1}^{\ell}L(v_{\tilde m},v_{\tilde m})-\ell L(v_{\ell+1},v_{\ell+1})\right)v_m\\
&=\tfrac{1}{\sqrt{2\ell(\ell+1)\tau}}h\left(L(v_j,v_j),\sum_{\tilde m=1}^{\ell}L(v_{\tilde m},v_{\tilde m})-\ell L(v_{\ell+1},v_{\ell+1})\right)v_j,
\end{aligned}
$$

and

$$
\begin{aligned}
K(v_j,w_{j_1j_2})&=\tfrac{1}{\sqrt{\tau}}\,K(v_j,L(v_{j_1},v_{j_2}))\\
&=\tfrac{1}{\sqrt{\tau}}\sum_{m=1}^{r-1}h(L(v_j,v_m),L(v_{j_1},v_{j_2}))v_m.
\end{aligned}
$$

Then, by (5.1)–(5.5), we easily obtain the following

Lemma 5.4.

(i) *If* $j\le\ell$, *then* $K(v_j,w_\ell)=\frac{1}{\sqrt{2\ell(\ell+1)\tau}}\,\frac{(n+1)(2n-r+1)}{8(n-r)^2}\lambda_1^2v_j$.

(ii) $K(v_{\ell+1},w_\ell)=-\sqrt{\frac{\ell}{2(\ell+1)\tau}}\,\frac{(n+1)(2n-r+1)}{8(n-r)^2}\lambda_1^2v_{\ell+1}$.

(iii) *If* $j>\ell+1$, *then* $K(v_j,w_\ell)=0$.

(iv) *If* $j\ne j_1,j_2$, *then* $K(v_j,w_{j_1j_2})=0$.

(v) *If* $j<\ell$, *then* $K(v_j,w_{j\ell})=\sqrt{\tau}\,v_\ell$, $K(v_\ell,w_{j\ell})=\sqrt{\tau}\,v_j$.

Similarly, we use Lemma 4.4 to carry out tedious calculations for getting $K(w_j,w_\ell)$, $K(w_j,w_{k\ell})$, and $K(w_{jk},w_{pq})$ in terms of the basis (5.19). For simplicity, this will be omitted.

Now let us consider the standard embedding of $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)\to\mathbb{R}^{r(r+1)/2}$ with affine structure as stated in section 6 of [**HLSV**], which is a hyperbolic affine hypersphere with parallel cubic form. Denote E_{jk} the $r\times r$ matrix which has (j,k) entry 1 and all others 0. Then with respect to the metric $h(X,Y)=\frac{4}{r}\,\mathrm{tr}\,(XY)$ of $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)$ at I, we can choose an orthonormal basis as follows:

$$\tag{5.21}\begin{cases}\tilde e_1=\frac{1}{\sqrt{4(r-1)}}\big((r-1)E_{rr}-E_{11}-\cdots-E_{r-1,r-1}\big),\\ \tilde v_j=\sqrt{\frac{r}{8}}\big(E_{rj}+E_{jr}\big),\ \ 1\le j\le r-1;\\ \tilde w_j=\sqrt{\frac{r}{4j(j+1)}}\big(\sum\limits_{m=1}^{j}E_{mm}-jE_{j+1,j+1}\big),\ \ 1\le j\le r-2;\\ \tilde w_{j\ell}=\sqrt{\frac{r}{8}}\big(E_{\ell j}+E_{j\ell}\big),\ \ 1\le j<\ell\le r-1.\end{cases}$$

By using the formula $K(X,Y)=K_XY=XY+YX-\frac{2}{r}\,\mathrm{tr}\,(XY)I$ and $E_{jk}E_{pq}=E_{jq}\delta_{kp}$, it can be seen easily that, if we define

$$\begin{cases}\tilde L_1=L(\tilde v_1,\tilde v_1)-L(\tilde v_2,\tilde v_2)=K(\tilde v_1,\tilde v_1)-K(\tilde v_2,\tilde v_2),\\ \tilde L_2=L(\tilde v_1,\tilde v_1)+L(\tilde v_2,\tilde v_2)-2L(\tilde v_3,\tilde v_3)\\ \quad\ =K(\tilde v_1,\tilde v_1)+K(\tilde v_2,\tilde v_2)-2K(\tilde v_3,\tilde v_3),\qquad\dots\\ \tilde L_{r-2}=K(\tilde v_1,\tilde v_1)+\cdots+K(\tilde v_{r-2},\tilde v_{r-2})-(r-2)K(\tilde v_{r-1},\tilde v_{r-1}),\end{cases}$$

then the following relations hold:

$$\tag{5.22}\begin{cases}\tilde w_j=\frac{\tilde L_j}{\|\tilde L_j\|},\ \ 1\le j\le r-2;\\ \tilde w_{j\ell}=\frac{K(\tilde v_j,\tilde v_\ell)}{\|K(\tilde v_j,\tilde v_\ell)\|}=\frac{L(\tilde v_j,\tilde v_\ell)}{\|L(\tilde v_j,\tilde v_\ell)\|},\ \ 1\le j<\ell\le r-1.\end{cases}$$

Moreover, we have the calculation for the difference tensor at I:

$$\tag{5.23}\begin{cases}K_{\tilde e_1}\tilde e_1=\frac{r-2}{\sqrt{r-1}}\tilde e_1,\ K_{\tilde e_1}\tilde v_j=\frac{r-2}{2\sqrt{r-1}}\tilde v_j,\ 1\le j\le r-1;\\ K_{\tilde e_1}\tilde w_j=-\frac{1}{\sqrt{r-1}}\tilde w_j,\ 1\le j\le r-2;\\ K_{\tilde e_1}\tilde w_{j\ell}=-\frac{1}{\sqrt{r-1}}\tilde w_{j\ell},\ \ 1\le j<\ell\le r-1.\end{cases}$$

Since $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)\to\mathbb{R}^{r(r+1)/2}$ has parallel cubic form, if we identify $\{\tilde e_1;\tilde v_j\,|_{1\le j\le r-1};\ \tilde w_j\,|_{1\le j\le r-2};\tilde w_{j\ell}\,|_{1\le j<\ell\le r-1}\}$ in (5.21) with the basis (5.19) of $M^{r(r+1)/2-1}$, then due to the facts (5.22) and (5.23), we see that the difference tensor of $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)\to\mathbb{R}^{r(r+1)/2}$ is exactly the same as that of $M^{r(r+1)/2-1}\to\mathbb{R}^{r(r+1)/2}$ corresponding to $\lambda_1=\frac{r-2}{\sqrt{r-1}}$, or equivalently $\lambda=-1$.

Now for the locally strongly convex $\mathfrak{C}_r$ affine hypersphere $M^n\to\mathbb{R}^{n+1}$ with $\mathfrak{p}=0$ and $n=\frac{1}{2}r(r+1)-1$, we see from the above discussion that, by applying a homothetic transformation to make $\lambda=-1$, if necessary, $M^{r(r+1)/2-1}$ and the standard embedding $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)\to\mathbb{R}^{r(r+1)/2}$ have affine metric h and cubic form C with identically the same affine invariant properties. Applying Cartan's lemma (cf. lemma 1.35 of [**CE**]) together with the fundamental uniqueness theorem of affine differential geometry, we obtain that $M^{r(r+1)/2-1}$ and $\mathbf{SL}(r,\mathbb{R})/\mathbf{SO}(r)$ are locally affine equivalent.

This completes the proof of Theorem 5.1. q.e.d.

6. Hypersurfaces in $\mathbb{R}^{n+1}$ with $\mathfrak{p}=1$

In this section, we will prove the following theorem.

Theorem 6.1. *Let M^n be a locally strongly convex affine hypersurface of $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\dim \mathcal{D}_2 = r-1 = 2k_0 \geq 2$ and $\mathfrak{p}$ as determined in section 4 satisfies $\mathfrak{p}=1$, then $n \geq \frac{1}{4}(r+1)^2-1$.*

Moreover, if $k_0=1$, then M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form. If $k_0 \geq 2$, then either we have

(i) *$n=\frac{1}{4}(r+1)^2$, and M^n can be written as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point, or*

(ii) *$n>\frac{1}{4}(r+1)^2$, and M^n can be written as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form, or*

(iii) *$n=\frac{1}{4}(r+1)^2-1$, and up to a homothetic transformation, M^n is affine equivalent to an open part of the standard embedding $\mathbf{SL}(\frac{r+1}{2},\mathbb{C})/\mathbf{SU}(\frac{r+1}{2}) \hookrightarrow \mathbb{R}^{n+1}$.*

To prove the theorem, we first prove the following

Lemma 6.1. *Suppose $\dim \mathcal{D}_2 = r-1 \geq 3$ and $\mathfrak{p}=1$. Then from the decomposition (4.1) there exists unit vector $u_j \in V_{v_j}(0)$, $1 \leq j \leq k_0 = \frac{1}{2}(r-1)$, such that the orthonormal basis $\{v_1,u_1;\ldots;v_{k_0},u_{k_0}\}$ of $\mathcal{D}_2$ satisfies the relations*

$$L(v_j,u_\ell) = -L(u_j,v_\ell), \quad L(v_j,v_\ell) = L(u_j,u_\ell), \quad 1 \leq j,\ell \leq k_0. \tag{6.1}$$

Proof. We have the decomposition (4.1) with $\dim V_{v_j}(0)=1$, $1 \leq j \leq k_0$. Let $V_{v_2}(0)=\{u_2\}$; here u_2 is a unit vector.

According to Lemma 4.2, for each $j \neq 2$, we have a unique unit vector $u_j \in V_{v_j}(0)$ satisfying the relation

$$L(v_j,-u_2) = L(u_j,v_2), \quad 1 \leq j \leq k_0, \; j \neq 2. \tag{6.2}$$

Claim 6.1. *Based on the above definition, the relations*

$$L(u_j,u_2) = L(v_j,v_2), \; 1 \leq j \leq k_0 \tag{6.3}$$

hold.

In fact, if $j=2$, then $u_2 \in V_{v_2}(0)$ implies that $L(u_2,v_2)=0$ and thus $L(u_2,u_2)=L(v_2,v_2)$.

Next, for each $j \neq 2$, we use the fact $L(u_j, v_j) = 0$, (6.2), and (2.10) to see that

$$\begin{aligned} h(L(u_j, u_2), L(v_j, v_2)) &= h(L(v_j, -u_2), L(u_j, v_2)) \\ &= h(L(v_j, -u_2), L(v_j, -u_2)) = h(u_2, P(v_j)u_2) = \tau. \end{aligned}$$

On the other hand, $h(L(u_j, u_2), L(u_j, u_2)) = h(L(v_j, v_2), L(v_j, v_2)) = \tau$. Then by the Cauchy-Schwarz inequality we get $L(u_j, u_2) = L(v_j, v_2)$. This finishes the proof of the claim.

Claim 6.2. $L(u_j, u_\ell) = L(v_j, v_\ell),\ 1 \leq j,\ \ell \leq k_0,\quad j, \ell \neq 2.$

For $j = \ell$, the fact that $u_j \in V_{v_j}(0)$ implies $L(v_j, u_j) = 0$. It follows that $L(u_j, u_j) = L(v_j, v_j)$.

Next, for $\dim \mathcal{D}_2 \geq 6$, we fix $j, \ell \neq 2$ such that $j \neq \ell$. By Lemma 4.2, there exists a unique unit vector in $V_{v_j}(0)$, denoted $u_j(\ell)$, such that

$$L(v_j, u_\ell) = -L(u_j(\ell), v_\ell). \tag{6.4}$$

Since both unit vectors $u_j,\ u_j(\ell) \in V_{v_j}(0)$ and $\dim V_{v_j}(0) = 1$, we have two possibilities: $u_j(\ell) = u_j$ or $u_j(\ell) = -u_j$.

(1) If $u_j(\ell) = u_j$, then we have $L(v_j, u_\ell) = -L(u_j, v_\ell)$. By using (2.10) and $L(u_j, v_j) = 0$ we have

$$\begin{aligned} h(L(u_j, u_\ell), L(v_j, v_\ell)) &= h(L(v_j, -u_\ell), L(u_j, v_\ell)) \\ &= h(L(v_j, -u_\ell), L(v_j, -u_\ell)) = h(u_\ell, P(v_j)u_\ell) = \tau. \end{aligned}$$

On the other hand, $h(L(u_j, u_\ell), L(u_j, u_\ell)) = h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau$. Then by the Cauchy-Schwarz inequality the claim $L(u_j, u_\ell) = L(v_j, v_\ell)$ follows.

(2) If $u_j(\ell) = -u_j$, then we have $L(v_j, u_\ell) = L(u_j, v_\ell)$. We will show that this is impossible.

In fact, by (2.10) and $L(u_j, v_j) = 0$ we have

$$\begin{aligned} h(L(v_j, v_\ell), L(u_j, u_\ell)) &= -h(L(v_j, u_\ell), L(u_j, v_\ell)) \\ &= -h(L(v_j, u_\ell), L(v_j, u_\ell)) = -\tau. \end{aligned}$$

Since $h(L(u_j, u_\ell), L(u_j, u_\ell)) = h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau$, by the Cauchy-Schwarz inequality we obtain $L(u_j, u_\ell) = -L(v_j, v_\ell)$. Hence we have

$$K(L(u_j, u_\ell) + L(v_j, v_\ell), L(v_2, u_j)) = 0. \tag{6.5}$$

On the other hand, by using (6.2) and Lemma 4.4, we find that

$$K(L(u_j, u_\ell), L(v_2, u_j)) = \tau L(u_\ell, v_2) = -\tau L(v_\ell, u_2),$$

$$K(L(v_j, v_\ell), L(v_2, u_j)) = K(L(v_j, v_\ell), -L(v_j, u_2)) = -\tau L(v_\ell, u_2).$$

Combining the above with (6.5), we get $L(v_\ell, u_2) = 0$, which is a contradiction. So we complete the proof of Claim 6.2.

We have now completed the proof of Lemma 6.1. q.e.d.

Remark 6.1. For $\mathfrak{p} = 1$ we have $\dim \mathcal{D}_2 = 2k_0$. Denote $V_j = \{v_j\} \oplus V_{v_j}(0) = \{v_j\} \oplus \{u_j\},\ 1 \leq j \leq k_0$. For each $1 \leq j \leq k_0$, we define a linear map $J : V_j \to V_j$ by setting $Jv_j = u_j$, $Ju_j = -v_j$. Then $J : \mathcal{D}_2 \to \mathcal{D}_2$ is an almost complex structure and Lemma 6.1 shows that it satisfies the relations

$$L(Ju, v) = -L(u, Jv), \quad L(Ju, Jv) = L(u, v) \tag{6.6}$$

for all $u, v \in \mathcal{D}_2$.

Proof of Theorem 6.1. Let $r-1 = 2k_0$. If $k_0 = 1$, then the dimension of the image of L would be 1, which is a case we have treated in theorem 5.1 of [**HLSV**], and in that case it follows that M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form.

We now assume that $k_0 \geq 2$ and let $\{v_1, u_1; \ldots, v_{k_0}, u_{k_0}\}$ be the orthonormal basis of $\mathcal{D}_2$ as constructed in Lemma 6.1. According to Lemma 3.4 and that for $j \neq \ell$, $u_j, v_j \in V_{v_\ell}(\tau) = V_{u_\ell}(\tau)$, we have

$$h(L(v_j, u_\ell), L(v_j, u_\ell)) = h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau, \quad j \neq \ell, \tag{6.7}$$

$$\begin{aligned} h(L(u_j, v_{\ell_1}), L(u_j, v_{\ell_2})) &= h(L(v_j, u_{\ell_1}), L(v_j, u_{\ell_2})) \\ &= h(L(v_j, v_{\ell_1}), L(v_j, v_{\ell_2})) \\ &= 0, \quad j, \ell_1, \ell_2 \text{ distinct}, \end{aligned} \tag{6.8}$$

$$h(L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})) = 0, \quad j_1, j_2, j_3, j_4 \text{ distinct}, \tag{6.9}$$

$$h(L(v_j, v_\ell), L(v_{j_1}, u_{\ell_1})) = 0, \quad j \neq \ell \text{ and } j_1 \neq \ell_1. \tag{6.10}$$

Thus $\{\frac{1}{\sqrt{\tau}} L(v_j, v_\ell)\}_{1 \leq j < \ell \leq k_0} \cup \{\frac{1}{\sqrt{\tau}} L(v_j, u_\ell)\}_{1 \leq j < \ell \leq k_0}$ consists of $k_0(k_0 - 1) = \frac{1}{4}(r-1)(r-3)$ orthonormal vectors. For

$$\{L(v_j, v_j) = L(u_j, u_j)\}_{1 \leq j \leq k_0},$$

we note that

$$h(L(v_j, v_j), L(v_j, v_j)) = \tfrac{n+1}{4(n-r)} \lambda_1^2, \quad 1 \leq j \leq k_0, \tag{6.11}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_\ell, v_\ell)) &= \tfrac{n+1}{4(n-r)} \lambda_1^2 - 2\tau \\ &= -\tfrac{(n+1)(r+1)}{8(n-r)^2} \lambda_1^2, \quad 1 \leq j \neq \ell \leq k_0, \end{aligned} \tag{6.12}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_j, v_\ell)) &= h(L(v_j, v_j), L(v_j, u_\ell)) \\ &= 0, \quad 1 \leq j \neq \ell \leq k_0, \end{aligned} \tag{6.13}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_{\ell_1}, v_{\ell_2})) &= h(L(v_j, v_j), L(v_{\ell_1}, u_{\ell_2})) \\ &= 0, \quad 1 \leq j, \ell_1, \ell_2 \text{ distinct } \leq k_0. \end{aligned} \tag{6.14}$$

Similar to the previous section, we see that

$$\{L_j := L(v_1, v_1) + \cdots + L(v_j, v_j) - jL(v_{j+1}, v_{j+1})\}_{1 \leq j \leq k_0 - 1}$$

are $k_0-1=\frac{1}{2}(r-3)$ mutually orthogonal vectors which are orthogonal to all $L(v_j,v_\ell)$ and $L(v_j,u_\ell)$, $j\neq\ell$. Moreover, we easily have $h(L_j,L_j)=2j(j+1)\tau\neq 0$. Hence

$$\begin{cases} w_{j\ell}=\frac{1}{\sqrt{\tau}}L(v_j,v_\ell), & 1\le j<\ell\le k_0;\\ w'_{j\ell}=\frac{1}{\sqrt{\tau}}L(v_j,u_\ell), & 1\le j<\ell\le k_0;\\ w_j=\frac{1}{\sqrt{2j(j+1)\tau}}L_j, & 1\le j\le k_0-1 \end{cases} \tag{6.15}$$

are $\frac{1}{4}(r+1)(r-3)$ orthonormal vectors in $\mathrm{Im}\,(L)\subset\mathcal{D}_3$.

Finally, it is easily verified that $\frac{1}{2}\mathrm{Tr}\,L=L(v_1,v_1)+\cdots+L(v_{k_0},v_{k_0})$ is orthogonal to the above $(r+1)(r-3)/4$ vectors and satisfies

$$\begin{aligned} \tfrac{1}{4}h(\mathrm{Tr}\,L,\mathrm{Tr}\,L)&=\tfrac{(n+1)k_0(n-k_0(k_0+2))}{4(n-r)^2}\lambda_1^2\\ &=\tfrac{(n+1)(r-1)\lambda_1^2}{8(n-r)^2}\big[n-\tfrac{1}{4}(r+1)^2+1\big]. \end{aligned} \tag{6.16}$$

The above results imply that

$$n=1+\dim\mathcal{D}_2+\dim\mathcal{D}_3\ge 1+(r-1)+\tfrac{1}{4}(r+1)(r-3)=\tfrac{1}{4}(r+1)^2-1.$$

Moreover, from (6.16) we see that $\mathrm{Tr}\,L=0$ if and only if $n=\frac{1}{4}(r+1)^2-1$. Then, there are three cases to be considered: **(i)** $n=\frac{1}{4}(r+1)^2$; **(ii)** $n>\frac{1}{4}(r+1)^2$; **(iii)** $n=\frac{1}{4}(r+1)^2-1$.

For **Case (i)** and **Case (ii)**, we define a unit vector

$$t=\frac{4(n-r)}{\sqrt{2(n+1)(r-1)[4n-(r-1)(r+3)]}\,\lambda_1}\mathrm{Tr}\,L.$$

In **Case (i)**, the previous results show that $\{t;\ w_{j\ell},\ w'_{j\ell}\,|\,1\le j<\ell\le\frac{1}{2}(r-1);\ w_j\,|1\le j\le\frac{1}{2}(r-3)\}$ is an orthonormal basis of $\mathrm{Im}\,(L)=\mathcal{D}_3$. By direct calculations and applying Lemma 2.4, Lemma 4.4, and (6.7)–(6.14), we have

Lemma 6.2. *From the above assumptions, it follows that*

$$\begin{cases} K(t,e_1) = -\frac{r+1}{2(n-r)}\lambda_1 t,\\ K(t,t) = -\frac{r+1}{2(n-r)}\lambda_1 e_1+\frac{2n-(r-1)(r+2)}{n-r}\sqrt{\frac{2(n+1)}{(r-1)[4n-(r-1)(r+3)]}}\,\lambda_1 t,\\ K(t,v_j) = \frac{1}{2(n-r)}\sqrt{\frac{(n+1)[4n-(r-1)(r+3)]}{2(r-1)}}\,\lambda_1 v_j, \quad 1\le j\le\frac{1}{2}(r-1),\\ K(t,u_j) = \frac{1}{2(n-r)}\sqrt{\frac{(n+1)[4n-(r-1)(r+3)]}{2(r-1)}}\,\lambda_1 u_j, \quad 1\le j\le\frac{1}{2}(r-1),\\ K(t,w_j) = \frac{1}{n-r}\sqrt{\frac{(n+1)[4n-(r-1)(r+3)]}{2(r-1)}}\,\lambda_1 w_j, \quad 1\le j\le\frac{1}{2}(r-3),\\ K(t,w_{j\ell}) = \frac{1}{n-r}\sqrt{\frac{(n+1)[4n-(r-1)(r+3)]}{2(r-1)}}\,\lambda_1 w_{j\ell}, \quad 1\le j<\ell\le\frac{1}{2}(r-1),\\ K(t,w'_{j\ell}) = \frac{1}{n-r}\sqrt{\frac{(n+1)[4n-(r-1)(r+3)]}{2(r-1)}}\,\lambda_1 w'_{j\ell}, \quad 1\le j<\ell\le\frac{1}{2}(r-1). \end{cases} \tag{6.17}$$

Put $T=\alpha e_1+\beta t$, $T^*=-\beta e_1+\alpha t$, where

$$\alpha=\sqrt{\frac{4n-(r-1)(r+3)}{4n+(r-1)(2n-r-1)}},\quad \beta=\sqrt{\frac{2(n+1)(r-1)}{4n+(r-1)(2n-r-1)}}. \tag{6.18}$$

Then

$$\{T,\,T^*;\;v_j,u_j\,|_{1\le j\le(r-1)/2};\;w_m\,|_{1\le m\le(r-3)/2};\;w_{k\ell},w'_{k\ell}\,|_{1\le k<\ell\le(r-1)/2}\}$$

forms an orthonormal basis of $T_{x_0}M$. By Lemma 6.2 we easily obtain the following

Lemma 6.3. *From the above assumptions, we deduce that*

$$(6.19)\qquad \begin{cases} K(T,T)=\xi_1T,\;\;K(T,T^*)=\xi_2T^*,\\ K(T,v_j)=\xi_2v_j,\;\;K(T,u_j)=\xi_2u_j,\quad 1\le j\le\frac12(r-1),\\ K(T,w_j)=\xi_2w_j,\;\;1\le j\le\frac12(r-3),\\ K(T,w_{j\ell})=\xi_2w_{j\ell},\;\;K(T,w'_{j\ell})=\xi_2w'_{j\ell},\;\;1\le j<\ell\le\frac12(r-1),\end{cases}$$

where ξ_1 *and* ξ_2 *are defined by*

$$(6.20)\qquad \begin{cases}\xi_1=\frac{(n-r)[4n-(r-1)(r+3)]-(n+1)(r^2-1)}{(n-r)[4n-(r-1)(r+3)]}\sqrt{\frac{4n-(r-1)(r+3)}{4n+(r-1)(2n-r-1)}}\,\lambda_1,\\ \xi_2=\frac{2n-r+1}{2(n-r)}\sqrt{\frac{4n-(r-1)(r+3)}{4n+(r-1)(2n-r-1)}}\,\lambda_1,\end{cases}$$

which satisfy the relation

$$(6.21)\qquad \xi_1\xi_2-\xi_2^2=-\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2=\lambda.$$

Based on the conclusions of Lemma 6.3, we can again apply theorem 4 of [**HLV**] to conclude that in Case (i), M^n is decomposed as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point.

For **Case (ii)**, we see that $\{t;\;w_{j\ell},\,w'_{j\ell}\,|\,1\le j<\ell\le\frac12(r-1);\;w_j\,|1\le j\le\frac12(r-3)\}$ is still an orthonormal basis of $\mathrm{Im}\,(L)$. But now $\mathrm{Im}\,(L)\subsetneq\mathcal{D}_3$. Denote $\tilde n=n-\frac14(r+1)^2\ge1$ and choose $w'_1,\dots,w'_{\tilde n}$ in the orthogonal complement of $\mathrm{Im}\,(L)$ in $\mathcal{D}_3$, such that

$$\{t;\;w_{j\ell},\,w'_{j\ell}\,|\,1\le j<\ell\le\tfrac12(r-1);\;w_j\,|1\le j\le\tfrac12(r-3);\;w'_j\,|1\le j\le\tilde n\}$$

is an orthonormal basis of $\mathcal{D}_3$. Then, (6.17) together with (2) of Lemma 2.5 gives that

$$(6.22)\qquad K(t,w'_j)=-\tfrac{r+1}{2(n-r)}\sqrt{\tfrac{2(n+1)(r-1)}{4n-(r-1)(r+3)}}\lambda_1w'_j,\quad 1\le j\le\tilde n.$$

Now we define T and T^* as in Case (i). Similar to Lemma 6.3, we can establish the following

Lemma 6.4. *For Case (ii), we have that*

$$(6.23)\qquad \begin{cases} K(T,T)=\xi_1T,\qquad K(T,T^*)=\xi_2T^*,\\ K(T,v_j)=\xi_2v_j,\quad K(T,u_j)=\xi_2u_j,\quad 1\le j\le\frac12(r-1),\\ K(T,w_j)=\xi_2w_j,\quad 1\le j\le\frac12(r-3),\\ K(T,w_{j\ell})=\xi_2w_{j\ell},\quad K(T,w'_{j\ell})=\xi_2w'_{j\ell},\quad 1\le j<\ell\le\frac12(r-1),\\ K(T,w'_j)=\xi_3w'_j,\quad 1\le j\le\tilde n,\end{cases}$$

where ξ_1*,* ξ_2*, and* ξ_3 *are defined by* (6.20) *and*

$$\xi_3 = -\tfrac{r+1}{2(n-r)}\sqrt{\tfrac{4n+(r-1)(2n-r-1)}{4n-(r-1)(r+3)}}\,\lambda_1, \tag{6.24}$$

which satisfy the relations $\xi_2 \neq \xi_3,\ 2\xi_2 \neq \xi_1 \neq 2\xi_3$*, and*

$$\xi_1 = \xi_2 + \xi_3, \quad \xi_2\xi_3 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{6.25}$$

Based on the conclusions of Lemma 6.4, we can as in the previous section apply theorem 3 of [**HLV**] to conclude that in Case (ii), M^n is decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form.

Finally, for **Case (iii)**, we have $\operatorname{Tr} L = 0$ and thus

$$\mathcal{D}_3 = \{w_{j\ell}\}_{1\le j<\ell\le (r-1)/2} \oplus \{w'_{j\ell}\}_{1\le j<\ell\le (r-1)/2} \oplus \{w_j\}_{1\le j\le (r-3)/2}.$$

It follows that

$$\{e_1;\, v_j, u_j\,|_{1\le j\le (r-1)/2};\ w_{j\ell}, w'_{j\ell}\,|_{1\le j<\ell\le (r-1)/2};\ w_j\,|_{1\le j\le (r-3)/2}\} \tag{6.26}$$

is an orthonormal basis of $T_{x_0}M$. Now, applying Lemma 2.4, Lemma 4.4, (6.1), and the previous formulas from (6.7) up to (6.14), we can calculate all components of the difference tensor with respect to the basis (6.26).

Now, we look at the homogeneous space $\mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m)$ and recall its standard embedding into $\mathbb{R}^{m^2}$.

Let $\mathfrak{s}(m)$ be the set of Hermitian (m,m)-matrices, $\mathbf{SL}(m,\mathbb{C})$ be the set of complex (m,m)-matrices of determinant 1, and $\mathbf{SU}(m) = \{A \in \mathbf{SL}(m,\mathbb{C})\,|\,{}^t\bar{A}A = I_m\}$ be the set of unitary (m,m)-matrices with determinant 1. Let $\mathfrak{X}$ be the action of $\mathbf{SL}(m,\mathbb{C})$ on $\mathfrak{s}(m)$ defined as follows:

$$\mathfrak{X}:\ \mathbf{SL}(m,\mathbb{C}) \times \mathfrak{s}(m) \to \mathfrak{s}(m) \quad \text{s.t.} \quad (A,X) \mapsto \mathfrak{X}_A(X) = AX\,{}^t\bar{A}.$$

Let $F : \mathfrak{s}(m) \to \mathbb{C}$ be given by $F(X) := \det(X)$. Consider the hypersurface of $\mathfrak{s}(m)$ satisfying the equation $\det(X) = 1$; we take the connected component M that lies in the open set of $\mathfrak{s}(m)$ consisting of all Hermitian positive definite matrices. Then the mapping $f : \mathbf{SL}(m,\mathbb{C}) \to s(m)$, defined by $f(A) := A\,{}^t\bar{A}$, is a submersion onto M, and it satisfies $f(AB) = \mathfrak{X}_A(f(B))$; hence f is equivariant. M is the orbit of I under the action $\mathfrak{X}$. The isotropy group is $\mathbf{SU}(m)$. Hence M is diffeomorphic to $\mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m)$. This is an irreducible, homogeneous, symmetric space of non-compact type, and the involution at the identity matrix I is given by $A \mapsto {}^t\bar{A}^{-1}$. We denote this symmetric space by M'.

Clearly $f(A) = f(B)$ if and only if $B^{-1}A \in \mathbf{SU}(m)$ and therefore the map $f : \mathbf{SL}(m,\mathbb{C}) \to \mathfrak{s}(m)$ induces an embedding $f : \mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m) \to \mathfrak{s}(m)$. Let $\pi : \mathbf{SL}(m,\mathbb{C}) \to M'$ be the natural projection; then there is an immersion $f' : M' \to \mathfrak{s}(m)$ such that $f = f' \circ \pi$. Now we consider

$$f:\ \mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m) \to \mathbb{R}^{n+1} = \mathfrak{s}(m), \quad n+1 = m^2 \tag{6.27}$$

with a transversal vector field $\xi_A = f(A)$ for any $A \in \mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m)$. Then ξ is equiaffine and equivariant.

Consider the Cartan decomposition of the Lie algebra $\mathfrak{sl}(m,\mathbb{C}) = \mathfrak{s}_0 \oplus \mathfrak{su}(m)$, where $\mathfrak{su}(m)$ denotes the set of skew-Hermitian (m,m)-matrices and $\mathfrak{s}_0 := \{X \in \mathfrak{s}(m) \mid \operatorname{tr}(X) = 0\}$. If $X \in \mathfrak{s}_0$ then $f_*(X) = X$. Now $\mathfrak{s}_0$ can be considered as the tangent space of M' at $\pi(I)$.

Since f is equivariant, it is sufficient to compute the invariant objects of the immersed hypersurface M' in terms of $\mathfrak{s}_0$.

The embedding $f : \mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m) \to \mathbb{R}^{n+1} = \mathfrak{s}(m)$ with $\xi = f$ has a Blaschke structure that can be expressed algebraically in terms of the Lie algebra as follows (cf. [**BD**] for the case $m = 3$):

$$\begin{cases} K(X,Y) = XY + YX - \frac{2}{m}\operatorname{tr}(XY)I_m, \\ h(X,Y) = \frac{4}{m}\operatorname{tr}(XY), \qquad S = -I_m. \end{cases} \tag{6.28}$$

Here h is the natural Riemannian metric on the symmetric space M'; this implies that the Levi-Civita connection of h is given by $\hat{\nabla}_X Y = \frac{1}{2}[X,Y]$. From this it follows easily that the difference tensor K satisfies $\big(\hat{\nabla}_X K\big)(X,X) = 0$. As $M = f'(M')$ is an affine hypersphere, we get that $\hat{\nabla} K$ is totally symmetric [**BNS**]; then from $\big(\hat{\nabla}_X K\big)(X,X) = 0$ and polarization of the multilinear symmetric expression over $T_{x_0}(M)$ at $x_0 \in M$ it follows that $\hat{\nabla} K = 0$.

Now we choose $m = \frac{1}{2}(r+1)$. Denote E_{jk} (resp. E'_{jk}) the $m \times m$ matrix which has (j,k) entry 1 (resp. $\sqrt{-1}$) and all other entries 0. Then with respect to the metric $h(X,Y) = \frac{4}{m}\operatorname{tr}(XY)$ of $\mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m)$ at I, we can choose an orthonormal basis as follows:

(6.29)

$$\begin{cases} \tilde{e}_1 = \frac{1}{\sqrt{4(m-1)}}\big((m-1)E_{mm} - E_{11} - \cdots - E_{m-1,m-1}\big), \\ \tilde{v}_j = \sqrt{\frac{m}{8}}\big(E_{mj} + E_{jm}\big),\ \tilde{u}_j = \sqrt{\frac{m}{8}}\big(E'_{mj} - E'_{jm}\big),\ 1 \le j \le m-1; \\ \tilde{w}_j = \sqrt{\frac{m}{4j(j+1)}}\big(E_{11} + \cdots + E_{jj} - jE_{j+1,j+1}\big),\ 1 \le j \le m-2; \\ \tilde{w}_{j\ell} = \sqrt{\frac{m}{8}}\big(E_{\ell j} + E_{j\ell}\big),\ \tilde{w}'_{j\ell} = \sqrt{\frac{m}{8}}\big(E'_{j\ell} - E'_{\ell j}\big),\ 1 \le j < \ell \le m-1. \end{cases}$$

By using the formula $K(X,Y) = XY + YX - \frac{2}{m}\operatorname{tr}(XY)I_m$ and

$$E_{jk}E_{pq} = E_{jq}\delta_{kp}, \quad E'_{jk}E_{pq} = E_{jk}E'_{pq} = E'_{jq}\delta_{kp}, \quad E'_{jk}E'_{pq} = -E_{jq}\delta_{kp},$$

we can show that $L(X,Y) = K(X,Y) - \frac{1}{2}\lambda_1 h(X,Y)\tilde{e}_1$, where $\lambda_1 = \frac{m-2}{\sqrt{m-1}}$, satisfies

$$L(\tilde{v}_j,\tilde{u}_\ell) = -L(\tilde{u}_j,\tilde{v}_\ell),\ L(\tilde{v}_j,\tilde{v}_\ell) = L(\tilde{u}_j,\tilde{u}_\ell),\ 1 \le j,\ell \le m-1.$$

Furthermore, if we define

$$\begin{cases} \tilde{L}_1 = L(\tilde{v}_1, \tilde{v}_1) - L(\tilde{v}_2, \tilde{v}_2) = K(\tilde{v}_1, \tilde{v}_1) - K(\tilde{v}_2, \tilde{v}_2), \\ \tilde{L}_2 = L(\tilde{v}_1, \tilde{v}_1) + L(\tilde{v}_2, \tilde{v}_2) - 2L(\tilde{v}_3, \tilde{v}_3) \\ \quad = K(\tilde{v}_1, \tilde{v}_1) + K(\tilde{v}_2, \tilde{v}_2) - 2K(\tilde{v}_3, \tilde{v}_3), \qquad \dots \\ \tilde{L}_{m-2} = K(\tilde{v}_1, \tilde{v}_1) + \cdots + K(\tilde{v}_{m-2}, \tilde{v}_{m-2}) - (m-2)K(\tilde{v}_{m-1}, \tilde{v}_{m-1}), \end{cases}$$

then the relations hold:

$$\begin{cases} \tilde{w}_j = \frac{\tilde{L}_j}{\|\tilde{L}_j\|}, \quad 1 \le j \le m-2; \\ \tilde{w}_{j\ell} = \frac{K(\tilde{v}_j, \tilde{v}_\ell)}{\|K(\tilde{v}_j, \tilde{v}_\ell)\|} = \frac{L(\tilde{v}_j, \tilde{v}_\ell)}{\|L(\tilde{v}_j, \tilde{v}_\ell)\|}, \quad 1 \le j < \ell \le m-1; \\ \tilde{w}'_{j\ell} = \frac{K(\tilde{v}_j, \tilde{u}_\ell)}{\|K(\tilde{v}_j, \tilde{u}_\ell)\|} = \frac{L(\tilde{v}_j, \tilde{u}_\ell)}{\|L(\tilde{v}_j, \tilde{u}_\ell)\|}, \quad 1 \le j < \ell \le m-1. \end{cases} \tag{6.30}$$

Moreover, we have the following calculation for the difference tensor at I:

$$\begin{cases} K_{\tilde{e}_1}\tilde{e}_1 = \frac{m-2}{\sqrt{m-1}}\tilde{e}_1, \\ K_{\tilde{e}_1}\tilde{v}_j = \frac{m-2}{2\sqrt{m-1}}\tilde{v}_j, \ K_{\tilde{e}_1}\tilde{u}_j = \frac{m-2}{2\sqrt{m-1}}\tilde{u}_j, \ 1 \le j \le m-1; \\ K_{\tilde{e}_1}\tilde{w}_j = -\frac{1}{\sqrt{m-1}}\tilde{w}_j, \ 1 \le j \le m-2; \\ K_{\tilde{e}_1}\tilde{w}_{j\ell} = -\frac{1}{\sqrt{m-1}}\tilde{w}_{j\ell}, \ K_{\tilde{e}_1}\tilde{w}'_{j\ell} = -\frac{1}{\sqrt{m-1}}\tilde{w}'_{j\ell}, \ 1 \le j < \ell \le m-1. \end{cases} \tag{6.31}$$

Since $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m) \to \mathbb{R}^{m^2}$ has parallel cubic form, if we identify $\big\{\tilde{e}_1; \tilde{v}_j, \tilde{u}_j\,|_{1\le j\le m-1}; \tilde{w}_{j\ell}, \tilde{w}'_{j\ell}\,|_{1\le j<\ell\le m-1}; \tilde{w}_j\,|_{1\le j\le m-2}\big\}$ in (6.29) with the basis (6.26) of M^n, then due to the facts (6.30) and (6.31), we see that the difference tensor of $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m) \to \mathbb{R}^{m^2}$ is exactly the same as that of $M^n \to \mathbb{R}^{n+1}$ corresponding to $\lambda_1 = \frac{m-2}{\sqrt{m-1}}$, or equivalently $\lambda = -1$.

Now for the locally strongly convex $\mathfrak{C}_r$ affine hypersphere $M^n \to \mathbb{R}^{n+1}$ with $\mathfrak{p} = 1$ and $n = \frac{1}{4}(r+1)^2 - 1$, we see from the above discussion that, by applying a homothetic transformation to make $\lambda = -1$, if necessary, $M^{(r+1)^2/4-1}$ and the standard embedding $\mathbf{SL}(\frac{r+1}{2}, \mathbb{C})/\mathbf{SU}(\frac{r+1}{2}) \hookrightarrow \mathbb{R}^{(r+1)^2/4}$ have affine metric h and cubic form C with identically the same affine invariant properties. According to Cartan's lemma and the fundamental uniqueness theorem of affine differential geometry, we obtain that $M^{(r+1)^2/4-1}$ and $\mathbf{SL}(\frac{r+1}{2}, \mathbb{C})/\mathbf{SU}(\frac{r+1}{2})$ are locally affine equivalent.

We have completed the proof of Theorem 6.1. q.e.d.

7. Hypersurfaces in $\mathbb{R}^{n+1}$ with $\mathfrak{p} = 3$

In this section, we will prove the following theorem.

Theorem 7.1. *Let M^n be a locally strongly convex affine hypersurface of $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\dim \mathcal{D}_2 = r - 1 = 4k_0 \ge 4$ and $\mathfrak{p} = 3$, then $n \ge (r-1)(r+5)/8$. Moreover, if $k_0 = 1$, then M^n can be decomposed as the Calabi product of two* *hyperbolic affine hyperspheres both with parallel cubic form. If $k_0 \geq 2$, then either*

(i) *if $n = \frac{1}{8}(r-1)(r+5)+1$, then M^n is the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point, or*

(ii) *if $n \geq \frac{1}{8}(r-1)(r+5)+2$, then M^n is the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form, or*

(iii) *if $n = \frac{1}{8}(r-1)(r+5)$, then up to a homothetic transformation, M^n is affine equivalent to an open part of the standard embedding $\mathbf{SU}^*(\frac{r+3}{2})/\mathbf{Sp}(\frac{r+3}{4}) \hookrightarrow \mathbb{R}^{n+1}$.*

In order to prove the theorem, we first show the following

Lemma 7.1. *Suppose $\dim \mathcal{D}_2 = r-1 \geq 4$ and $\mathfrak{p} = 3$. Then from the decomposition (4.1) there exist unit orthogonal vectors $x_j, y_j, z_j \in V_{v_j}(0)$, $1 \leq j \leq k_0 = (r-1)/4$, such that the orthonormal basis $\{v_1, x_1, y_1, z_1; \ldots; v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$ of $\mathcal{D}_2$ satisfies the relations*

$$\tag{7.1} \begin{cases} L(x_j, x_\ell) = L(y_j, y_\ell) = L(z_j, z_\ell) = L(v_j, v_\ell), \\ L(v_j, x_\ell) = -L(x_j, v_\ell) = -L(y_j, z_\ell) = L(y_\ell, z_j), \\ L(v_j, y_\ell) = -L(y_j, v_\ell) = -L(z_j, x_\ell) = L(x_j, z_\ell), \\ L(v_j, z_\ell) = -L(z_j, v_\ell) = -L(x_j, y_\ell) = L(x_\ell, y_j) \end{cases}$$

for all $1 \leq j, \ell \leq k_0$.

Proof. If $k_0 = 1$, then by using Lemma 4.1 it is easily seen that we have the conclusion. Hence we will suppose $k_0 \geq 2$ and we have the decomposition (4.1) with $\dim V_{v_j}(0) = 3$, $1 \leq j \leq k_0$.

Denote $V_j = \{v_j\} \oplus V_{v_j}(0)$. Let us fix an orthonormal basis $\{x_1, y_1, z_1\}$ of $V_{v_1}(0)$. Then according to Lemma 4.2, for each $j \neq 1$, we have unique unit vectors $x_j, y_j, z_j \in V_{v_j}(0)$ satisfying the relations

$$\tag{7.2} \begin{cases} L(v_j, -x_1) = L(x_j, v_1), \quad L(v_j, -y_1) = L(y_j, v_1), \\ L(v_j, -z_1) = L(z_j, v_1), \qquad\qquad 2 \leq j \leq k_0. \end{cases}$$

Claim 7.1. *Based on their definitions in (7.2), $\{x_j, y_j, z_j\}$ is an orthonormal basis of $V_{v_j}(0)$.*

In fact, for $j \neq 1$, taking the inner products between the expressions in (7.2), together with the fact that $x_j, y_j, z_j \in V_{v_1}(\tau)$ and $x_1, y_1, z_1 \in V_{v_j}(\tau)$, gives the desired result. For example, the equations

$$\begin{aligned} \tau h(x_j, y_j) &= h(L(v_1, x_j), L(v_1, y_j)) \\ &= h(-L(v_j, x_1), -L(v_j, y_1)) = \tau h(x_1, y_1) = 0 \end{aligned}$$

give that $y_j \perp x_j$.

Claim 7.2. *Based on their definitions in (7.2), $\{x_j, y_j, z_j\}$ satisfies the relations*

$$\tag{7.3} L(x_j, x_1) = L(y_j, y_1) = L(z_j, z_1) = L(v_j, v_1),$$

$$
(7.4) \qquad \begin{aligned} &L(x_j, y_1) = -L(x_1, y_j),\ \ L(x_j, z_1) = -L(x_1, z_j), \\ &L(y_1, z_j) = -L(y_j, z_1) \end{aligned}
$$

for all $1 \le j \le k_0$.

In fact, as seen previously, we only need to consider the case $j \ne 1$.

First, we use the fact $L(x_j, v_j) = 0$ (see Lemma 4.1), (7.2), and (2.10) to see that

$$
\begin{aligned} h(L(x_j, x_1), L(v_j, v_1)) &= h(L(v_j, -x_1), L(x_j, v_1)) \\ &= h(L(v_j, -x_1), L(v_j, -x_1)) = h(x_1, P(v_j)x_1) = \tau. \end{aligned}
$$

On the other hand, from Lemma 4.3 we see that $x_j \in V_{v_1}(\tau) = V_{x_1}(\tau)$ and hence

$$
h(L(x_j, x_1), L(x_j, x_1)) = \tau = h(L(v_j, v_1), L(v_j, v_1)).
$$

Then the Cauchy-Schwarz inequality implies that $L(x_j, x_1) = L(v_j, v_1)$. Similar discussions will finish the proof for the other cases in (7.3).

To verify (7.4), we use the fact $L(x_j, y_j) = L(x_j, z_j) = L(y_j, z_j) = 0$, (7.3), and (2.10) to see that

$$
\begin{aligned} h(L(x_j, y_1), L(x_1, y_j)) &= -h(L(x_j, x_1), L(y_1, y_j)) \\ &= -h(L(v_j, v_1), L(v_1, v_j)) = -h(v_j, P(v_1)v_j) = -\tau, \end{aligned}
$$

$$
h(L(x_j, z_1), L(x_1, z_j)) = -h(L(x_j, x_1), L(z_1, z_j)) = -\tau,
$$

$$
h(L(z_j, y_1), L(z_1, y_j)) = -h(L(z_j, z_1), L(y_1, y_j)) = -\tau.
$$

On the other hand, from Lemma 4.3 we see that

$$
x_j, y_j, z_j \in V_{v_1}(\tau) = V_{x_1}(\tau) = V_{y_1}(\tau) = V_{z_1}(\tau)
$$

and hence

$$
h(L(x_j, y_1), L(x_j, y_1)) = \tau = h(L(x_1, y_j), L(x_1, y_j)),
$$

$$
h(L(x_j, z_1), L(x_j, z_1)) = \tau = h(L(x_1, z_j), L(x_1, z_j)),
$$

$$
h(L(z_j, y_1), L(z_j, y_1)) = \tau = h(L(z_1, y_j), L(z_1, y_j)).
$$

Now the Cauchy-Schwarz inequality implies that all equations in (7.4) hold.

Claim 7.3. *The orthonormal basis* $\{x_j, y_j, z_j\}_{1 \le j \le k_0}$ *can be chosen to satisfy*

$$
\begin{aligned} &L(x_j, y_1) = L(z_j, v_1),\ \ L(y_j, z_1) = L(x_j, v_1), \\ &L(z_j, x_1) = L(y_j, v_1), \qquad\quad 1 \le j \le k_0. \end{aligned}
$$

Again, we need only to consider the cases for $j \ne 1$. Note that by Lemma 4.3 we have

$$
V_1 = \{v_1\} \oplus V_{v_1}(0) = \{y_1\} \oplus V_{y_1}(0),\ V_j = \{v_j\} \oplus V_{v_j}(0) = \{x_j\} \oplus V_{x_j}(0).
$$

Then by Lemma 4.2, we have a unique unit vector $\alpha_j \in V_{y_j}(0) = \text{span}\{v_j,\, x_j,\, z_j\}$ such that $L(x_j, y_1) = L(v_1, \alpha_j)$. It is easily seen from

Claim 7.2 that $L(x_j, y_1) = -L(x_1, y_j)$ is orthogonal to $L(v_1, x_j)$ and $L(v_1, v_j)$. Hence we have

$$L(x_j, y_1) = \varepsilon_j L(v_1, z_j), \quad \varepsilon_j = \pm 1, \quad j \geq 2. \tag{7.5}$$

Similar arguments show that

$$L(y_j, z_1) = \epsilon_j L(x_j, v_1), \quad L(z_j, x_1) = \zeta_j L(y_j, v_1), \tag{7.6}$$

where $\epsilon_j = \pm 1$, $\zeta_j = \pm 1$, and $j \geq 2$.

On the other hand, from (7.5), (7.6), and making use of (2.10) and (7.4), we have the following results:

$$\begin{aligned} \epsilon_j \tau &= h(L(y_j, z_1), L(x_j, v_1)) = -h(L(y_1, z_j), L(x_j, v_1)) \\ &= h(L(y_1, x_j), L(z_j, v_1)) = \varepsilon_j h(L(v_1, z_j), L(z_j, v_1)) = \varepsilon_j \tau, \end{aligned}$$

$$\begin{aligned} \zeta_j \tau &= h(L(z_j, x_1), L(y_j, v_1)) = h(L(z_1, x_j), L(y_1, v_j)) \\ &= -h(L(z_1, v_j), L(x_j, y_1)) = -\varepsilon_j h(L(z_1, v_j), L(z_j, v_1)) = \varepsilon_j \tau. \end{aligned}$$

It follows that

$$\zeta_j = \epsilon_j = \varepsilon_j = \pm 1. \tag{7.7}$$

Now, for any $j \neq \ell$ and $j, \ell \geq 2$, we first use $L(x_j, y_1) = \varepsilon_j L(v_1, z_j)$ and Lemma 4.4 to see that

$$\begin{aligned} \varepsilon_\ell \tau L(x_j, x_\ell) &= K(L(x_j, y_1), \varepsilon_\ell L(y_1, x_\ell)) = K(L(x_j, y_1), L(v_1, z_\ell)) \\ &= K(\varepsilon_j L(v_1, z_j), L(v_1, z_\ell)) = \varepsilon_j \tau L(z_j, z_\ell). \end{aligned}$$

It follows that

$$L(x_j, x_\ell) = \varepsilon_j \varepsilon_\ell L(z_j, z_\ell). \tag{7.8}$$

Similarly, we use $L(y_j, z_1) = \varepsilon_j L(x_j, v_1)$, $L(z_j, x_1) = \varepsilon_j L(y_j, v_1)$ and Lemma 4.4 to get

$$\begin{aligned} \varepsilon_\ell \tau L(y_j, y_\ell) &= K(L(y_j, z_1), \varepsilon_\ell L(y_\ell, z_1)) = K(L(y_j, z_1), L(v_1, x_\ell)) \\ &= K(\varepsilon_j L(v_1, x_j), L(v_1, x_\ell)) = \varepsilon_j \tau L(x_j, x_\ell), \end{aligned}$$

$$\begin{aligned} \varepsilon_\ell \tau L(z_j, z_\ell) &= K(L(z_j, x_1), \varepsilon_\ell L(z_\ell, x_1)) = K(L(z_j, x_1), L(v_1, y_\ell)) \\ &= K(\varepsilon_j L(v_1, y_j), L(v_1, y_\ell)) = \varepsilon_j \tau L(y_j, y_\ell), \end{aligned}$$

and so we obtain

$$L(x_j, x_\ell) = \varepsilon_j \varepsilon_\ell L(y_j, y_\ell), \tag{7.9}$$

$$L(z_j, z_\ell) = \varepsilon_j \varepsilon_\ell L(y_j, y_\ell). \tag{7.10}$$

From (7.8), (7.9), and (7.10) we have proved that $\varepsilon_2 = \cdots = \varepsilon_{k_0}$. Therefore, if necessary by changing the sign of z_1 and hence the sign of all other z's, we may assume $\varepsilon_2 = 1$, and thus we have completed the proof of Claim 7.3.

Claim 7.4. $L(x_j, x_\ell) = L(y_j, y_\ell) = L(z_j, z_\ell) = L(v_j, v_\ell)$, $2 \leq j, \ell \leq k_0$.

In fact, for $j=\ell$, the fact that $x_j, y_j, z_j \in V_{v_j}(0)$ implies that $L(v_j, x_j) = L(v_j, y_j) = L(v_j, z_j) = 0$. It now follows that $L(x_j, x_j) = L(y_j, y_j) = L(z_j, z_j) = L(v_j, v_j)$.

Next, for $\dim \mathcal{D}_2 \geq 12$, we fix $j, \ell \geq 2$ such that $j \neq \ell$. Then from the proof of Claim 7.3 and (7.8)–(7.10), we have

$$L(x_j, x_\ell) = L(y_j, y_\ell) = L(z_j, z_\ell). \tag{7.11}$$

By Lemma 4.2, there exists a unique unit vector $\alpha \in V_{v_\ell}(0) = \operatorname{span}\{x_\ell, y_\ell, z_\ell\}$ such that $L(v_j, x_\ell) = L(x_j, \alpha)$. It can easily be seen from (2.10) and (7.11) that $L(v_j, x_\ell)$ is orthogonal to both $L(x_j, y_\ell)$ and $L(x_j, z_\ell)$. Hence considering that they have the same length, we may assume

$$L(v_j, x_\ell) = a_{j\ell} L(x_j, v_\ell), \quad a_{j\ell} = \pm 1. \tag{7.12}$$

From (7.12) and Lemma 4.4 we have the calculation

$$\begin{aligned}\tau L(x_1, x_\ell) &= K(L(v_j, x_\ell), L(v_j, x_1)) = a_{j\ell} K(L(x_j, v_\ell), L(v_j, x_1)) \\ &= a_{j\ell} K(L(x_j, v_\ell), -L(v_1, x_j)) = -\tau L(v_1, v_\ell) a_{j\ell},\end{aligned}$$

which together with Claim 7.2 shows that it must be the case $a_{j\ell} = -1$. This, together with very similar arguments, then shows that

$$\begin{aligned}&L(v_j, x_\ell) = -L(x_j, v_\ell), \quad L(v_j, y_\ell) = -L(y_j, v_\ell), \\ &L(v_j, z_\ell) = -L(z_j, v_\ell).\end{aligned} \tag{7.13}$$

Applying the conclusion $L(v_j, x_\ell) = -L(x_j, v_\ell)$ and (2.10), we have

$$\begin{aligned}h(L(x_j, x_\ell), L(v_j, v_\ell)) &= -h(L(x_j, v_\ell), L(v_j, x_\ell)) \\ &= h(L(x_j, v_\ell), L(x_j, v_\ell)) = \tau.\end{aligned}$$

Note also that

$$h(L(x_j, x_\ell), L(x_j, x_\ell)) = h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau,$$

so that by the Cauchy-Schwarz inequality we get $L(x_j, x_\ell) = L(v_j, v_\ell)$, which together with (7.11) gives Claim 7.4.

Claim 7.5. *With respect to the above chosen $\{x_j, y_j, z_j\}_{1 \leq j \leq k_0}$, we have*

$$\begin{aligned}&L(x_j, y_\ell) = L(z_j, v_\ell), \ L(y_j, z_\ell) = L(x_j, v_\ell), \\ &L(z_j, x_\ell) = L(y_j, v_\ell), \qquad 1 \leq j, \ \ell \leq k_0.\end{aligned}$$

In fact, according to Claims 7.2, 7.3 and (7.2), it is sufficient to consider the cases that $2 \leq j < \ell \leq k_0$.

By using Claims 7.2, 7.3 and Lemma 4.4, we have the following computations:

$$\begin{aligned}\tau L(x_j, y_\ell) &= K(L(y_1, x_j), L(y_1, y_\ell)) = K(L(v_1, z_j), L(y_1, y_\ell)) \\ &= K(L(v_1, z_j), L(v_1, v_\ell)) = \tau L(z_j, v_\ell), \\ \tau L(y_j, z_\ell) &= K(L(z_1, y_j), L(z_1, z_\ell)) = K(L(v_1, x_j), L(z_1, z_\ell)) \\ &= K(L(v_1, x_j), L(v_1, v_\ell)) = \tau L(x_j, v_\ell),\end{aligned}$$

$$\begin{aligned}\tau L(z_j, x_\ell) &= K(L(x_1, z_j), L(x_1, x_\ell)) = K(L(v_1, y_j), L(x_1, x_\ell)) \\ &= K(L(v_1, y_j), L(v_1, v_\ell)) = \tau L(y_j, v_\ell).\end{aligned}$$

From these results, Claim 7.5 immediately follows.

Combining the above Claims, we complete the proof of Lemma 7.1. q.e.d.

Remark 7.1. Having fixed the orthonormal basis of $\mathcal{D}_2$ satisfying (7.1), we can now define three almost complex structures $J_1, J_2, J_3 : \mathcal{D}_2 \to \mathcal{D}_2$ such that for all $1 \le j \le k_0$,

$$\begin{aligned}&J_1 v_j = x_j, \ J_1 x_j = -v_j; \quad J_2 v_j = y_j, \\ &J_2 y_j = -v_j; \quad J_3 v_j = z_j, \ J_3 z_j = -v_j,\end{aligned} \tag{7.14}$$

and furthermore J_1, J_2, and J_3 satisfy

$$J_1 \circ J_1 = J_2 \circ J_2 = J_3 \circ J_3 = -id, \quad J_1 J_2 = -J_2 J_1 = J_3. \tag{7.15}$$

Then we define a quaternionic structure $\{J_1, J_2, J_3\}$ on $\mathcal{D}_2$. It is important to remark that (7.1) is equivalent to the following relations:

$$L(Ju, v) = -L(u, Jv), \quad L(Ju, Jv) = L(u, v) \tag{7.16}$$

for all $J = J_1, J_2, J_3$ and $u, v \in \mathcal{D}_2$.

Proof of Theorem 7.1. Let $r - 1 = 4k_0$. If $k_0 = 1$, then similar reasoning as in the proof of Theorem 6.1 shows that M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form. Hence in the remaining part of the proof we assume that $k_0 \ge 2$ and let

$$\{v_1, x_1, y_1, z_1; \ldots; v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$$

be the orthonormal basis of $\mathcal{D}_2$ as constructed in Lemma 7.1. Applying Lemma 3.4 and the fact that for $j \neq \ell$, $v_j, x_j, y_j, z_j \in V_{v_\ell}(\tau) = V_{x_\ell}(\tau) = V_{y_\ell}(\tau) = V_{z_\ell}(\tau)$, we easily show that

$$\begin{aligned}&h(L(v_j, x_\ell), L(v_j, x_\ell)) = h(L(v_j, y_\ell), L(v_j, y_\ell)) \\ &= h(L(v_j, z_\ell), L(v_j, z_\ell)) = h(L(v_j, v_\ell), L(v_j, v_\ell)) = \tau, \quad j \neq \ell,\end{aligned} \tag{7.17}$$

$$\begin{aligned}&h(L(x_j, v_{\ell_1}), L(x_j, v_{\ell_2})) = h(L(v_j, x_{\ell_1}), L(v_j, x_{\ell_2})) \\ &= h(L(y_j, v_{\ell_1}), L(y_j, v_{\ell_2})) = h(L(v_j, y_{\ell_1}), L(v_j, y_{\ell_2})) \\ &= h(L(z_j, v_{\ell_1}), L(z_j, v_{\ell_2})) = h(L(v_j, z_{\ell_1}), L(v_j, z_{\ell_2})) \\ &= h(L(v_j, v_{\ell_1}), L(v_j, v_{\ell_2})) = 0, \quad j, \ell_1, \ell_2 \text{ distinct},\end{aligned} \tag{7.18}$$

$$\begin{aligned}&h(L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})) = h(L(v_{j_1}, x_{j_2}), L(v_{j_3}, x_{j_4})) \\ &= h(L(v_{j_1}, y_{j_2}), L(v_{j_3}, y_{j_4})) = h(L(v_{j_1}, z_{j_2}), L(v_{j_3}, z_{j_4})) \\ &= 0, \qquad j_1, j_2, j_3, j_4 \text{ distinct},\end{aligned} \tag{7.19}$$

$$\begin{aligned}&h(L(v_j, v_\ell), L(v_{j_1}, x_{\ell_1})) = h(L(v_j, v_\ell), L(v_{j_1}, y_{\ell_1})) \\ &= h(L(v_j, v_\ell), L(v_{j_1}, z_{\ell_1})) = 0, \quad j \neq \ell \text{ and } j_1 \neq \ell_1.\end{aligned} \tag{7.20}$$

For $\{L(v_j, v_j) = L(x_j, x_j) = L(y_j, y_j) = L(z_j, z_j)\}_{1\le j\le k_0}$, we note that

$$h(L(v_j, v_j), L(v_j, v_j)) = \tfrac{n+1}{4(n-r)}\lambda_1^2, \quad 1 \le j \le k_0, \tag{7.21}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_\ell, v_\ell)) &= \tfrac{n+1}{4(n-r)}\lambda_1^2 - 2\tau \\ &= -\tfrac{(n+1)(r+1)}{8(n-r)^2}\lambda_1^2, \quad 1 \le j \ne \ell \le k_0, \end{aligned} \tag{7.22}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_j, v_\ell)) &= h(L(v_j, v_j), L(v_j, u_\ell)) \\ &= 0, \quad 1 \le j \ne \ell \le k_0, \end{aligned} \tag{7.23}$$

$$\begin{aligned} h(L(v_j, v_j), L(v_{\ell_1}, v_{\ell_2})) = h(L(v_j, v_j), L(v_{\ell_1}, u_{\ell_2})) = 0, \\ 1 \le j, \ell_1, \ell_2 \text{ distinct and } \le k_0. \end{aligned} \tag{7.24}$$

Similar to the previous section, we deduce that

$$\{L_j := L(v_1, v_1) + \cdots + L(v_j, v_j) - jL(v_{j+1}, v_{j+1})\}_{1\le j\le k_0-1}$$

are $k_0 - 1 = \frac{1}{4}(r-5)$ mutually orthogonal vectors which are orthogonal to all of the vectors

$$L(v_j, v_\ell), \; L(v_j, x_\ell), L(v_j, y_\ell), \; L(v_j, z_\ell), \quad j \ne \ell.$$

Also, we have $h(L_j, L_j) = 2j(j+1)\tau \ne 0$. Hence, the following vectors

$$\begin{cases} w_{j\ell} = \frac{1}{\sqrt{\tau}} L(v_j, v_\ell), \quad w'_{j\ell} = \frac{1}{\sqrt{\tau}} L(v_j, x_\ell), \\ w''_{j\ell} = \frac{1}{\sqrt{\tau}} L(v_j, y_\ell), \quad w'''_{j\ell} = \frac{1}{\sqrt{\tau}} L(v_j, z_\ell), \quad 1 \le j < \ell \le k_0; \\ w_j = \frac{1}{\sqrt{2j(j+1)\tau}} L_j, \quad 1 \le j \le k_0 - 1 \end{cases} \tag{7.25}$$

consist of $2k_0(k_0-1) + k_0 - 1 = \frac{1}{8}(r+1)(r-5)$ orthonormal vectors in $\mathrm{Im}\,(L) \subset \mathcal{D}_3$.

Finally, from Lemma 7.1, (7.21), and (7.22) it is easily known that the vector

$$\mathrm{Tr}\, L = 4\big(L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})\big)$$

is orthogonal to the above $(r+1)(r-5)/8$ vectors and it satisfies

$$h(\mathrm{Tr}\, L, \mathrm{Tr}\, L) = \tfrac{(n+1)(r-1)}{8(n-r)^2}\lambda_1^2\big[8n - (r-1)(r+5)\big]. \tag{7.26}$$

The above results imply that

$$n = 1 + \dim \mathcal{D}_2 + \dim \mathcal{D}_3 \ge 1 + (r-1) + \tfrac{1}{8}(r+1)(r-5) = \tfrac{1}{8}(r-1)(r+5).$$

Moreover, from (7.26) we see that $\mathrm{Tr}\, L = 0$ if and only if it holds $n = \frac{1}{8}(r-1)(r+5)$.

Now, as in the previous sections, we separate the discussion into the following three cases: **(i)** $n = \frac{1}{8}(r-1)(r+5) + 1$; **(ii)** $n > \frac{1}{8}(r-1)(r+5) + 1$; **(iii)** $n = \frac{1}{8}(r-1)(r+5)$.

For **Case (i)** and **Case (ii)**, we define a unit vector

$$t=\frac{4(n-r)}{\sqrt{2(n+1)(r-1)[8n-(r-1)(r+5)]}\,\lambda_1}\operatorname{Tr}L.$$

In **Case (i)**, the previous results show that

$$\big\{t;\ w_{j\ell},\ w'_{j\ell},\ w''_{j\ell},\ w'''_{j\ell}\,|_{1\le j<\ell\le(r-1)/4};\ w_j\,|_{1\le j\le(r-5)/4}\big\}$$

is an orthonormal basis of $\mathrm{Im}\,(L)=\mathcal{D}_3$. By direct calculations with application of Lemma 4.4, (7.1) and (7.17)–(7.24), we easily verify the following

Lemma 7.2. *Under the above notations, we have*

$$\begin{cases} K(t,e_1)=-\frac{r+1}{2(n-r)}\lambda_1 t,\\ K(t,t)=-\frac{r+1}{2(n-r)}\lambda_1 e_1+\frac{4n-(r-1)(r+3)}{n-r}\sqrt{\frac{2(n+1)}{(r-1)[8n-(r-1)(r+5)]}}\,\lambda_1 t,\\ K(t,u)=\frac{1}{4(n-r)}\sqrt{\frac{2(n+1)[8n-(r-1)(r+5)]}{r-1}}\,\lambda_1 u,\\ K(t,w)=\frac{1}{2(n-r)}\sqrt{\frac{2(n+1)[8n-(r-1)(r+5)]}{r-1}}\,\lambda_1 w,\end{cases}\tag{7.27}$$

where

$$\begin{cases} u=v_j,x_j,y_j,z_j,\quad 1\le j\le\frac14(r-1);\\ w=w_m,w_{j\ell},w'_{j\ell},w''_{j\ell},w'''_{j\ell},\\ \qquad 1\le m\le\frac14(r-5),\ 1\le j<\ell\le\frac14(r-1).\end{cases}\tag{7.28}$$

Put $T=\alpha e_1+\beta t,\ T^*=-\beta e_1+\alpha t$, where

$$\alpha=\sqrt{\frac{8n-(r-1)(r+5)}{8n+(r-1)(2n-r-3)}},\quad \beta=\sqrt{\frac{2(n+1)(r-1)}{8n+(r-1)(2n-r-3)}}.\tag{7.29}$$

Then

$$\big\{T,T^*;\ v_j,x_j,y_j,z_j\,|_{1\le j\le(r-1)/4};\ w_m\,|_{1\le m\le(r-5)/4}\big\}\cup$$
$$\cup\,\big\{w_{k\ell},w'_{k\ell},w''_{k\ell},w'''_{k\ell}\,|_{1\le k<\ell\le(r-1)/4}\big\}$$

forms an orthonormal basis of $T_{x_0}M$. Moreover, by Lemma 7.2 we easily obtain the following

Lemma 7.3. *Under the above notations, it holds*

$$K(T,T)=\nu_1 T,\qquad K(T,u)=\nu_2 u,\tag{7.30}$$

where ν_1 and ν_2 are defined by

$$\begin{cases}\nu_1=\frac{(n-r)[8n-(r-1)(r+5)]-(n+1)(r^2-1)}{(n-r)[8n-(r-1)(r+5)]}\sqrt{\frac{8n-(r-1)(r+5)}{8n+(r-1)(2n-r-3)}}\,\lambda_1,\\ \nu_2=\frac{2n-r+1}{2(n-r)}\sqrt{\frac{8n-(r-1)(r+5)}{8n+(r-1)(2n-r-3)}}\,\lambda_1,\end{cases}\tag{7.31}$$

and $u=T^*,v_j,x_j,y_j,z_j,w_m,w_{k\ell},w'_{k\ell},w''_{k\ell},w'''_{k\ell}$ with indices satisfying

$$1\le m\le\tfrac14(r-5);\ 1\le j\le\tfrac14(r-1);\ 1\le k<\ell\le\tfrac14(r-1).$$

From (7.31), we can verify the relation

$$\nu_1\nu_2 - \nu_2^2 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{7.32}$$

Then, based on the conclusions of Lemma 7.3, we can apply theorem 4 of [**HLV**] to conclude that in Case (i), M^n is decomposed as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point.

For **Case (ii)**, we see that

$$\big\{t;\ w_{j\ell},\ w'_{j\ell},\ w''_{j\ell},\ w'''_{j\ell}\,|_{1\le j<\ell\le (r-1)/4};\ w_j\,|_{1\le j\le (r-5)/4}\big\}$$

is still an orthonormal basis of $\operatorname{Im}(L)$. But now $\operatorname{Im}(L)$ does not coincide with $\mathcal{D}_3$. Denote $\tilde n = n - \frac{1}{8}(r-1)(r+5) - 1 \ge 1$ and choose $w'_1, \dots, w'_{\tilde n}$ in the orthogonal complement of $\operatorname{Im}(L)$ in $\mathcal{D}_3$ such that

$$\big\{t;\ w_{j\ell},\ w'_{j\ell},\ w''_{j\ell},\ w'''_{j\ell}\,|_{1\le j<\ell\le (r-1)/4};\ w_j\,|_{1\le j\le (r-5)/4};\ w'_j\,|_{1\le j\le \tilde n}\big\}$$

is an orthonormal basis of $\mathcal{D}_3$. Then, besides (7.27), we further use (4) of Lemma 2.5 to get

$$K(t, w'_j) = -\tfrac{r+1}{2(n-r)}\sqrt{\tfrac{2(n+1)(r-1)}{8n-(r-1)(r+5)}}\lambda_1 w'_j, \quad 1 \le j \le \tilde n. \tag{7.33}$$

Now we define T and T^* the same as in Case (i). Similar to Lemma 7.3, we have the following

Lemma 7.4. *For Case (ii), we have that*

$$K(T,T) = \nu_1 T,\ \ K(T,u) = \nu_2 u,\ \ K(T, w'_j) = \nu_3 w'_j,\ \ 1 \le j \le \tilde n, \tag{7.34}$$

where $\nu_1,\ \nu_2$*, and* ν_3 *are defined by* (7.31) *and*

$$\nu_3 = -\tfrac{r+1}{2(n-r)}\sqrt{\tfrac{8n+(r-1)(2n-r-3)}{8n-(r-1)(r+5)}}\,\lambda_1, \tag{7.35}$$

and $u = T^*, v_j, x_j, y_j, z_j, w_m, w_{k\ell}, w'_{k\ell}, w''_{k\ell}, w'''_{k\ell}$ *with indices varying as follows:*

$$1 \le m \le \tfrac{1}{4}(r-5);\ \ 1 \le j \le \tfrac{1}{4}(r-1);\ \ 1 \le k < \ell \le \tfrac{1}{4}(r-1).$$

It can be easily seen that $\nu_2 \ne \nu_3,\ 2\nu_2 \ne \nu_1 \ne 2\nu_3$, and

$$\nu_1 = \nu_2 + \nu_3, \quad \nu_2\nu_3 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{7.36}$$

Thus, based on the conclusions of Lemma 7.4, we can apply theorem 3 of [**HLV**] to conclude that in Case (ii), M^n is decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form.

Finally, for **Case (iii)**, we have $\operatorname{Tr} L = 0$ and thus

$$\mathcal{D}_3 = \{w_{j\ell},\ w'_{j\ell}, w''_{k\ell}, w'''_{k\ell}\}_{1\le j<\ell\le (r-1)/4} \oplus \{w_m\}_{1\le m\le (r-5)/4}.$$

It follows that

$$\begin{aligned}(7.37)\qquad &\{e_1;\, v_j, x_j, y_j, z_j\,|_{1\le j\le (r-1)/4}\}\cup\\ &\cup\{w_{j\ell}, w'_{j\ell}, w''_{k\ell}, w'''_{k\ell}\,|_{1\le j<\ell\le (r-1)/4};\ w_m\,|_{1\le m\le (r-5)/4}\}\end{aligned}$$

is an orthonormal basis of $T_{x_0}M$. Now, applying Lemma 2.4, Lemma 4.4, (7.1), and the previous formulas from (7.17) up to (7.24), we can calculate all the components of the difference tensor with respect to the basis (7.37).

Now, for $m \geq 3$, we look at the homogeneous space $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$ and recall its standard embedding into $\mathbb{R}^{2m^2-m}$.

Let $\mathbb{H}$ be the quaternion field over $\mathbb{R}$. Then the quaternionic general linear group $\mathbf{GL}(m, \mathbb{H})$ has a well-known complex representation

$$\mathbf{U}^*(2m) = \{A \in \mathbf{GL}(2m, \mathbb{C}) \mid AJ = J\bar{A}\},\ \text{where } J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

and I_m is the $m \times m$ identity matrix. The Lie algebra $\mathfrak{u}^*(2m)$ of $\mathbf{U}^*(2m)$ is given by

$$\mathfrak{u}^*(2m) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathfrak{gl}(m, \mathbb{C}) \right\}.$$

The quaternionic analogue of the set of Hermitian (m, m)-matrices and that of the complex special linear group $\mathbf{SL}(m, \mathbb{C})$ are given by

$$\mathbf{S}^*(m) = \{A \in \mathbf{U}^*(2m) \mid \bar{A} = {}^t\!A\}$$

and

$$\mathbf{SU}^*(2m) = \{A \in \mathbf{SL}(2m, \mathbb{C}) \mid AJ = J\bar{A}\} = \mathbf{SL}(2m, \mathbb{C}) \cap \mathbf{U}^*(2m).$$

The compact Lie subgroup $\mathbf{Sp}(m)$, which is also called the quaternion unitary group, is defined by

$$\mathbf{Sp}(m) = \{A \in \mathbf{SU}(2m) \mid AJ = J\bar{A}\} = \mathbf{SU}^*(2m) \cap \mathbf{SU}(2m).$$

The Lie algebra $\mathfrak{u}^*(2m)$ has a direct sum decomposition

$$\mathfrak{u}^*(2m) = \mathfrak{sp}(m) \oplus \mathfrak{s}^*(m),$$

where

$$\begin{aligned}\mathfrak{sp}(m) &= \{Y \in \mathfrak{u}^*(2m) \mid Y^* + Y = 0\}\\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha^* + \alpha = 0,\ {}^t\beta - \beta = 0 \right\},\\ \mathfrak{s}^*(m) &= \{X \in \mathfrak{u}^*(2m) \mid X^* - X = 0\}\\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha^* - \alpha = 0,\ {}^t\beta + \beta = 0 \right\}\end{aligned}$$

are the Lie algebras of $\mathbf{Sp}(m)$ and $\mathbf{S}^*(m)$, respectively.

Let ψ be the action of $\mathbf{SU}^*(2m)$ on $\mathbf{S}^*(m)$ as follows:

$$\psi:\ \mathbf{SU}^*(2m) \times \mathbf{S}^*(m) \to \mathbf{S}^*(m)\quad \text{s.t.}\quad (A, X) \mapsto \psi_A(X) = AX\,{}^t\bar{A}.$$

Let $F : \mathbf{S}^*(m) \to \mathbb{C}$ be given by $F(X) := \det(X)$.

Consider the hypersurface of $\mathbf{S}^*(m)$ satisfying the equation $\det(X) = 1$; we take the connected component M that lies in the open set of $\mathbf{S}^*(m)$ consisting of all Hermitian positive definite matrices. Then the mapping $f : \mathbf{SU}^*(2m) \to \mathbf{S}^*(m)$, defined by $f(A) := A\,{}^t\bar{A}$, is a submersion onto M, and it satisfies $f(AB) = \psi_A(f(B))$; hence f is equivariant. M is the orbit of I under the action ψ. The isotropy group is $\mathbf{Sp}(m)$. Hence M is diffeomorphic to $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$. This is an irreducible, homogeneous, symmetric space of non-compact type, and the involution at the identity matrix I is given by $A \mapsto {}^t\bar{A}^{-1}$. We denote this symmetric space by M'.

Clearly $f(A) = f(B)$ if and only if $B^{-1}A \in \mathbf{Sp}(m)$; then the map $f : \mathbf{SU}^*(2m) \to \mathbf{S}^*(m)$ induces an embedding $f : \mathbf{SU}^*(2m)/\mathbf{Sp}(m) \to \mathbf{S}^*(m)$. Let $\pi : \mathbf{SU}^*(2m) \to M'$ be the natural projection; then there is an immersion $f' : M' \to \mathbf{S}^*(m)$ such that $f = f' \circ \pi$. Now we consider

$$(7.38) \quad f : \ \mathbf{SU}^*(2m)/\mathbf{Sp}(m) \to \mathbb{R}^{n+1} = \mathbf{S}^*(m), \quad n+1 = 2m^2 - m$$

with a transversal vector field $\xi_A = f(A)$ for any $A \in \mathbf{SU}^*(2m)/\mathbf{Sp}(m)$. Then ξ is equiaffine and equivariant.

Consider the Cartan decomposition of the Lie algebra $\mathfrak{su}^*(2m) = \mathfrak{s}_0^*(m) \oplus \mathfrak{sp}_0(m)$, where $\mathfrak{s}_0^*(m) = \{X \in \mathfrak{s}^*(m) \,|\, \mathrm{tr}\,(X) = 0\}$ and $\mathfrak{sp}_0^*(m) = \{X \in \mathfrak{sp}(m) \,|\, \mathrm{tr}\,(X) = 0\}$. If $X \in \mathfrak{s}_0^*(m)$, then $f_*(X) = X$. Now $\mathfrak{s}_0^*(m)$ can be considered as the tangent space of M' at $\pi(I)$.

Since f is equivariant, it is sufficient to compute the invariant objects of the immersed hypersurface M' in terms of $\mathfrak{s}_0^*(m)$.

The embedding $f : \ \mathbf{SU}^*(2m)/\mathbf{Sp}(m) \to \mathbb{R}^{n+1} = \mathbf{S}^*(m)$ with $\xi = f$ has a Blaschke structure that can be expressed algebraically in terms of the Lie algebra as follows (cf. [**BD**] for the case $m = 3$):

$$(7.39) \qquad \begin{cases} K(X,Y) = XY + YX - \frac{1}{m}\,\mathrm{tr}\,(XY) I_{2m}, \\ h(X,Y) = \frac{2}{m}\,\mathrm{tr}\,(XY), \qquad S = -I_{2m}. \end{cases}$$

Here h is the natural Riemannian metric on the symmetric space M'; this implies that the Levi-Civita connection of h is given by $\hat{\nabla}_X Y = \frac{1}{2}[X,Y]$. From this it follows easily that the difference tensor K satisfies $\big(\hat{\nabla}_X K\big)(X,X) = 0$. As $M = f'(M')$ is an affine hypersphere, we get that $\hat{\nabla} K$ is totally symmetric [**BNS**]; then from $\big(\hat{\nabla}_X K\big)(X,X) = 0$ and polarization of the multilinear symmetric expression over $T_p(M)$ at $p \in M$, we obtain $\hat{\nabla} K = 0$.

Now we choose $m = \frac{1}{4}(r+3)$. Denote E_{jk} (resp. E'_{jk}) the $2m \times 2m$ matrix which has (j,k) entry 1 (resp. $\sqrt{-1}$) and all other entries 0. Then with respect to the metric $h(X,Y) = \frac{2}{m}\,\mathrm{tr}\,(XY)$ of $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$ at

I_{2m}, we can choose an orthonormal basis as follows:

(7.40)

$$\begin{cases} \tilde{e}_1 = \frac{1}{\sqrt{4(m-1)}}\Big[m(E_{2m,2m}+E_{mm}) - \sum_{j=1}^{2m} E_{jj}\Big]; \\ \tilde{v}_j = \sqrt{\frac{m}{8}}\big(E_{mj}+E_{jm}+E_{2m,m+j}+E_{m+j,2m}\big), \quad 1 \le j \le m-1, \\ \tilde{x}_j = \sqrt{\frac{m}{8}}\big(E'_{mj}-E'_{jm}-E'_{2m,m+j}+E'_{m+j,2m}\big), \quad 1 \le j \le m-1, \\ \tilde{y}_j = \sqrt{\frac{m}{8}}\big(E_{m,m+j}-E_{j,2m}+E_{m+j,m}-E_{2m,j}\big), \quad 1 \le j \le m-1, \\ \tilde{z}_j = \sqrt{\frac{m}{8}}\big(E'_{2m,j}-E'_{j,2m}-E'_{m+j,m}+E'_{m,m+j}\big), \quad 1 \le j \le m-1; \\ \tilde{w}_{j\ell} = \sqrt{\frac{m}{8}}\big(E_{j\ell}+E_{\ell j}+E_{m+\ell,m+j}+E_{m+j,m+\ell}\big), \quad 1 \le j < \ell \le m-1, \\ \tilde{w}'_{j\ell} = \sqrt{\frac{m}{8}}\big(E'_{j\ell}-E'_{\ell j}+E'_{m+\ell,m+j}-E'_{m+j,m+\ell}\big), \quad 1 \le j < \ell \le m-1, \\ \tilde{w}''_{j\ell} = \sqrt{\frac{m}{8}}\big(E_{j,m+\ell}-E_{\ell,m+j}-E_{m+j,\ell}+E_{m+\ell,j}\big), \quad 1 \le j < \ell \le m-1, \\ \tilde{w}'''_{j\ell} = \sqrt{\frac{m}{8}}\big(E'_{j,m+\ell}-E'_{\ell,m+j}+E'_{m+j,\ell}-E'_{m+\ell,j}\big), \quad 1 \le j < \ell \le m-1; \\ \tilde{w}_j = \sqrt{\frac{m}{4j(j+1)}}\,\Big[\sum_{\ell=1}^{j} E_{\ell\ell} + \sum_{\ell=m+1}^{m+j} E_{\ell\ell} \\ \qquad\qquad - j(E_{j+1,j+1}+E_{m+j+1.m+j+1})\Big], \quad 1 \le j \le m-2. \end{cases}$$

By using the formula $K(X,Y) = XY + YX - \frac{1}{m}\,\mathrm{tr}\,(XY)I_{2m}$ and

$$E_{jk}E_{pq} = E_{jq}\delta_{kp}, \quad E'_{jk}E_{pq} = E_{jk}E'_{pq} = E'_{jq}\delta_{kp}, \quad E'_{jk}E'_{pq} = -E_{jq}\delta_{kp},$$

we can show that, for $\lambda_1 = \frac{m-2}{\sqrt{m-1}}$

$$L(X,Y) = K(X,Y) - \tfrac{1}{2}\lambda_1 h(X,Y)\tilde{e}_1$$

satisfies

$$\begin{cases} L(\tilde{x}_j,\tilde{x}_\ell) = L(\tilde{y}_j,\tilde{y}_\ell) = L(\tilde{z}_j,\tilde{z}_\ell) = L(\tilde{v}_j,\tilde{v}_\ell), \\ L(\tilde{v}_j,\tilde{x}_\ell) = -L(\tilde{x}_j,\tilde{v}_\ell) = -L(\tilde{y}_j,\tilde{z}_\ell) = L(\tilde{y}_\ell,\tilde{z}_j), \\ L(\tilde{v}_j,\tilde{y}_\ell) = -L(\tilde{y}_j,\tilde{v}_\ell) = -L(\tilde{z}_j,\tilde{x}_\ell) = L(\tilde{x}_j,\tilde{z}_\ell), \\ L(\tilde{v}_j,\tilde{z}_\ell) = -L(\tilde{z}_j,\tilde{v}_\ell) = -L(\tilde{x}_j,\tilde{y}_\ell) = L(\tilde{x}_\ell,\tilde{y}_j) \end{cases} \tag{7.41}$$

for all $1 \le j, \ell \le m-1$. If we define

$$\begin{cases} \tilde{L}_1 = L(\tilde{v}_1,\tilde{v}_1) - L(\tilde{v}_2,\tilde{v}_2) = K(\tilde{v}_1,\tilde{v}_1) - K(\tilde{v}_2,\tilde{v}_2), \\ \tilde{L}_2 = L(\tilde{v}_1,\tilde{v}_1) + L(\tilde{v}_2,\tilde{v}_2) - 2L(\tilde{v}_3,\tilde{v}_3) \\ \quad = K(\tilde{v}_1,\tilde{v}_1) + K(\tilde{v}_2,\tilde{v}_2) - 2K(\tilde{v}_3,\tilde{v}_3), \qquad \dots \\ \tilde{L}_{m-2} = K(\tilde{v}_1,\tilde{v}_1) + \cdots + K(\tilde{v}_{m-2},\tilde{v}_{m-2}) - (m-2)K(\tilde{v}_{m-1},\tilde{v}_{m-1}), \end{cases}$$

then by the definition in (7.40) we have

$$\tilde{w}_j = \tilde{L}_j/\|\tilde{L}_j\|, \quad 1 \le j \le m-2. \tag{7.42}$$

Further direct calculations also show that

$$
(7.43)\qquad \begin{cases} \tilde w_{j\ell}=\frac{K(\tilde v_j,\tilde v_\ell)}{\|K(\tilde v_j,\tilde v_\ell)\|}=\frac{L(\tilde v_j,\tilde v_\ell)}{\|L(\tilde v_j,\tilde v_\ell)\|},\\ \tilde w'_{j\ell}=\frac{K(\tilde v_j,\tilde x_\ell)}{\|K(\tilde v_j,\tilde x_\ell)\|}=\frac{L(\tilde v_j,\tilde x_\ell)}{\|L(\tilde v_j,\tilde x_\ell)\|},\\ \tilde w''_{j\ell}=\frac{K(\tilde v_j,\tilde y_\ell)}{\|K(\tilde v_j,\tilde y_\ell)\|}=\frac{L(\tilde v_j,\tilde y_\ell)}{\|L(\tilde v_j,\tilde y_\ell)\|},\\ \tilde w'''_{j\ell}=\frac{K(\tilde v_j,\tilde z_\ell)}{\|K(\tilde v_j,\tilde z_\ell)\|}=\frac{L(\tilde v_j,\tilde z_\ell)}{\|L(\tilde v_j,\tilde z_\ell)\|}.\end{cases}, \quad 1\le j<\ell\le m-1,
$$

Moreover, we have the following calculation for the difference tensor at I_{2m}:

(7.44)

$$
\begin{cases} K_{\tilde e_1}\tilde e_1=\frac{m-2}{\sqrt{m-1}}\tilde e_1;\ K_{\tilde e_1}\tilde w_j=-\frac{1}{\sqrt{m-1}}\tilde w_j,\ 1\le j\le m-2;\\ K_{\tilde e_1}u=\frac{m-2}{2\sqrt{m-1}}u,\quad \text{for } u=\tilde v_j,\ \tilde x_j,\ \tilde y_j,\ \tilde z_j,\ 1\le j\le m-1;\\ K_{\tilde e_1}w=-\frac{1}{\sqrt{m-1}}w,\quad \text{for } w=\tilde w_{j\ell},\tilde w'_{j\ell},\tilde w''_{j\ell},\tilde w'''_{j\ell},\ 1\le j<\ell\le m-1.\end{cases}
$$

Since $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)\to\mathbf{S}^*(m)=\mathbb{R}^{2m^2-m}$ is of parallel cubic form, if we identify

$$
\big\{\tilde e_1;\tilde v_j,\tilde x_j,\tilde y_j,\tilde z_j\,|_{1\le j\le m-1};\tilde w_{j\ell},\tilde w'_{j\ell},\tilde w''_{j\ell},\tilde w'''_{j\ell}\,|_{1\le j<\ell\le m-1};\tilde w_j\,|_{1\le j\le m-2}\big\}
$$

in (7.40) with the basis (7.37) of M^n, then due to the facts (7.42), (7.43), and (7.44), we see that the difference tensor of $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)\to\mathbb{R}^{2m^2-m}$ is exactly the same as that of $M^n\to\mathbb{R}^{n+1}$ corresponding to $\lambda_1=\frac{m-2}{\sqrt{m-1}}$, or equivalently $\lambda=-1$.

Now for the locally strongly convex $\mathfrak{C}_r$ affine hypersphere $M^n\to\mathbb{R}^{n+1}$ with $\mathfrak{p}=3$ and $n=(r-1)(r+5)/8=(r+1)(r+3)/8-1$, we see from the above discussion that, by applying a homothetic transformation to make $\lambda=-1$, if necessary, $M^{(r+1)(r+3)/8-1}$ and the standard embedding $\mathbf{SU}^*(\frac{r+3}{2})/\mathbf{Sp}(\frac{r+3}{4})\hookrightarrow\mathbb{R}^{(r+1)(r+3)/8}$ has affine metric h and cubic form C with identically the same affine invariant properties. According to Cartan's lemma and the fundamental uniqueness theorem of affine differential geometry, we obtain that $M^{(r+1)(r+3)/8-1}$ and $\mathbf{SU}^*(\frac{r+3}{2})/\mathbf{Sp}(\frac{r+3}{4})$ are locally affine equivalent.

We have completed the proof of Theorem 7.1. q.e.d.

8. Hypersurfaces in $\mathbb{R}^{n+1}$ with $\mathfrak{p}=7$

In this section, we will prove the following theorem.

Theorem 8.1. *Let M^n be a locally strongly convex affine hypersurface of $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\dim\mathcal{D}_2=r-1=8k_0\ge 8$ and $\mathfrak{p}$ that is determined by the previous section satisfies $\mathfrak{p}=7$, then $k_0=(r-1)/8\le 2$.*

Moreover, if $k_0 = 1$, then M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form. If $k_0 = 2$, then $n \geq 26$ and either

(i) *$n = 27$, M^n is the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point, or*

(ii) *$n \geq 28$, M^n is the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form, or*

(iii) *$n = 26$, up to a homothetic transformation, M^n is affine equivalent to an open part of the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$.*

Similar to the previous sections, we first prove the following

Lemma 8.1. *Suppose $\dim \mathcal{D}_2 = r - 1 = 8k_0$ and $\mathfrak{p} = 7$. Then from the decomposition (4.1), if $k_0 \geq 2$, we can choose orthonormal basis $\{x_j\}_{1\leq j\leq 7}$ for $V_{v_1}(0)$ and orthonormal basis $\{y_j\}_{1\leq j\leq 7}$ for $V_{v_2}(0)$ so that by identifying $e_j(v_1) = x_j$ and $e_j(v_2) = y_j$, we have the following relations:*

$$L(e_j(v_1), e_\ell(v_2)) = -L(v_1, e_j e_\ell(v_2)) = -L(e_\ell e_j(v_1), v_2), \quad 1 \leq j, \ell \leq 7, \tag{8.1}$$

where $e_j e_\ell$ denotes a product defined by the following multiplication table.

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\mathrm{id}$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-\mathrm{id}$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-\mathrm{id}$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-\mathrm{id}$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-\mathrm{id}$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-\mathrm{id}$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-\mathrm{id}$

More precisely, (8.1) *can be equivalently written out in the following form:*

$$L(v_1, v_2) = L(x_1, y_1) = L(x_2, y_2) = L(x_3, y_3) = L(x_4, y_4) = L(x_5, y_5) = L(x_6, y_6) = L(x_7, y_7), \tag{8.1^1}$$

$$L(x_1, v_2) = -L(v_1, y_1) = L(x_2, y_3) = -L(x_3, y_2) = L(x_4, y_5) = -L(x_5, y_4) = -L(x_6, y_7) = L(x_7, y_6), \tag{8.1^2}$$

$$L(x_2, v_2) = -L(v_1, y_2) = -L(x_1, y_3) = L(x_3, y_1) = L(x_4, y_6) = L(x_5, y_7) = -L(x_6, y_4) = -L(x_7, y_5), \tag{8.1^3}$$

$$L(x_3, v_2) = -L(v_1, y_3) = L(x_1, y_2) = -L(x_2, y_1) = L(x_4, y_7) = -L(x_5, y_6) = L(x_6, y_5) = -L(x_7, y_4), \tag{8.1^4}$$

$$\begin{aligned} L(x_4,v_2) &= -L(v_1,y_4) = -L(x_1,y_5) = -L(x_2,y_6) \\ &= -L(x_3,y_7) = L(x_5,y_1) = L(x_6,y_2) = L(x_7,y_3), \end{aligned} \tag{8.1^5}$$

$$\begin{aligned} L(x_5,v_2) &= -L(v_1,y_5) = L(x_1,y_4) = -L(x_2,y_7) = L(x_3,y_6) \\ &= -L(x_4,y_1) = -L(x_6,y_3) = L(x_7,y_2), \end{aligned} \tag{8.1^6}$$

$$\begin{aligned} L(x_6,v_2) &= -L(v_1,y_6) = L(x_1,y_7) = L(x_2,y_4) = -L(x_3,y_5) \\ &= -L(x_4,y_2) = L(x_5,y_3) = -L(x_7,y_1), \end{aligned} \tag{8.1^7}$$

$$\begin{aligned} L(x_7,v_2) &= -L(v_1,y_7) = -L(x_1,y_6) = L(x_2,y_5) = L(x_3,y_4) \\ &= -L(x_4,y_3) = -L(x_5,y_2) = L(x_6,y_1). \end{aligned} \tag{8.1^8}$$

Proof. Let $k_0 \geq 2$ and that we have the decomposition (4.1) with $\dim V_{v_j}(0) = 7,\ 1 \leq j \leq k_0$.

Denote $V_j = \{v_j\} \oplus V_{v_j}(0)$. First we choose arbitrary orthonormal vectors $x_1, x_2 \in V_{v_1}(0)$. Next we can use Lemma 4.2 and Lemma 4.3 to consecutively find unit vectors $y_1, y_2 \in V_{v_2}(0)$, $x_3 \in V_{v_1}(0)$, and $y_3 \in V_{v_2}(0)$ satisfying the following:

$$L(y_1,v_1) = -L(x_1,v_2), \quad L(y_2,v_1) = -L(x_2,v_2), \tag{8.2}$$

$$L(y_1,x_2) = -L(v_2,x_3), \quad L(y_3,v_1) = -L(x_3,v_2). \tag{8.3}$$

Now we pick an arbitrary unit vector $x_4 \in V_{v_1}(0)$ so that it is orthogonal to all x_1, x_2, and x_3. Then we can take unit vectors $x_5, x_6, x_7 \in V_{v_1}(0)$ and unit vectors $y_4, y_5, y_6, y_7 \in V_{v_2}(0)$ inductively such that the following hold:

$$L(x_4,y_1) = -L(y_4,x_1) = -L(v_2,x_5) = L(v_1,y_5), \tag{8.4}$$

$$\begin{aligned} L(x_4,y_2) &= -L(v_2,x_6) = L(v_1,y_6), \\ L(x_4,y_3) &= -L(v_2,x_7) = L(v_1,y_7). \end{aligned} \tag{8.5}$$

As before we first have

Claim 8.1. *Based on the above definition, $x_3 \perp x_1, x_3 \perp x_2$; y_1, y_2, y_3 are mutually orthonormal.*

In fact, by (8.2) and (8.3) we have the computation

$$\begin{aligned} \tau h(x_3,x_1) &= h(L(v_2,x_3), L(v_2,x_1)) = h(-L(y_1,x_2), L(v_1,y_1)) \\ &= -h(x_2, P(y_1)v_1) = -\tau h(x_2,v_1) = 0, \end{aligned}$$

hence we get $x_3 \perp x_1$. Similarly, we have $x_3 \perp x_2$ and $y_2 \perp y_1 \perp y_3 \perp y_2$.

Next from (8.2) and (8.3), using Claim 8.1 and an argument similar to Claim 7.2, we get the relation

$$L(x_1,y_1) = L(x_2,y_2) = L(x_3,y_3) = L(v_1,v_2). \tag{8.6}$$

Now we exploit the condition that $L(y_1, x_2) = -L(v_2, x_3)$. The equation (8.6), the Cauchy Schwartz inequality, and the isotropy conditions imply the following relations:

$$L(y_1, x_3) = -L(x_1, y_3) = L(x_2, v_2), \tag{8.7}$$

$$L(x_1, y_2) = -L(y_1, x_2) = L(v_2, x_3), \tag{8.8}$$

$$L(x_2, y_3) = -L(y_2, x_3) = -L(y_1, v_1). \tag{8.9}$$

Firstly, from (8.4) and the computation

$$\begin{aligned}\tau h(y_4, y_1) &= h(L(y_4, x_1), L(y_1, x_1)) = h(-L(x_4, y_1), L(y_1, x_1)) \\ &= -h(x_4, P(y_1)x_1) = -\tau h(x_4, x_1) = 0,\end{aligned}$$

we have the following

Claim 8.2. $y_4 \perp y_1$.

Moreover, we have

Claim 8.3. *The vectors* $\{x_4, x_5, x_6, x_7\}$ *are mutually orthogonal. A similar conclusion holds for the vectors* $\{y_4, y_5, y_6, y_7\}$.

In fact, from (8.4) we get

$$\begin{aligned}\tau h(x_5, x_4) &= h(L(x_5, y_1), L(x_4, y_1)) = h(L(x_5, y_1), -L(v_2, x_5)) \\ &= -h(y_1, P(x_5)v_2) = -\tau h(y_1, v_2) = 0,\end{aligned}$$

which shows that $x_5 \perp x_4$. Similarly, we can prove $x_6 \perp \{x_4, x_5\}$ and $x_7 \perp \{x_4, x_5, x_6\}$.

From the above conclusions, we can use (8.4) again to see that

$$\begin{aligned}\tau h(y_5, y_4) &= h(L(y_5, v_1), L(y_4, v_1)) = h(-L(x_5, v_2), -L(x_4, v_2)) \\ &= h(x_5, P(v_2)x_4) = \tau h(x_5, x_4) = 0.\end{aligned}$$

This proves $y_5 \perp y_4$. Similarly, we can prove $y_6 \perp \{y_4, y_5\}$ and $y_7 \perp \{y_4, y_5, y_6\}$.

As direct consequences of Claim 8.2 and Claim 8.3, we can use the Cauchy-Schwarz inequality, $L(x_4, y_1) = -L(y_4, x_1)$, and the isotropy condition to obtain

$$L(x_4, y_4) = L(x_1, y_1) = L(v_1, v_2),$$

which implies

$$L(v_1, y_4) = -L(v_2, x_4). \tag{8.10}$$

Furthermore, the equations in (8.4)–(8.6), the Cauchy-Schwarz inequality, and the isotropy condition yield the following relations:

$$L(x_4, y_4) = L(x_5, y_5) = L(x_6, y_6) = L(x_7, y_7) = L(v_1, v_2), \tag{8.11}$$

$$\begin{aligned}&L(v_1, y_5) = -L(v_2, x_5), \;\; L(v_1, y_6) = -L(v_2, x_6), \\ &L(v_1, y_7) = -L(v_2, x_7),\end{aligned} \tag{8.12}$$

$$L(x_4,y_5)=-L(y_4,x_5),\ L(x_4,y_6)=-L(y_4,x_6),\quad L(x_4,y_7)=-L(y_4,x_7), \tag{8.13}$$

$$L(x_5,y_6)=-L(y_5,x_6),\ L(x_5,y_7)=-L(y_5,x_7),\quad L(x_6,y_7)=-L(y_6,x_7). \tag{8.14}$$

Moreover, using the equations in (8.2)–(8.6) and (8.7)–(8.10), similar arguments to those above will give the following orthonormality of the vectors:

$$\{x_1,x_2,x_3\}\perp\{x_4,x_5,x_6,x_7\},\qquad \{y_1,y_2,y_3\}\perp\{y_4,y_5,y_6,y_7\}.$$

In conclusion, we have shown that $\{x_j\}_{1\le j\le 7}$ is an orthonormal basis of $V_{v_1}(0)$ and $\{y_j\}_{1\le j\le 7}$ is an orthonormal basis of $V_{v_2}(0)$. Then using (8.6) and (8.11), the Cauchy-Schwarz inequality, and the isotropy conditions, as before we further get the relations

$$L(y_4,x_2)=-L(x_4,y_2),\ L(y_4,x_3)=-L(x_4,y_3), \tag{8.15}$$

$$L(y_5,x_1)=-L(x_5,y_1),\ L(y_5,x_2)=-L(x_5,y_2),\quad L(y_5,x_3)=-L(x_5,y_3), \tag{8.16}$$

$$L(y_6,x_1)=-L(x_6,y_1),\ L(y_6,x_2)=-L(x_6,y_2),\quad L(y_6,x_3)=-L(x_6,y_3), \tag{8.17}$$

$$L(y_7,x_1)=-L(x_7,y_1),\ L(y_7,x_2)=-L(x_7,y_2),\quad L(y_7,x_3)=-L(x_7,y_3). \tag{8.18}$$

Finally, based on the above relations from (8.2) to (8.18), using the Cauchy-Schwarz inequality and the isotropy conditions, we can establish the following relations:

$$L(x_4,y_5)=-L(v_1,y_1),\ L(x_4,y_6)=-L(v_1,y_2),\quad L(x_4,y_7)=-L(v_1,y_3), \tag{8.19}$$

$$L(x_5,y_1)=-L(v_1,y_4),\ L(x_5,y_2)=L(v_1,y_7),\quad L(x_5,y_3)=-L(v_1,y_6), \tag{8.20}$$

$$L(x_5,y_6)=L(v_1,y_3),\ L(x_5,y_7)=-L(v_1,y_2),\quad L(x_6,y_1)=-L(v_1,y_7), \tag{8.21}$$

$$L(x_6,y_2)=-L(v_1,y_4),\ L(x_6,y_3)=L(v_1,y_5),\quad L(x_6,y_7)=L(v_1,y_1), \tag{8.22}$$

$$L(x_7,y_1)=L(v_1,y_6),\ L(x_7,y_2)=-L(v_1,y_5),\quad L(x_7,y_3)=-L(v_1,y_4). \tag{8.23}$$

For example, from the computation

$$\begin{aligned} h(L(x_4,y_5),L(v_1,y_1)) &\overset{(8.2)}{=\!=} -h(L(x_4,y_5),L(v_2,x_1)) \\ &\overset{(2.10)}{=\!=} h(L(x_4,v_2),L(y_5,x_1)) \overset{(8.16)}{=\!=} -h(L(x_4,v_2),L(x_5,y_1)) \\ &\overset{(2.10)}{=\!=} h(L(x_4,y_1),L(x_5,v_2)) \overset{(8.4)}{=\!=} h(L(x_4,y_1),-L(x_4,y_1)) = -\tau \end{aligned}$$

and the Cauchy-Schwarz inequality, we obtain $L(x_4,y_5) = -L(v_1,y_1)$.

In order to provide another example, from the computation

$$\begin{aligned} h(L(x_6,y_1),L(v_1,y_7)) &\overset{(8.5)}{=\!=} h(L(x_6,y_1),L(x_4,y_3)) \\ &\overset{(8.17)}{=\!=} -h(L(x_1,y_6),L(x_4,y_3)) \overset{(2.10)}{=\!=} h(L(x_1,y_3),L(x_4,y_6)) \\ &\overset{(8.7)}{=\!=} -h(L(x_2,v_2),L(x_4,y_6)) \overset{(8.13)}{=\!=} h(L(x_2,v_2),L(y_4,x_6)) \\ &\overset{(2.10)}{=\!=} -h(L(x_2,y_4),L(v_2,x_6)) \\ &\overset{(8.5),(8.15)}{=\!=\!=\!=} -h(-L(x_4,y_2),-L(x_4,y_2)) = -\tau \end{aligned}$$

and the Cauchy-Schwarz inequality, we obtain $L(x_6,y_1) = -L(v_1,y_7)$.

Finally, from the computation

$$\begin{aligned} h(L(x_5,y_6),L(v_1,y_3)) &\overset{(8.3)}{=\!=} -h(L(x_5,y_6),L(v_2,x_3)) \\ &\overset{(2.10)}{=\!=} h(L(x_5,v_2),L(y_6,x_3)) \overset{(8.4)}{=\!=} -h(L(x_4,y_1),L(y_6,x_3)) \\ &\overset{(2.10)}{=\!=} h(L(x_4,y_6),L(y_1,x_3)) \overset{(8.7)}{=\!=} h(L(x_4,y_6),L(x_2,v_2)) \\ &\overset{(8.2)}{=\!=} -h(L(x_4,y_6),L(y_2,v_1)) \overset{(2.10)}{=\!=} h(L(x_4,y_2),L(v_1,y_6)) \overset{(8.5)}{=\!=} \tau \end{aligned}$$

and the Cauchy-Schwarz inequality, we immediately get $L(x_5,y_6) = L(v_1,y_3)$.

Following the above procedure, we can prove all of the relations in (8.19)–(8.23), and thus we complete the proof of Lemma 8.1. q.e.d.

Lemma 8.2. *Suppose* $\dim \mathcal{D}_2 = r-1 \geq 8$ *and* $\mathfrak{p} = 7$. *Then for the decomposition* (4.1), *if* $k_0 \geq 2$, *it must be the case that* $k_0 = 2$.

Proof. Suppose on the contrary that $k_0 \geq 3$. To choose a basis for $V_{v_3}(0)$, we follow the same ideas as in Lemma 8.1 for $V_{v_1}(0)$ and $V_{v_2}(0)$. Let x_1, x_2, x_3 be given as in Lemma 8.1; then we have unique unit vectors $z_1, z_2 \in V_{v_3}(0)$ and $\tilde{x}_3 \in V_{v_1}(0)$ with the relations

$$\begin{aligned} &L(z_1,v_1) = -L(x_1,v_3),\ L(z_2,v_1) = -L(x_2,v_3), \\ &L(z_1,x_2) = -L(v_3,\tilde{x}_3). \end{aligned} \tag{8.24}$$

Now we pick an arbitrary unit vector $x_4 \in V_{v_1}(0)$ so that it is orthogonal to x_1, x_2, x_3 and $\tilde{x}_3$. Then we can choose unit vectors $\tilde{x}_5, \tilde{x}_6, \tilde{x}_7 \in$

$V_{v_1}(0)$ and $z_3, z_4, z_5, z_6, z_7 \in V_{v_3}(0)$ inductively by the following conditions:

$$\begin{aligned} &L(z_3, v_1) = -L(\tilde{x}_3, v_3), \\ &L(x_4, z_1) = -L(z_4, x_1) = -L(v_3, \tilde{x}_5) = L(v_1, z_5), \end{aligned} \tag{8.25}$$

$$\begin{aligned} &L(x_4, z_2) = -L(v_3, \tilde{x}_6) = L(v_1, z_6), \\ &L(x_4, z_3) = -L(v_3, \tilde{x}_7) = L(v_1, z_7). \end{aligned} \tag{8.26}$$

Then, similar to the proof of Lemma 8.1, it can be shown that $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$ forms an orthonormal basis of $V_{v_3}(0)$.

On the other hand, while

$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\} \quad \text{and} \quad \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$$

are orthonormal bases of $V_{v_2}(0)$ and $V_{v_3}(0)$, respectively, we have two orthonormal bases for $V_{v_1}(0)$, namely,

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \quad \text{and} \quad \{x_1, x_2, \tilde{x}_3, x_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7\}.$$

Obviously, the bases x_3, x_5, x_6, x_7 and $\tilde{x}_3, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7$ are related by a matrix belonging to $SO(4)$. Let us write

$$\begin{pmatrix} x_3 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \end{pmatrix} := B \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \end{pmatrix},$$

where $B \in SO(4)$.

Now, having all the relations from (8.2) to (8.26), a similar method to that in the proof of Lemma 8.1 will give the following further relations:

$$\begin{aligned} L(x_1, v_3) &= -L(v_1, z_1) = L(x_2, z_3) = -L(\tilde{x}_3, z_2) = L(x_4, z_5) \\ &= -L(\tilde{x}_5, z_4) = -L(\tilde{x}_6, z_7) = L(\tilde{x}_7, z_6), \end{aligned} \tag{8.27}$$

$$\begin{aligned} L(x_2, v_3) &= -L(v_1, z_2) = -L(x_1, z_3) = L(\tilde{x}_3, z_1) = L(x_4, z_6) \\ &= L(\tilde{x}_5, z_7) = -L(\tilde{x}_6, z_4) = -L(\tilde{x}_7, z_5), \end{aligned} \tag{8.28}$$

$$\begin{aligned} L(\tilde{x}_3, v_3) &= -L(v_1, z_3) = L(x_1, z_2) = -L(x_2, z_1) = L(x_4, z_7) \\ &= -L(\tilde{x}_5, z_6) = L(\tilde{x}_6, z_5) = -L(\tilde{x}_7, z_4), \end{aligned} \tag{8.29}$$

$$\begin{aligned} L(x_4, v_3) &= -L(v_1, z_4) = -L(x_1, z_5) = -L(x_2, z_6) \\ &= -L(\tilde{x}_3, z_7) = L(\tilde{x}_5, z_1) = L(\tilde{x}_6, z_2) = L(\tilde{x}_7, z_3), \end{aligned} \tag{8.30}$$

$$\begin{aligned} L(\tilde{x}_5, v_3) &= -L(v_1, z_5) = L(x_1, z_4) = -L(x_2, z_7) = L(\tilde{x}_3, z_6) \\ &= -L(x_4, z_1) = -L(\tilde{x}_6, z_3) = L(\tilde{x}_7, z_2), \end{aligned} \tag{8.31}$$

$$\begin{aligned} L(\tilde{x}_6, v_3) &= -L(v_1, z_6) = L(x_1, z_7) = L(x_2, z_4) = -L(\tilde{x}_3, z_5) \\ &= -L(x_4, z_2) = L(\tilde{x}_5, z_3) = -L(\tilde{x}_7, z_1), \end{aligned} \tag{8.32}$$

$$\begin{aligned}L(\tilde{x}_7, v_3) &= -L(v_1, z_7) = -L(x_1, z_6) = L(x_2, z_5) = L(\tilde{x}_3, z_4)\\ &= -L(x_4, z_3) = -L(\tilde{x}_5, z_2) = L(\tilde{x}_6, z_1).\end{aligned} \tag{8.33}$$

To see relations between the above given bases of $V_{v_2}(0)$ and $V_{v_3}(0)$, we first have

Claim 8.4. $L(v_2, v_3) = L(y_1, z_1) = L(y_2, z_2) = L(y_3, z_3) = L(y_4, z_4) = L(y_5, z_5) = L(y_6, z_6) = L(y_7, z_7)$.

In fact, by Lemma 4.4 and the already established relations, we have

$$\begin{aligned}\tau L(y_1, z_1) &= K(L(y_1, x_2), L(z_1, x_2))\\ &\overset{(8.2),(8.8),(8.29)}{=\!=\!=} K(L(v_1, y_3), L(v_1, z_3)) = \tau L(y_3, z_3),\end{aligned}$$

$$\begin{aligned}\tau L(y_2, z_2) &= K(L(v_1, y_2), L(v_1, z_2))\\ &\overset{(8.2),(8.7),(8.28)}{=\!=\!=} K(L(x_1, y_3), L(x_1, z_3)) = \tau L(y_3, z_3),\end{aligned}$$

$$\begin{aligned}\tau L(v_2, v_3) &= K(L(v_2, x_1), L(v_3, x_1))\\ &\overset{(8.2),(8.9),(8.27)}{=\!=\!=} K(L(x_2, y_3), L(x_2, z_3)) = \tau L(y_3, z_3),\end{aligned}$$

$$\begin{aligned}\tau L(y_4, z_4) &= K(L(v_1, y_4), L(v_1, z_4))\\ &\overset{(8.17),(8.22),(8.30)}{=\!=\!=} K(L(x_2, y_6), L(x_2, z_6)) = \tau L(y_6, z_6),\end{aligned}$$

$$\begin{aligned}\tau L(y_6, z_6) &= K(L(v_1, y_6), L(v_1, z_6))\\ &\overset{(8.5),(8.26)}{=\!=\!=} K(L(x_4, y_2), L(x_4, z_2)) = \tau L(y_2, z_2),\end{aligned}$$

$$\begin{aligned}\tau L(y_5, z_5) &= K(L(v_1, y_5), L(v_1, z_5))\\ &\overset{(8.4),(8.25)}{=\!=\!=} K(L(x_4, y_1), L(x_4, z_1)) = \tau L(y_1, z_1),\end{aligned}$$

$$\begin{aligned}\tau L(y_7, z_7) &= K(L(v_1, y_7), L(v_1, z_7))\\ &\overset{(8.5),(8.26)}{=\!=\!=} K(L(x_4, y_3), L(x_4, z_3)) = \tau L(y_3, z_3).\end{aligned}$$

Then Claim 8.4 immediately follows.

As direct consequences of Claim 8.4, we have

Claim 8.5. $b_{k\ell} = \delta_{k\ell}$, *i.e.*, $\tilde{x}_3 = x_3,\ \tilde{x}_5 = x_5,\ \tilde{x}_6 = x_6,\ \tilde{x}_7 = x_7$.

In fact, by using Lemma 4.4 we get the following orthogonal decomposition:

$$\begin{aligned}\tau L(y_4, z_4) &= K(L(v_1, y_4), L(v_1, z_4)) \overset{(8.23)}{=\!=} K(-L(x_7, y_3), L(v_1, z_4))\\ &\overset{(8.30)}{=\!=} -b_{41}K(L(\tilde{x}_3, y_3), L(\tilde{x}_3, z_7)) + b_{42}K(L(\tilde{x}_5, y_3), L(\tilde{x}_5, z_1))\\ &\qquad + b_{43}K(L(\tilde{x}_6, y_3), L(\tilde{x}_6, z_2)) + b_{44}K(L(\tilde{x}_7, y_3), L(\tilde{x}_7, z_3))\\ &= -b_{41}\tau L(y_3, z_7) + b_{42}\tau L(y_3, z_1) + b_{43}\tau L(y_3, z_2) + b_{44}\tau L(y_3, z_3);\end{aligned}$$

$$\begin{aligned}\tau L(y_4,z_4) &= K(L(v_1,y_4),L(v_1,z_4)) \overset{(8.22)}{=\!=} K(-L(x_6,y_2),L(v_1,z_4))\\ &\overset{(8.30)}{=\!=} -b_{31}K(L(\tilde{x}_3,y_2),L(\tilde{x}_3,z_7)) + b_{32}K(L(\tilde{x}_5,y_2),L(\tilde{x}_5,z_1))\\ &\qquad + b_{33}K(L(\tilde{x}_6,y_2),L(\tilde{x}_6,z_2)) + b_{34}K(L(\tilde{x}_7,y_2),L(\tilde{x}_7,z_3))\\ &= -b_{31}\tau L(y_2,z_7) + b_{32}\tau L(y_2,z_1) + b_{33}\tau L(y_2,z_2) + b_{34}\tau L(y_2,z_3);\end{aligned}$$

$$\begin{aligned}\tau L(y_4,z_4) &= K(L(v_1,y_4),L(v_1,z_4)) \overset{(8.20)}{=\!=} K(-L(x_5,y_1),L(v_1,z_4))\\ &\overset{(8.30)}{=\!=} -b_{21}K(L(\tilde{x}_3,y_1),L(\tilde{x}_3,z_7)) + b_{22}K(L(\tilde{x}_5,y_1),L(\tilde{x}_5,z_1))\\ &\qquad + b_{23}K(L(\tilde{x}_6,y_1),L(\tilde{x}_6,z_2)) + b_{24}K(L(\tilde{x}_7,y_1),L(\tilde{x}_7,z_3))\\ &= -b_{21}\tau L(y_1,z_7) + b_{22}\tau L(y_1,z_1) + b_{23}\tau L(y_1,z_2) + b_{24}\tau L(y_1,z_3);\end{aligned}$$

$$\begin{aligned}\tau L(y_4,z_4) &= K(L(v_1,y_4),L(v_1,z_4)) \overset{(8.18),(8.23)}{=\!=\!=\!=} K(L(x_3,y_7),L(v_1,z_4))\\ &\overset{(8.30)}{=\!=} b_{11}K(L(\tilde{x}_3,y_7),L(\tilde{x}_3,z_7)) - b_{12}K(L(\tilde{x}_5,y_7),L(\tilde{x}_5,z_1))\\ &\qquad - b_{13}K(L(\tilde{x}_6,y_7),L(\tilde{x}_6,z_2)) - b_{14}K(L(\tilde{x}_4,y_4),L(\tilde{x}_7,z_3))\\ &= b_{11}\tau L(y_4,z_4) - b_{12}\tau L(y_7,z_1) - b_{13}\tau L(y_7,z_2) - b_{14}\tau L(y_7,z_3).\end{aligned}$$

Applying Claim 8.4, we then get the claim immediately.

Claim 8.6. *For the above bases of $V_{v_2}(0)$ and $V_{v_3}(0)$, there hold the following relations:*

$$L(z_1,y_2) = -L(v_3,y_3), \quad L(y_4,z_1) = L(v_2,z_5), \tag{8.34}$$

$$L(y_4,z_2) = L(v_2,z_6), \quad L(y_4,z_3) = L(v_2,z_7), \tag{8.35}$$

$$L(y_4,z_6) = -L(v_2,z_2), \quad L(y_4,z_7) = -L(v_2,z_3), \tag{8.36}$$

$$L(y_6,z_2) = -L(v_2,z_4) = L(y_7,z_3). \tag{8.37}$$

In fact, using Lemma 4.4 we see that (8.34) and (8.35) follow from the following calculations:

$$\begin{aligned}\tau L(z_1,y_2) &= K(L(v_1,z_1),L(v_1,y_2))\\ &\overset{(8.2),(8.7),(8.27)}{=\!=\!=\!=\!=} K(-L(v_3,x_1),L(x_1,y_3)) = -\tau L(v_3,y_3),\\ \tau L(y_4,z_1) &= K(L(v_1,y_4),L(v_1,z_1))\\ &\overset{(8.10),(8.27)}{=\!=\!=\!=} K(-L(v_2,x_4),-L(x_4,z_5)) = \tau L(v_2,z_5),\\ \tau L(y_4,z_2) &= K(L(v_1,y_4),L(v_1,z_2))\\ &\overset{(8.10),(8.28)}{=\!=\!=\!=} K(-L(v_2,x_4),-L(x_4,z_6)) = \tau L(v_2,z_6),\\ \tau L(y_4,z_3) &= K(L(v_1,y_4),L(v_1,z_3))\\ &\overset{(8.10),(8.29)}{=\!=\!=\!=} K(-L(v_2,x_4),-L(x_4,z_7)) = \tau L(v_2,z_7).\end{aligned}$$

Moreover, from Claim 8.4 we can get $L(v_3, y_6) = -L(v_2, z_6)$ and $L(z_4, y_2) = -L(y_4, z_2)$. Then from the calculations

$$\begin{aligned} h(L(y_4, z_6), L(v_2, z_2)) &= h(-L(z_4, y_6), -L(v_3, y_2)) \\ &\overset{(2.10))}{=\!=} -h(L(z_4, y_2), L(v_3, y_6)) = -h(-L(y_4, z_2), -L(v_2, z_6)) \\ &\overset{(8.35)}{=\!=} -h(L(y_4, z_2), L(y_4, z_2)) = -\tau, \end{aligned}$$

$$\begin{aligned} h(L(y_4, z_7), L(v_2, z_3)) &\overset{(2.10)}{=\!=} -h(L(y_4, z_3), L(v_2, z_7)) \\ &\overset{(8.35)}{=\!=} -h(L(v_2, z_7), L(v_2, z_7)) = -\tau \end{aligned}$$

and the Cauchy-Schwarz inequality, we immediately get (8.36).

Similarly, by using Claim 8.4 we have the following equations:

$$\begin{aligned} h(L(y_6, z_2), L(v_2, z_4)) &\overset{(2.10)}{=\!=} -h(L(y_6, z_4), L(v_2, z_2)) \\ &= h(L(y_4, z_6), L(v_2, z_2)) \overset{(8.36)}{=\!=} -\tau, \end{aligned}$$

$$\begin{aligned} h(L(y_7, z_3), L(v_2, z_4)) &\overset{(2.10)}{=\!=} -h(L(z_4, y_7), L(v_2, z_3)) \\ &\overset{(8.36)}{=\!=} h(L(z_4, y_7), L(y_4, z_7)) = -\tau. \end{aligned}$$

Then the Cauchy-Schwarz inequality implies the relations in (8.37).

To complete the proof of Lemma 8.2, we notice that from (8.23) and (8.18), we have

$$K(L(v_1, y_6) + L(x_1, y_7), L(x_2, v_3)) = 0. \tag{8.38}$$

On the other hand, by using Lemma 4.4, we have the following results:

$$K(L(v_1, y_6), L(x_2, v_3)) \overset{(8.28)}{=\!=\!=} K(L(v_1, y_6), -L(v_1, z_2)) = -\tau L(y_6, z_2),$$

$$K(L(x_1, y_7), L(x_2, v_3)) \overset{(8.28)}{=\!=\!=} K(L(x_1, y_7), -L(x_1, z_3)) = -\tau L(y_7, z_3).$$

These together with (8.38) give that

$$L(y_6, z_2) + L(y_7, z_3) = 0.$$

Combining the above equation with (8.37), we obtain $L(y_6, z_2) = L(y_7, z_3) = 0$, which contradicts the fact that $h(L(y_6, z_2), L(y_6, z_2)) = \tau$. This completes the proof of Lemma 8.2. q.e.d.

Proof of Theorem 8.1. If $k_0 = 1$, then as in the previous theorems, we know that M^n can be decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form. Hence we are left to consider the case that $k_0 = 2$, $r = 8k_0 + 1 = 17$, and $\dim \mathcal{D}_2 = 16$.

Let $\{v_1, v_2; x_j, y_j|_{1\le j\le 7}\}$ be the orthonormal basis of $\mathcal{D}_2$ as determined in Lemma 8.1, which satisfies the relations (8.1^1)–(8.1^8). Then we easily see that the image of L is spanned by

$$\big\{L(v_1, v_1),\ L(v_1, v_2),\ L(v_2, v_2);\ L(v_1, y_j)\,|_{1\le j\le 7}\big\}.$$

Noting that $\operatorname{Tr} L = 8\big[L(v_1, v_1) + L(v_2, v_2)\big]$, which is orthogonal to the above ten vectors, and by using (2.6) and (2.8), obviously satisfies

$$\begin{aligned} h(\operatorname{Tr} L, \operatorname{Tr} L) &= 128\big[h(L(v_1, v_1), L(v_1, v_1)) \\ &\qquad + h(L(v_1, v_1), L(v_2, v_2))\big] \\ &= \tfrac{32(n+1)(n-26)}{(n-17)^2}\lambda_1^2. \end{aligned} \tag{8.39}$$

Define $L_1 = L(v_1, v_1) - L(v_2, v_2)$; then direct calculation shows that

$$h(L_1, L_1) = 4\tau \neq 0. \tag{8.40}$$

We now easily see that the following nine vectors

$$w_0 = \tfrac{1}{2\sqrt{\tau}}L_1,\ w_1 = \tfrac{1}{\sqrt{\tau}}L(v_1, v_2);\ w_{j+1} = \tfrac{1}{\sqrt{\tau}}L(v_1, y_j)|_{1\le j\le 7} \tag{8.41}$$

consist of orthonormal vectors in $\operatorname{Im}(L) \subset \mathcal{D}_3$. Then we have the conclusion

$$n = 1 + \dim \mathcal{D}_2 + \dim \mathcal{D}_3 \ge 1 + 16 + 9 = 26,$$

and from (8.39) we see that $n = 26$ if and only if $\operatorname{Tr} L = 0$.

Now, we separate the discussions into the following three cases:
(i) $n = 27$; **(ii)** $n \ge 28$; **(iii)** $n = 26$.

For **Case (i)** and **Case (ii)**, we will define a unit vector

$$t = \frac{n-17}{\sqrt{32(n+1)(n-26)}\,\lambda_1}\operatorname{Tr} L.$$

First for **Case (i)**, the previous results show that $\big\{t;\ w_j,\ 0 \le j \le 8\big\}$ is an orthonormal basis of $\operatorname{Im}(L) = \mathcal{D}_3$. By direct calculations with application of Lemma 2.4, Lemma 4.4, and (8.1^1)–(8.1^8), we easily verify the following

Lemma 8.3. *For Case (i) and under the above notations, we deduce that*

$$\begin{cases} K(t, e_1) = -\frac{9}{n-17}\lambda_1 t, \\ K(t, t) = -\frac{9}{n-17}\lambda_1 e_1 + \frac{n-44}{2(n-17)}\sqrt{\frac{2(n+1)}{n-26}}\,\lambda_1 t, \\ K(t, u) = \frac{\sqrt{2(n+1)(n-26)}}{4(n-17)}\,\lambda_1 u, \\ K(t, w) = \frac{\sqrt{2(n+1)(n-26)}}{2(n-17)}\,\lambda_1 w, \end{cases} \tag{8.42}$$

where $u \in \mathcal{D}_2$ and $w = w_j$, $0 \le j \le 8$.

Denote $T = \alpha e_1 + \beta t,\ T^* = -\beta e_1 + \alpha t$, where

$$\alpha = \sqrt{\tfrac{n-26}{3(n-8)}},\quad \beta = \sqrt{\tfrac{2(n+1)}{3(n-8)}}. \tag{8.43}$$

Then for **Case (i)**, we see that

$$\Big\{T, T^*;\ v_1, v_2; x_j, y_j \,|\, {}_{1\le j\le 7};\ w_\ell \,|\, {}_{0\le \ell\le 8}\Big\}$$

forms an orthornormal basis of $T_{x_0}M$. Moreover, by Lemma 8.3 we easily obtain the following

Lemma 8.4. *For Case (i) and under the above notations, we find that*

$$K(T,T) = \nu_1 T, \quad K(T,u) = \nu_2 u, \tag{8.44}$$

where ν_1 *and* ν_2 *are defined by*

$$\begin{cases} \nu_1 = \frac{(n-8)(n-53)}{(n-17)(n-26)}\sqrt{\frac{n-26}{3(n-8)}}\,\lambda_1, \\ \nu_2 = \frac{n-8}{n-17}\sqrt{\frac{n-26}{3(n-8)}}\,\lambda_1, \end{cases} \tag{8.45}$$

and $u \in \big\{T^*;\ v_1, v_2; x_j, y_j \,|\, {}_{1\le j\le 7};\ w_\ell \,|\, {}_{0\le \ell\le 8}\big\}$.

From (8.45) and noting that $r = 17, n = 27$, we can verify the relation

$$\nu_1\nu_2 - \nu_2^2 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{8.46}$$

Then, based on the conclusions of Lemma 8.4, we can apply theorem 4 of [**HLV**] to conclude that in Case (i), M^{27} is decomposed as the Calabi product of a hyperbolic affine hypersphere with parallel cubic form and a point.

For **Case (ii)**, we see that $\big\{t;\ w_j,\ 0 \le j \le 8\big\}$ is still an orthonormal basis of $\mathrm{Im}\,(L)$. But now $\mathrm{Im}\,(L) \subsetneq \mathcal{D}_3$. Denote $\tilde n = n - 27 \ge 1$ and choose $w'_1, \dots, w'_{\tilde n}$ in the orthogonal complement of $\mathrm{Im}\,(L)$ in $\mathcal{D}_3$ such that

$$\{t;\ w_j \,\big|\, {}_{0\le j\le 8};\ w'_j \,|\, {}_{1\le j\le \tilde n}\}$$

is an orthonormal basis of $\mathcal{D}_3$. Then, besides (8.42), we further use (2) of Lemma 2.5 to get

$$K(t, w'_j) = -\tfrac{9}{n-17}\sqrt{\tfrac{2(n+1)}{n-26}}\lambda_1 w'_j, \quad 1 \le j \le \tilde n. \tag{8.47}$$

Now we define T and T^* to be the same as in Case (i). Similar to Lemma 8.4, we have the following

Lemma 8.5. *For Case (ii), we have that*

$$K(T,T) = \nu_1 T,\ K(T,u) = \nu_2 u,\ K(T,w'_j) = \nu_3 w'_j,\ 1 \le j \le \tilde n, \tag{8.48}$$

where $\nu_1,\ \nu_2,$ *and* ν_3 *are defined by* (8.45) *and*

$$\nu_3 = -\tfrac{9}{n-17}\sqrt{\tfrac{3(n-8)}{n-26}}\lambda_1, \tag{8.49}$$

and

$$u \in \{T^*;\ v_1,\ v_2;\ x_j, y_j\,|_{1\le j\le 7};\ w_j\,|_{0\le j\le 8}\}.$$

It is easily seen that $\nu_2 \ne \nu_3, 2\nu_2 \ne \nu_1 \ne 2\nu_3$, and

$$\nu_1 = \nu_2 + \nu_3, \quad \nu_2\nu_3 = -\tfrac{(r+1)(2n-r+1)}{4(n-r)^2}\lambda_1^2 = \lambda. \tag{8.50}$$

Thus, based on the conclusions of Lemma 8.5, we can apply theorem 3 of [**HLV**] to conclude that in Case (ii), M^n is decomposed as the Calabi product of two hyperbolic affine hyperspheres both with parallel cubic form.

Finally, for **Case (iii)**, we have $\operatorname{Tr} L = 0$ and thus

$$\mathcal{D}_3 = \big\{w_0,\, w_1,\, w_2,\, w_3,\, w_4,\, w_5,\, w_6,\, w_7,\, w_8\big\}.$$

It follows that

$$\big\{e_1;\ v_1, v_2;\ x_j, y_j\,|_{\,1\le j\le 7};\ w_\ell\,|_{\,0\le \ell\le 8}\big\} \tag{8.51}$$

is an orthonormal basis of $T_{x_0}M$. Now, applying Lemma 2.4, Lemma 4.4, (8.1^1)–(8.1^8), we can calculate all components of the difference tensor with respect to the basis (8.51).

To complete the proof of Theorem 8.1 for this case, we will first review the definitions of the exceptional non-compact Lie groups $\mathbf{E}_{6(-26)}$ and the compact Lie group $\mathbf{F}_4$. For references, among many others, we refer to Baez [**Ba**] and Yokota [**Y**]. Then we will recall an explicit embedding of $\mathbf{E}_{6(-26)}/\mathbf{F}_4$ into $\mathbb{R}^{27}$, which was recently discovered by Birembaux and Djoric [**BD**], and we will call it the *standard embedding* hereafter.

Let $\mathbb{O}$ be the octonions, i.e., the division Cayley algebra over the field $\mathbb{R}$ of real numbers, which is an 8-dimensional $\mathbb{R}$-vector space with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and define a multiplication between them, with $e_0 = 1$ being the unit, as in the following table.

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We note that here we choose the same basis as in Wood [**W**] (see also [**DV3**]). It is interesting to point out that the above table is closely related to the one that appears in Lemma 8.1. We also note that to get the basis in Baez ([**Ba**], p. 150) or in Yokota [**Y**] (the latter is implicitly used in [**BD**]), we should have the permutation as

$$e_1e_2e_3e_4e_5e_6e_7 \to e_6e_1e_5e_7e_2(-e_3)e_4|_{\mathrm{Baez}},$$
$$e_1e_2e_3e_4e_5e_6e_7 \to e_1e_2e_3e_4e_5(-e_6)e_7|_{\mathrm{Yokota}}.$$

In $\mathbb{O}$, the conjugate $\bar{x}$, the real part $R(x)$, an inner product (x, y), and the length $|x|$ are defined respectively by

$$\overline{\sum_{j=0}^{7} a_j e_j} = a_0 e_0 - \sum_{j=1}^{7} a_j e_j, \quad R\Big(\sum_{i=0}^{7} a_j e_j\Big) = a_0,$$

$$\Big(\sum_{j=0}^{7} a_j e_j, \sum_{j=0}^{7} b_j e_j\Big) = \sum_{j=0}^{7} a_j b_j = R(x\bar{y}), \qquad |x| = \sqrt{(x,x)} = \sqrt{x\bar{x}},$$

where $x = \sum_{j=0}^{7} a_j e_j$, $y = \sum_{j=0}^{7} b_j e_j$, and a_j, $b_j \in \mathbb{R}$. It is easily seen that $\overline{xy} = \bar{y}\bar{x}$ for $x, y \in \mathbb{O}$.

Let $\mathcal{M}_3(\mathbb{O})$ be the vector space of all 3×3 matrices with entries in $\mathbb{O}$ and $\mathfrak{y}_3(\mathbb{O})$ be the subset of all Hermitian matrices with entries in $\mathbb{O}$:

$$\mathfrak{y}_3(\mathbb{O}) = \{X \in \mathcal{M}_3(\mathbb{O}) \,|\, X^* = X\},$$

where $X^* = {}^t\bar{X}$ denotes the conjugate transpose of X. Any element $X \in \mathfrak{y}_3(\mathbb{O})$ is of the form

$$X = X(\xi, \eta) = \begin{bmatrix} \xi_1 & \eta_3 & \bar{\eta}_2 \\ \bar{\eta}_3 & \xi_2 & \eta_1 \\ \eta_2 & \bar{\eta}_1 & \xi_3 \end{bmatrix}, \quad \xi_i \in \mathbb{R},\ \eta_i \in \mathbb{O}.$$

By identifying $X = X(\xi, \eta) \in \mathfrak{y}_3(\mathbb{O})$ with $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3) \in \mathbb{R}^{27}$, we see that $\mathfrak{y}_3(\mathbb{O}) \simeq \mathbb{R}^{27}$ is a 27-dimensional $\mathbb{R}$-vector space. In $\mathfrak{y}_3(\mathbb{O})$, the multiplication $X \circ Y$, called the Jordan multiplication, is defined by

$$X \circ Y = \tfrac{1}{2}(XY + YX).$$

$\mathfrak{y}_3(\mathbb{O})$ equipped with the product $\circ$ is a real Jordan algebra. In $\mathfrak{y}_3(\mathbb{O})$, we also define the trace $\mathrm{tr}(X)$ and a symmetric inner product (X, Y) respectively by

$$\mathrm{tr}(X) = \xi_1 + \xi_2 + \xi_3; \qquad (X, Y) = \mathrm{tr}(X \circ Y).$$

Moreover, in $\mathfrak{y}_3(\mathbb{O})$ there is a symmetric cross product $X \times Y$, called the Freudenthal multiplication, defined by

$$X \times Y = \tfrac{1}{2}\Big[2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(X \circ Y))I\Big]$$

(where I is the 3×3 unit matrix) and a totally symmetric trilinear form (X, Y, Z) by

$$(X, Y, Z) = (X \times Y, Z) = (X, Y \times Z).$$

Despite noncommutativity and nonassociativity, the determinant of a matrix in $\mathfrak{y}_3(\mathbb{O})$ can be well defined by:

$$\det X = \tfrac{1}{3}(X, X, X).$$

For $X = X(\xi, \eta) \in \mathfrak{y}_3(\mathbb{O})$, noting that $X \circ X = X^2$, $X \circ X \circ X = X^3$, and

$$R(\eta_1(\eta_2\eta_3)) = R(\eta_2(\eta_3\eta_1)) = R(\eta_3(\eta_1\eta_2)) \ \ (= R(\eta_1\eta_2\eta_3)),$$

we have the calculation

$$\begin{aligned}\det X &= \tfrac{1}{3}\mathrm{tr}(X^3) - \tfrac{1}{2}\mathrm{tr}(X)\,\mathrm{tr}(X^2) + \tfrac{1}{6}(\mathrm{tr}(X))^3 \\ &= \xi_1\xi_2\xi_3 - \xi_1\eta_1\bar{\eta}_1 - \xi_2\eta_2\bar{\eta}_2 - \xi_3\eta_3\bar{\eta}_3 + 2R(\eta_1\eta_2\eta_3).\end{aligned} \tag{8.52}$$

This shows that the determinant is invariant under all automorphisms of $\mathfrak{h}_3(\mathbb{O})$. However, as stated in Baez ([**Ba**], p. 182) and Yokota ([**Y**], section 3.15), the determinant is invariant under an even bigger group of linear transformations, which is a 78-dimensional non-compact real form of the exceptional Lie group $\mathbf{E}_6$. More precisely, the group of determinant-preserving linear transformations of $\mathfrak{h}_3(\mathbb{O})$ turns out to be a non-compact real form of $\mathbf{E}_6$. This real form is sometimes called $\mathbf{E}_{6(-26)}$, because its Killing form has signature -26. Hence we have

$$\mathbf{E}_{6(-26)} = \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))\,|\,\det(\alpha X) = \det(X)\},$$

where $\mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))$ denotes all $\mathbb{R}$-linear isomorphisms of $\mathfrak{h}_3(\mathbb{O})$.

Recall that $\mathbf{F}_4$ denotes the full automorphism group of the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$:

$$\mathbf{F}_4 = \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))\,|\,\alpha(X \circ Y) = \alpha X \circ \alpha Y\}.$$

Using lemma 2.2.4 of Yokota [**Y**], we conclude

$$\begin{aligned}\mathbf{F}_4 &= \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))\,|\,\det(\alpha X) = \det(X), \alpha I = I\} \\ &= \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))\,|\,\det(\alpha X) = \det(X), (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \mathbf{E}_{6(-26)}\,|\,\alpha I = I\}.\end{aligned}$$

Hence, we get the inclusion $\mathbf{F}_4 \to \mathbf{E}_{6(-26)}$ and that, within $\mathbf{E}_{6(-26)}$, $\mathbf{F}_4$ is the stabilizer of I in $\mathfrak{h}_3(\mathbb{O})$. Moreover, as a closed subgroup of the orthogonal group

$$O(27) = O(\mathfrak{h}_3(\mathbb{O})) = \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))\,|\,(\alpha X, \alpha Y) = (X, Y)\},$$

$\mathbf{F}_4$ is a compact Lie group. This shows that $\mathbf{F}_4$ is a compact subgroup of $\mathbf{E}_{6(-26)}$. From Baez ([**Ba**], p. 196), it is in fact maximal. It follows that the Killing form of the Lie algebra $\mathfrak{e}_{6(-26)}$ is negative definite on its 52-dimensional maximal compact Lie algebra $\mathfrak{f}_4$ and positive definite on the complementary 26-dimensional subspace, giving a signature of $26 - 52 = -26$.

Hence, using Yokota ([**Y**], theorem 3.15.1), we have a non-compact homogeneous space $\tilde{M}^{26} = \mathbf{E}_{6(-26)}/\mathbf{F}_4 \simeq \mathbb{R}^{26}$. To obtain $T_I\tilde{M}^{26}$, we now consider the decomposition of the Lie algebra $\mathfrak{e}_{6(-26)}$ of $\mathbf{E}_{6(-26)}$. From Baez ([**Ba**], p. 191, theorem 5), we have

$$\mathfrak{f}_4 = \mathfrak{d}er(\mathbb{O}) \oplus \{X \in \mathcal{M}_3(\mathbb{O})\,|\,X^* = -X, \mathrm{tr}(X) = 0\},$$

where $\mathfrak{d}er(\mathbb{O})$ is the derivations of the octonions and is the Lie algebra $\mathfrak{g}_2$ of the 14-dimensional automorphism group $\mathbf{G}_2$ of the octonions. It follows that

$$
T_I\tilde{M}^{26} \simeq \{X \in \mathcal{M}_3(\mathbb{O}) \,|\, X^* = X, \mathrm{tr}(X) = 0\} = \{X \in \mathfrak{y}_3(\mathbb{O}) \,|\, \mathrm{tr}(X) = 0\}. \tag{8.53}
$$

Then $\tilde{M}^{26}$ is locally isomorphic to (will be identified hereafter) the connected component $\mathcal{A}$ of I in

$$
\{X \in \mathcal{M}_3(\mathbb{O}) \,|\, X^* = X, \det(X) = 1\},
$$

which can be naturally immersed into $\mathfrak{y}_3(\mathbb{O}) \simeq \mathbb{R}^{27}$.

To choose local coordinates of $\tilde{M}^{26} = \mathbf{E}_{6(-26)}/\mathbf{F}_4 \simeq \mathcal{A}$ around the unit matrix, we look at $X = X(\xi, \eta) \in \mathfrak{y}_3(\mathbb{O})$ and choose 26 real numbers $u_1, \dots, u_{26}$ such that

$$
\begin{cases} \xi_1 = 1 + \frac{\sqrt{2}}{2}u_1 + \frac{\sqrt{6}}{2}u_2,\ \xi_2 = 1 + \frac{\sqrt{2}}{2}u_1 - \frac{\sqrt{6}}{2}u_2,\ \xi_3 = 1 - \sqrt{2}u_1, \\ \eta_1 = \frac{\sqrt{6}}{2}\sum\limits_{j=0}^{7} u_{3+j}e_j,\ \eta_2 = \frac{\sqrt{6}}{2}\sum\limits_{j=0}^{7} u_{11+j}e_j,\ \eta_3 = \frac{\sqrt{6}}{2}\sum\limits_{j=0}^{7} u_{19+j}e_j. \end{cases} \tag{8.54}
$$

For $u = (u_1, \dots, u_{26})$ around the origin $(0, 0, 0, \dots, 0)$, $\det X \neq 0$ and $\det^{-\frac{1}{3}}(X)X \in \mathcal{A}$. Write $g(u) = \det X(\xi, \eta)$; then using (8.52) and the octonion multiplication table, we have the computation

$$
\begin{aligned}
g(u) &= (1 + \tfrac{\sqrt{2}}{2}u_1 + \tfrac{\sqrt{6}}{2}u_2)(1 + \tfrac{\sqrt{2}}{2}u_1 - \tfrac{\sqrt{6}}{2}u_2)(1 - \sqrt{2}u_1) \\
&- \tfrac{3}{2}(1 - \sqrt{2}u_1)\sum_{j=19}^{26} u_j^2 - \tfrac{3}{2}(1 + \tfrac{\sqrt{2}}{2}u_1 + \tfrac{\sqrt{6}}{2}u_2)\sum_{j=3}^{10} u_j^2 \\
&- \tfrac{3}{2}(1 + \tfrac{\sqrt{2}}{2}u_1 - \tfrac{\sqrt{6}}{2}u_2)\sum_{j=11}^{18} u_j^2 \\
&+ \tfrac{3\sqrt{6}}{2}\Big[u_{19}(u_3u_{11} - u_4u_{12} - u_5u_{13} - u_6u_{14} - u_7u_{15} - u_8u_{16} - u_9u_{17} - u_{10}u_{18}) \\
&\quad - u_{20}(u_3u_{12} + u_4u_{11} + u_5u_{14} - u_6u_{13} + u_7u_{16} - u_8u_{15} - u_9u_{18} + u_{10}u_{17}) \\
&\quad - u_{21}(u_3u_{13} - u_4u_{14} + u_5u_{11} + u_6u_{12} + u_7u_{17} + u_8u_{18} - u_9u_{15} - u_{10}u_{16}) \\
&\quad - u_{22}(u_3u_{14} + u_4u_{13} - u_5u_{12} + u_6u_{11} + u_7u_{18} - u_8u_{17} + u_9u_{16} - u_{10}u_{15}) \\
&\quad - u_{23}(u_3u_{15} - u_4u_{16} - u_5u_{17} - u_6u_{18} + u_7u_{11} + u_8u_{12} + u_9u_{13} + u_{10}u_{14}) \\
&\quad - u_{24}(u_3u_{16} + u_4u_{15} - u_5u_{18} + u_6u_{17} - u_7u_{12} + u_8u_{11} - u_9u_{14} + u_{10}u_{13}) \\
&\quad - u_{25}(u_3u_{17} + u_4u_{18} + u_5u_{15} - u_6u_{16} - u_7u_{13} + u_8u_{14} + u_9u_{11} - u_{10}u_{12}) \\
&\quad - u_{26}(u_3u_{18} - u_4u_{17} + u_5u_{16} + u_6u_{15} - u_7u_{14} - u_8u_{13} + u_9u_{12} + u_{10}u_{11})\Big].
\end{aligned} \tag{8.55}
$$

Adopting the above notations and the local coordinates (here we change slightly the coordinates introduced by Birembaux and Djoric in order to obtain an orthonormal basis at $u = 0$), Birembaux and

Djoric [**BD**] defined the hypersurface $F:\ \mathbb{R}^{26}\to\mathbb{R}^{27}$ by

$$(8.56)\qquad \begin{aligned} y\mapsto F(u)=g^{-1/3}(u)\Big(1+\tfrac{\sqrt2}{2}u_1+\tfrac{\sqrt6}{2}u_2, 1+\tfrac{\sqrt2}{2}u_1-\tfrac{\sqrt6}{2}u_2,\\ 1-\sqrt2 u_1, \tfrac{\sqrt6}{2}u_3,\dots,\tfrac{\sqrt6}{2}u_{26}\Big).\end{aligned}$$

To calculate the affine invariants of the above hypersurface, which obviously gives a local embedding of $\mathcal{A}$ into $\mathbb{R}^{27}$, we set the notation: $\frac{\partial F}{\partial u_j}=F_j$, $\frac{\partial g}{\partial u_j}=g_j$, and $f_j\in\mathbb{R}^{27}$ is a point where only the j-th coordinate is 1 and the others are 0. Then we have $\det(F,F_1,\dots,F_{26})=-\frac{3^{27/2}}{2^{12}}g^{-9}(u)\neq 0$.

Let us take $F(u)$ as a local transversal vector field of the hypersurface. Then we decompose $F_{jk}=D_{F_j}F_k$ as

$$(8.57)\qquad F_{jk}(u)=h(F_j,F_k)(u)F(u)+\nabla_{F_j}F_k(u),$$

where ∇ is the induced connection and h is the second fundamental form (the affine metric).

From (8.56) we have the results:

$$(8.58)\qquad \begin{cases} F_1=-\frac13 g^{-1}(u)g_1(u)F(u)+\frac{\sqrt2}{2}g^{-1/3}(u)(f_1+f_2-2f_3),\\ F_2=-\frac13 g^{-1}(u)g_2(u)F(u)+\frac{\sqrt6}{2}g^{-1/3}(u)(f_1-f_2),\\ F_j=-\frac13 g^{-1}(u)g_j(u)F(u)+\frac{\sqrt6}{2}g^{-1/3}(u)f_{j+1},\quad j=3,\dots,26,\end{cases}$$

and

$$(8.59)\qquad \begin{aligned} F_{jk}(u)=\Big(\tfrac29 g^{-2}(u)g_j(u)g_k(u)-\tfrac13 g^{-1}(u)g_{jk}(u)\Big)F(u)\\ -\tfrac13 g^{-1}(u)\big[g_j(u)F_k(u)+g_k(u)F_j(u)\big],\quad j,k=1,2,\dots,26.\end{aligned}$$

Comparing with (8.57), we conclude that, for $j,k=1,\dots,26$,

$$(8.60)\qquad h(F_j,F_k)(u)=\tfrac29 g^{-2}(u)g_j(u)g_k(u)-\tfrac13 g^{-1}(u)g_{jk}(u),$$

$$(8.61)\qquad \nabla_{F_j}F_k(u)=-\tfrac13 g^{-1}(u)\big[g_j(u)F_k(u)+g_k(u)F_j(u)\big].$$

It then follows that

$$(8.62)\qquad \begin{aligned} &(\nabla h)(F_j,F_k,F_\ell)(u)\\ &=\tfrac{\partial}{\partial u_j}(h(F_k,F_\ell))(u)-h(\nabla_{F_j}F_k,F_\ell)(u)-h(F_k,\nabla_{F_j}F_\ell)(u)\\ &=\tfrac19 g^{-2}(u)\big[g_j(u)g_{k\ell}(u)+g_k(u)g_{j\ell}(u)+g_\ell(u)g_{jk}(u)\big]\\ &\quad-\tfrac{4}{27}g^{-3}(u)g_j(u)g_k(u)g_\ell(u)-\tfrac13 g^{-1}(u)g_{jk\ell}(u).\end{aligned}$$

Using (1.1), namely

$$(\nabla h)(F_j,F_k,F_\ell)(u)=-2h\big(K(F_k,F_\ell),F_j\big)(u),$$

we have the computation

$$K(F_k, F_\ell)(u) = -\tfrac{1}{2}\sum_{j,\,m=1}^{26}(\nabla h)(F_j, F_k, F_\ell)(u)h^{jm}(u)F_m(u), \tag{8.63}$$

where we set $h_{jk}(u) = h(F_j, F_k)(u)$ and denote by $(h^{jk}(u))$ the inverse of the matrix $(h_{jk}(u))$.

From (8.60), noting that $g_j(0) = 0$, $g_{jk}(0) = -3\delta_{jk}$ for all j, k, we easily get

$$h_{jk}(0) = h^{jk}(0) = \delta_{jk}, \quad \forall j,\ k, \tag{8.64}$$

that is, $\{F_j(0)\}_{1\le j\le 26}$ is an orthonormal basis of $T_I\tilde{M}$ with respect to the affine metric h.

Using (8.62), (8.63), and (8.64), we have the computation, at $u = 0$,

$$\begin{aligned} K(F_k, F_\ell)(0) &= -\tfrac{1}{2}\sum_{j=1}^{26}(\nabla h)(F_j, F_k, F_\ell)(0)F_j(0) \\ &= \tfrac{1}{6}\sum_{j=1}^{26} g_{jk\ell}(0)F_j(0). \end{aligned} \tag{8.65}$$

Using (8.55), we have the following calculations:

$$g_{111}(0) = -g_{221}(0) = -3\sqrt{2}, \quad g_{112}(0) = g_{222}(0) = 0;$$

$$g_{12j}(0) = g_{jjk}(0) = 0, \quad \text{if } 3 \le j, k \le 26,$$

$$g_{jj1}(0) = \begin{cases} -\frac{3\sqrt{2}}{2}, & \text{if } 3 \le j \le 18, \\ 3\sqrt{2}, & \text{if } 19 \le j \le 26, \end{cases}$$

$$g_{jj2}(0) = \begin{cases} -\frac{3\sqrt{6}}{2}, & \text{if } 3 \le j \le 10, \\ \frac{3\sqrt{6}}{2}, & \text{if } 11 \le j \le 18, \\ 0, & \text{if } 19 \le j \le 26. \end{cases}$$

Then by (8.65) we obtain

$$\begin{aligned} &K(F_1, F_1)(0) = -\tfrac{\sqrt{2}}{2}F_1(0),\ K(F_2, F_2)(0) = \tfrac{\sqrt{2}}{2}F_1(0), \\ &K(F_1, F_2)(0) = \tfrac{\sqrt{2}}{2}F_2(0), \end{aligned} \tag{8.66}$$

$$K(F_j, F_j)(0) = \begin{cases} -\frac{\sqrt{2}}{4}F_1(0) - \frac{\sqrt{6}}{4}F_2(0), & \text{if } 3 \le j \le 10, \\ -\frac{\sqrt{2}}{4}F_1(0) + \frac{\sqrt{6}}{4}F_2(0), & \text{if } 11 \le j \le 18, \\ \frac{\sqrt{2}}{2}F_1(0), & \text{if } 19 \le j \le 26. \end{cases} \tag{8.67}$$

Now, besides (8.66) and (8.67), we using (8.65) and (8.55) to write down the other components of the difference tensor:

$$K(F_1, F_j)(0) = \tfrac{1}{6}g_{1jj}(0)F_j(0) = \begin{cases} -\frac{\sqrt{2}}{4}F_j(0), & \text{if } 3 \le j \le 18, \\ \frac{\sqrt{2}}{2}F_j(0), & \text{if } 19 \le j \le 26; \end{cases} \tag{8.68}$$

$$K(F_2,F_j)(0)=\tfrac{1}{6}g_{2jj}(0)F_j(0)=\begin{cases}-\frac{\sqrt{6}}{4}F_j(0), & \text{if } 3\le j\le 10,\\ \frac{\sqrt{6}}{4}F_j(0), & \text{if } 11\le j\le 18,\\ 0, & \text{if } 19\le j\le 26;\end{cases}\tag{8.69}$$

$$\begin{aligned}&K(F_3,F_{11})(0)=\tfrac{\sqrt{6}}{4}F_{19}(0),\\ &K(F_3,F_j)(0)=-\tfrac{\sqrt{6}}{4}F_{j+8}(0),\ 12\le j\le 18;\end{aligned}\tag{8.70}$$

$$K(F_4,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{20}(0),\quad K(F_4,F_{12})(0)=-\tfrac{\sqrt{6}}{4}F_{19}(0),\tag{8.71}$$

$$K(F_4,F_{13})(0)=-\tfrac{\sqrt{6}}{4}F_{22}(0),\quad K(F_4,F_{14})(0)=\tfrac{\sqrt{6}}{4}F_{21}(0),\tag{8.72}$$

$$K(F_4,F_{15})(0)=-\tfrac{\sqrt{6}}{4}F_{24}(0),\quad K(F_4,F_{16})(0)=\tfrac{\sqrt{6}}{4}F_{23}(0),\tag{8.73}$$

$$K(F_4,F_{17})(0)=\tfrac{\sqrt{6}}{4}F_{26}(0),\quad K(F_4,F_{18})(0)=-\tfrac{\sqrt{6}}{4}F_{25}(0);\tag{8.74}$$

$$K(F_5,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{21}(0),\quad K(F_5,F_{12})(0)=\tfrac{\sqrt{6}}{4}F_{22}(0),\tag{8.75}$$

$$K(F_5,F_{13})(0)=-\tfrac{\sqrt{6}}{4}F_{19}(0),\quad K(F_5,F_{14})(0)=-\tfrac{\sqrt{6}}{4}F_{20}(0),\tag{8.76}$$

$$K(F_5,F_{15})(0)=-\tfrac{\sqrt{6}}{4}F_{25}(0),\quad K(F_5,F_{16})(0)=-\tfrac{\sqrt{6}}{4}F_{26}(0),\tag{8.77}$$

$$K(F_5,F_{17})(0)=\tfrac{\sqrt{6}}{4}F_{23}(0),\quad K(F_5,F_{18})(0)=\tfrac{\sqrt{6}}{4}F_{24}(0);\tag{8.78}$$

$$K(F_6,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{22}(0),\quad K(F_6,F_{12})(0)=-\tfrac{\sqrt{6}}{4}F_{21}(0),\tag{8.79}$$

$$K(F_6,F_{13})(0)=\tfrac{\sqrt{6}}{4}F_{20}(0),\quad K(F_6,F_{14})(0)=-\tfrac{\sqrt{6}}{4}F_{19}(0),\tag{8.80}$$

$$K(F_6,F_{15})(0)=-\tfrac{\sqrt{6}}{4}F_{26}(0),\quad K(F_6,F_{16})(0)=\tfrac{\sqrt{6}}{4}F_{25}(0),\tag{8.81}$$

$$K(F_6,F_{17})(0)=-\tfrac{\sqrt{6}}{4}F_{24}(0),\quad K(F_6,F_{18})(0)=\tfrac{\sqrt{6}}{4}F_{23}(0);\tag{8.82}$$

$$K(F_7,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{23}(0),\quad K(F_7,F_{12})(0)=\tfrac{\sqrt{6}}{4}F_{24}(0),\tag{8.83}$$

$$K(F_7,F_{13})(0)=\tfrac{\sqrt{6}}{4}F_{25}(0),\quad K(F_7,F_{14})(0)=\tfrac{\sqrt{6}}{4}F_{26}(0),\tag{8.84}$$

$$K(F_7,F_{15})(0)=-\tfrac{\sqrt{6}}{4}F_{19}(0),\quad K(F_7,F_{16})(0)=-\tfrac{\sqrt{6}}{4}F_{20}(0),\tag{8.85}$$

$$K(F_7,F_{17})(0)=-\tfrac{\sqrt{6}}{4}F_{21}(0),\quad K(F_7,F_{18})(0)=-\tfrac{\sqrt{6}}{4}F_{22}(0);\tag{8.86}$$

$$K(F_8,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{24}(0),\quad K(F_8,F_{12})(0)=-\tfrac{\sqrt{6}}{4}F_{23}(0),\tag{8.87}$$

$$K(F_8,F_{13})(0)=\tfrac{\sqrt{6}}{4}F_{26}(0),\quad K(F_8,F_{14})(0)=-\tfrac{\sqrt{6}}{4}F_{25}(0),\tag{8.88}$$

$$K(F_8,F_{15})(0)=\tfrac{\sqrt{6}}{4}F_{20}(0),\quad K(F_8,F_{16})(0)=-\tfrac{\sqrt{6}}{4}F_{19}(0),\tag{8.89}$$

$$K(F_8,F_{17})(0)=\tfrac{\sqrt{6}}{4}F_{22}(0),\quad K(F_8,F_{18})(0)=-\tfrac{\sqrt{6}}{4}F_{21}(0);\tag{8.90}$$

$$K(F_9,F_{11})(0)=-\tfrac{\sqrt{6}}{4}F_{25}(0),\quad K(F_9,F_{12})(0)=-\tfrac{\sqrt{6}}{4}F_{26}(0),\tag{8.91}$$

(8.92) $$K(F_9, F_{13})(0) = -\tfrac{\sqrt{6}}{4}F_{23}(0), \quad K(F_9, F_{14})(0) = \tfrac{\sqrt{6}}{4}F_{24}(0),$$

(8.93) $$K(F_9, F_{15})(0) = \tfrac{\sqrt{6}}{4}F_{21}(0), \quad K(F_9, F_{16})(0) = -\tfrac{\sqrt{6}}{4}F_{22}(0),$$

(8.94) $$K(F_9, F_{17})(0) = -\tfrac{\sqrt{6}}{4}F_{19}(0), \quad K(F_9, F_{18})(0) = \tfrac{\sqrt{6}}{4}F_{20}(0);$$

(8.95) $$K(F_{10}, F_{11})(0) = -\tfrac{\sqrt{6}}{4}F_{26}(0), \quad K(F_{10}, F_{12})(0) = \tfrac{\sqrt{6}}{4}F_{25}(0),$$

(8.96) $$K(F_{10}, F_{13})(0) = -\tfrac{\sqrt{6}}{4}F_{24}(0), \quad K(F_{10}, F_{14})(0) = \tfrac{\sqrt{6}}{4}F_{23}(0),$$

(8.97) $$K(F_{10}, F_{15})(0) = \tfrac{\sqrt{6}}{4}F_{22}(0), \quad K(F_{10}, F_{16})(0) = \tfrac{\sqrt{6}}{4}F_{21}(0),$$

(8.98) $$K(F_{10}, F_{17})(0) = -\tfrac{\sqrt{6}}{4}F_{20}(0), \quad K(F_{10}, F_{18})(0) = -\tfrac{\sqrt{6}}{4}F_{19}(0).$$

Finally, if we denote $\mathcal{A} = \{3, \dots, 10\}, \mathcal{B} = \{11, \dots, 18\}, \mathcal{C} = \{19, \dots, 26\}$, then we have

(8.99) $$\begin{cases} K(F_j, F_k)(0) = 0, \\ \text{if } j \neq k \text{ and either } j, k \in \mathcal{A} \text{ or } j, k \in \mathcal{B} \text{ or } j, k \in \mathcal{C}. \end{cases}$$

Moreover, we note that in all the above equations from (8.70) up to (8.98), the three subindexes can be arbitrarily permutated; for example, from the first equation of (8.70), we also have

$$K(F_3, F_{19})(0) = \tfrac{\sqrt{6}}{4}F_{11}(0), \quad K(F_{11}, F_{19})(0) = \tfrac{\sqrt{6}}{4}F_3(0).$$

It then follows that all components of the difference tensor are described.

From (8.66) and (8.67), we can see immediately that $\mathrm{Trace}_h K = 0$ at $u = 0$. Consequently, $F(0)$ is the affine normal at 0 and by homogeneity of the hypersurface we conclude that $\tilde{M}$ is a locally strongly convex affine hypersphere. Moreover, as has been straightforwardly calculated by Birembaux and Djoric [**BD**], the hypersurfaces are in fact isotropic; namely, in the present case we have

(8.100) $$h(K(v, v), K(v, v)) = \tfrac{1}{2}h(v, v)h(v, v)$$

for all $v = \sum_j v_j F_j$. According to proposition 4 of Birembaux and Djoric [**BD**], $\tilde{M}$ is an affine hypersphere with affine mean curvature $\lambda = -1$ and its cubic form is parallel with respect to Levi-Civita connection of the affine metric h.

To simplify the above complicated formulas of calculating the difference tensor at $u = 0$, we have the crucial observation that similar results still hold as for the standard embeddings $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m) \hookrightarrow \mathbb{R}^{m(m+1)/2}$,

$\mathbf{SL}(m,\mathbb{C})/\mathbf{SU}(m) \hookrightarrow \mathbb{R}^{m^2}$, and $\mathbf{SU}^*(2m)/\mathbf{Sp}(m) \hookrightarrow \mathbb{R}^{2m^2-m}$. To present the details, we use (8.58) and the identification $\mathfrak{y}_3(\mathbb{O}) \simeq \mathbb{R}^{27}$ given by

$$\mathfrak{y}_3(\mathbb{O}) \ni \begin{bmatrix} \xi_1 & \eta_3 & \bar{\eta}_2 \\ \bar{\eta}_3 & \xi_2 & \eta_1 \\ \eta_2 & \bar{\eta}_1 & \xi_3 \end{bmatrix} \mapsto (\xi_1, \xi_2, \xi_3, u_3, \ldots, u_{26}) \in \mathbb{R}^{27},$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$; $\eta_1 = \sum_{j=0}^{7} u_{3+j} e_j$, $\eta_2 = \sum_{j=0}^{7} u_{11+j} e_j$, $\eta_3 = \sum_{j=0}^{7} u_{19+j} e_j \in \mathbb{O}$.

Let E_{jk} denote the 3×3 matrix which has (j,k) entry 1 and all other entries 0. With the identification we notice the following relations, and then we set

$$F_1(0) = \tfrac{\sqrt{2}}{2}(f_1 + f_2 - 2f_3) \simeq \tfrac{\sqrt{2}}{2}(E_{11} + E_{22} - 2E_{33}) = \tilde{e}_1, \tag{8.101}$$

$$F_2(0) = \tfrac{\sqrt{6}}{2}(f_1 - f_2) \simeq \tfrac{\sqrt{6}}{2}(E_{11} - E_{22}) = -\tilde{w}_0, \tag{8.102}$$

$$F_3(0) = \tfrac{\sqrt{6}}{2} f_4 \simeq \tfrac{\sqrt{6}}{2}(E_{23} + E_{32}) = -\tilde{v}_1, \tag{8.103}$$

$$F_{j+3}(0) = \tfrac{\sqrt{6}}{2} f_{j+4} \simeq \tfrac{\sqrt{6}}{2}(E_{23} - E_{32}) e_j = \tilde{x}_j, \quad 1 \le j \le 7, \tag{8.104}$$

$$F_{11}(0) = \tfrac{\sqrt{6}}{2} f_{12} \simeq \tfrac{\sqrt{6}}{2}(E_{13} + E_{31}) = -\tilde{v}_2, \tag{8.105}$$

$$F_{j+11}(0) = \tfrac{\sqrt{6}}{2} f_{j+12} \simeq \tfrac{\sqrt{6}}{2}(E_{31} - E_{13}) e_j = -\tilde{y}_j, \quad 1 \le j \le 7, \tag{8.106}$$

$$F_{19}(0) = \tfrac{\sqrt{6}}{2} f_{20} \simeq \tfrac{\sqrt{6}}{2}(E_{12} + E_{21}) = \tilde{w}_1, \tag{8.107}$$

$$F_{j+19}(0) = \tfrac{\sqrt{6}}{2} f_{j+20} \simeq \tfrac{\sqrt{6}}{2}(E_{12} - E_{21}) e_j = -\tilde{w}_{j+1}, \quad 1 \le j \le 7. \tag{8.108}$$

Then direct calculations show that, equivalently, the standard embedding $F: \mathbf{E}_{6(-26)}/\mathbf{F}_4 \to \mathbb{R}^{27}$ with affine normal F has a Blaschke structure, at I, that can be expressed algebraically in terms of the Lie algebra as follows:

$$\begin{cases} K(X,Y) = \frac{1}{2}(XY + YX) - \frac{1}{3}\,\mathrm{tr}\,(XY) I, \\ h(X,Y) = \frac{1}{3}\,\mathrm{tr}\,(XY), \qquad S = -Id. \end{cases} \tag{8.109}$$

Remark 8.1. Having the above expressions, the isotropic condition (8.100) is then changed equivalently to the relation

$$\big(\mathrm{tr}(X^2)\big)^2 = 2\,\mathrm{tr}(X^2 \circ X^2) \tag{8.110}$$

for all $X \in \mathfrak{y}_3(\mathbb{O})$ with $\mathrm{tr}(X) = 0$. By using properties of the octonions multiplications, this can be verified easily by direct computations.

Completion of the proof of Theorem 8.1 for Case (iii).

From the definition (8.101)–(8.108), using (8.109) we can verify that all the relations from (8.1^1) to (8.1^8), and (8.41), still hold for the above $\tilde{}$ notations with $\tau = \frac{3}{8}$ or equivalently $\lambda_1 = \frac{1}{\sqrt{2}}$, i.e., $\lambda = -1$. Since the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$ is of parallel cubic form, if we identify the following orthonormal basis of $T_I\tilde{M}^{26}$:

$$\left\{\tilde{e}_1;\ \tilde{v}_1, \tilde{v}_2;\ \tilde{x}_j, \tilde{y}_j \,|_{\,1\le j\le 7};\ \tilde{w}_j \,|_{0\le j\le 8}\right\}$$

defined in (8.101)–(8.108) with the basis (8.51) of $T_{x_0}M^{26}$, then applying Lemma 2.4 and Lemma 4.4, we can verify that the difference tensor of the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$ is exactly the same as that of $M^{26} \to \mathbb{R}^{27}$ corresponding to $\lambda_1 = \frac{1}{\sqrt{2}}$, or equivalently $\lambda = -1$.

Now for the locally strongly convex $\mathfrak{C}_{17}$ affine hypersphere $M^{26} \to \mathbb{R}^{27}$ with $\mathfrak{p} = 7$, we see from the above discussion that, by applying a homothetic transformation to make $\lambda = -1$, if necessary, M^{26} and the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$ has affine metric h and cubic form C with identically the same affine invariant properties. According to Cartan's lemma and the fundamental uniqueness theorem of affine differential geometry, we obtain that M^{26} and the standard embedding $\mathbf{E}_{6(-26)}/\mathbf{F}_4 \hookrightarrow \mathbb{R}^{27}$ are locally affine equivalent.

This completes the proof of Theorem 8.1. q.e.d.

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Department of Mathematics
Zhengzhou University
Zhengzhou 450052
People's Republic of China

E-mail address: huzj@zzu.edu.cn

Department of Mathematical Sciences
Tsinghua University
Beijing 100084
People's Republic of China

E-mail address: hli@math.tsinghua.edu.cn

Univ. Lille Nord de France
F-59000 Lille France
UVHC, LAMAV, F-59313 Valenciennes, France
and
Katholieke Universiteit Leuven
Departement Wiskunde
BE-3001 Leuven, Belgium
E-mail address: luc.vrancken@univ-valenciennes.fr