J. DIFFERENTIAL GEOMETRY
86 (2010) 303-354

FUNCTORIAL RELATIONSHIPS BETWEEN $QH^*(G/B)$ AND $QH^*(G/P)$

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Abstract

We give a natural filtration $\mathcal{F}$ on $QH^*(G/B)$, which respects the quantum product structure. Its associated graded algebra $Gr^{\mathcal{F}}(QH^*(G/B))$ is isomorphic to the tensor product of $QH^*(G/P)$ and a corresponding graded algebra of $QH^*(P/B)$ after localization. When the quantum parameter goes to zero, this specializes to the filtration on $H^*(G/B)$ from the Leray spectral sequence associated to the fibration $P/B \to G/B \longrightarrow G/P$.

1. Introduction

Let G be a simply connected complex simple Lie group, B be a Borel subgroup, and $P \supset B$ be a parabolic subgroup of G. The natural fibration $P/B \to G/B \longrightarrow G/P$ of homogeneous varieties gives rise to a $\mathbb{Z}^2$-filtration $\mathcal{F}$ on $H^*(G/B)$ over $\mathbb{Q}$ (or $\mathbb{C}$) such that $Gr^{\mathcal{F}}(H^*(G/B)) \cong H^*(P/B) \otimes H^*(G/P)$ as graded algebras by the Leray-Serre spectral sequence. Given another parabolic subgroup P' with $B \subset P' \subset P$, we obtain the corresponding natural fibration $P'/B \to P/B \longrightarrow P/P'$. Combining it with the former one, we obtain a $\mathbb{Z}^3$-filtration on $H^*(G/B)$. We can continue this procedure to obtain a (maximal) $\mathbb{Z}^{r+1}$-filtration.

In the present paper, we study the small quantum cohomology rings $QH^*(G/P)$'s of homogeneous varieties G/P's, which are deformations of the ring structures on $H^*(G/P)$'s by incorporating genus zero 3-pointed Gromov-Witten invariants of G/P's into the cup product. We show the "*functorial relationships*" between $QH^*(G/B)$ and $QH^*(G/P)$ in the sense that the $\mathbb{Z}^{r+1}$-filtration on $H^*(G/B)$ can be generalized to give a $\mathbb{Z}^{r+1}$-filtration on $QH^*(G/B)$ and there exist canonical maps between quantum cohomologies, in analog with the classical ones. We begin with a toy example to illustrate our results.

Example 1.1. When $G = SL(3, \mathbb{C})$, $G/B = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim_{\mathbb{C}} V_i = 1, i = 1, 2\} =: F\ell_3$ is a complete flag variety. Given a maximal parabolic subgroup $P \supset B$, we have $P/B = \mathbb{P}^1$ and $G/P = \mathbb{P}^2$ together with a natural fibration $\mathbb{P}^1 \overset{i}{\hookrightarrow} F\ell_3 \overset{\pi}{\longrightarrow} \mathbb{P}^2$. The quantum cohomology ring

Received 3/1/2010.

$QH^*(G/B)$ has a basis consisting of Schubert classes σ^w's over $\mathbb{Q}[q_1, q_2]$, indexed by the Weyl group $W = S_3 = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$. To obtain the $\mathbb{Z}^2$-filtration $\mathcal{F}$ on $QH^*(G/B)$, we need a deformation gr of the classical grading map which satisfies $gr(q_1^a q_2^b \sigma^w) = agr(q_1) + bgr(q_2) + gr(\sigma^w)$. In this example, gr is given explicitly by Table 1.

Table 1. $gr(q_1^a q_2^b \sigma^w) = (i, j)$ with $-2 \le i \le 4, 0 \le j \le 6$

4	q_1^2	$q_1^2\sigma^{s_2}$	$q_1^2\sigma^{s_1s_2}$	$q_1^2q_2\sigma^{s_1}$	$q_1^2q_2\sigma^{s_2s_1}$	$q_1^2q_2\sigma^{s_1s_2s_1}$	$q_1^3q_2^2$
3	$q_1\sigma^{s_1}$	$q_1\sigma^{s_2s_1}$	$q_1\sigma^{s_1s_2s_1}$	$q_1^2q_2$	$q_1^2q_2\sigma^{s_2}$	$q_1^2q_2\sigma^{s_1s_2}$	$q_1^2q_2^2\sigma^{s_1}$
2	q_1	$q_1\sigma^{s_2}$	$q_1\sigma^{s_1s_2}$	$q_1q_2\sigma^{s_1}$	$q_1q_2\sigma^{s_2s_1}$	$q_1q_2\sigma^{s_1s_2s_1}$	$q_1^2q_2^2$
1	σ^{s_1}	$\sigma^{s_2s_1}$	$\sigma^{s_1s_2s_1}$	q_1q_2	$q_1q_2\sigma^{s_2}$	$q_1q_2\sigma^{s_1s_2}$	$q_1q_2^2\sigma^{s_1}$
0	1	σ^{s_2}	$\sigma^{s_1s_2}$	$q_2\sigma^{s_1}$	$q_2\sigma^{s_2s_1}$	$q_2\sigma^{s_1s_2s_1}$	$q_1q_2^2$
−1	0	0	0	q_2	$q_2\sigma^{s_2}$	$q_2\sigma^{s_1s_2}$	$q_2^2\sigma^{s_1}$
−2	0	0	0	0	0	0	q_2^2
i / j	0	1	2	3	4	5	6

This determines a $\mathbb{Z}^2$-filtration $\mathcal{F} = \{F_{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{Z}^2}$ on $QH^*(G/B)$. The main point is that this filtration respects the quantum multiplication, i.e., $F_{\mathbf{c}}F_{\mathbf{d}} \subset F_{\mathbf{c}+\mathbf{d}}$. Indeed, this can be easily checked with the following well known quantum products for $F\ell_3$:

$$\begin{aligned}
\sigma^{s_1} \star \sigma^{s_1} &= \sigma^{s_2s_1} + q_1, & \sigma^{s_1} \star \sigma^{s_1s_2} &= \sigma^{s_1s_2s_1},\\
\sigma^{s_1} \star \sigma^{s_2s_1} &= q_1\sigma^{s_2}, & \sigma^{s_2} \star \sigma^{s_2} &= \sigma^{s_1s_2} + q_2,\\
\sigma^{s_2} \star \sigma^{s_2s_1} &= \sigma^{s_1s_2s_1}, & \sigma^{s_2} \star \sigma^{s_1s_2} &= q_2\sigma^{s_1},\\
\sigma^{s_1s_2} \star \sigma^{s_1s_2s_1} &= q_1q_2\sigma^{s_2}, & \sigma^{s_1s_2} \star \sigma^{s_1s_2} &= q_2\sigma^{s_2s_1},\\
\sigma^{s_1} \star \sigma^{s_1s_2s_1} &= q_1\sigma^{s_1s_2} + q_1q_2, & \sigma^{s_2s_1} \star \sigma^{s_1s_2s_1} &= q_1q_2\sigma^{s_1},\\
\sigma^{s_2s_1} \star \sigma^{s_2s_1} &= q_1\sigma^{s_1s_2}, & \sigma^{s_2} \star \sigma^{s_1s_2s_1} &= q_2\sigma^{s_2s_1} + q_1q_2,\\
\sigma^{s_1} \star \sigma^{s_2} &= \sigma^{s_1s_2} + \sigma^{s_2s_1}, & \sigma^{s_1s_2} \star \sigma^{s_2s_1} &= q_1q_2,\\
\sigma^{s_1s_2s_1} \star \sigma^{s_1s_2s_1} &= q_1q_2\, \sigma^{s_1} \star \sigma^{s_2}.
\end{aligned}$$

This $\mathbb{Z}^2$-filtration of the algebra structure on $QH^*(G/B)$ is a **q**-deformation of the classical one on $H^*(G/B)$, which comes from the Leray spectral sequence. Due to the existence of such a filtration, we can easily check that there are **algebra** isomorphisms $\bar{\varphi} : QH^*(G/B)/\mathcal{I} \longrightarrow$

$QH^*(P/B)$ and $\bar{\psi} : QH^*(G/P) \longrightarrow \mathcal{A}/\mathcal{J}$, where $\mathcal{A} := \bigcup_{j\geq 0} F_{(0,j)}$ is a subalgebra of $QH^*(G/B)$, $\mathcal{J} := F_{(0,-1)}$ is an ideal of $\mathcal{A}$, and $\mathcal{I}$ is the ideal in $QH^*(G/B)$ spanned by those $q_1^a q_2^b \sigma^w$'s with their gradings (d_1, d_2) satisfying $d_2 > 0$. Here $\bar{\varphi}$ sends $q_1^a q_2^b \sigma^w + \mathcal{I}$ to y^j where $q_1^a q_2^b \sigma^w$ is the (unique) one among such expressions with its grading equal to $(j, 0)$, and $\bar{\psi}$ sends x^j to the (unique) $q_1^a q_2^b \sigma^w \in QH^*(G/B)$ whose grading equals $(0, j)$, in which we have taken the well known isomorphisms $QH^*(P/B) \cong \mathbb{Q}[y]$ and $QH^*(G/P) \cong \mathbb{Q}[x]$. In particular, $QH^*(G/P)$ is the quotient of the subalgebra $\mathcal{A}$ generated by $\{q_2\sigma^{s_1}, \sigma^{s_2}, q_1 q_2^2, q_2\}$ by the ideal $\mathcal{J} = q_2\mathcal{A}$. (We remark that in this case $QH^*(G/B)$ itself is generated by $\{\sigma^{s_1}, \sigma^{s_2}, q_1, q_2\}$.)

These algebra isomorphisms generalize the classical ones in an obvious way, namely, $\mathcal{A}$, $\mathcal{J}$, and $\mathcal{I}$ are $\mathbf{q}$-deformations of $A := \pi^*(H^*(G/P))$, $J = 0$, and $I = \mathbb{Q}\{\sigma^{s_2}, \sigma^{s_1 s_2}, \sigma^{s_2 s_1}, \sigma^{s_1 s_2 s_1}\}$, respectively.

All the above descriptions for $G = SL(3, \mathbb{C})$ will be generalized to arbitrary complex semi-simple Lie groups. For simplicity, we assume P/B is irreducible. (Note that any homogeneous variety splits into a direct product of irreducible ones.) All the results can be easily generalized for reducible P/B's, and we will describe such generalizations in section 5. Note that P/B is again a complete flag variety, isomorphic to G'/B' for some other complex simple Lie group G'. Then we denote by r the rank of G', which depends only on P/B. Since we exclude the trivial cases, namely, P equals B or G, we always have $r = 1$ for $G = SL(3, \mathbb{C})$.

In general, we consider a special iterated fibration $\{P_{j-1}/P_0 \to P_j/P_0 \longrightarrow P_j/P_{j-1}\}_{j=2}^{r+1}$ in which P_j's are parabolic subgroups with $B = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r = P \subsetneq G = P_{r+1}$. Consequently, we obtain a canonical $\mathbb{Z}^{r+1}$-filtration on $H^*(G/B)$. Note that $QH^*(G/B)$ also has a natural basis of Schubert classes σ^w's over $\mathbb{Q}[\mathbf{q}]$. As we will see in section 2.2, there exists a grading map gr giving gradings $gr(q_\lambda\sigma^w) \in \mathbb{Z}^{r+1}$ for the ($\mathbb{Q}$-)basis $q_\lambda\sigma^w$'s. The Peterson-Woodward comparison formula in [**32**] plays a key role in defining gr. It is the only known formula that characterizes the relations of genus zero 3-pointed Gromov-Witten invariants between G/B and a general G/P explicitly. gr defines a $\mathbb{Z}^{r+1}$-filtration $\mathcal{F} = \{F_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^{r+1}}$ of subspaces in $QH^*(G/B)$, generalizing Example 1.1. The next theorem says that $\mathcal{F}$ respects the quantum product structure.

Theorem 1.2. *$QH^*(G/B)$ is a $\mathbb{Z}^{r+1}$-filtered algebra with filtration $\mathcal{F}$.*

We can obtain several important consequences as below.

Theorem 1.3. *The vector subspace $\mathcal{I}$, spanned by those $q_\lambda\sigma^w$'s with their gradings $(d_1, \cdots, d_{r+1})$ satisfying $d_{r+1} > 0$, is an ideal of $QH^*(G/B)$. Furthermore, there is a canonical algebra isomorphism*

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\simeq} QH^*(P/B).$$

Since $QH^*(G/B)$ has a $\mathbb{Z}^{r+1}$-filtration $\mathcal{F}$, we obtain an associated $\mathbb{Z}^{r+1}$-graded algebra $Gr^{\mathcal{F}}(QH^*(G/B)) = \bigoplus_{\mathbf{a}\in\mathbb{Z}^{r+1}} Gr^{\mathcal{F}}_{\mathbf{a}}$, where $Gr^{\mathcal{F}}_{\mathbf{a}} := F_{\mathbf{a}}\big/\cup_{\mathbf{b}<\mathbf{a}} F_{\mathbf{b}}$. For each j, we denote $Gr^{\mathcal{F}}_{(j)}(QH^*(G/B)) := \bigoplus_{i\in\mathbb{Z}} Gr^{\mathcal{F}}_{i\mathbf{e}_j}$.

Theorem 1.4. *For each $1 \leq j \leq r$, there exists a canonical algebra isomorphism*

$$\Psi_j : QH^*(P_j/P_{j-1}) \xrightarrow{\simeq} Gr^{\mathcal{F}}_{(j)}(QH^*(G/B)).$$

Furthermore, if $P/B \cong F\ell_{r+1}$, then there exists a canonical algebra isomorphism

$$\Psi_{r+1} : QH^*(G/P) \xrightarrow{\simeq} Gr^{\mathcal{F}}_{(r+1)}(QH^*(G/B)).$$

As a consequence, we have the following results for any G.

Theorem 1.5. *Suppose $P/B \cong F\ell_{r+1}$. Then there exists a subalgebra $\mathcal{A}$ of $QH^*(G/B)$ together with an ideal $\mathcal{J}$ of $\mathcal{A}$, such that $QH^*(G/P)$ is canonically isomorphic to $\mathcal{A}/\mathcal{J}$ as algebras.*

Theorem 1.6. *Suppose $P/B \cong F\ell_{r+1}$. Then as graded algebras $Gr^{\mathcal{F}}(QH^*(G/B))$ is isomorphic to $QH^*(\mathbb{P}^1)\otimes\cdots\otimes QH^*(\mathbb{P}^r)\otimes QH^*(G/P)$ after localization.*

We should point out that the requirement "$P/B \cong F\ell_{r+1}$" in Theorem 1.5 and Theorem 1.6 is not a strong assumption, because both of theorems can be easily generalized to the case "P/B is isomorphic to a product of $F\ell_k$'s" (see section 5). As a consequence, all G/P's for G being of A-type or G_2-type satisfy this assumption. Furthermore, for each remaining type, more than half of the homogeneous varieties G/P's also satisfy this. (We could also show Theorem 1.5 holds for any G/P with $G = Sp(2n, \mathbb{C})$.)

As we saw in Example 1.1, the gradings of elements in $QH^*(F\ell_3)$ only form a proper sub-semigroup S of $\mathbb{Z}^2$, which looks like stairs, so that the $\mathbb{Z}^2$-filtration comes from an S-filtration. In general, the $\mathbb{Z}^{r+1}$-filtration comes from a similar filtration. For this reason, we need localization to obtain the analog of graded-algebra isomorphism (in Theorem 1.6). In section 4, we will restate theorems 1.4, 1.5, and 1.6 more concretely. As we will see later, all the relevant maps generalize the classical ones in an obvious way, as in Example 1.1.

Our results relate the quantum cohomologies of the total space and the base space of the fibration $P/B \to G/B \longrightarrow G/P$. Similar structures occur when one studies the relationships of J-functions between an abelian quotient and a nonabelian quotient. Such relations were studied by Bertram, Ciocan-Fontanine, and Kim in [**3**] and [**4**]. (See also [**33**].) There were also relevant studies by Liu-Liu-Yau [**23**] and Paksoy [**27**] by using mirror principle [**22**] .

Let us mention two more important problems on the study of $QH^*(G/P)$, for which our theorems may also be helpful. One can see the excellent survey [**9**] and references therein for more details on the developments. As mentioned before, the (small) quantum cohomology ring $QH^*(G/P)$ has a basis of Schubert classes σ^w's over $\mathbb{Q}[\mathbf{q}]$. In order to understand $QH^*(G/P)$, one would like to have (i) a (good) presentation of the ring structure on $QH^*(G/P)$ and (ii) a (nice) formula (or algorithm) for the quantum Schubert structure constants $N_{u,v}^{w,\lambda_P}$'s in the quantum product $\sigma^u \star \sigma^v = \sum_{w,\lambda_P} N_{u,v}^{w,\lambda_P} q_{\lambda_P} \sigma^w$. For classical cohomology $H^*(G/P)$, these natural and important problems have been solved in [**5**] for (i) and in [**17**] and [**7**] for (ii). However, for quantum cohomology $QH^*(G/P)$, the answer to (i) is only known in certain cases—for instance, when G is of A-type (see [**15**], [**1**]) or $P = B$ is a Borel subgroup [**16**]. For problem (ii), there were early studies for a few cases, including complex Grassmannians (see the survey [**9**]) and complete flag varieties of A-type [**8**], besides the quantum Chevalley formula [**11**], which works for all cases. Recently, Mihalcea [**26**] has given an algorithm, and the authors ([**20**], [**21**]) have given a combinatorial formula for these structure constants.

All these problems were discussed in the unpublished work [**28**] by Dale Peterson. In [**32**], Woodward proves a comparison formula of Peterson. The Peterson-Woodward comparison formula explicitly characterizes the relations of the quantum Schubert structure constants between $QH^*(G/P)$ and $QH^*(G/B)$. However, it does not tell us the relations of the algebra structures between them. Along Peterson's approach, Lam and Shimozono [**18**] show that the torus-equivariant extension of $QH^*(G/P)$ is isomorphic to a quotient of the torus-equivariant homology of a based loop group after localization. In [**29**], K. Rietsch discusses the relationships between Peterson's work and mirror symmetry. In [**28**], Peterson had also claimed there was an analogous isomorphism for the (un-iterated) fibration $P/B \to G/B \longrightarrow G/P$ in terms of torus-equivariant homology of based loop groups after localization. We were motivated by his claim and the results by Woodward and Lam-Shimozono. We succeeded in obtaining natural generalizations of the classical isomorphisms. It is interesting to compare our results with Peterson's claim. It is also interesting to compare our Theorem 1.5 with Theorem 10.16 of [**18**] by Lam and Shimozono. As commented by Thomas Lam, our results should be related to the discussions in section 10.4 of [**18**].

We hope our results could be used to solve problem (i) by combining with Kim's early work [**16**], where a nice presentation of the ring structure on the complexified quantum cohomology $QH^*(G/B)$ was given.

This paper is organized as follows. In section 2, we define a grading map and prove our main result, Theorem 1.2, assuming the Key Lemma. Then we devote the whole section 3 to the proof of the Key Lemma. In section 4, we prove the remaining theorems discussed as above. In section 5, we show how to generalize our results to the general case when P/B is reducible. Finally in section 6, we give an appendix which deals with exceptional cases in the proof of the Key Lemma. Our proofs are combinatorial in nature. We hope to find nice geometrical explanations of them later.

Acknowledgements. The authors thank Baohua Fu, Bumsig Kim, Thomas Lam, Augustin-Liviu Mare, and Leonardo Constantin Mihalcea for useful discussions. We also thank the referee for valuable suggestions. The first author is supported in part by a RGC research grant from the Hong Kong government. The second author is supported in part by KRF-2007-341-C00006.

2. A filtration on $QH^*(G/B)$

2.1. Preliminaries. We recall some basic notions and fix the notation. See, for example, [**12**, **13**] for more details on Lie theory.

Let G be a simply-connected complex simple Lie group of rank n, $B \subset G$ be a Borel subgroup and $P \supset B$ be a proper parabolic subgroup of G. Fix a basis of simple roots $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ (with respect to (G, B)). Then P corresponds canonically to a proper subset Δ_P of Δ. (In particular, B corresponds to the empty subset $\emptyset$.) Let $\mathfrak{h}$ denote the corresponding Cartan subalgebra; then $\mathfrak{h}^* = \bigoplus_{i=1}^n \mathbb{C}\alpha_i$. Let $\{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the fundamental coroots and $\{\chi_1, \cdots, \chi_n\} \subset \mathfrak{h}^*$ be the fundamental weights. For any $1 \leq i, j \leq n$, we have $\langle \chi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ with respect to the natural pairing $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. Furthermore, we have $\rho := \frac{1}{2}\sum_{\gamma \in R^+} \gamma = \sum_{i=1}^n \chi_i$. For each $1 \leq i \leq n$, the simple reflection $s_i := s_{\alpha_i}$ acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ by

$$s_i(\lambda) = \lambda - \langle \alpha_i, \lambda \rangle \alpha_i^\vee, \text{ for } \lambda \in \mathfrak{h}; \quad s_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i, \text{ for } \beta \in \mathfrak{h}^*.$$

The Weyl group W, which is generated by $\{s_1, \cdots, s_n\}$, acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ and preserves the natural pairing. The root system is given by $R = W \cdot \Delta = R^+ \sqcup (-R^+)$, where $R^+ = R \cap \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ is the set of positive roots. Thus each root $\gamma \in R$ is given by $\gamma = w(\alpha_i)$ for some $w \in W$ and $1 \leq i \leq n$. Then we define $\gamma^\vee = w(\alpha_i^\vee)$ and $s_\gamma = w s_i w^{-1} \in W$, which is independent of the expressions of γ.

The length $\ell(w)$ of $w \in W$ (with respect to Δ) is defined by $\ell(1) \triangleq 0$ and $\ell(w) \triangleq \min\{k \mid w = s_{i_1} \cdots s_{i_k}\}$ for $w \neq 1$. An expression $w = s_{i_1} \cdots s_{i_\ell}$ is called **reduced** if $\ell = \ell(w)$. Let $\tilde{P} = P_{\tilde{\Delta}}$ denote the (standard) parabolic subgroup corresponding to a subset $\tilde{\Delta} \subset \Delta$, $W_{\tilde{P}}$ denote the subgroup generated by $\{s_j \mid \alpha_j \in \tilde{\Delta}\}$, and $\omega_{\tilde{P}}$ denote the longest element in $W_{\tilde{P}}$. For $\bar{\Delta} \subset \tilde{\Delta}$ with $\bar{P} := P_{\bar{\Delta}}$, we denote $W_{\tilde{P}}^{\bar{P}} := \{w \in W_{\tilde{P}} | \ell(w) \leq \ell(v), \forall v \in wW_{\bar{P}}\}$. Each coset in $W_{\tilde{P}}/W_{\bar{P}}$ has a unique (minimal length) representative in $W_{\tilde{P}}^{\bar{P}} \subset W_{\tilde{P}} \subset W$. In particular, we have $P_\Delta = G$ and $W_G = W$, and simply denote $W^{\bar{P}} := W_G^{\bar{P}}$ and $\omega := \omega_G$.

The (co)homology of a homogeneous variety $X = G/P$ has an additive basis of Schubert (co)homology classes indexed by W^P: $H_*(X, \mathbb{Z}) = \bigoplus_{v \in W^P} \mathbb{Z}\sigma_v$, $\quad H^*(X, \mathbb{Z}) = \bigoplus_{u \in W^P} \mathbb{Z}\sigma^u$ with $\langle \sigma^u, \sigma_v \rangle = \delta_{u,v}$ for any $u, v \in W^P$ [**2**]. In particular, $H_2(X, \mathbb{Z}) = \bigoplus_{\alpha_i \in \Delta \setminus \Delta_P} \mathbb{Z}\sigma_{s_i}$. Set $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ and $Q_P^\vee = \bigoplus_{\alpha_i \in \Delta_P} \mathbb{Z}\alpha_i^\vee$. Then we can identify $H_2(X, \mathbb{Z})$ with $Q^\vee / Q_P^\vee$ canonically, by mapping $\sum_{\alpha_j \in \Delta \setminus \Delta_P} a_j \sigma_{s_j}$ to $\lambda_P = \sum_{\alpha_j \in \Delta \setminus \Delta_P} a_j \alpha_j^\vee + Q_P^\vee$. For each $\alpha_j \in \Delta \setminus \Delta_P$, we introduce a formal variable $q_{\alpha_j^\vee + Q_P^\vee}$. For such λ_P, we denote $q_{\lambda_P} = \prod_{\alpha_j \in \Delta \setminus \Delta_P} q_{\alpha_j^\vee + Q_P^\vee}^{a_j}$.

Let $\overline{\mathcal{M}}_{0,m}(X, \lambda_P)$ be the moduli space of stable maps of degree $\lambda_P \in H_2(X, \mathbb{Z})$ of m-pointed genus zero curves into X [**10**], and ev_i denote the ith canonical evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,m}(X, \lambda_P) \to X$ given by $\mathrm{ev}_i([f : C \to X; p_1, \cdots, p_m]) = f(p_i)$. The genus zero Gromov-Witten invariant for $\gamma_1, \cdots, \gamma_m \in H^*(X) = H^*(X, \mathbb{Q})$ is defined as $I_{0,m,\lambda_P}(\gamma_1, \cdots, \gamma_m) = \int_{\overline{\mathcal{M}}_{0,m}(X, \lambda_P)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_m^*(\gamma_m)$. The (small) quantum product for $a, b \in H^*(X)$ is a deformation of the cup product, defined by

$$a \star b \triangleq \sum_{u \in W^P, \lambda_P \in H_2(X, \mathbb{Z})} I_{0,3,\lambda_P}(a, b, (\sigma^u)^\sharp) \sigma^u q_{\lambda_P},$$

where $\{(\sigma^u)^\sharp \mid u \in W^P\}$ are the elements in $H^*(X)$ satisfying $\int_X (\sigma^u)^\sharp \cup \sigma^v = \delta_{u,v}$ for any $u, v \in W^P$. The quantum product $\star$ is associative, making $(H^*(X) \otimes \mathbb{Q}[\mathbf{q}], \star)$ a commutative ring. This ring is denoted as $QH^*(X)$ and called the (**small**) **quantum cohomology ring** of X. The same Schubert classes $\sigma^u = \sigma^u \otimes 1$ form a basis for $QH^*(X)$ over $\mathbb{Q}[\mathbf{q}]$, and we write

$$\sigma^u \star \sigma^v = \sum_{w \in W^P, \lambda_P \in Q^\vee / Q_P^\vee} N_{u,v}^{w,\lambda_P} q_{\lambda_P} \sigma^w.$$

The coefficients $N_{u,v}^{w,\lambda_P}$'s are called the *quantum Schubert structure constants.* They generalize the well known Littlewood-Richardson coefficients when $X = Gr(k, n+1)$ is a complex Grassmannian. It is also well known that the quantum Schubert structure constants are non-negative.

When $P = B$, we have $Q_P^\vee = 0$, $W_P = \{1\}$ and $W^P = W$. In this case, we simply denote $\lambda = \lambda_P$ and $q_j = q_{\alpha_j^\vee}$. A combinatorial formula for $N_{u,v}^{w,\lambda}$'s has been given by the authors recently [**20**]. As a consequence, we can obtain the combinatorial formula for $N_{u,v}^{w,\lambda_P}$'s for general G/P, due to the following comparison formula.

Proposition 2.1 (Peterson-Woodward comparison formula [**32**]; see also [**18**])**.**

1) *Let* $\lambda_P \in Q^\vee/Q_P^\vee$. *Then there is a unique* $\lambda_B \in Q^\vee$ *such that* $\lambda_P = \lambda_B + Q_P^\vee$ *and* $\langle \alpha, \lambda_B \rangle \in \{0, -1\}$ *for all* $\alpha \in R_P^+$ $(= R^+ \cap \bigoplus_{\alpha_j \in \Delta_P} \mathbb{Z}\alpha_j)$.
2) *For every* $u, v, w \in W^P$, *we have*
$$N_{u,v}^{w,\lambda_P} = N_{u,v}^{w\omega_P\omega',\lambda_B},$$
where $\omega' = \omega_{P'}$ *with* $\Delta_{P'} = \{\alpha_i \in \Delta_P \mid \langle \alpha_i, \lambda_B \rangle = 0\}$.

Thanks to Proposition 2.1, we have canonical representatives of $W/W_P \times Q^\vee/Q_P^\vee$ in $W \times Q^\vee$ with respect to the pair (Δ, Δ_P), which is a generalization of the case $W/W_P \overset{\cong}{\to} W^P \subset W$. We will discuss them in more details in the next subsection.

When v is a simple reflection s_i, we have the following (Peterson's) quantum Chevalley formula for $\sigma^u \star \sigma^{s_i}$, which has been proved earlier in [**11**].

Proposition 2.2 (Quantum Chevalley Formula for G/B)**.** *For* $u \in W, 1 \le i \le n$,
$$\sigma^u \star \sigma^{s_i} = \sum_\gamma \langle \chi_i, \gamma^\vee \rangle \sigma^{us_\gamma} + \sum_\gamma \langle \chi_i, \gamma^\vee \rangle q_{\gamma^\vee} \sigma^{us_\gamma},$$
where the first sum is over roots γ *in* R^+ *for which* $\ell(us_\gamma) = \ell(u) + 1$, *and the second sum is over roots* γ *in* R^+ *for which* $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle$.

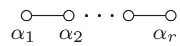
Note that we have fixed a base $\Delta = \{\alpha_1, \cdots, \alpha_n\}$. As a subset of Δ, we can write $\Delta_P = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$. Then giving an order on Δ_P is equivalent to giving a permutation of Δ_P. Once such a permutation Υ is given, we denote $\alpha'_j = \Upsilon(\alpha_{i_j})$ for each $1 \le j \le r$ and then naturally rewrite the remaining simple roots so that $\Delta = \{\alpha'_1, \cdots, \alpha'_n\}$. In the present paper, we always keep the information on the order, whenever referring to (Δ_P, Υ) or (an ordered set) $\Delta_P = (\alpha'_1, \cdots, \alpha'_r)$. Furthermore for convenience, we simply denote α'_j's by α_j's. (In other words, we take $\Upsilon = \mathrm{id}_{\Delta_P}$ under the assumption in the beginning that $\Delta_P = \{\alpha_1, \cdots, \alpha_r\}$ satisfies certain properties on its associated Dynkin diagram.)

Notation 2.3. Let (Δ_P, Υ) be given with $\Delta_P = (\alpha_1, \cdots, \alpha_r)$.

For any integers k, m with $1 \le k \le m \le r$, we denote $u_{[k,m]}^{\Delta_P,(r)} := s_{\alpha_k} s_{\alpha_{k+1}} \cdots s_{\alpha_m}$. If $k > m$, then we just denote $u_{[k,m]}^{\Delta_P,(r)} := 1$. Furthermore, we define $u_{[k,m]}^{\Delta_P,(m)} = u_{[k,m]}^{\Delta_P,(r)}$ and denote $u_i^{\Delta_P,(m)} := u_{[m-i+1,m]}^{\Delta_P,(m)}$ for $i = 0, 1, \cdots, m$. Whenever there is no confusion, we simply denote
$$s_j := s_{\alpha_j}, \quad u_{[k,m]}^{(r)} := u_{[k,m]}^{\Delta_P,(r)}, \quad \text{and} \quad u_i^{(m)} := u_i^{\Delta_P,(m)}.$$

Let $\Delta_j := \{\alpha_1, \cdots, \alpha_j\}$ and $P_j := P_{\Delta_j}$ for each $1 \leq j \leq r$. Denote $P_0 = B$ and $P_{r+1} = G$. A **decomposition of** $w \in W$ **associated to** (Δ_P, Υ) is an expression $w = v_{r+1} \cdots v_1$ with $v_i \in W_{P_i}^{P_{i-1}}$ for each $1 \leq i \leq r+1$, where $W_{P_1}^{P_0} = W_{P_1}$. By the **iterated fibration associated to** (Δ_P, Υ), we mean the family of fibrations of homogeneous varieties, given by $\{P_{j-1}/P_0 \to P_j/P_0 \longrightarrow P_j/P_{j-1}\}_{j=2}^{r+1}$.

We denote by $Dyn(\Delta')$ the Dynkin diagram associated to a base Δ'.

Example 2.4. Suppose $Dyn(\Delta_P)$ is given by  . Consider the iterated fibration $\{P_{j-1}/P_0 \to P_j/P_0 \longrightarrow P_j/P_{j-1}\}_{j=2}^{r+1}$ associated to $\Delta_P = (\alpha_1, \cdots, \alpha_r)$. Then we have $P_{r+1}/P_r = G/P$ and $P_j/P_{j-1} = \mathbb{P}^j$ for each $1 \leq j \leq r$. Furthermore, the natural inclusion $\{\alpha_1, \cdots, \alpha_{r-1}\} \hookrightarrow \Delta_P$ (or $SL(r, \mathbb{C}) \hookrightarrow SL(r+1, \mathbb{C})$) induces a canonical embedding $P_{r-1}/B = F\ell_{r-1} \hookrightarrow F\ell_r = P/B$ of complete flag varieties, which maps a flag $V_1 \leqslant \cdots \leqslant V_{r-1}$ in $\mathbb{C}^r$ to the flag $V_1 \leqslant \cdots \leqslant V_{r-1} \leqslant \mathbb{C}^r$ in $\mathbb{C}^{r+1}$.

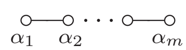
Due to the following well known lemma (see e.g., [**14**]), we obtain Corollary 2.6.

Lemma 2.5. *Let $\gamma \in R^+$ and $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression of $w \in W$.*

1) *$w \in W^P$ if and only if $w(\alpha) \in R^+$ for any $\alpha \in \Delta_P$.*
2) *If $\ell(ws_\gamma) < \ell(w)$, then $w(\gamma) \in -R^+$ and there is a unique $1 \leq k \leq \ell$ such that*

$$s_{i_k} \cdots s_{i_\ell} s_\gamma = s_{i_{k+1}} \cdots s_{i_\ell} \quad \text{and} \quad \gamma = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}).$$

Furthermore for $1 \leq j \leq n$, $\ell(ws_j) = \ell(w) - 1$ if and only if $w(\alpha_j) \in -R^+$.

Corollary 2.6. *For each $w \in W$, there exists a unique decomposition $w = v_{r+1} \cdots v_1$ associated to $\Delta_P = (\alpha_1, \cdots, \alpha_r)$. Furthermore, we assume that $Dyn(\{\alpha_1, \cdots, \alpha_m\})$ is given by*  *, where $m \leq r$. Then for each $1 \leq j \leq m$, $\ell(v_j) = i_j$ if and only if $v_j = u_{i_j}^{(j)}$. (In particular, the expression $u_{i_j}^{(j)}$ itself is reduced.)*

Proof. The former is also well known (see e.g., [**14**]). The latter statement is a direct consequence of Lemma 2.5, by noting $|W_{P_j}^{P_{j-1}}| = j+1$ and $u_0^{(j)}, \cdots, u_j^{(j)}$ are distinct elements of W_{P_j} for which (1) of Lemma 2.5 can be applied. q.e.d.

The following lemma should also be well known.

Lemma 2.7. *Let $\bar{\Delta} \subset \tilde{\Delta} \subset \Delta$, $\tilde{P} = P_{\tilde{\Delta}}$ and $\bar{P} = P_{\bar{\Delta}}$. Let $w = vu$ with $u \in W_{\bar{P}}$ and $v = s_{i_1} \cdots s_{i_m}$ being a reduced expression of $v \in W_{\tilde{P}}^{\bar{P}}$. For any $1 \le j \le m$, we have $v' := s_{i_{j+1}} \cdots s_{i_m} \in W_{\tilde{P}}^{\bar{P}}$ and $(v'u)^{-1}(\alpha_{i_j}) \in R_{\tilde{P}}^{+} \setminus R_{\bar{P}}$.*

Proof. Assume that the set $\{a \mid s_{i_{a+1}} s_{i_{a+2}} \cdots s_{i_m} \notin W_{\tilde{P}}^{\bar{P}}, 1 \le a \le m\}$ is non-empty. Then we can take the minimum k of this set. Consequently, $w' := s_{i_{k+1}} \cdots s_{i_m} \notin W_{\tilde{P}}^{\bar{P}}$ and $s_{i_k} w' \in W_{\tilde{P}}^{\bar{P}}$. Hence, there exists $\alpha \in \bar{\Delta}$ such that $w'(\alpha) \in -R^+$ and $s_{i_k} w'(\alpha) \in R^+$. Since s_{i_k} preserves $-R^+ \setminus \{-\alpha_{i_k}\}$, we have $w'(\alpha) = -\alpha_{i_k}$. Thus $w' s_\alpha w'^{-1} = s_{-\alpha_{i_k}} = s_{i_k}$ so that $\ell(w' s_\alpha) = \ell(s_{i_k} w') = \ell(w') + 1$. This implies $w'(\alpha) \in R^+$ by Lemma 2.5 and therefore deduces a contradiction. Hence, for any $1 \le j \le m$, we have $v' := s_{i_{j+1}} \cdots s_{i_m} \in W_{\tilde{P}}^{\bar{P}}$.

Note that $(v'u)^{-1}(\alpha_{i_j}) \in R_{\tilde{P}}^{+}$ and $\gamma := v'^{-1}(\alpha_{i_j}) \in R_{\tilde{P}}^{+}$. We claim $\gamma \notin R_{\bar{P}}$; otherwise we would conclude $v' s_\gamma(\gamma) = -v'(\gamma) \in -R_{\bar{P}}^{+}$, contrary to $v' s_\gamma(\gamma) = s_{i_j} v'(\gamma) \in R^+$. Since $u \in W_{\bar{P}}$, we have $(v'u)^{-1}(\alpha_{i_j}) = u^{-1}(\gamma) \notin R_{\bar{P}}$. q.e.d.

2.2. Definition of gradings. In this subsection, we define a grading map gr with respect to an ordered set (Δ_P, Υ), which is used for constructing a filtration on $QH^*(G/B)$. In order to obtain gr, we first define "PW-lifting" (Peterson-Woodward lifting) as follows.

Definition 2.8. Given (Δ_P, Υ) with $\Delta_P = (\alpha_1, \cdots, \alpha_r)$, we denote $\Delta_j = \{\alpha_1, \cdots, \alpha_j\}$, $P_j = P_{\Delta_j}$, and $Q_j^\vee = \bigoplus_{i=1}^{j} \mathbb{Z}\alpha_i^\vee$ for each $j \le r$. By the **PW-lifting** associated to (Δ_P, Υ), we mean the family $\{\psi_{\Delta_{j+1}, \Delta_j}\}_{j=1}^{r}$ of injective maps defined as follows. (We denote $Q_{r+1}^\vee = Q^\vee, \Delta_{r+1} = \Delta$ and $P_{r+1} = G$.) For each $1 \le j \le r$, the map

$$\psi_{\Delta_{j+1}, \Delta_j} : W_{P_{j+1}}^{P_j} \times Q_{j+1}^\vee / Q_j^\vee \longrightarrow W \times Q^\vee$$

is defined by sending $(v, \bar{\lambda})$ to its associated elements $(v\omega_{P_j}\omega_{P_j'}, \lambda')$ as described by the Peterson-Woodward comparison formula (see Proposition 2.1) with respect to (Δ_{j+1}, Δ_j). That is, λ' is the unique element in $Q_{j+1}^\vee \subset Q^\vee$ satisfying $\bar{\lambda} = \lambda' + Q_j^\vee$ and $\langle \alpha, \lambda' \rangle \in \{0, -1\}$ for all $\alpha \in R^+ \cap \bigoplus_{i=1}^{j} \mathbb{Z}\alpha_i$; $\Delta_{P_j'} = \{\alpha \in \Delta_j \mid \langle \alpha, \lambda' \rangle = 0\}$.

Remark 2.9. Each $\psi_{\Delta_{j+1}, \Delta_j}$ also defines an injective map in the canonical way:

$$QH^*(P_{j+1}/P_j) \longrightarrow QH^*(P_{j+1}/B); q_{\bar{\lambda}}\sigma^v \mapsto q_{\lambda'}\sigma^{v\omega_{P_j}\omega_{P_j'}}.$$

Recall that a natural basis of $QH^*(G/B)[q_1^{-1}, \cdots, q_n^{-1}]$ is given by $q_\lambda\sigma^w$'s labeled by $(w, \lambda) \in W \times Q^\vee$. We simply denote both of them as

$q_\lambda w$ (or wq_λ) by abuse of notation. Note that $q_\lambda w \in QH^*(G/B)$ if and only if $q_\lambda \in \mathbb{Q}[\mathbf{q}]$ is a polynomial.

Definition 2.8 (continued). *Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r+1}\}$ be the standard basis of $\mathbb{Z}^{r+1}$. We define a* ***grading map*** $gr : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1}$ *associated to (Δ_P, Υ) as follows.*

1) *For $w \in W$, we take its (unique) decomposition $w = v_{r+1} \cdots v_1$ associated to (Δ_P, Υ). Then we define $gr(w) := gr(w, 0) = \sum_{j=1}^{r+1} \ell(v_j)\mathbf{e}_j$.*
2) *For all $\alpha \in \Delta$, we simply denote $gr(q_{\alpha^\vee}) := gr(1, q_{\alpha^\vee})$. Using the PW-lifing associated to (Δ_P, Υ), we can define all $gr(q_j)$'s recursively in the following way. Define $gr(q_1) = 2\mathbf{e}_1$; for any $\alpha \in \Delta_{j+1} \setminus \Delta_j$, we define*

$$gr(q_{\alpha^\vee}) = \big(\ell(\omega_{P_j}\omega_{P'_j})+2+\sum\nolimits_{i=1}^{j} 2a_i\big)\mathbf{e}_{j+1} - gr(\omega_{P_j}\omega_{P'_j}) - \sum\nolimits_{i=1}^{j} a_i gr(q_i),$$

where $\omega_{P_j}\omega_{P'_j}$ and a_i's satisfy $(\omega_{P_j}\omega_{P'_j}, \alpha^\vee + \sum_{i=1}^{j} a_i\alpha_i^\vee) = \psi_{\Delta_{j+1},\Delta_j}(1, \alpha^\vee + Q_j^\vee)$.

3) *In general, $x = w\prod_{k=1}^{n} q_k^{b_k}$; then we define $gr(x) = gr(w) + \sum_{k=1}^{n} b_k gr(q_k)$.*

Furthermore for $1 \le k \le m \le |\Delta_P|$, we define

$$gr_m := gr_{[1,m]}, \quad \text{with} \quad gr_{[k,m]} : W \times Q^\vee \to \mathbb{Z}^{m-k+1}$$

being the composition of the natural projection map and the grading map gr. Precisely, write $gr(q_\lambda w) = \sum_{i=1}^{r+1} d_i\mathbf{e}_i$; then we define $gr_{[k,m]}(q_\lambda w) = \sum_{i=k}^{m} d_i\mathbf{e}_i$.

Recall that the *inversion set* of $w \in W$ is defined to be

$$\mathrm{Inv}(w) = \{\gamma \in R^+ \mid w(\gamma) \in -R^+\}.$$

It is well known that $\ell(w) = |\mathrm{Inv}(w)|$ (see e.g., [**14**]). Take the decomposition $w = v_{r+1} \cdots v_1$ of w associated to (Δ_P, Υ). For each k, we note $v_{r+1} \cdots v_{k+1} \in W^{P_k}$ and $v_k \cdots v_1 \in W_{P_k}$. Thus for $\gamma \in R_{P_k}$, $v_k \cdots v_1(\gamma) \in -R^+$ if and only if $w(\gamma) \in -R^+$. Consequently, $\ell(v_k \cdots v_1) = |\{\gamma \in R^+_{P_k} \mid w(\gamma) \in -R^+\}| = |\mathrm{Inv}(w) \cap R^+_{P_k}|$. Note that $\ell(v_k \cdots v_1) = \sum_{i=1}^{k} \ell(v_k)$. Hence, we have

$$gr(w) = \sum_{k=1}^{r+1} |\mathrm{Inv}(w) \cap (R^+_{P_k} \setminus R^+_{P_{k-1}})|\mathbf{e}_k.$$

Remark 2.10. We would like to thank the referee for reminding us of the above expression of $gr(w)$. Following the suggestions of the referee, the proof of Proposition 3.1 has been simplified substantially in the present version. In type A, the vector $gr(w)$ is essentially what is known as an "inversion table" (see e.g., [**31**]). The referee has also made the following conjecture:

$$gr(q_{\gamma^\vee}) = \sum_{k=1}^{r+1} \langle \sum_{\beta \in R^+_{P_k} \setminus R^+_{P_{k-1}}} \beta, \gamma^\vee \rangle \mathbf{e}_k.$$

If it is true, the proofs of our main results might also be simplified substantially.

In Proposition 3.10, Proposition 3.12, Lemma 3.26, and (the proof of) Lemma 3.27, we will explicitly describe all the gradings $gr(q_j)$'s with respect to a fixed (Δ_P, Υ). In particular, we will see that $gr(q_j) = (1-j)\mathbf{e}_{j-1} + (1+j)\mathbf{e}_j$ for $2 \le j \le r-1$ (which also holds for $j = r$ if Δ_P is of A-type).

2.3. Proof of Theorem 1.2. Assuming $Dyn(\Delta_P)$ is connected, we always consider (Δ_P, Υ) with the fixed order $\Delta_P = (\alpha_1, \cdots, \alpha_r)$ in a special way that will be explained in section 2.4. In this subsection, we construct a filtration on $QH^*(G/B)$ with respect to a totally ordered sub-semigroup S of $\mathbb{Z}^{r+1}$ and prove Theorem 1.2, which is the most essential part of our main results.

Unless otherwise stated, we will always use the **lexicographical order** whenever referring to a partial order on (a sub-semigroup of) $\mathbb{Z}^m$ in the present paper. (Recall that $\mathbf{a} < \mathbf{b}$, where $\mathbf{a} = (a_1, \cdots, a_m)$ and $\mathbf{b} = (b_1, \cdots, b_m)$, if and only if there is $1 \le j \le m$ such that $a_j < b_j$ and $a_k = b_k$ for each $1 \le k < j$.)

Definition 2.11. We define a subset S of $\mathbb{Z}^{r+1}$ and a family $\mathcal{F} = \{F_{\mathbf{a}}\}_{\mathbf{a} \in S}$ of subspaces of $QH^*(G/B)$ as follows:

$$S \triangleq \{gr(q_\lambda w) \mid q_\lambda w \in QH^*(G/B)\}; \;\; F_{\mathbf{a}} \triangleq \bigoplus_{gr(q_\lambda w) \le \mathbf{a}} \mathbb{Q} q_\lambda w \subset QH^*(G/B).$$

As will be shown in section 4, we have:

Lemma 2.12. *S is a totally-ordered sub-semigroup of $\mathbb{Z}^{r+1}$.*

Now we can state Theorem 1.2 more explicitly as follows.

Theorem 1.2. *$QH^*(G/B)$ is an S-filtered algebra with filtration $\mathcal{F}$. Furthermore, this S-filtered algebra structure is naturally extended to a $\mathbb{Z}^{r+1}$-filtered algebra structure on $QH^*(G/B)$.*

That is, we need to show $F_{\mathbf{a}} F_{\mathbf{b}} \subset F_{\mathbf{a}+\mathbf{b}}$ for any $\mathbf{a}, \mathbf{b} \in S$. In order to prove it, we need to assume the following Key Lemma first.

Key Lemma. *Let $u \in W$ and $\gamma \in R^+$.*

a) *If $\ell(us_\gamma) = \ell(u) + 1$, then we have $gr(us_\gamma) \le gr(u) + gr(s_i)$ whenever the fundamental weight χ_i satisfies $\langle \chi_i, \gamma^\vee \rangle \neq 0$.*

b) *If* $\ell(us_\gamma)=\ell(u)+1-\langle 2\rho,\gamma^\vee\rangle$, *then we have* $gr(q_{\gamma^\vee}us_\gamma)\leq gr(u)+gr(s_i)$ *whenever* $\langle\chi_i,\gamma^\vee\rangle\neq 0$.

Lemma 2.13. *For any* $1\neq w\in W$, *there exist* $w'\in W$ *and* $1\leq j\leq n$ *such that* $gr(w)=gr(w')+gr(s_j)$ *and the quantum structure constant* $N_{s_j,w'}^{w,0}$ *is positive.*

Proof. Take the decomposition $w=v_{r+1}\cdots v_1$ of w associated to (Δ_P,Υ). Since $w\neq 1$, the set $\{i\mid \ell(v_i)>0\}$ is non-empty, so that we can take the minimum k of this set. Thus we have $v_1=\cdots=v_{k-1}=1$ and $v_k=s_p\bar{v}$ with $\ell(s_p\bar{v})=1+\ell(\bar{v})$. Note that $\gamma:=\bar{v}^{-1}(\alpha_p)\in R^+_{P_k}$ and $\bar{v}s_\gamma=v_k$. Consequently, for $w':=v_{r+1}\cdots v_{k+1}\bar{v}$, we have $w=w's_\gamma$ and $\ell(w's_\gamma)=\ell(w)+1$. By Lemma 2.7, we have $\bar{v}\in W^{P_{k-1}}_{P_k}$ and $\gamma\notin R^+_{P_{k-1}}$. Hence, there exists $1\leq j\leq n$ with $\alpha_j\in\Delta_k\setminus\Delta_{k-1}$ such that $\langle\chi_j,\gamma^\vee\rangle>0$. For any one such j, by Proposition 2.2 we have $N^{w,0}_{s_j,w'}=N^{w's_\gamma,0}_{s_j,w'}=\langle\chi_j,\gamma^\vee\rangle>0$. Furthermore, we have $gr(w)=\sum_{i=k}^{r+1}\ell(v_i)\mathbf{e}_i=(\ell(\bar{v})\mathbf{e}_k+\sum_{i=k+1}^{r+1}\ell(v_i)\mathbf{e}_i)+\mathbf{e}_k=gr(w')+gr(s_j)$. q.e.d.

Proof of Theorem 1.2. For the first half of the statements, it suffices to show $\sigma^w\star q_\lambda\sigma^u\in F_{\mathbf{a}+\mathbf{b}}$, for any $\sigma^w,q_\lambda\sigma^u\in QH^*(G/B)$ with $\mathbf{a}=gr(w)$ and $\mathbf{b}=gr(q_\lambda u)$. We use induction on $\ell(w)$.

If $\ell(w)=0$, then σ^w is the unit and it is done. If $\ell(w)=1$, then $w=s_j$ and consequently we have $\sigma^{s_j}\star\sigma^u\in F_{gr(s_j)+gr(u)}=F_{\mathbf{a}+\mathbf{b}-gr(q_\lambda)}$, by using Proposition 2.2 and the Key Lemma. Thus we have $\sigma^w\star q_\lambda\sigma^u\in F_{\mathbf{a}+\mathbf{b}}$ in this case. Assume $\ell(w)>1$. By Lemma 2.13, there exist $w'\in W$ and $1\leq j\leq n$ such that $gr(w)=gr(w')+gr(s_j)$ and $\sigma^{w'}\star\sigma^{s_j}=c\sigma^w+\sum_{v,\mu}c_{v,\mu}q_\mu\sigma^v$, where $c=N^{w,0}_{w',s_j}>0$ and the summation is only over finitely many non-zero terms for which $c_{v,\mu}>0$. In particular, we have $\ell(w')=\ell(w)-1$. Using the induction hypothesis, we have $\sigma^{w'}\star q_\lambda u\in F_{gr(w')+\mathbf{b}}$. Thus $(c\sigma^w+\sum_{v,\mu}c_{v,\mu}q_\mu\sigma^v)\star q_\lambda\sigma^u=\sigma^{s_j}\star(\sigma^{w'}\star q_\lambda\sigma^u)\in F_{gr(s_j)+gr(w')+\mathbf{b}}=F_{\mathbf{a}+\mathbf{b}}$. Since all the quantum Schubert structure constants are non-negative, there is no cancellation in the summation on the left-hand side of the equality. Hence, we conclude $\sigma^w\star q_\lambda\sigma^u\in F_{\mathbf{a}+\mathbf{b}}$, by noting $c>0$.

The second half is a direct consequence of the first half. Indeed, $QH^*(G/B)$ has a $\mathbb{Z}^{r+1}$-filtration $\{F_\mathbf{a}\}_{\mathbf{a}\in\mathbb{Z}^{r+1}}$, which is a natural extension of $\mathcal{F}$. Here we just need to set $F_\mathbf{a}:=\bigcup_{\mathbf{b}\leq\mathbf{a},\mathbf{b}\in S}F_\mathbf{a}$ for any $\mathbf{a}\in\mathbb{Z}^{r+1}\setminus S$ (note S is sub-semigroup of $\mathbb{Z}^{r+1}$). q.e.d.

The next proposition follows directly from Definition 2.8.

Proposition 2.14. *The evaluation of* $\mathbf{q}$ *at* $\mathbf{0}$ *reduces the* $\mathbb{Z}^{r+1}$*-filtration on* $QH^*(G/B)$ *to the classical* $\mathbb{Z}^{r+1}$*-filtration on* $H^*(G/B)$, *which comes from the iterated fibration* $\{P_{j-1}/P_0\to P_j/P_0\longrightarrow P_j/P_{j-1}\}_{j=2}^{r+1}$. *(Recall that* $P_0=B$ *and* $P_{r+1}=G$.*)*

2.4. A canonical order (Δ_P, Υ). When referring to (Δ_P, Υ), we have already given an order on Δ_P via the permutation Υ. It is done if $r = 1$, since $\Upsilon = \mathrm{id}_{\Delta_P}$ is the only permutation map. In this subsection, we introduce the special choice of the orders for $r \geq 2$ as mentioned at the beginning of section 2.3. We will use this special order throughout the present paper, which is in fact obtained in a canonical way. We introduce it first for a subbase of A-type and then for others by reducing them to the case for A-type.

Suppose Δ_P is of A_r-type. We rewrite the simple roots so that $\Delta = \{\beta_1, \cdots, \beta_n\}$ and $Dyn(\Delta)$ is given by one of the cases in Table 2. In terms of the order $(\beta_1, \cdots, \beta_n)$, we obtain a canonical order (Δ_P, Υ), in the sense that $Dyn(\Delta_P)$ is inside $Dyn(\Delta \setminus \{\text{marked points}\})$ in a natural way. That is, we require the condition $(*)$ to be satisfied.

$$(*): \qquad \text{there exists } o \geq 0 \text{ such that } \alpha_j = \beta_{o+j} \text{ for each } 1 \leq j \leq r.$$

Furthermore, the additional conditions in Table 2 tell us the information on the starting point $\alpha_1 (= \beta_{o+1})$ and the ending point $\alpha_r (= \beta_\kappa = \beta_{o+r})$. For instance, any one case of C8), C9), and C10) implies that $o = 0$ and $\Delta_P = (\alpha_1, \alpha_2) = (\beta_1, \beta_2)$. That is, the order of $\Delta_P = \{\alpha_1, \alpha_2\}$ is expressed in terms of the order of $\{\beta_1, \beta_2\}$ with respect to the corresponding case.

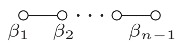
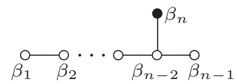
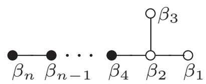
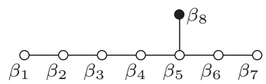
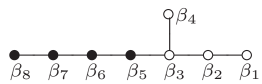
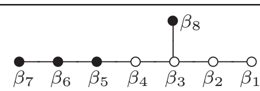
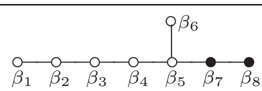
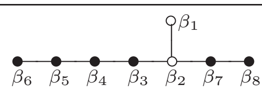
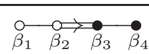
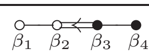
Remark 2.15. In Table 2, we have treated bases of type E_6 and E_7 as subsets of a base of type E_8 canonically. Because of our assumption $2 \leq r < n = |\Delta|$, a base of G_2-type does not occur there.

Remark 2.16. Intrinsically, we obtain the canonical order (Δ_P, Υ) as follows. Δ_P admits canonical orders in the sense that $Dyn(\Delta_P)$ is given by $\underset{\alpha_1}{\circ}\!-\!\underset{\alpha_2}{\circ} \cdots \underset{}{\circ}\!-\!\underset{\alpha_r}{\circ}$. There are two ways to denote an ending point (by α_1 or α_r). We fix one in the following way. There is at most one root in Δ_P, say, α, such that the Dynkin diagram of $\Delta_P \cup \{\alpha_k \in \Delta \setminus \Delta_P \mid \langle \alpha_k, \alpha^\vee \rangle \neq 0\}$ is not of A-type. We denote an ending point by α_1 such that both the ending point and the connected component of $\Delta \setminus \Delta_P$ adjacent to it are as far away from α as possible.

Comparing it with Table 2, we can easily see that Δ_P must occur in at least one case of Table 2 (together with condition $(*)$ being satisfied). If it occurs in more than one case, then we just choose any one of these cases. The choice does not affect all the results, since all the relevant statements hold with respect to all cases in Table 2, as we will see later.

Remark 2.17. If (Δ, Δ_P) occurs in more than one case in Table 2 (for instance, in case C2) and C3) with respect to the condition $\kappa = r = n - 1 = 3$), then the corresponding orders (Δ_P, Υ) and (Δ_P, Υ') are isomorphic. That is, there exists an isometry of $\phi : \Delta \to \Delta$ such that $\phi(\Delta_P) = \Delta_P$ and $\Upsilon \circ \phi = \Upsilon'$.

Table 2. (Δ_P, Υ) when $r \geq 2$

	Dynkin diagram of Δ	Additional conditions ($\kappa := o + r$)
C1)	 , β_n is adjacent to β_{n-1}	$\kappa \leq n-1$; Δ is of type A_n, B_n or C_n
C2)		$\kappa \leq n-2$ or $\begin{cases} r \geq 3 \\ \kappa = n-1 \end{cases}$
C3)		$\kappa = r \leq 3$
C4)		$\kappa \leq 5$ or $\begin{cases} r \geq 3 \\ \kappa = 6 \end{cases}$ or $\begin{cases} r \geq 5 \\ \kappa = 7 \end{cases}$
C5)		$\kappa \leq 3$ or $\begin{cases} r \geq 3 \\ \kappa = 4 \end{cases}$
C6)		$\kappa = r = 4$
C7)		$\kappa = 6, r \geq 3$
C8)		$\kappa = 2$
C9)		$\kappa = 2$
C10)		$\kappa = 2$

Now we assume Δ_P is not of A-type and denote $\varsigma := r - 1$. Note that there always exists $\alpha \in \Delta_P$ such that $Dyn(\Delta_P \setminus \{\alpha\})$ is of A_ς-type. Thus when $r > 2$, we obtain a canonical order (Δ_P, Υ) by requiring:

a) the restriction of Δ_P to $\Delta_\varsigma = (\alpha_1, \cdots, \alpha_\varsigma)$ is the canonical order obtained by directly replacing r with ς in Table 2;
b) $\alpha_r = \beta_{o+r}$ (note that $\alpha_{r-1} = \beta_{o+r-1}$ once a) holds).

Precisely, Δ_P fulfills one and only one of the followings (note that $\kappa = o + \varsigma$ and condition $(*)$ is satisfied):

1) Δ_P is not of D-type. It occurs in a unique case (among C1), C4) for $\kappa = 7$, C9) and C10)) in Table 2.
2) Δ_P and Δ are both of D-type. It occurs in case C2).

3) Δ_P is of D-type and Δ is of E-type. It occurs in either of cases C5), C7).

As a consequence, the canonical order (Δ_P, Υ) is determined by the corresponding case in which Δ_P occurs. For convenience, if Δ_P occurs in both C5) and C7), then we always choose case C7) for use.

When $r = 2$, we can still give an order on Δ_P so that it is compatible with our arrangements for $r > 2$. Indeed, we do this as follows. Since Δ_P is a proper subset of Δ_P, the case of G_2-type does not occur. Since Δ_P is not of A-type, Δ must be of type B, C, or F. We take (α_1, α_2) to be (β_{n-1}, β_n) for the former two cases, or (β_2, β_3) in C10) for the last case.

Remark 2.18. Δ_P occurs in case C5) other than in case C7) only if $r = 5$ and Δ is of E_7-type or E_8-type.

3. Proof of the Key Lemma

This whole section is devoted to the proof of the Key Lemma. Readers who wish to see more concrete statements of our theorems as well as their proofs can skip this section by assuming the Key Lemma and two consequences (Proposition 3.23 and Proposition 3.24) of a special case of it first. For emphasis, we restate the Key Lemma as follows.

Key Lemma. *Let $u \in W$ and $\gamma \in R^+$.*

a) *If $\ell(us_\gamma) = \ell(u) + 1$, then we have $gr(us_\gamma) \leq gr(u) + gr(s_i)$ whenever the fundamental weight χ_i satisfies $\langle \chi_i, \gamma^\vee \rangle \neq 0$.*
b) *If $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle$, then we have $gr(q_{\gamma^\vee} us_\gamma) \leq gr(u) + gr(s_i)$ whenever the fundamental weight χ_i satisfies $\langle \chi_i, \gamma^\vee \rangle \neq 0$.*

We first do some preparations in section 3.1 and section 3.2. Then we prove the Key Lemma for the special case when Δ_P is of A-type in section 3.3, and we obtain two consequences in section 3.4. Finally in section 3.5, we prove the Key Lemma for general cases. In addition, we also give the explicit descriptions of all $gr(q_j)$'s in section 3.2 and section 3.5.

We would like to remind our readers of the notation $gr(w) = \sum_{j=1}^{r+1} i_j \mathbf{e}_j = (i_1, \cdots, i_{r+1})$ for $w \in W$ and the notions "gr_m", "$gr_{[k,m]}$" in Definition 2.8. Furthermore, we assume Δ_P *to be of A-type throughout this section except subsection 3.5.* As a consequence, we have $w = u_{i_r}^{(r)} \cdots u_{i_1}^{(1)} \in W_P$ by Corollary 2.6, once assuming $i_{r+1} = 0$. Unless otherwise stated, by $w = vu_{i_r}^{(r)} \cdots u_{i_1}^{(1)}$ we always mean the decomposition of w associated (Δ_P, Υ) when Δ_P is of A-type; equivalently, we have $v \in W^P$.

3.1. Some properties on W. The main results of this subsection are Proposition 3.1 and Proposition 3.4, which compare the gradings of certain elements in W.

Proposition 3.1. *Let $\gamma \in R^+$ satisfy $\langle \alpha, \gamma^\vee \rangle = 0$ for all $\alpha \in \Delta_{\tilde{P}} = \Delta_P \setminus \{\alpha_a\}$, where $1 \leq a \leq r$. For any $w \in W$, we have*

$$gr_{a-1}(ws_\gamma) = gr_{a-1}(w) \text{ and } gr_r(ws_\gamma) \leq gr_r(w) + \sum\nolimits_{k=a}^{r} a\mathbf{e}_k.$$

Lemma 3.2. *Let $\gamma \in R, \Delta_{\tilde{P}} \subset \Delta$ and $w = vu$ with $v \in W^{\tilde{P}}$ and $u \in W_{\tilde{P}}$. If $\langle \alpha_j, \gamma^\vee \rangle = 0$ for all $\alpha_j \in \Delta_{\tilde{P}}$, then $ws_\gamma = \tilde{v}u$ with $\tilde{v} \in W^{\tilde{P}}$. In particular, if $\Delta_{\tilde{P}} = \{\alpha_1, \cdots, \alpha_a\}$ where $a \leq r$, then $gr_a(ws_\gamma) = gr_a(w)$.*

Proof. Let $ws_\gamma = \tilde{v}\tilde{u}$ where $\tilde{v} \in W^{\tilde{P}}$ and $\tilde{u} \in W_{\tilde{P}}$. By the assumption, we conclude $s_\gamma(\alpha_j) = \alpha_j$ and $s_j s_\gamma = s_\gamma s_j$ for any $\alpha_j \in \Delta_{\tilde{P}}$. Hence, $us_\gamma = s_\gamma u$ and consequently we have $\tilde{v}\tilde{u}u^{-1} = ws_\gamma u^{-1} = wu^{-1}s_\gamma = vs_\gamma$. If $\tilde{u} \neq u$, then there exists $\beta \in R^+_{\tilde{P}}$ such that $\tilde{\beta} := \tilde{u}u^{-1}(\beta) \in -R^+_{\tilde{P}}$. Hence, we conclude $vs_\gamma(\beta) = v(\beta) \in R^+$, contrary to $vs_\gamma(\beta) = \tilde{v}\tilde{u}u^{-1}(\beta) = \tilde{v}(\tilde{\beta}) \in -R^+$. The latter statement becomes a direct consequence. q.e.d.

Proof of Proposition 3.1. Write $gr(w) = \sum_{k=1}^{r+1} i_k \mathbf{e}_k$ and $gr(ws_\gamma) = \sum_{k=1}^{r+1} \tilde{i}_k \mathbf{e}_k$. By Lemma 3.2, we conclude $gr_{a-1}(ws_\gamma) = gr_{a-1}(w)$. That is, $\tilde{i}_k = i_k$ for $1 \leq k \leq a-1$.

Clearly, $\tilde{i}_a \leq a \leq i_a + a$. For $a+1 \leq k \leq r$, we note that $R^+_{P_k} \setminus R^+_{P_{k-1}} = \{\sum_{t=j}^{k} \alpha_t \mid 1 \leq j \leq k\}$. In addition, we have $i_k = |\mathrm{Inv}(w) \cap (R^+_{P_k} \setminus R^+_{P_{k-1}})|$ and $\tilde{i}_k = |\mathrm{Inv}(ws_\gamma) \cap (R^+_{P_k} \setminus R^+_{P_{k-1}})|$. Since $\langle \alpha_t, \gamma^\vee \rangle = 0$ for any $a+1 \leq t \leq r$, we have $ws_\gamma(\sum_{t=j}^{k} \alpha_t) = w(\sum_{t=j}^{k} \alpha_t)$ whenever $j \geq a+1$. Hence, $\tilde{i}_k - i_k \leq |\{\sum_{t=j}^{k} \alpha_t \mid 1 \leq j \leq a\}| = a$.

Hence, we have $gr_r(ws_\gamma) \leq gr_r(w) + \sum_{k=a}^{r} a\mathbf{e}_k$. q.e.d.

Lemma 3.3. *For any $1 \leq i \leq j \leq m \leq r$ and $1 \leq k \leq m$, we have*

$$u^{(m)}_{[i,j]} u^{(m)}_{[k,m]} = \begin{cases} u^{(m)}_{[k,m]} u^{(m)}_{[i,j]}, & \text{if } k \geq j+2 \\ u^{(m)}_{[i,m]}, & \text{if } k = j+1 \\ u^{(m)}_{[k+1,m]} u^{(m)}_{[i,j-1]}, & \text{if } i \leq k \leq j \\ u^{(m)}_{[k,m]} u^{(m)}_{[i-1,j-1]}, & \text{if } k < i \end{cases}.$$

For the above lemma, we recall that $u^{(m)}_{[i,j]} = s_i s_{i+1} \cdots s_j$. As a direct consequence, we obtain the following grading comparisons.

Proposition 3.4. *Let $w = u^{(r)}_{i_r} \cdots u^{(1)}_{i_1}$. Suppose $j \leq m \leq r$.*

a) *If $\ell(u^{(m)}_j w) = j + \ell(w)$, then $gr(u^{(m)}_j w) = gr(w) + j\mathbf{e}_k$ for a unique $1 \leq k \leq r$.*
b) *If $\ell(s_j w) = \ell(w) - 1$, then $gr(s_j w) = gr(w) - \mathbf{e}_k$ for a unique $1 \leq k \leq r$.*

c) $\ell(ws_j) = \ell(w) - 1$ *if and only if* $i_j \geq i_{j-1} + 1$ *(where* $i_0 := 0$*). When this happens, we have* $gr(ws_j) = \sum_{k=1}^{j-2} i_k\mathbf{e}_k + (i_j - 1)\mathbf{e}_{j-1} + i_{j-1}\mathbf{e}_j + \sum_{k=j+1}^{r} i_k\mathbf{e}_k$.

Furthermore, if $w' \in W_P$ *satisfies* $\ell(w'w) = \ell(w) \pm \ell(w')$*, then there exist non-negative integers* p_k*'s such that* $\sum_{k=1}^{r} p_k = \ell(w')$ *and* $gr(w'w) = gr(w) \pm \sum_{k=1}^{r} p_k\mathbf{e}_k$.

Proof. Note that $u_j^{(m)}u_{i_r}^{(r)} = u_{[m-j+1,m]}^{(m)}u_{[r-i_r+1,r]}^{(r)} = u_{[m-j+1,m]}^{(r)} \cdot u_{[r-i_r+1,r]}^{(r)}$. By Lemma 3.3, there are exactly four possibilities for this product. Since $\ell(u_j^{(m)}w) = j + \ell(w)$, the (third) case $m - j + 1 \leq r - i_r + 1 \leq m$ cannot occur. If $m = r - i_r$ (i.e., the second case occurs), then it is done by taking $\mathbf{e}_k = \mathbf{e}_r$. If $r - i_r + 1 \geq m + 2$, we have $m \leq r - 1$ and $u_j^{(m)}u_{i_r}^{(r)} = u_{[r-i_r+1,r]}^{(r)}u_{[m-j+1,m]}^{(r)} = u_{i_r}^{(r)}u_j^{(m)}$; if $r - i_r + 1 < m - j + 1$, we have $u_j^{(m)}u_{i_r}^{(r)} = u_{[r-i_r+1,r]}^{(r)}u_{[m-j,m-1]}^{(r)} = u_{[r-i_r+1,r]}^{(r)}u_{[m-j,m-1]}^{(m-1)} = u_{i_r}^{(r)}u_j^{(m-1)}$. That is, in either of the remaining two cases, we always have $u_j^{(m)}w = u_{i_r}^{(r)}u_j^{(m')}w'$ in which $\ell(u_j^{(m')}w') = j + \ell(w')$ with $m' \leq r - 1$ and $w' = u_{i_{r-1}}^{(r-1)} \cdots u_{i_1}^{(1)}$. Hence, a) follows by induction.

The arguments for the remaining parts of the statement are also easy and similar, which we leave to the readers. q.e.d.

Proof of Lemma 3.3. Note that $s_js_k = s_ks_j$ if $|j - k| \geq 2$, and $s_ks_js_k = s_js_ks_j$ if $|j - k| = 1$. The first two cases are trivial. For $1 \leq k < b \leq m$, we have

$$\begin{aligned} s_b \cdot u_{[k,m]}^{(m)} = s_b \cdot (s_k \cdots s_m) &= s_k \cdots s_{b-2}s_bs_{b-1}s_bs_{b+1} \cdots s_m \\ &= s_k \cdots s_{b-2}s_{b-1}s_bs_{b-1}s_{b+1} \cdots s_m \\ &= (s_k \cdots s_m) \cdot s_{b-1} = u_{[k,m]}^{(m)}s_{b-1}. \end{aligned}$$

Thus if $k < i$, then $u_{[i,j]}^{(m)}u_{[k,m]}^{(m)} = u_{[k,m]}^{(m)}u_{[i-1,j-1]}^{(m)}$. If $i \leq k \leq j$, then

$$\begin{aligned} u_{[i,j]}^{(m)}u_{[k,m]}^{(m)} &= (s_i \cdots s_k)(s_{k+1} \cdots s_j) \cdot (s_k \cdots s_m) \\ &= (s_i \cdots s_k) \cdot (s_k \cdots s_m)(s_k \cdots s_{j-1}) \\ &= (s_i \cdots s_{k-1})(s_{k+1} \cdots s_m)(s_k \cdots s_{j-1}) \\ &= (s_{k+1} \cdots s_m)(s_i \cdots s_{j-1}) = u_{[k+1,m]}^{(m)}u_{[i,j-1]}^{(m)}. \end{aligned}$$

q.e.d.

Let us recall the following well known fact, which holds in general.

Lemma 3.5. *Let* $\bar{P} \subset \tilde{P}$ *be parabolic subgroups of* G*. If* $w \in W_{\bar{P}}^{\tilde{P}}$*, then* $\ell(w) \leq \ell(\omega_{\tilde{P}}\omega_{\bar{P}})$*. Furthermore, the equality holds if and only if* $w = \omega_{\tilde{P}}\omega_{\bar{P}}$.

Proof. Note that $\omega_{\bar{P}}$ sends positive roots $R^+_{\bar{P}}$ to negative roots $-R^+_{\bar{P}} \subset -R^+_{\tilde{P}}$ and $\omega_{\tilde{P}}$ sends $-R^+_{\tilde{P}}$ to $R^+_{\tilde{P}}$. Hence $\omega_{\tilde{P}}\omega_{\bar{P}}(R^+_{\bar{P}}) \subset R^+$, implying $\omega_{\tilde{P}}\omega_{\bar{P}} \in W^{\bar{P}}_{\tilde{P}}$. Hence the statement follows, by noting $\omega_{\tilde{P}}$ is the unique longest element in $W_{\tilde{P}}$ and $\ell(\omega_{\tilde{P}}v) = \ell(\omega_{\tilde{P}}) - \ell(v)$ for any $v \in W_{\bar{P}}$. q.e.d.

Lemma 3.6. *For $\Delta_{\tilde{P}} = \Delta_P \setminus \{\alpha_k\}$ where $1 \le k \le r$, both of the following hold. a) $gr(\omega_P\omega_{\tilde{P}}) = \sum_{p=k}^r k\mathbf{e}_p$; b) for any $v \in W^{\tilde{P}}_P$, $gr(v) = \sum_{p=k}^r j_p\mathbf{e}_p$ with $j_r \le \cdots \le j_k \le k$.*

Proof. Write $gr(v) = \sum_{p=1}^r j_p\mathbf{e}_p$ and set $j_0 = 0$. For each $\alpha_p \in \Delta_{\tilde{P}}$, we have $\ell(vs_p) = \ell(v) + 1$ by Lemma 2.5. This implies $j_p \le j_{p-1}$ by Proposition 3.4. That is, $j_r \le \cdots \le j_{k+1} \le j_k \le k$ and $0 \le j_{k-1} \le \cdots \le j_1 \le j_0 = 0$. Thus b) follows.

Let $w = u_k^{(r)} \cdots u_k^{(k)}$. Note that $w \in W^{\tilde{P}}_P$ and $\ell(w) = k(r-k+1) = |R^+_P| - |R^+_{\tilde{P}}| = \ell(\omega_P\omega_{\tilde{P}})$. By Lemma 3.5, we have $w = \omega_P\omega_{\tilde{P}}$. That is, a) follows. q.e.d.

In addition, we introduce the next three useful lemmas.

Lemma 3.7 (see e.g., [**24**]). *Let $\gamma \in R^+$. Then $\ell(s_\gamma) \le \langle 2\rho, \gamma^\vee\rangle - 1$.*

Lemma 3.8. *Let $\gamma \in R^+ \setminus \Delta$ satisfy $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee\rangle - 1$. For any $1 \le j \le n$ with $\langle \alpha_j, \gamma^\vee\rangle > 0$, we have $\langle \alpha_j, \gamma^\vee\rangle = 1$. Furthermore, for $\beta := s_j(\gamma)$, we have $\beta^\vee = \gamma^\vee - \alpha_j^\vee$ and $\ell(s_\beta) = \ell(s_\gamma) - 2 = \langle 2\rho, \beta^\vee\rangle - 1$.*

Lemma 3.9. *Let $\gamma \in R^+ \setminus \Delta$. If $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee\rangle$ where $u \in W$, then $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee\rangle - 1$. Furthermore, we take any $1 \le j \le n$ with $\langle \alpha_j, \gamma^\vee\rangle > 0$ and set $\beta := s_j(\gamma)$. Then all the following hold:*

$$\begin{aligned}&\ell(us_j) = \ell(u) - 1, \ell(us_js_\beta)\\ &\qquad = \ell(us_j) - \ell(s_\beta), \ell(us_\gamma) = \ell(us_js_\beta s_j) = \ell(us_js_\beta) - 1.\end{aligned}$$

Proof. We prove all these three statements together, including the proof of Lemma 3.7 from [**24**] by induction on $\ell(s_\gamma)$.

If $\ell(s_\gamma) = 1$, then $\gamma \in \Delta$ and consequently $\ell(s_\gamma) = 1 = 2\langle \rho, \gamma^\vee\rangle - 1$. Now we assume $\gamma \in R^+ \setminus \Delta$. Take any $1 \le j \le n$ such that $\langle \gamma, \alpha_j^\vee\rangle > 0$ (such j does exist; otherwise, we would conclude $2 = \langle \gamma, \gamma^\vee\rangle \le 0$). Consequently, $\langle \alpha_j, \gamma^\vee\rangle > 0$. Thus $s_\gamma(\alpha_j) = \alpha_j - \langle \alpha_j, \gamma^\vee\rangle\gamma \in -R^+$. Also, $s_js_\gamma(\alpha_j) = (\langle \gamma, \alpha_j^\vee\rangle\langle \alpha_j, \gamma^\vee\rangle - 1)\alpha_j - \langle \alpha_j, \gamma^\vee\rangle\gamma$ is a negative root. By Lemma 2.5, we have $\ell(s_js_\gamma s_j) = \ell(s_\gamma) - 2$. Because $s_j(\gamma)^\vee = s_j(\gamma^\vee) = \gamma^\vee - \langle \alpha_j, \gamma^\vee\rangle\alpha_j^\vee$, we have $\langle \rho, s_j(\gamma)^\vee\rangle = \langle \rho, \gamma^\vee\rangle - \langle \alpha_j, \gamma^\vee\rangle$. By the induction hypothesis, we conclude the following:

$$\begin{aligned}&\ell(s_\gamma) = \ell(s_js_\gamma s_j) + 2 \le 2\langle \rho, s_j(\gamma)^\vee\rangle - 1 + 2 && (3.1)\\ &\qquad = 2\langle \rho, \gamma^\vee\rangle - 1 + 2(1 - \langle \alpha_j, \gamma^\vee\rangle) && (3.2)\\ &\qquad \le \langle 2\rho, \gamma^\vee\rangle - 1. && (3.3)\end{aligned}$$

If $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee\rangle - 1$, then both (2.1) and (2.3) must be equalities. In particular, we conclude $\langle \alpha_j, \gamma^\vee\rangle = 1$, $\beta^\vee = \gamma^\vee - \alpha_j^\vee$ and $\ell(s_\beta) = \ell(s_\gamma) - 2 = \langle 2\rho, \beta^\vee\rangle - 1$.

It remains to show Lemma 3.9. Indeed, we have

$$\ell(u) - \ell(s_\gamma) \le \ell(us_\gamma) = \ell(u) - (\langle 2\rho, \gamma^\vee\rangle - 1) \le \ell(u) - \ell(s_\gamma).$$

Hence, both inequalities become equalities. Thus $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee\rangle - 1$.

Furthermore, we note $\ell(us_js_\beta s_j) = \ell(us_\gamma) = \ell(u) - \ell(s_\gamma)$ and

$$\begin{aligned}\ell(us_js_\beta s_j) &\ge \ell(us_js_\beta) - 1\\ &\ge \ell(us_j) - \ell(s_\beta) - 1 \ge \ell(u) - 1 - \ell(s_\beta) - 1 = \ell(u) - \ell(s_\gamma).\end{aligned}$$

Hence, the statements in Lemma 3.9 also follow. q.e.d.

3.2. Explicit gradings of q_j's. The main results of this subsection are Proposition 3.10 and Proposition 3.12, giving explicit formulas for gradings $gr(q_j)$'s.

Proposition 3.10. *Let $2 \le j \le r$. Following the notation in Definition 2.8, we have $\psi_{\Delta_j, \Delta_{j-1}}(1, \alpha_j^\vee + Q_{j-1}^\vee) = (u_{j-1}^{(j-1)}, \alpha_j^\vee)$ and $gr(q_j) = (1-j)\mathbf{e}_{j-1} + (1+j)\mathbf{e}_j$.*

Proof. Note that $\Delta_{j-1} \subset \Delta_j$ with $\Delta_j \setminus \Delta_{j-1} = \{\alpha_j\}$. Clearly, $\langle \alpha, \alpha_j^\vee\rangle \in \{0, -1\}$ for all $\alpha \in R^+ \cap \bigoplus_{i=1}^{j-1} \mathbb{Z}\alpha_i$. Hence, we have $\Delta_{P'_{j-1}} = \{\alpha \in \Delta_{j-1} \mid \langle \alpha, \alpha_j^\vee\rangle = 0\} = \Delta_{j-1} \setminus \{\alpha_{j-1}\}$. Therefore we conclude $gr(\omega_{P_{j-1}}\omega_{P'_{j-1}}) = (j-1)\mathbf{e}_{j-1}$ by using Lemma 3.6 (with respect to Δ_{j-1}). Thus the former equality holds. Consequently, the latter equality follows by Definition 2.8. q.e.d.

The next lemma works in general, namely, we do not need to assume Δ_P to be of A-type.

Lemma 3.11. *Let $u \in W$ and $\lambda \in Q^\vee$.*

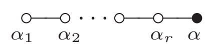
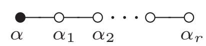
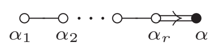
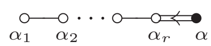
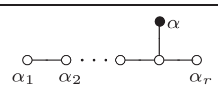
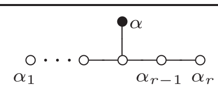
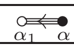
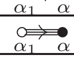
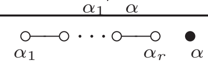
1) *Write $gr(q_\lambda u) = (j_1, \cdots, j_{r+1})$. Then $\sum_{k=1}^{r+1} j_k = \ell(u) + \langle 2\rho, \lambda\rangle$.*
2) *Let $\gamma \in R^+$ satisfy $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee\rangle$. For any $1 \le p \le n$, $gr(q_{\gamma^\vee} us_\gamma) \le gr(u) + gr(s_p)$ if and only if $gr_r(q_{\gamma^\vee} us_\gamma) \le gr_r(u) + gr_r(s_p)$.*

Proof. Denote $|(a_1, \cdots, a_{r+1})| = \sum_{k=1}^{r+1} a_k$. Note that $\ell(u) = |gr(u)|$. Furthermore, we conclude $|gr(q_{\alpha^\vee})| = 2$ for all $\alpha \in \Delta$ by induction. Thus (1) follows.

Write $gr(u) + gr(s_p) = (i_1, \cdots, i_{r+1})$ and $gr(q_{\gamma^\vee} us_\gamma) = (\tilde{i}_1, \cdots, \tilde{i}_{r+1})$. Assume $(\tilde{i}_1, \cdots, \tilde{i}_{r+1}) \le (i_1, \cdots, i_{r+1})$; then we have $(\tilde{i}_1, \cdots, \tilde{i}_r) \le (i_1, \cdots, i_r)$ by definition. Assume $(\tilde{i}_1, \cdots, \tilde{i}_r) \le (i_1, \cdots, i_r)$. If "<" holds, then it is already done by the definition of the lexicographical order. If "=" holds, then we conclude $\tilde{i}_{r+1} = i_{r+1}$, by noting $\sum_{k=1}^{r+1} \tilde{i}_k = |gr(q_{\gamma^\vee} us_\gamma)| = \ell(us_\gamma) + \langle 2\rho, \gamma^\vee\rangle = \ell(u) + 1 = \sum_{k=1}^{r+1} i_k$. Thus (2) follows. q.e.d.

Proposition 3.12. *For any $\alpha \in \Delta \setminus \Delta_P$, one and only one of the cases in Table* 3 *occurs, where we require $r \geq 2$ (resp.* 3 *and* 5*) for case* b) *(resp.* e) *and* f)*).*

Table 3. Explicit grading $gr(q_{\alpha^\vee})$ for $\alpha \in \Delta \setminus \Delta_P$

	$Dyn(\Delta_P \cup \{\alpha\})$	$\psi_{\Delta,\Delta_P}(1, \alpha^\vee + Q_P^\vee)$	$gr(q_{\alpha^\vee})$
a)		$q_{\alpha^\vee} u_r^{(r)}$	$(r+2)\mathbf{e}_{r+1} - r\mathbf{e}_r$
b)		$q_{\alpha^\vee} u_1^{(r)} \cdots u_1^{(1)}$	$(r+2)\mathbf{e}_{r+1} - \sum_{j=1}^r \mathbf{e}_j$
c)		$q_{\alpha^\vee} q_r u_{r-1}^{(r)} u_{r-1}^{(r-1)}$	$(2r+2)\mathbf{e}_{r+1} - 2r\mathbf{e}_r$
d)		$q_{\alpha^\vee} u_r^{(r)}$	$(r+2)\mathbf{e}_{r+1} - r\mathbf{e}_r$
e)		$q_{\alpha^\vee} u_{r-1}^{(r)} u_{r-1}^{(r-1)}$	$2r\mathbf{e}_{r+1} + (1-r)(\mathbf{e}_r + \mathbf{e}_{r-1})$
f)		$q_{\alpha^\vee} u_{r-2}^{(r)} u_{r-2}^{(r-1)} u_{r-2}^{(r-2)}$	$(3r-4)\mathbf{e}_{r+1} + (2-r) \sum_{j=r-2}^{r} \mathbf{e}_j$
g)		$q_{\alpha^\vee} s_1$	$(-1, 3)$
h)		$q_{\alpha^\vee} q_1 s_1$	$(-3, 5)$
i)		$q_{\alpha^\vee}$	$2\mathbf{e}_{r+1}$

Proof. Clearly, $Dyn(\Delta_P \cup \{\alpha\})$ is given by a unique case in Table 3.

Let $\lambda_P = \alpha^\vee + Q_P^\vee$ and $\psi_{\Delta,\Delta_P}(1, \lambda_P) = q_{\lambda_B}\omega_P\omega_{P'}$. Here $\lambda_B \in Q^\vee$ is the (unique) element satisfying $\langle \beta, \lambda_B \rangle \in \{0, -1\}$ for all $\beta \in R_P^+$. Since Δ_P is of A-type, this is equivalent to requiring $\langle \alpha_j, \lambda_B \rangle = 0$ for all $\alpha_j \in \Delta_P$ but at most 1, and if such unique α_j exists, then $\langle \alpha_j, \lambda_B \rangle = -1$. For each case in Table 3, it is easy to see that the element λ_B as provided does satisfy this property. Consequently, $\Delta_{P'} = \{\alpha_i \in \Delta_P \mid \langle \alpha_i, \lambda_B \rangle = 0\} = \Delta_P \setminus \{\alpha_k\}$ for a certain $1 \leq k \leq r+1$. Hence, we can directly write down $\omega_P\omega_{P'}$ by using Lemma 3.6. Finally, we obtain $gr(q_{\alpha^\vee})$ as is listed in Table 3, by direct calculations (with Definition 2.8 and Proposition 3.10). q.e.d.

The next corollary follows directly from Table 3.

Corollary 3.13. *If $r = 1$, then $gr(q_j) = (a, -a+2)$ with $a = \langle \alpha_1, \alpha_j^\vee \rangle$ for each j. Consequently, for any $\lambda \in Q^\vee$, we have $gr(q_\lambda) = (\langle \alpha_1, \lambda \rangle, \langle 2\rho - \alpha_1, \lambda \rangle)$.*

As we will see later, we use induction on $\ell(s_\gamma)$ to prove the Key Lemma. The next proposition shows the special case of the Key Lemma when $\ell(s_\gamma) = 1$.

Proposition 3.14. *Let $u \in W$ and $1 \le j \le n$. If $\ell(us_j) = \ell(u) - 1$, then $gr(q_j us_j) \le gr(u) + gr(s_j)$.*

Proof. Let $gr(u) = (i_1, \cdots, i_{r+1})$ and $gr(us_j) = (\tilde{i}_1, \cdots, \tilde{i}_{r+1})$. When $\alpha_j \in \Delta_P$, we have $1 \le j \le r$. If $j = 1$, then we have $i_1 = 1$ and $gr(us_1) = (0, i_2, \cdots, i_{r+1})$. Hence, $gr(q_1 us_1) = gr(q_1) + gr(us_1) = (2, 0, \cdots, 0) + (0, i_2, \cdots, i_{r+1}) = gr(u) + gr(s_1)$. If $2 \le j \le r$, then by Proposition 3.4 and Proposition 3.10, we conclude

$$\begin{aligned} &gr(q_j us_j) - gr(u) - gr(s_j) \\ &\quad = ((1-j) + i_j - 1 - i_{j-1})\mathbf{e}_{j-1} + ((1+j) + i_{j-1} - i_j - 1)\mathbf{e}_j. \end{aligned}$$

Thus we have $gr(q_j us_j) \le gr(u) + gr(s_j)$, by noting $0 \le i_{j-1} < i_j \le j$.

When $\alpha_j \in \Delta \setminus \Delta_P$, we note that $gr(s_j) = \mathbf{e}_{r+1}$. By Lemma 3.11, it suffices to show $gr_r(q_j us_j) \le gr_r(u)$. Write $gr(q_j us_j) = (\hat{i}_1, \cdots, \hat{i}_{r+1})$. and $\psi_{\Delta,\Delta_P}(1, \alpha_j^\vee + Q_P^\vee) = \lambda_B \omega_P \omega'$. We first assume $\lambda_B = \alpha_j^\vee$. Then $\Delta_{P'} = \{\alpha \in \Delta_P \mid \langle \alpha, \alpha_j^\vee \rangle = 0\}$. If $\Delta_{P'} = \Delta_P$ (i.e., case i) of Table 3 occurs), then we have $gr_r(q_j) = \mathbf{0}$ and $gr_r(us_j) = gr_r(u)$ (by Lemma 3.2). Thus it is done in this case. Otherwise, we conclude $\Delta_{P'} = \Delta_P \setminus \{\alpha_a\}$ for a unique $1 \le a \le r$ (from Table 3). Consequently, we have $gr_r(q_j) = -gr_r(\omega_P \omega')$ by definition, $gr_r(us_j) \le gr_r(u) + \sum_{k=a}^r a\mathbf{e}_k$ by Proposition 3.1, and $gr(\omega_P \omega') = \sum_{k=a}^r a\mathbf{e}_k$ by Lemma 3.6. Hence, we do have $gr_r(q_j us_j) \le gr_r(u)$ in this case. Now we assume $\lambda_B \ne \alpha_j^\vee$. Due to Table 3, it remains to consider case c) and h). If case h) occurs, then $n = 2, r = 1, gr(q_j) = (-3, 5)$ and we do have $\hat{i}_1 = -3 + \tilde{i}_1 \le -2 < i_1$. If case c) occurs, then $gr_r(q_j) = -2r\mathbf{e}_r$ and we have $gr_{r-1}(us_j) = gr_{r-1}(u)$ by Lemma 3.2. Hence, $gr_r(q_j us_j) - gr_r(u) = (-2r + \tilde{i}_r - i_r)\mathbf{e}_r \le (-2r + r - 0)\mathbf{e}_r < \mathbf{0}$. Hence, the statement follows. q.e.d.

3.3. Proof of the Key Lemma when Δ_P is of A-type.

Recall that we have assumed Δ_P to be of A-type in this subsection.

Proposition 3.15. *Part* a) *of the Key Lemma holds.*

Proof. Write $us_\gamma = v_{r+1} v_r \cdots v_1$, where $v_{r+1} \in W^P$ and $v_k = u_{i_k}^{(k)}$ for $1 \le k \le r$. Thus $gr(us_\gamma) = (i_1, \cdots, i_r, \ell(v_{r+1}))$. Fix a reduced expression of v_{r+1}. Since $\ell(u) = \ell(us_\gamma s_\gamma) < \ell(us_\gamma)$, by Lemma 2.5 we have $u = v_{r+1} \cdots v_{m+1} \bar{v}_m v_{m-1} \cdots v_1$ for some $1 \le m \le r+1$, in which $\bar{v}_m$ is the element obtained by with deleting a (unique) simple reflection from v_m. Since $\ell(u) = \ell(us_\gamma) - 1$, the induced expression of u is also reduced. Hence, $\ell(\bar{v}_m) = \ell(v_m) - 1$, and if we write $\bar{v}_m = v'w$ with $v' \in W_{P_m}^{P_{m-1}}$ and $w \in W_{P_{m-1}}$, then $\ell(\bar{v}_m) = \ell(v') + \ell(w)$ and $\ell(wv_{m-1} \cdots v_1) = \ell(w) + \ell(v_{m-1} \cdots v_1)$. By Proposition 3.4, there exist non-negative integers p_k's such that $gr(u) = (i_1 + p_1, \cdots, i_{m-1} + p_{m-1}, \ell(v'), i_{m+1}, \cdots, i_r, \ell(v_{r+1}))$ with $\sum_{k=1}^{m-1} p_k = \ell(w)$. On the other hand, by Lemma 2.7 we conclude $\gamma \in R_{P_m}^+ \setminus R_{P_{m-1}}$, so that $\min\{gr(s_i) \mid \langle \chi_i, \gamma^\vee \rangle \ne 0\} = \mathbf{e}_m$. Hence, we have $gr(us_\gamma) = (i_1, \cdots, i_r, \ell(v_{r+1})) \leq (i_1+p_1, \cdots, i_{m-1}+p_{m-1}, \ell(v')+1, i_{m+1}, \cdots, i_r, \ell(v_{r+1})) = gr(u) + \mathbf{e}_m$, by noting $\ell(v_{r+1}) + \sum_{k=1}^r i_k = \ell(us_\gamma) = \ell(u) + 1 = \ell(v') + \sum_{k=1}^r i_k + \sum_{k=1}^{m-1} p_k + 1$. q.e.d.

The remaining part of this subsection is devoted to a proof of the following.

Proposition 3.16. *Part* b) *of the Key Lemma holds. That is, for any $u \in W$ and $\gamma \in R^+$, if* **(L1)**: $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle$, *then we have* **(L2)**: $gr(q_{\gamma^\vee} us_\gamma) \leq gr(u) + \min\{gr(s_i) \mid \langle \chi_i, \gamma^\vee \rangle \neq 0\}$.

Lemma 3.17. *Part* b) *of the Key Lemma holds when $\Delta_P = \{\alpha_1\}$.*

Proof. We use induction on $\ell(s_\gamma)$. If $\ell(s_\gamma) = 1$, then $\gamma \in \Delta$ and consequently (L2) follows from Proposition 3.14. Now we assume $\gamma \in R^+ \backslash \Delta$. Write $gr(u) = (i_1, i_2)$, $gr(us_\gamma) = (j_1, j_2)$, and $gr(q_{\gamma^\vee}) = (k_1, k_2)$, in which $k_1 = \langle \alpha_1, \gamma^\vee \rangle$ by Corollary 3.13. If $k_1 < 0$, then $j_1 + k_1 \leq 1 + k_1 \leq 0$. If $k_1 = 0$, then we have $i_1 = j_1$ by Lemma 3.2. In either of the cases, we conclude $j_1 + k_1 \leq i_1$. Thus (L2) holds by Lemma 3.11. Otherwise, $\langle \alpha_1, \gamma^\vee \rangle = k_1 > 0$. Then by Lemma 3.8 and Lemma 3.9, we conclude that for $\beta := s_1(\gamma)$ the following holds: $\beta^\vee = \gamma^\vee - \alpha_1^\vee$; $gr(q_1) + gr(us_1) \leq gr(u) + gr(s_1)$ (by Proposition 3.14); $gr(q_{\beta^\vee}) + gr(us_1 s_\beta) \leq gr(us_1) + \mathbf{e}_c$ (by the induction hypothesis), where we denote $\mathbf{e}_c := \min\{gr(s_i) \mid \langle \chi_i, \beta^\vee \rangle \neq 0\}$; $gr(us_1 s_\beta s_1) = gr(us_1 s_\beta) - gr(s_1)$ (by Proposition 3.4). Hence, we conclude (L2) holds, by noting $\mathbf{e}_c = \min\{\mathbf{e}_c, gr(s_1)\} = \min\{gr(s_i) \mid \langle \chi_i, \gamma^\vee \rangle \neq 0\}$. q.e.d.

When Δ is also of A-type, it is easy to obtain λ_B and $gr(q_{\lambda_B})$ associated to a given $\lambda_P \in Q^\vee/Q_P^\vee$. For instance, by direct calculations we conclude the following lemma. (Recall that $\alpha_j = \beta_{o+j}$ for $1 \leq j \leq r$ in Table 2.)

Lemma 3.18. *Let Δ be of A-type and $m \leq r+1$. Following the notation in case* C1)*, we set $\lambda = \sum_{k=1}^m k\beta^\vee_{o+r+1-m+k}$. Then the following holds (where $0 \cdot \mathbf{e}_0 := \mathbf{0}$).*

1) *If $m = r+1$, then $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_P$; if $m < r+1$, then for any $\alpha \in \Delta_P$, $\langle \alpha, \lambda \rangle$ is equal to -1 if $\alpha = \alpha_{r+1-m}$, or equal to 0 otherwise. In particular, λ is the element associated to $m\beta^\vee_{o+r+1} + Q_P^\vee$ via PW-lifing.*
2) *$gr(q_\lambda) = m(r+2)\mathbf{e}_{r+1} - (r+1-m)\sum_{k=r+1-m}^r \mathbf{e}_k$. In particular, if $m = r+1$, then $gr_r(q_\lambda) = \mathbf{0}$.*

Furthermore, we have $gr_r(\sum_{k=o}^{o+r+1} \beta_k^\vee) = \mathbf{0}$, whenever $o \geq 1$.

Since the case of A-type is relatively easy to handle, we would like to compare all relevant information for Δ being of general type with those when Δ is of A-type. Due to Lemma 3.11, we only need to care about $gr_r(q_\lambda w)$. For these purposes, we bring in a base $\dot{\Delta}$ of A-type and introduce the notion of "*virtual coroot*" as below for $r \geq 2$.

Let $\dot{\Delta} = \{\dot{\beta}_1, \cdots, \dot{\beta}_n\}$ be a base with $Dyn(\dot{\Delta})$ given by $\underset{\dot\beta_1}{\circ}\!-\!\underset{\dot\beta_2}{\circ}\cdots\circ\!-\!\underset{\dot\beta_n}{\circ}$. Denote $\dot{\alpha}_i = \dot{\beta}_{o+i}$ for each $1 \leq i \leq r$ (the notation "o" is the same one as in Table 2). Set $\dot{\Delta}_P = \{\dot{\alpha}_1, \cdots, \dot{\alpha}_r\}$. Following Definition 2.8, we can obtain a grading map with respect to $(\dot{\Delta}_P, \mathrm{Id}_{\dot{\Delta}_P})$, which we also denote as gr by abuse of notation. Clearly, $\dot{\beta}_j \mapsto \beta_j$ extends to an isometry $\dot{\Delta} \setminus \{\dot{\beta}_\eta\} \to \Delta \setminus \{\beta_\eta\}$ of bases, where η is given in Table 4. Denote $\dot{Q}^\vee = \bigoplus_{i=1}^n \mathbb{Z}\dot{\beta}_i^\vee$.

Definition 3.19. Let $\lambda \in Q^\vee$. We call $\dot{\lambda} \in \dot{Q}^\vee$ a **virtual coroot** of λ (at level η) if $\dot{\lambda}$ satisfies both $gr_r(q_{\dot{\lambda}}) = gr_r(q_\lambda)$ and $\langle \dot{\alpha}_i, \dot{\lambda} \rangle = \langle \alpha_i, \lambda \rangle$ for $1 \leq i \leq r$.

Lemma 3.20. *For each case in Table* 2 *(where we have assumed $r \geq 2$), there is a virtual coroot $\dot{\lambda}$ of $\lambda = \sum_{j=1}^n c_j \beta_j^\vee$ (at level η), given by Table* 4.

Table 4. Virtual coroot $\dot{\lambda} = c_\eta \dot{\mu} + \sum_{j=1}^{\eta-1} c_j \dot{\beta}_j^\vee$

	C1)	C9)	C10)	C2)	C3)	C5)	C7)	C4)	C6)	C8)
η	n	3	3	n	4	5	7	8	8	7
$\dot{\mu}$	$-\langle \beta_{\eta-1}, \beta_\eta^\vee \rangle \dot{\beta}_\eta^\vee$			$\dot{\beta}_{\eta-1}^\vee + 2\dot{\beta}_\eta^\vee$				$\sum_{j=1}^{3} j\dot{\beta}_{5+j}^\vee$	$\sum_{j=1}^{5} j\dot{\beta}_{3+j}^\vee$	$\sum_{j=1}^{5} j\dot{\beta}_{2+j}^\vee$

Proof. Note that $\dot{\Delta} \setminus \{\dot{\beta}_\eta\}$ is canonically isomorphic to $\Delta \setminus \{\beta_\eta\}$ as bases and that $\Delta_P \subset \{\beta_1, \cdots, \beta_{\eta-1}\}$. It is easy to see $\dot{\beta}_j^\vee$ is a virtual coroot of $\beta_j^\vee$ (resp. 0) for each $j \leq \eta - 1$ (resp. $j \geq \eta + 1$). Combining Table 2 and Table 3, we conclude that $gr_r(q_{\dot{\mu}}) = gr_r(q_{\beta_\eta^\vee})$ and $\langle \dot{\alpha}_i, \dot{\mu} \rangle = \langle \alpha_i, \beta_\eta^\vee \rangle$ for $1 \leq i \leq r$. That is, $\dot{\mu}$ is a virtual coroot of $\beta_\eta^\vee$. Hence, the statement follows. q.e.d.

Remark 3.21. Lemma 3.20 tells us about the existence of a virtual coroot. Due to Lemma 3.18, we note that the uniqueness does not hold: if $\dot{\lambda}$ is a virtual coroot of λ, so is $\dot{\lambda} + \sum_{j=1}^{\eta} j\dot{\beta}_j^\vee$.

Due to Lemma 3.17, it remains to care about the case when $r \geq 2$. The next proposition shows that we can describe most of the coroots uniformly with the help of the notion of "virtual coroot."

Proposition 3.22. *Assume $r \geq 2$. Let $\gamma \in R^+ \setminus \Delta$ satisfy $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee \rangle - 1$. Then one and only one of the following holds.*

1) *There exists a virtual coroot $\dot{\gamma}^\vee = \sum_{p=1}^{r+1} \dot{c}_p \dot{\alpha}_p^\vee$ of $\gamma^\vee$, where $\dot{\alpha}_{r+1} := \dot{\beta}_{o+r+1}$, $\dot{c}_{r+1} \leq r$ and $\dot{c}_p - 1 \leq \dot{c}_{p-1} \leq \dot{c}_p$ for each $p \in \{1, \cdots, r+1\}$ (where $\dot{c}_0 := 0$).*
2) *$\gamma^\vee = \sum_{p=d}^{m} \beta_p^\vee$ where $o \leq m \leq o + r$ and $d < m$.*

3) *Case* C9) *occurs and* $\gamma^\vee = \beta_3^\vee + \beta_4^\vee$.

Proof. Let $\gamma^\vee=\sum_{j=1}^n c_j\beta_j^\vee$, which has a virtual coroot $\sum_{j=1}^\eta \tilde{c}_j\dot{\beta}_j^\vee$ by Lemma 3.20.

We first assume $c_\eta \neq 0$. Set $(c_1', \cdots, c_\eta') := (c_1, \cdots, c_\eta) - [\frac{c_\eta}{\eta}](1, \cdots, \eta)$. Then we obtain another virtual coroot $\sum_{j=1}^\eta c_j'\dot{\beta}_j^\vee$ of $\gamma^\vee$ by noting that $\sum_{j=1}^\eta j\dot{\beta}_j^\vee$ is a virtual coroot of 0. We claim $c_j' - 1 \le c_{j-1}' \le c_j'$ for each i (where $c_0' := 0$) and show this by discussing all possible coroots with respect to the type of Δ.

When Δ is of A-type, clearly it is done (by noting $(c_1', \cdots, c_\eta') = (\tilde{c}_1, \cdots, \tilde{c}_\eta) = (0, \cdots, 0, 1, \cdots, 1)$). When Δ is of D-type, either C2) or C3) will occur. For the former case, we have $\eta = n$ and $c_{n-2} \in \{1, 2\}$. If $c_{n-2} = 2$, then we have $c_{n-1} = 1$ and $(c_1', \cdots, c_n') = (\tilde{c}_1, \cdots, \tilde{c}_n) = (0, \cdots, 0, 1, \cdots, 1, 2, \cdots, 2)$. If $c_{n-2} = 1$, then $\gamma^\vee = \beta_n^\vee + \sum_{p=a}^b \beta_p^\vee$ for some $a \le n-2 \le b \le n-1$. Hence, $(c_1', \cdots, c_n') = (\tilde{c}_1, \cdots, \tilde{c}_n) = (0, \cdots, 0, 1, \cdots, 1, 1 + \delta_{b,n-1}, 2)$. Thus our claim holds. For the latter case, we have $\eta = 4$ and can show our claim with similar arguments. When Δ is of E-type, there are only finite coroots which are listed in Plate V, VI and VII of [**6**]. In this case, our claim still holds by direct calculations.

When Δ is of type B_n (resp. C_n), then our claim follows immediately from Plate III (resp. II) of [**6**], except for the following coroots.

	C1) for type B_n	C1) for type C_n
$\gamma^\vee$	$\beta_n^\vee + 2\sum\limits_{i\leqslant p<n} \beta_p^\vee \quad (1 \le i < n)$	$\sum\limits_{i\leqslant p<j} \beta_p^\vee + 2\sum\limits_{j\leqslant p\leqslant n} \beta_p^\vee \quad (1 \le i < j \le n)$

However, none of the above coroots satisfies our condition: $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee\rangle - 1$. Indeed if they satisfied this condition, then for the former case we would have $\langle \beta_i, \gamma^\vee\rangle = 2 > 1$, contrary to Lemma 3.8. For the latter case, we denote $\gamma_k^\vee = \sum_{i\le p<k} \beta_p^\vee + 2\sum_{k\le p\le n} \beta_p^\vee$ for $j \le k \le n$. Note that $\gamma_j^\vee = \gamma^\vee$ and $\langle \beta_k, \gamma_k^\vee\rangle > 0$ for all k. By Lemma 3.8, we have $\gamma_{j+1}^\vee = \gamma_j^\vee - \beta_j^\vee$ and $\ell(s_{\gamma_{j+1}}) = \langle 2\rho, \gamma_{j+1}^\vee\rangle - 1$. Thus by induction we conclude $\ell(s_{\gamma_n}) = \langle 2\rho, \gamma_n^\vee\rangle - 1$. However, $\langle \beta_n, \gamma_n^\vee\rangle = \langle \beta_n, \beta_{n-1}^\vee + 2\beta_n^\vee\rangle = 2 > 1$, contrary to Lemma 3.8 again. Hence, our claim holds in this case. When Δ is of type F_4, which is the remaining case we need to consider since $r \ge 2$, case C10) or C9) must occur. When C10) occurs, our claim follows immediately from Plate VIII of [**6**] and Table 4. When C9) occurs, we denote $M := \max\{c_1, c_2, c_3, c_4\}$. If $M > 1$, then there are 14 coroots in total (see Plate VIII of [**6**]), only 5 coroots among which satisfy our condition on the length. Explicitly, $(c_1, c_2, c_3, c_4) = (1, 2, 1, 0), (1, 2, 1, 1), (1, 2, 2, 1), (1, 3, 2, 1)$, or $(2, 3, 2, 1)$. If $M = 1$, then $\gamma^\vee = \sum_{p=a}^b \beta_p^\vee$ for some $1 \le a < b \le 4$. Clearly, our claim follows, except for the coroot $\gamma^\vee = \beta_3^\vee + \beta_4^\vee$.

Note that $\Delta_P \subset \{\beta_1, \cdots, \beta_{\eta-1}\}$. We conclude $\dot{\beta}_j^\vee$ is a virtual coroot of 0 whenever $j < o$ or $j > o+1$. In particular, we set $\dot{c}_i = c'_{o+i} - c'_o$ for each $0 \le i \le r+1$. Then we obtain a virtual coroot $\dot{\gamma}^\vee = \sum_{p=1}^{r+1} \dot{c}_p \dot{\alpha}_p^\vee$ of $\gamma^\vee$ satisfying $\dot{c}_p - 1 \le \dot{c}_{p-1} \le \dot{c}_p$ for each p, whenever $c_\eta \neq 0$ except when the case of statement (3) occurs. Furthermore, we note that $\dot{c}_{r+1} \le r+1$ and if "=" holds then we must have $\dot{\gamma}^\vee = \sum_{p=1}^{r+1} p\dot{\alpha}_p^\vee$, which is still a virtual coroot of 0. In this case, we just replace $\dot{\gamma}^\vee$ with $0 = \sum_{p=1}^{r+1} 0 \cdot \dot{\alpha}_p^\vee$.

Now we assume $c_\eta = 0$. Note that $Dyn(\{\beta_1, \cdots, \beta_{\eta-1}\})$ is of A-type and that $o + r + 1 \le \eta$. Thus if 0 is not a virtual coroot of $\gamma^\vee$, then we must have $\gamma^\vee = \sum_{p=d}^{m} \beta_p^\vee$ for some $1 \le d < m \le \eta - 1$. Hence, one of the following must hold: (i) $m < o$; (ii) $m \ge o + r + 1$ and $d \le o$; (iii) $m \ge o + r + 1$ and $d > o$; (iv) $o \le m \le o + r$. If either (i) or (ii) held, then 0 would be a virtual coroot of $\gamma^\vee$. If (iii) holds, then $\sum_{p=d}^{r+1} \dot{\beta}^\vee$ is a virtual coroot of $\gamma^\vee$, so that statement (1) holds. If (iv) holds, then statement (2) holds. q.e.d.

Proof of Proposition 3.16. Due to Lemma 3.17, we assume $r \ge 2$ and then use induction on $\ell(s_\gamma)$.

If $\ell(s_\gamma) = 1$, then $\gamma \in \Delta$ and consequently (L2) follows from Proposition 3.14.

Now we assume $\gamma \in R^+ \setminus \Delta$. Take any $1 \le j \le n$ with $\langle \alpha_j, \gamma^\vee \rangle > 0$. Write $\beta = s_j(\gamma)$, $gr(q_{\beta^\vee}) = (\lambda_1, \cdots, \lambda_{r+1}), \min\{gr(s_i) \mid \langle \chi_i, \beta^\vee \rangle \neq 0\} = \mathbf{e}_c$ and

$$\begin{aligned} gr(q_j) + gr(us_j) &= gr(u) + (a_1, \cdots, a_{r+1}), \\ gr(q_{\beta^\vee}) + gr(us_js_\beta) &= gr(us_j) + \mathbf{e}_c + (\mu_1, \cdots, \mu_{r+1}), \\ gr(us_js_\beta s_j) &= gr(us_js_\beta) + (b_1, \cdots, b_{r+1}). \end{aligned}$$

Thus we have $gr(q_{\gamma^\vee} us_\gamma) = gr(u) + \mathbf{e}_c + \sum_{p=1}^{r+1} (a_p + b_p + \mu_p)\mathbf{e}_p$, taking the summation of the last three equalities. Due to Lemma 3.8 and Lemma 3.9, we conclude $\min\{gr(s_i) | \langle \chi_i, \gamma^\vee \rangle \neq 0\} = \min\{\mathbf{e}_c, gr(s_j)\}$ and $(\mu_1, \cdots, \mu_{r+1}) \le \mathbf{0}$ by the induction hypothesis. Furthermore, we have $(a_1, \cdots, a_{r+1}) \le gr(s_j)$ by Proposition 3.14. We first make several observations as follows.

- (Ob1) Assume $\mathbf{e}_c \le gr(s_j)$ and $\langle \alpha, \alpha_j^\vee \rangle = 0$ for all $\alpha \in \Delta_P$. Then $(b_1, \cdots, b_{r+1}) = -\mathbf{e}_{r+1} = -gr(s_j)$ (by Lemma 3.2). Consequently, (L2) follows.
- (Ob2) Assume $j = 1$. Then we have $\mathbf{e}_c \le gr(s_1)$, $(b_1, \cdots, b_{r+1}) = -gr(s_1)$ by Proposition 3.4 and consequently (L2) follows.
- (Ob3) Assume $2 \le j \le r$ and $\mathbf{e}_c < \mathbf{e}_j$. Then $gr(q_j) = (j+1)\mathbf{e}_j - (j-1)\mathbf{e}_{j-1}$. Write $gr(u) = \sum_{p=1}^{r+1} i_p \mathbf{e}_p$ and $gr(us_js_\beta) = \sum_{p=1}^{r+1} k_p \mathbf{e}_p$. Note that $\ell(us_j) = \ell(u) - 1$ and $\ell(us_js_\beta s_j) = \ell(us_js_\beta) - 1$. By

Proposition 3.4, we have

$$(a_1, \cdots, a_{r+1}) = (i_j - i_{j-1} - j)\mathbf{e}_{j-1} + (j + 1 + i_{j-1} - i_j)\mathbf{e}_j,$$
$$(b_1, \cdots, b_{r+1}) = (k_j - k_{j-1} - 1)\mathbf{e}_{j-1} + (k_{j-1} - k_j)\mathbf{e}_j,$$

$k_j = i_{j-1} + \mu_j - \lambda_j$ and $k_{j-1} = i_j - 1 + \mu_{j-1} - \lambda_{j-1}$.

As a consequence, we have $(a_1+b_1+\mu_1, \cdots, a_{r+1}+b_{r+1}+\mu_{r+1}) = (\mu_1, \cdots, \mu_{j-2}, \mu_j + M, \mu_{j-1} - M, \mu_{j+1}, \cdots, \mu_{r+1})$, where $M := \lambda_{j-1} - \lambda_j - j$. Thus if $M = 0$ and $(\mu_1, \cdots, \mu_{j-2}, \mu_j, \mu_{j-1}) \leq (0, \cdots, 0)$, then (L2) follows.

Now we begin to discuss all possibilities for $\gamma^\vee$, using Proposition 3.22.

When case (1) of Proposition 3.22 holds, there exists a virtual coroot $\dot\gamma^\vee = \sum_{p=1}^{r+1} \dot c_p \dot\alpha_p^\vee$ of $\gamma^\vee$ such that $\dot c_{r+1} \leq r$ and $\dot c_p - 1 \leq \dot c_{p-1} \leq \dot c_p$ for each p (recall that $\dot\alpha_p := \dot\beta_{o+p}$ and $\dot c_0 = 0$). Clearly, if a): $\dot c_{r+1} = 0$, then all $\dot c_p$'s are equal to 0. If b): $1 \leq \dot c_{r+1} \leq r$ and any two non-zero $\dot c_p$ and $\dot c_{p'}$ are distinct, then we have $\sum_{p=1}^{r+1} \dot c_p \dot\alpha_p = \sum_{p=1}^{m+1} p\dot\alpha_{r-m+p}$ where $0 \leq m < r$. Otherwise, we have c): $1 \leq \dot c_{r+1} \leq r$ and there exist distinct $p < p'$ such that $\dot c_p = \dot c_{p'} \neq 0$. This must imply $\dot c_p = \dot c_{p+1}$ since $(\dot c_1, \cdots, \dot c_{r+1})$ is a non-decreasing sequence. Corresponding to these three cases, we have the following conclusions.

a) $(\dot c_1, \cdots, \dot c_{r+1}) = (0, \cdots, 0)$. Then we have $gr_r(q_{\gamma^\vee}) = \mathbf{0}$ and $gr_r(us_\gamma) = gr_r(u)$ by Lemma 3.2. Thus (L2) holds by Lemma 3.11.

b) $\dot\gamma^\vee = \sum_{p=1}^{m+1} p\dot\alpha_{r-m+p}$, where $0 \leq m < r$. Hence, we have $gr_r(q_{\gamma^\vee}) = gr_r(q_{\dot\gamma^\vee}) = (m-r)\sum_{p=r-m}^{r} \mathbf{e}_p$ (by Lemma 3.18) and $\langle \alpha_p, \gamma^\vee\rangle = \langle \dot\alpha_p, \dot\gamma^\vee\rangle = 0$ for $p \in \{1, \cdots, r\} \setminus \{r - m\}$. By Proposition 3.1, we have $gr_r(us_\gamma) \leq gr_r(u) + (r - m)\sum_{p=r-m}^{r} \mathbf{e}_p$. Thus (L2) holds by Lemma 3.11.

c) In this case, we can take $j := \min\{p \mid 1 \leq p \leq r, \dot c_p = \dot c_{p+1} \neq 0\}$. That is, $\sum_{p=1}^{j+1} \dot c_p \dot\alpha_p = m\dot\alpha_{j+1} + \sum_{p=1}^{m} p\dot\alpha_{j-m+p}$, where $1 \leq m \leq j \leq r$. Consequently, we have $\langle \alpha_j, \gamma^\vee\rangle = \langle \dot\alpha_j, \dot\gamma^\vee\rangle > 0$, $\dot\beta^\vee := \dot\gamma^\vee - \dot\alpha_j^\vee$ is virtual coroot of $\beta^\vee (= \gamma^\vee - \alpha_j^\vee)$, and $\mathbf{e}_c < \mathbf{e}_j$. If $j = 1$, then we are done by (Ob2). If $j \geq 2$, then we use (Ob3). Note that $gr_r(q_{\beta^\vee}) = gr_r(q_{\dot\beta^\vee})$. By using Lemma 3.18, we conclude $(\lambda_1, \cdots, \lambda_{j-2}) = (m - j)\sum_{p=j-m}^{j-2} \mathbf{e}_p$, $\lambda_{j-1} = (m - j) - (-j + 1) = m - 1$ and $\lambda_j = m(j + 1) - mj - (j + 1) = m - 1 - j$. Hence, $M = \lambda_{j-1} - \lambda_j - j = 0$.

By the induction hypothesis, we have $\sum_{p=1}^{j-2} \mu_p\mathbf{e}_p \leq \mathbf{0}$. If "<" holds, already done. If "=" holds, we have $\mu_1 = \cdots = \mu_{j-2} = 0$ and consequently $\mu_{j-1} \leq 0$. Write $gr(us_\gamma) = (\tilde k_1, \cdots, \tilde k_{r+1})$, $gr(q_{\gamma^\vee}) = (\tilde\lambda_1, \cdots, \tilde\lambda_{r+1})$ and $gr(us_j) = (\tilde i_1, \cdots, \tilde i_{r+1})$. Then $\lambda_p + k_p = \tilde i_p + \mu_p$, $\tilde\lambda_p = \lambda_p$, $\tilde i_p = i_p$ and $\tilde k_p = k_p$ for $1 \leq p \leq j - 2$.

Hence, $(\tilde{k}_1, \cdots, \tilde{k}_{j-2}) = (i_1, \cdots, i_{j-2}) + (j-m)\sum_{p=j-m}^{j-2} \mathbf{e}_p$. Since $\langle \alpha_p, \gamma^\vee \rangle = \langle \dot{\alpha}_p, \dot{\gamma}^\vee \rangle = 0$ for all $p \in \{1, 2, \cdots, j-1\} \setminus \{j-m\}$, we conclude $(\tilde{k}_1, \cdots, \tilde{k}_{j-1}) \le (i_1, \cdots, i_{j-1}) + (j-m)\sum_{p=j-m}^{j-1} \mathbf{e}_p$, by using Proposition 3.1 with respect to $(\Delta, \Delta_{\bar{P}})$ with $\Delta_{\bar{P}} = (\alpha_1, \cdots, \alpha_{j-1})$. Thus we have $\tilde{k}_{j-1} \le i_{j-1} + (j-m)$. Since $\ell(us_j) = \ell(u) - 1$ and $\ell(us_\gamma) = \ell(us_j s_\beta s_j) = \ell(us_j s_\beta) - 1$, we have $\tilde{i}_j = i_{j-1}$ and $\tilde{k}_{j-1} = k_j - 1$. Hence, $\mu_j = k_j + \lambda_j - \tilde{i}_j = \tilde{k}_{j-1} + 1 + (m-1-j) - i_{j-1} \le 0$.

Therefore, we conclude $(\mu_1, \cdots, \mu_{j-2}, \mu_j, \mu_{j-1}) \le (0, \cdots, 0)$ and consequently (L2) holds by (Ob3).

When case (2) of Proposition 3.22 holds, we have $\gamma^\vee = \sum_{p=d}^{m} \beta_p^\vee$ where $o \le m \le o+r$ and $d < m$. If $m = o$, then $d < o$; consequently, we take $\alpha_j = \beta_d$ and use (Ob1). If $m = o+1$ or $d = o+1$, then we take $j = 1$ and use (Ob2). Otherwise, we have either $d \le o < m = o+j$ or $o+j = d < m \le o+r$, where $2 \le j \le r$. Then we take such j and use (Ob3). Note that $\beta^\vee = \gamma^\vee - \alpha_j^\vee$ and $\mathbf{e}_c < \mathbf{e}_j$. For the former case, we have $\lambda_{j-1} = j-1$ and $\lambda_j = -1$; for the latter case, we have $\lambda_{j-1} = 0$ and $\lambda_j = -j$. Hence, we always have $M = \lambda_{j-1} - \lambda_j - j = 0$. By the induction hypothesis again, we have $\sum_{p=1}^{j-2} \mu_p \mathbf{e}_p \le \mathbf{0}$. If "$<$" holds, it is already done. If "$=$" holds, we have $\mu_1 = \cdots = \mu_{j-2} = 0$ and consequently $\mu_{j-1} \le 0$. For the former case, we conclude $\mu_{j-1} = 0$ and consequently $\mu_j \le 0$, by noting $0 \ge \mu_{j-1} = k_{j-1} + (j-1) - (i_j - 1) \ge 0$. For the latter case, we have $\mu_j = k_j - i_{j-1} + (-j) \le 0$. Hence, we always have $(\mu_1, \cdots, \mu_{j-2}, \mu_j, \mu_{j-1}) \le (0, \cdots, 0)$. Thus (L2) holds.

It remains to consider the case when statement (3) of Proposition 3.22 holds. That is, C9) occurs and $\gamma^\vee = \beta_3^\vee + \beta_4^\vee$. Then we just take $\alpha_j = \beta_4$ and use (Ob1). Thus (L2) still holds. q.e.d.

3.4. Two consequences. In this subsection, we derive two propositions with the help of our notion of virtual coroot.

Proposition 3.23. *Let $u \in W^P$ and $1 \le j \le r$. Then $\sigma^u \star \sigma^{s_j} = \sigma^{us_j} + \sum_{w,\lambda} b_{w,\lambda} q_\lambda \sigma^w$ with $gr(q_\lambda w) < gr(us_j)$ whenever $b_{w,\lambda} \neq 0$.*

Proof. Clearly, $\ell(us_j) = \ell(u) + 1$. Thus $N_{u,s_j}^{us_j,0} = \langle \chi_j, \alpha_j^\vee \rangle = 1$ by quantum Chevalley formula (Proposition 2.2). We need to analyze the remaining non-zero terms.

If $\ell(us_\gamma) = \ell(u)+1$ and $\langle \chi_j, \gamma^\vee \rangle \neq 0$, then we have $gr(us_\gamma) \le gr(u) + gr(s_j)$ by part a) of the Key Lemma. Note that $gr(us_j) = \mathbf{e}_j + \ell(u)\mathbf{e}_{r+1}$. If the equality holds, then we have $us_\gamma = vs_j$ where $\ell(v) = \ell(u)$ and $v \in W^P$. By Lemma 2.5, an expression of $u \in W^P$ is obtained by deleting a simple reflection from a (fixed) reduced expression of vs_j. Note that this simple reflection cannot come from v. Otherwise, we denote by $\bar{v}$ the element obtained by deleting such simple reflection from

v. Then $u = \bar{v}s_j$ and we would deduce a contradiction, say, $1 + \ell(u) = \ell(us_j) = \ell(\bar{v}) < \ell(v) = \ell(u)$. Thus $u = v$.

If $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee\rangle$ and $\langle \chi_j, \gamma^\vee\rangle \neq 0$, then we have $gr(q_{\gamma^\vee} us_\gamma) \leq gr(u) + gr(s_j)$ by part b) of the Key Lemma. Furthermore, we have $\ell(us_\alpha) = \ell(u) - 1$ whenever $\langle \alpha, \gamma^\vee\rangle > 0$, by Lemma 3.8 and Lemma 3.9. Since $u \in W^P$, $\ell(us_p) = \ell(u) + 1$ for any $\alpha_p \in \Delta_P$. If the equality held, then we would deduce a contradiction as follows (mainly by finding $\alpha \in \Delta_P$ satisfying $\langle \alpha, \gamma^\vee\rangle > 0$).

Note that $\ell(s_\gamma) > 1$ (otherwise, we would conclude $\gamma = \alpha_j \in \Delta_P$).

We first assume $r \geq 2$ and write $gr(us_\gamma) = (\tilde{k}_1, \cdots, \tilde{k}_{r+1})$ and $gr(q_{\gamma^\vee}) = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{r+1})$. Since the grading equality holds, we have $\tilde{k}_p + \tilde{\lambda}_p = \delta_{p,j}$ for each $1 \leq p \leq r$. As before, we discuss all possible coroots by using Proposition 3.22.

When (1) of Proposition 3.22 holds, $\gamma^\vee$ has a virtual coroot $\dot{\gamma}^\vee = \sum_{p=1}^{r+1} \dot{c}_p \dot{\alpha}_p^\vee$ satisfying one and only one of the followings (from the proof of Proposition 3.16).

a) $(\dot{c}_1, \cdots, \dot{c}_{r+1}) = (0, \cdots, 0)$. In this case, we have $gr_r(q_{\gamma^\vee}) = \mathbf{0}$ and $gr_r(us_\gamma) = gr_r(u)$. In particular, we have $\tilde{k}_j = \tilde{\lambda}_j = 0$, deducing a contradiction: $1 = \delta_{j,j} = \tilde{k}_j + \tilde{\lambda}_j = 0 + 0$.

b) $\dot{\gamma}^\vee = \sum_{p=1}^{m+1} p\dot{\alpha}_{r-m+p}$, where $0 \leq m < r$. Then we have $gr_r(q_{\gamma^\vee}) = gr_r(q_{\dot{\gamma}^\vee}) = (m-r)\sum_{p=r-m}^{r} \mathbf{e}_p$ and $\langle \alpha_p, \gamma^\vee\rangle = 0$ for $p \in \{1, \cdots, r\}\setminus\{r-m\}$. Note that $gr_{r-m-1}(q_{\gamma^\vee}) = \mathbf{0}$ and $gr_{r-m-1}(us_\gamma) = gr_{r-m-1}(u) = \mathbf{0}$. If $j \leq r - m - 1$, then we would deduce the contradiction $1 = \delta_{j,j} = \tilde{k}_j + \tilde{\lambda}_j = 0 + 0$ again. If $j = r - m$, then we still deduce a contradiction: $1 = \delta_{j,j} = \tilde{k}_j + \tilde{\lambda}_j = \tilde{k}_{r-m} + m - r \leq 0$. Hence, we conclude $j > r - m$. Then we have $r \geq j > j - 1 \geq r - m > 0$ and $\langle \alpha_j, \gamma^\vee\rangle = 0$. Thus we have $\tilde{k}_j = \delta_{j,j} - \tilde{\lambda}_j = 1 + r - m$, $\tilde{k}_{j-1} = \delta_{j,j-1} - \tilde{\lambda}_{j-1} = r - m$ and consequently $\tilde{k}_j = \tilde{k}_{j-1} + 1$. Hence, we have $\ell(us_\gamma s_j) = \ell(us_\gamma) - 1$ by Proposition 3.4. Then by Lemma 2.5, we conclude $us_\gamma(\alpha_j) \in -R^+$, contrary to $us_\gamma(\alpha_j) = u(\alpha_j) \in R^+$.

c) $\sum_{p=1}^{i+1} \dot{c}_p \dot{\alpha}_p = m\dot{\alpha}_{i+1} + \sum_{p=1}^{m} p\dot{\alpha}_{i-m+p}$, where $1 \leq m \leq i \leq r$. Then we have $\langle \alpha_i, \gamma^\vee\rangle = \langle \dot{\alpha}_i, \dot{\gamma}^\vee\rangle > 0$ and therefore deduce a contradiction.

When (2) of Proposition 3.22 holds, we have $\gamma^\vee = \sum_{p=d}^{m} \beta_p^\vee$ where $o \leq m \leq o + r$ and $d < m$. Since $\langle \chi_j, \gamma^\vee\rangle \neq 0$, we conclude $m \geq o + r$. Thus we find $\alpha = \beta_m \in \Delta_P$ that satisfies $\langle \alpha, \gamma^\vee\rangle > 0$. Hence, we deduce a contradiction in this case.

It remains to consider the case when (3) of Proposition 3.22 holds. That is, C9) occurs and $\gamma^\vee = \beta_3^\vee + \beta_4^\vee$. In this case, we note $r = 2$ and deduce a contradiction, say, $-4 = \tilde{\lambda}_2 = \delta_{2,j} - \tilde{k}_2 \geq 0 - 2 = -2$.

Hence, our assumption that the grading equality holds is not true, when $r \geq 2$.

Now we assume $r = 1$. Then $\alpha_j = \alpha_1$ and we have $\tilde{\lambda}_1 = \langle \alpha_1, \gamma^\vee \rangle$ by Corollary 3.13. If $\tilde{\lambda}_1 > 0$, then we find a contradiction by taking $\alpha = \alpha_1 \in \Delta_P$. If $\tilde{\lambda}_1 < 0$, then $gr_1(q_{\gamma^\vee} us_\gamma) < 0 + \tilde{k}_1 \leq 1 = gr_1(us_1)$ and consequently the grading equality does not hold. If $\tilde{\lambda}_1 = 0$, then $\tilde{k}_1 = gr_1(us_\gamma) = gr_1(u) = 0$ and consequently we deduce the contradiction $1 = \delta_{j,1} = \tilde{k}_1 + \tilde{\lambda}_1 = 0 + 0$.

Due to the quantum Chevalley formula, we have discussed all the non-zero terms for the quantum product $\sigma^u \star \sigma^{s_j}$. Hence, the statement follows. q.e.d.

By Theorem 1.2, we obtain a filtered-algebra structure on $QH^*(G/B)$, which induces an associated graded subalgebra along the $\mathbb{Z}\mathbf{e}_{r+1}$ direction. Thanks to the Peterson-Woodward comparison formula and our definition of $gr(q_j)$'s (with the help of PW-lifting), we wish to obtain an algebra isomorphism between $QH^*(G/P)$ and (at least a subalgebra of) this graded subalgebra. For this it is necessary that the gradings of $\psi_{\Delta,\Delta_P}(1, q_{\lambda_P})$'s, which are canonical candidates in $QH^*(G/B)$ playing the role of the polynomials q_{λ_P}'s in $QH^*(G/P)$, are in $\mathbb{Z}\mathbf{e}_{r+1}$. Indeed, the Peterson-Woodward comparison formula, together with our definition of $gr(q_j)$'s, has shown that $gr_r(\psi_{\Delta,\Delta_P}(1, q_{\lambda_P})) = \mathbf{0}$ whenever $q_{\lambda_P} \in QH^*(G/P)$ occurs in the quantum product $\sigma^u \star_P \sigma^v$ for $\sigma^u, \sigma^v \in QH^*(G/P)$. However, apparently it does not tell us about the behavior when the degree of q_{λ_P} is large. Therefore we need the following proposition for later use.

Proposition 3.24. $gr_r(\psi_{\Delta,\Delta_P}(1, q_{\lambda_P})) = \mathbf{0}$ *whenever* $q_{\lambda_P} \in QH^*(G/P)$.

The idea of the proof is as follows. We write $\psi_{\Delta,\Delta_P}(1, q_{\lambda_P}) = q_{\lambda_B}\omega_P\omega'$ as before. The case when $r = 1$ is easy to handle. When $r \geq 2$, we can use our notion of virtual coroot to obtain λ_B and consequently $\omega_P\omega'$ and $gr_r(q_{\lambda_B})$. More precisely, we write $\lambda_P = \lambda' + Q_P^\vee$ with $\lambda' = \sum_{\alpha \notin \Delta_P} a_\alpha \alpha$. Consider a virtual coroot $\dot{\lambda}'$ of λ'; then we can easily write down the element $\dot{\lambda}' + \sum_{i=1}^r a_i \dot{\alpha}_i^\vee$ associated to $\dot{\lambda}' + \dot{Q}_P^\vee \in \dot{Q}^\vee / \dot{Q}_P^\vee$, where $\dot{Q}_P^\vee := \bigoplus_{i=1}^r \mathbb{Z}\dot{\alpha}_i^\vee$. For instance, the case of $m\dot{\beta}_{o+r+1}^\vee + \dot{Q}_P^\vee$ has been studied in Lemma 3.18. By our definition of virtual coroot, we conclude $\lambda' + \sum_{i=1}^r a_i \alpha_i^\vee$ is the element that we expect. In addition, we also show the a_i's are indeed non-negative so that $q_{\lambda' + \sum_{i=1}^r a_i \alpha_i^\vee} \in QH^*(G/B)$.

Proof of Proposition 3.24. Write $\psi_{\Delta,\Delta_P}(1, q_{\lambda_P}) = q_{\lambda_B}\omega_P\omega'$. When $r = 1$, we have $gr_1(q_{\lambda_B}) = \langle \alpha_1, \lambda_B \rangle =: k_1$ by Corollary 3.13. Thus $k_1 \in \{0, -1\}$ following from the definition of λ_B. If $k_1 = 0$, then $\Delta_{P'} = \{\alpha_1\}$, implying $\omega' = s_1 = \omega_P$ and $\omega_P\omega' = 1$. Thus $gr_1(q_{\lambda_B}) + gr_1(\omega_P\omega') = 0 + 0 = 0$. If, $k_1 = -1$. Then we have $\Delta_{P'} = \emptyset$, implying $\omega_P\omega' = s_1 \cdot 1 = s_1$.

Thus $gr_1(q_{\lambda_B}) + gr_1(\omega_P\omega') = -1 + 1 = 0$. Hence, the statement holds when $r = 1$.

Now we assume $r \geq 2$. We consider the virtual coroots and introduce some special elements in $\dot{Q}^\vee$ first. Denote $\mu_m = \sum_{k=1}^{m} k\dot{\beta}^\vee_{o+r+1-m+k}$. Whenever $o > 0$, we denote $\nu_m = \sum_{k=0}^{m-1}(m-k)\dot{\beta}^\vee_{o+k}$ and $\varrho = \sum_{k=o}^{o+r+1}\dot{\beta}^\vee_k$ where $1 \leq m \leq r+1$. By direct calculations, we conclude $gr_r(q_x) = \mathbf{0}$ and $\langle\dot{\alpha}_i, x\rangle = 0$ for all $1 \leq i \leq r$ whenever $x = \mu_{r+1}$, ν_{r+1} or ϱ. That is, μ_{r+1}, ν_{r+1} and ϱ are all virtual coroots of $0 \in Q^\vee$. Furthermore, for $1 \leq m \leq r$, we have

$$gr_r(q_{\nu_m}) = -m\sum_{j=1}^{r}\mathbf{e}_j + \sum_{k=1}^{m-1}(m-k)\big((k+1)\mathbf{e}_k - (k-1)\mathbf{e}_{k-1}\big) = -m\sum_{k=m}^{r}\mathbf{e}_k,$$

and $gr_r(q_{\mu_m}) = -(r+1-m)\sum_{k=r+1-m}^{r}\mathbf{e}_k$ (by Lemma 3.18).

Write $\lambda_P = \lambda' + Q^\vee_P$ with $\lambda' = \sum_{j=1}^{o} b_j\beta^\vee_j + \sum_{j=o+r+1}^{n} b_j\beta^\vee_j$. From Table 4, we obtain a virtual coroot $\dot{\lambda}' = \sum_{j=1}^{\eta}\tilde{b}_j\dot{\beta}^\vee_j$ of λ' in which we note $\tilde{b}_o = b_o$ and $o+r+1 \leq \eta$. If $\tilde{b}_o \leq \tilde{b}_{o+r+1}$, we set $y = \tilde{b}_o\varrho + a\mu_{r+1} + \mu_m$ where $\mu_0 := 0$ and $\tilde{b}_{o+r+1} - \tilde{b}_o = a(r+1) + m$ with $0 \leq m \leq r$ and $a \geq 0$. Similarly, if $\tilde{b}_o > \tilde{b}_{o+r+1}$, we set $y = \tilde{b}_{o+r+1}\varrho + a\nu_{r+1} + \nu_m$ where $\nu_0 := 0$ and $\tilde{b}_o - \tilde{b}_{o+r+1} = a(r+1) + m$ with $0 \leq m \leq r$ and $a \geq 0$. Clearly, we can write $y = \tilde{b}_o\dot{\beta}_o + \tilde{b}_{o+r+1}\dot{\beta}_{o+r+1} + \sum_{i=1}^{r} d_i\dot{\alpha}_i$. Note that $\dot{\beta}^\vee_p$ is a virtual coroot of $0 \in Q^\vee$ whenever $p < o$ or $p > o+r+1$. Thus we conclude y is virtual coroot of $\lambda_B := \lambda' + \sum_{i=1}^{r}(d_i - \tilde{b}_{o+i})\alpha_i$. Furthermore, we note that for all $\alpha_i \in \Delta_P$ we have $\langle\alpha_i, \lambda_B\rangle = \langle\dot{\alpha}_i, y\rangle =$ $\langle\dot{\alpha}_i, x\rangle = \begin{cases} -1, & \text{if } \dot{\alpha}_i = \dot{\alpha}_m \text{ (resp. } \dot{\alpha}_{r+1-m}) \\ 0, & \text{otherwise} \end{cases}$ where $x = \nu_m$ (resp. μ_m), if $\tilde{b}_o > \tilde{b}_{o+r+1}$ (resp. $\tilde{b}_o \leq \tilde{b}_{o+r+1}$). Hence, λ_B is the very one associated to λ_P that we are expecting. Correspondingly, we can directly write down $\Delta_{P'}$ as well as $gr(\omega_P\omega') = -gr_r(q_x)$ by Lemma 3.6. Note that $gr_r(q_{\lambda_B}) = gr_r(q_x)$. Hence, $gr_r(q_{\lambda_B}\omega_P\omega') = \mathbf{0}$.

Since $q_{\lambda_P} \in QH^*(G/P)$, $b_j \geq 0$ for each j. It remains to show $q_{\lambda_B} \in QH^*(G/B)$. That is, we need to show $d_i - \tilde{b}_{o+i}$ is non-negative for each $1 \leq i \leq r$. Clearly, only the part $\sum_{j=o+r+1}^{n} b_j\beta^\vee_j$ of λ' make contributions for the part $\sum_{i=1}^{r}\tilde{b}_i\dot{\beta}^\vee_{o+r}$ of the coroot $\dot{\lambda}'$ of λ'. From Table 4 we see that $\tilde{b}_{o+1} \leq \tilde{b}_{o+2} \leq \cdots \leq \tilde{b}_{o+r+1}$. Thus if $\tilde{b}_o \geq \tilde{b}_{o+r+1}$, then we have $d_i - \tilde{b}_{o+i} \geq \tilde{b}_o - \tilde{b}_{o+r+1} \geq 0$ for each $1 \leq i \leq r$. Now we consider the case when $(0 \leq b_o =)\tilde{b}_o < \tilde{b}_{o+r+1}$ and then note that all d_i's are non-negative from the way we obtain them. From Table 4 and Table 2, we can make the following observations. (i) If case C1), C9), or C10) occurs, then the virtual coroot $\dot{\lambda}'$ does not make contributions on these $\tilde{b}_{o+i}$'s. That is, we have $d_i - \tilde{b}_{o+r} = d_i \geq 0$ for each $1 \leq i \leq r$. (ii) For the remaining cases, we have $\tilde{b}_{o+1} = \cdots = \tilde{b}_{o+r-1} = 0 \leq 2\tilde{b}_{o+r} \leq \tilde{b}_{o+r+1}$, except for the case when C4) occurs with $r \geq 5$ and $o + r = 7$. (iii) For the only exceptional case, we have $\tilde{b}_{o+1} = \tilde{b}_{o+2} = \cdots = \tilde{b}_5 = 0$, $\tilde{b}_6 = b_8$, $\tilde{b}_7 = 2b_8$ and $\tilde{b}_8 = 3b_8 > 0$. Recall that $\sum_{i=1}^{r} d_i\dot{\beta}_{o+i} = \tilde{b}_o\varrho + a\mu_{r+1} + \mu_m - (\tilde{b}_o\dot{\beta}_o + \tilde{b}_{o+r+1}\dot{\beta}_{o+r+1})$ in which $\tilde{b}_{o+r+1} = \tilde{b}_o + a(r+1) + m$. When (ii) holds, we have $d_r - \tilde{b}_{o+r} \geq \tilde{b}_o + ar + (m-1) - \tilde{b}_{o+r} \geq \tilde{b}_o + ar + m - 1 - [\frac{\tilde{b}_{o+r+1}}{2}] \geq ar + m - 1 - [\frac{\tilde{b}_{o+r+1} - \tilde{b}_o}{2}] = ar + m - 1 - [\frac{a(r+1)+m}{2}] \geq 0$, and note $d_i - \tilde{b}_{o+i} = d_i \geq 0$ for $1 \leq i \leq r-1$. When (iii) holds, we have $\tilde{b}_8 \geq 0 + a(5+1) + 0$ so that $a \leq \frac{b_8}{2}$. Since a is an integer, $a \leq [\frac{b_8}{2}]$. Thus we have $d_r - \tilde{b}_{o+r} \geq \tilde{b}_o + ar + m - 1 - 2b_8 = \tilde{b}_8 - a - 1 - 2b_8 = b_8 - a - 1 \geq b_8 - [\frac{b_8}{2}] - 1 \geq 0$ and $d_{r-1} - \tilde{b}_{o+r-1} \geq \tilde{b}_o + a(r-1) + m - 2 - b_8 = \tilde{b}_8 - 2a - 2 - b_8 = 2(b_8 - a - 1) \geq 0$. For $1 \leq i \leq r-2$, we have $d_i - \tilde{b}_{o+i} = d_i \geq 0$. Hence, we do show $d_i - \tilde{b}_{o+i} \geq 0$ for $1 \leq i \leq r$ for all cases. q.e.d.

Remark 3.25. In [**18**], Lam and Shimozono have given a combinatorial description of λ_B. In our case when Δ_P is of A-type, we obtain another way to describe λ_B and to show the property $q_{\lambda_B} \in QH^*(G/B)$ in the above proof.

3.5. Proof of the Key Lemma for general Δ_P. In this subsection, we assume Δ_P is not of A-type. We give the proof of the Key Lemma, after describing the formulas for the gradings of all q_j's. Recall that $\varsigma = r - 1$ and that in this case we have replaced r with ς in Table 2, in order to fix the order (Δ_P, Υ). In particular, we have $\kappa = o + \varsigma$ in this subsection.

Using Definition 2.8 with respect to the ordered subset $\Delta_\varsigma = (\alpha_1, \ldots, \alpha_\varsigma)$ of (Δ_P, Υ), we obtain a grading map

$$\tilde{gr} = gr_{\Delta_\varsigma} : W \times Q^\vee \longrightarrow \mathbb{Z}^{\varsigma+1}.$$

That is, we define $\tilde{gr}(w) = \sum_{j=1}^{\varsigma+1} \ell(v_j)\mathbf{e}_j$ using the decomposition $w = v_{\varsigma+1}v_\varsigma \cdots v_1$ of $w \in W$ associated to ordered subset $\Delta_\varsigma = (\alpha_1, \cdots, \alpha_\varsigma)$, define $\tilde{gr}(q_1) = 2\mathbf{e}_1$, and define the remaining $\tilde{gr}(q_j)$'s recursively with the help of PW-lifting $\{\psi_{\Delta_2,\Delta_1}, \psi_{\Delta_3,\Delta_2}, \cdots, \psi_{\Delta_\varsigma,\Delta_{\varsigma-1}}, \psi_{\Delta,\Delta_\varsigma}\}$. Let $\iota : \mathbb{Z}^{\varsigma+1} = \mathbb{Z}^r \hookrightarrow \mathbb{Z}^{r+1}$ be the natural inclusion. Thus we obtain a map $\iota \circ \tilde{gr} : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1}$, which we simply denote as $\tilde{gr}$ whenever there is no confusion.

As a direct consequence of the definition of $\tilde{gr}$, we can apply Proposition 3.10 and Proposition 3.12 with respect to the ordered subset Δ_ς, so that we have:

Lemma 3.26. $gr(q_j) = \tilde{gr}(q_j)$ *for each* $1 \leq j \leq \varsigma + 1$. *Precisely,* $gr(q_1) = 2\mathbf{e}_1$; $gr(q_j) = (1-j)\mathbf{e}_{j-1} + (1+j)\mathbf{e}_j$ *for* $2 \leq j \leq \varsigma$; $gr(q_{\varsigma+1})$ *is obtained by directly replacing* r *with* ς. *(Only case* c), d), e) *or* f) *in Table* 3 *can occur.)*

Furthermore, we note from Table 2 that either $o \geq 1$ or $\kappa + 2 \leq n$ must hold. For any $\alpha \in \Delta \setminus (\Delta_P \cup \{\beta_o, \beta_{\kappa+2}\})$, we have $gr(q_{\alpha^\vee}) = 2\mathbf{e}_{r+1} <$

$2\mathbf{e}_{\varsigma+1} = \tilde{gr}(q_{\alpha^\vee})$. For $\lambda_P \in Q^\vee/Q^\vee_P$, we write $q_{\lambda_B}\omega_P\omega' = \psi_{\Delta,\Delta_P}(1,\lambda_P)$ as before.

Lemma 3.27. *Suppose $p \in \{o, \kappa+2\}\cap\{1,\cdots,n\}$. Set $\lambda_P = \beta_p^\vee + Q_P^\vee$. Then we have $\lambda_B = \beta_p^\vee$ except for either of the following cases.*

1) *$p = o$ and $\{\beta_o\}\cup\Delta_P$ is of C-type. In this case, $\lambda_B = \beta_o^\vee + \sum_{j=1}^r \alpha_j^\vee$.*
2) *$p = \kappa+2$ and C9) occurs. In this case, $\lambda_B = \beta_{\kappa+2}^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee$.*

Furthermore, we write $gr(q_{\beta_p^\vee}) = \sum_{j=1}^{r+1} d_j\mathbf{e}_j$ and $\tilde{gr}(q_{\beta_p^\vee}) = \sum_{j=1}^{r+1}\tilde{d}_j\mathbf{e}_j$, and denote $\Delta_{\tilde{P}} = \{\alpha\in\Delta_P \mid \langle\alpha,\beta_p^\vee\rangle = 0\}$. Then $d_j = \tilde{d}_j$ for $1\le j\le\varsigma$ and we have

a) $gr(q_{\beta_p^\vee}) < \tilde{gr}(q_{\beta_p^\vee})$; b) $d_{r+1} \le \ell(\omega\omega_P)+1$; c) $\sum_{j=1}^r d_j \le -\ell(\omega_P\omega_{\tilde{P}})$.

Proof. Let $\theta_P = \sum_{j=1}^r a_j\alpha_j$ denote the highest root in R_P. Note that $\ell(\omega_P\omega_{\tilde{P}}) = |R_P^+| - |R_{\tilde{P}}^+|$, $\ell(\omega\omega_P) = |R^+| - |R_P^+|$ and $\Delta_{P'} = \{\alpha\in\Delta_P \mid \langle\alpha,\lambda_B\rangle = 0\}$.

We first assume $p = o$ and note that $\tilde{gr}(q_{\beta_o^\vee}) = (\varsigma+2)\mathbf{e}_{\varsigma+1} - \sum_{j=1}^{\varsigma}\mathbf{e}_j$.

Whenever $\{\beta_o\}\cup\Delta_P$ is not of C-type, we note (Table 2 and [**12**]) that $a_1 = 1$, $\langle\alpha_1,\beta_o^\vee\rangle = -1$, and $\langle\alpha_j,\beta_o^\vee\rangle = 0$ for $2\le j\le r$. Hence, we conclude $\lambda_B = \beta_o^\vee$, $\Delta_{P'} = \Delta_{\tilde{P}} = \Delta_P\setminus\{\alpha_1\}$ and consequently we have $\omega' = \omega_{\tilde{P}}$ and $gr(q_{\beta_p^\vee}) = (\ell(\omega_P\omega')+2)\mathbf{e}_{r+1} - gr(\omega_P\omega')$. Hence, $d_{r+1} = \ell(\omega_P\omega')+2 \le \ell(\omega\omega_P)+1$ by direct calculations. Write $\omega_P\omega' = v_r u$, where $v_r \in W_P^{P_\varsigma}$ and $u = W_{P_\varsigma}$. Then we have $u(\alpha_j)\in R_{P_\varsigma}^+$ for all $\alpha_j\in\Delta_\varsigma\cap\Delta_{P'} = \Delta_\varsigma\setminus\{\alpha_1\}$ (otherwise, we would conclude $v_r u(\alpha_j)\in -R^+$, contrary to $\omega_P\omega'\in W_P^{P'}$). Noting $u(\alpha_1)\in -R^+$, we deduce $u = s_k\cdots s_2 s_1$ for some $1\le k\le\varsigma$, by Lemma 3.6. If Δ_P is of B-type (resp. D-type), then we conclude $\omega_P\omega' = v_r s_\varsigma\cdots s_1$ with $v_r = s_1\cdots s_r$ (resp. $v_r = s_1\cdots s_{r-2}s_r$) by easily checking such element satisfies the condition in Lemma 3.5. If Δ_P is of E-type, we note that $s_\varsigma\cdots s_1(\alpha_r) = u(\alpha_r) + \sum_{j=k+1}^{\varsigma} b_j\alpha_j$ for non-negative integers b_j's. Consequently, we have $v_r s_\varsigma\cdots s_1 \in W_P^{P'}$ and $\ell(v_r s_\varsigma\cdots s_1) = \ell(v_r u) + \varsigma - k = \ell(\omega_P) - \ell(\omega') + \varsigma - k$. Thus $k = \varsigma$ by Lemma 3.5, implying $\ell(v_r) = \ell(\omega_P\omega') - \varsigma$. For all these cases, we deduce $gr(\omega_P\omega') = (\ell(\omega_P\omega')-\varsigma)\mathbf{e}_{\varsigma+1} + \sum_{j=1}^{\varsigma}\mathbf{e}_j$. Hence, $d_j = -1$ for $1\le j\le\varsigma$, $d_r = \varsigma - \ell(\omega_P\omega')$, and $gr(q_{\beta_o^\vee}) < \tilde{gr}(q_{\beta_o^\vee})$. Thus $\sum_{j=1}^r d_j = -\ell(\omega_P\omega') = -\ell(\omega_P\omega_{\tilde{P}})$.

Assume $\{\beta_o\}\cup\Delta_P$ is of C-type, in which case there are only two possibilities, say, (i) case C1) with Δ being of C-type and (ii) case C10) with $r = 2$. Then Δ_P itself is of C-type. Thus we have $\ell(\omega_P\omega_{\tilde{P}}) = r^2 - (r-1)^2 = 2r-1$ and $\ell(\omega\omega_P) = n^2 - r^2 \ge (r+1)^2 - r^2 = 2r+1$. Furthermore, we conclude $\lambda_B = \beta_o^\vee + \sum_{j=1}^r \alpha_j^\vee$, by noting such element satisfies $\langle\alpha_j,\lambda_B\rangle = 0$ for each $1\le j\le r$. Thus $\Delta_{P'} = \Delta_P$, $\omega_P\omega' = 1$ and then $gr(q_{\beta_o^\vee}) = (2\varsigma+4)\mathbf{e}_{\varsigma+2} - (\varsigma+2)\mathbf{e}_{\varsigma+1} - \sum_{j=1}^{\varsigma}\mathbf{e}_j$ by definition.

In particular, we have $gr(q_{\beta_o^\vee}) < \tilde{gr}(q_{\beta_o^\vee})$, $d_{r+1} = 2\varsigma + 4 = 2r + 2 \leq \ell(\omega\omega_P) + 1$ and $\sum_{j=1}^r d_j = d_r - \varsigma = -2\varsigma - 2 = -2r < -\ell(\omega_P\omega_{\tilde{P}})$.

Now we assume $p = \kappa + 2$, which holds only if case C5), C7), C9), or C10) in Table 2 occurs. Note that $\tilde{gr}(q_{\beta_{\kappa+2}^\vee}) = 2\mathbf{e}_{\varsigma+1}$. If C9) does not occur, then we conclude $a_r = 1$, $\langle\alpha_r, \beta_{\kappa+2}^\vee\rangle = -1$ and $\langle\alpha_j, \beta_{\kappa+2}^\vee\rangle = 0$ for $1 \leq j \leq \varsigma$. Hence, $\lambda_B = \beta_{\kappa+2}^\vee$, $\Delta_{P'} = \Delta_{\tilde{P}} = \Delta_\varsigma$ and consequently we have $gr(q_{\beta_{\kappa+2}^\vee}) = (\ell(\omega_P\omega_{\tilde{P}}) + 2)\mathbf{e}_{\varsigma+2} - \ell(\omega_P\omega_{\tilde{P}})\mathbf{e}_{\varsigma+1}$. Therefore, a) and c) follow, so does b) by direct calculations. If C9) occurs, then $|R^+| = 24$, $|R_P^+| = r^2$, $|\Delta_{\tilde{P}}| = \frac{r(r+1)}{2}$, $n = 4 = \kappa + 2$, and $r \in \{2, 3\}$. By direct calculations, we conclude $\lambda_B = \beta_4^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee$. Furthermore, if $r = 2$, then $\Delta_{P'} = \Delta_P$ and consequently we have $gr(q_{\beta_4^\vee}) = 6\mathbf{e}_{\varsigma+2} - 4\mathbf{e}_{\varsigma+1}$. If $r = 3$, then $\Delta_{P'} = \{\alpha_2, \alpha_3\}$. Consequently, $\omega_P\omega' = s_1s_2s_3s_2s_1$ with $gr(\omega_P\omega') = (1, 1, 3, 0)$. Hence, we have $gr(q_{\beta_4^\vee}) = 11\mathbf{e}_{\varsigma+2} - 9\mathbf{e}_{\varsigma+1}$. For either of the cases, it is easy to check all the statements hold. q.e.d.

From the above discussions, we note that $gr_\varsigma(q_j) = \tilde{gr}_\varsigma(q_j)$ for all j. Using these discussions together with Lemma 3.11, we obtain the following immediately.

Lemma 3.28. *Let $\gamma \in R^+$. Write $gr(q_{\gamma^\vee}) = \sum_{j=1}^{r+1} d_j\mathbf{e}_j$ and $\tilde{gr}(q_{\gamma^\vee}) = \sum_{j=1}^{r+1} \tilde{d}_j\mathbf{e}_j$. Then we have $d_r + d_{r+1} = \tilde{d}_r + \tilde{d}_{r+1} = \tilde{d}_r$ and $d_j = \tilde{d}_j$ for each $1 \leq j \leq \varsigma$.*

Now we give the proof of the Key Lemma as follows.

Proof of the Key Lemma. Let $w \in W$ and take its decomposition $w = v_{r+1}\cdots v_1$ associated to (Δ_P, Υ). Suppose $\ell(ws_\gamma) < \ell(w)$; then by Lemma 2.5 we conclude $ws_\gamma = v_{r+1}\cdots v_{m+1}\bar{v}_m v_{m-1}\cdots v_1$ for a unique $1 \leq m \leq r + 1$, in which $\bar{v}_m$ is obtained by deleting a unique simple reflection from (a fixed reduced expression of) v_m. Set $D := \big(gr(ws_\gamma) - gr(w)\big) - \big(\tilde{gr}(ws_\gamma) - \tilde{gr}(w)\big)$. If $1 \leq m \leq r$, then we have $D = 0$ and $\gamma \in R_P$. Furthermore, we have $gr(q_\gamma) = \tilde{gr}(q_\gamma)$ and $gr(s_i) = \tilde{gr}(s_i)$ whenever $\langle\chi_i, \gamma^\vee\rangle \neq 0$. In particular, the Key Lemma holds for such γ, by using Proposition 3.15 and Proposition 3.16 with respect to the ordered subset Δ_ς. If $m = r + 1$, we write $\bar{v}_{r+1}v_r = \tilde{v}_{r+1}\tilde{v}_r u'$ with $\tilde{v}_{r+1} \in W^P$, $\tilde{v}_r \in W_P^{P_\varsigma}$, and $u' \in W_{P_\varsigma}$. Thus $ws_\gamma = \tilde{v}_{r+1}\cdots\tilde{v}_1$ with $\tilde{v}_j \in W_{P_j}^{P_{j-1}}$ for each $1 \leq j \leq r + 1$.

In order to show a), it remains to consider the case when $m = r + 1$. Set $w := us_\gamma$ and note that $\tilde{gr}(us_\gamma) = \sum_{j=1}^\varsigma \ell(v_j)\mathbf{e}_j + (\ell(v_r) + \ell(v_{r+1}))\mathbf{e}_r$. Thus we have $-D = (\ell(v_{r+1}) - \ell(\tilde{v}_{r+1}))\mathbf{e}_{r+1} + (\ell(v_r) - \ell(\tilde{v}_r) - \ell(v_{r+1}v_r) + \ell(\tilde{v}_{r+1}\tilde{v}_r))\mathbf{e}_r = (\ell(v_{r+1}) - \ell(\tilde{v}_{r+1}))(\mathbf{e}_{r+1} - \mathbf{e}_r) \leq \mathbf{e}_{r+1} - \mathbf{e}_r$. Note that $\gamma \in R \setminus R_P$ (by Lemma 2.7). Therefore we have $gr(us_\gamma) - gr(u) = -D + \big(\tilde{gr}(us_\gamma) - \tilde{gr}(u)\big) \leq \mathbf{e}_{r+1} - \mathbf{e}_r + \min\{\tilde{gr}(s_i) \mid \langle\chi_i, \gamma^\vee\rangle \neq 0\} = \mathbf{e}_{r+1} = \min\{gr(s_i) \mid \langle\chi_i, \gamma^\vee\rangle \neq 0\}$. Thus a) follows.

To show b), we set $w := u$ in the rest of the proof and use induction on $\ell(s_\gamma)$.

First we observe that $gr(q_j) + gr(us_j) \leq gr(u) + gr(s_j)$ for any $1 \leq j \leq n$. Indeed, this inequality holds if $1 \leq j \leq r$ with the discussion in the beginning. If $\langle \alpha, \alpha_j^\vee \rangle = 0$ for all $\alpha \in \Delta_P$, then for $\gamma = \alpha_j$ we have $m = r + 1$, $\bar{v}_{r+1} \in W^P$ (by Lemma 3.2) and consequently $gr(us_j) - gr(u) = -\mathbf{e}_{r+1} = -gr(q_j) + gr(s_j)$. Otherwise, we much have $\alpha_j = \beta_p$ with $p \in \{o, \kappa+2\} \cap \{1, \cdots, n\}$. Then the inequality still holds by using Lemma 3.29 a) (with $\gamma = \beta_p$) and Lemma 3.27 c).

Now we assume $\gamma \notin \Delta$. Take any simple root α_j satisfying $\langle \alpha_j, \gamma^\vee \rangle > 0$ and write $\beta = s_j(\gamma)$, $gr(q_{\beta^\vee}) = (\lambda_1, \cdots, \lambda_{r+1}), \min\{gr(s_i) \mid \langle \chi_i, \beta^\vee \rangle \neq 0\} = \mathbf{e}_c$, and

$$\begin{aligned} gr(q_j) + gr(us_j) &= gr(u) + (a_1, \cdots, a_{r+1}), \\ gr(q_{\beta^\vee}) + gr(us_j s_\beta) &= gr(us_j) + \mathbf{e}_c + (\mu_1, \cdots, \mu_{r+1}), \\ gr(us_j s_\beta s_j) &= gr(us_j s_\beta) + (b_1, \cdots, b_{r+1}). \end{aligned}$$

In addition, we use the notations $\tilde{c}$, $\tilde{a}_j$'s, $\tilde{b}_j$'s, and $\tilde{\mu}_j$'s, whenever replacing "gr" with "$\tilde{gr}$" in the above three equalities. Then we have $\min\{gr(s_i) | \langle \chi_i, \gamma^\vee \rangle \neq 0\} = \min\{\mathbf{e}_c, gr(s_j)\}$ and by the induction hypothesis $(\mu_1, \cdots, \mu_{r+1}) \leq (0, \cdots, 0)$. Due to Lemma 3.11, it suffices to show $\sum_{i=1}^r (a_i + b_i + \mu_i)\mathbf{e}_i \leq \mathbf{0}$. Furthermore, we note that $\tilde{a}_{r+1} = \tilde{b}_{r+1} = \tilde{\mu}_{r+1} = 0, \mu_r + \mu_{r+1} = \tilde{\mu}_r, \mathbf{e}_{\tilde{c}} \geq \mathbf{e}_c$ and $\tilde{a}_k = a_k, \tilde{b}_k = b_k, \tilde{\mu}_k = \mu_k$ for each $1 \leq k \leq \varsigma$. Clearly, either of the following must hold.

(i) There is $\beta_p \in \Delta$ such that $p \in \{1 \cdots, n\} \setminus \{o, \kappa+1, \kappa+2\}$ and $\langle \beta_p, \gamma^\vee \rangle > 0$.
(ii) Whenever $\beta_p \in \Delta$ satisfies $\langle \beta_p, \gamma^\vee \rangle > 0$, we have $p \in \{o, \kappa+1, \kappa+2\}$. In this case, we note the constrain $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee \rangle - 1$ on γ, which is deduced from our assumption by using Lemma 3.9.

Suppose (i) holds. Then we just take any one such $\alpha_j = \beta_p$. If $p \notin \{o, o+1, \cdots, \kappa+2\}$, then $\langle \alpha, \alpha_j^\vee \rangle = 0$ for all $\alpha \in \Delta_P$; consequently, it is done by noting $a_k = b_k = 0$ for $1 \leq k \leq r$ (using Lemma 3.2) and $\mathbf{e}_c = \mathbf{e}_{r+1}$. Otherwise, there exists $o+1 \leq p \leq o+\varsigma$ such that $\langle \beta_p, \gamma^\vee \rangle > 0$. Recall that $\alpha_i = \beta_{o+i}$ for each $1 \leq i \leq r$. For any one choice $\alpha_j = \beta_p$ among such roots, we always have $a_i = b_i = 0$ for $i \notin \{j-1, j\}$ and consequently $\sum_{i=j+1}^r (a_i + b_i + \mu_i)\mathbf{e}_i = \sum_{i=j+1}^r \mu_i \mathbf{e}_i$. In addition from the proof of Proposition 3.16, we can always take a certain $\alpha_j = \beta_p$ among such roots such that both $\mathbf{e}_{\tilde{c}} < \mathbf{e}_j$ and $\sum_{i=1}^j (\tilde{a}_i + \tilde{b}_i + \tilde{\mu}_i)\mathbf{e}_i \leq \mathbf{0}$ hold by considering $\tilde{gr}$. Since $j \leq r-1$, we have $\tilde{a}_i = a_i$, $\tilde{b}_i = b_i$ and $\tilde{\mu}_i = \mu_i$ for each $1 \leq i \leq j$. Thus for such a choice $\alpha_j = \beta_p$, both $\mathbf{e}_c \leq \mathbf{e}_{\tilde{c}} < \mathbf{e}_j$ and $\sum_{i=1}^j (a_i + b_i + \mu_i)\mathbf{e}_i \leq \mathbf{0}$ hold. Hence, the Key Lemma holds for such γ by using the induction hypothesis.

Suppose (ii) holds. Then the constrains are so strong that there are only very few roots. We discuss all such roots with respect to each type of Δ and label the method we will use.

Assume Δ is of B-type. (That is, part of case C1) in Table 1 occurs.) There are only two coroots satisfying the conditions, say, $\beta_{o-1}^\vee + 2\sum_{i=o}^{n-1}\beta_i^\vee + \beta_n^\vee$ (with $o \geq 2$) or $\sum_{i=o}^{n}\beta_i^\vee$. (See the proof of Proposition 3.22.)

(M1): For the former coroot, we note that $\Delta_{\bar{P}} = \{\alpha \in \Delta_P \mid \langle \alpha, \gamma^\vee\rangle = 0\} = \Delta$ and $gr(q_{\gamma^\vee}) = \sum_{i=1}^{r+1} d_i\mathbf{e}_i = d_{r+1}$ by direct calculations. Hence, $\sum_{i=1}^{r} d_i = 0 = -\ell(\omega_P\omega_{\bar{P}})$. Thus the inequality holds by using Lemma 3.29 a).

(M2): For the latter coroot, we take $\alpha_j = \beta_n$; that is, $\alpha_j = \alpha_r$. Then $\beta^\vee = \gamma^\vee - \beta_n^\vee$ and $gr(q_{\beta^\vee}) = d_{r+1}\mathbf{e}_{r+1} - r\mathbf{e}_r + \varsigma\mathbf{e}_\varsigma$. Write $gr(u) = (i_1, \cdots, i_{r+1}), gr(us_r) = (i'_1, \cdots, i'_{r+1}), gr(us_rs_\beta) = (k_1, \cdots, k_{r+1})$, and $gr(us_\gamma) = gr(us_rs_\beta s_r) = (k'_1, \cdots, k'_{r+1})$. Noting that $\langle \alpha_t, \alpha_r^\vee\rangle = \langle \alpha_t, \beta^\vee\rangle = 0$ for $1 \leq t \leq \varsigma - 1$, we conclude $a_t = b_t = \mu_t = 0$ for $t \leq \varsigma - 1$ by Lemma 3.2 and consequently $\mu_\varsigma \leq 0$ by the induction hypothesis. Furthermore, we note that $a_\varsigma + a_r = 1$, $b_\varsigma + b_r = -1$, $a_\varsigma + b_\varsigma = -2\varsigma + i'_\varsigma - i_\varsigma + k'_\varsigma - k_\varsigma \leq 0$. If $a_\varsigma + b_\varsigma + \mu_\varsigma < 0$, then it is done. Otherwise, we conclude $i'_\varsigma = k'_\varsigma = \varsigma$ and $i_\varsigma = k_\varsigma = \mu_\varsigma = 0$. Consequently, we have $a_r + b_r + \mu_r = \mu_r = \mu_{\varsigma+1} \leq 0$ and it is done.

Assume Δ is of C-type. (That is, part of case C1) in Table 1 occurs.) There are only one such coroots, say, $\gamma^\vee = \sum_{i=o}^{n}\beta_i^\vee$. Thus the inequality holds by (M1).

Assume Δ is of D-type. (Case C2) is used.) There are only two such coroots, say, $\beta_{o-1}^\vee + 2\sum_{i=o}^{n-2}\beta_i^\vee + \beta_{n-1}^\vee + \beta_n^\vee$ (with $o \geq 2$) or $\sum_{i=o}^{n-2}\beta_i^\vee + \beta_n^\vee$. For the former coroot, the inequality holds by using (M1).

(M3): For the latter coroot, we have $gr(q_{\gamma^\vee}) = 2r\mathbf{e}_{r+1} + \varsigma\mathbf{e}_r - \varsigma\mathbf{e}_\varsigma$ by direct calculation. Using the notation of Lemma 3.29, we conclude $\sum_{i=1}^{r} d_i = 0$, $\Delta_{\hat{P}} = \{\beta_o, \beta_{o+1}, \cdots, \beta_n\}$, $\Xi_1 = \{\beta_n\} \cup \{\sum_{i=k}^{r-2}\beta_{o+i} + \beta_n \mid 1 \leq k \leq r-2\}$, and $\Xi_2 := \{\beta_o + \sum_{i=o+k}^{n-2}\beta_i + \beta_{n-1} + \beta_n \mid o+1 \leq k \leq n-2\} \cup \{\sum_{i=o}^{n}\beta_i\}$. Note that $n = o + r$ in this case. Hence, we have $|\Xi_1| - |\Xi_2| = r - 1 - (r-1) = 0 = \sum_{i=1}^{r} d_i$. Hence, the Key Lemma holds for $\gamma = \sum\limits_{i=o}^{n-2}\beta_i^\vee + \beta_n^\vee$ by using Lemma 3.29 b).

It remains to discuss the cases when Δ is of either E-type or F-type. Since there are only finite exceptional types (among which only a few roots satisfy (ii)) and the arguments are similar, we leave the details in the appendix (see section 6).

Hence, the statement follows. q.e.d.

It remains to show the following, which was used in the proof of the Key Lemma.

Lemma 3.29. *Let $u \in W$ and $\gamma \in R^+ \setminus R_P$. Write $gr(q_{\gamma^\vee}) = \sum_{j=1}^{r+1} d_j \mathbf{e}_j$. Then Key Lemma* b) *holds, if either of the following holds.*

a) $\sum_{j=1}^{r} d_j \leq -\ell(\omega_P \omega_{\bar{P}})$, *where* $\Delta_{\bar{P}} := \{\alpha \in \Delta_P \mid \langle \alpha, \gamma^\vee \rangle = 0\}$.
b) $\sum_{j=1}^{r} d_j \leq |\Xi_1| - |\Xi_2|$, *where* $\Xi_1 := \{\alpha \in R_P^+ \mid \langle \alpha, \gamma^\vee \rangle > 0\}$ *and* $\Xi_2 := \{\alpha \in R_{\hat{P}}^+ \setminus R_P \mid \alpha - \gamma \in R^+, \langle \alpha, \gamma^\vee \rangle > 0\}$ *with* $\Delta_{\hat{P}} := \Delta_P \cup \{\alpha_i \in \Delta \mid \langle \chi_i, \gamma^\vee \rangle \neq 0\}$.

Proof. Let $u = v_{r+1} \cdots v_1$ (resp. $us_\gamma = \tilde{v}_{r+1} \cdots \tilde{v}_1$) be its decomposition associated to (Δ_P, Υ). Since $\gamma \in R^+ \setminus R_P$, we have $\min\{gr(s_i) \mid \langle \chi_i, \gamma^\vee \rangle \neq 0\} = \mathbf{e}_{r+1}$. Note that $gr_\varsigma(q_\lambda w) = \tilde{gr}_\varsigma(q_\lambda w)$ for any $q_\lambda w$. Applying Proposition 3.16 with respect to Δ_ς, we have $\sum_{j=1}^{r-1}(d_j + \ell(\tilde{v}_j))\mathbf{e}_j \leq \sum_{j=1}^{r-1} \ell(v_j)\mathbf{e}_j$. If "<" holds, it is already done. If "=" holds, we conclude $\sum_{j=1}^{r-1} d_j + \ell(\tilde{v}_{r-1} \cdots \tilde{v}_1) = \ell(v_{r-1} \cdots v_1)$. Due to Lemma 3.11, we have $\sum_{j=1}^{r+1}(d_j + \ell(\tilde{v}_j)) = \langle 2\rho, \gamma^\vee \rangle + \ell(us_\gamma) = 1 + \ell(u) = 1 + \sum_{j=1}^{r+1} \ell(v_j)$. It remains to show $d_r + \ell(\tilde{v}_r) \leq \ell(v_r)$, or equivalently to show $\ell(v_{r+1}) \leq \ell(\tilde{v}_{r+1}) + d_{r+1} - 1 = \ell(\tilde{v}_{r+1}) + (\sum_{j=1}^{r+1} d_j - 1) - \sum_{j=1}^{r} d_j = \ell(\tilde{v}_{r+1}) + \ell(s_\gamma) - \sum_{j=1}^{r} d_j$.

a) Since $v_{r+1}, \tilde{v}_{r+1} \in W^P$, we conclude $\ell(v_r \cdots v_1) = |A_1|$ and $\ell(\tilde{v}_r \cdots \tilde{v}_1) = |A_2|$ where $A_1 := \{\beta \in R_P^+ \mid u(\beta) \in -R^+\}$ and $A_2 := \{\beta \in R_P^+ \mid us_\gamma(\beta) \in -R^+\}$. Note that $us_\gamma(\beta) = u(\beta)$ for $\beta \in R_{\bar{P}}$. Hence, $\beta \in A_2 \setminus A_1$ only if $\beta \in R_P^+ \setminus R_{\bar{P}}$. Thus $|A_2| - |A_1| \leq |A_2 \setminus A_1| \leq |R_P^+ \setminus R_{\bar{P}}| = \ell(\omega_P \omega_{\bar{P}})$. Consequently, we have $\ell(\tilde{v}_r) - \ell(v_r) = \big(\ell(\tilde{v}_r \cdots \tilde{v}_1) - \ell(v_r \cdots v_1)\big) + \sum_{j=1}^{r-1} d_j = |A_2| - |A_1| + \sum_{j=1}^{r-1} d_j \leq \ell(\omega_P \omega_{\bar{P}}) + \sum_{j=1}^{r-1} d_j \leq -d_r$.

b) Since $v_{r+1} \in W^P$ and $\Delta_{\hat{P}} \supset \Delta_P$, we can write $v_{r+1} = v'_{r+2} v'_{r+1}$ in which $v'_{r+2} \in W^{\hat{P}}$ and $v'_{r+1} \in W_{\hat{P}}^P \subset W^P$. Similarly, we write $\tilde{v}_{r+1} = \tilde{v}'_{r+2} \tilde{v}'_{r+1}$ with $\tilde{v}'_{r+2} \in W^{\hat{P}}$ and $\tilde{v}'_{r+1} \in W_{\hat{P}}^P$. Note that $\ell(v'_{r+1}) = |A_3|$ and $\ell(\tilde{v}'_{r+1}) = |A_4|$, where $A_3 := \{\alpha \in R_{\hat{P}}^+ \setminus R_P \mid u(\alpha) \in -R^+\}$ and $A_4 := \{\alpha \in R_{\hat{P}}^+ \setminus R_P \mid us_\gamma(\alpha) \in -R^+\}$. We claim A_3 can be written as a disjoint union $B_1 \sqcup B_2 \sqcup B_3$ such that $|B_1| \leq \ell(s_\gamma) - |\Xi_1|$, $B_2 \subset \Xi_2$, and $B_3 \subset A_4$. Hence, $\ell(v'_{r+1}) = |A_3| = |B_1| + |B_2| + |B_3| \leq \ell(s_\gamma) - |\Xi_1| + |\Xi_2| + \ell(\tilde{v}'_{r+1})$. Since $\gamma \in R_{\hat{P}}$, we have $\tilde{v}'_{r+2} = v'_{r+2}$. Therefore $\ell(v_{r+1}) - \ell(\tilde{v}_{r+1}) = \ell(v'_{r+1}) - \ell(\tilde{v}'_{r+1}) \leq \ell(s_\gamma) - |\Xi_1| + |\Xi_2| \leq \ell(s_\gamma) - \sum_{j=1}^{r} d_j$.

It remains to show our claim. Clearly, A_3 is a disjoint union of B_i's in which the corresponding sets are given by $B_1 := \big\{\alpha \in R_{\hat{P}}^+ \setminus R_P \;\big|\; u(\alpha) \in$

$-R^+, s_\gamma(\alpha) \in -R^+\big\}$,

$$B_2 := \left\{ \alpha \in R^+_{\hat{P}} \setminus R_P \,\middle|\, \begin{array}{l} u(\alpha) \in -R^+, \\ s_\gamma(\alpha) \in R^+, \\ \langle \alpha, \gamma^\vee \rangle > 0 \end{array} \right\},$$

$$B_3 := \left\{ \alpha \in R^+_{\hat{P}} \setminus R_P \,\middle|\, \begin{array}{l} u(\alpha) \in -R^+, \\ s_\gamma(\alpha) \in R^+, \\ \langle \alpha, \gamma^\vee \rangle \leq 0 \end{array} \right\}.$$

Note that $B_1 \subset \big\{\alpha \in R^+_{\hat{P}} \setminus R_P \bigm| s_\gamma(\alpha) \in -R^+\big\} = \big\{\alpha \in R^+_{\hat{P}} \bigm| s_\gamma(\alpha) \in -R^+\big\} - \big\{\alpha \in R^+_P \bigm| s_\gamma(\alpha) \in -R^+\big\}$. Since $\gamma \notin R_P$, for any $\alpha \in R^+_P$ we conclude that $s_\gamma(\alpha) = \alpha - \langle \alpha, \gamma^\vee \rangle \gamma \in -R^+$ if and only if $\langle \alpha, \gamma^\vee \rangle > 0$. Hence, $|B_1| \leq \big|\{\alpha \in R^+_{\hat{P}} \mid s_\gamma(\alpha) \in -R^+\}\big| - \big|\{\alpha \in R^+_P \mid s_\gamma(\alpha) \in -R^+\}\big| = \ell(s_\gamma) - |\Xi_1|$. It is obvious that $B_2 \subset \Xi_2$. Since $\ell(us_\gamma) < \ell(u)$, we have $u(\gamma) \in -R^+$ by Lemma 2.5. Consequently, if $\alpha \in B_3$, then we have $us_\gamma(\alpha) = u(\alpha) + (-\langle \alpha, \gamma^\vee \rangle)u(\gamma) \in -R^+$, implying $\alpha \in A_4$. Hence, $B_3 \subset A_4$. q.e.d.

4. Proofs of main results

In this section, we prove all the theorems mentioned in the introduction. Recall that the proof of Theorem 1.2 has been given in section 2.3.

When Δ_P is not of A-type, we have denoted $\varsigma = r - 1$. For convenience, we denote $\varsigma = r$ if Δ_P is of A-type. Recall that $\Delta_\varsigma = \{\alpha_1, \cdots, \alpha_\varsigma\}$, $P_\varsigma = P_{\Delta_\varsigma}$ and $Q^\vee_\varsigma = \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\alpha_i^\vee$. (In particular when $\varsigma = r$, we have $P_\varsigma = P$ and $Q^\vee_\varsigma = Q^\vee_P$.)

Lemma 4.1. *$gr(W_{P_\varsigma} \times Q^\vee_\varsigma) = \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\mathbf{e}_i$, where we have treated $W_{P_\varsigma} \times Q^\vee_\varsigma$ as a subset of $W \times Q^\vee$ naturally. Furthermore, for any $\mathbf{d} = \bigoplus_{i=1}^{\varsigma} d_i \mathbf{e}_i$, we have*

1) *$\mathbf{d} = gr(wq_\lambda)$ for a unique $wq_\lambda \in W \times Q^\vee$. In fact, $wq_\lambda \in W_{P_\varsigma} \times Q^\vee_\varsigma$.*
2) *Take the unique wq_λ as in (1). Then $wq_\lambda \in QH^*(G/B)$ if $d_i \geq 0$ for all i.*

Proof. Define a matrix $M = \big(m_{i,j}\big)_{\varsigma \times \varsigma}$ by using the gradings $gr(q_i)$'s. That is, we define $\sum_{j=1}^{\varsigma} m_{i,j}\mathbf{e}_j = (1-i)\mathbf{e}_{i-1} + (1+i)\mathbf{e}_i (= gr(q_i))$ for each $1 \leq i \leq \varsigma$. Note that M is a lower-triangular matrix. Hence, there exist unique sequences $\mathbf{a} = (a_1, \cdots, a_\varsigma), \mathbf{b} = (b_1, \cdots, b_\varsigma)$ of integers such that $\mathbf{d} = \mathbf{a}M + \mathbf{b}$ and $0 \leq b_i \leq m_{i,i} - 1 = i$ for $1 \leq i \leq \varsigma$. Furthermore, if $d_i \geq 0$ for all i, then we conclude $a_i \geq 0$ for all i, by noting $m_{i,j} \leq 0$ whenever $j < i$. Since $W_{P_i}^{P_{i-1}} = \{u_k^{(i)} \mid 0 \leq k \leq i\}$, each $0 \leq b_i \leq i$ corresponds to a unique element in $W_{P_i}^{P_{i-1}}$, say, $u_{b_i}^{(i)}$. Hence, we find a unique $(w, \lambda) := (u_{b_\varsigma}^{(\varsigma)} \cdots u_{b_1}^{(1)}, \sum_{i=1}^{\varsigma} a_i \alpha_i^\vee) \in W_{P_\varsigma} \times Q^\vee_\varsigma$ such that $gr(wq_\lambda) = \mathbf{d}$; furthermore, $wq_\lambda \in QH^*(G/B)$ whenever $d_i \geq 0$ for all i.

It remains to show $gr(uq_\mu) \notin \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\mathbf{e}_i$ whenever $(u, \mu) \notin W_{P_\varsigma} \times Q_\varsigma^\vee$. Indeed, it follows directly from Definition 2.8 that $gr_{[k,r+1]}(q_{\alpha^\vee}) = x_k\mathbf{e}_k$ with $x_k \geq 2$, whenever $\alpha \in \Delta_k \setminus \Delta_{k-1}$. Take the decomposition $u = v_{r+1} \cdots v_1$ of u associated to (Δ_P, Υ) and note that $gr(uq_\mu) = \bigoplus_{i=1}^{r+1} \ell(v_i)\mathbf{e}_i + gr(q_\mu)$. Thus if $gr(uq_\mu) \in \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\mathbf{e}_i$, then we have $\ell(v_{r+1}) = 0$ and $\mu \in Q_P^\vee (= Q_r^\vee)$. When $\varsigma = r$, it is done. When $\varsigma = r-1$, we proceed to conclude $\ell(v_r) = 0$ and $\mu \notin Q_r^\vee \setminus Q_\varsigma^\vee$. Thus $u \in W_{P_\varsigma}$ and $\mu \in Q_\varsigma^\vee$. q.e.d.

Proof of Lemma 2.12. We need to show for any $q_\mu u, q_\nu v \in QH^*(G/B)$ there exists $q_\lambda w \in QH^*(G/B)$ such that $gr(q_\mu u) + gr(q_\nu v) = gr(q_\lambda w)$. Note that $gr(q_\mu) + gr(q_\nu) = gr(q_{\mu+\nu})$ and $gr(q_\mu) + gr(u) = gr(q_\mu u)$. Thus it remains to show $gr(u) + gr(v) \in S$. It suffices to show $\mathbf{x} = (x_1, \cdots, x_{r+1}) \in S$ for any $\mathbf{x} \in (\mathbb{Z}_{\geq 0})^{r+1}$.

We first assume $\varsigma = r$. Take any simple root in $\Delta_{\varsigma+1} \setminus \Delta_\varsigma (= \Delta \setminus \Delta_P)$, say, α. From Table 3, we conclude $gr(q_{\alpha^\vee}) = \sum_{i=1}^{\varsigma+1} d_i\mathbf{e}_i$ with $2 \leq d_{\varsigma+1} \leq 1 + \ell(\omega_{P_{\varsigma+1}}\omega_{P_\varsigma}) (= \ell(\omega\omega_P) + 1)$ and $d_i \leq 0$ for $i \leq \varsigma$. Since $x_{\varsigma+1} \geq 0$, we can write $x_{\varsigma+1} = a_{\varsigma+1}d_{\varsigma+1} + b_{\varsigma+1}$ for unique $a_{\varsigma+1} \geq 0$ and $0 \leq b_{\varsigma+1} \leq \ell(\omega_{P_{\varsigma+1}}\omega_{P_\varsigma})$. Note that $\ell(\omega_{P_{\varsigma+1}}\omega_{P_\varsigma}) = \max\{\ell(v) \mid v \in W_{P_{\varsigma+1}}^{P_\varsigma}\}$ so that we can choose $v_{\varsigma+1} \in W_{P_{\varsigma+1}}^{P_\varsigma}$ satisfying $\ell(v_{\varsigma+1}) = b_{\varsigma+1}$. Furthermore, $(x_1 - a_{\varsigma+1}d_1, \cdots, x_\varsigma - a_{\varsigma+1}d_\varsigma)$ is again a sequence of non-negative integers. Thus it is the grading of a unique $(w', \lambda') \in W_{P_\varsigma} \times Q_\varsigma^\vee$ with $q_{\lambda'} \in QH^*(G/B)$ by Lemma 4.1. Set $w = v_{\varsigma+1}w'$ and $\lambda = a_{\varsigma+1}\alpha^\vee + \lambda'$. Then $wq_\lambda \in QH^*(G/B)$ is the element as required.

Now we assume $\varsigma = r-1$. By Lemma 3.27 there exists $\alpha' \in \Delta \setminus \Delta_P$ such that $gr(q_{\alpha'^\vee}) = \sum_{i=1}^{r+1} d_i'\mathbf{e}_i$ with $2 \leq d_{r+1}' \leq 1 + \ell(\omega\omega_P)$ and $d_i' \leq 0$ for $i \leq r = \varsigma + 1$. Repeating the above discussions, we can reduce it to the question of finding a element in $q_\lambda w \in W_{P_{\varsigma+1}} \times Q_{\varsigma+1}^\vee$ with $q_\lambda w \in QH^*(G/B)$ and the grading of it being equal to $\sum_{i=1}^{\varsigma+1} x_i'\mathbf{e}_i$ for given non-negative integers x_i''s. Thus the statement follows by using the same arguments again. q.e.d.

Remark 4.2. $\mathbb{Z}_{\geq 0}\mathbf{e}_{r+1}$ is a sub-semigroup of S. Indeed, we can take $\alpha \in \Delta$ such that $gr_{[r+1,r+1]}(q_{\alpha^\vee}) = d_{r+1}\mathbf{e}_{r+1}$ with $2 \leq d_{r+1} \leq 1 + \ell(\omega\omega_P)$, from the above proof. For any $c \in \mathbb{Z}_{\geq 0}$, $c = ad_{r+1} + b$ with $0 \leq b \leq d_{r+1} - 1$. Then we can choose $v \in W^P$ such that $\ell(v) = b$. Note that $-gr_r(q_{a\alpha^\vee}) = \sum_{i=1}^r x_i\mathbf{e}_i$ with x_i's being in $\mathbb{Z}_{\geq 0}$. Hence, it is a grading of certain element $q_\lambda u \in QH^*(G/B)$ where $(u, \lambda) \in W_P \times Q_P^\vee$. Then $q_{a\alpha^\vee+\lambda}vu \in QH^*(G/B)$ and its grading is equal to $c\mathbf{e}_{r+1}$.

The next lemma proves the first half of Theorem 1.3.

Lemma 4.3. *The subspace $\mathcal{I}$ defined in Theorem* 1.3 *is an ideal of* $QH^*(G/B)$.

Proof. We need to show for any $q_\mu u \in \mathcal{I}$ and $q_\nu v \in QH^*(G/B)$, the product $q_\mu u \star q_\nu v = \sum_{w,\lambda} N_{u,v}^{w,\lambda} q_{\lambda+\mu+\nu} w$ also lies in $\mathcal{I}$. That is, we need to show $d_{r+1} \geq 1$ where $gr_{[r+1,r+1]}(q_{\lambda+\mu+\nu}w) = d_{r+1}\mathbf{e}_{r+1}$, whenever $N_{u,v}^{w,\lambda} \neq 0$. Clearly, this is true if either μ or ν lies in $Q^\vee \setminus Q_P^\vee$, which follows directly from Definition 2.8. When $\mu, \nu \in Q_P^\vee$, we must have $u \in W \setminus W_P$. Then we shall show $gr_{[r+1,r+1]}(q_\lambda w) \geq \mathbf{e}_{r+1}$ whenever $N_{u,v}^{w,\lambda} \neq 0$, by using induction on $\ell(v)$.

If $\ell(v) = 0$, then $v = \text{id}$ and it is done. If $\ell(v) = 1$, then v is a simple reflection and therefore we can use the quantum Chevalley formula (Proposition 2.2). When $\ell(us_\gamma) = \ell(u) + 1$, we take the decomposition $us_\gamma = v_{r+1} \cdots v_1$ associated to (Δ_P, Υ) and note u is obtained by deleting a reflection in some v_m. Since $u \notin W_P$, we conclude $v_{r+1} \neq 1$. In particular, we have $gr_{[r+1,r+1]}(us_\gamma) \geq \mathbf{e}_{r+1}$. When $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle$, then we also conclude $gr_{[r+1,r+1]}(q_{\gamma^\vee} us_\gamma) \geq \mathbf{e}_{r+1}$, by noting $gr_{[r+1,r+1]}(q_{\gamma^\vee} us_\gamma) \geq gr_{[r+1,r+1]}(us_\gamma) = gr_{[r+1,r+1]}(u) \geq \mathbf{e}_{r+1}$ if $\gamma \in Q_P^\vee$, and $gr_{[r+1,r+1]}(q_{\gamma^\vee} us_\gamma) \geq gr_{[r+1,r+1]}(q_{\gamma^\vee}) \geq 2\mathbf{e}_{r+1}$ if $\gamma \notin Q_P^\vee$. Thus if $\ell(v) = 1$, then $\sigma^v \star \mathcal{I} \subset \mathcal{I}$. Now we assume $\ell(v) > 1$. By Lemma 2.13, there exist $v' \in W$ and $1 \leq j \leq n$ such that $gr(v) = gr(v') + gr(s_j)$ and $\sigma^{v'} \star \sigma^{s_j} = c\sigma^v + \sum_{w,\lambda} c_{w,\lambda} q_\lambda w$, where $c > 0$ and the summation is only over finitely many non-zero terms for which $c_{w,\lambda} > 0$. In particular, we have $\ell(v') = \ell(v) - 1$. Using the induction hypothesis, we have $\sigma^{v'} \star \mathcal{I} \subset \mathcal{I}$. Thus $(c\sigma^v + \sum_{w,\lambda} c_{w,\lambda} q_\lambda w) \star \mathcal{I} = \sigma^{s_j} \star (\sigma^{v'} \star \mathcal{I}) \subset \mathcal{I}$. Since all the structure constants are non-negative, there is no cancellation in the summation on the left-hand side of the equality. Hence, we conclude $\sigma^v \star \mathcal{I} \subset \mathcal{I}$. q.e.d.

It remains to show the second half of Theorem 1.3. There are combinatorial characterizations of $QH^*(G/B)$ (see e.g., [**25**]), or more generally on its torus-equivariant extension [**26**]. In particular, intuitively, $QH^*(G/B)$ should also have a non-equivariant version of Mihalcea's criterion [**26**] for torus-equivariant quantum cohomology of G/B. That is, an algebra $(\bigoplus_{w \in W} \mathbb{Q}[\mathbf{q}]\sigma^w, *)$ should be isomorphic to $QH^*(G/B)$ as algebras, if it satisfies the quantum Chevalley formula together with some natural properties (e.g., commutativity and associativity). However, we did not find any explicit reference for this. In our case, we obtain a natural algebra of this form which has one more (strong) property saying that $(\bigoplus_{w \in W} \mathbb{Q}[\mathbf{q}]\sigma^w, *)|_{\mathbf{q}=\mathbf{0}}$ is canonically isomorphic to $H^*(G/B)$. Thus it becomes easy to show the algebra isomorphism (by using induction). We would like to thank A.-L. Mare and L. C. Mihalcea for their comments for such a criterion and the proof.

Proof of Theorem 1.3. Due to Lemma 4.3, it remains to show $QH^*(P/B)$ is canonically isomorphic to $QH^*(G/B)/\mathcal{I}$. Note that P/B is isomorphic to the complete flag variety determined by the pair $(\Delta_P, \emptyset)$. Hence,

$QH^*(P/B)$ has a natural basis of Schubert classes $\{\sigma^w \mid w \in W_P\}$ over $\mathbb{Q}[q_1, \cdots, q_r]$, and the formula of $\sigma^u \star_f \sigma^{s_i}$ (where $u \in W_P$ and $\alpha_i \in \Delta_P$) is directly obtained from Proposition 2.2 by restriction of $\gamma \in \Delta$ to $\gamma \in \Delta_P$ in the summation. Here we denote the quantum product of $QH^*(P/B)$ by $\star_f$, in order to distinguish it with the quantum product $\star$ of $QH^*(G/B)$. On the other hand, $QH^*(G/B)/\mathcal{I}$ has a natural algebra structure induced from $QH^*(G/B)$. Thus it is also commutative and associative, and we denote the product of it by the same $\star$ by abuse of notation.

It is clear that for any $wq_\lambda \in W \times Q^\vee$, $gr_{[r+1,r+1]}(wq_\gamma) = \mathbf{0}$, i.e., $wq_\lambda \notin \mathcal{I}$, if and only if $wq_\lambda \in W_P \times Q_P^\vee$. We define a map $\varphi : QH^*(G/B) \longrightarrow QH^*(P/B)$, given by $\varphi(q_\lambda) = q_\lambda$ if $wq_\lambda \notin \mathcal{I}$, or 0 if $wq_\lambda \in \mathcal{I}$. Clearly, φ induces a natural isomorphism $\bar{\varphi}$ of vector spaces, $\bar{\varphi} : QH^*(G/B)/\mathcal{I} \longrightarrow QH^*(P/B)$, given by $\bar{\varphi}(\overline{wq_\lambda}) := \varphi(wq_\lambda)$. In particular, it is easy to check $\bar{\varphi}(\overline{\sigma^{s_i}} \star \overline{\sigma^{s_j}}) = \sigma^{s_i} \star_f \sigma^{s_j}$ for any $\alpha_i, \alpha_j \in \Delta_P$. It is a well known fact that $QH^*(P/B)$ is generated by $\{\sigma^{s_\alpha} \mid \alpha \in \Delta_P\}$ over $\mathbb{Q}[q_1, \cdots, q_r]$. Thus it is sufficient to show $QH^*(G/B)/\mathcal{I}$ is generated by $\{\overline{\sigma^{s_\alpha}} \mid \alpha \in \Delta_P\}$. Since our filtration on $QH^*(G/B)$ generalizes the classical filtration on $H^*(G/B)$ (by Proposition 2.14) naturally, $QH^*(G/B)/\mathcal{I}\big|_{\overline{q_\lambda}=0}$ is canonically isomorphic to $H^*(P/B)$ as algebras. In particular, it is generated by $\{\overline{\sigma^{s_1}}, \cdots, \overline{\sigma^{s_r}}\}$ with respect to the induced cup product. Hence, the statement follows by using the quantum Chevalley formula and induction (for instance one can follow the proof of Lemma 2.1 of [**30**] exactly). q.e.d.

Remark 4.4. For the classical case, the induced map $i^* : H^*(G/B) \longrightarrow H^*(P/B)$ is given by $i^*(\sigma^w) = \sigma^w$ if $w \in W_P$, or 0 otherwise. And the ideal I is given by $I = \mathbb{Q}\{\sigma^w \mid w = vu \text{ with } u \in W_P, v \in W^P, v \neq 1\}$. Note that for any $w \in W, w \notin I$ if and only if $gr_{[r+1,r+1]}(\sigma^w) = \mathbf{0}$. Clearly, $\mathcal{I}$ is a $\mathbf{q}$-deformation of I and φ is a natural generalization of i^*.

Lemma 4.5. *Let $w = v_{r+1} \cdots v_1$ be the decomposition of $w \in W$ associated to (Δ_P, Υ). For any $1 \le m \le \varsigma$, the following holds.*

1) *If $\ell(v_m) < m$, then there exists $\gamma \in R^+$ such that $\langle \chi_m, \gamma^\vee \rangle = 1$, $\ell(ws_\gamma) = \ell(w) + 1$ and $gr(ws_\gamma) = gr(w) + \mathbf{e}_m$.*
2) *If $\ell(v_m) = m$, then there exists $\gamma \in R^+$ such that $\langle \chi_m, \gamma^\vee \rangle = 1$, $\ell(ws_\gamma) = \ell(w) + 1 - \langle 2\rho, \gamma^\vee \rangle$ and $gr(q_{\gamma^\vee} ws_\gamma) = gr(w) + \mathbf{e}_m$.*

Proof. Note that $Dyn(\{\alpha_1, \cdots, \alpha_\varsigma\})$ is of A-type. We have $v_k = u_{i_k}^{(k)}$ with $i_k = \ell(v_k)$ whenever $1 \le k \le \varsigma$.

(1) If $i_m < m$, we set $\gamma := (v_{m-1} \cdots v_1)^{-1}(\alpha_{m-i_m} + \alpha_{m-i_m+1} + \cdots + \alpha_m)$. Then γ is of the form $\alpha_m + \sum_{j=1}^{m-1} a_j \alpha_j \in R$. Thus $\gamma \in R^+$ and $\langle \chi_m, \gamma^\vee \rangle = 1$. Moreover, we conclude $ws_\gamma = v_{r+1} \cdots v_{m+1} u_{i_m+1}^{(m)} v_{m-1} \cdots v_1$. Thus (1) follows.

(2) Denote $k = 1 + \max\{j \mid i_j = 0, 0 \le j \le m-1\}$, where $i_0 := 0$, and set $\gamma = \alpha_k + \alpha_{k+1} + \cdots + \alpha_m \in R^+$. Clearly, $\langle \chi_m, \gamma^\vee\rangle = 1$. For each $j \le m$, we denote $\gamma_j = \alpha_j + \alpha_{j+1} + \cdots + \alpha_m$. Then $\gamma_j = \gamma_{j+1} + \alpha_j = s_j(\gamma_{j+1})$. Thus for any $i_j \ge 1$, we have $u^{(j)}_{i_j} s_{\gamma_j} = u^{(j)}_{i_j} s_j s_{\gamma_{j+1}} s_j = u^{(j-1)}_{i_j-1} s_{\gamma_{j+1}} s_j = s_{\gamma_{j+1}} u^{(j-1)}_{i_j-1} s_j = s_{\gamma_{j+1}} u^{(j)}_{i_j}$. Furthermore, we have $s_1 s_2 \cdots s_j u^{(j)}_{i_j} = u^{(j)}_{i_j-1} s_1 s_2 \cdots s_{j-1}$ by Lemma 3.3. Note that $\gamma = \gamma_k, \gamma_m = \alpha_m$ and denote $u = v_m \cdots v_1$. Hence,

$$\begin{aligned} u s_\gamma &= u^{(m)}_{i_m} \cdots u^{(k)}_{i_k} s_{\gamma_k} u^{(k-2)}_{i_{k-2}} \cdots u^{(1)}_{i_1} \\ &= u^{(m)}_{i_m} \cdots u^{(k+1)}_{i_{k+1}} s_{\gamma_{k+1}} u^{(k)}_{i_k} s_\gamma u^{(k-2)}_{i_{k-2}} \cdots u^{(1)}_{i_1} \\ &= u^{(m)}_{i_m} s_{\gamma_m} u^{(m-1)}_{i_{m-1}} \cdots u^{(k)}_{i_k} u^{(k-2)}_{i_{k-2}} \cdots u^{(1)}_{i_1} \\ &= s_1 s_2 \cdots s_{m-1} u^{(m-1)}_{i_{m-1}} \cdots u^{(k)}_{i_k} u^{(k-2)}_{i_{k-2}} \cdots u^{(1)}_{i_1} \\ &= u^{(m-1)}_{i_{m-1}-1} \cdots u^{(k)}_{i_k-1} (s_1 \cdots s_{k-1}) u^{(k-2)}_{i_{k-2}} \cdots u^{(1)}_{i_1}. \end{aligned}$$

Note that $i_m = m$ and $i_j = \ell(v_j)$ for $j \le \varsigma$. Thus,

$$\begin{aligned} \ell(w s_\gamma) &= \sum\nolimits_{p=m+1}^{r+1} \ell(v_p) + \big(\sum\nolimits_{j=k}^{m-1} (i_j - 1)\big) + k - 1 + \sum\nolimits_{j=1}^{k-2} i_j \\ &= \sum\nolimits_{p=1}^{r+1} \ell(v_p) - (m-k) + k - 1 - m \\ &= \ell(w) + 1 - 2\langle \rho, (\alpha_k + \cdots + \alpha_m)^\vee\rangle = \ell(w) + 1 - \langle 2\rho, \gamma^\vee\rangle. \end{aligned}$$

Furthermore, we conclude $gr(q_{\gamma^\vee} w s_\gamma) = gr(w) + \mathbf{e}_m$, by noting $i_m = m$, $i_{k-1} = 0$, and $gr(q_{\gamma^\vee}) = (1-k)\mathbf{e}_{k-1} + (m+1)\mathbf{e}_m + \sum_{j=k}^{m-1} \mathbf{e}_j$. q.e.d.

Since $QH^*(G/B)$ has an S-filtration $\mathcal{F}$, we obtain an associated S-graded algebra $Gr^{\mathcal{F}}(QH^*(G/B)) = \bigoplus_{\mathbf{a}\in S} Gr^{\mathcal{F}}_{\mathbf{a}}$, where $Gr^{\mathcal{F}}_{\mathbf{a}} := F_{\mathbf{a}} / \cup_{\mathbf{b}<\mathbf{a}} F_{\mathbf{b}}$. For each $j \le r+1$, we denote $Gr^{\mathcal{F}}_{(j)}(QH^*(G/B)) := \bigoplus_{i\ge 0} Gr^{\mathcal{F}}_{i\mathbf{e}_j}$. Note that for the iterated fibration $\{P_{j-1}/P_0 \to P_j/P_0 \longrightarrow P_j/P_{j-1}\}_{j=2}^{r+1}$ associated to (Δ_P, Υ), we have $P_{r+1}/P_r = G/P$ and $P_j/P_{j-1} \cong \mathbb{P}^j$ whenever $j \le \varsigma$. Take the canonical isomorphism $QH^*(\mathbb{P}^k) \cong \frac{\mathbb{Q}[x_k, t_k]}{\langle x_k^{k+1} - t_k\rangle}$ for each $k \le \varsigma$. Then we can state Theorem 1.4 more concretely as follows (in which we denote $u^{(0)}_0 := 1$).

Theorem 1.4. *There exist canonical isomorphisms Ψ_k's of algebras:*
For each $k \le \varsigma$, $\Psi_k : QH^*(\mathbb{P}^k) \longrightarrow Gr^{\mathcal{F}}_{(k)}(QH^*(G/B))$;

$$x_k \mapsto \overline{u^{(k)}_1},\ t_k \mapsto \overline{q_k u^{(k-1)}_{k-1}}.$$

For $k = \varsigma + 1$, $\Psi_{\varsigma+1} : QH^*(P_{\varsigma+1}/P_\varsigma) \longrightarrow Gr^{\mathcal{F}}_{(\varsigma+1)}(QH^*(G/B))$;

$$q_{\lambda_{P_\varsigma}} \sigma^v \mapsto \overline{\psi_{\Delta_{\varsigma+1}, \Delta_\varsigma}(q_{\lambda_{P_\varsigma}} \sigma^v)}.$$

In particular if Δ_P is of A-type (i.e., if $\varsigma = r$), then we have $P_{\varsigma+1}/P_\varsigma = G/P$, $\Delta_{r+1} = \Delta$ and $\Delta_\varsigma = \Delta_P$. Thus in this case, Theorem 1.4 gives an isomorphism $QH^*(G/P) \xrightarrow{\simeq} Gr^{\mathcal{F}}_{(r+1)}(QH^*(G/B))$.

Proof of Theorem 1.4. By Lemma 4.1, for any $\mathbf{a} \in \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\mathbf{e}_i$ there exists a unique $q_\lambda u \in QH^*(G/B)$ such that $gr(q_\lambda u) = \mathbf{a}$. Thus $\dim_{\mathbb{Q}} Gr^{\mathcal{F}}_{\mathbf{a}} = 1$ and $Gr^{\mathcal{F}}_{\mathbf{a}} = \mathbb{Q}\overline{q_\lambda u}$. Then we simply denote $A_{\mathbf{a}} := \overline{q_\lambda u}$. In particular, we conclude $A_{\mathbf{a}} \star A_{\mathbf{e}_j} = c_j A_{\mathbf{a}+\mathbf{e}_j}$ whenever $j \leq \varsigma$. Furthermore, we have $c_j = 1$ by Lemma 4.5. When $\ell(v) > 1$, there exists $v' \in W_{P_\varsigma}$ satisfying $gr(v') + gr(s_p) = gr(v)$ with $p \leq \varsigma$ by using Lemma 4.5 again. Thus by induction on $\ell(v)$, we conclude $A_{\mathbf{a}} \star A_{gr(v)} = A_{\mathbf{a}+gr(v)}$ for any $v \in W_{P_\varsigma}$. Hence, $A_{\mathbf{a}} \star A_{\mathbf{b}} = A_{\mathbf{a}+\mathbf{b}}$ for any $\mathbf{a}, \mathbf{b} \in \bigoplus_{i=1}^{\varsigma} \mathbb{Z}\mathbf{e}_i$. As a consequence, we obtain a canonical isomorphism $QH^*(\mathbb{P}^k) \cong Gr^{\mathcal{F}}_{(k)}(QH^*(G/B))$ for each $1 \leq k \leq \varsigma$, given by $x_k \mapsto \overline{u_1^{(k)}}$ and $t_k \mapsto \overline{q_k u_{k-1}^{(k-1)}}$.

In order to analyze $\Psi_{\varsigma+1}$, we need to compare the algebra structure of $QH^*(P_{\varsigma+1}/P_\varsigma)$ with the filtered-algebra structure of $QH^*(G/B)$. Note that if $\varsigma = r-1$, then $P_{\varsigma+1}/B = P/B$. Due to Theorem 1.3, essentially we need to compare $QH^*(P_{\varsigma+1}/P_\varsigma)$ with $QH^*(P/B)$ by using the Peterson-Woodward comparison formula in this case. Thus without loss of generality, we can assume $\varsigma = r$ in the rest, which is of main interest to us and can bring convenience on the notation.

Denote the quantum product of $QH^*(G/P)$ by $\star_P$. Write $\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^v) = q_{\lambda_B}\sigma^{v\omega_P\omega'}$, where $q_{\lambda_P}\sigma^v \in QH^*(G/P)$. Then we have $gr_r(q_{\lambda_B}\sigma^{v\omega_P\omega'}) = \mathbf{0}$ by Proposition 3.24. On the other hand, if $gr_r(q_\lambda\sigma^{vu}) = \mathbf{0}$ with $\lambda_P = \lambda + Q_P^\vee$ and $u \in W_P$, then we conclude $gr(q_{\lambda_B-\lambda}\omega_P\omega') = gr(u)$ where $\lambda_B - \lambda \in Q_P^\vee$. By the uniqueness (from Lemma 4.1), we conclude $\lambda_B = \lambda$ and $\omega_P\omega' = u$. Hence, Ψ_{r+1} is an isomorphism of vector spaces.

By Proposition 2.1, we have $\Psi_{r+1}(\sigma^u \star_P \sigma^v) = \Psi_{r+1}(\sigma^u) \star \Psi_{r+1}(\sigma^v)$ for $u, v \in W^P$. To show $\Psi_{\varsigma+1}$ is an algebra isomorphism, it remains to show (i) $\Psi_{r+1}(q_{\lambda_P} \star_P q_{\mu_P}) = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(q_{\mu_P})$ and (ii) $\Psi_{r+1}(q_{\lambda_P} \star_P \sigma^v) = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(\sigma^v)$. For (i), we write $\lambda_P = \lambda' + Q_P^\vee$ and $\mu_P = \mu' + Q_P^\vee$ where λ', μ' are elements in $\bigoplus_{\alpha \in \Delta \setminus \Delta_P} \mathbb{Z}\alpha^\vee$. Note that $gr_{[r+1,r+1]}(q_{\lambda'}) - gr_{[r+1,r+1]}(q_{\lambda_B}\omega_P\omega') = \mathbf{0}$. Hence, $q_{\lambda_B}\omega_P\omega' = q_{\lambda'}x$ with x being the unique element in $W_P \times Q_P^\vee$ determined by the grading $-gr_r(\lambda') =: \mathbf{a}$. Similarly, we have $\psi_{\Delta,\Delta_P}(q_{\mu_P}) = q_{\mu'}y$ and $\psi_{\Delta,\Delta_P}(q_{\lambda_P+\mu_P}) = q_{\lambda'+\mu'}z$ where $gr(y) = -gr_r(q_{\mu'}) =: \mathbf{b}$ and $gr(z) = -gr_r(q_{\lambda'+\mu'})$. Hence, $\Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(q_{\mu_P}) = \overline{q_{\lambda'}x} \star \overline{q_{\mu'}y} = \overline{q_{\lambda'+\mu'}} \star A_{\mathbf{a}} \star A_{\mathbf{b}} = \overline{q_{\lambda'+\mu'}} \star A_{\mathbf{a}+\mathbf{b}} = \Psi_{r+1}(q_{\lambda_P+\mu_P})$. For (ii), we first conclude $\overline{\sigma^{s_j}} \star \overline{\sigma^v} = \overline{\sigma^{vs_j}}$ where $1 \leq j \leq r$ and $v \in W^P$, by Proposition 3.23. Thus by induction on $\ell(u)$ where $u \in W_P$, we conclude $\overline{\sigma^u} \star \overline{\sigma^v} = \overline{\sigma^{vu}}$. As a consequence, (ii) follows. Hence, Ψ_{r+1} is an algebra isomorphism. q.e.d.

As a consequence, we obtain the following.

Theorem 1.5. *Denote* $\Gamma := \{gr(q_\lambda w) \mid gr(q_\lambda w) < \mathbf{0}, q_\lambda w \in QH^*(G/B)\}$. *Let* $\mathcal{A} = \bigoplus_{gr(q_\lambda\sigma^w)\in\mathbb{Z}\mathbf{e}_{r+1}\cup\Gamma} \mathbb{Q}q_\lambda\sigma^w$ *and* $\mathcal{J} = \bigoplus_{gr(q_\lambda\sigma^w)\in\Gamma} \mathbb{Q}q_\lambda\sigma^w$. *Then* $\mathcal{A}$ *is a subalgebra of* $QH^*(G/B)$ *and* $\mathcal{J}$ *is an ideal of* $\mathcal{A}$. *Furthermore, if* Δ_P *is of A-type, then there is a canonical algebra isomorphism*

$$QH^*(G/P) \xrightarrow{\simeq} \mathcal{A}/\mathcal{J};$$
$$q_{\lambda_P}\sigma^v \mapsto \psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^v) + \mathcal{J}.$$

Proof. Note that for any $q_\lambda\sigma^w \in QH^*(G/B)$, $gr(q_\lambda\sigma^w) \in \mathbb{Z}\mathbf{e}_{r+1}$ if and only if $gr(q_\lambda\sigma^w) \in \mathbb{Z}_{\geq 0}\mathbf{e}_{r+1}$. Clearly, $\mathbb{Z}_{\geq 0}\mathbf{e}_{r+1} \cup \Gamma$ is a sub-semigroup of S. Hence, $\mathcal{A}$ is a subalgebra of $QH^*(G/B)$, due to Theorem 1.2. From Definition 2.8, we note $gr_{[r+1,r+1]}(q_\lambda w) \geq \mathbf{0}$ whenever $q_\lambda w \in QH^*(G/B)$. Thus for such element, $gr(q_\lambda w) < \mathbf{0}$ if and only if $gr_r(q_\lambda w) < \mathbf{0}$. In particular, we conclude $\mathcal{J}$ is an ideal of $\mathcal{A}$, by use of Theorem 1.2 again. As a consequence, we obtain a natural isomorphism $\mathcal{A}/\mathcal{I} \xrightarrow{\simeq} Gr^{\mathcal{F}}_{(r+1)}(QH^*(G/B))$. Hence, the statement follows from Theorem 1.4. q.e.d.

Remark 4.6. In fact, $\mathcal{A} = \bigcup_{i\geq 0} F_{i\mathbf{e}_{r+1}}$. If we use the $\mathbb{Z}^{r+1}$-filtration on $QH^*(G/B)$ that is naturally extended from the S-filtration, then we have $\mathcal{J} = F_{-\mathbf{e}_{r+1}}$. Furthermore, it is obvious that $\mathcal{A}$, $\mathcal{J}$ are $\mathbf{q}$-deformations of $A = \pi^*(H^*(G/P))$ and $J = 0$, respectively. Note that $\pi^*(\sigma^v) = \sigma^v$ for any $v \in W^P$. ψ_{Δ,Δ_P} is a natural generalization of π^*.

Recall that in Definition 2.8, we have given the gradings for all $q_\lambda w$'s. Clearly, S is contained in $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$ as a sub-semigroup. Combining Lemma 4.1 and (part of) the proof of Lemma 2.12, we conclude that S is naturally extended to the whole $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$ with negative powers of $\{q_1, \cdots, q_r\}$ allowed. That is, $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0} = \{gr(q_\lambda\sigma^w) \mid q_\lambda\sigma^w \in QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}]\}$. Therefore we obtain a natural $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$-filtration on $QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}]$, making it a $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$-filtered algebra, due to Theorem 1.2. By abuse of notation, we also denote this filtration as $\mathcal{F}$. Consequently, we obtain a natural embedding of graded algebras $Gr^{\mathcal{F}}(QH^*(G/B)) \hookrightarrow Gr^{\mathcal{F}}(QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}])$. For simplicity, we assume Δ_P is of A-type. Then for each $1 \leq k \leq r$, we note $\Psi_k(t_k^k) = \overline{\prod_{i=1}^k q_i^i}$. By defining $t_k^{-1} \mapsto \Psi_k(t_k^{k-1}) \star \overline{\prod_{i=1}^k q_i^{-i}}$, we can extend the algebra isomorphism Ψ_k to a larger algebra isomorphism

$$QH^*(\mathbb{P}^k)[t_k^{-1}] \xrightarrow{\simeq} \bigoplus_{j\in\mathbb{Z}} Gr^{\mathcal{F}}_{j\mathbf{e}_k}(QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}]),$$

which we also simply denote as Ψ_k. Thus the next theorem follows as a direct consequence of Theorem 1.2 and Theorem 1.4.

Theorem 1.6. *$QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}]$ has a $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$-filtration $\mathcal{F}$. If Δ_P is of A-type, then combining Ψ_k's gives an isomorphism of $\mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$-graded algebras,*

$$\Psi: \ \Big(\bigotimes_{k=1}^{r} QH^*(\mathbb{P}^k)[t_k^{-1}]\Big) \bigotimes QH^*(G/P) \longrightarrow Gr^{\mathcal{F}}(QH^*(G/B)[q_1^{-1}, \cdots, q_r^{-1}]).$$

That is, $\Psi = \Psi_1 \star \cdots \star \Psi_{r+1} : \big(\otimes_{k=1}^r f_k\big) \otimes q_{\lambda_P}\sigma^v \longmapsto \big(\prod_{k=1}^r \Psi_k(f_k)\big) \star \Psi_{r+1}(q_{\lambda_P}\sigma^v)$.
(Note we have an isomorphism $H^*(\mathbb{P}^1) \otimes \cdots \otimes H^*(\mathbb{P}^r) \otimes H^*(G/P) \cong Gr^{\mathcal{F}}(H^*(G/B))$ of graded algebras, coming from the Leray spectral sequence.)

5. Conclusions

All the theorems in the induction can be easily generalized to all cases by dropping our assumption that Δ_P is connected. We give a brief description as follows.

Write $\Delta_P = \bigsqcup_{k=1}^m \Delta_{(k)}$ such that $Dyn(\Delta_{(k)})$ is a connected component of $Dyn(\Delta_P)$ for each k. Then the Weyl subgroup W_P also splits into direct product of W_k's which are the corresponding Weyl subgroups of $\Delta_{(k)}$'s. That is, $W_P = W_1 \times \cdots \times W_m$. Among these $\Delta_{(k)}$'s, there is at most one which is not of A-type. If such a subbase exists, then we just assume it to be the last one, say, $\Delta_{(m)}$. For each k, we denote $r_k = |\Delta_{(k)}|$. Set $M = \sum_{k=1}^m r_k$ and then take the standard basis $\{\mathbf{e}_{1,1}, \cdots, \mathbf{e}_{1,r_1}, \cdots, \mathbf{e}_{m,1}, \cdots, \mathbf{e}_{m,r_m}, \mathbf{e}_{m+1,1}\}$ of $\mathbb{Z}^{M+1}$.

For each k, we fix the canonical order $(\Delta_{(k)}, \Upsilon_k)$ as described in section 2.4. Then we obtain a grading map $gr_{\Delta_{(k)}} : W \times Q^\vee \longrightarrow \bigoplus_{i=1}^{r_k+1} \mathbb{Z}\mathbf{e}_{k,i}$, using Definition 2.8 with respect to $(\Delta_{(k)}, \Upsilon_k)$. In particular, for any $x \in W_k$ or $x = q_{\alpha^\vee}$ with $\alpha \in \Delta_{(k)}$, we have $gr_{\Delta_{(k)}}(x) \in \bigoplus_{i=1}^{r_k} \mathbb{Z}\mathbf{e}_{k,i} \hookrightarrow \mathbb{Z}^{M+1}$, which we treat as an element of $\mathbb{Z}^{M+1}$ naturally.

Definition 5.1. We define a grading map $gr : W \times Q^\vee \longrightarrow \mathbb{Z}^{M+1}$ associated to (Δ_P, Υ) as follows, where $\Upsilon = \prod_{k=1}^m \Upsilon_k$.

1) Write $w = v_{m+1}v_m \cdots v_1$ (uniquely), in which $(v_1, \cdots, v_m, v_{m+1}) \in W_1 \times \cdots \times W_m \times W^P$. Then $gr(w) \triangleq \ell(v_{m+1})\mathbf{e}_{m+1,1} + \sum_{k=1}^m gr_{\Delta_{(k)}}(v_k)$.

2) For each $\alpha_{k,i} \in \Delta_{(k)}$, $gr(q_{\alpha^\vee_{k,i}}) \triangleq gr_{\Delta_{(k)}}(q_{\alpha^\vee_{k,i}})$. For $\alpha \in \Delta \setminus \Delta_P$, we write $\psi_{\Delta,\Delta_P}(q_{\alpha^\vee + Q^\vee_P}) = \omega_P\omega' q_{\alpha^\vee} \prod_{k=1}^{m}\prod_{i=1}^{r_k} q_{\alpha^\vee_{k,i}}^{a_{k,i}}$ and then define

$$gr(q_{\alpha^\vee}) \triangleq \big(\ell(\omega_P\omega') + 2 + \sum_{k=1}^{m}\sum_{i=1}^{r_k} 2a_{k,i}\big)\mathbf{e}_{m+1,1} - gr(\omega_P\omega') - \sum_{k=1}^{m}\sum_{i=1}^{r_k} a_{k,i}\, gr(q_{\alpha^\vee_{k,i}}).$$

3) In general, $x = w\prod_{\alpha\in\Delta} q_{\alpha^\vee}^{b_\alpha}$, then $gr(x) \triangleq gr(w) + \sum_{\alpha\in\Delta} b_\alpha gr(q_{\alpha^\vee})$.

As in section 2.3, we can define a subset, consisting of the gradings of $q_\lambda w$'s in $QH^*(G/B)$. This subset also turns out to be a (totally-ordered) sub-semigroup of $\mathbb{Z}^{M+1}$, and we also simply denote it as S by abuse of notation. In addition, we obtain a family of subspaces of $QH^*(G/B)$ in the same way, which we also simply denote as $\mathcal{F} = \{\mathcal{F}_\mathbf{a}\}_{\mathbf{a}\in S}$ by abuse of notation. Then all the theorems in the introduction can be easily generalized. For instance, we state part of them in summary as follows.

Theorem 5.2.

1) *$QH^*(G/B)$ has an S-filtered algebra structure with filtration $\mathcal{F}$, which naturally extends to a $\mathbb{Z}^{M+1}$-filtered algebra structure on $QH^*(G/B)$.*
2) *There is a canonical algebra isomorphism*

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\simeq} QH^*(P/B)$$

for an ideal $\mathcal{I}$ (which is explicitly defined) of $QH^*(G/B)$.
3) *Assume P/B is isomorphic to product of $F\ell_{1+r_k}$'s (i.e., $\Delta_{(k)}$'s are of A-type).*
    a) *There exists a subalgebra $\mathcal{A}$ of $QH^*(G/B)$ together with an ideal $\mathcal{J}$ of $\mathcal{A}$, such that $QH^*(G/P)$ is canonically isomorphic to $\mathcal{A}/\mathcal{J}$ as algebras.*
    b) *As graded algebras, (after localization) $Gr^{\mathcal{F}}(QH^*(G/B))$ is isomorphic to $\big(\bigotimes_{k=1}^{m}\bigotimes_{i_k=1}^{r_k} QH^*(\mathbb{P}^{i_k})\big)\bigotimes QH^*(G/P)$.*

We would like to point out again that our assumption “ all $\Delta_{(k)}$'s are of A-type” is already general enough. This situation has covered all G/P's for G being of A-type or G_2-type, and more than half of G/P's for each remaining type. Unfortunately, Theorem 5.2 (3.b) is not true in a more general case when $\Delta_{(m)}$ is not of A-type. In fact in this case, $QH^*(G/P)$ is only canonically isomorphic to a proper subspace of $Gr^{\mathcal{F}}_{(M+1)}(QH^*(G/B)) = \bigoplus_{i\geq 0} F_{i\mathbf{e}_{M+1}}/\cup_{\mathbf{b}<i\mathbf{e}_M} F_\mathbf{b}$ as vector spaces. However, we could still expect the following.

Conjecture 5.3. *There exists a canonical algebra isomorphism between $QH^*(G/P)$ and a subalgebra of $Gr^{\mathcal{F}}_{(M+1)}(QH^*(G/B))$.*

As a direct consequence of Conjecture 5.3, we can conclude Theorem 5.2 (3.a) always holds for any G/P. Part of the points of the proof for this is to show (i) and (ii) in the proof of Theorem 1.4. That is, we need to show the behavior of $\overline{\psi_{\Delta,\Delta_P}(q_{\lambda_P})}$'s is like that of polynomials. Indeed, when Δ is of C-type, (i) and (ii) become trivial. (Precisely, we use the notation in case C1) in Table 2 and assume Δ_P to be of C-type. Then for any $\lambda_P = \sum_{j=1}^{o} b_j\beta_j^\vee + Q_P^\vee \in Q^\vee/Q_P^\vee$, we conclude $\overline{\psi_{\Delta,\Delta_P}(q_{\lambda_P})} = \overline{q_{\lambda_B}\cdot 1}$ with $\lambda_B = \sum_{j=1}^{o} b_j\beta_j^\vee + b_o\sum_{p=o+1}^{n}\beta_p^\vee$ by direct calculations.) In this case, we could still prove Conjecture 5.3 together with some other arguments. On the other hand, it is shown in [**18**] that after taking torus-equivariant extension and localization, Theorem 5.2 (3.a) is true in terms of the localization of equivariant homology of a based loop group. Hence, we believe that Theorem 5.2 (3.a) also holds without taking equivariant extension and localization. Both of these provide evidence for our conjecture.

In addition, we would like to ask the following.

Question 5.4. What is the difference between $QH^*(G/P)$ and $Gr^{\mathcal{F}}_{(M+1)}(QH^*(G/B))$?

The ring structure of $Gr^{\mathcal{F}}_{(M+1)}(QH^*(G/B))$, or equivalently $\mathcal{A}/\mathcal{J}$, which is defined in the same form as in Theorem 1.5, seems close to the ring structure of $QH^*(G/B)$. Especially, there might be one way to obtain a nice presentation of $Gr^{\mathcal{F}}_{(M+1)}(QH^*(G/B))$ from the presentation of $QH^*(G/B)$ [**16**]. Suppose there were such a way and we knew the answer to Question 5.4; then we would have a better understanding on $QH^*(G/P)$.

6. Appendix

In this section, we show the Key Lemma also holds for all the roots that satisfy condition (ii) in the proof of the Key Lemma in section 3.5 whenever Δ is of F_4-type or E-type. Since all the arguments are similar, we just list all such roots as well as the corresponding methods for them. One can see [**19**] for more details.

When Δ is of F_4-type, case C9) or C10) will occur. For instance for C9), $\gamma^\vee$ must be either of the form $\sum_{i\leq t\leq k}\beta_t^\vee$ or equal to one of the following five coroots: $\beta_1^\vee+2\beta_2^\vee+\beta_3^\vee, \beta_1^\vee+2\beta_2^\vee+\beta_3^\vee+\beta_4^\vee, \beta_1^\vee+2\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee, \beta_1^\vee+3\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee, 2\beta_1^\vee+3\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$, by noting $\ell(s_\gamma)=\langle 2\rho,\gamma^\vee\rangle-1$. Then we have:

Table for case C9)

Coroots	$r=2$	$r=3$
$\beta_2^\vee+\beta_3^\vee$	done ($\gamma\in R_P$)	
$\beta_3^\vee+\beta_4^\vee$	(M3)	
$\beta_1^\vee+\beta_2^\vee+\beta_3^\vee$	(M3)	done
$\beta_2^\vee+\beta_3^\vee+\beta_4^\vee$	(M1)	
$\beta_1^\vee+\beta_2^\vee+\beta_3^\vee+\beta_4^\vee$	(M1)	done
$\beta_1^\vee+2\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$	(M2)	
$2\beta_1^\vee+3\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$	(M1)	done

Table for case C10)

Coroots	$r=2$	$r=3$
$\beta_3^\vee+\beta_4^\vee$	(M3)	(M2)
$\beta_1^\vee+\beta_2^\vee+\beta_3^\vee$	(M1)	done
$\beta_1^\vee+\beta_2^\vee+\beta_3^\vee+\beta_4^\vee$	(M1)	done
$\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$	(M3)	(M2)
$\beta_1^\vee+\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$	(M2)	done
$\beta_1^\vee+2\beta_2^\vee+3\beta_3^\vee+\beta_4^\vee$	(M2)	
$\beta_1^\vee+2\beta_2^\vee+3\beta_3^\vee+2\beta_4^\vee$	(M1)	

We would like to make some comments for the tables in this section.

1) By "done," we mean that there exists $\alpha_j\in\{\alpha_1,\cdots,\alpha_{r-1}\}$ such that $\langle\alpha_j,\gamma^\vee\rangle>0$. Thus it is done by the arguments for condition (i) in the proof of the Key Lemma. By "done ($\gamma\in R_P$)," we mean $\gamma\in R_P$ and thus it is done by the arguments at the beginning of the proof of the Key Lemma.
2) By "(M1)" (resp. "(M3)"), we mean the corresponding method, especially the use of part a) (resp. b)) of Lemma 3.29.
3) By "(M2)", we mean the induction hypothesis is used. In fact, whenever referring to (M2) in the tables, we can use the same arguments as follows. For instance, we consider the case when C10) occurs, $\gamma^\vee=\beta_2^\vee+2\beta_3^\vee+\beta_4^\vee$, and $r=3$. In this case, we can take $\alpha_j=\beta_3(=\alpha_3)$. Then $\beta^\vee=s_j(\gamma^\vee)=\beta_2^\vee+\beta_3^\vee+\beta_4^\vee$ and consequently $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=gr(q_{\beta^\vee})=(-1,1,-2,8)$. Furthermore, we have $(a_1,a_2,a_3,a_4)=(0,a_2,a_3,0)$ and $(b_1,b_2,b_3,b_4)=(0,b_2,b_3,0)$ with $a_2+a_3=1$ and $b_2+b_3=-1$ by noting $\langle\alpha_1,\alpha_j^\vee\rangle=0$. If $\mu_1<0$, then it is done. If $\mu_1=0$, then by the induction hypothesis we have $\mu_2\leq 0$. We claim $\mu_2=0$. Thus $\mu_3\leq 0$ and $a_2+b_2=a_2+\mu_2+b_2\leq 0$ (by considering $\tilde{gr}$). Since $a_2+a_3+b_2+b_3=0$, if $a_2+b_2<0$ then it is done; otherwise, we have $a_3+b_3=-(a_2+b_2)=0$ so that $a_3+b_3+\mu_3=\mu_3\leq 0$. Thus it is done. It remains to show our claim. Indeed, we note that $\mu_2+i_2'=k_2+\lambda_2=k_2+1$. Since $\ell(us_js_\beta)<\ell(us_j)$, $us_j(\beta)\in -R^+$. Then if $\langle\alpha,\beta^\vee\rangle\leq 0$, $us_j(\alpha)\in -R^+$ implies $us_js_\beta(\alpha)\in -R^+$. Hence $i_2'=\sharp\{\alpha\in R^+_{P_2}\setminus R_{P_1}\mid us_j(\alpha)\in -R^+\}=\sharp\{\alpha\in R^+_{P_2}\setminus R_{P_1}\mid us_j(\alpha)\in -R^+,\langle\alpha,\beta^\vee\rangle\leq 0\}+\sharp\{\alpha\in R^+_{P_2}\setminus R_{P_1}\mid us_j(\alpha)\in -R^+,\langle\alpha,\beta^\vee\rangle>0\}\leq\sharp\{\alpha\in R^+_{P_2}\setminus R_{P_1}\mid us_js_\beta(\alpha)\in -R^+\}+\sharp\{\alpha\in R^+_{P_2}\setminus R_{P_1}\mid\langle\alpha,\beta^\vee\rangle>0\}=k_2+\sharp\{\beta_2\}$. Thus $\mu_2=k_2+1-i_2'\geq 0$ and consequently we have $\mu_2=0$.

Now we assume Δ is of E-type. Denote $\Xi:=\{\beta_i\mid\langle\beta_i,\gamma^\vee\rangle>0\}$. Recall that we should replace $\kappa=o+r$ with $\kappa=o+\varsigma(=o+r-1)$ in Table 2 when Δ_P is not of A-type. Note that any $\gamma\in R^+$ is of length $\langle 2\rho,\gamma^\vee\rangle-1$. It suffices to assume $n=8$. It remains to discuss at most the roots in the tables as below.

Table for case C4) with $r = 6$ or $r = 7$

Roots with $\Xi \subset \{\beta_1, \beta_2, \beta_8\}$	$r = 6$	$r = 7$
$\beta_1 + \beta_2$	done	done
$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_8$	(M3)	done
$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_8$	done	(M3)
$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_8$	(M1)	done
$\beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	done ($\gamma \in R_P$)	
$\beta_2 + \beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	(M3)	done
$\beta_1 + \beta_2 + \beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	done	(M3)
$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	(M2)	done
$\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 3\beta_6 + \beta_7 + 3\beta_8$	(M2)	(M3)
$\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 4\beta_6 + 2\beta_7 + 3\beta_8$	(M1)	done
$2\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 4\beta_6 + 2\beta_7 + 3\beta_8$	done	(M1)

Table for case C5) with $r = 5$

Roots with $\Xi \subset \{\beta_5, \beta_6\}$	Method
$\beta_5 + \beta_6$	(M3)
$\beta_2 + 2\beta_3 + \beta_4 + 2\beta_5 + \beta_6$	(M3)
$\beta_2 + 2\beta_3 + \beta_4 + 2\beta_5 + 2\beta_6 + \beta_7$	(M1)
$\beta_1 + 2\beta_2 + 3\beta_3 + \beta_4 + 3\beta_5 + 2\beta_6 + \beta_7$	(M3)
$\beta_1 + 2\beta_2 + 3\beta_3 + \beta_4 + 3\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M1)
$\beta_1 + 2\beta_2 + 4\beta_3 + 2\beta_4 + 4\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M2)
$2\beta_1 + 4\beta_2 + 6\beta_3 + 3\beta_4 + 5\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M2)
$2\beta_1 + 4\beta_2 + 6\beta_3 + 3\beta_4 + 5\beta_5 + 4\beta_6 + 2\beta_7 + \beta_8$	(M1)

Table for case C7) with $0 \leq o \leq 3$

	Roots with $\begin{cases} \Xi \subset \{\beta_1, \beta_2, \beta_3, \beta_7, \beta_8\} \\ \lvert\Xi \cap \{\beta_1, \beta_2, \beta_3\}\rvert \leq 1 \end{cases}$	Constraint	Method
1)	$\beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 3$	(M3)
2)	$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 2$	
3)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 1$	
4)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7$	$o = 3$	(M1)
5)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7$	$o = 2$	
6)	$\beta_7 + \beta_8$	$o \geq 0$	(M2,2,3,3)
7)	$\beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 3$	(M1)
8)	$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 2$	
9)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 1$	
10)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7 + \beta_8$	$o = 3$	
11)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7 + \beta_8$	$o = 2$	
12)	$\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o \geq 0$	(M2,2,3,3)
13)	$\beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	(M3)
14)	$\beta_2 + \beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 2$	(M2)
15)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 1$	
16)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	
17)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 2$	
18)	$\beta_1 + 2\beta_2 + 3\beta_3 + 3\beta_4 + 3\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	(M1)
19)	$\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o \geq 0$	(M2)
20)	$\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o = 1$	
21)	$\beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o = 2$	

22)	$\beta_1+2\beta_2+3\beta_3+3\beta_4+4\beta_5+2\beta_6+3\beta_7+\beta_8$	$o=3$	
23)	$\beta_2+2\beta_3+3\beta_4+4\beta_5+2\beta_6+3\beta_7+2\beta_8$	$o\geq 0$	(M1)
24)	$\beta_1+\beta_2+2\beta_3+3\beta_4+4\beta_5+2\beta_6+3\beta_7+2\beta_8$	$o=1$	
25)	$\beta_1+2\beta_2+2\beta_3+3\beta_4+4\beta_5+2\beta_6+3\beta_7+2\beta_8$	$o=2$	
26)	$\beta_1+2\beta_2+3\beta_3+3\beta_4+4\beta_5+2\beta_6+3\beta_7+2\beta_8$	$o=3$	
27)	$\beta_1+2\beta_2+3\beta_3+4\beta_4+5\beta_5+2\beta_6+4\beta_7+2\beta_8$	$o\geq 0$	(M3,2,2,2)
28)	$\beta_1+2\beta_2+4\beta_3+5\beta_4+6\beta_5+3\beta_6+4\beta_7+2\beta_8$	$o=3$	(M1)
29)	$\beta_1+3\beta_2+4\beta_3+5\beta_4+6\beta_5+3\beta_6+4\beta_7+2\beta_8$	$o=2$	
30)	$2\beta_1+3\beta_2+4\beta_3+5\beta_4+6\beta_5+3\beta_6+4\beta_7+2\beta_8$	$o=1$	

In the above table, by "(M2,2,3,3)" for the root $\beta_7+\beta_8$, we mean (M2) (resp. (M2), (M3), and (M3)) is used when $o=0$ (resp. 1, 2 and 3). Similar notations are used for the case no. 12) and no. 27).

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