J. DIFFERENTIAL GEOMETRY
85 (2010) 461-477

AN INTEGRAL FORMULA FOR THE VOLUME ENTROPY WITH APPLICATIONS TO RIGIDITY

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1. Introduction

Let M^n be a compact Riemannian manifold and $\pi : \widetilde{M} \to M$ its universal covering. The fundamental group $G = \pi_1(M)$ acts on $\widetilde{M}$ as isometries such that $M = \widetilde{M}/G$. Associated to $\widetilde{M}$ are several asymptotic invariants. In this paper we are primarily concerned with the volume entropy v defined by

$$v = \lim_{r\to\infty} \frac{\ln \mathrm{vol} B_{\widetilde{M}}(x,r)}{r},$$

where $B_{\widetilde{M}}(x,r)$ is the ball of radius r centered at x in $\widetilde{M}$. It is proved by Manning [**M**] and Freire-Mañé [**FM**] that

- the limit exists and is independent of the center $x \in \widetilde{M}$,
- $v \leq H$, the topological entropy of the geodesic flow on M,
- $v = H$ if M has no conjugate points.

There has been a lot of work on understanding the volume entropy of which we only mention the celebrated paper of Besson, Courtois, and Gallot [**BCG1**], where one can find other results and references. But the volume entropy still remains a subtle invariant. If M is negatively curved, it is better understood due to the existence of the so-called Patterson-Sullivan measure on the ideal boundary. Let $\partial\widetilde{M}$ be the ideal boundary of $\widetilde{M}$ defined as equivalence classes of geodesic rays. We fix a base point $o \in \widetilde{M}$ and for $\xi \in \partial\widetilde{M}$ we denote B_ξ the associated Busemann function, i.e.,

$$B_\xi(x) = \lim_{t\to\infty} d(x, \gamma(t)) - t,$$

where γ is the geodesic ray initiating from o and representing ξ. It is well known that B_ξ is smooth and its gradient is of length one. The Patterson-Sullivan measure [**P**, **S**, **K**] is a family $\left\{\nu_x : x \in \widetilde{M}\right\}$ of measures on $\partial\widetilde{M}$ such that

- for any pair $x, y \in \widetilde{M}$, the two measures ν_x, ν_y are equivalent with

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-v\left(B_\xi(x) - B_\xi(y)\right)};$$

Received 11/2/2009.

- for any $g \in G$,

$$g_* \nu_x = \nu_{gx}.$$

The Patterson-Sullivan measure contains a lot of information and plays an important role in [**BCG2**]. Moreover, it is proved by Knieper, Ledrappier, and Yue ([**K**, **L2**, **Y1**]) that the following integral formula for the volume entropy holds in terms of the Patterson-Sullivan measure:

$$v = \frac{1}{C} \int_M \left(\int_{\partial \widetilde{M}} \Delta B_\xi(x) \, d\nu_x(\xi) \right) dx, \tag{1.1}$$

where $C = \int_M \nu_x \left(\partial \widetilde{M} \right) dx$. (To interpret the formula properly, notice after integrating over $\partial \widetilde{M}$ we get a function on $\widetilde{M}$ which is G-invariant and hence descends to M.) This formula shows how v interacts with local geometry.

In this paper, we will extend the theory of the Patterson-Sullivan measure to any manifold without the negative curvature assumption. More generally, let $\pi : \widetilde{M} \to M$ be a regular Riemannian covering of a compact manifold M and G the discrete group of deck transformations. We will consider the Busemann compactification of $\widetilde{M}$, denoted by $\widehat{M}$. On the Busemann boundary $\partial \widehat{M}$ we will construct a Patterson-Sullivan measure which retains the essential features of the classical theory. Namely:

Theorem 1. *There exists a probability measure ν on the laminated space $X_M = \left(\widetilde{M} \times \partial \widehat{M} \right) / G$ such that for any continuous vector field Y on X_M which is C^1 along the leaves,*

$$\int \operatorname{div}^{\mathcal{W}} Y \, d\nu = v \int \left\langle Y, \nabla^{\mathcal{W}} \xi \right\rangle d\nu,$$

where $\operatorname{div}^{\mathcal{W}}$ *and* $\nabla^{\mathcal{W}}$ *are laminated divergence and gradient, respectively.*

As an application of the above theorem, we will prove the following rigidity theorem.

Theorem 2. *Let M^n be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi : \widetilde{M} \to M$ a regular covering. Then the volume entropy of $\widetilde{M}$ satisfies $v \leq (n-1)$ and equality holds if and only if M is hyperbolic.*

The inequality $v \leq (n-1)$ is of course well known and follows easily from the volume comparison theorem. What is new is the rigidity part. For some perspective on this result, recall another invariant: the bottom spectrum of the Laplacian on $\widetilde{M}$, denoted by λ_0 and defined as

$$\lambda_0 = \inf_{f \in C_c^1(\widetilde{M})} \frac{\int_{\widetilde{M}} |\nabla f|^2}{\int_{\widetilde{M}} f^2}.$$

It is a well-known fact that $\lambda_0 \leq v^2/4$. Therefore as an immediate corollary of Theorem 2 we have the following result previously proved by the second author [**W**].

Corollary 1. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi : \widetilde{M} \to M$ a regular covering. If $\lambda_0 = (n-1)^2/4$, then $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n$.*

Clearly the asymptotic invariant v is much weaker than λ_0. It is somewhat surprising that we still have a rigidity theorem for v. If M is negatively curved, Theorem 2 is proved by Knieper [**K**] using (1.1). The proof in the general case is more subtle due to the fact the Busemann functions are only Lipschitz. In fact, it is partly to prove this rigidity result that we are led to the construction of the measure ν and the formula in Theorem 1.

We will also discuss the Kähler and quaternionic Kähler analogue of Theorem 2. In the Kähler case, our method yields the following:

Theorem 3. *Let M be a compact Kähler manifold with $\dim_{\mathbb{C}} M = m$ and $\pi : \widetilde{M} \to M$ a regular covering. If the bisectional curvature $K_{\mathbb{C}} \geq -2$, then the volume entropy v satisfies $v \leq 2m$. Moreover, equality holds if and only if M is complex hyperbolic (normalized to have constant holomorphic sectional curvature -4).*

To clarify the statement, the condition $K_{\mathbb{C}} \geq -2$ means that for any two vectors X, Y,

$$R(X,Y,X,Y)+R(X,JY,X,JY) \geq -2\left(|X|^2|Y|^2 + \langle X,Y\rangle^2 + \langle X,JY\rangle^2\right),$$

where J is the complex structure.

In the quaternionic Kähler case we have:

Theorem 4. *Let M be a compact quaternionic Kähler manifold of $\dim = 4m$ with $m \geq 2$ and scalar curvature $-16m(m+2)$. Let $\pi : \widetilde{M} \to M$ be a regular covering. Then the volume entropy v satisfies $v \leq 2(2m+1)$. Moreover, equality holds if and only if M is quaternionic hyperbolic.*

The paper is organized as follows. In Section 2, we discuss the Busemann compactification and construct the Patterson-Sullivan measure and prove Theorem 1. Theorem 2 will be proved in Section 3. We will discuss the Kähler case and the quaternionic Kähler case in Section 4.

Acknowledgments. The first author was partially supported by NSF grant DMS-0801127. The second author was partially supported by NSF grant DMS-0905904.

2. Construction of the measure

Let $\widetilde{M}$ be a noncompact, complete Riemannian manifold. Fix a point $o \in \widetilde{M}$ and define, for $x \in \widetilde{M}$, the function $\xi_x(z)$ on $\widetilde{M}$ by

$$\xi_x(z) \;=\; d(x,z) - d(x,o).$$

The assignment $x \mapsto \xi_x$ is continuous, one-to-one, and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The Busemann compactification $\widehat{M}$ of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology. The space $\widehat{M}$ is a compact separable space. The *Busemann boundary* $\partial\widehat{M} := \widehat{M} \setminus \widetilde{M}$ is made of Lipschitz continuous functions ξ on $\widetilde{M}$ such that $\xi(o) = 0$. Elements of $\partial\widehat{M}$ are called *horofunctions*.

First we collect some general facts about horofunctions; see, e.g., [**SY**, **Pe**]. Suppose $\xi \in \widehat{M}$ is the limit of $\{a_k\} \subset \widetilde{M}$ with $d(o, a_k) \to \infty$, i.e.,

$$\xi(x) = \lim_{k\to\infty} f_k(x), \tag{2.1}$$

where $f_k(x) = \xi_{a_k}(x) = d(x, a_k) - d(o, a_k)$. The convergence is uniform over compact sets. We fix a point $p \in \widetilde{M}$ and for each k let γ_k be a minimizing geodesic from p to a_k. Passing to a subsequence, we can assume that γ_k converges to a geodesic ray γ starting from p. Let b_γ be the Busemann function associated to γ, i.e., $b_\gamma(x) = \lim_{s\to+\infty} d(x, \gamma(s)) - s$.

Lemma 1. *We have*

1) $\xi \circ \gamma(s) = \xi(p) - s$ *for* $s \geq 0$;
2) $\xi(x) \leq \xi(p) + d(x, \gamma(s)) - s$ *for* $s \geq 0$;
3) $\xi(x) \leq \xi(p) + b_\gamma(x)$.

Proof. For any $s > 0$ and $\varepsilon > 0$, we have $d(\gamma_k(s), \gamma(s)) \leq \varepsilon$ for k sufficiently large. Then

$$\begin{aligned} f_k \circ \gamma(s) - f_k(p) &= d(\gamma(s), a_k) - d(p, a_k) \\ &= d(\gamma(s), a_k) - d(\gamma_k(s), a_k) + d(\gamma_k(s), a_k) - d(p, a_k) \\ &\leq d(\gamma(s), \gamma_k(s)) + d(\gamma_k(s), a_k) - d(p, a_k) \\ &= d(\gamma(s), \gamma_k(s)) - s \\ &\leq \varepsilon - s. \end{aligned}$$

Taking limit yields $\xi \circ \gamma(s) - \xi(p) \leq \varepsilon - s$. Hence $\xi \circ \gamma(s) \leq \xi(p) - s$. On the other hand, we have the reversed inequality $\xi \circ \gamma(s) \geq \xi(p) - s$ as ξ is Lipschitz with Lipschitz constant 1.

To prove the second part, we have for $s \geq 0$,

$$\begin{aligned} f_k(x) &= d(x, a_k) - d(o, a_k) \\ &\leq d(x, \gamma(s)) + d(a_k, \gamma(s)) - d(o, a_k). \end{aligned}$$

Letting $k \to \infty$ yields

$$\begin{aligned}\xi(x) &\leq d(x, \gamma(s)) + \xi \circ \gamma(s) \\ &= d(x, \gamma(s)) - s + \xi(p).\end{aligned}$$

Taking limit as $s \to \infty$ yields the third part. q.e.d.

It follows that if ξ is differentiable at x, then $|\nabla \xi(x)| = 1$. Therefore $|\nabla \xi| = 1$ almost everywhere on $\widetilde{M}$.

Proposition 1. *$\widetilde{M}$ is open in its Busemann compactification $\widehat{M}$. Hence the Busemann boundary $\partial \widehat{M}$ is compact.*

Proof. Suppose otherwise and $p \in \widetilde{M}$ is the limit of a sequence $\{a_k\} \subset \widetilde{M}$ with $d(o, a_k) \to \infty$, i.e., $\xi_p(x) = \lim_{k\to\infty} \xi_{a_k}(x)$ and the convergence is uniform over compact sets. Then by Lemma 1 there is a geodesic ray γ starting from p such that $\xi_p \circ \gamma(s) = \xi_p(p) - s = -s - d(o, p)$ for $s \geq 0$. But

$$\begin{aligned}\xi_p \circ \gamma(s) &= d(\gamma(s), p) - d(o, p) \\ &= s - d(o, p),\end{aligned}$$

Clearly a contradiction. q.e.d.

We now further assume that $\widetilde{M}$ is a regular Riemannian covering of a compact manifold M, i.e., $\widetilde{M}$ is a Riemannian manifold and there is a discrete group G of isometries of $\widetilde{M}$ acting freely and such that the quotient $M = \widetilde{M}/G$ is a compact manifold. The quotient metric makes M a compact Riemannian manifold. We recall the construction of the laminated space X_M ([**L1**]). Observe that we may extend by continuity the action of G from $\widetilde{M}$ to $\widehat{M}$, in such a way that for ξ in $\widehat{M}$ and g in G,

$$g.\xi(z) \;=\; \xi(g^{-1}z) - \xi(g^{-1}o).$$

We define now the *horospheric suspension* X_M of M as the quotient of the space $\widetilde{M} \times \widehat{M}$ by the diagonal action of G. The projection onto the first component in $\widetilde{M} \times \widehat{M}$ factors into a projection from X_M to M so that the fibers are isometric to $\widehat{M}$. It is clear that the space X_M is metric compact. If $M_0 \subset \widetilde{M}$ is a fundamental domain for M, one can represent X_M as $M_0 \times \widehat{M}$ in a natural way.

To each point $\xi \in \widehat{M}$ is associated the projection W_ξ of $\widetilde{M} \times \{\xi\}$. As a subgroup of G, the stabilizer G_ξ of the point ξ acts discretely on $\widetilde{M}$ and the space W_ξ is homeomorphic to the quotient of $\widetilde{M}$ by G_ξ. We put on each W_ξ the smooth structure and the metric inherited from $\widetilde{M}$. The manifold W_ξ and its metric vary continuously on X_M. The collection of all $W_\xi, \xi \in \widehat{M}$ form a continuous lamination $\mathcal{W}_M$ with leaves which are manifolds locally modeled on $\widetilde{M}$. In particular, it makes

sense to differentiate along the leaves of the lamination, and we denote $\nabla^{\mathcal{W}}$ and $\mathrm{div}^{\mathcal{W}}$ the associated gradient and divergence operators: $\nabla^{\mathcal{W}}$ acts on continuous functions which are C^1 along the leaves of $\mathcal{W}$, $\mathrm{div}^{\mathcal{W}}$ on continuous vector fields in $T\mathcal{W}$ which are of class C^1 along the leaves of $\mathcal{W}$. We want to construct a measure on X_M which would behave as the Knieper measure $\nu = \int_M \left(\int_{\partial \widetilde{M}} d\nu_x(\xi)\right) dx$ in the negatively curved case. The construction follows Patterson's in the Fuchsian case.

Let v be the volume entropy of $\widetilde{M}$

$$v = \lim_{r\to\infty} \frac{\ln \mathrm{vol} B_{\widetilde{M}}(x,r)}{r},$$

where $B_{\widetilde{M}}(x,r)$ is the ball of radius r centered at x in $\widetilde{M}$.

Let us consider the Poincaré series of $\widetilde{M}$:

$$P(s) := \sum_{g\in G} e^{-sd(o,go)}.$$

Proposition 2. *The series $P(s)$ converges for $s > v$, and diverges to $+\infty$ for $s < v$.*

Proof. This is classical and for completeness we recall the proof. Take M_0 a fundamental domain in $\widetilde{M}$ containing o in its interior, and positive constants d, D such that $B(o,d) \subset M_0 \subset B(o,D)$.

Define $\pi(R) := \sharp\{g \in G : d(o,go) \le R\}$. We have $\pi(R+S) \le \pi(R+D)\pi(S+D)$, which implies $\pi(R+S+2D) \le \pi(R+2D)\pi(S+2D)$. It follows that the following limit exists:

$$\lim_{R\to\infty} \frac{1}{R} \ln \pi(R+2D) = \inf_R \frac{1}{R} \ln \pi(R+2D).$$

The above limit is the critical exponent of the Poincaré series. Since

$$\pi(R)\mathrm{vol}B(o,d) \le \mathrm{vol}B(o,R+d) \le \pi(R+D)\mathrm{vol}M,$$

the above limit is also $\lim_{R\to\infty} \frac{\ln \mathrm{vol} B_{\widetilde{M}}(x,R)}{R} = v$. q.e.d.

As in the classical case, a distinction has to be made between the case that $P(s)$ diverges at v and the case that it converges. The following lemma is due to Patterson [**P**] (see also [**N**]).

Lemma 2. *There exists a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous, non-decreasing, and*

1) *the series $P^*(s) := \sum_{g\in G} e^{-sd(o,go)} h\left(e^{d(o,go)}\right)$ converges for $s > v$ and diverges for $s \le v$,*
2) *if $\varepsilon > 0$ is given, there exists r_0 such that for $r > r_0, t > 1$, $h(rt) \le t^\varepsilon h(r)$.*

If $P(s)$ diverges at v, we will simply take h to be identically 1. As a consequence of property (2) above, we note that for t in a bounded interval

$$\frac{h\left(e^{r+t}\right)}{h\left(e^{r}\right)} \to 1$$

uniformly as $r \to \infty$.

For $x \in \widetilde{M}, s > v$, we define a finite measure $\nu_{x,s}$ by setting, for all f continuous on $\widehat{M}$,

$$\int f(\xi) d\nu_{x,s}(\xi) := \frac{1}{P^*(s)} \sum_{g \in G} e^{-sd(x,go)} h\left(e^{d(x,go)}\right) f(\xi_{go}).$$

Clearly, for $g \in G$, $g_* \nu_{x,s} = \nu_{gx,s}$, so that the measure $\widetilde{\nu}_s := \int \nu_{x,s} dx$ is G-invariant on $\widetilde{M} \times \widehat{M}$. We write ν_s for the corresponding measure on $X_M = \widetilde{M} \times \widehat{M}/G$. Choose a sequence $s_k > v$ and $s_k \to v$ as $k \to \infty$ such that the probability measures ν_{o,s_k} converge towards some probability measure ν_o. Since $\lim_{k\to\infty} P^*(s_k) = \infty$, the measure ν_o is supported on $\partial\widehat{M}$.

Proposition 3. *For any $x \in \widetilde{M}$, the measures ν_{x,s_k} converge to a measure ν_x on $\partial\widehat{M}$. Moreover,*

$$d\nu_x(\xi) = e^{-v\xi(x)} d\nu_o(\xi).$$

In particular, for any $g \in G$, we have

$$d(g_*\nu_o)(\xi) = d\nu_{go}(\xi) = e^{-v\xi(go)} d\nu_o(\xi), \quad d(g_*\nu_x)(\xi) = d\nu_{gx}(\xi),$$

and the limit of the measures ν_{s_k} on X_M is a measure ν on X_M which can be written, in the $M_0 \times \widehat{M}$ representation of X_M, as

$$\nu = e^{-v\xi(x)} d\nu_o(\xi)\, dx. \tag{2.2}$$

Proof. Observe first that for a fixed x, $\nu_{x,s}(\widehat{M}) \leq e^{(v+s)d(o,x)}$, so that the $\nu_{x,s}$ form a bounded family of measures on $\widehat{M}$. Let f be a continuous function on $\widehat{M}$. We may write

$$
\begin{aligned}
&\int f(\xi)\, e^{-v\xi(x)} d\nu_o(\xi) \\
&= \lim_{k\to\infty} \int f(\xi)\, e^{-v\xi(x)} d\nu_{o,s_k}(\xi) \\
&= \lim_{k\to\infty} \frac{1}{P^*(s_k)} \sum_{g\in G} f(\xi_{go})\, e^{-v(d(x,go)-d(o,go))} e^{-s_k d(o,go)} h\left(e^{d(o,go)}\right) \\
&= \lim_{k\to\infty} \frac{1}{P^*(s_k)} \sum_{g\in G} f(\xi_{go})\, e^{(s_k-v)\xi_{go}(x)} \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right) \\
&= \lim_{k\to\infty} \left(\int f(\xi)\, e^{(s_k-v)\xi(x)} d\nu_{x,s_k}(\xi) \;+\; \varepsilon_k \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_k = \frac{1}{P^*(s_k)} \sum_{g\in G} f(x,\xi_{go})\, e^{(s_k-v)\xi_{go}(x)} &\left(\frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} - 1 \right) \\
&e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right).
\end{aligned}
$$

Suppose $\lim \varepsilon_k = 0$. Then, for a fixed x, $e^{(s_k-v)\xi(x)}$ converges to 1 and, therefore, the limit exists and is $\int f e^{-v\xi(x)} d\nu_o$, as claimed. It only remains to show that $\lim \varepsilon_k = 0$. Indeed, for any $\delta > 0$ and any x, there exists a finite set $E \subset G$ such that for any $g \in G\backslash E$

$$
\left| \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} - 1 \right| < \delta.
$$

Then

$$
\begin{aligned}
|\varepsilon_k| &\le \frac{1}{P^*(s_k)} \sum_{g\in E} f(\xi_{go})\, e^{(s_k-v)\xi_{go}(x)} \left| \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} - 1 \right| e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right) \\
&\quad + \delta \frac{1}{P^*(s_k)} \sum_{g\in G\backslash E} f(\xi_{go})\, e^{(s_k-v)\xi_{go}(x)} e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right) \\
&\le \frac{1}{P^*(s_k)} \sum_{g\in E} f(\xi_{go})\, e^{(s_k-v)\xi_{go}(x)} \left| \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} - 1 \right| e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right) \\
&\quad + \delta \int f(\xi)\, e^{(s_k-v)\xi(x)} d\nu_{x,s_k}(\xi).
\end{aligned}
$$

Taking limit yields

$$
\lim_{k\to\infty} |\varepsilon_k| \le \delta \|f\|_\infty e^{vd(o,x)}.
$$

Therefore $\lim \varepsilon_k = 0$. q.e.d.

We can integrate by parts along each $M_0 \times \{\xi\}$, for ν_o-almost every ξ, and get the following for any function f which is C^2 along the leaves of the lamination $\mathcal{W}$ and has a support contained in $M_0 \times \widehat{M}$:

$$\begin{aligned}\int \Delta^{\mathcal{W}} f d\nu &= \int \left(\int_{M_0} \Delta^{\mathcal{W}} f e^{-v\xi(x)} dx \right) d\nu_o(\xi) \\ &= v \int \left(\int_{M_0} \left\langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \right\rangle e^{-v\xi(x)} dx \right) d\nu_o(\xi) \\ &= v \int \left\langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \right\rangle d\nu.\end{aligned}$$

The integral makes sense because $\nabla^{\mathcal{W}} \xi$ is defined Lebesgue almost everywhere on the leaves and because, by (2.2), the measure ν has absolutely continuous conditional measures along the leaves $\mathcal{W}$. By choosing the fundamental domain M_0, we get the same formula for any function which is C^2 along the leaves of the lamination $\mathcal{W}$ and has a small support. Using a partition of unity on M, we see that for all functions on X_M which are C^2 along the leaves of the lamination $\mathcal{W}$, we have

$$\int \Delta^{\mathcal{W}} f d\nu = v \int \left\langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \right\rangle d\nu.$$

In the same way, one gets the following for all continuous functions f_1, f_2 which are smooth along the leaves of the lamination $\mathcal{W}$:

$$\int \operatorname{div}^{\mathcal{W}}(f_1 \nabla^{\mathcal{W}} f_2) d\nu = v \int f_1 \left\langle \nabla^{\mathcal{W}} f_2, \nabla^{\mathcal{W}} \xi \right\rangle d\nu.$$

By approximation, we have for all $\mathcal{W}$ vector field Y which is C^1 along the leaves and globally continuous,

$$\int \operatorname{div}^{\mathcal{W}} Y d\nu = v \int \left\langle Y, \nabla^{\mathcal{W}} \xi \right\rangle d\nu. \tag{2.3}$$

Since the measure ν gives full measure to $\widetilde{M} \times \partial \widehat{M}$, Theorem 1 is proven.

3. The rigidity theorem

In this section we prove the rigidity theorem.

Theorem 5. *Let M^n be a compact Riemannian manifold with $Ric \geq -(n-1)$ and $\pi : \widetilde{M} \to M$ its universal covering. Then the volume entropy of $\widetilde{M}$ satisfies $v \leq (n-1)$ and equality holds if and only if M is hyperbolic.*

Observe that this proves Theorem 2, since the volume entropy of the universal covering is not smaller than the volume entropy of an intermediate covering space. First we have:

Proposition 4. *For any $\xi \in \partial\widehat{M}$, we have $\Delta\left(e^{-(n-1)\xi}\right) \geq 0$ in the sense of distribution.*

Proof. It is well known that $\Delta\xi \leq n-1$ in the distribution sense for any $\xi \in \partial\widehat{M}$. Indeed, suppose ξ is given as in formula (2.1). By the Laplacian comparison theorem,

$$\Delta f_k(x) \leq (n-1)\frac{\cosh(d(x,a_k))}{\sinh(d(x,a_k))}$$

in the distribution sense. Taking limit then yields $\Delta\xi \leq n-1$. Therefore

$$\begin{aligned}\Delta\left(e^{-(n-1)\xi}\right) &= -(n-1)e^{-(n-1)\xi}\left(\Delta\xi - (n-1)|\nabla\xi|^2\right)\\ &= -(n-1)e^{-(n-1)\xi}(\Delta\xi - (n-1))\\ &\geq 0,\end{aligned}$$

all understood in the sense of distribution. q.e.d.

Let $p_t(x,y)$ be the heat kernel on $\widetilde{M}$. For any function f on $\widetilde{M}$, we define

$$P_t f(x) = \int_{\widetilde{M}} p_t(x,y) f(y)\, dy.$$

We have $P_t(g\cdot f) = g\cdot P_t f$ for any $g \in G$.

We now proceed to prove Theorem 5. We consider the following vector field on $\widetilde{M}\times\widehat{M}$:

$$Y_t(x,\xi) = \nabla(P_t\xi)(x).$$

It is easy to see that Y_t descends to X_M, i.e., for any $g\in G$ we have $Y_t(gx, g\cdot\xi) = g_* Y_t(x,\xi)$. By Theorem 1,

$$\begin{aligned}&v\int_{X_M}\langle\nabla^w\xi, Y_t\rangle\, d\nu\\ &= \int_{X_M}\mathrm{div}^w Y_t\, d\nu\\ &= \int_M\left(\int_{\partial\widehat{M}}\mathrm{div}^w Y_t e^{-v\xi(x)} d\nu_o(\xi)\right)dx\\ &= \int_M\left(\int_{\partial\widehat{M}}\left(\mathrm{div}^w\left(Y_t e^{-v\xi(x)}\right) + v\langle\nabla^w\xi, Y_t\rangle e^{-v\xi(x)}\right)d\nu_o(\xi)\right)dx\\ &= \int_M\left(\int_{\partial\widehat{M}}\mathrm{div}^w\left(Y_t e^{-v\xi(x)}\right)d\nu_o(\xi)\right)dx + v\int_{X_M}\langle\nabla^w\xi, Y_t\rangle\, d\nu,\end{aligned}$$

whence

$$\int_M\left(\int_{\partial\widehat{M}}\mathrm{div}^w\left(Y_t e^{-v\xi(x)}\right)d\nu_o(\xi)\right)dx = 0.$$

We now cover M by finitely many open sets $\{U_i : 1\leq i\leq k\}$ such that each U_i is so small that $\pi^{-1}(U_i)$ is the disjoint union of open sets each diffeomorphic to U_i via π. Let $\{\chi_i\}$ be a partition of unity subordinating

to $\{U_i\}$. For each U_i, let $\widetilde{U}_i$ be one of the components of $\pi^{-1}(U_i)$ and let $\widetilde{\chi}_i$ be the lifting of χ_i to $\widetilde{U}_i$. Then

$$\begin{aligned}
0 &= \int_M \left(\int_{\partial \widehat{M}} \operatorname{div}^w \left(Y_t e^{-v\xi(x)} \right) d\nu_o(\xi) \right) dx \\
&= \sum_i \int_M \left(\int_{\partial \widehat{M}} \operatorname{div}^w \left(Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) \, d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{U_i} \left(\int_{\partial \widehat{M}} \operatorname{div}^w \left(Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) \, d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{\widetilde{U}_i} \left(\int_{\partial \widehat{M}} \operatorname{div}^w \left(Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) \, d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} \operatorname{div}^w \left(Y_t e^{-v\xi(x)} \right) \widetilde{\chi}_i dx \right) d\nu_o(\xi) \\
&= -\sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} \langle Y_t, \nabla \widetilde{\chi}_i \rangle \, e^{-v\xi(x)} dx \right) d\nu_o(\xi).
\end{aligned}$$

Letting $t \to 0$ yields

$$\sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} \langle \nabla \xi, \nabla \widetilde{\chi}_i \rangle \, e^{-v\xi(x)} dx \right) d\nu_o(\xi) = 0.$$

Integrating by parts again, we obtain

$$\begin{aligned}
&\sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} e^{-v\xi(x)} \Delta \widetilde{\chi}_i dx \right) d\nu_o(\xi) \\
&= -\sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} \left\langle \nabla \left(e^{-v\xi(x)} \right), \nabla \widetilde{\chi}_i \right\rangle dx \right) d\nu_o(\xi) \\
&= v \sum_i \int_{\partial \widehat{M}} \left(\int_{\widetilde{U}_i} \langle \nabla \xi, \nabla \widetilde{\chi}_i \rangle \, e^{-v\xi(x)} dx \right) d\nu_o(\xi).
\end{aligned}$$

Therefore

$$\int_{\partial \widehat{M}} \sum_i \left(\int_{\widetilde{U}_i} e^{-v\xi(x)} \Delta \widetilde{\chi}_i dx \right) d\nu_o(\xi) = 0.$$

We now assume $v = n-1$. By Proposition 4, $\Delta e^{-v\xi(x)} \geq 0$ in the sense of distribution for all $\xi \in \partial \widehat{M}$ and hence $\int_{\widetilde{U}_i} e^{-v\xi(x)} \Delta \widetilde{\chi}_i dx \geq 0$ for all i. Therefore we conclude for ν_o-a.e. $\xi \in \partial \widehat{M}$,

$$\int_{\widetilde{U}_i} e^{-v\xi(x)} \Delta \widetilde{\chi}_i dx = 0$$

for all i. In this discussion we can replace $\widetilde{U}_i$ by $g\widetilde{U}_i$ and $\widetilde{\chi}_i$ by $g\cdot\widetilde{\chi}_i$ for any $g\in G$. Since G is countable we conclude for ν_o-a.e. $\xi\in\widehat{M}$

$$\int_{g\widetilde{U}_i} e^{-v\xi(x)}\Delta\left(g\cdot\widetilde{\chi}_i\right)dx=0$$

for all i and $g\in G$.

We claim that $\Delta e^{-v\xi(x)}=0$ in the sense of distribution. Indeed, denote the distribution $\Delta e^{-v\xi(x)}$ simply by T, i.e.,

$$T\left(f\right)=\int_{\widetilde{M}} e^{-v\xi(x)}\Delta f\left(x\right)dx$$

for $f\in C_c^\infty\left(\widetilde{M}\right)$. We know that $T\left(f\right)\geq 0$ if $f\geq 0$. We observe that $\{g\cdot\widetilde{\chi}_i:1\leq i\leq k,g\in G\}$ is a partition of unity on $\widetilde{M}$ subordinating to the open cover$\left\{g\widetilde{U}_i:1\leq i\leq k,g\in G\right\}$. Hence for any $f\in C_c^\infty\left(\widetilde{M}\right)$ with $f\geq 0$ we have

$$0\leq f\leq C\sum_{g\widetilde{U}_i\cap\mathrm{spt}\,f\neq\emptyset} g\cdot\widetilde{\chi}_i,$$

with $C=\sup f$. Notice that the right-hand side is a finite sum as the support $\mathrm{spt}f$ is compact. Then

$$\begin{aligned}0\leq T\left(f\right)&\leq T\left(C\sum_{g\widetilde{U}_i\cap\mathrm{spt}\,f} g\cdot\widetilde{\chi}_i\right)\\&=C\sum_{g\widetilde{U}_i\cap\mathrm{spt}\,f\neq\emptyset} T\left(g\cdot\widetilde{\chi}_i\right)\\&=0.\end{aligned}$$

Hence $T\left(f\right)=0$, i.e., $\Delta e^{-v\xi(x)}=0$ in the sense of distribution. By elliptic regularity, $\phi=e^{-v\xi(x)}$ is then a smooth harmonic function and obviously $|\nabla\log\phi|=n-1$. The rigidity now follows from the following result.

Theorem 6. *Let N^n be a complete, simply connected Riemannian manifold such that*

1) *$Ric\geq-(n-1)$;*
2) *the sectional curvature is bounded.*

If there is a positive harmonic function ϕ on N such that $|\nabla\log\phi|=n-1$, then N is isometric to the hyperbolic space $\mathbb{H}^n$.

Remark 1. Without assuming bounded sectional curvature, the second author [**W**] proved that N is isometric to the hyperbolic space $\mathbb{H}^n$ provided that there are two such special harmonic functions. We thank Ovidiu Munteanu for pointing out that one such special harmonic function is enough if the sectional curvature is bounded.

Proof. The first step is to show that ϕ satisfies an over-determined system which then leads to the splitting of N as a warped product. This is standard and we outline the argument. Let $f = \log \phi$. We have $\Delta f = -|\nabla f|^2 = -(n-1)^2$. Since

$$D^2 f(\nabla f, \nabla f) = \frac{1}{2}\left\langle \nabla f, \nabla |\nabla f|^2 \right\rangle = 0,$$

we have by Cauchy-Schwarz

$$\left|D^2 f\right|^2 \geq \frac{(\Delta f)^2}{n-1} = (n-1)^3 \tag{3.1}$$

with equality if and only if

$$D^2 f = -(n-1)\left[g - \frac{1}{(n-1)^2} df \otimes df\right].$$

On the other hand, by the Bochner formula, we have

$$\begin{aligned} 0 &= \frac{1}{2}\Delta |\nabla f|^2 \\ &= \left|D^2 f\right|^2 + \langle \nabla f, \nabla \Delta f\rangle + Ric(\nabla f, \nabla f) \\ &\geq \left|D^2 f\right|^2 - (n-1)|\nabla f|^2 \\ &= \left|D^2 f\right|^2 - (n-1)^3, \end{aligned}$$

i.e., $\left|D^2 f\right|^2 \leq (n-1)^3$. This shows that (3.1) is in fact an equality. Therefore we have $D^2 f = -(n-1)\left[g - \frac{1}{(n-1)^2} df \otimes df\right]$. From this one can show that $N = \mathbb{R} \times \Sigma^{n-1}$ with the metric $g = dt^2 + e^{2t} h$, where h is a Riemannian metric on Σ. For more detail, see [**LW3**].

For any $p \in \Sigma$ let $\{e_i\}$ be an orthogonal basis on $(T_p\Sigma, h)$. By a simple calculation using the Gauss equation, the curvature of N is given by

$$R\left(e^{-t}e_i, e^{-t}e_j, e^{-t}e_i, e^{-t}e_j\right) = e^{-4t} R^h(e_i, e_j, e_i, e_j) - \left(\delta_{ii}\delta_{jj} - \delta_{ij}^2\right),$$

where R^h is the curvature tensor of (Σ, h). Since N has bounded sectional curvature, the left-hand side is bounded in t. Therefore $R^h = 0$, i.e., Σ is flat. Since Σ is also simply connected as N is simply connected, it is isometric to $\mathbb{R}^{n-1}$. It follows that N is the hyperbolic space. q.e.d.

4. The Kähler and quaternionic Kähler cases

In this Section, we first discuss the Kähler case.

Theorem 7. *Let M be a compact Kähler manifold with $\dim_{\mathbb{C}} M = m$ and $\pi : \widetilde{M} \to M$ the universal covering. If the bisectional curvature $K_{\mathbb{C}} \geq -2$, then the volume entropy v satisfies $v \leq 2m$. Moreover, equality holds if and only if M is complex hyperbolic (normalized to have constant holomorphic sectional curvature -4).*

The inequality follows from the comparison theorem in [**LW2**]. Indeed, under the curvature assumption $K_{\mathbb{C}} \geq -2$, Li and J. Wang [**LW2**] proved

$$\mathrm{vol}B_{\widetilde{M}}(x,r) \leq V_{\mathbb{CH}^m}(r) = \tau_{2m-1}\int_0^r \sinh(2t)\sinh^{2(m-1)}(t)\,dt,$$

where τ_{2m-1} is the volume of the unit sphere in $\mathbb{R}^{2m}$. It follows that $v \leq 2m$. Another consequence of the comparison theorem is that for $\xi \in \partial\widehat{M}$,

$$\Delta\xi \leq 2m$$

in the distribution sense. It follows as in the Riemannian case that $\Delta e^{-2m\xi} \geq 0$ in the distribution sense.

We now assume that $v = 2m$. By the argument in Section 3, we conclude that ξ is smooth and

$$\Delta\xi = 2m, |\nabla\xi| = 1$$

for ν_o-a.e. $\xi \in \partial\widehat{M}$. Take such a function ξ. We choose a local unitary frame $\left\{X_i, \overline{X}_i\right\}$.

Lemma 3. *We have*

$$\xi_{i\bar{j}} = \delta_{ij}, \xi_{ij} = -2\xi_i\xi_j.$$

Proof. We can assume that $X_1 = \left(\nabla\xi - \sqrt{-1}J\nabla\xi\right)/\sqrt{2}$ without loss of generality. Therefore

$$\xi_1 = \frac{1}{\sqrt{2}}, \xi_i = 0 \text{ for } i \geq 2.$$

Suppose ξ is given as in (2.1). Let $p \in \widetilde{M}$, and we use the construction preceding Lemma 1. By the second part of that Lemma, we see that for any $s > 0$ the function $u_s(x) = \xi(p) + d(x, \gamma(s)) - s$ is a support function for ξ from above at p. Moreover u_s is clearly smooth at p. Therefore at p we have

$$D^2\xi \leq D^2 u_s.$$

By the comparison theorem in [**LW2**], we have

$$\xi_{1\bar{1}} \leq (u_s)_{1\bar{1}} \leq \frac{\cosh 2s}{\sinh 2s},$$
$$\xi_{i\bar{i}} \leq (u_s)_{i\bar{i}} \leq \frac{\cosh s}{\sinh s} \text{ for } i \geq 2.$$

Taking limit as $s \to \infty$ yields $\xi_{i\bar{i}} \leq 1$. On the other hand, we have $\sum_{i=1}^m \xi_{i\bar{i}} = \frac{1}{2}\Delta\xi = m$. Therefore we must have

$$\xi_{i\bar{i}} = 1. \tag{4.1}$$

By the Bochner formula we have

$$0 = \frac{1}{2}\Delta\left|\nabla\xi\right|^2 = \left|D^2\xi\right|^2 + \langle\nabla\xi, \nabla\Delta\xi\rangle + Ric\left(\nabla\xi, \nabla\xi\right)$$
$$\geq \left|D^2\xi\right|^2 - 2\left(m+1\right).$$

Therefore

$$\left|\xi_{i\bar{j}}\right|^2 + \left|\xi_{ij}\right|^2 \leq m+1. \tag{4.2}$$

We have

$$0 \leq \left|\xi_{ij} + 2\xi_i\xi_j\right|^2 = \left|\xi_{ij}\right|^2 + 2\xi_{ij}\xi_{\bar{i}}\xi_{\bar{j}} + 2\xi_{\bar{i}\bar{j}}\xi_i\xi_j + 1.$$

By differentiating $\xi_j\xi_{\bar{j}} = \frac{1}{2}$, we obtain $\xi_{ij}\xi_{\bar{j}} + \xi_j\xi_{i\bar{j}} = 0, \xi_{\bar{i}j}\xi_{\bar{j}} + \xi_j\xi_{\bar{i}\bar{j}} = 0$. Hence from the previous inequality we obtain

$$\left|\xi_{ij}\right|^2 \geq 4\xi_{i\bar{j}}\xi_{\bar{i}}\xi_j - 1 = 2\xi_{1\bar{1}} - 1. \tag{4.3}$$

On the other hand,

$$\left|\xi_{i\bar{j}}\right|^2 \geq \frac{1}{m}\left(\frac{1}{2}\Delta\xi\right)^2 = m.$$

Combining this inequality with (4.2) and (4.3) yields

$$\xi_{1\bar{1}} \leq 1.$$

However, we already proved that equality holds (4.1). By inspecting the argument we conclude that ξ satisfies the following over-determined system:

$$\xi_{i\bar{j}} = \delta_{ij}, \xi_{ij} = -2\xi_i\xi_j.$$

q.e.d.

With such a function, it is proved by Li and J. Wang [**LW1**] that $\widetilde{M}$ is isometric to $\mathbb{R}\times N^{2m-1}$ with the metric

$$g = dt^2 + e^{-4t}\theta_0^2 + e^{-2t}\sum_{i=1}^{2(m-1)}\theta_i^2,$$

where $\left\{\theta_0, \theta_1, \cdots, \theta_{2(m-1)}\right\}$ is an orthonormal frame for T^*N. Moreover, since our $\widetilde{M}$ is simply connected and has bounded curvature, N is isometric to the Heisenberg group by their theorem. Therefore $\widetilde{M}$ is isometric to the complex hyperbolic space $\mathbb{CH}^m$.

Theorem 4 for quaternionic Kähler manifolds is proved in the same way, using the work of Kong, Li, and Zhou [**KLZ**] in which they proved a Laplacian comparison theorem for quaternionic Kähler manifolds.

We close with some remarks. An obvious question is whether Theorem 7 for Kähler manifolds remains true if the curvature condition is relaxed to $Ric \geq -2\left(m+1\right)$. This seems a very subtle question. It is quite unlikely that the comparison theorem for Kähler manifolds could

still hold in this case. On the other hand, it is conceivable that Theorem 7 will remain valid due to some global reason. This hope is partly based on the recent work of Munteanu [**Mu**] in which a sharp estimate for the Kaimanovich entropy is derived under the condition $Ric \geq -2(m+1)$.

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