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MEAN CURVATURE FLOW OF PINCHED SUBMANIFOLDS TO SPHERES

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Abstract

We consider compact submanifolds of dimension $n \geq 2$ in $\mathbb{R}^{n+k}$, with nonzero mean curvature vector everywhere, and such that the full norm of the second fundamental form is bounded by a fixed multiple (depending on n) of the length of the mean curvature vector at every point. We prove that the mean curvature flow deforms such a submanifold to a point in finite time, and that the solution is asymptotic to a shrinking sphere in some $(n+1)$-dimensional affine subspace of $\mathbb{R}^{n+k}$.

1. Introduction

The evolution of hypersurfaces by their mean curvature has been studied by many authors since the appearance of Gerhard Huisken's seminal paper [**Hu1**]. More recently, mean curvature flow of higher codimension submanifolds has also received attention. In this paper we prove a result analogous to that of [**Hu1**] for submanifolds of any codimension.

Let $F_0 : \Sigma^n \to \mathbb{R}^{n+k}$ be a smooth immersion of a compact manifold Σ. The mean curvature flow with initial condition F_0 is a smooth family of immersions $F : \Sigma \times [0, T) \to \mathbb{R}^{n+k}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} F(p,t) = H(p,t), \quad p \in \Sigma,\ t \geq 0, \\ F(\cdot, 0) = F_0, \end{cases} \tag{1}$$

where $H(p,t)$ is the mean curvature vector of the submanifold $\Sigma_t = F(\Sigma, t)$ at p. We use the abbreviation "MCF" for the system (1), and denote the second fundamental form by h. See section 2 for further details of our notation and conventions.

High codimension MCF is the steepest descent flow for the area functional, and so arises naturally in several contexts. For example, singular sets in harmonic map heat flow move by generalized mean curvature flow

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[**LT**]. Generalized solutions can be defined using geometric measure theory, minimal barriers or level sets [**B1**], [**AS1**], [**AS2**], [**BN**]. Huisken's monotonicity formula [**Hu3**] applies in any codimension [**Ha4**], relating singularity formation to self-similar solutions of the flow. However rather little is known about such self-similar solutions (see [**S3**] for some recent results in this direction).

Much of the previous work on high codimension mean curvature flow has used assumptions on the Gauss image, focussing on graphical submanifolds [**CLT**], [**LL2**], [**W2**], [**W4**] or symplectic or Lagrangian submanifolds [**SW**], [**CL2**], [**W1**], [**S2**], [**N**], or making use of the fact that convex subsets of the Grassmannian are preserved [**TW**], [**W3**], [**W5**].

In this paper we work with conditions on the extrinsic curvature (second fundamental form), which have the advantage of being invariant under rigid motions. Several difficulties arise in carrying out this program: First, in high codimension the second fundamental form has a much more complicated structure than in the hypersurface case. In particular, under MCF the second fundamental form evolves according to a reaction-diffusion system in which the reaction terms are rather complicated, whereas in the hypersurface case they are quite easily understood. Thus it can be extremely difficult to determine whether the reaction terms are favourable for preserving a given curvature condition. Second, there do not seem to be any useful invariant conditions on the extrinsic curvature which define convex subsets of the space of second fundamental forms. This lack of convexity is forced by the necessity for invariance under rotation of the normal bundle. This means that the vector bundle maximum principle formulated by Hamilton in [**Ha2**], which states that the reaction-diffusion system will preserve an invariant convex set if the reaction terms are favourable, cannot be applied. The latter maximum principle has been extremely effective in the Ricci flow in high dimensions [**BW**], [**BS**], [**B2**] where the algebraic complexity of the curvature tensor has presented similar difficulties. For arbitrary reaction-diffusion systems, the convexity condition is necessary for a maximum principle to apply. However, in our setting the Codazzi identity adds a constraint on the first derivatives of solutions which allows some non-convex sets to be preserved. A similar situation arose Huisken's work on evolving hypersurfaces in spheres [**Hu2**], where a non-convex condition was preserved. Our result is as follows:

Theorem 1. *Let $n \geq 2$, and suppose $\Sigma_0 = F_0(\Sigma^n)$ is a compact submanifold smoothly immersed in $\mathbb{R}^{n+k}$. If Σ_0 has $H \neq 0$ everywhere and satisfies $|h|^2 \leq c|H|^2$, where*

$$c \leq \begin{cases} \frac{4}{3n}, & \text{if } 2 \leq n \leq 4 \\ \frac{1}{n-1}, & \text{if } n \geq 4, \end{cases}$$

then MCF has a unique smooth solution $F: \Sigma \times [0,T) \to \mathbb{R}^{n+k}$ on a finite maximal time interval, and $F_t(.)$ converges uniformly to a point $q \in \mathbb{R}^{n+k}$ as $t \to T$. The rescaled maps $\tilde{F}_t = \frac{F_t - q}{\sqrt{2n(T-t)}}$ converge in C^∞ as $t \to T$ to a limiting embedding $\tilde{F}_T$ with image equal to a regular unit n-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+k}$.

In dimensions greater than or equal to four the pinching ratio in Theorem 1 is optimal, as the following example shows: Consider the submanifolds $\mathbb{S}^{n-1}(\epsilon) \times \mathbb{S}^1(1) \subset \mathbb{R}^n \times \mathbb{R}^2$, where ϵ is a small positive number. The second fundamental form is given by

$$h\big|_{(\varepsilon x, y)} = \begin{pmatrix} \frac{1}{\epsilon} & & & \\ & \ddots & & \\ & & \frac{1}{\epsilon} & \\ & & & 0 \end{pmatrix} (x,0) + \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} (0,y)$$

and so they satisfy $|h|^2 = \frac{1}{n-1}\left(1 + \frac{\epsilon^2(n-2)}{(n-1)^2+\epsilon^2}\right)|H|^2$. These submanifolds collapse to $\mathbb{S}^1$ under MCF and do not contract to points. In dimensions two and three the size of gradient and reaction terms of equation (29) prevent the optimal result from being achieved. This is similar to the situation in [**Hu2**], where in dimension two the difficulty in controlling the gradient terms prevents the optimal result from being obtained. We remark that contrary to the situation in [**Hu2**], one cannot expect to obtain such a result with $c = 1/(n-1) = 1$ in the case $n = 2$ in arbitrary codimension, since the Veronese surface provides a counterexample: This is a surface in $\mathbb{R}^5$ which satisfies $|h|^2 = \frac{5}{6}|H|^2$, but which contracts without changing shape under mean curvature flow. However we are not aware of any such counterexamples in the case $n = 3$ (there are none among minimal submanifolds of spheres [**CO**]).

Curvature pinching conditions similar to those in our theorem have appeared previously in a number of results for special classes of submanifolds: In [**O1**] Okumura shows that if a submanifold of Euclidean space with parallel mean curvature vector and flat normal bundle satisfies $|h|^2 < |H|^2/(n-1)$, then the submanifold is a sphere. The equivalent result for hypersurfaces of the sphere with $|h|^2 < \frac{1}{n-1}|H|^2 + 2$ (where the flat normal bundle condition is vacuous) was proved by Okumura in [**O2**]. Chen and Okumura [**CO**] later removed the assumption of flat normal bundle and so proved that if a submanifold of Euclidean space with parallel mean curvature vector satisfies $|h|^2 < 1/(n-1)|H|^2$, then the submanifold is a sphere (or, in the case $n = 2$, a minimal surface with positive intrinsic curvature in a sphere, such as the Veronese surface). A related series of results began with work of Chern, do Carmo and Kobayashi [**CdCK**]. They classified minimal submanifolds of the sphere that satisfy $|h|^2 \le n/(2-1/k)$, where k is the codimension. Two decades later do Carmo and Alencar [**AdC**] classified hypersurfaces of the sphere with constant mean curvature satisfying a certain pinching condition, and shortly afterwards Santos [**Sa**] extended the classification, under a suitable pinching condition, to submanifolds of the sphere with parallel mean curvature vector.

Our result is closely related to some of the above: In particular the results on minimal submanifolds of spheres relate to ours, since such submanifolds contract without change of shape under the mean curvature flow. The results for parallel mean curvature vector do not relate as directly, since such submanifolds do not behave simply under the mean curvature flow. Our theorem implies that the entire class of n-submanifolds satisfying the curvature pinching condition retracts onto the totally umbillic n-spheres, and thus onto the Grassmannian $G_{n+1,n+k}$ of $(n+1)$-dimensional subspaces of $\mathbb{R}^{n+k}$.

The broad structure of the proof of Theorem 1 is similar to that in [**Hu1**], which in turn, was inspired by Hamilton's seminal paper on Ricci flow [**Ha1**]. We first introduce our notation and some facts from submanifold geometry of high codimension. A key aspect of this is the machinery of connections on vector bundles which we employ extensively in deriving the evolution equations for geometric quantities. In particular we introduce connections on tangent and normal bundles defined over both space and time, which prove very useful in deriving evolution equations and allowing simple commutation of time and space derivatives. This connection also provides a natural interpretation of the 'Uhlenbeck trick' introduced in [**Ha2**] to take into account the change in length of spatial tangent vectors under the flow. The key step in our argument is to prove that curvature pinching is preserved. This plays a role similar to Huisken's estimate $h_{ij} - \varepsilon H g_{ij} \geq 0$ from [**Hu1**]. The Codazzi identity is used to derive an inequality on the derivatives of the second fundamental form analogous to that in [**Hu1**], in order to control the gradient terms which arise in the evolution equation. An inequality from [**LL1**] also appears in the argument to control the reaction terms, which in this setting are much more complicated than in the hypersurface case. A stronger pinching estimate is deduced using a Stampacchia iteration argument following the model of [**Hu1**], although again the curvature terms are considerably more complicated here and the argument to control them is quite involved. The subsequent analysis to prove convergence is again parallel to that in [**Hu1**], with only minor differences introduced by the high codimension setting.

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2. Notation and preliminary results

To a large extent our notations are compatible with those of [**Hu1**]. In order to work with the normal bundle we introduce some new vector bundle machinery suited to the analysis of evolution equations. The machinery we develop is useful and new even in the codimension one case, as we work with the tangent and normal bundles as vector bundles over the space-time domain, and introduce natural metrics and connections on these. In particular, the connection we introduce on the 'spatial' tangent bundle (as a bundle over spacetime) contains more information than the Levi-Civita connections of the metrics at each time, and this proves particularly useful in computing evolution equations for geometric quantities. We note that space-time connections such as these have been employed previously (see for example [**S1**, Section 3]). Here we give a detailed discussion of these connections, and derive their curvature tensors and structure equations. Although we only require this machinery in the setting of Euclidean background spaces for the results of this paper, we take the opportunity to present the machinery in the more general context of Riemannian backgrounds.

2.1. Vector bundles. We denote the space of smooth sections of a vector bundle E over a manifold N by $\Gamma(E)$, the dual bundle of E by E^*, and the tensor product of bundles E_1 and E_2 by $E_1 \otimes E_2$. Given a connection ∇ on E, we define its curvature $R_\nabla \in \Gamma(T^*N \otimes T^*N \otimes E^* \otimes E)$ by

$$R_\nabla(X,Y)\xi = \nabla_Y\left(\nabla_X \xi\right) - \nabla_X\left(\nabla_Y \xi\right) - \nabla_{[Y,X]}\xi.$$

We note that if ∇^i is a connection on E_i for $i = 1,2$, then the connection ∇ induced on $E_1^* \otimes E_2$ (E_2-valued tensors acting on E_1) is defined by

$$(\nabla_X S)(\xi) = \nabla^2_X\left(S(\xi)\right) - S\left(\nabla^1_X \xi\right), \tag{2}$$

and the curvature of this connection is given by

$$(R(X,Y)S)(\xi) = R_{\nabla^2}(X,Y)(S(\xi)) - S\left(R_{\nabla^1}(X,Y)\xi\right). \tag{3}$$

If M and N are smooth manifolds, E a vector bundle over N and f a smooth map from M to N, we denote by f^*E the pullback bundle of E over M. If $\xi \in \Gamma(E)$, then we denote by ξ_f the section of f^*E defined by $\xi_f(x) = \xi(f(x))$ for each $x \in M$ (called the *restriction* of ξ to f). If ∇ is a connection on E, we denote by ${}^f\nabla$ the *pullback connection*, which is the unique connection which satisfies ${}^f\nabla_u\left(X_f\right) = \nabla_{f_*(u)}X$ for any $u \in TM$ and $X \in \Gamma(E)$. We note some elementary properties of this:

Proposition 1. *If g is a metric on E and ∇ is a connection on E compatible with g, then ${}^f\nabla$ is compatible with the restriction metric g_f.*

Proposition 2. *The curvature of the pull-back connection is the pull-back of the curvature of the original connection. Here* $R_\nabla \in \Gamma(T^*N \otimes T^*N \otimes E^* \otimes E)$*, so that*

$$\begin{aligned} f^*(R_\nabla) &\in \Gamma(T^*M \otimes T^*M \otimes f^*(E^* \otimes E)) \\ &= \Gamma\left(T^*M \otimes T^*M \otimes (f^*E)^* \otimes f^*E\right). \end{aligned}$$

In the case of pulling back a tangent bundle, there is another important property:

Proposition 3. *If* ∇ *is a symmetric connection on* TN*, then the pull-back connection* ${}^f\nabla$ *on* f^*TN *is symmetric, in the sense that for any* $U, V \in \Gamma(TM)$*,*

$${}^f\nabla_U(f_*V) - {}^f\nabla_V(f_*U) = f_*([U,V]).$$

2.2. Subbundles. A *subbundle* K of a vector bundle E over M is a vector bundle K over M with an injective vector bundle homomorphism $\iota_K : K \to E$ covering the identity map on M. We consider complementary sub-bundles K and L, so that $E_x = \iota_K(K_x) \oplus \iota_L(L_x)$, and denote by π_K and π_L the corresponding projections onto K and L (so $\pi_K \circ \iota_K = \mathrm{Id}_K$, $\pi_L \circ \iota_L = \mathrm{Id}_L$, $\pi_K \circ \iota_L = 0$, $\pi_L \circ \iota_K = 0$, and $\iota_K \circ \pi_K + \iota_L \circ \pi_L = \mathrm{Id}_E$). If ∇ is a connection on E, we define a connection $\overset{K}{\nabla}$ on K and a tensor $h^K \in \Gamma(T^*M \otimes K^* \otimes L)$ (the second fundamental form of K) by

$$\overset{K}{\nabla}_u \xi = \pi_K\left(\nabla_u\left(\iota_K \xi\right)\right); \qquad h^K(u,\xi) = \pi_L\left(\nabla_u\left(\iota_K\xi\right)\right); \tag{4}$$

so that

$$\nabla_u\left(\iota_K\xi\right) = \iota_K\left(\overset{K}{\nabla}_u\xi\right) + \iota_L\left(h^K(u,\xi)\right) \tag{5}$$

for any $u \in TM$ and $\xi \in \Gamma(K) \subset \Gamma(E)$. The curvature R^K of $\overset{K}{\nabla}$ is related to the second fundamental form h^K and the curvature of ∇ via the Gauss equation:

$$R^K(u,v)\xi = \pi_K\left(R_\nabla(u,v)(\iota_K\xi)\right) + h^L(u, h^K(v,\xi)) - h^L(v, h^K(u,\xi)) \tag{6}$$

for all $u, v \in T_xM$ and $\xi \in \Gamma(K)$. The other important identity relating the second fundamental form to the curvature is the Codazzi identity, which states:

$$\begin{aligned} &\pi_L(R_\nabla(v,u)\iota_K\xi) \\ &= \overset{L}{\nabla}_u\big(h^K(v,\xi)\big) - \overset{L}{\nabla}_v\big(h^K(u,\xi)\big) - h^K(u, \overset{K}{\nabla}_v\xi) + h^K(v, \overset{K}{\nabla}_u\xi) - h^K([u,v],\xi). \end{aligned} \tag{7}$$

If we are supplied with an arbitrary symmetric connection on TM, then we can make sense of the covariant derivative ∇h^K of the second fundamental form h^K, and the Codazzi identity becomes

$$\nabla_u h^K(v,\xi) - \nabla_v h^K(u,\xi) = \pi_L\left(R_\nabla(v,u)(\iota_K\xi)\right). \tag{8}$$

An important case is where K and L are orthogonal with respect to a metric g on E compatible with ∇. Then $\overset{K}{\nabla}$ is compatible with the induced metric g^K, and h^K and h^L are related by the Weingarten relation:

$$g^L\left(h^K(u,\xi),\eta\right)+g^K\left(\xi,h^L(u,\eta)\right)=0 \tag{9}$$

for all $\xi\in\Gamma(K)$ and $\eta\in\Gamma(L)$.

2.3. The tangent and normal bundles of a time-dependent immersion. The machinery introduced above is familiar in the following setting: If $F:\ M^n\to N^{n+k}$ is an immersion, then $F_*:\ TM\to F^*TN$ defines the tangent sub-bundle of F^*TN, and its orthogonal complement is the normal bundle $NM=F_*(TM)^\perp$. If $\bar g$ is a metric on TN with Levi-Civita connection $\bar\nabla$, then the metric g^{TM} is the induced metric on M, and ∇^{TM} is its Levi-Civita connection, while $h^{TM}\in\Gamma(T^*M\otimes T^*M\otimes NM)$ is the second fundamental form, and h^{NM} is minus the Weingarten map. The Gauss identities (6) for TM are the usual Gauss equations for a submanifold, while those for NM are usually called the Ricci identities. The Codazzi identities for the two are equivalent to each other.

In this paper we want to apply the same machinery in a setting adapted to time-dependent immersions. If I is a real interval, then the tangent space $T(\Sigma\times I)$ splits into a direct product $\mathcal{H}\oplus\mathbb{R}\partial_t$, where $\mathcal{H}=\{u\in T(\Sigma\times I):\ dt(u)=0\}$ is the 'spatial' tangent bundle.

We consider a smooth map $F:\ \Sigma^n\times I\to N^{n+k}$ which is a time-dependent immersion, i.e. for each $t\in I$, $F(.,t):\ \Sigma\to N$ is an immersion. Then F^*TN is a vector bundle over $\Sigma\times I$, which we can equip with the restriction metric $\bar g_F$ and pullback connection ${}^F\bar\nabla$ coming from a Riemannian metric $\bar g$ on N and its Levi-Civita connection $\bar\nabla$. The map $F_*:\ \mathcal{H}\to F^*TN$ defines a sub-bundle of F^*TN of rank n. The orthogonal complement of $F_*(\mathcal{H})$ in F^*TN is a vector bundle of rank k which we denote by $\mathcal{N}$ and refer to as the (spacetime) normal bundle. We denote by π the orthogonal projection from F^*TN onto $\mathcal{H}$, and by $\overset{\perp}{\pi}$ the orthogonal projection onto $\mathcal{N}$, and by ι the inclusion of $\mathcal{N}$ in F^*TN. The restrictions of these bundles to each time t are the usual tangent and normal bundles of the immersion F_t.

The construction of the previous section gives a metric $g(u,v)=\bar g(F_*u,F_*v)$ and a connection $\nabla:=\pi\circ{}^F\bar\nabla\circ F_*$ on the bundle $\mathcal{H}$ over $\Sigma\times I$, which agrees with the Levi-Civita connection of g for each fixed t. We denote by $\overset{\perp}{g}$ the metric induced on $\mathcal{N}$, given by $\overset{\perp}{g}(\xi,\eta)=\bar g(\iota\xi,\iota\eta)$. The construction also gives a connection $\overset{\perp}{\nabla}:=\overset{\perp}{\pi}\circ{}^F\bar\nabla\circ\iota$ on $\mathcal{N}$. We denote by $h\in\Gamma\left(\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{N}\right)$ the restriction of $h^{\mathcal{H}}=\overset{\perp}{\pi}\circ{}^F\nabla\circ F_*$ to $\mathcal{H}$ in the first argument. Proposition 3 implies that h is a symmetric bilinear form on $\mathcal{H}$ with values in $\mathcal{N}$. The remaining components of $h^{\mathcal{H}}$

are given by

$$h^{\mathcal{H}}(\partial_t, v) = \overset{\perp}{\pi}\left({}^F\nabla_t F_* v\right) = \overset{\perp}{\pi}\left({}^F\nabla_v F_*\partial_t + F_*\left([\partial_t, v]\right)\right) = \overset{\perp}{\nabla}_v\left(\overset{\perp}{\pi}F_*\partial_t\right) + h(v, \pi F_*\partial_t) \tag{10}$$

where we used Proposition 3. Henceforward we restrict to normal variations (with $\pi F_*\partial_t = 0$), since this is the situation for the mean curvature flow. We also define $\mathcal{W} \in \Gamma\left(\mathcal{H}^* \otimes \mathcal{N} \otimes \mathcal{H}\right)$ by $\mathcal{W}(u,\xi) = -h^{\mathcal{N}}(u,\xi) = -\pi\left({}^F\nabla_u \iota\xi\right)$ for any $u \in \Gamma(\mathcal{H})$ and $\xi \in \Gamma(\mathcal{N})$ (we refer to this as the Weingarten map). The Weingarten relation (9) gives two identities:

$$\overset{\perp}{g}\left(h(u,v),\xi\right) = g\left(v, \mathcal{W}(u,\xi)\right); \tag{11}$$

$$g\left(h^{\mathcal{N}}(\partial_t,\xi), v\right) = -\overset{\perp}{g}\left(\overset{\perp}{\nabla}_v \overset{\perp}{\pi}F_*\partial_t, \xi\right) \tag{12}$$

where the latter identity used (10). The Gauss and Codazzi identities for $\mathcal{H}$ and $\mathcal{N}$ give the following identities for the second fundamental form: First, if u and v are in $\mathcal{H}$, then the Gauss equation (6) for $\mathcal{H}$ amounts to the usual Gauss equation at the fixed time, i.e.

$$R(u,v)w = \mathcal{W}(v, h(u,w)) - \mathcal{W}(u, h(v,w)) + \pi\left(\bar{R}(F_*u, F_*v)F_*w\right) \tag{13a}$$

$$R(u,v,w,z) = \overset{\perp}{g}\left(h(u,w), h(v,z)\right) - \overset{\perp}{g}\left(h(v,w), h(u,z)\right) + F^*\bar{R}\left(u,v,w,z\right). \tag{13b}$$

If $u = \partial_t$ but $v \in \mathcal{H}$, then we find:

$$R(\partial_t, v, w, z) = \overset{\perp}{g}\left(\overset{\perp}{\nabla}_w \overset{\perp}{\pi}F_*\partial_t, h(v,z)\right) - \overset{\perp}{g}\left(\overset{\perp}{\nabla}_z \overset{\perp}{\pi}F_*\partial_t, h(v,w)\right) + F^*\bar{R}(\partial_t, v, w, z). \tag{14}$$

The Gauss equation for the curvature $\overset{\perp}{R}$ of $\mathcal{N}$ also splits into two parts: If u and v are spatial these are simply the Ricci identities for the submanifold at a fixed time:

$$\overset{\perp}{R}(u,v)\xi = h\left(v, \mathcal{W}(u,\xi)\right) - h\left(u, \mathcal{W}(v,\xi)\right) + \overset{\perp}{\pi}\left(\bar{R}(F_*u, F_*v)(\iota\xi)\right); \tag{15}$$

while if $u = \partial_t$ and $v \in \mathcal{H}$, then we have the identity

$$\overset{\perp}{R}(\partial_t, v, \xi, \eta) = \bar{R}(F_*\partial_t, F_*v, \iota\xi, \iota\eta) - \overset{\perp}{g}\left(\overset{\perp}{\nabla}_{\mathcal{W}(v,\xi)} \overset{\perp}{\pi}F_*\partial_t, \eta\right) + \overset{\perp}{g}\left(\overset{\perp}{\nabla}_{\mathcal{W}(v,\eta)} \overset{\perp}{\pi}F_*\partial_t, \xi\right). \tag{16}$$

Finally, the Codazzi identities resolve into the tangential Codazzi identities, given by

$$\nabla_u h(v,w) - \nabla_v h(u,w) = \overset{\perp}{\pi}\left(\bar{R}(F_*v, F_*u)F_*w\right) \tag{17}$$

for all $u, v, w \in \Gamma(\mathcal{H})$, and the 'timelike' part, where $u = \partial_t$ and $v, w \in \Gamma(\mathcal{H})$:

$$\stackrel{\perp}{\pi}\left(\bar{R}(F_* v, F_*\partial_t)F_* w\right) \tag{18}$$
$$= \nabla_{\partial_t} h(v,w) - \nabla_v \nabla_w \left(\stackrel{\perp}{\pi} F_* \partial_t\right) - h\left(w, \mathcal{W}(v, \stackrel{\perp}{\pi} F_* \partial_t)\right).$$

Note that here $\nabla h \in \Gamma(T^*(\Sigma \times I) \otimes \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{N})$ is defined using the connection induced from ∇ and $\stackrel{\perp}{\nabla}$, that is $\nabla_{\partial_t} h(u,v) = \stackrel{\perp}{\nabla}_{\partial_t}(h(u,v)) - h(\nabla_{\partial_t} u, v) - h(u, \nabla_{\partial_t} v)$.

We remark that by construction we have $\nabla g = 0$ and $\nabla \stackrel{\perp}{g} = 0$. In contrast to the situation in other work on evolving hypersurfaces, we have $\nabla_{\partial_t} g = 0$. That is, the connections we have constructed automatically build in the so-called 'Uhlenbeck trick' [**Ha2**, Section 2].

Proposition 4. *The tensors $F_* \in \Gamma(\mathcal{H}^* \otimes F^*TN)$, $\iota \in \Gamma(\mathcal{N}^* \otimes F^*TN)$, $\pi \in \Gamma(F^*TN \otimes \mathcal{H})$ and $\stackrel{\perp}{\pi} \in \Gamma(F^*TN \otimes \mathcal{N})$ satisfy*

$$(\nabla_U F_*)(V) = \iota h(U,V) \tag{19}$$
$$(\nabla_U \iota)(\xi) = -F_* \mathcal{W}(U, \xi) \tag{20}$$
$$(\nabla_U \pi)(X) = \mathcal{W}(U, \stackrel{\perp}{\pi} X) \tag{21}$$
$$(\nabla_U \stackrel{\perp}{\pi})(X) = -h(U, \pi X) \tag{22}$$

for all $U, V \in \Gamma(\mathcal{H})$, $\xi \in \Gamma(\mathcal{N})$ and $X \in \Gamma(F^*TN)$.

Proof. For the first identity we have (since F_* is a F^*TN-valued tensor acting on $\mathcal{H}$)

$$(\nabla_U F_*)(V) = {}^F\nabla_U (F_* V) - F_*(\nabla_U V)$$
$$= F^*(\nabla_U V) + \iota h(U,V) - F_*(\nabla_U V) = \iota h(U,V),$$

where we used the definitions of h and ∇. The second identity is similar. For the third we have:

$$\begin{aligned}(\nabla_U \pi)(X) &= \nabla_U(\pi X) - \pi\left({}^F\nabla_U X\right)\\ &= \nabla_U(\pi X) - \pi\left({}^F\nabla_U\left(F_*\pi X + \iota \stackrel{\perp}{\pi} X\right)\right)\\ &= \nabla_U(\pi X) - \nabla_U(\pi X) + \mathcal{W}(U, \stackrel{\perp}{\pi} X)\\ &= \mathcal{W}(U, \stackrel{\perp}{\pi} X).\end{aligned}$$

The fourth identity is similar to the third. q.e.d.

We illustrate the application of the above identities in the proof of Simons' identity, which amounts to the statement that the second derivatives of the second fundamental form are totally symmetric, up to corrections involving second fundamental form and the curvature of N:

Proposition 5.

$$
\begin{aligned}
&\nabla_w\nabla_z h(u,v)-\nabla_u\nabla_v h(w,z)\\
&=h(v,\mathcal{W}(u,h(w,z)))-h(z,\mathcal{W}(w,h(u,v)))-h(u,\mathcal{W}(w,h(v,z)))\\
&\quad+h(w,\mathcal{W}(u,h(v,z)))+h(z,\mathcal{W}(u,h(w,v)))-h(v,\mathcal{W}(w,h(u,z)))\\
&\quad-h(u,\pi\bar{R}(F_*v,F_*w)F_*z)-h(w,\pi\bar{R}(F_*u,F_*z)F_*v)\\
&\quad-h(z,\pi\bar{R}(F_*u,F_*w)F_*v)-h(v,\pi\bar{R}(F_*u,F_*w)F_*z)\\
&\quad+\overset{\perp}{\pi}\bar{R}(\iota h(u,v),F_*w)F_*z-\overset{\perp}{\pi}\bar{R}(\iota h(w,z),F_*u)F_*v\\
&\quad+\overset{\perp}{\pi}\bar{R}(F_*u,F_*w)\iota h(v,z)+\overset{\perp}{\pi}\bar{R}(F_*v,F_*z)\iota h(u,w)\\
&\quad+\overset{\perp}{\pi}\bar{R}(F_*v,F_*w)\iota h(u,z)+\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)\iota h(v,w)\\
&\quad+\overset{\perp}{\pi}\bar{\nabla}_{F_*u}\bar{R}(F_*v,F_*w)F_*z-\overset{\perp}{\pi}\bar{\nabla}_{F_*w}\bar{R}(F_*z,F_*u)F_*v.
\end{aligned}
$$

Proof. Since the equation is tensorial, it suffices to work with $u,v,w,z\in\Gamma(\mathcal{H})$ for which $\nabla u=0$, etc, at a given point. Computing at that point we find

$$
\begin{aligned}
\nabla_w\nabla_z h(u,v) &= \nabla_w\left(\nabla_u h(z,v)+\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)F_*v\right)\\
&= \nabla_u\nabla_w h(v,z)+(R(u,w)h)(v,z)\\
&\quad+\nabla_w\left(\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)F_*v\right)\\
&= \nabla_u\Big(\nabla_v h(w,z)+\overset{\perp}{\pi}\bar{R}(F_*v,F_*w)F_*z\Big)\\
&\quad+(R(u,w)h)(v,z)+\nabla_w\Big(\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)F_*v\Big)\\
&= \nabla_u\nabla_v h(w,z)+(R(u,w)h)(v,z)\\
&\quad+\nabla_w\Big(\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)F_*v\Big)+\nabla_u\Big(\overset{\perp}{\pi}\bar{R}(F_*v,F_*w)F_*z\Big)
\end{aligned}
$$

where we used the Codazzi identity in the first and third lines, and the definition of curvature in the second. Since h is a $\mathcal{N}$-valued tensor with arguments in $\mathcal{H}$, the second term may be computed using the identity (3) to give

$$(R(u,w)h)(v,z)=\overset{\perp}{R}(u,w)(h(v,z))-h(R(u,w)v,z)-h(v,R(u,w)z).$$

This in turn can be expanded using the Gauss identity (13a) for R and the Ricci identity (15) for $\overset{\perp}{R}$. In the third term (and similarly the fourth) we observe the following:

$$
\begin{aligned}
&\nabla_w(\overset{\perp}{\pi}\bar{R}(F_*u,F_*z)F_*v)\\
&=\nabla_w\overset{\perp}{\pi}\left(\bar{R}(F_*u,F_*z)F_*v\right)+\overset{\perp}{\pi}\left({}^F\nabla_w\left(\bar{R}(F_*u,F_*z)F_*v\right)\right).
\end{aligned}
$$

In the first term here we apply the identity (22). In the second we can expand further as follows:

$${}^F\nabla_w\left(\bar{R}(F_*u,F_*z)F_*v\right)$$

$$= \left({}^F\nabla_w \bar{R}\right)(F_*u, F_*z)F_*v + \bar{R}((\nabla_w F_*)u, F_*z)F_*v \\ + \bar{R}(F_*u, \nabla_w F_*(z))F_*v + \bar{R}(F_*u, F_*z)(\nabla_u F_*(v)).$$

In the terms involving ∇F_* we apply (19), and we also observe that ${}^F\nabla_w \bar{R} = \bar{\nabla}_{F_*w}\bar{R}$ by the definition of the connection ${}^F\nabla$. Substituting these identities gives the required result. q.e.d.

In subsequent computations we often work in a local orthonormal frame $\{e_i\}$ for the spatial tangent bundle $\mathcal{H}$, and a local orthonormal frame $\{\nu_\alpha\}$ for the normal bundle $\mathcal{N}$. We use greek indices for the normal bundle, and latin ones for the tangent bundle. When working in such orthonormal frames we sum over repeated indices whether raised or lowered. For example the mean curvature vector $H \in \Gamma(\mathcal{N})$ may be written in the various forms

$$H = \mathrm{tr}_g h = g^{ij}h_{ij} = h_i{}^i = h_{ii} = g^{ij}h_{ij}{}^\alpha \nu_\alpha = h_{ii\alpha}\nu_\alpha.$$

Similarly, we write $|h|^2 = g^{ik}g^{jl}g^{\mathcal{N}}_{\alpha\beta}h_{ij}{}^\alpha h_{kl}{}^\beta = h_{ij\alpha}h_{ij\alpha}$. The Weingarten relation (11) becomes

$$\mathcal{W}(e_i, \nu_\alpha) = h_{iq\alpha}e_q,$$

while the Gauss equation (13a) becomes

$$R_{ijkl} = h_{ik\alpha}h_{jl\alpha} - h_{jk\alpha}h_{il\alpha} + \bar{R}_{ijkl},$$

where we denote $\bar{R}_{ijkl} = \bar{R}(F_*e_i, F_*e_j, F_*e_k, F_*e_l)$. The Ricci equations (15) give

$$\overset{\perp}{R}_{ij\alpha\beta} = h_{ip\alpha}h_{jp\beta} - h_{jp\alpha}h_{ip\beta} + \bar{R}_{ij\alpha\beta},$$

where $\bar{R}_{ij\alpha\beta} = \bar{R}(F_*e_i, F_*e_j, \iota\nu_\alpha, \iota\nu_\beta)$, and the Codazzi identity (17) gives

$$\nabla_i h_{jk} - \nabla_j h_{ik} = \bar{R}_{jik\alpha}\nu_\alpha.$$

In this notation the identity from Proposition 5 takes the following form:

$$\begin{aligned}\nabla_k\nabla_l h_{ij} = {} & \nabla_i\nabla_j h_{kl} + h_{kl\alpha}h_{ip\alpha}h_{jp} - h_{ij\alpha}h_{kp\alpha}h_{lp} \\ & + h_{jl\alpha}h_{ip\alpha}h_{kp} + h_{jk\alpha}h_{ip\alpha}h_{lp} - h_{il\alpha}h_{kp\alpha}h_{jp} - h_{jl\alpha}h_{kp\alpha}h_{ip} \\ & + h_{kl\alpha}\bar{R}_{i\alpha j\beta}\nu_\beta - h_{ij\alpha}\bar{R}_{k\alpha l\beta}\nu_\beta + \bar{R}_{kjlp}h_{ip} \\ & + \bar{R}_{kilp}h_{jp} - \bar{R}_{iljp}h_{kp} - \bar{R}_{ikjp}h_{lp} \\ & + h_{jl\alpha}\bar{R}_{ik\alpha\beta}\nu_\beta + h_{ik\alpha}\bar{R}_{jl\alpha\beta}\nu_\beta + h_{il\alpha}\bar{R}_{jk\alpha\beta}\nu_\beta + h_{jk\alpha}\bar{R}_{il\alpha\beta}\nu_\beta \\ & + \bar{\nabla}_i\bar{R}_{jkl\beta}\nu_\beta - \bar{\nabla}_k\bar{R}_{lij\beta}\nu_\beta.\end{aligned}$$

Particularly useful is the equation obtained by taking a trace of the above identity over k and l:

(23)
$$\begin{aligned}\Delta h_{ij} &= \nabla_i\nabla_j H + H\cdot h_{ip}h_{pj} - h_{ij}\cdot h_{pq}h_{pq}\\ &\quad + 2h_{jq}\cdot h_{ip}h_{pq} - h_{iq}\cdot h_{qp}h_{pj} - h_{jq}\cdot h_{qp}h_{pi}\\ &\quad + H_\alpha \bar{R}_{i\alpha j\beta}\nu_\beta - h_{ij\alpha}\bar{R}_{k\alpha k\beta}\nu_\beta + \bar{R}_{kjkp}h_{pi} + \bar{R}_{kikp}h_{pj} - 2\bar{R}_{ipjq}h_{pq}\\ &\quad + 2h_{jp\alpha}\bar{R}_{ip\alpha\beta}\nu_\beta + 2h_{ip\alpha}\bar{R}_{jp\alpha\beta}\nu_\beta + \bar{\nabla}_i\bar{R}_{jkk\beta}\nu_\beta - \bar{\nabla}_k\bar{R}_{kij\beta}\nu_\beta.\end{aligned}$$

Here the dots represent inner products in $\mathcal{N}$.

We finish this section with a brief comment about short time existence of MCF. It is well known that the geometric invariance of MCF introduces degeneracies into the principal symbol, so that standard parabolic theory does not immediately apply. This may be circumvented by including a tangential term corresponding to a harmonic map heat flow (the so called DeTurck trick). In [**Ha3**] Hamilton shows how to use the latter method to achieve short time existence to mean curvature flow of arbitrary codimension. The reader is referred there for the details.

3. Preservation of curvature pinching

In this section we prove the following key pinching estimate:

Theorem 2. *If a solution $F:\ \Sigma\times[0,T)\to\mathbb{R}^{n+k}$ of MCF satisfies $|h|^2+a<c|H|^2$ for some constants $c\le\frac{1}{n}+\frac{1}{3n}$ and $a>0$ at $t=0$, then this remains true for all $0\le t<T$.*

Note that under the conditions of Theorem 1 (at least in the case where the inequalities hold strictly), there exist constants $c<\frac{4}{3n}$ and $a>0$ such that the conditions of Theorem 2 hold. Thus the result implies both that H remains everywhere non-zero, and that the curvature pinching is preserved.

To prove Theorem 2 we derive evolution equations for the squared lengths of the second fundamental form and the mean curvature vector. The mean curvature flow amounts to the prescription $F_*\partial_t=\iota H$ in the notation of the previous section. The timelike Codazzi identity (18) gives an evolution equation for second fundamental form under MCF of a submanifold in an arbitrary background space N:

$$\nabla_{\partial_t}h(u,v)=\nabla_u\nabla_v H + h(v,\mathcal{W}(u,H)) + \overset{\perp}{\pi}\left(\bar{R}(F_*u,\iota H)F_*v\right), \tag{24}$$

or with respect to arbitrary local frames for the tangent and normal bundles

$$\nabla_{\partial_t}h_{ij}=\nabla_i\nabla_j H + H\cdot h_{ip}h_{pj} + H^\alpha \bar{R}_{i\alpha j}{}^{\beta}\nu_\beta.$$

This converts to a reaction-diffusion equation using the identity (23):

$$\begin{aligned}\nabla_{\partial_t} h_{ij} &= \Delta h_{ij} + h_{ij}\cdot h_{pq}h_{pq} + h_{iq}\cdot h_{qp}h_{pj} + h_{jq}\cdot h_{qp}h_{pi} - 2h_{ip}\cdot h_{jq}h_{pq}\\ &\quad + 2\bar{R}_{ipjq}h_{pq} - \bar{R}_{kjkp}h_{pi} - \bar{R}_{kikp}h_{pj} + h_{ij\alpha}\bar{R}_{k\alpha k\beta}\nu_\beta\\ &\quad - 2h_{jp\alpha}\bar{R}_{ip\alpha\beta}\nu_\beta - 2h_{ip\alpha}\bar{R}_{jp\alpha\beta}\nu_\beta + \bar{\nabla}_k\bar{R}_{kij\beta}\nu_\beta - \bar{\nabla}_i\bar{R}_{jkk\beta}\nu_\beta.\end{aligned}$$

This holds in an arbitrary Riemannian background space of arbitrary codimension. Henceforth we are concerned only with the case $N = \mathbb{R}^{n+k}$, in which case the equation becomes

$$\nabla_{\partial_t} h_{ij} = \Delta h_{ij} + h_{ij}\cdot h_{pq}h_{pq} + h_{iq}\cdot h_{qp}h_{pj} + h_{jq}\cdot h_{qp}h_{pi} - 2h_{ip}\cdot h_{jq}h_{pq}. \tag{25}$$

Taking the trace with respect to g we obtain an evolution equation for the mean curvature vector:

$$\nabla_{\partial_t} H = \Delta H + H\cdot h_{pq}h_{pq}. \tag{26}$$

The evolution equations for $|h|^2$ and $|H|^2$ follow from equations (25) and (26):

$$\begin{aligned}\frac{\partial}{\partial t}|h|^2 &= \Delta|h|^2 - 2|\nabla h|^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2\\ &\quad + 2\sum_{i,j,\alpha,\beta}\Big(\sum_p h_{ip\alpha}h_{jp\beta} - h_{jp\alpha}h_{ip\beta}\Big)^2\end{aligned} \tag{27}$$

$$\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\overset{\perp}{\nabla} H|^2 + 2\sum_{i,j}\Big(\sum_\alpha H_\alpha h_{ij\alpha}\Big)^2. \tag{28}$$

The last term in (27) is the length squared of the normal curvature, which we denote by $|\overset{\perp}{R}|^2$.

For future reference we label the reaction terms of the above evolution equations as follows:

$$\begin{aligned}R_1 &= \sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 + |\overset{\perp}{R}|^2\\ R_2 &= \sum_{i,j}\Big(\sum_\alpha H_\alpha h_{ij\alpha}\Big)^2.\end{aligned}$$

Consider now the quantity $\mathcal{Q} = |h|^2 + a - c|H|^2$, where c and a are positive constants. Combining the evolution equations for $|h|^2$ and $|H|^2$ we get

$$\frac{\partial}{\partial t}\mathcal{Q} = \Delta\mathcal{Q} - 2(|\nabla h|^2 - c|\nabla H|^2) + 2R_1 - 2cR_2. \tag{29}$$

By assumption this quantity is initially negative. We will prove that it remains negative by contradiction: Otherwise there is a first point and time where $\mathcal{Q}$ becomes zero, and at this point we necessarily have $\frac{\partial Q}{\partial t} \geq 0$ and $\Delta\mathcal{Q} \leq 0$. We will derive a contradiction by showing that the gradient terms on the right-hand side of equation (29) are non-positive, while the reaction terms are strictly negative.

We begin by estimating the gradient terms:

Proposition 6. *We have the estimates*

$$|\nabla h|^2 \geq \frac{3}{n+2}|\nabla H|^2 \tag{30a}$$

$$|\nabla h|^2 - \frac{1}{n}|\nabla H|^2 \geq \frac{2(n-1)}{3n}|\nabla h|^2. \tag{30b}$$

Proof. In exactly the same way as [**Hu1**] and [**Ha1**], we decompose the tensor ∇h into orthogonal components $\nabla_i h_{jk} = E_{ijk} + F_{ijk}$, where

$$E_{ijk} = \frac{1}{n+2}(g_{ij}\nabla_k H + g_{ik}\nabla_j H + g_{jk}\nabla_i H).$$

Then $|\nabla h|^2 \geq |E|^2 = \frac{3}{n+2}|\nabla H|^2$. The second estimate follows easily from the first. q.e.d.

Since $c < \frac{3}{n+2}$ under the assumption of Theorem 2, the gradient terms are non-positive.

In order to estimate the reaction terms of (29) it is convenient to work with the traceless part of second fundamental form $\mathring{h} = h - \frac{1}{n}gH$. The lengths of h and $\mathring{h}$ are related by $|\mathring{h}|^2 = |h|^2 - \frac{1}{n}|H|^2$.

At a point where $\mathcal{Q} = 0$, we certainly have $|H| \neq 0$, so we can choose a local orthonormal frame $\{\nu_\alpha,\ 1 \leq \alpha \leq k\}$ for $\mathcal{N}$, such that $\nu_1 = H/|H|$. With this choice the traceless second fundamental form takes the form

$$\begin{cases} \mathring{h}_1 = h_1 - \frac{|H|}{n} Id & \\ \mathring{h}_\alpha = h_\alpha & \alpha > 1, \end{cases}$$

and

$$\begin{cases} \operatorname{tr} h_1 = |H| & \\ \operatorname{tr} h_\alpha = 0 & \alpha > 1. \end{cases}$$

At a point we may choose a basis for the tangent space such that h_1 is diagonal. We denote by λ_i and $\mathring{\lambda}_i$ the diagonal entries of h_1 and $\mathring{h}_1$ respectively. We denote the norm of the $(\alpha \neq 1)$-directions of the second fundamental form by $|\mathring{h}_-|^2$, i.e. $|\mathring{h}|^2 = |\mathring{h}_1|^2 + |\mathring{h}_-|^2$. One final piece of notation we adopt from [**CdCK**]: For a matrix $A = (a_{ij})$, we denote

$$N(A) = \operatorname{tr}\left(A \cdot A^t\right) = \sum_{ij}(a_{ij})^2.$$

In particular, we have $\sum_{\alpha,\beta} N(\mathring{h}_\alpha \mathring{h}_\beta - \mathring{h}_\beta \mathring{h}_\alpha) = |\overset{\perp}{R}|^2$.

To estimate the reaction terms, we work with the bases described above and separate the $\alpha = 1$ components from the others. The reaction terms of (29) become

$$\begin{aligned}\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 \;&=\; |\mathring{h}_1|^4 + \frac{2}{n}|\mathring{h}_1|^2|H|^2 + \frac{1}{n^2}|H|^4 \\ &\quad + 2\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{h}_{ij1}\mathring{h}_{ij\alpha}\Big)^2 + \sum_{\alpha,\beta>1}\Big(\sum_{i,j}\mathring{h}_{ij\alpha}\mathring{h}_{ij\beta}\Big)^2;\end{aligned}$$

$$|R^\perp|^2 = 2\sum_{\alpha>1} N(h_1\mathring{h}_\alpha - \mathring{h}_\alpha h_1) + \sum_{\alpha,\beta>1} N(\mathring{h}_\alpha\mathring{h}_\beta - \mathring{h}_\beta\mathring{h}_\alpha);$$

$$\sum_{i,j}\Big(\sum_\alpha H_\alpha h_{ij\alpha}\Big)^2 = |\mathring{h}_1|^2|H|^2 + \frac{1}{n}|H|^4.$$

Writing out all the reaction terms we now have

(31)

$$\begin{aligned}2R_1 - 2cR_2 &= 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 + 2|\overset{\perp}{R}|^2 - 2c\sum_{i,j}\Big(\sum_\alpha H_\alpha h_{ij\alpha}\Big)^2 \\ &= 2|\mathring{h}_1|^4 - 2\left(c - \frac{2}{n}\right)|\mathring{h}_1|^2|H|^2 - \frac{2}{n}\left(c - \frac{1}{n}\right)|H|^4 \\ &\quad + 4\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{h}_{ij1}\mathring{h}_{ij\alpha}\Big)^2 + 4\sum_{\alpha>1} N(h_1\mathring{h}_\alpha - \mathring{h}_\alpha h_1) \\ &\quad + 2\sum_{\alpha,\beta>1}\Big(\sum_{i,j}\mathring{h}_{ij\alpha}\mathring{h}_{ij\beta}\Big)^2 + 2\sum_{\alpha,\beta>1} N(\mathring{h}_\alpha\mathring{h}_\beta - \mathring{h}_\beta\mathring{h}_\alpha).\end{aligned}$$

Now we use the fact that $\mathcal{Q} = 0$ to replace $\left(c - \frac{1}{n}\right)|H|^2$ by $|\mathring{h}|^2 + a$ in the first line of (31), giving

$$\begin{aligned}&2|\mathring{h}_1|^4 - 2\left(c - \frac{2}{n}\right)|\mathring{h}_1|^2|H|^2 - \frac{2}{n}\left(c - \frac{1}{n}\right)|H|^4 \\ &\qquad = 2|\mathring{h}_1|^4 - 2|\mathring{h}_1|^2\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 + a\right) \\ &\qquad\quad - \frac{2}{n(c-1/n)}\left(|\mathring{h}_-|^2 + a\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 + a\right) \\ &\qquad < -\frac{2c}{c-1/n}|\mathring{h}_1|^2|\mathring{h}_-|^2 - \frac{2}{n(c-1/n)}|\mathring{h}_-|^4,\end{aligned}$$

where we use the fact that all terms involving a are non-positive, and we have a strictly negative term $-\frac{2a^2}{n(c-1/n)}$. We need to control the last two lines of (31). In the second last line, we proceed by expanding the terms and using the fact that $\mathring{h}_1$ is diagonal:

$$\begin{aligned}\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{h}_{ij1}\mathring{h}_{ij\alpha}\Big)^2 &= \sum_{\alpha>1}\Big(\sum_i \mathring{\lambda}_i\mathring{h}_{ii\alpha}\Big)^2\\ &\le \Big(\sum_i \mathring{\lambda}_i^{\,2}\Big)\Big(\sum_{\substack{j\\ \alpha>1}}(\mathring{h}_{jj\alpha})^2\Big)\\ &= |\mathring{h}_1|^2\sum_{\substack{i\\ \alpha>1}}(\mathring{h}_{ii\alpha})^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{\alpha>1}N(h_1\mathring{h}_\alpha-\mathring{h}_\alpha h_1) &= \sum_{\substack{i\neq j\\ \alpha>1}}(\lambda_i-\lambda_j)^2(\mathring{h}_{ij\alpha})^2\\ &= \sum_{\substack{i\neq j\\ \alpha>1}}(\mathring{\lambda}_i-\mathring{\lambda}_j)^2(\mathring{h}_{ij\alpha})^2\\ &\le \sum_{\substack{i\neq j\\ \alpha>1}}2(\mathring{\lambda}_i^{\,2}+\mathring{\lambda}_j^{\,2})(\mathring{h}_{ij\alpha})^2\\ &\le 2|\mathring{h}_1|^2\sum_{\substack{i\neq j\\ \alpha>1}}(\mathring{h}_{ij\alpha})^2\\ &= 2|\mathring{h}_1|^2\Bigg(|\mathring{h}_-|^2-\sum_{\substack{i\\ \alpha>1}}(\mathring{h}_{ii\alpha})^2\Bigg),\end{aligned}$$

and so

$$\begin{aligned}&\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{h}_{ij1}\mathring{h}_{ij\alpha}\Big)^2+\sum_{\alpha>1}N(h_1\mathring{h}_\alpha-\mathring{h}_\alpha h_1)\\ &\quad\le 2|\mathring{h}_1|^2|\mathring{h}_-|^2-|\mathring{h}_1|^2\sum_{\substack{i\\ \alpha>1}}(\mathring{h}_{ii\alpha})^2\\ &\quad\le 2|\mathring{h}_1|^2|\mathring{h}_-|^2.\end{aligned}$$

To estimate the last line, we use an inequality first derived in [**CdCK**] and later improved to the version we use in [**LL1**]. In our notation we have

Lemma 1.

$$\sum_{\alpha,\beta>1}\Big(\sum_{i,j}\mathring{h}_{ij\alpha}\mathring{h}_{ij\beta}\Big)^2+\sum_{\alpha,\beta>1}N(\mathring{h}_\alpha\mathring{h}_\beta-\mathring{h}_\beta\mathring{h}_\alpha)\le\frac{3}{2}|\mathring{h}_-|^4.$$

Proof of Theorem 2. Using the above inequalities we estimate the reaction terms by

$$2R_1 - 2cR_2 < \left(6 - \frac{2}{n(c-1/n)}\right)|\mathring{h}_1|^2|\mathring{h}_-|^2 + \left(3 - \frac{2}{n(c-1/n)}\right)|\mathring{h}_-|^4.$$

The $|\mathring{h}_1|^2|\mathring{h}_-|^2$ terms are nonpositive for $c \leq \frac{1}{n} + \frac{1}{3n}$ and the $|\mathring{h}_-|^4$ terms are nonpositive for $c \leq \frac{1}{n} + \frac{2}{3n}$. The gradient terms are nonpositive for $c \leq \frac{3}{n+2}$, so the right-hand side of (29) is negative for $c \leq \frac{1}{n} + \frac{1}{3n}$, while the left-hand side is non-negative. This is a contradiction, so $\mathcal{Q}$ must remain negative. q.e.d.

To apply the pinching estimate in the case where equality holds in the assumptions of Theorem 1, we need the following result:

Proposition 7. *Suppose $\Sigma_0 = F_0(\Sigma^n)$ is a compact submanifold of $\mathbb{R}^{n+k}$ satisfying the conditions of Theorem* 1*, and let $F: \Sigma \times [0,T) \to \mathbb{R}^{n+k}$ be the solution of MCF with initial data F_0. Then for any sufficiently small $t > 0$ there exists $c \leq \frac{1}{n} + \frac{1}{3n}$ and $a > 0$ such that the conditions of Theorem* 2 *hold for Σ_t.*

Proof. We assume that Σ_0 is not a totally umbillic sphere, since in that case the conditions of Theorem 2 certainly apply. Since the solution is smooth, H remains non-zero on a short time interval. On this interval we can carry out the proof of Theorem 2 with $a = 0$, yielding

$$\begin{aligned}&\frac{\partial}{\partial t}\left(|h|^2 - c|H|^2\right)\\ &\leq \Delta\left(|h|^2 - c|H|^2\right) - 2\left(1 - \frac{c(n+2)}{3}\right)|\nabla h|^2 + \left(3 - \frac{2}{n(c-1/n)}\right)|\mathring{h}_-|^4.\end{aligned}$$

The coefficients of the last two terms are negative under the assumptions of Theorem 1. Therefore by the strong maximum principle, if $|h|^2 - c|H|^2$ does not immediately become negative, then $\nabla h \equiv 0$ and $\mathring{h}_- \equiv 0$. The latter implies that Σ_t lies in a $(n+1)$-subspace of $\mathbb{R}^{n+k}$, and then $\nabla h = 0$ implies that Σ_t is a product $\mathbb{S}^p \times \mathbb{R}^{n-p} \hookrightarrow \mathbb{R}^{n+k}$ (see [**L**], Theorem 4), and since Σ_0 is not a sphere we have $p < n$. But this is impossible since Σ_t is compact. Therefore for any small $t > 0$ there exists $a > 0$ such that $|h|^2 - c|H|^2 \leq -a$ on Σ_t, and Theorem 2 applies. q.e.d.

4. Higher derivative estimates and long time existence

Here we consider the long time behaviour of MCF and establish the existence of a solution on a finite maximal time interval determined by the blowup of the second fundamental form.

Theorem 3. *Under the assumptions of Theorem* 1*, MCF has a unique solution on a finite maximal time interval $0 \leq t < T < \infty$. Moreover, $\max_{\Sigma_t}|h|^2 \to \infty$ as $t \to T$.*

Lemma 2. *The maximal time of existence T is finite.*

Proof. This follows immediately from the evolution equation $\frac{\partial}{\partial t}|F|^2 = \Delta|F|^2 - 2n$, since this implies $|F(p,t)|^2 \le R^2 - 2nt$ and hence $T \le \frac{R^2}{2n}$, where $R = \sup\{|F_0(p)| : \; p \in \Sigma\}$. q.e.d.

We next want to prove interior-in-time higher derivative estimates for the second fundamental form. We use Hamilton's $*$ notation: For tensors S and T (that is, sections of bundles constructed from $\mathcal{H}$ and $\mathcal{N}$ by taking duals and tensor products) the product $S * T$ denotes any linear combination of contractions of S with T.

Proposition 8. *The evolution of the m-th covariant derivative of h is of the form*

$$\nabla_t \nabla^m h = \Delta \nabla^m h + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h.$$

Proof. We argue by induction on m. The case $m = 0$ is given by the evolution equation for the second fundamental form. Now suppose that the result holds up to $m-1$. Differentiating the m-th covariant derivative of h in time and using the timelike Gauss and Ricci equations to interchange derivatives we find

$$\begin{aligned}
\nabla_t \nabla^m h &= \nabla \nabla_t \nabla^{m-1} h + \nabla^{m-1} h * h * \nabla h \\
&= \nabla \Big(\Delta \nabla^{m-1} h + \sum_{i+j+k=m-1} \nabla^p h * \nabla^q h * \nabla^r h \Big) + \nabla^{m-1} h * h * \nabla h \\
&= \nabla \Delta \nabla^{m-1} h + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h.
\end{aligned}$$

The formula for commuting the Laplacian and gradient of a normal-valued tensor is given by:

$$\Delta \nabla_k T = \nabla_k \Delta T + \nabla_m \big(R(\partial_k, \partial_m) T \big) + \big(\big(R(\partial_k, \partial_m)(\nabla T) \big) \big) (\partial_m).$$

Since T and ∇T are $\mathcal{N}$-valued tensors acting on $\mathcal{H}$, equation (3) gives expressions for $R(\partial_k, \partial_m)T$ as $R * T + \overset{\perp}{R} * T$, and similarly $R(\partial_k, \partial_m)\nabla T = R * \nabla T + \overset{\perp}{R} * \nabla T$, where R and $\overset{\perp}{R}$ are the curvature tensors on $\mathcal{H}$ and $\mathcal{N}$, which are both of the form $h * h$. The terms arising in commuting the gradient and Laplacian of $\nabla^{m-1} h$ are of the form $\sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h$, so we obtain

$$\nabla_t \nabla^m h = \Delta \nabla^m h + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h$$

as required. q.e.d.

Proposition 9. *The evolution of* $|\nabla^m h|^2$ *is of the form*

$$\frac{\partial}{\partial t}|\nabla^m h|^2 = \Delta|\nabla^m h|^2 - 2|\nabla^{m+1}h|^2 + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h * \nabla^m h.$$

Proof. Denoting by angle brackets the inner product on $\otimes^{m+2}\mathcal{H}^*\otimes\mathcal{N}$, which is compatible with the connection on the same bundle, we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m h|^2 &= \frac{\partial}{\partial t}\left\langle \nabla^m_p h, \nabla^m_p h\right\rangle \\ &= 2\left\langle \nabla^m_p h, \nabla_t \nabla^m_p h\right\rangle \\ &= 2\Big\langle \nabla^m_p h, \Delta\nabla^m_p h + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h\Big\rangle \\ &= \Delta|\nabla^m h|^2 - 2|\nabla^{m+1}h|^2 + \sum_{i+j+k=m} \nabla^i h * \nabla^j h * \nabla^k h * \nabla^m h\end{aligned}$$

as required. q.e.d.

Lemma 3. *Suppose that mean curvature flow of a given submanifold* Σ_0 *has a solution on a time interval* $t \in [0,\tau]$. *If* $|h|^2 \leq K$ *for all* $t \in [0,\tau]$, *then* $|\nabla^m h|^2 \leq C_m(1+1/t^m)$ *for all* $t \in (0,\tau]$, *where* C_m *is a constant that depends on* m, n *and* K.

The strength of this estimate is that assuming only a bound on the second fundamental form (and no information about its derivatives) we can bound all higher derivatives. The fact that these estimates blow up as t approaches zero poses no difficulty, since the short-time existence result bounds all derivatives of h for a short time. While not crucial here, the interior-in-time estimates are useful in the singularity analysis of Section 7.

Proof. The proof is by induction on m. We first prove the Lemma for $m = 1$. We consider the quantity $G = t|\nabla h|^2 + |h|^2$, which has a bound at $t = 0$ depending only on curvature. The strategy is now to use the good term from the evolution of $|h|^2$ to control the bad term in the evolution of $|\nabla h|^2$: Differentiating G we get

$$\begin{aligned}\frac{\partial G}{\partial t} &= |\nabla h|^2 + t\big(\Delta|\nabla h|^2 - 2|\nabla^2 h|^2 + h * h * \nabla h * \nabla h\big) \\ &\quad + \big(\Delta|h|^2 - 2|\nabla h|^2 + h * h * h * h\big) \\ &\leq \Delta G + (c_1 t|h|^2 - 1)|\nabla h|^2 + c_2|h|^4.\end{aligned}$$

for $t \leq 1/(c_1 K)$ we can estimate

$$\frac{\partial}{\partial t}G \leq \Delta G + c_2 K^2,$$

and the maximum principle implies $\max_{x,t} G \leq K + c_2K^2 t$. Then $|\nabla h|^2 \leq G/t \leq K/t + c_2K^2$ for $t \in (0, 1/(c_1K)]$. If $t > 1/(c_1K)$ we apply the same argument on the interval $[t - 1/(c_1K), t]$, yielding $|\nabla h|^2(t) \leq (c_1 + c_2)K^2$. This completes the proof for $m = 1$. Now suppose the estimate holds up to $m-1$, and consider $G = t^m|\nabla^m h|^2 + mt^{m-1}|\nabla^{m-1}h|^2$. Differentiating G gives

$$
\begin{aligned}
\frac{\partial}{\partial t}G &= mt^{m-1}|\nabla^m h|^2 \\
&\quad + t^m\Big\{\Delta|\nabla^m h|^2 - 2|\nabla^{m+1}h|^2 + \sum_{i+j+k=m}\nabla^i h * \nabla^j h * \nabla^k h * \nabla^m h\Big\} \\
&\quad + m\Big\{(m-1)t^{m-2}|\nabla^{m-1}h|^2 + t^{m-1}\big(\Delta|\nabla^{m-1}h|^2 - 2|\nabla^m h|^2 \\
&\qquad\qquad + \sum_{i+j+k=m-1}\nabla^i h * \nabla^j h * \nabla^k h * \nabla^{m-1}h\big).\Big\}
\end{aligned}
$$

Noticing that in the quartic reaction terms there can only be one or two occurrences of the highest order derivative, using Young's inequality we can estimate

$$
\begin{aligned}
\frac{\partial}{\partial t}G &\leq mt^{m-1}|\nabla^m h|^2 + t^m\Big\{\Delta|\nabla^m h|^2 + c_3|\nabla^m h|^2 + \frac{c_4}{t^m}\Big\} \\
&\quad + m\Big\{(m-1)t^{m-2}|\nabla^{m-1}h|^2 + t^{m-1}\big(\Delta|\nabla^{m-1}h|^2 - 2|\nabla^m h|^2 \\
&\qquad\qquad + c_5|\nabla^{m-1}h|^2 + \frac{c_6}{t^{m-1}}\big)\Big\}.
\end{aligned}
$$

We split the gradient term of order m out of the second line, and then since m is at least two, all other terms are bounded by the induction hypothesis for $t \leq 1$, giving

$$\frac{\partial}{\partial t}G \leq \Delta G + (c_3 t - m)t^{m-1}|\nabla^m h|^2 + c_7.$$

Thus $\frac{\partial}{\partial t}G \leq \Delta G + c_8$ if $t \leq \min\{1, m/c_3\}$, so by the maximum principle $|\nabla^m h|^2 \leq C/t^m$ for $t \leq \min\{1, m/c_3\}$. The same argument on later time intervals gives the result for larger t. q.e.d.

Proof of Theorem 3. Fix a smooth metric $\tilde{g}$ on Σ with Levi-Civita connection $\tilde{\nabla}$. $\tilde{g}$ extends to a time-independent metric on $\mathcal{H}$, and $\tilde{\nabla}$ extends to $\mathcal{H}$ by taking $\tilde{\nabla}_{\partial_t}u = 0$ whenever $[\partial_t, u] = 0$. The difference $T = \nabla - \tilde{\nabla}$ restricts to a section of $\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$. If S is a section of a bundle constructed from $\mathcal{H}$, $\mathcal{N}$ and F^*TN, $\tilde{\nabla}S$ denotes the derivative of S with the connection on this bundle induced by the connections $\tilde{\nabla}$ on $\mathcal{H}$, $\overset{\perp}{\tilde{\nabla}}$ on $\mathcal{N}$, and ${}^F\nabla$ on F^*TN, so that $\tilde{\nabla}S - \nabla S = S * T$.

To prove Theorem 3 we assume that $|h|$ remains bounded on the interval $[0, T)$, and derive a contradiction. This suffices to prove the Theorem, since if $|h|$ is bounded on any subsequence of times approaching T, then Equation (27) implies that $|h|$ is bounded on $\Sigma \times [0, T)$. Under this assumption the boundedness of $\tilde{\nabla}_t g = -2H \cdot h$ implies that the metric g remains comparable to $\tilde{g}$: We have for any non-zero vector $v \in T\Sigma$

$$\left|\frac{\partial}{\partial t}\left(\frac{g(v,v)}{\tilde{g}(v,v)}\right)\right| = \left|\frac{\tilde{\nabla}_t g(v,v)}{g(v,v)}\,\frac{g(v,v)}{\tilde{g}(v,v)}\right| \leq 2|H||h|_g\frac{g(v,v)}{\tilde{g}(v,v)},$$

so that the ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant c_9 such that

$$\frac{1}{c_9}\tilde{g} \leq g \leq c_9\tilde{g}. \tag{32}$$

Next we observe that covariant derivatives of all orders of F with respect to $\tilde{\nabla}$ can be expressed in terms of h and T and their derivatives: We prove by induction that

(33)

$$\begin{aligned}
\tilde{\nabla}^k F = {}& F_*\tilde{\nabla}^{k-2}T + \\
& F_*\left(\sum_{i_0+2i_1+\cdots+(k-2)i_{k-3}=k-1} T^{i_0} * \left(\tilde{\nabla}T\right)^{i_1} * \cdots * \left(\tilde{\nabla}^{k-3}T\right)^{i_{k-3}}\right) \\
& + (\iota + F_*) * \sum_{j=1}^{k-1}\left(\sum_{\sum(n+1)i_n=k-1-j}\prod_{n=0}^{k-2-j}\left(\tilde{\nabla}^n T\right)^{i_n}\right) \\
& * \left(\sum_{\sum(m+1)p_m=j}\prod_{m=0}^{j-1}(\nabla^m h)^{p_m}\right).
\end{aligned}$$

This is true for $k=2$, since

(34)

$$\tilde{\nabla}^2_{u,v}F = {}^F\nabla_u(F_*v) - F_*(\tilde{\nabla}_u v) = F_*(\nabla_u v - \tilde{\nabla}_u v) + \iota h_{u,v} = F_*T_{u,v} + \iota h_{u,v}.$$

To deduce the result for higher k by induction, we note that Equation (34) implies a formula for the derivative of F_*:

$$(\tilde{\nabla}F_*)(V) = F_*T(.,V) + \iota h(.,V) = F_*T * V + \iota h * V,$$

while Equation (20) gives

$$(\tilde{\nabla}\iota)(\xi) = -F_*\mathcal{W}(.,\xi) = F_*h * \xi.$$

The result for $k+1$ now follows by differentiating the expression (33), and writing $\tilde{\nabla}(\nabla^n h) = \nabla^{n+1}h + \nabla^n h * T$. It follows that if $|\tilde{\nabla}^j F|_{\tilde{g}}$ is bounded for $j = 1, \dots, k-1$, then

$$|\tilde{\nabla}^{k-2} T|_{\tilde{g}} \leq C \left(1 + |\tilde{\nabla}^k F|_{\tilde{g}}\right). \tag{35}$$

The above observations allow us to prove C^k convergence of F as $t \to T$ for every k: We have $\tilde{\nabla}_t F = \iota H$, so the boundedness of H implies that F remains bounded and converges uniformly as $t \to T$. Differentiating as above, we find by induction that

$$\begin{aligned} \tilde{\nabla}_t \tilde{\nabla}^k F = (F_* + \iota) * \sum_{j=0}^{k-1} & \left(\sum_{\sum (n+1) i_n = k-1-j} \prod_{n=0}^{k-2-j} \left(\tilde{\nabla}^n T\right)^{i_n} \right) \\ & * \left(\sum_{\sum (m+1) p_m = j+2} \prod_{m=0}^{j+1} (\nabla^m h)^{p_m} \right). \end{aligned} \tag{36}$$

Suppose we have established a bound on $|\tilde{\nabla}^j F|_{\tilde{g}}$ for $j \leq k-1$. Then using the estimate (35), the bounds on $|\nabla^n h|_g$ from Lemma 3, and the comparability of g and $\tilde{g}$ from (32) we can estimate

$$|\tilde{\nabla}_t \tilde{\nabla}^k F|_{\tilde{g}} \leq C \left(1 + |\tilde{\nabla}^{k-2} T|_{\tilde{g}}\right) \leq C \left(1 + |\tilde{\nabla}^k F|_{\tilde{g}}\right),$$

so that $|\tilde{\nabla}^k F|_{\tilde{g}}$ remains bounded, and $\tilde{\nabla}^k F$ converges uniformly as $t \to T$. This completes the induction, proving that $F(.,t)$ converges in C^∞ to a limit $F(.,T)$ which is an immersion.

Finally, applying the short time existence result with initial data $F(.,T)$, we deduce that the solution can be continued to a larger time interval, contradicting the maximality of T. This completes the proof of Theorem 3. q.e.d.

5. A pinching estimate for the traceless second fundamental form

In this section we prove a pinching estimate for the traceless second fundamental form. This is the key estimate that will imply that the submanifold is evolving to a "round" point.

Theorem 4. *Under the assumptions of Theorem* 1 *there exist constants* $C_0 < \infty$ *and* $\delta > 0$ *both depending only on* Σ_0 *such that for all time* $t \in [0, T)$ *we have the estimate*

$$|\mathring{h}|^2 \leq C_0 |H|^{2-\delta}.$$

We wish to bound the function $f_\sigma = (|h|^2 - 1/n|H|^2)/|H|^{2(1-\sigma)}$ for sufficiently small σ. As in the hypersurface case, a distinguishing feature of mean curvature flow when compared to Ricci flow is that this result cannot be proved by a maximum principle argument alone. Somewhat more technical integral estimates and a Stampacchia iteration procedure are required. We proceed by first deriving an evolution equation for f_σ.

Proposition 10. *For any $\sigma \in [0,1/2]$ we have the evolution equation*

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{4(1-\sigma)}{|H|}\big\langle \nabla_i |H|, \nabla_i f_\sigma \big\rangle - \frac{2\epsilon_\nabla}{|H|^{2(1-\sigma)}}|\nabla H|^2 + 2\sigma |h|^2 f_\sigma. \tag{37}$$

Proof. Differentiating f_σ in time and substituting in the evolutions equations for the squared lengths of the second fundamental form and mean curvature we get

$$\begin{aligned} \partial_t f_\sigma &= \frac{\Delta |h|^2 - 2|\nabla h|^2 + 2R_1}{(|H|^2)^{1-\sigma}} \\ &\quad - \frac{1}{n}\frac{(\Delta|H|^2 - 2|\nabla H|^2 + 2R_2)}{(|H|^2)^{1-\sigma}} \\ &\quad - \frac{(1-\sigma)(|h|^2 - 1/n|H|^2)}{(|H|^2)^{2-\sigma}}(\Delta |H|^2 - 2|\nabla H|^2 + 2R_2). \end{aligned} \tag{38}$$

Substituting in the Laplacian of f_σ:

$$\begin{aligned} \Delta f_\sigma = {} & \frac{\Delta(|h|^2 - 1/n|H|^2)}{(|H|^2)^{1-\sigma}} - \frac{2(1-\sigma)}{(|H|^2)^{2-\sigma}}\big\langle \nabla_i(|h|^2 - 1/n|H|^2), \nabla_i |H|^2 \big\rangle \\ & - \frac{(1-\sigma)(|h|^2 - 1/n|H|^2)}{(|H|^2)^{2-\sigma}}\Delta |H|^2 \\ & + \frac{(2-\sigma)(1-\sigma)(|h|^2 - 1/n|H|^2)}{(|H|^2)^{3-\sigma}}|\nabla |H|^2|^2 \end{aligned}$$

and using the following identity:

$$\begin{aligned} & -\frac{2(1-\sigma)}{(|H|^2)^{2-\sigma}}\big\langle \nabla_i(|h|^2 - 1/n|H|^2), \nabla_i |H|^2 \big\rangle \\ & \quad = -\frac{2(1-\sigma)}{|H|^2}\big\langle \nabla_i |H|^2, \nabla_i f_\sigma \big\rangle - \frac{8(1-\sigma)^2}{(|H|^2)^2} f_\sigma |H|^2 |\nabla |H||^2, \end{aligned}$$

equation (38) can be manipulated into the form

$$\begin{aligned} \partial_t f_\sigma = {} & \Delta f_\sigma + \frac{2(1-\sigma)}{|H|^2}\big\langle \nabla_i |H|^2, \nabla_i f_\sigma \big\rangle \\ & - \frac{2}{(|H|^2)^{1-\sigma}}\left(|\nabla h|^2 - \frac{|h|^2}{|H|^2}|\nabla H|^2 \right) + \frac{2\sigma R_2 f_\sigma}{|H|^2} \\ & - \frac{4\sigma(1-\sigma)}{|H|^4} f_\sigma |H|^2 |\nabla |H||^2 - \frac{2\sigma(|h|^2 - 1/n|H|^2)}{(|H|^2)^{2-\sigma}}|\nabla H|^2 \\ & + \frac{2}{(|H|^2)^{1-\sigma}}\left(R_1 - \frac{|h|^2}{|H|^2}R_2 \right). \end{aligned}$$

We discard the terms on the last line as these are nonpositive under our pinching assumption. The gradient terms on the first line may be estimated as follows:

$$-\frac{2}{(|H|^2)^{1-\sigma}}\left(|\nabla h|^2-\frac{|h|^2}{|H|^2}|\nabla H|^2\right)\leq -\frac{2}{(|H|^2)^{1-\sigma}}\left(\frac{3}{n+2}-c\right)|\nabla H|^2,$$

and also $R_2 \leq |h|^2|H|^2$. Importantly, observe that if $c \leq 4/(3n)$, then $\epsilon_\nabla := 3/(n+2) - c$ is strictly positive. q.e.d.

The reaction term $2\sigma|h|^2 f_\sigma$ in this evolution equation is positive and hence we cannot apply the maximum principle. As in the hypersurface case, we exploit the negative term involving the gradient of the mean curvature by integrating a suitable form of Simons' identity: Contracting equation (23) with the second fundamental form we obtain the following:

$$\frac{1}{2}\Delta|\mathring{h}|^2 = \mathring{h}_{ij}\cdot\nabla_i\nabla_j H + \left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2\right) + Z, \tag{39}$$

where

$$Z = -\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 - |\overset{\perp}{R}|^2 + \sum_{\substack{i,j,p\\ \alpha,\beta}} H_\alpha h_{ip\alpha}h_{ij\beta}h_{pj\beta}.$$

Lemma 4. *If Σ^n is a submanifold of $\mathbb{R}^{n+k}$ with $H \neq 0$ and $|h|^2 \leq c|H|^2$, where*

$$c < \begin{cases} \frac{3}{4}, & \text{if } n = 2\\ \frac{181}{384}, & \text{if } n = 3\\ \frac{1}{n-1}, & \text{if } n \geq 4,\end{cases}$$

then there exists $\epsilon > 0$ such that $Z \geq \epsilon|\mathring{h}|^2|H|^2$.

In dimension two and three our pinching condition is preserved for $c \leq 2/3$ and $c \leq 4/9$ respectively. As $2/3 < 3/4$ and $4/9 < 181/384$ we do not need to assume any pinching beyond that of the pinching lemma to prove Theorem 4. In dimensions greater than or equal to four, we will soon see that we need the stronger condition that $c < 1/(n-1)$.

Proof of Lemma 4. Working with the local orthonormal frames of Section 3 we expand Z to get

$$
\begin{aligned}
Z = &-|\mathring{h}_1|^4 + \frac{1}{n}|\mathring{h}_1|^2|H|^2 + \frac{1}{n}|\mathring{h}_-|^2|H|^2 - 2\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{\lambda}_i\mathring{h}_{ii\alpha}\Big)^2 \\
&- 2\sum_{\alpha>1} N(h_1\mathring{h}_\alpha - \mathring{h}_\alpha h_1) \\
&- \sum_{\alpha,\beta>1}\Big(\sum_{i,j}\mathring{h}_{ij\alpha}\mathring{h}_{ij\beta}\Big)^2 - \sum_{\alpha,\beta>1} N(\mathring{h}_\alpha\mathring{h}_\beta - \mathring{h}_\beta\mathring{h}_\alpha) \\
&+ \sum_{\substack{\alpha>1\\ i}}|H|\mathring{\lambda}_i^{\,3} + \sum_{\substack{\alpha>1\\ i}}|H|\mathring{\lambda}_i(\mathring{h}_{ii\alpha})^2 + \sum_{\substack{\alpha>1\\ i\neq j}}|H|\mathring{\lambda}_i(\mathring{h}_{ij\alpha})^2.
\end{aligned}
$$

For dimension $n = 2$ all the above terms can be computed explicitly to give $Z \geq c_1|\mathring{h}|^4$ for some $c_1 > 0$, provided $c < 3/4$. The Lemma then follows using the same argument we use below for the cases $n \geq 3$. For the cases $n \geq 3$ we cannot easily calculate the terms of Z explicitly and we proceed by estimating the various terms. We can estimate the first summation term on line one and the two terms on line two as before, that is

$$
\begin{aligned}
-2\sum_{\alpha>1}\Big(\sum_{i,j}\mathring{\lambda}_i\mathring{h}_{ii\alpha}\Big)^2 &\geq -2|\mathring{h}_1|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 \\
-\sum_{\alpha,\beta>1}\Big(\sum_{i,j}\mathring{h}_{ij\alpha}\mathring{h}_{ij\beta}\Big)^2 - \sum_{\alpha,\beta>1} N(\mathring{h}_\alpha\mathring{h}_\beta - \mathring{h}_\beta\mathring{h}_\alpha) &\geq -\frac{3}{2}|\mathring{h}_-|^4,
\end{aligned}
$$

however we need to work somewhat harder with the remaining summation terms.

Proposition 11. *For any $\eta \geq 8$ we have the following estimate*

$$
\begin{aligned}
&-2\sum_{\alpha>1} N(h_1\mathring{h}_\alpha - \mathring{h}_\alpha h_1) + \sum_{\substack{\alpha>1\\ i\neq j}}|H|\mathring{\lambda}_i(\mathring{h}_{ij\alpha})^2 \\
&\quad\geq -\frac{\eta}{2}|\mathring{h}_1|^2\Big(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\Big) - \frac{1}{4\eta}|H|^2\Big(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\Big).
\end{aligned}
$$

Proof. We estimate

$$
\begin{aligned}
&-2\sum_{\alpha>1} N(h_1\mathring{h}_\alpha - \mathring{h}_\alpha h_1) + \sum_{\substack{\alpha>1\\ i\neq j}} |H|\mathring{\lambda}_i(\mathring{h}_{ij\alpha})^2\\
&\quad = -\sum_{\substack{\alpha>1\\ i\neq j}} \left\{2(\mathring{\lambda}_i - \mathring{\lambda}_j)^2 - \frac{|H|}{2}(\mathring{\lambda}_i + \mathring{\lambda}_j)\right\}(\mathring{h}_{ij\alpha})^2\\
&\quad \geq -\sum_{\substack{\alpha>1\\ i\neq j}} \left\{2(\mathring{\lambda}_i - \mathring{\lambda}_j)^2 + \frac{\eta}{4}(\mathring{\lambda}_i + \mathring{\lambda}_j)^2\right\}(\mathring{h}_{ij\alpha})^2\\
&\qquad - \frac{1}{4\eta}|H|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right)\\
&\quad = -\sum_{\substack{\alpha>1\\ i\neq j}} \left\{\left(2+\frac{\eta}{4}\right)(\mathring{\lambda}_i{}^2 + \mathring{\lambda}_j{}^2) + \left(\frac{\eta}{2} - 4\right)\mathring{\lambda}_i\mathring{\lambda}_j\right\}(\mathring{h}_{ij\alpha})^2\\
&\qquad - \frac{1}{4\eta}|H|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right)\\
&\quad \geq -\sum_{\substack{\alpha>1\\ i\neq j}} \left\{\left(2+\frac{\eta}{4}\right)(\mathring{\lambda}_i{}^2 + \mathring{\lambda}_j{}^2) + \left(\frac{\eta}{4} - 2\right)(\mathring{\lambda}_i{}^2 + \mathring{\lambda}_j{}^2)\right\}(\mathring{h}_{ij\alpha})^2\\
&\qquad - \frac{1}{4\eta}|H|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right)\\
&\quad = -\frac{\eta}{2}\sum_{\substack{\alpha>1\\ i\neq j}}(\mathring{\lambda}_i{}^2 + \mathring{\lambda}_j{}^2)(\mathring{h}_{ij\alpha})^2 - \frac{1}{4\eta}|H|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right)\\
&\quad \geq -\frac{\eta}{2}|\mathring{h}_1|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right) - \frac{1}{4\eta}|H|^2\left(|\mathring{h}_-|^2 - \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\right)
\end{aligned}
$$

as required. q.e.d.

To estimate the remaining two terms we use the following two inequalities from [**AdC**] and [**Sa**]:

$$
\begin{aligned}
\sum_{\substack{\alpha>1\\ i}}|H|\mathring{\lambda}_i{}^3 &\geq -\frac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}_1|^3\\
\sum_{\substack{\alpha>1\\ i}}|H|\mathring{\lambda}_i(\mathring{h}_{ii\alpha})^2 &\geq -\frac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}_1|\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2,
\end{aligned}
$$

and further estimate using Peter-Paul to get

$$\sum_{\substack{\alpha>1\\ i}} |H|\mathring{\lambda}_i^3 \geq -\frac{\mu}{2}|\mathring{h}_1|^4 - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}|\mathring{h}_1|^2|H|^2$$

$$\sum_{\substack{\alpha>1\\ i}} |H|\mathring{\lambda}_i(\mathring{h}_{ii\alpha})^2 \geq -\rho|\mathring{h}_1|^2 \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 - \frac{1}{4\rho}\frac{(n-2)^2}{n(n-1)}|H|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2.$$

Putting everything together we obtain

$$\begin{aligned}
Z \geq &-|\mathring{h}_1|^4 + \frac{1}{n}|\mathring{h}_1|^2|H|^2 + \frac{1}{n}|\mathring{h}_-|^2|H|^2 - 2|\mathring{h}_1|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 - \frac{3}{2}|\mathring{h}_-|^4\\
&- \frac{\eta}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{\eta}{2}|\mathring{h}_1|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 - \frac{1}{4\eta}|H|^2|\mathring{h}_-|^2\\
&+ \frac{1}{4\eta}|H|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 - \frac{\mu}{2}|\mathring{h}_1|^4 - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}|\mathring{h}_1|^2|H|^2\\
&- \rho|\mathring{h}_1|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 - \frac{1}{4\rho}\frac{(n-2)^2}{n(n-1)}|H|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2.
\end{aligned}$$

We now need to choose the optimal values of the constants η, μ and ρ. First we choose μ to be equal to $n-2$. This is valid for all $n \geq 3$. We next choose ρ to cancel the $|\mathring{h}_1|^2 \sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2$ terms, that is we want

$$\left(\frac{\eta}{2} - 2 - \rho\right)|\mathring{h}_1|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2 = 0$$

and so we choose

$$\rho = \frac{\eta}{2} - 2 = \frac{1}{2}(\eta - 4).$$

This is always valid since $\eta \geq 8$. The only mildly troublesome term that remains is

$$\left(\frac{1}{4\eta} - \frac{1}{2(\eta-4)}\frac{(n-2)^2}{n(n-1)}\right)|H|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2.$$

The optimal choice for η is $n+2$, however we still also require that $\eta \geq 8$. Consequently in dimensions three, four and five we choose $\eta = 8$ and for all higher dimensions choose $\eta = n+2$. In dimension three our choice of η makes the above term positive and we discard it in this case.

In dimensions four and higher the term is negative and we estimate

$$
\begin{aligned}
-\bigg(\frac{1}{2(\eta-4)}\frac{(n-2)^2}{n(n-1)}-\frac{1}{4\eta}\bigg)|H|^2\sum_{\substack{\alpha>1\\ i}}(\mathring{h}_{ii\alpha})^2\\
\geq -\bigg(\frac{1}{2(\eta-4)}\frac{(n-2)^2}{n(n-1)}-\frac{1}{4\eta}\bigg)|H|^2|\mathring{h}_-|^2.
\end{aligned}
$$

After substituting in our choices for ρ, μ and η we have, in dimension three

$$
\begin{aligned}
Z \geq & -|\mathring{h}_1|^4+\frac{1}{n}|\mathring{h}_1|^2|H|^2+\frac{1}{n}|\mathring{h}_-|^2|H|^2-\frac{3}{2}|\mathring{h}_-|^4-4|\mathring{h}_1|^2|\mathring{h}_-|^2\\
& -\frac{1}{32}|H|^2|\mathring{h}_-|^2-\frac{n-2}{2}|\mathring{h}_1|^4-\frac{n-2}{2n(n-1)}|\mathring{h}_1|^2|H|^2,
\end{aligned}
$$

in dimensions four and five

$$
\begin{aligned}
Z \geq & -|\mathring{h}_1|^4+\frac{1}{n}|\mathring{h}_1|^2|H|^2+\frac{1}{n}|\mathring{h}_-|^2|H|^2-\frac{3}{2}|\mathring{h}_-|^4-4|\mathring{h}_1|^2|\mathring{h}_-|^2\\
& -\frac{n-2}{2}|\mathring{h}_1|^4-\frac{n-2}{2n(n-1)}|\mathring{h}_1|^2|H|^2-\frac{(n-2)^2}{8n(n-1)}|H|^2|\mathring{h}_-|^2,
\end{aligned}
$$

and in dimensions six and higher

$$
\begin{aligned}
Z \geq & -|\mathring{h}_1|^4+\frac{1}{n}|\mathring{h}_1|^2|H|^2+\frac{1}{n}|\mathring{h}_-|^2|H|^2-\frac{3}{2}|\mathring{h}_-|^4-\frac{n+2}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2\\
& -\frac{n-2}{2}|\mathring{h}_1|^4-\frac{n-2}{2n(n-1)}|\mathring{h}_1|^2|H|^2-\frac{n-2}{2n(n-1)}|H|^2|\mathring{h}_-|^2.
\end{aligned}
$$

We now group like terms, estimate $|H|^2$ from below by $|\mathring{h}|^2/(c-1/n)$ and calculate the maximum value of c permissable in each case such that the coefficients are all strictly positive. For $n=3$ we have

$$
\begin{aligned}
Z \geq & \bigg(-1+\frac{1}{n(c-\frac{1}{n})}-\frac{n-2}{2}-\frac{n-2}{2n(n-1)(c-\frac{1}{n})}\bigg)|\mathring{h}_1|^4\\
& +\bigg(\frac{2}{n(c-\frac{1}{n})}-4-\frac{1}{32(c-\frac{1}{n})}-\frac{n-2}{2n(n-1)(c-\frac{1}{n})}\bigg)|\mathring{h}_1|^2|\mathring{h}_-|^2\\
& +\bigg(\frac{1}{n(c-\frac{1}{n})}-\frac{3}{2}-\frac{1}{32(c-\frac{1}{n})}\bigg)|\mathring{h}_-|^4.
\end{aligned}
$$

The $|\mathring{h}_1|^4$ terms are strictly positive for $c<1/(n-1)=1/2$, the $|\mathring{h}_1|^2|\mathring{h}_-|^2$ terms for $c<181/384$ and the $|\mathring{h}_-|^4$ terms for $c<77/144$. Note that in this case the smallest term is the mixed term; this too is the case when $n=2$. The higher dimensional cases follow similarly however the smallest term in all these cases is now the $|\mathring{h}_1|^4$ term. Furthermore, we find this term is always identically zero for all dimensions $n\geq 4$ when $c=1/(n-1)$. We have now shown for the values of c stated in the Proposition and strictly positive constants c_2, c_3 and c_4 depending on Σ_0 that

$$\begin{aligned} Z &\geq c_2|\mathring{h}_1|^4 + c_3|\mathring{h}_1|^2|\mathring{h}_-|^2 + c_4|\mathring{h}_-|^4 \\ &\geq c_5|\mathring{h}|^4, \end{aligned} \tag{40}$$

where $c_5 = \min\{c_2, c_3/2, c_4\}$. To prove the desired estimate we note that by using Peter-Paul on various terms of Z we can estimate

$$Z \geq c_6|\mathring{h}|^2|H|^2 - c_7|\mathring{h}|^4.$$

Combining this with (40) gives for any $a \in [0,1]$ that

$$Z \geq a\left(c_6|\mathring{h}|^2|H|^2 - c_7|\mathring{h}|^4\right) + (1-a)c_5|\mathring{h}|^4.$$

Choosing $a = c_5/(c_5 + c_7)$ gives

$$Z \geq \frac{c_5 c_6}{c_5 + c_7}|\mathring{h}|^2|H|^2$$

and the Lemma is complete by setting $\epsilon = c_5c_6/(c_5+c_7)$. q.e.d.

Proposition 12. *For any $p \geq 2$ and $\eta > 0$ we have the estimate*

$$\int_\Sigma f_\sigma^p |H|^2\, dV_g$$

$$\leq \frac{(p\eta + 4)}{\epsilon}\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\, dV_g + \frac{p-1}{\epsilon\eta}\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\, dV_g.$$

Proof. Using the contracted form of Simons' identity and $\Delta|H|^2 = 2|H|\Delta|H| + 2|\nabla|H||^2$, the Laplacian of f_σ can be expressed as

$$\begin{aligned} \Delta f_\sigma = {} & \frac{2}{|H|^{2(1-\sigma)}}\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\rangle + \frac{2}{|H|^{2(1-\sigma)}} Z \\ & - \frac{4(1-\sigma)}{|H|}\langle \nabla_i|H|, \nabla_i f_\sigma\rangle - \frac{2(1-\sigma)}{|H|} f_\sigma \Delta|H| \\ & + \frac{2}{|H|^{2(1-\sigma)}}\left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2\right) - \frac{2(1-\sigma)(1-2\sigma)}{|H|^2} f_\sigma |\nabla|H||^2. \end{aligned}$$

The combination of the last two terms is non-negative and we discard them. We multiply the remaining terms by f_σ^{p-1} and integrate over Σ. On the left, and in the last term on line one we use Green's first identity, and in integrating the first term on the right we use the Divergence Theorem and the Codazzi equation. The term arising from integrating on the left is non-negative and we discard it. Two other terms arising from the integration combine, ultimately giving

$$\begin{aligned}
&2\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} Z\,dV_g\\
&\quad\le\ 2(p-1)\int_\Sigma \frac{f_\sigma^{p-2}}{|H|^{2(1-\sigma)}}\big\langle \nabla_i f\cdot \mathring{h}_{ij}, \nabla_j H\big\rangle\,dV_g\\
&\qquad - 4(1-\sigma)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(\sigma-1)+1}}\big\langle \nabla_i|H|\cdot \mathring{h}_{ij}, \nabla_j H\big\rangle\,dV_g\\
&\qquad + \frac{2(n-1)}{n}\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g\\
&\qquad - 2(1-\sigma)(p-2)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|}\big\langle \nabla_i|H|, \nabla_i f_\sigma\big\rangle\,dV_g\\
&\qquad + 2(1-\sigma)\int_\Sigma \frac{f_\sigma^{p}}{|H|^2}|\nabla|H||^2\,dV_g.
\end{aligned}$$

The terms with an inner product do not have a sign. Using the Cauchy-Schwarz and Young's inequality, the inequalities $f_\sigma \le c|H|^{2\sigma}$, $|\nabla|H||^2 \le |\nabla H|^2$, $1-\sigma\le 1$, $c\le 1$, and $|\mathring{h}|^2 = f_\sigma|H|^{2(1-\sigma)}$ we estimate each term as follows:

$$\begin{aligned}
&2(p-1)\int_\Sigma \frac{f_\sigma^{p-2}}{|H|^{2(1-\sigma)}}\big\langle \nabla_i f_\sigma\cdot \mathring{h}_{ij}, \nabla_j H\big\rangle\,dV_g\\
&\quad\le \frac{p-1}{\eta}\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g + (p-1)\eta\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g;\\
&\qquad - 4(1-\sigma)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)+1}}\big\langle \nabla_i|H|\cdot \mathring{h}_{ij}, \nabla_j H\big\rangle\,dV_g\\
&\quad\le 4\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g;\\
&\qquad - 2(1-\sigma)(p-2)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|}\big\langle \nabla_i|H|, \nabla f_\sigma\big\rangle\,dV_g\\
&\quad\le \frac{p-2}{\mu}\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g + (p-2)\mu\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g;\\
&\quad 2(1-\sigma)\int_\Sigma \frac{f_\sigma^{p}}{|H|^2}|\nabla|H||^2\,dV_g \le 2\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g.
\end{aligned}$$

Putting all the estimates together we obtain

$$
\begin{aligned}
&2\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} Z\, dV_g \\
&\qquad \le \Big(6 + \frac{2(n-1)}{n} + (p-1)\eta + (p-2)\mu\Big) \int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2\, dV_g \\
&\qquad\quad + \left(\frac{p-1}{\eta} + \frac{p-2}{\mu}\right) \int_\Sigma f_\sigma^{p-2} |\nabla f_\sigma|^2\, dV_g.
\end{aligned}
$$

Our use for this inequality will be to show that sufficiently high L^p norms of f_σ are bounded. We are not interested in finding optimal values of p and consequently we are going to be a little rough with the final estimates in order to put the Lemma into a convenient form. Setting $\mu = \eta$, and using $p-2 \le p-1 \le p$ and Lemma 4 we get

$$
\begin{aligned}
&2\epsilon \int_\Sigma f_\sigma^p |H|^2 \\
&\quad \le (2p\eta + 8) \int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2-\sigma}} |\nabla H|^2\, dV_g + \frac{2(p-1)}{\eta} \int_\Sigma f_\sigma^{p-2} |\nabla f_\sigma|^2\, dV_g.
\end{aligned}
$$

Dividing through by 2ϵ completes the Lemma. q.e.d.

Proposition 13. *For any $p \ge \max\{2, 8/(\epsilon_\nabla + 1)\}$ we have the estimate*

$$
\begin{aligned}
\frac{d}{dt}\int_\Sigma f_\sigma^p\, dV_g \le& -\frac{p(p-1)}{2} \int_\Sigma f_\sigma^{p-2} |\nabla f_\sigma|^2\, dV_g \\
& - p\epsilon_\nabla \int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2-\sigma}} |\nabla H|^2\, dV_g + 2p\sigma \int_\Sigma |H|^2 f_\sigma^p\, dV_g.
\end{aligned}
$$

Proof. Differentiating under the integral sign and substituting in the evolution equations for f_σ and the measure dV_g gives

(41)

$$
\begin{aligned}
&\frac{d}{dt}\int_\Sigma f_\sigma^p\, dV_g \\
&= \int_\Sigma \left(p f_\sigma^{p-1} \frac{\partial f_\sigma}{\partial t} - |H|^2 f_\sigma^p \right) dV_g \\
&\le \int_\Sigma p f_\sigma^{p-1} \frac{\partial f_\sigma}{\partial t}\, dV_g \\
&\le -p(p-1) \int_\Sigma f_\sigma^{p-2} |\nabla f_\sigma|^2\, dV_g + 4(1-\sigma)p \int_\Sigma \frac{f_\sigma^{p-1}}{|H|} |\nabla |H|| |\nabla f_\sigma|\, dV_g \\
&\quad - 2p\epsilon_\nabla \int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2\, dV_g + 2p\sigma \int_\Sigma |H|^2 f_\sigma^p\, dV_g.
\end{aligned}
$$

We estimate the second integral by

$$\begin{aligned}
&4(1-\sigma)p\int_\Sigma \frac{f_\sigma^{p-1}}{|H|}|\nabla|H|||\nabla f_\sigma|\,dV_g\\
&\quad\le \frac{2p}{\rho}\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g + 2p\rho\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g,
\end{aligned}$$

and then substituting this estimate back into (41) gives

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma f_\sigma^p\,dV_g \le \Big(-p(p-1)+\frac{2p}{\rho}\Big)&\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g\\
&-(2p\epsilon_\nabla - 2p\rho)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g\\
&+2p\sigma\int_\Sigma |H|^2 f_\sigma^p\,dV_g\\
=-p(p-1)&\Big(1-\frac{2}{\rho(p-1)}\Big)\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g\\
&-2p\epsilon_\nabla\Big(1-\frac{\rho}{\epsilon_\nabla}\Big)\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g\\
&+2p\sigma\int_\Sigma |H|^2 f_\sigma^p\,dV_g.
\end{aligned}$$

We now want to choose ρ so that $1-2/(\rho(p-1))\ge 1/2$ and p so that $1-\rho/\epsilon_\nabla \ge 1/2$. Choosing $\rho = 4/(p-1)$ and $p\ge \max\{2, 8/(\epsilon_\nabla+1)\}$ gives the result. q.e.d.

Lemma 5. *There exist constants c_8 and c_9 depending only on Σ_0 such that if $p\ge c_8$ and $\sigma\le c_9/\sqrt{p}$, then for all time $t\in[0,T)$ we have the estimate*

$$\frac{d}{dt}\int_\Sigma f_\sigma^p\,dV_g \le 0.$$

Proof. Combining Propositions 12 and 13 we get

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma f_\sigma^p\,dV_g \le -p(p-1)\Big(\frac{1}{2}-\frac{2\sigma}{\epsilon\eta}\Big)&\int_\Sigma f_\sigma^{p-2}|\nabla f_\sigma|^2\,dV_g\\
-\Big(p\epsilon_\nabla - \frac{2p\sigma(p\eta+4)}{\epsilon}\Big)&\int_\Sigma \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\,dV_g.
\end{aligned}$$

Suppose that

$$\sigma \le \frac{\epsilon}{8}\sqrt{\frac{\epsilon_\nabla}{p}}.$$

Set $\eta = 4\sigma/\epsilon$, then

$$\begin{cases} \frac{2\sigma}{\epsilon\eta} = \frac{1}{2}\\ \frac{2p\sigma(p\eta+4)}{\epsilon} \le \frac{1}{4}\sqrt{p\epsilon_\nabla}\left(\frac{1}{2}\sqrt{p\epsilon_\nabla}+4\right) \le \frac{p\epsilon_\nabla}{2}. \end{cases}$$

For the second last inequality to hold we must assume $p \geq 64/\epsilon_\nabla$. We conclude that

$$\frac{d}{dt}\int_\Sigma f_\sigma^p \, dV_g \leq 0.$$

The Lemma holds with $c_8 = \max\{2, 8/(\epsilon_\nabla+1), 64/\epsilon_\nabla\}$ and $c_9 = \epsilon\sqrt{\epsilon_\nabla}/8$.

q.e.d.

Lemma 5 shows that for σ small enough, high L^p norms of f_σ are bounded. We proceed as in [**Hu1**] to derive a bound on the supremum of f_σ by Stampacchia iteration. The argument follows line for line the argument in [**Hu1**] and the reader is referred there for the details (see also [**HS**]).

6. A gradient estimate for the mean curvature

In this section we derive a gradient estimate for the mean curvature. This will be used in the following section to compare the mean curvature of the submanifold at different points.

Theorem 5. *Under the assumptions of Theorem* 1*, for each* $\eta > 0$ *there exists a constant* C_η *depending only on* η *and* Σ_0 *such that the estimate*

$$|\nabla H|^2 \leq \eta |H|^4 + C_\eta$$

holds on $\Sigma \times [0, T)$.

We begin by deriving a number of evolution equations.

Proposition 14. *There exists a constant* A *depending only on* Σ_0 *such that*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + A|H|^2|\nabla h|^2.$$

Proof. Differentiating the length of the gradient squared in time gives

(42)
$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla H|^2 &= \frac{\partial}{\partial t}\langle \nabla H, \nabla H\rangle \\
&= 2\langle \nabla_t \nabla H, \nabla H\rangle \\
&= 2\overset{\perp}{g}(\nabla_k \nabla_t H + \overset{\perp}{R}(\partial_k, \partial_t)H, \nabla_k H) \\
&= 2\overset{\perp}{g}\big(\nabla_k(\Delta H + H \cdot h_{pq}h_{pq}), \nabla_k H\big) + 2\overset{\perp}{g}\big(\overset{\perp}{R}(\partial_k, \partial_t)H, \nabla_k H\big).
\end{aligned}$$

To manipulate the last line into the desired form we need the following two formulae:

$$\Delta |H|^2 = 2\overset{\perp}{g}(\Delta \nabla_k H, \nabla_k H) + 2|\nabla^2 H|^2$$

$$\Delta \nabla_k H = \nabla_k \Delta H + \nabla_p\big(\overset{\perp}{R}(\partial_k, \partial_p)H\big) + \overset{\perp}{R}(\partial_k, \partial_p)\overset{\perp}{\nabla}_p H + Rc_{pk}\overset{\perp}{\nabla}_p H.$$

Substituting these into (42) and observing that the Gauss equation (13a) and the Ricci equation (15) are of the form $R = h*h$ and $\overset{\perp}{R} = h*h$, and that the timelike Ricci equation (16) is of the form $\overset{\perp}{R}(\cdot, \partial_t) = h * \nabla h$, we find

$$\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + h*h*\nabla h*\nabla h.$$

The proposition now follows from the Cauchy-Schwarz inequality and the Pinching Lemma. q.e.d.

Proposition 15. *For any N_1, $N_2 > 0$ we have the estimates*

$$\frac{\partial}{\partial t}|H|^4 \geq \Delta|H|^2 - 12|H|^2|\nabla H|^2 + \frac{4}{n}|H|^6 \tag{43}$$

$$\begin{aligned}\frac{\partial}{\partial t}&\big((N_1 + N_2|H|^2)|\mathring{h}|^2\big)\\ &\leq \Delta\big((N_1 + N_2|H|^2)|\mathring{h}|^2\big) - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla h|^2\\ &\quad - \frac{4(n-1)}{3n}(N_1 - c_1(N_2))|\nabla h|^2 + c_2(N_1, N_2)|\mathring{h}|^2(|H|^4 + 1),\end{aligned} \tag{44}$$

where c_1 and c_2 depend only on Σ_0, N_1 and N_2.

Proof. The evolution equation for $|H|^4$ is easily derived from that of $|H|^2$:

$$\frac{\partial}{\partial t}|H|^4 = \Delta|H|^2 - 2|\nabla|H|^2|^2 - 4|H|^2|\nabla H|^2 + 4R_2|H|^2.$$

Equation (43) follows from the use of $|\nabla|H||^2 \leq |\nabla H|^2$ and $R_2 \geq 1/n|H|^4$. To prove (44), from the evolution equations for $|h|^2$ and $|H|^2$ we derive

$$\begin{aligned}\frac{\partial}{\partial t}&\big((N_1 + N_2|H|^2)|\mathring{h}|^2\big)\\ &= \Delta\big((N_1 + N_2|H|^2)|\mathring{h}|^2\big)\\ &\quad - 2N_2\big\langle\nabla_i|H|^2, \nabla_i|\mathring{h}|^2\big\rangle - 2N_2|\mathring{h}|^2|\nabla h|^2 + 2N_2R_2|\mathring{h}|^2\\ &\quad - 2(N_1 + N_2|H|^2)\left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2\right)\\ &\quad + 2(N_1 + N_2|H|^2)\left(R_1 - \frac{1}{n}R_2\right).\end{aligned}$$

We estimate the second term on the right as follows:

$$\begin{aligned}-2N_2\big\langle\nabla_i|H|^2, \nabla_i|\mathring{h}|^2\big\rangle &\leq 8N_2|h||\mathring{h}||\nabla H||\nabla h|\\ &\leq 8N_2|H|\sqrt{n}|\nabla h|^2\sqrt{C_0}|H|^{1-\delta/2}\\ &\leq \frac{4(n-1)}{3n}|H|^2|\nabla h|^2 + c_1(N_2)|\nabla h|^2.\end{aligned}$$

Using Young's inequality, $R_2 \leq |h|^2|H|^2$, and $R_1 - 1/n\, R_2 \leq 2|\overset{\circ}{h}|^2|H|^2$ we estimate

$$2N_2R_2|\overset{\circ}{h}|^2 + 2(N_1 + N_2|H|^2)\left(R_1 - \frac{1}{n}R_2\right) \leq c_2(N_1, N_2)|\overset{\circ}{h}|^2(|H|^4 + 1),$$

and equation (44) now follows. q.e.d.

Proof of Theorem 5. Consider $f := |\nabla H|^2 + (N_1 + N_2|H|^2)|\overset{\circ}{h}|^2$. From the evolution equations derived above we see f satisfies

$$\begin{aligned}\frac{\partial}{\partial t}f \leq \Delta f + A|H|^2|\nabla h|^2 - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla h|^2 \\ - \frac{4(n-1)}{3n}(N_1 - c_1(N_2))|\nabla h|^2 + c_2(N_1, N_2)|\overset{\circ}{h}|^2(|H|^4 + 1).\end{aligned}$$

Choose N_2 large enough to consume the positive term arising from the evolution equation for $|\nabla H|^2$. This leaves

$$\begin{aligned}\frac{\partial}{\partial t}f \leq \Delta f - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla h|^2 \\ - \frac{4(n-1)}{3n}(N_1 - c_1(N_2))|\nabla h|^2 + c_2(N_1, N_2)|\overset{\circ}{h}|^2(|H|^4 + 1).\end{aligned}$$

Now consider $g := f - \eta|H|^4$. From the above evolution equations we have

$$\begin{aligned}\frac{\partial}{\partial t}g \leq \Delta g - \frac{4(n-1)}{3n}(N_2 - 1)|H|^2|\nabla h|^2 - \frac{4(n-1)}{3n}(N_1 - c_1(N_2))|\nabla h|^2 \\ + c_2(N_1, N_2)|\overset{\circ}{h}|^2(|H|^4 + 1)\big) + 12\eta|H|^2|\nabla H|^2 - \frac{4\eta}{n}|H|^6.\end{aligned}$$

By choosing N_2 sufficiently large the gradient term on the last line can be absorbed, and then we choose N_1 larger again to make the $|\nabla h|^2$ term negative. We finally discard the negative gradient terms to get

$$\frac{\partial}{\partial t}g \leq \Delta g + c_2(N_1, N_2)|\overset{\circ}{h}|^2(|H|^4 + 1) - \frac{4\eta}{n}|H|^6.$$

Using Theorem 4 and Young's inequality we further estimate

$$\frac{\partial}{\partial t}g \leq \Delta g + c_3,$$

from which we conclude $g \leq c_4$. The gradient estimate now follows from the definition of g. q.e.d.

7. Contraction to a point and convergence

In Section 4 we established that MCF has a unique solution on a finite maximal time interval $0 \leq t < T$ determined by the blowup of the second fundamental form. With the results of the previous two sections in place, we can now show that the diameter of the submanifold approaches zero as $t \to T$, or put another away, the submanifold is shrinking to a point. This combined with Theorem 3 then completes the first part of the Main Theorem.

Theorem 6. *Under the conditions of Theorem* 1*,* $\operatorname{diam}\Sigma_t \to 0$ *as* $t \to T$.

The proof is of course motivated by Hamilton's idea in [**Ha1**] to use Myer's Theorem, however here our pinching condition gives a strictly positive lower bound on the sectional curvature of Σ_t, and we can use Bonnet's Theorem instead. The reader is referred to [**P**, page 170] for a proof of Bonnet's Theorem:

Theorem 7 (Bonnet, Hopf-Rinow, Myers)**.** *Let M be a complete Riemannian manifold and suppose that $x \in M$ such that the sectional curvature satisfies $K \geq K_{min} > 0$ along all geodesics of length $\pi/\sqrt{K_{min}}$ from x. Then M is compact and* $\operatorname{diam} M \leq \pi/\sqrt{K_{min}}$.

We will also need the following result due to Bang-Yen Chen:

Proposition 16. *For $n \geq 2$, if Σ^n is a submanifold of $\mathbb{R}^{n+k}$, then at each point $p \in \Sigma^n$ the smallest sectional curvature K_{min} satisfies*

$$K_{min}(p) \geq \frac{1}{2}\Big(\frac{1}{n-1}|H(p)|^2 - |h(p)|^2\Big).$$

Proof. The proof is a consequence of the Gauss equations and a custom-made inequality and can be found in [**C**, Lemma 3.2]. q.e.d.

Combining this with our pinching assumption we see

$$K_{\min}(p) \geq \frac{1}{2}\Big(\frac{1}{n-1} - c\Big)|H(p)|^2 := \epsilon^2|H(p)|^2 > 0. \tag{45}$$

Lemma 6. *The ratio $|H|_{max}/|H|_{min} \to 1$ as $t \to T$.*

Proof. From Theorem 5 we know that for any $\eta > 0$ there exists a constant $C(\eta)$ such that $|\nabla H| \leq \eta|H|^2 + C(\eta)$ on $0 \leq t < T$. Since $|H|_{\max} \to \infty$ as $t \to T$, there exists a $\tau(\eta)$ such that $C(\eta/2) \leq 1/2\eta|H|^2_{\max}$ for all $\tau \leq t < T$, and so $|\nabla H| \leq \eta|H|^2_{\max}$ for all $t \geq \tau$. For any $\sigma \in (0,1)$ we choose $\eta = \frac{\sigma(1-\sigma)\varepsilon}{\pi}$. Let $t \in [\tau(\eta), T)$, and let x be a point with $|H(x)| = |H|_{\max}$. Then along any geodesic of length $\frac{\pi}{\varepsilon\sigma H_{\max}}$ from x, we have $|H| \geq |H|_{\max} - \frac{\pi}{\varepsilon\sigma|H|_{\max}}\eta|H|^2_{\max} = \sigma|H|_{\max}$, and consequently the sectional curvatures satisfy $K \geq \varepsilon^2\sigma^2|H|^2_{\max}$. The Bonnet Theorem applies to prove that $\operatorname{diam} M \leq \frac{\pi}{\varepsilon\sigma H_{\max}}$, so that $|H|_{\min} \geq \sigma|H|_{\max}$ for $t \in [\tau(\eta), T)$. q.e.d.

Theorem 6 is now also proved in the last line of the proof above, so the first part of Theorem 1 is complete.

We now have all the necessary results in place to proceed as in sections 9 and 10 of [**Hu1**] (see also Section 17 of [**Ha1**]) to obtain smooth convergence of the rescaled maps to a sphere. The reader is referred to these sources for the details.

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