ANTI-SELF-DUAL BIHERMITIAN STRUCTURES ON INOUE SURFACES

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Abstract

In this article we show that any hyperbolic Inoue surface (also called Inoue-Hirzebruch surface of even type) admits anti-self-dual bihermitian structures. The same result holds for any of its small deformations as far as its anti-canonical system is non-empty. Similar results are obtained for parabolic Inoue surfaces. Our method also yields a family of anti-self-dual *hermitian* metrics on any half Inoue surface. We use the twistor method of Donaldson-Friedman [**13**] for the proof.

1. Introduction

Let M be a compact smooth oriented four-dimensional manifold. A *bihermitian structure* on M is a triple $\{[g], J_1, J_2\}$ consisting of a conformal class $[g]$ of a Riemannian metric g and two complex structures $J_i, i = 1, 2$, such that $([g], J_i)$ define a conformal hermitian structure on M, and J_i are compatible with the oritentation of M and are inequivalent to each other in the sense that $J_1 \neq \pm J_2$ when considered as integrable almost complex structures. It is called an *anti-self-dual bihermitian structure* if, further, $(M, [g])$ is an anti-self-dual structure in the sense of [**3**]. Note that such a structure is always (twisted) generalized Kähler in the sense of [**19**] if the hyperhermitian case is excluded, as we do in this paper.

The second-named author [**36**] with a supplement by Dloussky [**12**] has shown the following: Let M be a compact smooth oriented four-manifold admitting an anti-self-dual bihermitian structure $\{[g], J_1, J_2\}$ which is not hyperhermitian. Let $S = (M, J_i)$ be the associated compact complex surface. Then the anti-canonical system $|-K_S|$ admits a disconnected member. In particular if S is minimal, S is either a Hopf surface, a parabolic Inoue surface, or a hyperbolic Inoue surface. In general, S is obtained from its minimal model $\bar{S}$ by blowing up a finite number of points on a fixed anti-canonical divisor. Note that all these surfaces have the underlying C^∞ manifold $M = M[m] := (S^1 \times S^3) \# m\bar{\boldsymbol{P}}^2$ where $\bar{\boldsymbol{P}}^2$ denotes the complex projective plane with reversed oritentation.

In this paper we show the existence of anti-self-dual bihermitian structures on any hyperbolic Inoue surface and also on any of its small deformations preserving the unique anti-canonical divisor on it. This is thought of as a partial converse of the above result of [**36**]. One of our main results is more precisely as follows. Let $\{S, {}^{\mathbf{t}}S\}$ be any pair of hyperbolic Inoue surfaces S and its transposition ${}^{\mathbf{t}}S$ with second Betti number m. Then there exists a family of anti-self-dual bihermitian structures $\{[g]_t, J_t, {}^{\mathbf{t}}J_t\}$ on $M[m]$ with real smooth m-dimensional parameter t such that $(M[m], J_t) \cong S$ and $(M[m], {}^{\mathbf{t}}J_t) \cong {}^{\mathbf{t}}S$ independently of the parameter t (Theorem 7.5). The same result also holds for any properly blown-up hyperbolic Inoue surface (see §3) and for any of its small deformations with an effective (and disconnected) anti-canonical divisor (Theorem 7.6). Moreover, we prove similar results also for parabolic Inoue surfaces (Theorem 7.7). Finally, the same method also yields the existence of a real m-dimensional family of anti-self-dual *hermitian* structures on all (properly blown-up) half Inoue surfaces (Theorem 9.3), which never carry anti-self-dual bihermitian structures.

For the proof we use a variation of the twistor method due to Donaldson-Friedman [**13**] in the spirit of [**25**]. Namely, instead of an anti-self-dual bihermitian triple $\{[g], J_1, J_2\}$, we construct the twistor space corresponding to $[g]$ and two pairs of mutually conjugate elementary, i.e., degree-1, surfaces $\{S_i^+, S_i^-\}$ on it giving rise to $\pm J_i, i = 1, 2$. The twistor spaces associated to self-dual metrics on $m\boldsymbol{P}^2$ constructed by Joyce [**24**], studied in detail in [**15**], play a crucial role in our construction.

All our examples yield anti-self-dual (bi)hermitian—also (twisted) generalized Kähler—and locally conformally Kähler structures, which are new except possibly for the parabolic Inoue case and its deformations, which should be compared with the examples of LeBrun [23].

We now give a brief description of each section: after some preliminaries on deformation theory of pairs of complex spaces in Section 2, we recall in Section 3 basic properties of hyperbolic, half, and parabolic Inoue surfaces and study the Kuranishi family of deformations of associated pairs. All the Inoue surfaces are known to be obtained as a deformation of a singular toric surface $\hat{S}$ [**32**]. In Section 4 we formulate the result in terms of the Kuranishi family of deformations of a pair. This concludes the first half of the paper and will be used constantly in the following sections where we take up our construction of twistor spaces.

First, in Section 5 we recall the basic properties of a Joyce twistor space Z according to [**15**] and explain how to obtain its singular model $\hat{Z}$ together with the natural anti-canonical divisor $\hat{S}$ by analytic modifications. Then, in Section 6 we study the local structures of the singularities of the pair $(\hat{Z}, \hat{S})$ and its automorphism group. We state our main results in Section 7, and their proof will be subsequently provided in

Section 8. The main point is to show the vanishing of the obstructions for smoothing of the pairs $(\hat{Z}, \hat{S})$. Technically, this section is the most delicate part of the paper. In Section 9 we prove that a twisted version of our previous construction yields anti-self-dual *hermitian* metrics on half Inoue surfaces. Finally, in Section 10 we summarize differential geometric implications of our results, including the relations with generalized Kähler and locally conformally Kähler structures.

2. Preliminaries

Sheaves of logarithmic forms. For a complex space X we shall denote by Ω_X the sheaf of germs of holomorphic 1-forms on X and by Θ_X the sheaf of germs of holomorphic vector fields on X. Θ_X is the dual of Ω_X.

Now let Y be a reduced Cartier divisor on X. In our case of interest X is smooth or with at worst normal crossings singularities along a smooth connected hypersuface D and Y has only mild singulariteis.

We define the *sheaf $\Omega_X(\log Y)$ of logarithmic 1-forms* on X along Y to be the sheaf of germs of meromorphic 1-forms ω on X such that for any local equation $f = 0$ of Y in X, both $f\omega$ and $fd\omega$ are holomorphic 1-forms on X. On the other hand, we define the sheaf $\Theta_X(-\log Y)$ of logarithmic vector fields on X along Y to be the sheaf of germs of holomorphic vector fields v on X such that $v(f)/f$ is again holomorphic with f as above, namely, germs of those holomorphic vector fields which are tangent to Y. It is easy to see that they are coherent analytic sheaves on X. At a smooth point of X at which Y has at worst normal crossings singularities, both the sheaves $\Omega_X(\log Y)$ and $\Theta_X(-\log Y)$ are free (see [**9**]).

We also consider the subsheaf

$$\tag{1} \Omega'_X(\log Y)$$

of $\Omega_X(\log Y)$ locally generated by Ω_X and the element df/f for a defining equation $f = 0$ of Y in X.

2.1. Deformation theory. Let X and Y be as above and suppose that they are compact. A *deformation* of the pair (X, Y) is a triple

$$\tag{2} f : (\mathcal{X}, \mathcal{Y}) \to T, \ X_o = X, \ o \in T$$

where f is a flat morphism $\mathcal{X} \to T$ of complex spaces which induces a flat morphism $\mathcal{Y} \to T$; especially $\mathcal{Y}$ is a Cartier divisor on $\mathcal{X}$. A *log-deformation* of the pair (X, Y) is a deformation of (X, Y) such that any local Cartier irreducible component of Y remains locally irreducible under deformations. Especially the number of irreducible components of Y remains the same under deformations.

As usual we have the notions of the Kuranishi family (semiuniversal family) and the universal family of such deformations. Suppose that (2) is a Kuranishi family of log-deformations (resp. deformations) of (X, Y). In this case we call the base space T the associated *Kuranishi space*. The cohomological description of the infinitesimal deformation spaces of (X, Y), the obstrcution space for the deformations and the space of infinitesimal automorphisms, are given by the following:

Proposition 2.1. *The Kuranishi space T is smooth if $Ext^2_{O_X}(\Omega_X (\log Y), O_X) = 0$ (resp. $Ext^2_{O_X}(\Omega'_X(\log Y), O_X) = 0$). $Ext^1_{O_X}(\Omega_X(\log Y), O_X)$ (resp. $Ext^1_{O_X}(\Omega'_X(\log Y), O_X)$) is naturally identified with the tangent space of T at the reference point. Moreover, if X is weakly normal, i.e., the Riemann extention theorem holds on X, we have*

$$Ext^0_{O_X}(\Omega_X(\log Y), O_X) = Ext^0_{O_X}(\Omega'_X(\log Y), O_X) = H^0(X, \Theta_X(-\log Y)),$$

and if these vector spaces vanish, the Kuranishi family (2) is universal in a small neighborhood of o.

In order to determine these vector spaces we use the local to global spectral sequence for Ext functors. For instance in the case of log-deformations this takes the form

(3)
$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q_{O_X}(\Omega_X(\log Y), O_X)) \Longrightarrow Ext^{p+q}_{O_X}(\Omega_X(\log Y), O_X),$$

giving rise to the five-term exact sequence in case X is weakly normal:

$$
\begin{aligned}
(4) \qquad 0 \;\to\;\; & H^1(X, \Theta_X(-\log Y)) \to Ext^1_{O_X}(\Omega_X(\log Y), O_X) \\
\to\;\; & H^0(X, \mathcal{E}xt^1_{O_X}(\Omega_X(\log Y), O_X)) \\
\to\;\; & H^2(X, \Theta_X(-\log Y)) \xrightarrow{c} Ext^2_{O_X}(\Omega_X(\log Y), O_X).
\end{aligned}
$$

2.2. A general lemma on Ext. Let X be a complex space and Y a Cartier divisor on X. Let $N = [Y]|Y$ be the normal bundle of Y in X. Then for any coherent analytic sheaf F on Y we have the following comparison theorem of Ext.

Lemma 2.2. *The notations being as above we have the following natural isomorphisms:*

$$(5) \qquad\qquad Ext^i_{O_X}(F \otimes N, O_X) \cong Ext^{i-1}_{O_Y}(F, O_Y), \quad i \geq 0,$$

$$(6) \qquad\qquad \mathcal{E}xt^i_{O_X}(F \otimes N, O_X) \cong \mathcal{E}xt^{i-1}_{O_Y}(F, O_Y), \quad i \geq 0,$$

where the second is an isomorphism of O_X-modules.

Proof. Applying [**1**, p.72,Proposition 2.9] for $E = N$ and $G = O_X$ in the notation there, we obtain a spectral sequence

$$(7) \quad E_2^{p,q} := Ext_{O_Y}^p(F, \mathcal{E}xt_{O_X}^q(N, O_X)) \Longrightarrow Ext_{O_X}^{p+q}(F \otimes N, O_X).$$

Since Y is a Cartier divisor in X, we have $\mathcal{E}xt_{O_X}^q(N, O_X) = 0, q \neq 1$, and for $q = 1$,

$$\mathcal{E}xt_{O_X}^1(N, O_X) \cong \mathcal{E}xt_{O_X}^1(O_X(Y) \otimes_{O_X} O_Y, O_X)$$

$$\cong \mathcal{E}xt_{O_X}^1(O_Y, O_X) \otimes_{O_X} O_X(-Y) \cong N \otimes_{O_X} O_X(-Y) \cong O_Y$$

(see [**1**, p.74,Proposition 3.4] for the third isomorphism). Thus (7) yields the desired results.   q.e.d.

2.3. Automorphism groups. For a compact complex space X and its subspace Y, denote by $\mathrm{Aut}(X, Y)$ the group of automorphisms of X preserving Y. This has a natural structure of a complex Lie group with its Lie algebra naturally identified with $H^0(\Theta_X(-\log Y))$. We denote by $\mathrm{Aut}_0(X, Y)$ the identity component of $Aut(X, Y)$.

2.4. Cycle of rational curves. A *cycle of rational curves* on a smooth surface is a compact connected curve C which either is an irreducible rational curve with a single node or is a reducible curve with k nodes whose irreducible components are nonsingular rational curves $C_i, 1 \leq i \leq k, k \geq 2$, such that C_i and C_{i+1} intersect at a single point and there exist no other intersections, where $C_{k+1} = C_1$ by convention. We write such a C as $C = C_1 + \cdots + C_k$. Then the sequence $(b_1, \ldots, b_k)$ of opposite self-intersection numbers $b_i = -(C_i)^2$ is called the *weight sequence* of C, which are considered modulo cyclic permutations and reversing of the order.

2.5. Toric surfaces. Let $G = \boldsymbol{C}^{*2}$ be the algebraic two-torus. Let S be a projective toric surface with fixed G-action with open orbit U. The complement $C := S - U$ forms a cycle of rational curves $C = C_1 + \cdots + C_k$ and is an element of the anti-canonical system $|-K_S|$ of S. We shall call C the *anti-canonical cycle* of S and denote the toric surface also by the pair (S, C). In this case $\Theta_S(-\log C)$ is free, i.e.,

$$\Theta_S(-\log C) \cong O_S^2.$$

The weight sequence of C is also called the weight sequence of S.

2.6. Notation. For a sheaf F on a complex space X we set $h^i(X, F) = \dim H^i(X, F)$ for any integer i.

3. Inoue surfaces

Let S be a compact connected complex surface. It is called a surface *of class VII* if its first Betti number $b_1 = 1$ and its Kodaira dimension $\kappa = -\infty$. It is called *of class VII$_0$* if it is further minimal, i.e., contains no (-1)-curves. (A (-1)-*curve* is a nonsingular rational curve with self-intersection number -1.)

3.1. Hopf surfaces. S is called a *Hopf surface* if its universal covering is isomorphic to $\boldsymbol{C}^2 - 0$. A Hopf surface is a surface of class VII_0 with second Betti number $b_2 = 0$. It is called a *diagonal Hopf surface* if it is isomorphic to the quotient of $\boldsymbol{C}^2 - 0$ by an infinite cyclic group generated by a transformation of the form $(z, w) \to (\alpha z, \beta w)$, $0 < |\alpha|, |\beta| < 1$. Such a surface is diffeomorphic to the product of spheres $S^1 \times S^3$.

In this case the images of $\{z = 0\}$ and $\{w = 0\}$ give two nonsingular elliptic curves E_1 and E_2 on S. We use the following characterization of a diagonal Hopf surface, which is due to Kato-Nakamura [**32**, Theorem 5.2] when the algebraic dimension $a(S) = 0$, and is due to Kodaira [**27**, Theorems 28, 31] when $a(S) = 1$. (Note the difference of the definition of class VII_0 here and the one originally given by Kodaira.)

Lemma 3.1. *Let S be a surface of class VII_0 with infinite cyclic fundamental group. If S contains two smooth elliptic curves, S is a diagonal Hopf surface.*

3.2. Inoue surfaces. Denote by VII_0^+ the class of surfaces of class VII_0 with positive second Betti number. The first examples of surfaces of class VII_0^+ were discovered by Inoue [**22**, **23**], which we shall divide into three classes according to Nakamura [**32**] as those of hyperbolic, half, and parabolic Inoue surfaces. (The first two surfaces are also called *Inoue-Hirzebruch surfaces*.) For the purpose of this paper it is most convenient to use the characterization of these surfaces similar to Lemma 3.1 due to Nakamura [**32**, (8.1), (7.1), (9.2)]) as their definitions.

Definition. Let S be a surface of class VII_0^+ with $m := b_2 > 0$. S is called a *hyperbolic* (resp. *parabolic*) *Inoue surface* if S contains two cycles of rational curves (resp. one cycle of rational curves and a nonsingular elliptic curve E). In both cases we denote by C the union of all these curves. S is called a *half Inoue surface* if it contains a cycle C of rational curves with $C^2 < 0$ such that the number of its irreducible components equals m.

In all cases there are no other curves on the surface and C is the unique maximal (reduced) curve on S. In the hyperbolic or half Inoue case the number of irreducible components in C equals m, and in the parabolic case the number of irreducible components of the unique cycle equals m.

All the known examples of surfaces of class VII_0^+ including Inoue surfaces are orientation-preservingly diffeomorphic to

$$M[m] := (S^1 \times S^3) \# m \bar{\boldsymbol{P}}^2$$

where $\bar{\boldsymbol{P}}^2$ is the complex projective plane with orientation reversed; in fact they are all obtained as complex-analytic deformations of a blown-up Hopf surface. In particular they all have infinite cyclic fundamental groups.

Starting from any of the Inoue surfaces, we obtain other surfaces of class VII by blowing up successively the nodes of the cycles of rational curves on them. These surfaces again contain one or two cycles of rational curves, or one cycle of rational curves and a nonsingular elliptic curve. We call such surfaces *properly blown-up* hyperbolic, half, or parabolic Inoue surfaces. (We include the case where the blowing-down is trivial; thus in this terminology a hyperbolic Inoue surface is also a properly blown-up hyperbolic Inoue surface.) The reason why we consider these surfaces as well is that they arise equally naturally in our construction in Section 5, while from the viewpoint of anti-self-dual or bihermitian structures the minimality of the surfaces should be irrelevant.

3.3. Anti-canonical curves. We call a member C of the anti-canonical system $|-K_S|$ of a surface S an *anti-canonical curve*. In this case we write simply $-K_S = C$ or $K_S + C = 0$. For a diagonal Hopf surface S we have by [**27**, (96)]

$$(8) \qquad\qquad -K_S = E_1 + E_2$$

in the previous notation and for a properly blown-up hyperbolic or parabolic Inoue surface

$$(9) \qquad\qquad -K_S = C$$

for the unique maximal curve C on it (see [**32**] for the minimal case; the general case is immedeately deduced from this case.) In particular any diagonal Hopf surface or hyperbolic or parabolic Inoue surface admits a disconnected anti-canonical curve. We may speak of *the* anti-canonical curve on a hyperbolic or parabolic Inoue surface. In the half Inoue case $|-K_S|$ is empty, but C becomes the unique member of the system $|-(K_S+L)|$, where L is the unique non-trivial holomorphic line bundle on S with $L^2(=2L)$ trivial. (We call C L-*twisted* anti-canonical curve.)

The following result, originally due to Nakamura, is crucial for our whole investigation. (See Section 10 for a proof.)

Lemma 3.2. *Let S be a compact complex surface of class VII_0 with infinite cyclic fundamental group. Suppose that there exists a disconnected anti-canonical curve on S. Then S is either a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface.*

Let S be as in the lemma and C an anti-canonical curve on it. Let $h : \tilde{S} \to S$ be the blow-up of a finite number of points on C with exceptional curve B. Then by the adjunction formula

$$(10) \qquad\qquad -K_{\tilde{S}} = h^*(-K_S) - B.$$

$\tilde{S}$ also admits a disconnected anti-canonical curve $\tilde{C}$ which is mapped surjectively onto C. From (10) we also deduce the following lemma, which extends Lemma 3.2 to the non-minimal case (see [**36**, Cor.3.14]).

Lemma 3.3. *Let S be a compact complex surface of class VII with infinite cyclic fundamental group. Suppose that there exists a disconnected anti-canonical curve C on S. Then the minimal model $\bar{S}$ of S is either a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface, and $S \to \bar{S}$ is obtained by blowing up $\bar{S}$ at a finite number of points (possibly infinitely near) of the image $\bar{C}$ of C. Moreover, $\bar{C}$ is an anti-canonical curve on $\bar{S}$.*

Proof. Let $h : S \to \bar{S}$ be the blowing-down map to the minimal model. We show that $\bar{C}$ is a disconnected anti-canonical curve on $\bar{S}$. We have only to prove this in the case of one point blown-up; the general case then follows by induction. By (10) we get $|-K_{\tilde{S}}| = |-K_S(-x)|$, i.e., an anti-canonical curve on $\bar{S}$ is identified with a anti-canonical curve on S which passes through the blown-up point x. In particular $\bar{C}$ is an anti-canonical curve and $x \in \bar{C}$ and $C = \bar{C}$ in the above correspondence. Suppose that $\bar{C}$ is connected. If x is a smooth point of $\bar{C}, C$ is a proper transform of $\bar{C}$ and is connected. If x is a singular point of $\bar{C}$, B is in the support of C and intersects with the proper transform of every branch passing through x. Thus C is again connected. This shows that under our assumption $\bar{C}$ must be disconnected. Then applying Lemma 3.3 we obtain the possible structures of the surface $\bar{S}$.   q.e.d.

We next state an analogue of the above result in the half Inoue case. Let S be a compact complex surface of class VII with infinite cyclic fundamental group. Then there exists a unique non-trivial holomorphic line bundle L with $L^2 = 1$. We denote this line bundle by $L = L_S$ in what follows. We have the associated unramified double covering $u : \tilde{S} \to S$ such that u^*L is trivial.

Lemma 3.4. *Suppose that S contains an L-twisted connected anti-canonical curve C such that $u^{-1}(C)$ is disconnected in $\tilde{S}$. Then the minimal model $\bar{S}$ of S is a half Inoue surface or a diagonal Hopf surface, and S is obtained by blowing up $\bar{S}$ at a finite number of points (possibly infinitely near) on the image $\bar{C}$ of C.*

The proof is easily obtained by passing to $\tilde{S}$ associated to L and then applying Lemma 3.3. But we refer the detailed proof with more precise structures of S in this special case to the short note [**18**].

3.4. Transpositions of hyperbolic Inoue surfaces. In [**45**] Zaffran defined for any hyperbolic Inoue surface S its transposition $^{\mathbf{t}}S$, which is again a hyperbolic Inoue surface with $^{\mathbf{t}}(^{\mathbf{t}}S) = S$. We shall recall its definition and basic properties.

Let S be a hyperbolic Inoue surface. Let $C^{\alpha}, \alpha = 1, 2$, be the two cycles of rational curves on S. Then there is a geometric way of choosing one of the two (cyclic) numberings of the irreducible components of each

C^α up to cyclic permutations due to Dloussky [**10**], which we shall now explain. We shall call this a *canonical* numbering for C^α.

In general, a domain D in S is called a *spherical shell* if it is isomorphic to a domain in $\boldsymbol{C}^2$ bounded by two concentric spheres. It is called *global* if $S - D$ is connected. D has thus two boundaries $\partial_+ D$ and $\partial_- D$ which are strictly pseudoconvex and pseudoconcave, respectively.

We use the characterization of the canonical numbering in the form of the following lemma (see [**10**]). Let $B_1 + \cdots + B_{h_\alpha}$ be a cyclic numbering of the irreducible components of C^α. We assume that $h_\alpha > 1$ since otherwise the numbering in question is unique.

Lemma 3.5. *Suppose that there exists a global spherical shell D in S which intersects with C^α in a domain U in B_1. Among the two connected components of $B_1 - U$, let V be the component which contains $B_1 \cap B_2$. Then the numbering above is canonical if $\partial V = \partial D_- \cap B_1$ (instead of $\partial D_+ \cap B_1$).*

Accordingly we may also speak of the *canonical* weight sequence of each C^α up to cyclic permutations. We further recall the following facts:

(a) Let S and S' be hyperbolic Inoue surfaces with two cycles of rational curves C^α and C'^α respectively, $\alpha = 1, 2$. Then S and S' are isomorphic if and only if the canonical weight sequences of C^α and C'^α coincide up to cyclic permutations for one (and then both) of $\alpha = 1, 2$, after interchanging C^1 and C^2 if necessary. (See Remark 1.1 of [**45**].)

(b) For any hyperbolic Inoue surface S there exists up to isomorphisms a unique hyperbolic Inoue surface S' such that the canonical weight sequences of C^α and C'^α for one (and then both) of $\alpha = 1, 2$ are reverse to each other up to cyclic permutations, after interchanging C^1 and C^2 if necessary. S' is called the *transposition* of S and is denote by $^{\mathbf{t}}S$. (See [**45**].)

The *transposition* of a half Inoue surface is defined similarly, considering only the unique cycle instead of two cycles. By reducing to the corresponding minimal model we can also speak of the notion of transpositions of properly blown-up hyperbolic or half Inoue surfaces.

3.5. Weight sequences. Let S be a hyperbolic Inoue surface with second Betti number m, and $C^\alpha, \alpha = 1, 2$, the two cycles of rational curves on S. The weight sequences of C^1 and C^2 are of the following form up to cyclic permutations and the interchange of C^α:

(11) $$(k_1 + 2, [k_2 - 1], \ldots, k_{2n-1} + 2, [k_{2n} - 1]),$$

(12) $$([k_1 - 1], k_2 + 2, \ldots, [k_{2n-1} - 1], k_{2n} + 2),$$

where n and $k_i, 1 \le i \le 2n$, are positive integers and for a positive integer l, $[l]$ stands for the sequence $(2, \ldots, 2)$ (l times), while $[0]$ denotes the empty sequence [**32**, (6.8) Theorem](see [**45**, (2) and (3)]). However, the case $n = 1$ and k_1 (resp. k_2) $= 1$ is exceptional; in this case we

should replace $k_2 + 2$ (resp. $k_1 + 2$) by k_2 (resp. k_1) [**32**, (1.4)]. Note that $m = \sum_{1 \leq i \leq 2n} k_i$. Conversely, given n and k_i arbitrarily as above, there exists a hyperbolic Inoue surface with $b_2 = m$ and with the above weight sequences. Hyperbolic Inoue surfaces are determined by the pair of weight sequences as above up to at most two non-isomorphic surfaces which are transpositions of each other. (The last statements holds true also for the properly blown-up case.) In particular there are only a countable number of hyperbolic surfaces up to isomorphisms.

3.6. Isomorphism classes of parabolic Inoue surfaces. For a parabolic Inoue surface S with second Betti number m, the weight sequence of its unique cycle is given by $[m]$ for $m > 1$ and 0 for $m = 1$ in the above notation, while the elliptic curve E on it has the self-intersection number $E^2 = -m$.

Parabolic Inoue surfaces with fixed second Betti number are parametrized by the punctured unit disc $D^* = \{|d| < 1\}$ (see [**32**, (1.1)]). So we may write $S = S_d$ for some $d \in D^*$. The parameter d is geometrically interpreted as follows. Let $u : U \to S$ be the universal covering of S. Then for the unique elliptic curve E in S, we get an infinite cyclic covering $v : \tilde{E} := u^{-1}(E) \to E$. Let γ be a fixed generator of the covering transformation group. Then there exists a unique complex number α with $0 < \alpha < 1$ such that with respect to an isomorphism $w : \tilde{E} \xrightarrow{\sim} \boldsymbol{C}^* = \boldsymbol{C}^*(s)$, γ takes the form $\gamma(s) = \alpha s$. This number α is independent of the choice of γ, and the isomorphism w and depends only on the isomorphism class of v. In fact, if $S = S_d$, the construction [**32**] clearly shows that $d = \alpha$. In particular we get:

Lemma 3.6. *Let S be a parabolic Inoue surface with fixed second Betti number m, and $v : \tilde{E} \to E$ as above. Then the isomorphism class of S is determined by the isomorphism class of the infinite cyclic covering v. D^* is thus the moduli space of parabolic Inoue surface.*

3.7. Real structure on Inoue surfaces. Let J be a complex structure on a smooth manifold M. Let $S := (M, J)$ be the resulting complex manifold and $\bar{S} := (M, -J)$ its complex conjugate. Then S and $\bar{S}$ are biholomorphic if and only if S admits an anti-holomorphic diffeomorphism; in particular a real structure, i.e., an anti-holomorphic involution. In this context we note the following:

Lemma 3.7. *Any hyperbolic, half, or parabolic Inoue surface S has a natural real structure. The same is true for a proper blowing-up of any such surface.*

Proof. The universal covering U of S is covered by coordinate neighborhoods, such that in the intersection of any two of them the two coordinates are related by Laurent *monomials* (see [**32**, §1]). Hence the complex conjugations with respect to each such coordinates are compatible in the intersections and give a real structure $\tilde{\mu}$ on U. Moreover,

a generating covering transformation of $U \to S$ is also given by Laurent monomials (see [**32**, §1]) and hence the real structure $\tilde{\mu}$ descends to a real structure μ on S. Moreover, since the nodes of the anti-canonical divisors of these surfaces are fixed points of the real structure, μ lifts to its proper blowing-ups.   q.e.d.

3.8. Deformations of Inoue surfaces. Let S be a properly blown-up hyperbolic, half, or parabolic Inoue surface. For brevity we refer to the hyperbolic (resp. half, resp. parabolic) case as *Case-H* (resp. *Case-H′*, resp. *Case-P*) in what follows. Let C be the unique maximal curve on S. In other words, C is the unique anti-canonical curve in Case-H and -P, while it is the unique L-twisted anti-canonical curve in Case H′ where $L = L_S$.

Let $n : \tilde{C} := \coprod_{1 \le d \le b} \tilde{C}_d \to C$ be the normalization of C, where $\tilde{C}_d$ are the normalizations of the irreducible components of C, and $b = m$ in Case-H or -H′ and $= m+1$ in Case-P. It is known that $\mathrm{Aut}_0(S, C) = \{e\}$ in Case-H or -H′ and $\boldsymbol{C}^*$ in Case-P (see [**11**, Prop.2.5] and [**35**] for Case-H and -H′ and [**20**] for Case-P).

Lemma 3.8. *In Case-H or -H′ we have $h^i(\Theta_S(-\log C)) = 0$ for $i = 0, 1, 2$, while in Case-P we have $h^i(\Theta_S(-\log C)) = 1$ for $i = 0, 1$ and $= 0$ for $i = 2$.*

Proof. First we consider Case-H and -P. We have the exact sequence

$$(13) \quad 0 \longrightarrow \Theta_S(-C) \longrightarrow \Theta_S(-\log C) \longrightarrow \Theta_C \longrightarrow 0$$

and the isomorphisms

$$\Theta_C \cong \oplus_d n_* \Theta_{\tilde{C}_d}(-(0_d + \infty_d))) \cong \oplus_d n_* O_{C_d}$$

where 0_d and ∞_d are the inverse images of the nodes of C in C_d. Since $C = -K_S$ by (9), we have $\Theta_S(-C) \cong \Omega_S$ so that $h^i(\Theta_S(-C)) = h^i(\Omega_S) = h^{1,i}$, where $h^{p,q}$ denote the Hodge numbers. Hence we get $h^i(\Theta_S(-C)) = 0$ for $i = 0, 2$ and $h^1(\Theta_S(-C)) = h^{1,1} = b_2(S) = m$. Thus taking the long exact sequence associated to (13), we obtain $h^2(\Theta_S(-\log C)) = 0$ (see [**33**, Th.1.3]) and the exact sequence
$$(14)$$
$$0 \longrightarrow H^0(\Theta_S(-\log C)) \longrightarrow \boldsymbol{C}^b \longrightarrow \boldsymbol{C}^m \longrightarrow H^1(\Theta_S(-\log C))$$

$$\longrightarrow \qquad \boldsymbol{C}^k \qquad \longrightarrow \quad 0$$

where $k = 0$ in Case-H and $= 1$ in Case-P. On the other hand, as we noted before the lemma $h^0(\Theta_S(-\log C)) = 0$ in Case-H and $= 1$ in Case-P. Thus the lemma is proved in these cases.

In Case-H′ take the canonical unramified double covering $(\tilde{S}, \tilde{C}) \to (S, C)$ with covering involution ι. Then $H^i(\Theta_S(-C))$ are naturally identified with the subspaces $H^i(\Theta_{\tilde{S}}(-\tilde{C}))^\iota$ of ι-fixed elements for all i. From this the results follow from those in Case-H.   q.e.d.

As follows from the above proof in Case-P, the restriction map $\Theta_S(-\log C) \to \Theta_E$ induces a natural isomorphism

$$(15) \qquad H^1(\Theta_S(-\log C)) \cong H^1(\Theta_E)$$

where E is the elliptic component of C.

The two sheaves $\Omega'_S(\log C)$ (see (1)) and $\Omega_S(\log C)$ coincide except at the nodes of C. Let B be the set of nodes of C. Then more precisely we have the following:

Lemma 3.9. *We have an exact sequence*

$$(16) \qquad 0 \to \Omega'_S(\log C) \to \Omega_S(\log C) \to \oplus_{p \in B} \boldsymbol{C}_p \to 0,$$

where $\boldsymbol{C}_p$ is the skyscraper sheaf with support p and with fiber $\boldsymbol{C}$.

Proof. The fact that the quotient $\Omega_S(\log C)/\Omega'_S(\log C)$ is isomorphic to $\boldsymbol{C}_p$ at each $p \in B$ is seen by checking the image of $\Omega'_S(\log C)$ by the Poincare residue map $P : \Omega_S(\log C) \to O_1 \oplus O_2$, where $O_s, s = 1, 2$, are the structure sheaves of the two irreducible components of C at p. In the local model $(\boldsymbol{C}^2(x, y), xy = 0)$ of (S, C), P takes the form $a(x, y)dx/x + b(x, y)dy/y \to (a(x, 0), b(0, y)) \in O_1 \oplus O_2$, while the elements $\Omega'_S(\log C)$ are of the form $f(x, y)(dx/x + dy/y)$ so that their images are given by $(f(x, 0), f(0, y))$. The assertion thus holds.   q.e.d.

Since $\mathcal{E}xt^i_{O_S}(\boldsymbol{C}_p, O_S) = 0$ for $i \neq 2$ and $= \boldsymbol{C}$ for $i = 2$, by applying $\mathcal{E}xt^i_{O_S}(-, O_S)$ to the sequence (16) we have the following:

Corollary 3.10. *(1)* $\mathcal{H}om_{O_S}(\Omega'_S(\log C), O_S) \cong \Theta_S(-\log C)$.
(2) There is a natural isomorphism

$$(17) \qquad \mathcal{E}xt^1_{O_S}(\Omega'_S(\log C), O_S) \cong \oplus_{p \in B} \boldsymbol{C}_p.$$

(3) $\mathcal{E}xt^i_{O_S}(\Omega'_S(\log C), O_S) = 0$ *for $i \geq 2$.*

In view of Corollary 3.10 and Lemma 3.8, the local-to-global spectral sequence for $Ext^i_{O_S}(\Omega'_S(\log C), O_S)$ yields the following:

Lemma 3.11. *We have*

$$(18) \qquad Ext^2_{O_S}(\Omega'_S(\log C), O_S) = 0,$$

$$(19)$$
$$0 \to H^1(\Theta_S(-\log C)) \to Ext^1_{O_S}(\Omega'_S(\log C), O_S) \to \oplus_{p \in B} \boldsymbol{C}_p \to 0,$$

$$(20) \qquad Ext^0_{O_S}(\Omega'_S(\log C), O_S) = H^0(\Theta_S(-\log C)),$$

where the sequence (19) is exact.

Let

$$(21) \qquad g : (\mathcal{S}, \mathcal{C}) \to T, \ (S_o, C_o) = (S, C), \ o \in T$$

be the Kuranishi family of deformations of the pair (S, C). Also, for any $p \in B$, we denote by

$$g_p : \mathcal{C}(p) \to T_p, \ (C_o, o) \cong (C, p), \ o \in T_p$$

the Kuranishi family of deformations of the isolated singularity (C, p), where T_p is smooth of dimension one. Any deformation of (S, C) induces a deformation of (C, p), and correspondingly we have a versal map $\tau_p : T \to T_p$. The fiber $T(p) := \tau_p^{-1}(o)$ is uniquely determined independently of the choice of τ_p. Thus a point $t \in T$ is outside of $T(p)$ precisely when the two irreducible components of C passing through p are merged together in C_t to become one smooth curve locally.

In Case-P we also consider the Kuranishi family

$$e : \mathcal{E} \to T_E, \ E_o = E, \ o \in T_E$$

of the elliptic curve E in S, where T_E is smooth of dimension one. Since a deformation of (S, C) induces a deformation of E, we have the (unique) versal map $\tau_E : T \to T_E$.

In Case-H and -P we write C as the disjoint union $C = C^1 \cup C^2$ of two curves, where $C^\alpha, \alpha = 1, 2$, are cycles of rational curves in Case-H, and C^1 is a cycle of rational curves and $C^2 = E$ in Case-P. The fiber $C_t, t \in T$, is similarly a disjoint union $C_t = C_t^1 \cup C_t^2$, where C_t^α is either a cycle of rational curves or a smooth elliptic curve. Recall that $\#B = m$.

Proposition 3.12. *Let (S, C) be as above. Then the Kuranishi space T is smooth of dimension m (resp. $m + 1$) in Case-H or -H′ (resp. -P). Moreover, in each case we have the following:*

(a) Case-H or -H′: The product map

$$\Pi_p \tau_p : T \to \Pi_{p \in B} T_p$$

is isomorphic; in particular $T(p)$ is a smooth hypersuface in T for each $p \in B$. The family is universal at each point of T. Accordingly, $\dim Ext^1_{O_{S_t}}(\Omega'_{S_t}(\log C_t), O_{S_t}) = m$ independently of $t \in T$.

(b) Case-P: The product map

$$\Pi_p \tau_p \times \tau_E : T \to \Pi_p T_p \times T_E$$

is isomorphic. The family is not universal. In fact, $h^0(\Theta_{S_t}(-\log C_t)) = 1$ (resp. $= 0$) and $\dim Ext^1_{O_{S_t}}(\Omega'_{S_t}(\log C_t), O_{S_t}) = m + 1$ (resp. $= m$) for for $t \in I$ (resp. $\notin I$), where $I := (\Pi_p \tau_p)^{-1}(o), o \in T_E$, is a submanifold of dimension one.

Proof. The smoothness of T follows from (18). We have $\dim T = \dim Ext^1_{O_S}(\Omega'_S(\log C), O_S)$, and the latter is identified with the claimed value by Lemmas 3.8 and 3.11. By Lemma 3.8 and the upper semicontinuity of cohomology, we have $h^0(\Theta_{S_t}(-\log C_t)) = 0$ independently of t in Case-H or -H′. Hence this family is universal at each point of T.

The third arrow of (19) is identified with the differential of $\Pi_p \tau_p$ at the base point. The first assertion of (a) follows from this.

In Case-P, $\Pi_p \tau_p$ is a submersion and the inverse image I of the reference point is identified with the local moduli space of S as a parabolic Inoue surface, whose tangent space is identified with $H^1(\Theta_S(-\log C))$. The differential of the restriction of τ_E to I is identified with the isomorphism (15). From this we get the first assertion in (b). The rest follows from the fact that $\mathrm{Aut}_0(S_t, C_t) = \{e\}$ for any $t \notin I$.   q.e.d.

Next, we assume that S is a properly blown-up hyperbolic or parabolic Inoue surface. We write the set B of nodes as the disjoint union $B = B_1 \cup B_2$ in the obvious way, where $B_2 = \emptyset$ in Case-P. Then on the structure of the surfaces S_t in the family we have the following proposition.

Proposition 3.13. *Let S be as above.*
(1) For any $t \in T$, C_t is the unique anti-canonical curve on S_t.
(2) If $t \notin T(p)$ for some $p \in B$ (i.e., $t \neq o$ in Case-H), S_t is not minimal and its minimal model $\bar{S}_t$ is either a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface.
(3) $\bar{S}_t$ is a diagonal Hopf surface if and only if $t \notin T(p)$ for any $p \in B$. $\bar{S}_t$ is a a parabolic Inoue surface if and only if $t \notin T(p)$ for any $p \in B_\alpha$ for one of α $(= 1$ or $2)$, but not for both in Case-H (resp. $t \in T(p)$ for some $p \in B_1$ in Case-P).

Remark 3.1. We may call (S, C) an *anti-canonical pair* in the sense that C is an anti-canonical curve on S. The above lemma implies that the Kuranishi family (21) of (S, C) is actually a Kuranishi family of (S, C) as an anti-canonical pair. Thus we can identify our Kuranishi family with the family constructed by Nakamura in Lemma 5.7 of [**32**]. (Indeed, we can show that the family (21) is realized as a subfamily of the Kuranishi family of S itself.) However, in [**32**] neither the smoothness of T nor the precise structures of T as above is clear.

Proof. (1) Consider the short exact sequence

$$0 \longrightarrow O_{S_t}(-C_t) \longrightarrow O_{S_t} \longrightarrow O_{C_t} \longrightarrow 0$$

and the associated long exact sequence

$$\longrightarrow H^1(O_{S_t}) \longrightarrow H^1(O_{C_t}) \longrightarrow H^2(O_{S_t}(-C_t)) \longrightarrow .$$

Together with Serre duality and the upper semicontinuoity of cohomology, this yields

$$2 = h^1(O_{C_t}) \leq h^1(O_{S_t}) + h^0(K_t + C_t) \leq 1 + h^0(K + C) = 2.$$

Since $h^1(O_{S_t}) = 1$, we get that $h^0(K_t + C_t) = 1$ for all t. Then any non-vanishing element u_0 of $H^0(K + C)$ extends locally to elements u_t of $H^0(K_t + C_t)$, which is again non-vanishing since so is u_0. Thus

$K_t + C_t = 0$ as desired. The uniqueness then follows from the inequality $h^0(-K_t) \le h^0(-K) = 1$.

Assertion (2) in Proposition 3.13 is then a consequence of Lemma 3.3.

(3) $C_t^\alpha, \alpha = 1, 2$, is a smooth elliptic curve if and only if all the nodes of C^α are smoothed in the deformation C_t^α, and this is precisely the condition that $t \notin T(p)$ for all $p \in B_\alpha$. From this the conclusion follows from Lemma 3.1.   q.e.d.

When S is a properly blown-up half Inoue surface, the statement analogous to Proposition 3.13 is given as follows:

Proposition 3.14. *Suppose that S is a properly blown-up half Inoue surface. Then:*

(1) For any $t \in T$, C_t is the unique L_t-twsited anti-canonical curve on S_t, where $L_t = L_{S_t}$.

(2) If $t \ne o$, S_t is not minimal and its minimal model $\bar{S}_t$ is either a half Inoue surface or a diagonal Hopf surface.

(3) $\bar{S}_t$ is a diagonal Hopf surface if and only if $t \notin T(p)$ for any $p \in B$.

As in the case of Lemma 3.4, we refer the proof of this proposition to the short note [**18**].

4. Deformations of rational surface with a nodal curve

Let $\tilde{S}$ be a projective toric surface acted by $G := \boldsymbol{C}^{*2}$, and $\tilde{C} = \tilde{C}_1 + \cdots + \tilde{C}_{k+2}$ the anti-canonical cycle on $\tilde{S}$, where we assume for simplicity that $k > 2$. We put $\infty_i = 0_{i+1} = \tilde{C}_i \cap \tilde{C}_{i+1}$, where the subscripts are considered cyclically modulo $k + 2$.

Suppose that there exist disjoint irreducible components H and E of $\tilde{C}$ with $H^2 = 1$ and $E^2 = -1$, respectively. (In this case we call the toric surface (S, C) *admissible*.) Take an isomorphism

$$(22) \quad \varphi : (H, 0_H, \infty_H) \to (E, \infty_E, 0_E) \text{ or } (H, 0_H, \infty_H) \to (E, 0_E, \infty_E)$$

where $0_H = 0_i$ if $H = \tilde{C}_i$, etc. In the latter case we call φ *of twisted type* and in the former case *of untwisted type*. Let $\hat{S}$ be the non-normal surface obtained by identifying the points $x \in H$ with $\varphi(x) \in E$.

Let $n : \tilde{S} \to \hat{S}$ be the natural map and denote the singular locus of $\hat{S}$ by $\hat{F} = n(H) = n(E) \cong \boldsymbol{P}$. Let $\tilde{C}^\alpha, \alpha = 1, 2$, be the connected components of the union of the irreducible components of $\tilde{C}$ other than H and E. Denote by $\hat{C}^\alpha$ their images in $\hat{S}$, and put $\hat{C} = \hat{C}^1 \cup \hat{C}^2$. When φ is of untwisted type, $\hat{C}^\alpha$ are disjoint and each forms a cycle of rational curves on $\hat{S}$, while when φ is of twisted type, $\hat{C}$ is connected and forms a single cycle of rational curves on $\hat{S}$. In both cases $\hat{C}$ is an anti-canonical curve on $\hat{S}$ (see Lemma 6.1 below).

In this section we study the smoothing of the singular surface $\hat{S}$ under deformations. This subject was studied extensively by Nakamura in [**31**, **32**, **33**]. However, since we treat it from a little different point of view, we will describe some details here.

4.1. Automorphism groups. We compute the identity component of the automorphism group of $(\hat{S}, \hat{C})$.

Lemma 4.1. *We have*

$$\mathrm{Aut}_0(\hat{\mathrm{S}}, \hat{\mathrm{C}}) \cong \boldsymbol{C}^*.$$

Proof. Fix an affine coordinate z_i on each $\tilde{C}_i$ with 0_i and ∞_i corresponding to 0 and ∞. With respect to this coordinate the action of G on $\tilde{C}_i$ is written as $z_i \to \chi_i(g)z_i, g \in G$, for a unique character χ_i of G, which is independent of the choice of z_i.

In view of the natural inclusion of algebraic groups $\mathrm{Aut}_0(\hat{S}, \hat{C}) \hookrightarrow \mathrm{Aut}_0(\tilde{S}, \tilde{C})$, it suffices to show that there exists a unique one-dimensional subgroup of $\mathrm{Aut}_0(\tilde{S}, \tilde{C}) \cong G$ which descends to an automorphism group of $(\hat{S}, \hat{C})$. A one-parameter subgroup ρ of G induces a $\boldsymbol{C}^*$-action on $\hat{S}$ if and only if φ is ρ-equivariant with respect to the induced ρ-actions on H and E. When φ is of untwisted (resp. twisted) type, φ is written as $z_E = \varphi(z_H) = a/z_H$ (resp. $= az_H$) for some $a \neq 0$, where $z_H = z_i$ etc. as before. Thus the condition becomes

(23) $\chi_H \rho = -\chi_E \rho$ (resp. $\chi_H \rho = \chi_E \rho$)

if φ is of untwisted (resp. twisted) type, where $\chi_H = \chi_i$ if $H = C_i$ etc. Under our assumption that $k > 2$, we easily see that $\chi_H \neq \pm\chi_E$, and hence (23) defines a unique one-parameter subgroup as desired.   q.e.d.

It is convenient to formulate the above result in a more formal way as follows. Let $M := \boldsymbol{Z}^2$ be the free abelian group consisting of all the characters $\chi : G \to \boldsymbol{C}^*$ of G and $N := \boldsymbol{Z}^2$ the free abelian group consisting of all the one-parameter subgroups $\rho : \boldsymbol{C}^* \to G$ of G, written additively. We have a natural perfect pairing $\langle,\rangle : M \times N \to \boldsymbol{Z}$, where $\langle \rho, \chi \rangle = l$ if $\chi\rho(t) = t^l, l \in \boldsymbol{Z}, t \in \boldsymbol{C}^*$. We can define an orientation of M by the condition that $-\chi_{i-1}, \chi_i$ form an oriented basis of M for any i.

For $\chi \in M$ let $\chi^\perp$ be the unique element of N such that it is orthogonal to χ, has the same length as χ, and such that $\chi^\perp$ and χ define the positive oritentation on $\boldsymbol{Z}^2$. We have $(\chi + \chi')^\perp = \chi^\perp + \chi'^\perp$. On the other hand, $\langle \chi_i, \rho_i \rangle = 0$, $\langle \chi_{i-1}^\perp, \rho_i \rangle > 0$ and hence $\chi_i^\perp = \rho_i$. Thus we obtain the following supplement to Lemma 4.1 giving the explicit description of the one-parameter group in question.

Lemma 4.2. $Aut_0(\hat{S}, \hat{C}) \cong \boldsymbol{C}^*$, *being induced by the one-parameter subgroup* $\rho_H + \rho_E$ *(resp.* $\rho_H - \rho_E$*) of G if φ is of untwisted (resp. twisted) type.*

4.2. Computation of Ext groups. Since $\hat{C}$ is a curve with normal crossings, the following is well-known:

$$(24) \qquad Ext^1_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}}) \cong \oplus_{p \in \bar{B}} \boldsymbol{C}_p,$$

$$(25) \qquad H^1(\Theta_{\hat{C}}) = H^2(\Theta_{\hat{C}}) = Ext^2_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}}) = 0,$$

where $\bar{B}$ is the set of nodes of $\hat{C}$ with $\#\bar{B} = k$ and $\boldsymbol{C}_p$ is the skyscraper sheaf at p with fiber $\boldsymbol{C}$. Similarly, we know that

$$(26) \qquad \mathcal{E}xt^i_{O_{\hat{S}}}(\Omega_{\hat{S}}, O_{\hat{S}}) = 0, \ i \geq 2,$$

$$(27) \qquad \mathcal{E}xt^1_{O_{\hat{S}}}(\Omega_{\hat{S}}, O_{\hat{S}}) \cong O_{\hat{F}}$$

(see [**14**]), and hence the local to global spectral sequence yields the exact sequence:

$$(28) \qquad 0 \to H^1(\Theta_{\hat{S}}) \to Ext^1_{O_{\hat{S}}}(\Omega_{\hat{S}}, O_{\hat{S}}) \to H^0(O_{\hat{F}}) \to 0.$$

We shall next show the following:

Lemma 4.3.

$$(29) \qquad h^0(\Theta_{\hat{S}}(-\log \hat{C})) = 1 \ \text{and} \ \ h^q(\Theta_{\hat{S}}(-\log \hat{C})) = 0, \ q \geq 1.$$

Proof. The first one follows from Lemma 4.1. Let $\hat{D}$ be the set of the two intersection points, say r_α, $\alpha = 1, 2$, of $\hat{F}$ and $\hat{C}$. Take the normalization exact sequence for $n : \tilde{S} \to \hat{S}$:

$$(30) \qquad \begin{aligned} 0 \longrightarrow \ & \Theta_{\hat{S}}(-\log \hat{C}) \ \longrightarrow \ n_*(\Theta_{\tilde{S}}(-\log \tilde{C})) \\ & \longrightarrow \ \Theta_{\hat{F}}(-\log \hat{D}) \ \longrightarrow \qquad 0. \end{aligned}$$

Since $\Theta_{\tilde{S}}(-\log \tilde{C}) \cong O_{\tilde{S}}^{\oplus 2}$ and $\Theta_{\hat{F}}(-\log \hat{D}) \cong O_{\hat{F}}$, from the long exact sequence associated to (30) we get the vanishing of $H^2(\Theta_{\hat{S}}(-\log \hat{C}))$ and the exact sequence

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \boldsymbol{C}^2 \longrightarrow \boldsymbol{C} \longrightarrow H^1(\Theta_{\hat{S}}(-\log \hat{C})) \longrightarrow 0.$$

The lemma follows from this. q.e.d.

Next we prove:

Lemma 4.4. $\mathcal{H}om_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \cong \Theta_{\hat{S}}(-\log \hat{C})$ *and* $\mathcal{E}xt^i_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \cong \mathcal{E}xt^i_{O_{\hat{S}}}(\Omega_{\hat{S}}, O_{\hat{S}})$ *for* $i \geq 1$.

Proof. The first isomorphism is well-known (see Proposition 7.1 below). For the second isomorphism we observe the sheaf exact sequence

$$(31) \qquad 0 \to \Omega_{\hat{S}} \to \Omega_{\hat{S}}(\log \hat{C}) \xrightarrow{P} \oplus_l O_{C'_l} \to 0$$

where the last direct sum is over Cartier irreducible components C'_l of $\hat{C}$, and P is the Poincare residue map. Here irreducible components C_i of $\hat{C}$ with $C_i \cap \hat{F} = \emptyset$ are Cartier irreducible components and the unions

of two irreducible components passing through $r_\alpha, \alpha = 1, 2$, are the remaining Cartier irreducible components. Since $\mathcal{E}xt^i_{O_{\hat{S}}}(O_{\hat{C}'_l}, O_{\hat{S}}) \cong N'_l$ for $i = 1$ and vanish otherwise, applying $\mathcal{E}xt^i_{O_{\hat{S}}}(-, O_{\hat{S}})$ to the above sequence we obtain the exact sequence of sheaves

(32)
$$0 \longrightarrow \Theta_{\hat{S}}(-\log \hat{C}) \longrightarrow \Theta_{\hat{S}} \longrightarrow \oplus_l N'_l$$
$$\longrightarrow \mathcal{E}xt^1_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \longrightarrow \mathcal{E}xt^1_{O_{\hat{S}}}(\Omega_{\hat{S}}, O_{\hat{S}}) \longrightarrow 0$$

where N'_l is the normal bundle of C'_l in $\hat{S}_l$. The lemma follows easily from this.   q.e.d.

Together with (4), (26), (27), and Lemma 4.3 the lemma implies the following:

Lemma 4.5. *We have*

(33)
$$Ext^2_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) = 0 \quad and$$

(34)
$$Ext^1_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \cong H^0(O_{\hat{F}}) \cong \boldsymbol{C}.$$

Let $\hat{B} := \bar{B} - \hat{D}$ be the set of nodes of $\hat{C}$ outside $\hat{F}$ with $\#\hat{B} = m := k - 2$. $\bar{B} - \hat{B}$ consists of the two points $r_\alpha, \alpha = 1, 2$, as above.

The two sheaves $\Omega'_{\hat{S}}(\log \hat{C})$ (see (1)) and $\Omega_{\hat{S}}(\log \hat{C})$ coincide except at points in $\hat{B}$. Thus by Lemma 3.9 we get a natural exact sequence:

(35)
$$0 \to \Omega'_{\hat{S}}(\log \hat{C}) \to \Omega_{\hat{S}}(\log \hat{C}) \to \oplus_{p \in \hat{B}} \boldsymbol{C}_p \to 0.$$

Then similarly to Corollary 3.10 we get

Lemma 4.6. *(1)* $\mathcal{H}om_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \cong \Theta_{\hat{S}}(-\log \hat{C})$.
(2) There is a natural exact sequence

(36)
$$0 \to O_{\hat{F}} \to \mathcal{E}xt^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \to \oplus_{p \in \hat{B}} \boldsymbol{C}_p \to 0.$$

(3) $\mathcal{E}xt^i_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) = 0$ *for* $i \geq 2$.

In view of Lemmas 4.3 and 4.6 the local to global spectral sequence for $Ext^i_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}})$ yields the first assertion of the following:

Proposition 4.7. *(1) We have*

(37)
$$Ext^2_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) = 0.$$

(2) There exists a natural isomorphism

(38)
$$c : Ext^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \xrightarrow{\sim} \boldsymbol{C}_r \oplus (\oplus_{p \in \hat{B}} \boldsymbol{C}_p),$$

where r is any one of $r_\alpha, \alpha = 1, 2$. In particular,

$$\dim Ext^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) = m + 1.$$

Proof. We shall show assertion (2). We first prove that the following sequence is exact:

$$(39) \qquad 0 \to \Omega'_{\hat{S}}(\log \hat{C}) \xrightarrow{a} \Omega_{\hat{S}}(\hat{C}) \xrightarrow{b} \Omega_{\hat{C}} \otimes \hat{N} \to 0,$$

where $\hat{N}$ is the normal bundle of $\hat{C}$ in $\hat{S}$. Indeed, the restriction map b is given locally by $[dx/xy, dy/xy \to dx|\hat{C}, dy|\hat{C}]$, where $|\hat{C}$ denotes the restriction to $\hat{C}$. Note that the element $dx = -dy$, which generates $\Omega_{\hat{S}}(\log \hat{C})/\Omega'_{\hat{S}}(\log \hat{C})$, generates locally the torsion part τ of $\Omega_{\hat{C}} \cong \Omega_{\hat{C}} \otimes \hat{N}$, thus inducing the isomorphism $\Omega_{\hat{S}}(\log \hat{C})/\Omega'_{\hat{S}}(\log \hat{C}) \cong \tau$. From this the assertion follows easily.

We apply Lemma 2.2 to the pair $(\hat{S}, \hat{C})$ and $F = \Omega_{\hat{C}}$, and obtain the isomorphism

$$(40) \qquad Ext^i_{O_{\hat{S}}}(\Omega_{\hat{C}} \otimes \hat{N}, O_{\hat{S}}) \cong Ext^{i-1}_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}}).$$

Substituting this isomorphism for $i = 1$ to Ext homomorphism obtained by applying $Ext^1_{O_{\hat{S}}}(-, O_{\hat{S}})$ to b in (39), we get a map $Ext^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \xrightarrow{\beta} Ext^1_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}})$, and its sheaf version $\mathcal{E}xt^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \xrightarrow{\beta'} \mathcal{E}xt^1_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}})$, which is surjective since $\mathcal{E}xt^2_{O_{\hat{S}}}(\Omega_{\hat{S}}(\hat{C}), O_{\hat{S}}) = 0$. These fit into the following commutative diagram:

$$
\begin{array}{ccc}
Ext^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) & \xrightarrow{\beta} & Ext^1_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}}) \\
\Big\downarrow e & & \Big\downarrow d \\
H^0(\mathcal{E}xt^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}})) & \xrightarrow{v} & H^0(\mathcal{E}xt^1_{O_{\hat{C}}}(\Omega_{\hat{C}}, O_{\hat{C}})) \cong \oplus_{p \in \bar{B}} \boldsymbol{C}_p \longrightarrow 0 \\
& & \Big\downarrow p \\
& & \boldsymbol{C}_r \oplus_{p \in \hat{B}} \boldsymbol{C}_p
\end{array}
$$

where $v = H^0(\beta')$, d is the isomorphism (24), and p is the natural projection. Note that e also is isomorphic by Lemma 29. Then we put $c := pd\beta = pve$. In view of the exact sequence (36), in order to prove that c is isomorphic it suffices to show that $pv : H^0(\mathcal{E}xt^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}})) \to \boldsymbol{C}_r \oplus_{p \in \hat{B}} \boldsymbol{C}_p$ gives an isomorphism $u : H^0(O_{\hat{F}}) \to \boldsymbol{C}_r$ when restricted to the subspace $H^0(O_{\hat{F}})$ (see (36)). Indeed, along $\hat{F}$, β' becomes the natural sheaf surjection $O_{\hat{F}} \to \boldsymbol{C}_{r_1} \oplus \boldsymbol{C}_{r_2}$ and u is just the associated map $H^0(O_{\hat{F}}) \to \boldsymbol{C}_r$, which is isomorphic. q.e.d.

Finally, we also record the following exact sequence deduced from (35):

$$(41) \quad
\begin{array}{ccccc}
0 & \longrightarrow & Ext^1_{O_{\hat{S}}}(\Omega_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) & \longrightarrow & Ext^1_{O_{\hat{S}}}(\Omega'_{\hat{S}}(\log \hat{C}), O_{\hat{S}}) \\
& \longrightarrow & \oplus_{p \in \hat{B}} \boldsymbol{C}_p & \longrightarrow & 0.
\end{array}
$$

4.3. Properties of the Kuranishi families. Let

$$(42) \qquad \hat{g} : (\hat{\mathcal{S}}, \hat{\mathcal{C}}) \to \hat{T}, \ (\hat{S}_o, \hat{C}_o) = (\hat{S}, \hat{C}), \ o \in \hat{T}$$

be the Kuranishi family of deformations of the pair $(\hat{S}, \hat{C})$. For any $p \in \bar{B}$, we denote by

$$\hat{g}_p : \hat{\mathcal{C}}(p) \to T_p, \ (\hat{C}_o, p_o) \cong (\hat{C}, p), \ o \in \hat{T}_p,$$

the Kuranishi family of deformations of the isolated singularity $(\hat{C}, p)$. $\hat{T}_p$ is smooth of dimension one. Any deformation of $(\hat{S}, \hat{C})$ induces a deformation of $(\hat{C}, p)$, and correspondingly we have a versal map $\hat{\tau}_p : \hat{T} \to \hat{T}_p$. The fiber $\hat{T}(p) := \hat{\tau}_p^{-1}(o)$ is a hypersuface which is uniquely determined independently of the choice of $\hat{\tau}_p$.

Recall that $m = k - 2$ and $\#\hat{B} = m$.

Proposition 4.8. *Let the notations be as above. Then the Kuranishi space $\hat{T}$ is smooth of dimension $m + 1$. The product map $\hat{\tau} := \hat{\tau}_r \times \Pi_{p \in \hat{B}} \hat{\tau}_p : \hat{T} \to \Pi_p \hat{T}_p$ is isomorphic, where $r = r_\alpha, \alpha = 1$ or 2.*

Proof. $\hat{\tau}$ is identified with the map c in assertion (2) of Proposition 4.7. Thus the proposition follows from that proposition. q.e.d.

Remark 4.1. The proof shows that the singularities of $\hat{C}$ at the two points r_α, $\alpha = 1, 2$, and the singularities of $\hat{S}$ along $\hat{F}$ are simultaneously smoothed.

Define $A := \hat{T}(r)$. This is a smooth hypersurface by the above proposition, and is independent of the choice of $r = r_\alpha$ by the above remark. Also we consider the subspace $I := \cap_{p \in \hat{B}} \hat{T}(p)$, which is a one-dimensional smooth subspace of $\hat{T}$ by the above proposition.

Lemma 4.9. *The restriction of the family to I is identified with the Kuranishi family of log-deformations of $(\hat{S}, \hat{C})$.*

Proof. In view of Lemma 4.5 and (41), for the Kuranishi family $(h : (\hat{\mathcal{S}}', \hat{\mathcal{C}}') \to \hat{I}, \ (S'_o, C'_o) = (\hat{S}, \hat{C}), \ o \in \hat{I})$ of log-deformations of $(\hat{S}, \hat{C})$, the base space $\hat{I}$ is smooth of dimension one and is realized as a subspace of $\hat{T}$. It is in fact a subspace of $I \subseteq \hat{T}$ since I is the maximal subspace of $\hat{T}$ parametrizing log-deformations of $(\hat{S}, \hat{C})$. Then we must have $I = \hat{I}$ since both are smooth of dimension one. q.e.d.

For later purpose (see Lemma 4.15) we also give a more explicit construction of the above one-dimensional deformation by the method of Nakamura [**31**, (4.2)].

First, we construct a local model of the deformations of the pair $(\hat{S}, \hat{C})$ along $\hat{F}$. For $m \in \mathbf{Z}$ let L_m be the holomorphic line bundle of degree m on the complex projective line $\mathbf{P}$, identified with its total space. Let f be the holomorphic function on $V := L_1 \oplus L_{-1}$ given by the

composition $V \to L_0 \to \boldsymbol{C}$, where the first arrow is given by the natural pairing and the second arrow is the natural projection from the product $L_0 = \boldsymbol{P} \times \boldsymbol{C}$. The fiber $V_t := f^{-1}(t), t \neq 0$, then gives a smoothing of the pair

$$(V_0, C_{0,0} \cup C_{0,\infty})) := (L_1 \cup L_{-1}, (L_{1,0} \cup L_{-1,0}) \cup (L_{1,\infty} \cup L_{-1,\infty}))$$

to $(V_t, C_{t,0} \cup C_{t,\infty})$, where $C_{t,*} = p^{-1}(*) \cap V_t, * = 0, \infty$, with $p : V \to \boldsymbol{P}$ the natural projection. For any $t \neq 0$ the projection $q_t : V_t \to L_1 - 0$ is isomorphic and sends $C_{t,*}$ to $L_{1,*} - 0$, where 0 is the zero section.

$(\hat{S}, \hat{C})$ and $(V_0, C_{0,0} \cup C_{0,\infty})$ are then isomorphic as germs along $\hat{F}$ and along the zero section, respectively. (For this one uses the fact that $(\tilde{S}, \tilde{C})$ is obtained from the toric projective plane with three fixed lines in general position, $(\boldsymbol{P}^2, l_0 \cup l_1 \cup l_2)$, by blowing up successively nodes on the anti-canonical cycle over the node $l_1 \cap l_2$.)

Since the deformation above is trivial off the zero section, the induced deformation of the germ $(\hat{S}, \hat{C})$ along $\hat{F}$ extends to a global log-deformation of $(\hat{S}, \hat{C})$ which is trivial outside a neighborhood of $\hat{F}$. Call a deformation obtained in this way a *standard* family of deformations of $(\hat{S}, \hat{C})$.

Lemma 4.10. *A standard family of deformations of $(\hat{S}, \hat{C})$ is a Kuranishi family of log-deformations of $(\hat{S}, \hat{C})$.*

Proof. It suffices to show that the induced versal map $\tilde{\tau} : (\boldsymbol{C}, 0) \to (I, o)$ is isomorphic. This is true if the composite map $\hat{\tau}_r \tau : (\boldsymbol{C}, 0) \to (\hat{T}_r, o)$ is isomorphic. But the latter is clear by the above construction.
$$\text{q.e.d.}$$

We call the original pair $(\tilde{S}, \tilde{C})$ *minimal* if each irreducible component D of $\tilde{C}$ with $D^2 = -1$ intersects either H or E. For $t \in \hat{T}$ we shall identify the fibers (S_t, C_t) of $\hat{g}$ over t in the next proposition. In order to state it we distinguish three cases:

Case-H: $\hat{C}^\alpha$, $\alpha = 1, 2$, are disjoint, and both of $\hat{C}^\alpha$ are reducible.

Case-H′: $\hat{C}$ is connected, i.e., the case where φ is of twisted type.

Case-P: $\hat{C}^\alpha$ are disjoint, but one of $\hat{C}^\alpha$, say, $\alpha = 1$, is irreducible.

Note that by our assumption $k > 2$ at most one of $\hat{C}^\alpha$ is irreducible.

Proposition 4.11. *(1) Suppose that $t \notin A$. Then S_t is a smooth surface of class VII with second Betti number m. In Case-P and -H (resp. Case-H′) $C_t, t \notin A$, is the unique anti-canonical curve (resp. L_t-twisted anti-canonical curve with $L_t = L_{S_t}$) on S_t. The minimal model $\bar{S}_t$ of S_t is isomorphic to one of the following surfaces in each case:*

Case-H: a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface

Case-H′: a half Inoue surface or a diagonal Hopf surface

Case-P: a parabolic Inoue surface or a diagonal Hopf surface

(2) The restriction of the family to I is identified with the Kuranishi family of log-deformations of $(\hat{S}, \hat{C})$. Let $t \in I$ with $t \neq o$. Then the surface S_t is a properly blown-up hyperbolic (resp. half, resp. parabolic) Inoue surface in Case-H (resp. Case-H′, resp. Case-P). The isomorphism class of S_t is independent of t in Case-H or -H′. If $(\tilde{S}, \tilde{C})$ is minimal, then S_t also is minimal, namely, S_t is a hyperbolic (resp. half, resp. parabolic) Inoue surface.

Remark 4.2. (1) By Lemmas 3.8 and 4.1 we have $h^0(\Theta_{\hat{S}_t}(-\log \hat{C}_t)) = 1$ for $t = o$, and $= 0$ otherwise in Case-H and -H′, while $h^0(\Theta_{\hat{S}_t}(-\log \hat{C}_t)) = 1$ for all $t \in I$ in Case-P. Correspondingly, the Kuranishi family of log-deformations over I above is universal in Case-P and not in Case-H or -H′.

(2) $\bar{S}_t$ is a diagonal Hopf surface if and only if $t \notin \hat{T}(p)$ for any $p \in \hat{B}$ by Lemma 3.1.

Proof of Proposition 4.11. By the definition of A it is clear that S_t is smooth if and only if $t \notin A$ (see Remark 4.1).

First we consider the restriction of $\hat{g}$ to I, which may be identified with the Kuranishi family of log-deformations of $(\hat{S}, \hat{C})$ by Lemma 4.9 and also with the standard family constructed before, by Lemma 4.10. Note that $I \cap A = \{o\}$. In this case by [**28**, Theorem 44] the general fiber $S_t, t \in I - o$ is, topologically, obtained from $\hat{S}$ by a spherical modification (see [**31**, (3.3)]). Then by [**31**, (3.4)] S_t has infinite cyclic fundamental group and has $m = k - 2$ as the second Betti number. Moreover, since $\hat{C}$ is an anti-canonical curve on $\hat{S}$ as we have already noted, $h^0(-\hat{K}) \geq 1$ and hence $h^0(\hat{K}^n) = 0$ for all $n > 0$, where $\hat{K} = K_{\hat{S}}$. Thus by the upper semicontinuity we get that $S_t, t \neq o$, all have Kodaira dimension $-\infty$. Hence S_t are surfaces of class VII for all $t \neq o$ and hence for all $t \notin A$.

Indeed, more precisely, when $(\hat{S}, \hat{C})$ is minimal in the sense defined above, S_t is a hyperbolic Inoue surface for any $t \in I - 0$ by the more precise computation of the self-intersection numbers of the irreducible components of C_t in S_t due to [**33**, (5.13) and (5.14)] (see also below). In fact, from Lemma 4.10 we may identify our family g' with that used by Nakamura in [**33**]. Using the unique extension theorem of (-1)-curves [**26**] the general case can easily be reduced to the minimal case by contracting all the (-1)-curves contained in the irreducible components of $C_t, t \in I$, successively and simultaneously to the points. This shows assertion (2).

Next we show assertion (1). Suppose first that we are in Case-H or -P. We show that $K_t + C_t$ is trivial for all $t \notin A$, where K_t is the canonical bundle of S_t. Note first that the family is versal at any point of $\hat{T}$ by the openness of versality (see [**6**]). Thus by assertion (1) of

Proposition 3.13 for some open neighborhood U of $I - \{o\}$ in $\hat{T}$, this is true. Then by analytic continuation $K_t + C_t$ are trivial for all $t \in \hat{T} - A$. Indeed, the component of the identity section $P := \mathrm{Pic}_0 \hat{\mathcal{S}}/\hat{T} \to \hat{T}$ of the relative Picard variety associated to $\hat{g}$ is a principal $\boldsymbol{C}^*$-bundle at least over $\hat{T} - A$ and $K_t + C_t$ defines a holomorphic section of P over $\hat{T} - A$ which is trivial over U, and hence over the whole $\hat{T} - A$. (In fact, P itself can be shown to be seperated as the singular fiber $\hat{S}$ is irreducible and reduced.)

Once this is proved, noting that C_t is disconnected we obtain the structure of the surface $\bar{S}_t$ as stated in the proposition by Lemma 3.3. This finishes the proof of assertion (1) in Case-H and -P. Case-H′ is treated similarly by using assertion (1) of Proposition 3.14 and Lemma 3.4. \hfill q.e.d.

We now restrict to Case-H. For $t \in I - o$ we consider the log-deformations (S_t, C_t) of $(\hat{S}, \hat{C})$. Fixing $\alpha, \alpha = 1, 2$, write $\hat{C}^\alpha = \hat{B}_0 + \cdots + \hat{B}_{h_\alpha}$ cyclically such that $\tilde{B}_0 \cap \hat{H} \neq \emptyset \neq \tilde{B}_{h_\alpha} \cap \hat{E}$, where $\tilde{B}_i = n^{-1}(\hat{B}_i)$. Then the cycle C_t^α which is a deformation of $\hat{C}^\alpha$ is written as $B_{0,t} + \cdots + B_{h_\alpha - 1, t}$, where $B_{i,t}$ are deformations of $\hat{B}_i$ for $i \neq 0$, and $B_{0,t}$ is one of $\hat{B}_{h_\alpha} + \hat{B}_0$. This also gives a natural numbering of the cycle C_t^α, which we call tentatively a *toric* numbering. Clearly for $i \neq 0$ we have $(B_{i,t})^2 = \tilde{B}_i^2$ for the self-intersection numbers. For $B_{0,t}$ we note that $(B_{0,t})^2 = (\tilde{B}_{h_\alpha} + \tilde{B}_0)^2$ if $h_\alpha > 0$ and $(B_{0,t})^2 = (\tilde{B}_0)^2$ if $h_\alpha = 0$ since the intersection number is invariant under normalization and deformation. Thus we get the following:

Lemma 4.12.

$$(43) \qquad (B_{0,t})^2 = \tilde{B}_0^2, \quad \text{if} \quad h_\alpha = 0,$$

$$(44) \qquad (B_{0,t})^2 = \tilde{B}_0^2 + \tilde{B}_1^2 + 2, \quad \text{if} \quad h_\alpha = 1,$$

$$(45) \qquad (B_{0,t})^2 = \tilde{B}_0^2 + \tilde{B}_{h_\alpha}^2, \quad \text{if} \quad h_\alpha > 1.$$

This lemma is a prerequisite for proving the following result due to Nakamura [**33**, (5.13) and (5.14)].

Proposition 4.13. *Any properly blown-up hyperbolic, half, or parabolic Inoue surface (S, C) is obtained by log-deformations of a rational surface $(\hat{S}, \hat{C})$ with a nodal curve obtained from an admissible toric surface $(\tilde{S}, \tilde{C})$ as above.*

Although in [**33**] only the minimal case has been treated, the generalization to the blown-up case is immediate by the simultaneous blowing up of the family of deformations in the minimal case.

A toric surface which gives rise to the given hyperbolic or half Inoue surface is not unique. In fact, we note in the following result that Nakamura's construction actually gives rise to m such toric surfaces

according to which pair of irreducible components $C_t^\alpha, \alpha = 1, 2$, is to be bent and broken.

Proposition 4.14. *Let S be any properly blown-up hyperbolic Inoue surface with second Betti number m. Then there exist in general $\bar{m}$ admissible toric surfaces $(\tilde{S}, \tilde{C})$ which give rise to S as a deformation S_t of $\hat{S}$ as above, where $\bar{m}$ is the second Betti number of the minimal model of S.*

Proof. Suppose first that S is minimal and that the canonical weight sequences of S are given by (11) and (12) with n and k_i with $n > 0$, $k_i \geq 1$, and $1 \leq i \leq 2n$ fixed. Let $(\boldsymbol{P}^2, l_0 \cup l_1 \cup l_2)$ be a toric projective plane with three fixed lines in general position. Then our toric surface $(\tilde{S}, \tilde{C})$ is given by a finite succession of blowning-ups such that the center of each blowing-up is mapped to the point $l_1 \cap l_2$ and such that the proper transform of l_2 corresponds to the weight 0 in (47) below, where the last condition comes from the minimality assumption. Consider the following k_{2n} canonical weight sequences of $\tilde{C}^1, \tilde{C}^2$ considered modulo interchanging $\tilde{C}^1$ and $\tilde{C}^2$ with the same notational convention as in (11) (12):

$$(46) \qquad k_1, \quad [k_2 - 1], k_3 + 2, \ldots, k_{2n-1} + 2, [k_{2n}],$$

$$(47) \qquad 0, [k_1 - 1], k_2 + 2, \ldots, [k_{2n-1} - 1], k_{2n} + 2$$

and

$$(48) \qquad 0, [\kappa - 1], k_1 + 2, [k_2 - 1], \ldots, k_{2n-1} + 2, [k_{2n} - \kappa],$$

$$(49) \qquad \kappa, \quad [k_1 - 1], \ldots, [k_{2n-1} - 1], k_{2n} - \kappa + 2$$

where $0 < \kappa < k_{2n}$. Then for each of the above pairs of weight sequences we can find a unique minimal admissible toric surface $(\tilde{S}, \tilde{C})$ having this sequence as its canonical weight sequences. Since the admissible toric surfaces are determined completely by the canonical weight sequences, the uniqueness is clear, but in fact, one can show that there exists a unique way to obtain $(\tilde{S}, \tilde{C})$ from $(\boldsymbol{P}^2, l_0 \cup l_1 \cup l_2)$ by a finite succession of the blowing-ups as above. (The details are omitted.)

By Lemma 4.12 it is immediate to see that any of the above toric surfaces gives rise to the hyperbolic Inoue surface with canonical weight sequences (11) and (12).

Finally, the canonical weight sequences of hyperbolic Inoue surfaces of the above form are determined up to cyclic permutations. After taking into account all the sequences after such permutations and recalling that $\sum_{1 \leq i \leq 2n} k_i = m$, in all we obtain m minimal admissible toric surfaces. From these we conclude the proof of the proposition in the minimal case.

Suppose next that S is not minimal and is obtained as a log-deformation $S = S_t$ of an admissible toric surface $\hat{S}$. As follows easily from Lemma

4.12, no (-1)-curve in C is ever an amalgamated deformation of the union of two irreducible components of $\hat{C}$ intersecting along $\hat{F}$. Thus any (-1)-curve in C is a deformation of a (-1)-curve in $\hat{C}$ contained in the smooth locus of $\hat{S}$. Then by blowing down these (-1)-curves simultaneously we get in a canonical way the minimal model $\bar{S}$ of S as a log-deformation of the corresponding "minimal model" of $\hat{S}$. This reduces the number in question for S to the corresponding number for its minimal model.   q.e.d.

Remark 4.3. The m admissible toric surfaces obtained in the proposition would all be distinct if the weight sequences are general enough. On the other hand, for a parabolic Inoue surface S with second Betti number m the admissible toric surface which gives rise to S by the above process is uniquely given by the following weight sequences:

$$(50) \qquad\qquad\qquad\qquad m,$$

$$(51) \qquad\qquad\qquad\qquad 0\,[m].$$

Let S be a properly blown-up hyperbolic Inoue surface. By the previous results (S, C) is obtained as a log-deformation (S_t, C_t) of some admissible toric surface $(\hat{S}, \hat{C})$. In this case, for each cycle C^α we have the toric numbering for its irreducible components as defined before Lemma 4.12. We show that this toric numbering indeed coincides with the canonical numbering as characterized by Lemma 3.5. This implies also that the toric numbering is independent of the initial datum $\hat{S}$.

Lemma 4.15. *The canonical numbering and toric numbering coincide in the sense explained above.*

Proof. By Lemma 4.10 we may consider the standard family of deformations of $(\hat{S}, \hat{C})$. Then by using the notations there we may assume that $C_t^1 = C_{t,0}$ and $C_t^2 = C_{t,\infty}$ in a neighborhood W of $\hat{F}$ (in the total space of the family). For example, we consider C_t^1. S_t contains the global spherical shell U_t (identifying S_t with V_t in W), and we have $U_t \cap C_t^1 \subseteq B_{0,t}$. Then from the fact that tubular neighborhoods of the zero sections in L_1 and L_{-1} are strongly pseudoconcave and are strongly pseudoconvex, respectively, and by the definition of toric numbering, we immediately see by Lemma 3.5 that the two numberings coincide. q.e.d.

5. Construction of a singular twistor space

5.1. Twistor spaces. Let M be an oriented compact C^∞ 4-manifold and $[g]$ a self-dual structure, i.e., the conformal class of a self-dual metric, on M. Denote by Z the twistor space associated to the self-dual manifold $(M, [g])$ with the twistor fibration $t : Z \to M$, which is a $\boldsymbol{P}$-bundle where $\boldsymbol{P}$ is the complex projective line [3].

Any fiber of t is called a *twistor line.* There exists an anti-holomorphic involution σ of Z, called the *real structure* of Z, which is fixed-point-free and preserves each twistor line. A complex surface S on Z is called *elementary* if the intersection number $LS = 1$ for any twistor line L on Z. If S is an elementary surface, then its *conjugate* $\bar{S} := \sigma(S)$ is again an elementary surface.

There are two cases to consider:

> Case 1: S contains a twistor line.
> Case 2: S contains no twistor lines.

In Case 1 the structure of S is as follows (see [**38**, Lemmas 1.9, 1.10]):

Lemma 5.1. *Let S be an elementary surface in Case 1. Then S contains precisely one twistor line, say, L, S and $\bar{S}$ intersect transversally along L, and S is obtained from a complex projective plane $\boldsymbol{P}^2$ by a succession of blowing-ups such that the total blowing-down $S \to \boldsymbol{P}^2$ maps a neighborhood of L isomorphically onto a neighborhood of a line on $\boldsymbol{P}^2$. In particular $L^2 = 1$ in S. M is diffeomorphic to $m\boldsymbol{P}^2$ with $m = b_2(M)$, and the restriction of the twistor fibration t to S is nothing but the smooth contraction of L to a point $t(L)$ of $M = m\boldsymbol{P}^2$.*

On the other hand, in Case 2 we easily check the following:

Lemma 5.2. *In Case 2, $S \cap \bar{S} = \emptyset$ and they are mapped diffeomorphically onto M. If J and $\bar{J}$ are, respectively, the complex structures on M induced from S and $\bar{S}$ via this diffeomorphism, then $\bar{J} = -J$. Moreover, the self-dual structure $[g]$ is compatible with $\pm J$ and gives the anti-self-dual hermitian surfaces $(M, \pm J, [g])$ which are complex conjugate to each other.*

5.2. Joyce twistor space Z. Let m be a positive integer and $m\boldsymbol{P}^2$ the connected sum of m copies of complex projective plane $\boldsymbol{P}^2$. Fix any such m and write $M = m\boldsymbol{P}^2$. Denote by $K := S^1 \times S^1$ the real 2-torus. Then M admits a finite number $\psi(m)$ of smooth effective K-actions on M up to diffeomorphisms. (For instance, $\psi(m) = 1$ for $m = 1, 2$ and $\psi(3) = 3$.) For each such smooth K-action on M, Joyce [**24**] constructed a smooth connected family of K-invariant self-dual conformal structures on M, depending on $(m-1)$ real smooth parameter.

Fix any such K-invariant self-dual structure on M and denote it by $[g]$. Let Z be the associated twistor space with natural projection $t : Z \to M$, making Z a smooth $\boldsymbol{P}$-bundle over M. Denote the real structure of Z by σ as before. In general we call such a Z a *Joyce twistor space.*

The K-action on M naturally lifts to a holomorphic K-action on Z. This action on Z then extends to a holomorphic action of the complexification $G = \boldsymbol{C}^* \times \boldsymbol{C}^*$ of K, which is an algebraic torus of dimension two.

The structure of Z with this G-action has been studied in detail in [**15**, §4, §6]. We explain some of the structures of Z which are important for us in what follows. We set $k = m + 2 \geq 2$. Then Z admits exactly k pairs of G-invariant elementary surfaces $\{(S_i^+, S_i^-)\}, 1 \leq i \leq k$, with $\sigma(S_i^\pm) = S_i^\mp$. $S_i^\pm$ are projective smooth toric surfaces with respect to the induced G-action, and S_i^+ and S_i^- intersect transversally along a G-invariant twistor line L_i (see [**15**, Proposition 6.12]). The self-intersection numbers of L_i in $S_i^\pm$ both equal one:

$$L_i^2 = 1.$$

The point $p_i := t(L_i)$ is a fixed point of K on M, and this sets up bijective correspondences among the set of pairs of G-invariant elementary surfaces on Z, the set of G-invariant twistor lines, and the set of K-fixed points on M.

The union $S_i := S_i^+ \cup S_i^-, 1 \leq i \leq k$, all belong to the fundamental system $|-\frac{1}{2}K|$, where $\frac{1}{2}K$ is the canonical square root of the canonical bundle K of Z. The subspace $H^0(Z, -\frac{1}{2}K)^G$ of $H^0(Z, -\frac{1}{2}K)$ of G-invariant elements is two-dimensional, and the associated pencil $|-\frac{1}{2}K|^G$ is important for the study of the structure of Z. The base locus C of $|-\frac{1}{2}K|^G$ is a cycle of rational curves which are both G- and σ-invariant and is of the form

$$(52) \qquad C = C_1^+ + \cdots + C_k^+ + C_1^- + \cdots + C_k^-, \quad C_i^\pm \cong \boldsymbol{P},$$

with $\sigma(C_i^\pm) = C_i^\mp$. The G-action is free outside $C \cup (\cup_i L_i)$ (see [**15**, Proposition 4.4]).

A general member S_0 of $|-\frac{1}{2}K|^G$ is a smooth toric surface with respect to the induced G-action with anti-canonical cycle C. In fact, any smooth members are all isomorphic to each other with the same weight sequence of the form

$$(53) \qquad\qquad (a_1, \ldots, a_k, a_1, \ldots, a_k).$$

There exist precisely k singular members of the pencil $|-\frac{1}{2}K|^G$. They are the surfaces S_i above with two irreducible components $S_i^\pm$.

We put $p_i^\pm = C_i^\pm \cap C_{i+1}^\pm$, $1 \leq i \leq k$, with the convention that $C_{k+1}^\pm = C_1^\pm$. Thus $\sigma(p_i^\pm) = p_i^\mp$. $S_i^\pm$ contains exactly half of the cycle C, i.e., $C \subseteq S_i$ and the intersection $C_{(i)}^\pm := S_i^\pm \cap C$ is a chain of rational curves given by

$$C_{(i)}^\pm = C_{i+1}^\mp + \cdots + C_k^\mp + C_1^\pm + \cdots + C_i^\pm.$$

Then the anti-canonical cycle $B_i^\pm$ of the toric surface $S_i^\pm$ is written as

$$B_i^\pm = L_i + C_{(i)}^\pm.$$

Moreover, for each $j \neq i$, L_i intersects with $S_j^\pm$ transversally at the unique points $p_i^\pm$ and $L_i \cap C = \{p_i^\pm\}$. The weight sequence of $S_i^\pm$ is

then given by

$$(54) \qquad (1, a_{i+1} + 1, a_{i+2}, \ldots, a_k, a_1, \ldots, a_i + 1),$$

independently of $\pm$ (see [**15**, (13)]). We call the pair (i, j) *minimal*, if for any d, $a_d = 1$ implies that $d = l$ or $l + 1$ where $l = i, j$. The next lemma is used in proving that our construction covers all the hyperbolic Inoue surfaces.

Lemma 5.3. *For any projective toric surface S with a $(+1)$-curve H in its anti-canonical cycle there exist a Joyce twistor space Z as above and an index $i, 1 \leq i \leq k$, such that $S_i^{\pm}$ are both isomorphic to S.*

Proof. Consider the induced K-action on S. Then we may K-equivariantly contract the curve H to a point x of a smooth K-manifold M, which is necessarily diffeomorphic to $m\boldsymbol{P}^2$, where $m + 1$ is the second Betti number of S. The point x is one of the k fixed points of the K-action on M. Take any K-invariant self-dual structure on M of Joyce and in the associated twistor space take the pair $\{S_i^{\pm}\}$ of elementary surfaces corresponding to x in the sense mentioned above. By Lemma 5.1 and the above description we see that S and $S_i^{\pm}$ are K-diffeomorphic with respect to the induced K-action. Since the induced K-action determines the toric surface as a complex surface, we are done.   q.e.d.

5.3. Blown-up twistor space $\tilde{Z}$. Fix i, j with $1 \leq i < j \leq k$ and write l for i and/or j. (Note however that since we consider i and j cyclically modulo k, the roles of i and j are symmetric.)

Let $\mu : \tilde{Z} \to Z$ be the blowing-up with center the disjoint union $L_i \cup L_j$ and with exceptional divisors $Q_l := \mu^{-1}(L_l)$, $l = i, j$. Q_l are isomorphic to the product $\boldsymbol{P} \times \boldsymbol{P}$ with $\mu|Q_l : Q_l \to L_l$ identified with the projection $\boldsymbol{P} \times \boldsymbol{P} \to \boldsymbol{P}$, say, to the first factor. Then the normal bundle $N_{Q_l/\tilde{Z}}$ of Q_l in $\tilde{Z}$ is isomorphic to the line bundle $O(1, -1)$ of bidegree $(1, -1)$ on $Q_l \cong \boldsymbol{P} \times \boldsymbol{P}$.

Let $\tilde{S}_l^{\pm}$ be the proper transforms of $S_l^{\pm}$ in $\tilde{Z}$. From the construction we see that $\tilde{S}_l^{\pm}$ are disjoint, but the intersection of any other pairs from the four surfaces $\tilde{S}_l^{\pm}, l = i, j$, is non-empty and consists of a chain of rational curves, which is a connected component of the proper transform in $\tilde{Z}$ of the cycle C. (See the formulae for its image in $\hat{Z}$ in (57) and (58) below.) Now write $\tilde{S}_l$ for the disjoint union $\tilde{S}_l^+ \cup \tilde{S}_l^-$ and set $\tilde{S} := \tilde{S}_i \cup \tilde{S}_j$. The latter is a connected surface with four irreducible components.

The actions of G and σ naturally lift to $\tilde{Z}$ with $\sigma(\tilde{S}_l^{\pm}) = \tilde{S}_l^{\mp}$ and $\sigma(Q_l) = Q_l$. $\tilde{S}_l^{\pm}$ and Q_l are G-invariant and will be considered as toric surfaces with respect to the induced G-action.

We put $H_l^{\pm} := \tilde{S}_l^{\pm} \cap Q_l, E_l^{\pm} := \tilde{S}_l^{\pm} \cap Q_{l'}$, where $\{l, l'\} = \{i, j\}$. $H_l^{\pm}$ is mapped isomorphically onto L_l by μ, and $\tilde{S}_l^{\pm} \to S_l^{\pm}$ is the blowing-up

of $p_j^\mp$ if $l = i$ and of $p_i^\pm$ if $l = j$ with exceptional curve $E_l^\pm$. Thus we get

$$(H_l^\pm)^2 = 1 \quad \text{and} \quad (E_l^\pm)^2 = -1 \quad \text{in} \quad \tilde{S}_l^\pm$$

while $(H_l^\pm)^2 = (E_l^\pm)^2 = 0$ in Q_l. In addition, the anti-canonical cycles $\tilde{B}_l^\pm$ of $\tilde{S}_l^\pm$ is given, e.g., when $l = i$, by

$$\tilde{B}_i^\pm = H_i^\pm + \tilde{C}_{i+1}^\mp + \cdots + \tilde{C}_j^\mp + E_i^\pm + \tilde{C}_{j+1}^\mp + \cdots + \tilde{C}_i^\pm$$

with the same weight sequence (independently of $\pm$)

$$(55) \quad (1, a_{i+1} + 1, a_{i+2}, \ldots, a_j - 1, -1, a_{j+1} - 1, \ldots, a_k, a_1, \ldots, a_i + 1)$$

where $\tilde{C}_d^\pm$ is the proper transform of $C_d^\pm$ in $\tilde{Z}, 1 \leq d \leq k$. Similarly, the anti-canonical cycles F_l of Q_l are given by

$$F_l = H_l^+ + E_{l'}^+ + H_l^- + E_{l'}^-.$$

5.4. Singular twistor space $\hat{Z}$. Now we choose and fix an isomorphism of the pairs

$$(56) \qquad\qquad \varphi : (Q_i, F_i) \to (Q_j, F_j)$$

which maps $H_i^\pm$ (resp. $E_j^\pm$) to $E_i^\pm$ (resp. $H_j^\pm$), thus interchanging the horizontal and vertical directions. Let $\hat{Z}$ be the complex space obtained by identifying in $\tilde{Z}$ the subspaces Q_i and Q_j via φ. Let $\nu : \tilde{Z} \to \hat{Z}$ be the quotient map, which is considered as the normalization map of $\hat{Z}$. Let $\hat{Q} := \nu(Q_i) = \nu(Q_j)$ be the singular locus of $\hat{Z}$ and $\hat{S}_l^\pm := \nu(\tilde{S}_l^\pm)$ the image of $\tilde{S}_l^\pm$ in $\hat{Z}$. Then $\hat{S}_l^\pm$ is a non-normal surface with singular locus $\hat{F}_l^\pm := \nu(H_l^\pm) = \nu(E_l^\pm)$ $(= \hat{Q} \cap \hat{S}_l^\pm)$. The image

$$\hat{F} := \hat{F}_i^+ + \hat{F}_j^+ + \hat{F}_i^- + \hat{F}_j^-$$

of $\nu(F_i) = \nu(F_j)$ in $\hat{Q}$ belongs to the anti-canonical system on $\hat{Q}$ and shall be called the anti-canonical cycle of $\hat{Q}$. (Note that since the G-action is not φ-equivariant, $\hat{Q}$ has no natural structure of a toric surface in general (see Proposition 6.3 below).)

Let $\hat{S}_l = \hat{S}_l^+ \cup \hat{S}_l^-$. Then $\hat{S} := \hat{S}_i \cup \hat{S}_j = \nu(\tilde{S})$ is a surface in $\hat{Z}$ consisiting of four irreducible components $\hat{S}_l^\pm, l = i, j$. By our construction φ maps the intersection points $\tilde{C}_i^\pm \cap H_i^\pm$ and $\tilde{C}_{i+1}^\mp \cap H_i^\pm$ to $\tilde{C}_{j+1}^\mp \cap E_i^\pm$ and $\tilde{C}_j^\mp \cap E_i^\pm$, respectively. This implies that if we set $\hat{C}_d^\pm = \nu(\tilde{C}_d^\pm)$, the curves

$$\hat{C}_{j+1}^\mp + \cdots + \hat{C}_i^\pm \quad \text{and} \quad \hat{C}_{i+1}^\mp + \cdots + \hat{C}_j^\mp$$

form (four disjoint) cycles of rational curves on $\hat{Z}$. Moreover, we have

$$(57) \qquad\qquad \hat{S}_i^\pm \cap \hat{S}_j^\pm = \hat{C}_{j+1}^\mp + \cdots + \hat{C}_i^\pm,$$

$$(58) \qquad\qquad \hat{S}_i^\pm \cap \hat{S}_j^\mp = \hat{C}_{i+1}^\mp + \cdots + \hat{C}_j^\mp.$$

In this way we see that each of the four surfaces $\hat{S}_l^\pm$ contains a pair of disjoint cycles of rational curves. (Thus our choice of φ in (56) amounts to assuming that the restrictions of φ to all surfaces $\tilde{S}_l^\pm$ (see (22)) are of untwisted type in the sense defined there.)

We denote by $\hat{C}_{(l)}^\pm$ the union of these cycles on $\hat{S}_l^\pm$. Note also that $\hat{S}_l^+$ and $\hat{S}_l^-$ are disjoint and no three of $\hat{S}_l^\pm$ have common points.

In what follows it is convenient to distinguish the following two cases:

Case-P (parabolic case) $j = i + 1$, or $(i, j) = (1, k)$

Case-H (hyperbolic case) otherwise

In other words, when we consider i, j cyclically modulo k, Case-P is precisely the case where i and j are adjacent. In fact, precisely in this case one of the intersections $\hat{S}_i^\pm \cap \hat{S}_j^\pm = \hat{C}_j^\mp$ or $\hat{S}_j^\mp \cap \hat{S}_i^\pm = \hat{C}_i^\pm$ becomes irreducible and is a single rational curve with a node.

Actually, the constructions in Section 6 and the arguments in Section 8 below for Case-H all apply to the twisted cases where (some of) the restrictions become of twisted type. However, the results in terms of the bihermitian structures are somewhat different. For this reason and also for the simplicity of exposition we treat the twisted cases separately in Section 9.

6. Structure of a singular twistor space

$\hat{Z}$ and $\hat{S}_l^\pm$ are complex spaces with normal crossing singularities. Therefore we may speak of the canonical bundles $\hat{K}$ and $K_{\hat{S}_l^\pm}$ of the respective spaces, corresponding to the dualizing sheaves.

We also note that $\hat{S}$ and $\hat{S}^\pm$ are both Cartier divisors on $\hat{Z}$, and similarly $C_{(l)}^\pm$ is a Cartier divisor on $\hat{S}_l^\pm$ (see (b) below).

6.1. Anti-canonical system. We first identify the anti-canonical divisors of $\hat{Z}$ and $\hat{S}_l^\pm$.

Lemma 6.1. *Let $\hat{K}$ and $K_{\hat{S}_l^\pm}$ be the canonical bundle on $\hat{Z}$ and $\hat{S}_l^\pm$, respectively. Then we have*

$$-\hat{K} = \hat{S} \quad and \quad -K_{\hat{S}_l^\pm} = \hat{C}_{(l)}^\pm.$$

Proof. Let $S := S_i^+ + S_i^- + S_j^+ + S_j^-$. S is a member of the anti-canonical system $|-K|$ of Z. Then by the adjunction formula for the blowing-up μ we get

$$(59) \quad \tilde{K} = \mu^* K + Q_i + Q_j = -\mu^*(S) + Q_i + Q_j = -\tilde{S} - (Q_i + Q_j).$$

where $\tilde{K}$ is the canonical bundle of $\tilde{Z}$. Then $\hat{K}$ is obtained by identifying $(\tilde{K} + Q_i)|Q_i \cong K_{Q_i}$ and $(\tilde{K} + Q_j)|Q_j \cong K_{Q_j}$ along $\hat{Q}$ by the defining isomorphism φ of $\hat{Z}$ (see [**14**, (2.11)]). On the other hand, by (59) and the adjunction formula we have $(\tilde{K} + Q_l)|Q_l = -\tilde{S} \cap Q_l = F_l$ and φ

induces an isomorphism $F_i \xrightarrow{\sim} F_j$. Thus $\hat{S} = \nu(\tilde{S})$ is a member of $-\hat{K}$, giving the first equality. The proof of the second equality is similar. q.e.d.

6.2. Local structure of $(\hat{Z}, \hat{S})$. (a) *Tangential points*

We put $\hat{p}_n^{\pm} = \hat{C}_n^{\pm} \cap \hat{C}_{n+1}^{\pm}$ for $1 \le n \le k$ with the convention $\hat{C}_{k+1}^{\pm} = \hat{C}_1^{\mp}$. Outside $\hat{Q}$ the intersections $\hat{S}_i^{\pm} \cap \hat{S}_j^{\pm}$ (resp. $\hat{S}_j^{\mp} \cap \hat{S}_i^{\pm}$) are transversal except at the points $\hat{p}_n^{\pm}$ with $j + 1 \le n \le k$ and $\hat{p}_n^{\mp}$ with $1 \le n \le i - 1$ (resp. $\hat{p}_n^{\mp}$ with $i < n < j$), where the corresponding two components of $\hat{S}$ intersect and are tangent to each other; in fact, locally, with respect to suitable local coordinates x, y, z at such a point of $\hat{Z}$, $\hat{S}$ has a local equation

$$(60) \qquad\qquad z(z - xy) = 0.$$

Note that there exist in all $2m$ $(= (2k - 4))$ such tangential points.

(b) *Points of $\hat{F}$*

Let r be any one of the four singular points of $\hat{F}$, i.e., the intersection points of the irreducible components $\hat{F}_l^{\pm}$.

The local structure of the pair $(\hat{Z}, \hat{S})$ at r is described as follows. Let $X = \boldsymbol{C}^2(u, v)$ and $Y = \boldsymbol{C}^2(x, y)$. Let A and D be the curves in X and Y defined by $uv = 0$ and $xy = 0$, respectively. Also denote by D_1 and D_2 the irreducible components of D defined by $x = 0$ and $y = 0$, respectively. If we identify all these spaces with the germs at the origin they define, we have an isomorphism

$$(61) \qquad\qquad (\hat{Z}, \hat{S}) \cong (A \times Y, A \times D)$$

with $A \times D_s, s = 1, 2$, corresponding to the germs of the two (global) irreducible components of $\hat{S}$ at r. The structure of $(\hat{Z}, \hat{S})$ at a smooth point of $\hat{F}$ is given by the germ at any point outside the origin. We conclude that $\hat{S}$ is a Cartier divisor in $\hat{Z}$ since xy is not a zero divisor on $A \times Y$. We can then consider the logarithmic 1-forms on $\hat{Z}$ along $\hat{S}$ (see Section 7).

6.3. Automorphism group. We determine the identity component of the automorphism group of $(\hat{Z}, \hat{S})$. We first recall the following:

Lemma 6.2. *Let Z be a Joyce twistor space associated to a K-invariant self-dual structure on $m\boldsymbol{P}^2$ with $m \ge 1$ and $S := S_i^+ + S_i^- + S_j^+ + S_j^-$ as before. Then $\mathrm{Aut}_0(Z, S) \cong G := \boldsymbol{C}^{*2}$. For the blowing-up $\tilde{Z}$ of Z, we have a natural isomorphism $\mathrm{Aut}_0(\tilde{Z}, \tilde{S} \cup \tilde{Q}) \cong \mathrm{Aut}_0(Z, S)$, where $\tilde{Q} = Q_i \cup Q_j$.*

Proof. See, e.g., [**16**, Proposition 5.5] for the first assertion when $m > 1$. A direct computation yields also the result when $m = 1$, the details being omitted. Since the center of the blowing-up μ is G-invariant, the second assertion is obvious. q.e.d.

Using the above lemma we shall show the corresponding result for the pair $(\hat{Z}, \hat{S})$:

Proposition 6.3. *We have $Aut_0(\hat{Z}, \hat{S}) = \{e\}$ in Case-H and $\cong \boldsymbol{C}^*$ in Case-P.*

First, we recall that $\varphi : Q_i \to Q_j$ induces isomorphisms $\varphi_i^{\pm} : H_i^{\pm} \to E_i^{\pm}$ and $\varphi_j^{\pm} : E_j^{\pm} \to H_j^{\pm}$. Then, with respect to to the natural G-equivariant isomorphisms $Q_l \cong H_l^+ \times E_{l'}^+$, $l = i, j$, we may write $\varphi = \varphi_i^+ \times \varphi_j^+$.

Hence, given one-parameter subgroup $\rho : \boldsymbol{C}^* \to G$ with the induced $\boldsymbol{C}^*$-actions on $(Q_l, H_l^+, E_l^+), l = i, j$, the following two conditions are equivalent:

(1) $\varphi : Q_i \to Q_j$ is ρ-equivariant

(2) $\varphi_l^+, l = i, j$, are both ρ-equivariant

In this case the one-parameter subgroup corresponding to the curve H_l^+ in $\tilde{S}_l^+$ is the one-parameter subgroup $\mu_l^+ := -\rho_l + \rho_{l+1}$ corresponding to L_l, where ρ_d is the one-parameter subgroup corresponding to $\boldsymbol{C}_d$ (see [**15**, Proposition 6.12 and p. 241 (10)]). On the other hand, the one-parameter subgroup ν_l^+ corresponding to E_l^+ is given by $\nu_l^+ = \mp(-\rho_l - \rho_{l+1})$, where we take $-$-sign (resp. $+$-sign) for $l = i$ (resp. j) (see [**15**, p. 235 (5)]). Hence by Lemma 4.2 the one-parameter subgroup which makes φ_i^+ (resp. φ_j^+) equivariant is

$$-\rho_i + \rho_{i+1} - \rho_j - \rho_{j+1} \quad (\text{resp.} - \rho_i + \rho_{i+1} + \rho_j + \rho_{j+1})$$

up to signs. (The assumption $k > 2$ made in Lemma 4.2 corresponds to the condition $m > 0$ here.) Hence the equivariancy of φ is given by the coincidence of these two subgroups. Namely, $\rho_j = \rho_{i+1}$ or $\rho_i = -\rho_{j+1}$. Since $i < j$, this implies that $j = i + 1$ or $i = k + j + 1$, and the latter holds only when $j = k$ and $i = 1$. Namely, in the cyclic sense we have $j = i + 1, 1 \le i \le k$.

From this we get the following:

Lemma 6.4. *Let G_1 be the maximal connected subgroup of G such that φ is G_1-equivariant. In Case-H, G_1 reduces to the identity, and in Case-P, $G_1 \cong \boldsymbol{C}^*$.*

Proof of Proposition 6.3. Since ν is the normalization, we have the natural inclusion $Aut_0(\hat{Z}, \hat{S}) \subseteq Aut_0(\tilde{Z}, \tilde{S} \cup \tilde{Q})$, and the latter is isomorphic to G by Lemma 6.2. On the other hand, with respect to this inclusion an element $g \in G$ is contained in $Aut_0(\hat{Z}, \hat{S})$ if and only if g commutes with φ. Thus the proposition follows from Lemma 6.4.   q.e.d.

Proposition 6.3 has the following implication.

Proposition 6.5. *In Case-H the isomorphism class of $(\hat{Z}, \hat{S})$ is independent of the choice of φ. In Case-P there exists a one-parameter family of isomorphisms $\varphi_t : (Q_1, F_1) \to (Q_2, F_2), t \in \boldsymbol{C}^*$, such that the corresponding pairs $(\hat{Z}_t, \hat{S}_t)$ form a non-trivial family of log-deformations of $(\hat{Z}, \hat{S})$ and exhausts all non-isomorphic pairs obtained from some φ.*

Proof. Let $I = Isom((Q_1, F_1), (Q_2, F_2))$ be the space of isomorphisms of (Q_1, F_1) to (Q_2, F_2). I has a natural structure of an algebraic principal homogeneous space of G. Then with respect to the algebraic action of G on I defined by $\psi \to g\psi g^{-1}, g \in G$, the identity component of the stabilizer group at φ is precisely identified with the algebraic subgroup G_1 of G in Lemma 6.4. Thus by Proposition 6.3 the first assertion is immediate from this since the G-action on I above is then transitive.

Similarly, in Case-P the G-action on I is not transitive and admits a one-dimensional quotient isomorphic to $\boldsymbol{C}^*$. Then any closed one-dimensional subspace of I which is mapped surjectively to this quotient parametrizes the family of isomorphisms φ with the properties of the proposition.   q.e.d.

7. Statement of main theorems

We retain the notations of the previous sections. Moreover, we denote by U the smooth locus $\hat{Z}_{reg} = \hat{Z} - \hat{Q}$ of $\hat{Z}$. Let $\hat{B}$ be the set of tangential points $\hat{p}_n^{\pm}$ and set $V = U - \hat{B}$. Thus $\hat{S}$ is a divisor with normal crossings on V.

The structure of the sheaf $\Omega_{\hat{Z}}(\log \hat{S})$ (see §2) along the singular locus $\hat{Q}$ of $\hat{Z}$ can be read from its structure at any of the singular points r of $\hat{F}$. In the notations of (61) let $p : A \times Y \to A$ and $q : A \times Y \to Y$ be the natural projections. Then we have

(62) $\qquad \Omega_{\hat{Z}} \cong p^*\Omega_A \oplus q^*\Omega_Y$ and $\Omega_{\hat{Z}}(\log S) \cong p^*\Omega_A \oplus q^*\Omega_Y(\log D)$,

such that the natural inclusion $\iota_{\hat{Z}} : \Omega_{\hat{Z}} \to \Omega_{\hat{Z}}(\log \hat{S})$ is given locally by

(63) $\qquad id_A \oplus \iota_Y : p^*\Omega_A \oplus q^*\Omega_Y \to p^*\Omega_A \oplus q^*\Omega_Y(\log D)$,

where id_A denotes the identity of A and ι_Y the natural inclusion on Y.

Proposition 7.1. *(1) $\Omega_{\hat{Z}}(\log \hat{S})$ and $\Theta_{\hat{Z}}(-\log \hat{S})$ are locally free on V and reflexive on U. In particular both sheaves have homological codimension ≥ 2 on U.*

(2) $\Theta_{\hat{Z}}(-\log \hat{S})$ is isomorphic to the dual module of $\Omega_{\hat{Z}}(\log \hat{S})$ on the whole $\hat{Z}$.

(3) There exists an exact sequence of $O_{\hat{Z}}$-modules

(64) $\qquad 0 \to \Omega_{\hat{Z}} \to \Omega_{\hat{Z}}(\log \hat{S}) \xrightarrow{b} \oplus_{\{\pm, l=i, j\}} O_{\hat{S}_l^{\pm}} \to 0$

where b is the (Poincare) residue homomorphism.

Proof. On U the assertions are all special cases of the results due to K. Saito [**39**] (see (1.6), (1.7), and (2.9) of [**39**]). In particular assertion (1) is true (see [**41**, (1.21) Corollary]). Moreover, he showed that there exists a natural perfect pairing on U

$$\alpha_U : \Omega_U(\log \hat{S}) \times \Theta_U(-\log \hat{S}) \to O_U$$

making $\Omega_U(\log \hat{S})$ and $\Theta_U(-\log \hat{S})$ the dual O_U-modules of each other.

We shall show that the pairing α_U and the exact sequence (64) both extend to the whole $\hat{Z}$. First we note the following two properties of $\Theta_{\hat{Z}}(-\log \hat{S})$ on the whole $\hat{Z}$:

(a) There exist no local sections of $\Theta_{\hat{Z}}(-\log \hat{S})$ whose support is dimension ≤ 2.

(b) Any local section of $\Theta_{\hat{Z}}(-\log \hat{S})$ defined outside an analytic subset, say, J, of codimension ≥ 2 extends holomorphically across J.

In fact, e.g., (b) follows from the fact that in the notation of the above definition the quotient $v(f)/f$, which is holomorphic outside J, extends to a holomorphic function across J.

Now as for the extension of α_U, α_U extends trivially to α_W on $W := \hat{Z} - \hat{F} = U \cup (\hat{Q} - \hat{F})$ since at the points of $\hat{Q} - \hat{F}$, $\Omega_{\hat{Z}}(\log \hat{S}) = \Omega_{\hat{Z}}$, $\Theta_{\hat{Z}}(-\log \hat{S}) = \Theta_{\hat{Z}}$ and $\Theta_{\hat{Z}}$ is the dual of $\Omega_{\hat{Z}}$. Since $\hat{F}$ is of codimension ≥ 2 in $\hat{Z}$, the weak normality of $\hat{Z}$ implies that α_W further extends to yield a natural pairing

$$\alpha_{\hat{Z}} : \Omega_{\hat{Z}}(\log \hat{S}) \times \Theta_{\hat{Z}}(-\log \hat{S}) \to O_{\hat{Z}}$$

on the whole $\hat{Z}$. Moreover, by the above properties of $\Theta_{\hat{Z}}(-\log \hat{S})$ we see that the induced map $\Theta_{\hat{Z}}(-\log \hat{S}) \to \Omega_{\hat{Z}}(\log \hat{S})^*$ is isomorphic since it is already isomorphic on W. In particular assertion (2) is proved.

Finally, from the local description of the inclusion $\iota_{\hat{Z}}$ (63) and from the standard Poincare residue exact sequence

$$(65) \qquad 0 \to \Omega_Y \to \Omega_Y(\log D) \overset{b_Y}{\to} \oplus_{s=1,2} O_{D_s} \to 0$$

for (Y, D), we readily obtain the exact sequence (64) extending the one obtained outside $\hat{Q}$ by [**39**]. q.e.d.

7.1. Log-deformations of $(\hat{Z}, \hat{S})$.

We consider the log-deformations of the pair $(\hat{Z}, \hat{S})$. This amounts to considering deformations of the pair $(\hat{Z}, \hat{S})$ which induce deformations of each irreducible components $\hat{S}_l^{\pm}$ of $\hat{S}$. Let

$$(66) \qquad g : (\mathcal{Z}, \mathcal{S}) \to T, \ (Z_o, S_o) = (\hat{Z}, \hat{S}), \ o \in T,$$

be the Kuranishi family of log-deformations of the pair $(\hat{Z}, \hat{S})$. For any $t \in T$, Z_t and S_t shall denote, respectively, the fibers over t of the projections $\mathcal{Z} \to T$ and $\mathcal{S} \to T$. S_t consists of four irreducible components $S_{l,t}^{\pm}$ which are deformations of $\hat{S}_l^{\pm}$, respectively. For a fixed l, $S_{l,t}^{\pm}$ are

mutually disjoint since this is true at $t = o$. Similarly, consider the Kuranishi family of deformations of the pair $(\hat{Z}, \hat{S})$ which are locally trivial at each point of $\hat{Z}$ (resp. at each point of $\hat{Q}$, resp. at each tangential point $p = \hat{p}_n^{\pm}$ of $U = \hat{Z}_{\text{reg}}$). These are subfamilies h (resp. $h_{\hat{Q}}$, resp. g_p) of g for a unique subspace A ((resp. $A(\hat{Q})$, resp. $T(p)$) of T. Clearly we have $A = A(\hat{Q}) \cap (\cap_{p \in \hat{B}} T(p))$.

First we note the following:

Theorem 7.2. *Let $g : (\mathcal{Z}, \mathcal{S}) \to T$ be the Kuranishi family of $(\hat{Z}, \hat{S})$ as above. Then the following hold:*

*(1) $\mathcal{Z}$ is smooth and $g : \mathcal{Z} \to T$ is a standard deformation of $\hat{Z}$ in the sense of [**13**, Definition 4.1].*

(2) T is smooth of dimension $3m$ in Case-H (resp. $3m+1$ in Case-P).

(3) $A(\hat{Q})$ and $T(p)$ are smooth hypersufaces of T passing through o such that $D := A(\hat{Q}) \cup (\cup_{p \in \hat{B}} T(p))$ is a divisor with normal crossings in T. In particular, $I := \cap_{p \in \hat{B}} T(p)$ is a smooth subspace of T of dimension m in Case-H (resp. $m + 1$ in Case-P) and A is a smooth hypersuface of I.

We set $C_{l,t}^{\pm} := S_{l,t}^{\pm} \cap (S_{l',t}^{+} \cup S_{l',t}^{-})$ with $\{l, l'\} = \{i, j\}$. As for the structure of the surfaces $S_{l,t}^{\pm}$, $l = i, j$, for $t \in T - A(\hat{Q})$ we shall show the following:

Theorem 7.3. *(1) Assume that $t \in T - A(\hat{Q})$. Then the fibers Z_t and $S_{l,t}^{\pm}$ are all smooth, and $S_{l,t}^{\pm}$ are surfaces of class VII. In Case-H (resp. Case-P) their minimal models $\bar{S}_{l,t}^{\pm}$ are either a hyperbolic or parabolic (resp. a parabolic) Inoue surface or a diagonal Hopf surface. Each $S_{l,t}^{\pm}$ is obtained from $\bar{S}_{l,t}^{\pm}$ by blowing up, as described in Lemma 3.3, a finite number of (possibly infinitely near) points on the image $\bar{C}_{l,t}^{\pm}$ of $C_{l,t}^{\pm}$ in $\bar{S}_{l,t}^{\pm}$.*

(2) Assume that $t \in I - A(\hat{Q})$. Then in Case-H (resp. Case-P), $S_{l,t}^{\pm}$ are all properly blown-up hyperbolic (resp. parabolic) Inoue surfaces. In Case-H the isomorphism class of $S_{l,t}^{\pm}$ is independent of t, $S_{l,t}^{+}$ and $S_{l,t}^{-}$ are isomorphic to each other, and $S_{i,t}^{\pm}$ and $S_{j,t}^{\pm}$ are transpositions of each other. If (i, j) is minimal, they are hyperbolic (resp. parabolic) Inoue surfaces. If $t \in T - D$, $S_{l,t}^{\pm}$ are blown-up diagonal Hopf surfaces.

(3) In Case-H the Kuranishi family g is universal.

The proofs of Theorem 7.2 and Theorem 7.3 will be given in Section 8.

7.2. Real deformations and twistor spaces. In the construction above, suppose that we have taken $\varphi : Q_i \to Q_j$ to be σ-equivariant,

which is always possible. Then $(\hat{Z}, \hat{S})$ has the induced real structure (denoted by the same letter σ) which interchanges $\hat{S}_l^+$ and $\hat{S}_l^-$, $l = i, j$.

First we consider Case-H, namely, we assume that $|j-i| > 1$. Then by Theorem 7.3 g is universal and the real structure σ on $\hat{Z} = Z_o$ extends canonically to the family $g : (\mathcal{Z}, \mathcal{S}) \to T$. Denote by T^σ the set of fixed points of σ, which is a real submanifold of T of real dimension $3m$. It is not contained in any proper analytic subset of T. For any point $t \in T^\sigma$, the fiber (Z_t, S_t) has the induced real structure σ_t. Recall that we set $M[m] = (S^1 \times S^3) \# m\bar{\boldsymbol{P}}^2$.

Theorem 7.4. *For any $t \in T^\sigma - A(\hat{Q})$, the fiber Z_t, together with the induced real structure σ_t, is a twistor space of an anti-self-dual biher-mitian structure $([g]_t, J_{1,t}, J_{2,t})$ on the smooth oriented manifold $M[m]$ such that $(M[m], \pm J_{1,t}) \cong S_{i,t}^\pm$ and $(M[m], \pm J_{2,t}) \cong S_{j,t}^\pm$. The structure of the surfaces $S_{l,t}^\pm$ are given by Theorem 7.3, part (1). Moreover, this family gives a universal family of anti-self-dual bihermitian structures on $M[m]$ at each point of $t \in T^\sigma - A(\hat{Q})$ with $3m$ real parameters.*

Proof. The fact that Z_t is a twistor space associated to a self-dual structure on the C^∞ 4-manifold $(S^1 \times S^3) \# m\boldsymbol{P}^2$, or equivalently, to an anti-self-dual structure on $M = M[m]$, is similar to [**13**, 4.2] except that in this case M is a "self-connected sum" of $m\boldsymbol{P}^2$ as we have identified Q_i and Q_j in a single manifold $\tilde{Z}$. Here the degrees of the surfaces $S_{l,t}^\pm$ are all equal to one as well as the original surfaces $S_l^\pm$ in Z. Thus $\{S_{l,t}^+, S_{l,t}^-\}, l = i, j$, are two pairs of elementary surfaces in Z_t. By Lemmas 5.1 and 5.2, they are in Case 2, and hence give an anti-self-dual bihermitian structure on $M[m]$. The rest follows immediately from Theorem 7.3.   q.e.d.

There exist nice subfamilies of this universal family. Most typically, when $t \in (T^\sigma \cap I) - A$, we know by Theorem 7.3 that $S_{i,t}^\pm$ and $S_{j,t}^\pm$ are both properly blown-up hyperbolic Inoue surfaces whose isomorphism class is independent of t. More precisely, we shall show the following:

Theorem 7.5. *Let S be an arbitrary properly blown-up hyperbolic Inoue surface. Let m be the second Betti number of S and $\bar{m}$ that of its minimal model. Then there exist $\bar{m}$ families of anti-self-dual bihermi-tian structures $([g]_t; J_{1,t}, J_{2,t})$ on $M[m]$ with real smooth m-dimensional parameters t such that $(M[m], \pm J_{1,t})$ and $(M[m], \pm J_{2,t})$ are biholomor-phic, respectively, to S and to its transposition ${}^t S$, independently of t.*

Proof of Theorem 7.5. By Proposition 4.14 there exists an admissible toric surface $\tilde{S}$ such that the given hyperbolic Inoue surface S is obtained by smoothing the rational surface $\hat{S}$ with a nodal curve obtained from $\tilde{S}$ via the procedure of Section 4. On the other hand, let $\bar{S}$ be the surface obtained from $\tilde{S}$ by blowing down its (-1)-curve E. Then by Lemma 5.3 there exists a Joyce twistor space Z which contains $\bar{S}$ as one of its

G-invariant elementary surfaces. We may take $\bar{S} = S_i^+$. Then we get a unique number j such that the twistor line L_j passes thruough the point $p \in \bar{S} = S_i^+$ which is the image of E. (Changing the numbering cyclically we can assume that $i < j$.)

Now starting from this Joyce twistor space Z and the pair of indices (i, j), we perform the construction of Section 5 and consider the universal family in Theorem 7.4. We restrict the family to $I \cap T^\sigma - A(\hat{Q})$ and obtain a real m-dimensional family of bihermitian structures on $M[m]$. By Theorem 7.3 the corresponding pairs of elementary surfaces are as described in the theorem. Finally, for the given S, according to Proposition 4.14 we have actually $\bar{m}$ choices of admissible toric surfaces $\tilde{S}$ and correspondingly we get $\bar{m}$ such families.   q.e.d.

Remark 7.1. The parameter t in the above theorem belongs to a complement of a real hyperplane in $\boldsymbol{R}^m$ in a neighborhood of the origin. In this sense the parameter space has actually two connected components. However, our construction depends on the initial Joyce self-dual metrics. Once the K-action is fixed, they form a connected $(m-1)$-dimensional family parametrized by $m+2$ points on the real projective line $\boldsymbol{RP}^1$ up to the action of $PSL(2, \boldsymbol{R})$. It should still be checked if the global parameter space is connected or not. On the other hand, the choice of K-action and of the index (i, j) gives discrete invariants for our construction. The $\bar{m}$ families in the theorem refer to $\bar{m}$ families with different discrete invariants but giving one and the same properly blown-up hyperbolic Inoue surface. Basically, similar remarks apply also to the other theorems. (See [**17**].)

Next, for a surface which is obtained from a properly blown-up hyperbolic Inoue surface by a small deformation, we can show a similar but weaker resut:

Theorem 7.6. *Let S be an arbitrary properly blown-up hyperbolic Inoue surface and C the unique anti-canonical curve on it. Let m be the second Betti number of S and $\bar{m}$ that of its minimal model. Let (S', C') be any fixed sufficiently small deformation of (S, C) in the Kuranishi family (21). Then S' admits $\bar{m}$ m-dimensional families of anti-self-dual bihermitian structures. Namely, there exist $\bar{m}$ families of anti-self-dual bihermitian structures $([g]_t; J_{1,t}, J_{2,t})$ on $M[m]$ with real and smooth m-dimensional parameters t such that $(M[m], J_{1,t})$ is biholomorphic to S' independently of t.*

Remark 7.2. As noted in Remark 3.1, (S', C') is a deformation of (S, C) as an anti-canonical pair with C' disconnected as well as C. By [**36**, Th.4.1] the existence of a disconnected anti-canonical curve is a necessary condition for the existence of an anti-self-dual bihermitian

structure. The above theorem is our strongest result toward the sufficiency of this condition, although the converse is in general not true, as the diagonal Hopf surface case already shows.

We shall give proofs of this and the next theorem in the next section. In Case-P our result is less complete. We state the result only in the minimal case for simplicity.

Theorem 7.7. *For any $m > 0$ there exists a real one-parameter family of parabolic Inoue surfaces with second Betti number m such that for any member S of this family we have a family of anti-self-dual bihermitian structures $\{([g]_t; J_{1,t}, J_{2,t})\}$ on $M[m]$ with real and smooth m-dimensional parameter t such that $(M[m], J_{1,t})$ and $(M[m], J_{2,t})$ are both biholomorphic to S.*

Remark 7.3. (1) It remains open to identify the parabolic Inoue surfaces which corresponds to the points of $I \cap T^\sigma - A(\hat{Q})$.

(2) For each fixed m a Joyce twistor space Z which produces Case-P in the minimal case is uniquely characterized, up to deformations, by the three equivalent conditions of Proposition 6.14 of [**15**]. (We call these *LeBrun-Joyce twistor spaces.*) In this case the weight sequence (53) of S_0 is given by $(a_1, \ldots, a_k) = (1, m, 1, 2, \ldots, 2)$, and then $(i, j) = (1, 2)$ is the unique choice of the indices.

Remark 7.4. One interesting probelm is to compare the above anti-self-dual bihermitian structures on parabolic Inoue surfaces with those constructed by LeBrun [**29**]. For instance, we can ask if both coincide at least for some parameters.

8. Proof of theorems

In this section we prove Theorems 7.2, 7.3, 7.6, and 7.7. The main part of the proof consists in showing the following theorem, which is the corresponding cohomological computations of relevant Ext and cohomology groups.

Theorem 8.1. *Both $H^2(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S}))$ and $Ext^2_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$ vanish. We have a natural short exact sequence*

$$(67) \qquad 0 \to H^1(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S})) \to Ext^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$$

$$\xrightarrow{\hat{c}} H^0(O_{\hat{Q}}) \oplus (\oplus_{p \in \hat{B}} \boldsymbol{C}_p) \to 0.$$

In Case-H and Case-P we have, respectively,

(68)
$$\dim H^1(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S})) = m - 1, \text{ and } \dim \mathrm{Ext}^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 3m \text{ and}$$

(69)
$$\dim H^1(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S})) = m, \text{ and } \dim \mathrm{Ext}^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 3m + 1.$$

Finally,

(70) $\dim Ext^0_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 0$ *in Case-H and* $= 1$ *in Case-P.*

Recall here that $\hat{B}$ is the set of tangential points $p_n^{\pm}$ on $\hat{Z}$. We start by determining the structure of $\mathcal{E}xt^i_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$.

Lemma 8.2. *(1)* $\mathcal{E}xt^0_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) \cong \Theta_{\hat{Z}}(-\log \hat{S})$ *and hence* $Ext^0_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) \cong H^0(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S}))$.
(2) $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) \cong \mathcal{E}^1_{\hat{Q}} \oplus \mathcal{E}^1_{\hat{B}}$, *where* $\mathcal{E}^1_Y$ *has support in* Y *(*$Y = \hat{Q}, \hat{B}$*). Moreover* $\mathcal{E}^1_{\hat{Q}} \cong O_{\hat{Q}}$ *and* $\mathcal{E}^1_{\hat{B}} \cong \oplus_{p \in \hat{B}} \mathbf{C}_p$, *where* $\mathbf{C}_p$ *is the skyscraper sheaf at* p.
(3) $\mathcal{E}xt^2_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 0$.

Proof. Since $\mathcal{E}xt^0 \cong \mathcal{H}om^0$, assertion (1) follows from Proposition 7.1, assertion (2). Applying $\mathcal{E}xt(-, O_{\hat{Z}})$ to the exact sequence (64), we obtain a long sheaf exact sequence

(71) $0 \to \Theta_{\hat{Z}}(-\log \hat{S}) \to \Theta_{\hat{Z}} \xrightarrow{b} \oplus_{\{\pm, l=i,j\}} \mathcal{E}xt^1_{O_{\hat{Z}}}(O_{\hat{S}_l^{\pm}}, O_{\hat{Z}})$

$$\to \mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) \xrightarrow{a} \mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}, O_{\hat{Z}}).$$

On the other hand, since $\Omega_{\hat{Z}}(\log \hat{S})$ is locally free outside $\hat{Q} \cup \hat{B}$, the support of $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$ is contained in $\hat{Q} \cup \hat{B}$. Locally along $\hat{Q}$ the map a is induced from the map in (63). Since $q^*\Omega_Y(\log D)$ and $q^*\Omega_Y$ are locally free and hence their $\mathcal{E}xt^1$'s vanish, a is locally isomorphic to the identity $\mathcal{E}xt^1_{O_{\hat{Z}}}(p^*\Omega_A, O_{\hat{Z}}) \to \mathcal{E}xt^1_{O_{\hat{Z}}}(p^*\Omega_A, O_{\hat{Z}})$. In particular a is isomorphic along $\hat{Q}$. On the other hand, by Friedman [**14**, Corollary 2.4] $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}, O_{\hat{Z}}) \cong O_{\hat{Q}}$ along $\hat{Q}$. Also we see from (63) that $\Omega_{\hat{Z}}(\log \hat{S})$ and $\Omega_{\hat{Z}}$ are locally isomorphic. Hence $\mathcal{E}xt^2_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$ vanishes since so does $\mathcal{E}xt^2_{O_{\hat{Z}}}(\Omega_{\hat{Z}}, O_{\hat{Z}})$ along $\hat{Q}$ (see [**14**, Section 2]).

Thus assertions (2) and (3) are shown along $\hat{Q}$. It remains to check these assertions at each point p of $\hat{B}$. First of all, since the homological codimension of $\Omega_{\hat{Z}}(\log \hat{S})$ is two at p by Proposition 7.1, assertion (3) is true there. We shall compute $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})$ at p. Since $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}, O_{\hat{Z}}) = 0$ at p, it is the cokernel of b in (71). Since $\hat{S}_l^{\pm}$ are Cartier divisors, we have $\mathcal{E}xt^1_{O_{\hat{Z}}}(O_{\hat{S}_l^{\pm}}, O_{\hat{Z}})) \cong N_l^{\pm}$, where $N_l^{\pm} = [\hat{S}_l^{\pm}]|\hat{S}_l^{\pm}$ is the normal bundle of $\hat{S}_l^{\pm}$ in $\hat{Z}$.

Now we work in the local model (60) so that we may put $\hat{Z} = \mathbf{C}^3(x, y, z)$. Let $D_m, m = 1, 2$, be the irreducible components of $\hat{S}$ at p defined by the local equations $f_1 := z = 0$ and $f_2 := z - xy = 0$, respectively. Let $N_m = Hom(I_m, O_{D_m})$ be the normal sheaves of

D_m in $\hat{Z}$, where I_m is the ideal sheaf of D_m. Then b is given by $b(\theta) = (\theta(f_1)|D_1, \theta(f_2)|D_2)$ in terms of the above identifications. Then for $\theta = \partial/\partial x, \partial/\partial y, \partial/\partial z$ we obtain $(0, -y), (0, -x), (1, 1)$ restricted to (D_1, D_2). We conclude immediately that the cokernel of b, which has support in p, is in fact one-dimensional. This shows assertion (2) at p and the lemma. q.e.d.

Remark 8.1. For $p \in \hat{B}$, $\mathcal{E}xt^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}})_p (\cong \boldsymbol{C})$ is considered to be the tangent space of the local versal log-deformation of the pair $(\hat{Z}, \hat{S})$ considered as a germ at p. The versal deformation $(\hat{Z}_t, \hat{S}_t) = (\boldsymbol{C}^3, \hat{S}_t)$ is given explicitly by the defining equation $z(z - xy + t) = 0$ of $\hat{S}_t$. For $t \neq 0$, the intersection of the two irreducible components $D_{1,t}$ and $D_{2,t}$ of $\hat{S}_t$ is now smooth, along which $D_{m,t}$ are transversal. We shall denote this versal family by $g(p) : (\mathcal{Z}(p), \mathcal{S}(p)) \to T(p)$, $(Z(p)_o, S(p)_o) = (\hat{Z}(p), \hat{S}(p))$, $o \in T(p)$. The original Kuranishi family induces a log-deformation of the germ $(\hat{Z}(p), \hat{S}(p))$ and we have a versal map $\tau(p) : T \to T(p)$.

We next compute the cohomology groups $H^i(\Theta_{\hat{Z}}(-\log \hat{S}))$ by relating them to the corresponding cohomology groups of the blown-up twistor space $(\tilde{Z}, \tilde{S})$ and of the original Joyce twistor space (Z, S), where $\tilde{S} = \tilde{S}_i \cup \tilde{S}_j$ and $S = S_i \cup S_j$. We first record the infinitesimal form of the results of Lemma 6.2 and Proposition 6.3.

Proposition 8.3. We have $h^0(\Theta_Z(-\log S)) = 2$, $h^0(\Theta_{\tilde{Z}}(-\log(\tilde{S} + \tilde{Q}))) = 2$, and $h^0(\Theta_{\hat{Z}}(-\log \hat{S})) = 0$ in Case-H and $= 1$ in Case-P.

Now we compare the cohomology groups of $(\hat{Z}, \hat{S})$ with those of $(\tilde{Z}, \tilde{S})$ via the normalization exact sequence

$$0 \to \Theta_{\hat{Z}}(-\log \hat{S}) \to \Theta_{\tilde{Z}}(-\log(\tilde{S} + \tilde{Q})) \to \Theta_{\hat{Q}}(-\log \hat{F}) \to 0$$

with associated long exact sequence

$$0 \to H^0(\Theta_{\hat{Z}}(-\log \hat{S})) \to H^0(\Theta_{\tilde{Z}}(-\log(\tilde{S} + \tilde{Q}))) \xrightarrow{a} H^0(\Theta_{\hat{Q}}(-\log \hat{F})) \to$$

(72)

$$\to H^1(\Theta_{\hat{Z}}(-\log \hat{S})) \to H^1(\Theta_{\tilde{Z}}(-\log(\tilde{S} + \tilde{Q}))) \to H^1(\Theta_{\hat{Q}}(-\log \hat{F})) \to$$

$$\to H^2(\Theta_{\hat{Z}}(-\log \hat{S})) \to H^2(\Theta_{\tilde{Z}}(-\log(\tilde{S} + \tilde{Q}))) \to H^2(\Theta_{\hat{Q}}(-\log \hat{F})) \to .$$

Here $\hat{F}$ is the anti-canonical cycle of the toric surface $\hat{Q}$, and hence we have $\Theta_{\hat{Q}}(-\log \hat{F}) = O_{\hat{Q}}^2$; thus $H^i(\Theta_{\hat{Q}}(-\log \hat{F})) = 0$ for $i = 1, 2$ and $H^0(\Theta_{\hat{Q}}(-\log \hat{F})) = \boldsymbol{C}^2$. In view of Proposition 8.3 this implies that a is isomorphic in Case-H and has one-dimensional image in Case-P. Thus

the above exact sequence reduces in Case-H to

$$(73) \qquad H^0(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))) \cong H^0(\Theta_{\hat{Q}}(-\log \hat{F})) \cong \boldsymbol{C}^2,$$

$$(74) \qquad H^1(\Theta_{\hat{Z}}(-\log \hat{S})) \cong H^1(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))),$$

and in Case-P to the two short exact sequences

$$(75) \qquad 0 \to \boldsymbol{C} \to H^0(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))) \overset{a}{\to} \boldsymbol{C} \to 0,$$

$$(76) \qquad 0 \to \boldsymbol{C} \to H^1(\Theta_{\hat{Z}}(-\log \hat{S})) \to H^1(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))) \to 0.$$

In both cases we have

$$(77) \qquad H^2(\Theta_{\hat{Z}}(-\log \hat{S})) \cong H^2(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))).$$

Next, we compare the cohomology groups $H^i(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q})))$ with those of (Z, S). Namely, we have:

Lemma 8.4. *We get natural isomorphisms*

$$(78) \qquad H^q(\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q}))) \cong H^q(Z, \Theta_Z(-\log S)), \;\; q \geq 0.$$

Proof. There exists a natural sheaf isomorphism

$$\Theta_{\tilde{Z}}(-\log(\tilde{S}+\tilde{Q})) \cong \mu^*\Theta_Z(-\log S)$$

induced by μ_*. Together with the Leray spectral sequence for

$$E_2^{p,q} := H^p(R^q\mu_*\mu^*\Theta_Z(-\log S)) \Rightarrow H^{p+q}(\mu^*\Theta_Z(-\log S))$$

and the fact that $R^q\mu_*O = 0$ for $q > 0$, we get the desired isomorphisms. (Note that in a neighborhood of L_i and L_j the sheaf $\Omega_Z(\log S)$ is locally free so that the projection formula

$$R^q\mu_*\mu^*\Theta_Z(-\log S) \cong R^q\mu_*O_{\tilde{Z}} \otimes \Theta_Z(-\log S)$$

holds.) $\hspace{6cm}$ q.e.d.

We have thus reduced our computation to that of the cohomology groups on Z. Consider now the short exact sequence of O_Z-modules

$$0 \to \Theta_Z(-S) \to \Theta_Z(-\log S) \to \Theta_S \to 0$$

and the associated cohomology exact sequence

$$
\begin{aligned}
& 0 \to H^0(\Theta_Z(-S)) \to H^0(\Theta_Z(-\log S)) \to H^0(\Theta_S) \to \\
(79) \quad & \to H^1(\Theta_Z(-S)) \to H^1(\Theta_Z(-\log S)) \to H^1(\Theta_S) \to \\
& \to H^2(\Theta_Z(-S)) \to H^2(\Theta_Z(-\log S)) \to H^2(\Theta_S) \to .
\end{aligned}
$$

We compute the dimensions of the spaces as follows.

Lemma 8.5. *We have*

$$(80) \qquad h^0(\Theta_Z(-S)) = 0, \ h^1(\Theta_Z(-S)) = 0, \ h^2(\Theta_Z(-S)) = m+1,$$
$$h^3(\Theta_Z(-S)) = 0,$$

$$(81) \qquad h^0(\Theta_Z(-\log S)) = 2, \ h^3(\Theta_Z(-\log S)) = 0,$$

$$(82) \qquad h^0(\Theta_S) = 2, \ h^2(\Theta_S) = 0, \ h^3(\Theta_S) = 0.$$

Proof. Since $K = -S$, we have $h^i(\Theta_Z(-S)) = h^{3-i}(\Omega_Z) = h^{1,3-i}(Z)$, where $h^{p,q}$ denotes the Hodge numbers. We know that $h^{k,0}(Z) = 0$ for any twistor space and hence also $h^{0,k}(Z) = 0$ since Z is Moishezon. By the same reason, we have $b_k(Z) = \sum_{p+q=k} h^{p,q}(Z)$ for the Betti numbers $b_k(Z) = b_k(M) + b_{k-2}(M)$, where $M = m\mathbf{P}^2$. In particular for odd k, we have $b_k(Z) = 0$. Thus $h^i(\Theta_Z(-S)) = 0$ if i is odd. Also from $h^{1,1} = b_2 = m+1$, we have $h^2(\Theta_Z(-S)) = m+1$.

Further, we show that $h^0(\Theta_Z(-S)) = 0$. Take any twistor line L and consider the standard short exact sequence

$$0 \to \Theta_L \to \Theta_Z|L \to N \to 0$$

where $N \cong O(1) \oplus O(1)$ is the normal bundle of L in Z. Since $K|L$ is of degree -4, we get that $h^0(\Theta_L(-S)) = h^0(N(-S)) = 0$. Hence the above exact sequence tensored with K yields $h^0(L, \Theta_Z(-S)|L) = 0$, from which follows the desired vanishing since L is arbitrary. Thus (80) is proved.

The identity component of the automorphism group of Z is $\mathbf{C}^{*2}$ for $m \geq 2$ and it preserves S. Also, in case $m = 1, \mathbf{C}^{*2}$ is the maximal connected automorphism group of (Z, S). Thus $h^0(\Theta_Z(-\log S)) = 2$. We get $h^3(\Theta_S) = 0$ since dim $S = 2$. Then together with (80) we deduce $h^3(\Theta_Z(-\log S)) = 0$ from the exact sequence (79). This shows (81).

We show that $h^2(\Theta_S) = 0$. Let $\omega_S := K[S]|S = O_S$ be the dualizing sheaf of S. By Serre duality $H^2(\Theta_S)$ is dual to $Ext^0(\Theta_S, \omega_S) \cong Ext^0(\Theta_S, O_S) \cong H^0(\Omega_S^{**})$, where Ω_S^{**} denotes the double dual of Ω_S. (See [**14**, Lemma (2.9)] for the structure of Ω_S^{**} outside tangential points.) The last space injects into $\oplus_{\pm,l} H^0(\Omega_{S_l^\pm})$ which vanishes. Hence $h^2(\Theta_S) = 0$. From the exact sequence (79) together with (80) and (81), we get $h^0(\Theta_S) = 2$. (82) is proved. q.e.d.

The main part of the exact sequence (79) now reduces to

$$(83) \qquad 0 \to H^1(\Theta_Z(-\log S)) \to H^1(\Theta_S) \xrightarrow{\delta} H^2(\Theta_Z(-S))$$
$$\to H^2(\Theta_Z(-\log S)) \to 0.$$

Lemma 8.6. $h^1(\Theta_S) = 2m$.

Proof. We consider the normalization exact sequence

$$(84) \qquad 0 \to \Theta_S \to \oplus_{l,\pm} \Theta_{S_l^\pm}(-\log B_l^\pm) \xrightarrow{a} \oplus_\alpha \Theta_{B_\alpha}((0+\infty)) \to 0$$

where $B_l^{\pm}$ is the anti-canonical cycle of $S_l^{\pm}$, and B_α are the irreducible components of the curve $B := C \cup L_i \cup L_j$; $0 = 0_\alpha$ and $\infty = \infty_\alpha$ are the two points of intersection of B_α and the other irreducible components of B (see (52)). (The surjectivity at the tangential points of a may be shown by using two sections $z\partial/\partial z - x\partial/\partial x, z\partial/\partial z - y\partial/\partial y$ of $\Theta_Z(-\log S)$ in the local notations of (60).)

Note that B has $2k + 2$ irreducible components. Let

$$0 \to H^0(\Theta_S) \to \oplus_{l,\pm} H^0(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \to \oplus_\alpha H^0(\Theta_{B_\alpha}(0+\infty)) \to$$

$$(85) \quad \to H^1(\Theta_S) \to \oplus_{l,\pm} H^1(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \oplus_\alpha H^1(\Theta_{B_\alpha}(0+\infty))$$

$$\to H^2(\Theta_S) \to \oplus_{l,\pm} H^2(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \oplus_\alpha H^2(\Theta_{B_\alpha}(0+\infty)) \to$$

be the associated long exact sequence.

Since $\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \cong O_{S_l^{\pm}}^2$, we have

$$h^0(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) = 2 \quad \text{and} \quad h^i(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) = 0, i > 0.$$

Similarly, we have $\Theta_{B_\alpha}((0_+\infty)) \cong O_{B_\alpha}$ and hence

$$h^0(\Theta_{B_\alpha}(0+\infty)) = 1 \quad \text{and} \quad h^i(\Theta_{B_\alpha}(0+\infty)), i > 0.$$

Thus we get $H^2(\Theta_S) = 0$ (deduced above by a different method), and the exact sequence

$$(86) \quad 0 \to H^0(\Theta_S) \to \oplus_{l,\pm} H^0(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \to \oplus_\alpha H^0(\Theta_{B_\alpha}(0+\infty))$$

$$\to H^1(\Theta_S) \to 0$$

with $\oplus_{l,\pm} H^0(\Theta_{S_l^{\pm}}(-\log B_l^{\pm})) \cong \boldsymbol{C}^8$ and $\oplus_\alpha H^0(\Theta_{B_\alpha}(0+\infty)) \cong \boldsymbol{C}^{2k+2}$. Together with (82) we get $h^1(\Theta_S) = 2m$.   q.e.d.

We next prove a lemma which will be used in the proof of Proposition 8.8 below.

Lemma 8.7. $\mathcal{E}xt_{O_S}^1(\Theta_S, O_S) = 0.$

Proof. First we prove this at a non-tangential point, i.e., at a point p where S has only normal crossings singularities. We apply $\mathcal{E}xt_{O_S}^1(-, O_S)$ to the sequence (84) and obtain

$$\to \oplus_{l,\pm} \mathcal{E}xt_{O_S}^1(\Theta_{S_l^{\pm}}(-\log B_l^{\pm}), O_S) \to \mathcal{E}xt_{O_S}^1(\Theta_S, O_S)$$

$$\to \oplus_\alpha \mathcal{E}xt_{O_S}^2(\Theta_{B_\alpha}(0+\infty), O_S).$$

It suffices to show that $\mathcal{E}xt_{O_S}^1(\Theta_{S_l^{\pm}}(-\log B_l^{\pm}), O_S) = 0$ and $\mathcal{E}xt_{O_S}^2(\Theta_{B_\alpha}(0+\infty), O_S) = 0$ at p. We prove this when p is a general point, leaving similar arguments to the reader at four triple points. Let $S_\alpha, \alpha = 1, 2$, be the irreducible components of S passing through p with structure sheaves O_α and put $D = S_1 \cap S_2$, the singular locus of S at p. Locally at p, $\Theta_{S_l^{\pm}}(-\log B_l^{\pm}) \cong O_1 \oplus O_2 \oplus O_S$ and $\Theta_{B_\alpha}((0+\infty)) \cong \Theta_{B_\alpha} \cong O_D, D =$

$S_1 \cap S_2$. Therefore what we have to check is that $\mathcal{E}xt^1_{O_S}(O_{S_\alpha}, O_S) = 0$ and $\mathcal{E}xt^2_{O_S}(O_D, O_S) = 0$. For this we consider the short exact sequences

(87) $$0 \to I_\alpha \to O_S \to O_\alpha \to 0,$$

(88) $$0 \to I_D \to O_S \to O_D \to 0,$$

and the associated long $\mathcal{E}xt$-exact sequences. The desired assertion then follows from the following facts:

(1) $\mathcal{H}om_{O_S}(O_S, O_S) \to \mathcal{H}om_{O_S}(O_\alpha, O_S)$ is surjective, being isomorphic to the quotient map $O_S \to O'_\alpha, \{\alpha, \alpha', \} = \{1, 2\}$.

(2) $\mathcal{E}xt^i_{O_S}(I_D, O_S) = 0, i \geq 1$ (see [**14**, Lemma 2.8]).

Thus $\mathcal{E}xt^1_{O_S}(\Theta_S, O_S)$ has support in the tangential points. But at any of these points p we can find an exact sequence

$$0 \to O_S^2 \to \Theta_S \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ has support in p. Then it is easily seen that $\mathcal{E}xt^1_{O_S}(\mathcal{Q}, O_S) = 0$. (Take an exact sequence

$$0 \to F' \to F \to \mathcal{Q} \to 0$$

with some coherent O_S-modules with F free. Then $\mathcal{H}om_{O_S}(F, O_S) \to \mathcal{H}om_{O_S}(F', O_S)$ is surjective since $\mathcal{Q}$ has support in p and S is weakly normal.) By applying $\mathcal{E}xt^1(-, O_S)$ to the above exact sequence we get the desired vanishing of $\mathcal{E}xt^1_{O_S}(\Theta_S, O_S)$. q.e.d.

Proposition 8.8. *The map δ in (83) is surjective, and we have* $H^2(\Theta_Z(-\log S)) = 0$.

Proof. By Serre dualty it suffices to show that the dual map $\gamma : H^1(\Omega_Z) \to Ext^1(\Theta_S, O_S)$ of δ is injective, where we have used the isomorphism $\omega_S \cong O_S$. Consider the exact sequence

$$0 \to H^1(\Omega_S^{**}) \to Ext^1_{O_S}(\Theta_S, O_S) \to H^0(\mathcal{E}xt^1_{O_S}(\Theta_S, O_S))$$

similar to (4). By Lemma 8.7 we may identify $Ext^1_{O_S}(\Theta_S, O_S)$ with $H^1(\Omega_S^{**})$ and γ with the natural map $\gamma' : H^1(\Omega_Z) \to H^1(\Omega_S^{**})$. Taking any of the irreducible components of S, say, S_i^+, we obtain a natural map $H^1(\Omega_S^{**}) \to H^1(\Omega_{S_i^+}^{**}) \cong H^1(\Omega_{S_i^+})$. Composing γ' with this, we obtain the natural map $H^1(\Omega_Z) \to H^1(\Omega_{S_i^+})$, which in turn is identified with the restriction map of corresponding complex cohomology groups $r : H^2(Z, \boldsymbol{C}) \to H^2(S_i^+, \boldsymbol{C})$.

In fact we can prove the injectivity of r precisely as in the proof of [**16**, Lemma 5.4], where we showed the injectivity of $H^2(Z, \boldsymbol{C}) \to H^2(S, \boldsymbol{C})$ for a smooth member S of $|K^{-\frac{1}{2}}|$. Since in our case $S = S_i^+$ is an elementary surface, we have only to note the following: S_i^+ is obtained as an m-times blowing-up of $\boldsymbol{P}^2$ so that $H^2(S_i^+, \boldsymbol{C})$ is spanned by the exceptional curves and by the first Chern class. Thus the proposition

is proved. (Note that the argument above is in principle similar to that for the vanishing of $H^2(Z, \Theta_Z(-\log D))$ for a Joyce twistor space and a smooth element D of $K^{-\frac{1}{2}}$ (see [**16**, Theorem 5.1])). q.e.d.

By (79) and Lemmas 8.5 and 8.6 we get:

Corollary 8.9. $h^1(\Theta_Z(-\log S)) = m - 1$.

Now we are in a position to prove Theorem 8.1.

Proof of Theorem 8.1. By Lemma 8.2, 3) $H^0(\mathcal{E}xt^2_{O_Z}(\Omega_{\hat{Z}}(\log \hat{S}), O_Z)) = 0$. By (2) of the same lemma we get $H^1(\mathcal{E}xt^1_{O_Z}(\Omega_{\hat{Z}}(\log \hat{S}), O_Z)) \cong H^1(O_{\hat{Q}}) = 0$. Thus in view of (3) for $(X, Y) = (\hat{Z}, \hat{S})$ the last arrow c in (4) is surjective. Since $H^2(\Theta_Z(-\log S)) = 0$ by Proposition 8.8, we have the first two vanishings by using the isomorphisms (78) and (77). The sequence (4) reduces to (67) above, again by using Lemma 8.2. Finally, from (73)–(76) we get the dimensional counts of (68) and (69), and the final assertion comes from Proposition 8.3. q.e.d.

We still have to compare the deformations of the pair $(\hat{Z}, \hat{S})$ with those of the subspaces $(\hat{S}_l, \hat{C}_l)$. We start from the following:

Lemma 8.10. *There exists a natural exact sequence of $O_{\hat{Z}}$-modules:*

$$(89) \qquad 0 \to \Omega_{\hat{Z}}(\log \hat{S}) \xrightarrow{a} \Omega_{\hat{Z}}(\log \hat{S}_{l'})(\hat{S}_l) \xrightarrow{b} \Omega'_{\hat{S}_l}(\log \hat{C}_l) \otimes \hat{N}_l \to 0,$$

where $\hat{N}_l := N_{\hat{S}_l/\hat{Z}}$ is the normal bundle of $\hat{S}_l$ in $\hat{Z}$.

Proof. The map a is the natural inclusion. The map b is given by the tensor product of the natural restriction maps $\Omega_{\hat{Z}}(\log \hat{S}_{l'}) \to \Omega'_{\hat{S}_l}(\log \hat{C}_l)$ and $[\hat{S}_l] \to \hat{N}_l$. Note that a defining equation of $\hat{S}_{l'}$ in $\hat{Z}$ restricts one of $\hat{C}_l(= \hat{S}_l \cap \hat{S}_{l'})$ in $\hat{S}_l$ so that the first restriction makes sense. This remark also implies that b is surjective. Now what we have to show is that Ker b = Im a.

Indeed, locally at points which are regular for both $\hat{Z}$ and $\hat{C}_l$, the map b takes the form

$$dx/xy, \; dy/y, \; dz/y \to dx/x|\hat{S}_l, \; dy|\hat{S}_l = 0, \; dz|\hat{S}_l,$$

while the image of a is generated by dx/x, dy/y, dz, where $x = 0$ (resp. $y = 0$) is the local equation of $\hat{S}_{l'}$ (resp. $\hat{S}_l$). Here we consider sections of $\Omega_{\hat{Z}}(\log \hat{S}_{l'})(\hat{S}_l)$ as meromorphic 1-forms on $\hat{Z}$ and identify those of $\Omega'_{\hat{S}_l}(\log \hat{C}_l) \otimes \hat{N}_l$ as sections of $\Omega'_{\hat{S}_l}(\log \hat{C}_l)$ regarding $(1/y)|\hat{S}_l$ as giving the trivialization of N_l. The desired assertion is now obvious. By applying the product principle (62) and (63), the same consideration applies also at singular points of $\hat{Z}$.

It only remains to consider the tangential points. If a local section of $\Omega_{\hat{Z}}(\log \hat{S}_{l'})(\hat{S}_l)$ at such a point p is mapped to zero by b, by what we

have proved above it is in the image of a section s of $\Omega_{\hat{Z}}(\log \hat{S})$ outside p. But by Proposition 7.1 the latter sheaf is reflexive at p and hence s extends across p as a section of the same sheaf. The assertion is thus proved on the whole $\hat{Z}$.   q.e.d.

In the same way as we get the isomorphism (40) by using Lemma 2.2, we obtain the isomorphisms

$$(90) \qquad Ext^i_{O_{\hat{Z}}}(\Omega'_{\hat{S}_l}(\log \hat{C}_l) \otimes \hat{N}_l, O_{\hat{Z}}) \cong Ext^{i-1}_{O_{\hat{S}_l}}(\Omega'_{\hat{S}_l}(\log \hat{C}_l), O_{\hat{S}_l}).$$

Comparing with the Ext sequences associated with (89), we obtain a natural map

$$(91) \qquad Ext^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) \xrightarrow{\alpha} Ext^1_{O_{\hat{S}_l}}(\Omega'_{\hat{S}_l}(\log \hat{C}_l), O_{\hat{S}_l}).$$

Together with part of the local to global sequences, this fits into the following commutative diagram:

$$
(92) \quad
\begin{array}{ccc}
H^1(\Theta_{\hat{Z}}(-\log \hat{S})) & \hookrightarrow & \\
Ext^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) & \xrightarrow{\hat{c}} & H^0(O_{\hat{Q}}) \oplus (\oplus_{p \in B} \boldsymbol{C}_p) \\
\alpha \downarrow & & u \downarrow \\
Ext^1_{O_{\hat{S}_l}}(\Omega'_{\hat{S}_l}(\log \hat{C}_l), O_{\hat{S}_l}) & \stackrel{c}{\cong} & H^0(O_{\hat{F}_l}) \oplus (\oplus_{p \in \hat{B}_l} \boldsymbol{C}_p) \\
\beta \downarrow & & v \| \\
Ext^1_{O_{\hat{C}_l}}(\Omega_{\hat{C}_l}, O_{\hat{C}_l}) & \stackrel{d}{\cong} & (\oplus_{p \in \bar{B}_l - \hat{B}_l} \boldsymbol{C}_p) \oplus (\oplus_{p \in \hat{B}_l} \boldsymbol{C}_p),
\end{array}
$$

recalling the isomorphisms c in (38) and d in (24), where the top sequence is nothing but (67) and $\bar{B}_l = \bar{B}_l^+ \cup \bar{B}_l^-$ and $\hat{B}_l = \hat{B}_l^+ \cup \hat{B}_l^-$ with $\bar{B}_l^{\pm}$ and $\hat{B}_l^{\pm}$ defined for each $\hat{S}_l^{\pm}$ as $\bar{B}$ and $\hat{B}$ are defined for $\hat{C}$ in Section 4. Note also that c and d are the direct sum of two isomorphisms

$$(93) \qquad c_l^{\pm} : Ext^1_{O_{\hat{S}_l^{\pm}}}(\Omega'_{\hat{S}_l^{\pm}}(\log \hat{C}_l^{\pm}), O_{\hat{S}_l^{\pm}}) \cong H^0(O_{\hat{F}_l^{\pm}}) \oplus (\oplus_{p \in \hat{B}_l^{\pm}} \boldsymbol{C}_p)$$

and

$$(94) \qquad d_l^{\pm} : Ext^1_{O_{\hat{C}_l^{\pm}}}(\Omega_{\hat{C}_l^{\pm}}, O_{\hat{C}_l^{\pm}}) \cong (\oplus_{p \in \bar{B}_l^{\pm} - \hat{B}_l^{\pm}} \boldsymbol{C}_p) \oplus (\oplus_{p \in \hat{B}_l^{\pm}} \boldsymbol{C}_p).$$

By construction we have the natural identification of sets: $\hat{B}_l = B$. In particular the right vertical maps u and v give isomorphisms (identifications) of the second factors, while on the first factors these maps are isomorphic to the diagonal embeddings

$$(95) \qquad\qquad\qquad \boldsymbol{C} \to \boldsymbol{C}^2 \to \boldsymbol{C}^4.$$

Proof of Theorem 7.2. assertions (1) and (2) are immediate consequences of Theorem 8.1 in view of Proposition 2.1. For (3), first note that α in (91) is identified with the differential of the versal map from the Kuranishi space T of deformations of $(\hat{Z}, \hat{S})$ to the Kuranishi space $\hat{T}_l$ of deformations of the pair $(\hat{S}_l, \hat{C}_l)$, which is the disjoint union of $(\hat{S}_l^{\pm}, \hat{C}_l^{\pm})$. Let

$v_{\hat{Q}}$ and $v_p, p \in B$, be the natural projections from $H^0(O_{\hat{Q}}) \oplus (\oplus_{p \in B} \boldsymbol{C}_p)$ to $H^0(O_{\hat{Q}}) \cong \boldsymbol{C}$ and to $\boldsymbol{C}_p$, respectively. Then the Zariski tangent spaces of $A(\hat{Q})$ and $T(p)$ are naturally identified with the kernels of $v_{\hat{Q}} \hat{c}$ and $v_p \hat{c}$, respectively, which are of codimension one.

For $p \in \bar{B}_l$ let T'_p be the Kuranishi space of deformations of the isolated singularity (C_l, p), which is smooth of dimension one. Then we have a versal map $\tau'_p : (T, o) \to (T'_p, o)$, whose differential is identified with $d\beta\alpha$ composed with the projection to $\boldsymbol{C}_p$, which is a surjection. The inverse image $\tau'^{-1}_p(o)$, which is independent of the choice of τ'_p, is easily identified with $T(p)$ when $p \in \hat{B} = \hat{B}_l$ and with $A(\hat{Q})$ when $p \in \bar{B}_l - \hat{B}_l$ (independently of p). Thus by the properties of the diagram (92) we see easily that $T(p)$ and $A(\hat{Q})$ are both smooth of codimension one and D has the properties stated in assertion (3). The rest of the assertions in assertion (3) is obvious.   q.e.d.

Proof of Theorem 7.3. First of all, 3) is a consequence of general theory [**34**] in view of (70) in Theorem 8.1. The Kuranishi family of $(\hat{Z}, \hat{S})$ induces a deformation of each of the pairs $(\hat{S}^{\pm}_l, \hat{C}^{\pm}_l), l = i, j$, and the induced versal maps $\tau^{\pm}_l : T \to \hat{T}^{\pm}_l$ between the corresponding Kuranishi spaces T and $\hat{T}^{\pm}_l$ are submersions with $(\tau^{\pm}_l)^{-1}(A^{\pm}_l) = A(\hat{Q})$ by the diagram (92) in view of (93) and (95), where $A^{\pm}_l$ are the subspaces corresponding to A in Proposition 4.11. Thus Z_t and $S^{\pm}_{l,t}$ are smooth for $t \in T - A(\hat{Q})$ and the structure of the surfaces $S^{\pm}_{l,t}$ as stated in parts (1) and (2) of the theorem is obtained from Proposition 4.11 and part (2) of Remark 4.2. It only remains to prove the relations among $(S^{\pm}_{l,t})$, which is given in the next lemma.   q.e.d.

Lemma 8.11. $S^+_{l,t}$ *and* $S^-_{l,t}$ *are isomorphic.* $S^{\pm}_{i,t}$ *and* $S^{\pm}_{j,t}$ *are transpositions to each other.*

Proof. The weight sequence of $\tilde{S}^{\pm}_i$ is given by (55) and similarly for $\tilde{S}^{\pm}_j$. From the two chains between 1 and -1 in (55) arise the two cycles of $\hat{S}^{\pm}_l$, which in turn produces the two cycles $C^{\pm,\alpha}_{l,t}, \alpha = 1, 2$, on $S^{\pm}_{l,t}$ via smoothing. The weight sequence of $C^{\pm,\alpha}_{l,t}$ are computed by (55) and the formulae in Lemma 4.12. From this already follows the first isomorphy. For instance the chain $\tilde{C}^{\mp}_{i+1} + \cdots + \tilde{C}^{\mp}_j$ with weight sequence (55) gives rise to that of $C^{\pm,\alpha}_{l,t}$, and its toric numbering is determined by the conditions $\tilde{C}^{\mp}_{i+1} \cap H^{\pm}_i \neq \emptyset$ and $\tilde{C}^{\mp}_j \cap E^{\pm}_i \neq \emptyset$. Since by Lemma 4.15 toric numbering and canonical numbering coincide, we see that $S^+_{l,t}$ and $S^-_{l,t}$ are isomorphic, as we recalled in Section 3.

The second assertion is proved similarly as follows. Consider the intersections $S^*_{i,t} \cap S^{*'}_{j,t}$, where $\{*, *'\} = \{+, -\}$. They are precisely one

of the cycles contained in either of the surfaces. Consider for instance $S_{i,t}^+ \cap S_{j,t}^-$, which is the cycle coming from the chain $\tilde{S}_i^+ \cap \tilde{S}_j^- = \tilde{C}_{i+1}^- + \cdots + \tilde{C}_j^-$. Here, we have $\tilde{C}_{i+1}^- \cap H_i^+ \neq \emptyset$ and $\tilde{C}_j^- \cap E_i^+ \neq \emptyset$, while $\tilde{C}_{i+1}^- \cap E_j^- \neq \emptyset$ and $\tilde{C}_j^- \cap H_j^- \neq \emptyset$. This implies that that the toric numberings of $S_{i,t}^+ \cap S_{j,t}^-$ as a cycle in $S_{i,t}^+$ and in $S_{j,t}^-$ are reverse to each other. Thus again by Lemma 4.15 and by Section 3 we conclude that $S_{i,t}^+$ and of $S_{j,t}^-$ are transpositions to each other.   q.e.d.

Proof of Theorem 7.6. First proceed in the same way as in the proof of Theorem 7.5 and consider the universal family of log-deformations of $(\hat{Z}, \hat{S})$ such that the given surface S is realized as a fiber $S_u, u \in I \cap T^\sigma - A(\hat{Q})$. Then we may realize S' as S_t for some $t \in T - A(\hat{Q})$ which is sufficiently close to u. Note that this family is universal at any point t of T since $\dim Ext^0_{O_{\hat{Z}_t}}(\Omega_{\hat{Z}_t}(\log \hat{S}_t), O_{\hat{Z}_t}) = 0$ by the upper semicontinuity of Ext (see [**4**]). Now we consider this family as a germ at u and consider as in the proof of Theorem 7.3 the versal maps $\tau_i^\pm : T \to \hat{T}_i^\pm$, but at u instead of at o. It suffices to show that τ_i^+ maps T^σ submersively onto $\hat{T}_i^+$ at u with (smooth) fiber of real dimension m.

$\hat{T}_i^+ \times \hat{T}_i^-$ may be considered as the Kuranishi space of the universal deformations of the disjoint union $S_{i,u}^+ \cup S_{i,u}^-$, where the universality is due to Proposition 3.12. The real structure on $\hat{Z}$ interchanges $S_{i,u}^\pm$ and therefore defines a real structure on $S_{i,u}^+ \cup S_{i,u}^-$. Since the family over $\hat{T}_i^+ \times \hat{T}_i^-$ is universal, this real structure extends to the real structure on the total family over $\hat{T}_i^+ \times \hat{T}_i^-$. The fixed point set D of this action is clearly a real submanifold of dimension m of $\hat{T}_i^+ \times \hat{T}_i^-$ which is mapped diffeomorphically onto each factor by the natural projections. Moreover, $\tau_i^+ \times \tau_i^-$ becomes real in the sense that it commutes with the real structure; in particular $\tau_i^+ \times \tau_i^-$ induces a smooth map $\delta : T^\sigma \to D$.

The differential of $\tau_i^+ \times \tau_i^-$ at u is given by α in the following commutative diagram similar to (92):

(96)
$$
\begin{array}{ccccc}
H^1(\Theta_{Z_u}(-\log S_u)) & \hookrightarrow & Ext^1_{O_{Z_u}}(\Omega_{Z_u}(\log S_u), O_{Z_u}) & \twoheadrightarrow & \oplus_{p \in B} \boldsymbol{C}_p \\
& & \alpha \downarrow & & \| \\
& & Ext^1_{O_{S_{i,u}}}(\Omega'_{S_{i,u}}(\log C_{i,u}), O_{S_{i,u}}) & \cong & \oplus_{p \in B_i} \boldsymbol{C}_p
\end{array}
$$

where B is the set of tangential points of S_u and B_i is the set of nodes of $C_{i,u}$; they are naturally identified. Here each term admits a natural real structure and each map is real. From this we immediately see that δ is a submersion and T^σ is mapped submersively onto $\hat{T}_i^+$ with fiber of real dimension equal to $\dim_{\boldsymbol{C}} H^1(\Theta_{Z_u}(-\log S_u)) =$

$\dim_{\boldsymbol{C}} Ext^1_{O_{Z_u}}(\Omega_{Z_u}(\log S_u), O_{Z_u}) - 2m = m$ (see Proposition 3.12) as desired, where we have used the constancy of $\dim_{\boldsymbol{C}} Ext^1_{O_{Z_t}}(\Omega_{Z_t}(\log S_t), O_{Z_t})$. In fact, since the base space is smooth, it is immediate to see that $\Omega_{\mathcal{Z}/\mathcal{T}}(\log \mathcal{S})$ is flat ovet T. Then by the upper semicontinuity of relative Ext (see [**4**]), we have the vanishing of $Ext^i_{O_{Z_t}}(\Omega_{Z_t}(\log S_t), O_{Z_t})$ for $i = 0, 2$, and then by the invariance of the alternating sum of the dimensions of Ext^i, we get the constancy of the dimension of Ext^1 [**4**].

<div align="right">q.e.d.</div>

Proof of Theorem 7.7. We start from any of the LeBrun-Joyce twistor spaces Z with the distinguished choice of (i, j) as mentioned in Remark 7.3 and get the singular twistor space $(\hat{Z}, \hat{S})$ with real structure σ. First of all, in order to get the universal family we fix as usual (see [**13**]) any twistor line L on Z other than $L_l, 1 \le l \le k$, and consider its proper transform $\hat{L}$ in $\hat{Z}$. We then consider the Kuranishi family of log-deformations of the triple $(\hat{Z}, \hat{S}, \hat{L})$, "log" referring only to the deformations of the pair $(\hat{Z}, \hat{S})$, which is universal since $\mathrm{Aut}_0(\hat{Z}, \hat{S}, \hat{L})$ now reduces to the identity. The Kuranishi space $T(L)$ is a smooth fiber space over the original Kuranishi space T for the deformations of the pair $(\hat{Z}, \hat{S})$ with four-dimensional fibers. We then restrict the family over the inverse image $I(L)$ of $I \subseteq T$. Then σ induces a canonical real structure on $T(L)$ preserving $I(L)$. The restriction to $I(L)^\sigma$ of the family of anti-self-dual bihermitian structures of Theorem 7.4 has as its underlying complex structures all parabolic Inoue surfaces with Betti number m.

We consider the universal family of *log*-deformations of $(\hat{S}^\pm_i, \hat{C}^\pm_i)$ constructed in assertion (2) of Proposition 4.11. Then the product $\hat{T}^+_i \times \hat{T}^-_i$ of the corresponding Kuranishi spaces $\hat{T}^\pm_i$ is naturally considered as parametrizing the universal family of deformations of the disjoint union of $(\hat{S}^\pm_i, \hat{C}^\pm_i)$. Since this family is universal and σ interchanges both pairs inducing a real structure on this disjoint union, there exists a natural action of σ on $\hat{T}^+_i \times \hat{T}^-_i$ interchanging the two factors. Let D be the associated real part.

Now as in the proof of Theorem 7.6 we get a versal map $\tau : I(L) \to \hat{T}^+_i \times \hat{T}^-_i$ which induces a smooth map of the real part: $I^\sigma(L) \to D$. The map is of rank one at the origin o, and the image of its differential is mapped surjectively onto both factors as in the proof of Theorem 7.3. Therefore, τ has rank at least one also at all nearby points u of o and submersive onto the both factors. This implies that the image of $\tau | I(L)^\sigma$ contains a (local) real smooth curve K contained in $D \cap \tau(I(L))$ whose images on both factors of $\hat{T}^\pm_i$ again are real smooth curves $K^\pm_i$. Let S be any parabolic Inoue surface corresponding to a point κ of $K^\pm_i$. Then the family of anti-self-dual bihermitian structures on $M[m]$

restricted to a suitable m-dimensional submanifold of $\tau^{-1}(\kappa)$ has the desired properties.

Finally, we show that S_i^+ and S_j^+ are isomorphic. We may assume that the intersection $E_t := S_{i,t}^+ \cap S_{j,t}^+$ is an elliptic curve. (Otherwise we have only to replace $S_{j,t}^+$ by $S_{j,t}^-$ and define $J_{2,t}$ via $S_{j,t}^-$.) The twistor fibration $Z_t \to M[m]$ induces the isomorphism of the fundamental groups of these spaces. Since the induced projection $S_{l,t}^+ \to M[m], l = i, j,$ is diffeomorphic, the inclusion $S_{l,t}^+ \hookrightarrow Z_t$ also gives the isomorphism of fundamental groups. Thus, if $r : \tilde{Z}_t \to Z_t$ is the universal covering, the induced map $\tilde{S}_{l,t}^+ = r^{-1}(S_{l,t}^+) \to S_{l,t}^+$ is the universal covering of $S_{l,t}^+$ also, and $\tilde{E}_t := r^{-1}(E_t) \to E_t$ gives the common infinite cyclic unramified covering of E_t for both of $S_{l,t}^+, l = i, j$. Hence by Lemma 3.6 we conclude that $S_i^{\pm}$ are isomorphic. q.e.d.

9. Anti-self-dual hermitian structures on half Inoue surfaces

In the construction of the pair $(\hat{Z}, \hat{S})$ in Section 5 using the identification map φ of (56), we may also use a map φ of *twisted type* of three kinds in the sense that φ maps $(H_i^{\pm}, E_j^{\pm})$ to $(E_i^{\pm}, H_j^{\mp})$ (resp. $(E_i^{\mp}, H_j^{\pm})$, resp. $(E_i^{\mp}, H_j^{\mp})$) (instead of to $(E_i^{\pm}, H_j^{\pm})$). To be consistent with the terminology in Section 4 we call such a φ *i-twisted* (resp. *j-twisted*, resp. *bi-twisted*). The main geometric implication of these variations is that if, e.g., φ is i-twisted, $\hat{S}_j$ becomes connected, while $\hat{S}_i$ consists of two connected components $\hat{S}_i^{\pm}$ as before. Similarly, if φ is bi-twisted, both $\hat{S}_i$ and $\hat{S}_j$ are connected.

In this case we are led to anti-self-dual hermitian structures on half Inoue surfaces and to anti-self-dual bihermitian structures on hyperbolic Inoue surfaces on their unramified double coverings which are hyperbolic Inoue surfaces. (There are no "parabolic" case, unlike in the untwisted case.)

To explain this, we first note that most of the constructions and results in Case-H in Section 7 are also valid for this case without any change. (The relevancy of Case-H comes from Proposition 6.3.) Indeed, the cohomological computations in Section 8 are either local along $\hat{Q}$ or those on Z or $\tilde{Z}$ for which φ plays no role, and hence we get the same result also in this case. In particular, the obstruction for the log-deformations for the pair $(\hat{Z}, \hat{S})$ vanishes and we get the Kuranishi family

$$(97) \qquad g : (\mathcal{Z}, \mathcal{S}) \to T, \ (Z_o, S_o) = (\hat{Z}, \hat{S}), \ o \in T$$

of log-deformations of $(\hat{Z}, \hat{S})$ with the properties in Case-H of Theorem 7.2. In particular T is smooth of dimension $3m$. The main difference now lies in the structure of the surfaces S_t for $t \in T - A(\hat{Q})$. (Here

and in what follows we use the notations of Section 7.) For $l = i$ or j, denote by l' the complementary index with $\{l, l'\} = \{i, j\}$ as before. Then also in this case, by the arguments in [**31**, §3 and §4] we get easily the following:

Lemma 9.1. *Let t be any point of $T - A(\hat{Q})$. If φ is l-twisted, then the deformation $S_{l',t}$ of $\hat{S}_{l'} = \hat{S}_{l'}^+ \cup \hat{S}_{l'}^-$ is a connected smooth surface of class VII with second Betti number $2m$, while the deformation $S_{l,t}^{\pm}$ of $\hat{S}_l^{\pm}$ are smooth disjoint surfaces of class VII with second Betti number m. Similarly, if φ is bi-twsited, the conclusion for $\hat{S}_{l'}$ above holds for both $S_{i,t}$ and $S_{j,t}$.*

Using this lemma and Proposition 4.11 in Case-H', the more precise structure of the surfaces $S_{l,t}$ and $S_{l,t}^{\pm}$ are deduced as in the proof of Theorem 7.3. Namely, using the notations of Section 7, we have the following:

Theorem 9.2. *(1) Assume that $t \in T - A(\hat{Q})$. In the l-twisted case the fibers Z_t, $S_{l,t}^{\pm}$, and $S_{l',t}$ are all smooth. The minimal model $\bar{S}_{l,t}^{\pm}$ of $S_{l,t}^{\pm}$ is either a half Inoue surface or a diagonal Hopf surface, while the minimal model $\bar{S}_{l',t}$ of $S_{l',t}$ is either a hyperbolic Inoue surfaces or a diagonal Hopf surface. The diagonal Hopf case occurs if and only if $t \in T - D$. In the bi-twisted case the statements for $S_{l',t}$ above hold for both $S_{i,t}$ and $S_{j,t}$.*

(2) Assume that $t \in I - A$. If φ is l-twisted, $S_{l,t}^{\pm}$ is a properly blown-up half Inoue surface, while $S_{l',t}$ is a properly blown-up hyperbolic Inoue surface which is the unique double covering of the transposition ${}^{\mathsf{t}}S_{l,t}^{\pm}$ of $S_{l,t}^{\pm}$. If φ is bi-twisted, $S_{i,t}$ and $S_{j,t}$ are transpositions of each other. Moreover, the complex surfaces above are independent of t up to isomorphisms.

(3) The Kuranishi family g is universal.

By restricting the family obtained in the above theorem to the real parts T^{σ} of T and I^{σ} of I, respectively, we can immediately deduce the conclusions similar to Theorem 7.4, 7.5, and 7.6 in the same way as we obtained these theorems from Theorem 7.3. Here we state only an analogue of Theorem 7.5, leaving the reader to formulate the analogues of Theorems 7.4 and 7.6. In fact, considering the deformations $S_{l,t}^{\pm}$ of $\hat{S}_l^{\pm}$ over I^{σ} in the l-twisted case, we now obtain:

Theorem 9.3. *Let S be an arbitrary properly blown-up half Inoue surface with second Betti number m. Then there exists a real m-dimensional family of anti-self-dual hermitian structures on S.*

For the statement of the next result we introduce the following terminology. Let $a : M[2m] \to M[m]$ be the unique unramified double

covering. Denote the covering involution by κ. A *bihermitian structure with anti-holomorphic involutions* on the pair $\langle M[2m], \kappa \rangle$ is by definition a bihermitian structure $([g], J_1, J_2)$ on $M[2m]$ such that on each $S_i := (M[2m], J_i)$, κ is anti-holomorphic. If instead κ is holomorphic on S_1 and anti-holomorphic on S_2, it is called a *bihermitian structure with holomorphic and anti-holomorphic involutions*. Then from the deformations of $S_{l,t}$ of $\hat{S}_l$, in the bi-twisted (resp. l-twisted) case we have the following:

Theorem 9.4. *Let S be any properly blown-up half Inoue surface and $\tilde{S}$ the properly blown-up hyperbolic Inoue surface which is an unramified double covering of S with Galois involution ι. Then there exists a real m-dimensional family $([g]_t, J_{1,t}, J_{2,t})$ of anti-self-dual bihermitian structures with anti-holomorphic (resp. holomorphic and anti-holomorphic) involutions on $\langle M[2m], \kappa \rangle$ such that $(M[2m], J_{1,t}) \cong \tilde{S}$ (resp. $\langle (M[2m], J_{1,t}), \kappa \rangle = \langle \tilde{S}, \iota \rangle$) and $(M[2m], J_{2,t}) \cong {}^{\mathbf{t}}\tilde{S}$, the transposition of $\tilde{S}$, independently of t.*

Proof. We consider the family obtained from S as in Theorem 9.2. Assuming that φ is bi-twisted, we shall show the existence of the family in the case of anti-holomorphic involutions, the other case being shown similarly by starting with l-twisted φ. Now if we restrict the obtained family to $U := I^\sigma - A(\hat{Q})$, we get a smooth family $\{[g]_t\}_{t \in U}$ of anti-self-dual structures on $M = M[m]$, and the associated family of twistor spaces $\{Z_t\}_{t \in U}$. By Lemma 9.1 we have surfaces $S_{l,u}$, $l = i, j$, in Z_t which are σ-invariant hyperbolic Inoue surfaces and are transpositions to each other such that the restriction of the twistor fibration $b_t : Z_t \to M$ makes $S_{l,u}$ smooth unramified double coverings of M.

Take the natural fibered product $\tilde{b}_t : \tilde{Z}_t := Z_t \times_M \tilde{M} \to \tilde{M} := M[2m]$. $\tilde{Z}_t$ is an unramified double covering of Z_t which is the twistor space of the induced anti-self-dual structure $(\tilde{M}, [\tilde{g}_t])$. Moreover, the inverse image of $S_{l,t}$ in $\tilde{Z}_t$ is a disjoint union of two copies of $S_{l,t}$, denoted by $\tilde{S}_{l,t}^\pm$. These are $\tilde{\sigma}$-conjugate to each other and are mapped diffeomorphically on to $\tilde{M}$, where $\tilde{\sigma} = \tilde{\sigma}_t$ is the real structure of $\tilde{Z}_t$. Moreover, $\tilde{S}_{j,t}^\pm$ is the transposition of $\tilde{S}_{i,t}^\pm$ by Lemma 8.11. Thus in $\tilde{Z}_t$ we get two pairs of $\tilde{\sigma}$-invariant elementary surfaces $\{\tilde{S}_{l,t}^\pm\}$, $l = i, j$, giving rise to anti-self-dual bihermitian structures on $(\tilde{M}, [\tilde{g}]_t)$.

It remains to see that these latter structures are actually those on $(\tilde{M}, \kappa)$. In fact, since κ preserves the anti-self-dual structures $[\tilde{g}]_t$, it lifts to a biholomorphic automorphism $\tilde{\kappa}$ of $\tilde{Z}_t$ which interchanges $\tilde{S}_{l,t}^\pm$. Then the compsition $\tilde{\sigma}\tilde{\kappa}$ preserves $\tilde{S}_{l,t}^\pm$ and induces an anti-holomorphic involution on each, which is also a lift of κ. Since b_t induces an $(\tilde{\sigma}\tilde{\kappa}, \kappa)$-equivariant isomorphism, $\tilde{S}_{l,t}^+ \cong (\tilde{M}, \tilde{J}_{l,t})$, where $\tilde{J}_{l,t}$ is the pull-back of $J_{l,t}$ to $\tilde{M}$. Then by the definition of $\tilde{J}_{l,t}$, we are done.   q.e.d.

Remark 9.1. (1) Using Proposition 4.11 in Case-H′, we can also obtain an analogue of Theorem 7.6 above, in which we get a family of anti-self-dual structures on certain blown-up half Inoue surfaces and blown-up diagonal Hopf surfaces.

(2) As in Case-H, for a fixed S as in Theorem 9.3, we can construct other families with the same properties by starting from suitable other choice of K-actions on $m\boldsymbol{P}^2$ and pairs (i,j) in the construction of Section 5.

(3) The family of anti-self-dual bihermitian structures on $M[2m]$ in Theorem 9.4 could possibly be connected by deformations to those obtained in part (2) of Theorem 7.3 with m replaced by $2m$ there. This type of relation would deserve further study.

(4) Let $\tilde{S} \to S$ and ι be as in the theorem. The theorem implies that $\tilde{S}$ always admits a fixed point free anti-holomorphic involution, which could be identified with the anti-holomorphic involution $\iota\mu$, where μ is the real structure of $\tilde{S}$ defined in Lemma 3.7.

10. Differential geometric consequences

Our interest in bihermitian metrics comes from anti-self-duality and this case was first treated in [**36**] motivated by questions of Salamon [**40**] concerning existence of orthogonal (integrable) complex structures in a given conformal class $[g]$; here we explicitly want to exclude the well-known case of hyperhermitian structures.

More impetus came from the work of [**2**], who considered the general four-dimensional case; new examples have been recently found by Hitchin [**21**] on del Pezzo surfaces in the context of generalized Kähler manifolds as introduced by Gualtieri [**19**]. In dimension four this condition amounts to say that there is a metric $g \in [g]$ which is Gauduchon for both J_i, $i = 1, 2$, and for which the sum of the Lee forms θ_i vanishes:

$$\theta_1 + \theta_2 = 0 \quad \text{and} \quad \delta\theta_1 = 0 = \delta\theta_2.$$

The above two equations always hold for a bihermitian metric on a compact anti-self-dual 4-manifold (we excluded the hyperhermitian case) [**36**, Proposition 3.5] and their twistor correspondence is that the *real* degree-4 divisor defined by $\{\pm J_i, i = 1, 2\}$ is an anticanonical divisor of Z [**36**, Lemma 3.4.].

By Lemmas 3.2 and 3.3 an anti-self-dual bihermitian surface S with odd first Betti number must be a blow-up of either a hyperbolic or parabolic Inoue or Hopf surface. This condition is therefore a necessary condition for the existence of anti-self-dual bihermitian metrics.

However, Lemma 3.2 was known to Nakamura (unpublished), and the same statement holds in fact on any compact generalized Kähler four manifold with $b_1 = 1$ [**2**, II,2] (in their proof Lemma (2.8) should be replaced by Lemma (2.7)); it also appears in [**12**, 2.29], where a

more general result is proved concerning surfaces with a global spherical shell. For all these reasons we would like to give here a complete proof following the work of Nakamura.

Proof of Lemma 3.2. When $b_2(S) = 0$ Bombieri-Inoue surfaces have no curves, therefore by Bogomolov theorem [**30**, **42**] S is a Hopf surface and is diagonal because $-K$ is disconnected. We can therefore assume that $S \in \text{VII}_0^+$ and by [**32**, 12.4], S contains a cycle C of rational curves. Suppose that [**33**, 2.2] holds (i.e., S has a branch); then [**33**, 3.1] applies, with $m = 1$ and F the trivial line bundle, and Nakamura concludes that anti-canonical divisor itself is connected, which is absurd. The only other possibility is that [**33**, 2.2] does not hold, in which case S is an Enoki surface (but $-K$ has no divisor in this case) or a half Inoue surface (but $-K$ is connected in this case) or else S must be parabolic or hyperbolic Inoue.   q.e.d.

The results of Section 7 show that the above mentioned necessary condition is actually sufficient at least for properly blown-up hyperbolic Inoue surfaces and also for some parabolic Inoue surfaces. To summarize we have the following:

Proposition 10.1. *All metrics constructed in Section 7 on (blown-up) hyperbolic or parabolic Inoue surfaces or Hopf surfaces are (twisted) generalized Kähler.*

We also have applications to an old and basic question of Vaisman, who asked [**44**]: Which compact complex surfaces S can admit locally conformally Kähler (l.c.K.) metrics? (see [**37**] for more details.)

By a result of Tricerri [**43**] one can assume that S is minimal, and after the work of Belgun [**5**], the answer is positive for all the locally homogeneous surfaces, i.e., all surfaces in class VI, Kodaira surfaces, and surfaces with $b_2 = 0 = b_1 - 1$, except for the complement of a real line in a complex 1-dimensional family of certain Bombieri-Inoue surfaces which do not admit l.c.K. metrics at all. Therefore, Vaisman's question remains open only for surfaces in class- VII_0^+.

Now, by a theorem of Boyer [**7**] anti-self-dual hermitian metrics are automatically l.c.K. on a compact complex surface. Therefore we have:

Theorem 10.2. *All the surfaces of class VII_0^+ in Theorems 7.5, 7.6, 7.7, 9.3, and 9.4 have l.c.K. metrics.*

We conclude with the following:

Remark 10.1. Except for the anti-self-dual hermitian metrics on parabolic Inoue surfaces by LeBrun [**29**], all the examples in the proposition and the theorem above are new and are the only known examples in class-VII_0^+.

Note. While undergoing the final draft of this work M. Brunella communicated to us that he constructed l.c.K. metrics on all Enoki surfaces [**8**].

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