

INSUFFICIENT CONVERGENCE OF INVERSE MEAN CURVATURE FLOW ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract

We construct a solution to inverse mean curvature flow on an asymptotically hyperbolic 3-manifold which does not have the convergence properties needed in order to prove a Penrose-type inequality. This contrasts sharply with the asymptotically flat case. The main idea consists in combining inverse mean curvature flow with work done by Shi–Tam regarding boundary behavior of compact manifolds. Assuming the Penrose inequality holds, we also derive a nontrivial inequality for functions on S^2 .

1. Introduction

A Penrose inequality for asymptotically flat 3-manifolds was proven independently by Huisken–Ilmanen [8], using inverse mean curvature flow, and Hugh Bray [1], using a conformal deformation of the ambient metric. Recently, Hugh Bray and Dan Lee [1] extended Bray’s approach and prove a Penrose inequality for dimensions less than 8.

In this paper we investigate an analogous problem for (M, g) , an asymptotically hyperbolic 3-manifold with scalar curvature $R \geq -6$.

We start by mentioning that these manifolds can arise in General Relativity in two ways: As spacelike hypersurfaces in space-time with a cosmological constant $\Lambda = -3$ or as “hyperboloidal hypersurface” in space-time (i.e., second fundamental form h in space-time satisfies $h = g$) with cosmological constant $\Lambda = 0$. In both cases, the dominant energy condition translates into

$$R \geq -6.$$

We also recall that the mean curvature of an apparent horizon Σ is given by $H = \text{tr}_\Sigma h$. Hence, in the first case ($\Lambda = -3$) and assuming that $h = 0$ (time-symmetric hypothesis), we have that apparent horizons are minimal surfaces, while in the second case ($\Lambda = 0$) apparent horizons correspond to $H = 2$ surfaces.

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Broadly speaking, the Penrose inequality says that, assuming the dominant energy condition, the presence of outermost apparent horizons implies (conjecturally) a lower bound on the “mass” of M . Following the presentation of [2], the lower bound should be given by

$$Mass \geq \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \frac{4}{3}\Lambda|\Sigma| \right),$$

where Σ is a sphere which is an outermost apparent horizon (see next subsection for definitions) and Λ is the cosmological constant.

Xiadong Wang in [16] proposed a definition of mass (see also [5] and [6] for a less restrictive definition) and conjectured the following version of the Penrose inequality for $\Lambda = 0$. The definition of mass M will be recalled in the next subsection.

Conjecture. *If Σ_0 is an outermost sphere with $H(\Sigma_0) = 2$, then*

$$M \geq \left(\frac{|\Sigma_0|}{16\pi} \right)^{1/2}.$$

If equality holds then (M, g) is isometric to an Anti-de Sitter–Schwarzschild manifold outside Σ_0 .

The main purpose of this paper is to show that, contrarily to what was suggested in [16], the inverse mean curvature flow does not have the necessary convergence properties needed to prove this conjecture.

As it was pointed out by the referee, our argument does not carry to the case $\Lambda = -3$, i.e., where Σ_0 is an outermost minimal surface. At the end of the proof of Theorem 1.2, in Section 5, we suggest a modification in our argument that would handle that case.

We should also point out that, even if it is geometrically natural, there is no physical evidence supporting the choice of the mass considered in [16]. An evidence-based choice for the “correct” mass term in the Penrose inequality was suggested by Chruściel and Simon in [4] (see also [2]) which we now briefly describe. Like before, we refer the reader to the next subsection for the relevant definitions.

Assume that there is a smooth function u defined on M so that $\partial\{u < 0\}$ coincides with the outermost horizon Σ_0 , and the surfaces $\Sigma_t = \partial\{u < t\}$ give a smooth solution for inverse mean curvature flow. Set the mass to be the limit of the Hawking masses

$$Mass = \lim_{t \rightarrow \infty} m_H(\Sigma_t).$$

Then, it is a standard fact that

$$Mass \geq m_H(\Sigma_0) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \frac{4}{3}\Lambda|\Sigma| \right).$$

Even if Σ_0 does not admit a smooth solution to inverse mean curvature flow, the work done by Huisken and Ilmanen in [8] implies with

no difficulty that if Σ_0 is an outermost horizon, then there is a weak solution $(\Sigma_t)_{t \geq 0}$ to inverse mean curvature flow such that $m_H(\Sigma_t)$ is non-decreasing and so

$$Mass = \lim_{t \rightarrow \infty} m_H(\Sigma_t) \geq m_H(\Sigma_0) = \frac{|\Sigma_0|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \frac{4}{3} \Lambda |\Sigma_0| \right).$$

1.1. Notation and Definitions. Given a complete noncompact Riemannian 3-manifold (M, g) , we denote its connection by D , the Ricci curvature by Rc , and the scalar curvature by R . The induced connection on a surface $\Sigma \subset M$ is denoted by ∇ , the exterior unit normal by ν (whenever its defined), the mean curvature by H , the trace free part of the second fundamental form by \mathring{A} , and the surface area by $|\Sigma|$.

A sphere $\Sigma \subset M$ with mean curvature $H(\Sigma) = 2$ is said to be *outermost* if it is the boundary of a compact set and its outside region contains no other spheres with $H = 2$. We say that Σ is *outer minimizing* if every compact perturbation lying outside of Σ has bigger surface area.

In what follows g_0 denotes the standard metric on S^2 .

Definition 1.1. A complete noncompact Riemannian 3-manifold (M, g) is said to be asymptotically hyperbolic if the following are true:

- (i) There is a compact set $K \subset\subset M$ such that $M \setminus K$ is diffeomorphic to \mathbb{R}^3 minus an open ball.
- (ii) With respect to the spherical coordinates induced by the above diffeomorphism, the metric can be written as

$$g = dr^2 + \sinh^2 r g_0 + h/(3 \sinh r) + Q$$

where h is a symmetric 2-tensor on S^2 and

$$|Q| + |DQ| + |D^2Q| + |D^3Q| \leq C \exp(-4r)$$

for some constant C .

For simplicity, the manifolds we consider have only one end. The above definition is stated differently from the one given in [16] (see also [6]). Nonetheless, using a simple substitution of variable

$$t = \ln \left(\frac{\sinh(r/2)}{\cosh(r/2)} \right),$$

they can be seen to be equivalent.

Note that a given coordinate system on $M \setminus K$ induces a radial function $r(x)$ on $M \setminus K$. With respect to this coordinate system, we define the *inner radius* and *outer radius* of a surface $\Sigma \subset M \setminus K$ to be

$$\underline{r} = \sup\{r \mid B_r(0) \subset \Sigma\} \quad \text{and} \quad \bar{r} = \inf\{r \mid \Sigma \subset B_r(0)\}$$

respectively. Furthermore, we denote the coordinate spheres induced by a coordinate system by

$$\{|x| = r\} := \{x \in M \setminus K \mid r(x) = r\}$$

and the radial vector by ∂_r . We stress that the radial function $r(x)$ depends on the coordinate system chosen. If γ is an isometry of \mathbb{H}^3 , the radial function $s(x)$ induced by this new coordinate system is such that

$$|s(x) - r(x)| \leq C \quad \text{for all } x \in M \setminus K,$$

where C depends only on the distance from γ to the identity. We denote by $\bar{}$ and $\underline{}$ the correspondent quantities defined with respect to this new coordinate system.

The mass M of an asymptotically hyperbolic manifold (M, g) with $R \geq -6$ is given by

$$M = \frac{1}{16\pi} \left[\left(\int_{S^2} \text{tr}_{g_0} h d\mu_0 \right)^2 - \sum_{i=1}^3 \left(\int_{S^2} \text{tr}_{g_0} h x_i d\mu_0 \right)^2 \right]^{1/2},$$

where (x_1, x_2, x_3) are the standard coordinates on $S^2 \subset \mathbb{R}^3$. This quantity is well defined (i.e. independent of the coordinate system chosen for $M \setminus K$) by [16] (see also [5] and [6] for a less restrictive definition).

The Anti-de Sitter-Schwarzschild metric $(S^2 \times [t_0, +\infty), g_m)$ is given by

$$g_m = \frac{dt^2}{1 + t^2 - m/t} + t^2 g_0,$$

where we choose t_0 so that the mean curvature of the coordinate sphere $\Sigma_0 = \{|x| = t_0\}$ is 2. A change of variable (see [16, page 294]) shows that the metric can be written as

$$g = dr^2 + (\sinh^2 r + m/(3 \sinh r))g_0 + P,$$

where P is term with order $\exp(-5r)$. An explicit computation reveals that the scalar curvature equals -6 and that

$$M = \frac{m}{2} = \left(\frac{|\Sigma_0|}{16\pi} \right)^{1/2}.$$

1.2. Statement of the main results. We start by briefly describing how inverse mean curvature flow could prove the conjecture. Find a family of surfaces $(\Sigma_t)_{t \geq 0}$ with initial condition Σ_0 such that

$$\frac{dx}{dt} = \frac{\nu}{H(\Sigma_t)}.$$

Note that the existence theory for a weak solution developed in [8, Section 3] can be used in the current setting. Moreover, the same arguments in [8, Section 5] show that the quantity (called the *Hawking mass*)

$$m_H(\Sigma_t) := \frac{|\Sigma_t|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{\Sigma_t} H^2 - 4 d\mu_t \right)$$

is monotone nondecreasing along the flow. Therefore,

$$\left(\frac{|\Sigma_0|}{16\pi}\right)^{1/2} = m_H(\Sigma_0) \leq \lim_{t \rightarrow \infty} m_H(\Sigma_t).$$

The result would follow if one could show that the limit of the Hawking mass is not bigger than M .

In the asymptotically flat case, Huisken and Ilmanen [8, Section 7] showed this by proving that

$$\liminf_{t \rightarrow \infty} \frac{\text{area}(B_{\bar{r}_t}(0))}{\text{area}(B_{\underline{r}_t}(0))} = \liminf_{t \rightarrow \infty} \frac{\bar{r}_t}{\underline{r}_t} = 1,$$

where \bar{r}_t and \underline{r}_t denote the outer radius and inner radius of Σ_t respectively. In our setting, it is not hard to see that in order for the limit of the Hawking mass to be smaller than M we need to find an isometry γ of \mathbb{H}^3 such that, with respect to the induced coordinate system, the following two properties hold:

1)

$$\liminf_{t \rightarrow \infty} \frac{|B_{\bar{s}_t}(0)|}{|B_{\underline{s}_t}(0)|} = \liminf_{t \rightarrow \infty} (\bar{s}_t - \underline{s}_t) = 0,$$

where \bar{s}_t and \underline{s}_t denote, respectively, the outer radius and inner radius of Σ_t with respect to the radial function $s(x)$ induced by γ ;

2) If the metric with respect to the coordinates induced by γ is written as

$$g = ds^2 + \sinh^2 s g_0 + h^\gamma / (3 \sinh s) + P,$$

then

$$\int_{S^2} x_i \text{tr}_{g_0} h^\gamma d\mu_0 = 0 \quad \text{for } i = 1, 2, 3,$$

where x_i denotes the coordinate functions of the unit sphere in \mathbb{R}^3 .

If these properties do not hold, it is impossible to compare the limit of the Hawking mass with the mass of the manifold. More precisely, we can construct a family of spheres $(\Sigma_r)_{r \geq 0}$ and an asymptotically hyperbolic metric (M, g) where

$$m_H(\Sigma_\infty) := \lim_{r \rightarrow \infty} m_H(\Sigma_r)$$

is bigger or smaller than the mass M . This comes from the following observation. Pick a 2-tensor h on S^2 such that

$$\int_{S^2} x_i \text{tr}_{g_0} h d\mu_0 = 0 \quad \text{for } i = 1, 2, 3.$$

Then the mass is given by

$$M = \frac{1}{4} \left(\int_{S^2} \text{tr}_{g_0} h d\mu_0 \right)$$

and, according to Proposition 2.1 e), given any function f in $C^\infty(S^2)$ we can construct a family of spheres $(\Sigma_r)_{r \geq 0}$ such that

$$m_H(\Sigma_\infty) = \frac{1}{4} \left(\int_{S^2} \exp(2f) d\mu_0 \right)^{1/2} \int_{S^2} \text{tr}_{g_0} h \exp(-f) d\mu_0.$$

It is simple to recognize that one can choose h positive definite and some function f for which $m_H(\Sigma_\infty) < M$. Moreover, if one chooses f to be zero but

$$\int_{S^2} x_i \text{tr}_{g_0} h d\mu_0 \neq 0 \quad \text{for } i = 1, 2, 3.$$

then we would have in this case

$$m_H(\Sigma_\infty) = \frac{1}{4} \int_{S^2} \text{tr}_{g_0} h d\mu_0 > M.$$

We can now state the main theorem.

Theorem 1.2. *There is an asymptotically hyperbolic 3-manifold (M, g) with scalar curvature -6 and for which its boundary Σ_0 is an outer-minimizing sphere with $H(\Sigma_0) = 2$ satisfying the following property.*

There is a smooth solution to inverse mean curvature flow $(\Sigma_t)_{t \geq 0}$ with initial condition Σ_0 such that for every coordinate system we have

$$\liminf_{t \rightarrow \infty} (\bar{s}_t - \underline{s}_t) > 0.$$

- Remark 1.3.**
- i) We note that the sphere Σ_0 might not be outermost, i.e., there could be another $H = 2$ sphere enclosing Σ_0 . If this is the case, the area of this sphere has to be bigger than $|\Sigma_0|$. We point out that the Penrose inequality in the asymptotically flat case also holds with an outer-minimizing minimal sphere instead of an outermost minimal sphere.
 - ii) The author does not know whether the manifold constructed constitutes a counterexample to the Penrose inequality. The reason is that it is hard to compute explicitly the mass of the manifold (M, g) . In the last section of the paper and assuming that the manifolds we construct do not violate the Penrose inequality, we deduce a nontrivial inequality for functions on S^2 (see Section 5.1). Jointly with Alice Chang, we have verified that such inequality indeed holds.

The strategy of the proof is the following. We first construct a solution to inverse mean curvature flow $(\Sigma_t)_{t \geq 0}$ on an Anti-De Sitter-Schwarzschild metric g_m with mass $m/2$ such that

$$\lim_{t \rightarrow \infty} m_H(\Sigma_t) > m/2.$$

This is done in Sections 2 and 3, where we prove a long time existence result for inverse mean curvature flow on Anti-de Sitter-Schwarzschild space. It is important that the estimates in this section do not depend on

the area of our initial condition and this requires a careful bookkeeping. Unfortunately the initial condition for this solution is not a sphere with $H = 2$. To fix that, we find a function u such that the metric

$$g = \frac{u^2}{H^2} dt^2 + g_t$$

has

$$R = -6 \text{ and } u_{\Sigma_0} = H(\Sigma_0)/2.$$

Note that, with respect to this metric, Σ_0 has $H = 2$. This is done in Section 4 where we adapt the work of Shi-Tam [14] and Wang-Yau [15] to prove the existence of such function u . A simple computation will show that $(\Sigma_t)_{t \geq 0}$ is also a solution to inverse mean curvature flow with respect to the metric g . We are left to show that this solution has the desired asymptotic behavior. Using Proposition 2.1 f) we will see that the Gaussian curvature \widehat{K}_t of Σ_t with respect to the normalized metric

$$\hat{g}_t := (4\pi)|\Sigma_t|^{-1} g_t$$

does not converge to one when t goes to infinity. Note that it does not matter whether we choose g or g_m in this step because the intrinsic geometry is the same. This fact and Proposition 2.1 f) imply that the metric g is such that *for every* coordinate system we have

$$\liminf_{t \rightarrow \infty} (\bar{s}_t - \underline{s}_t) > 0.$$

The proof of Theorem 1.2 is done in Section 5.

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2. Basic properties of graphical surfaces on asymptotically hyperbolic 3-manifolds

In this section (M, g) denotes an asymptotically hyperbolic manifold with some given coordinate system on $M \setminus K$. Given a function f on S^2 we consider the surfaces

$$\Sigma(q_0) = \{(q_0 + f(\theta), \theta) \mid \theta \in S^2\} \subset M \setminus K.$$

The function f satisfies hypothesis (I) if there are constants V, V_0 such that

$$(I) \quad \begin{cases} |f| \leq V, \\ |\nabla_0 f| \leq V_0, \end{cases}$$

where ∇_0 denotes the connection with respect to the round metric on S^2 . We denote by $\bar{s}(q_0)$ and $\underline{s}(q_0)$, respectively, the outer radius and

inner radius of $\Sigma(q_0)$, where $s(x)$ is the radial function induced by some coordinate system γ .

Given any geometric quantity T defined on $\Sigma(q_0)$, we use the notation

$$T = O(\exp(-kr))$$

when we can find a constant $C = C(g, V, V_0)$ for which

$$|T| \leq C \exp(-kr).$$

The next proposition collects some properties for the surfaces $\Sigma(q_0)$ when q_0 is very large.

Proposition 2.1. *Assume that f satisfies hypothesis (I). The following properties hold:*

a) *When q_0 goes to infinity, the normalized metrics*

$$\hat{g}(q_0) := 4\pi |\Sigma(q_0)|^{-1} g_{\Sigma(q_0)}$$

converge to

$$\hat{g} := \left(\int_{S^2} \exp(2f) d\mu_0 \right)^{-1} \exp(2f) g_0.$$

b) *There is a constant $C = C(g, V, V_0)$ such that, for all $q_0 \geq 1$,*

$$|\Sigma(q_0)| |H - 2| + |\Sigma(q_0)| |\dot{A}| \leq C + C \sup_{S^2} |\nabla_0^2 f|$$

and

$$\sup_{S^2} |\nabla_0^2 f| \leq C(|\Sigma(q_0)| |H - 2| + |\Sigma(q_0)| |\dot{A}|) + C;$$

c) *Assume that*

$$\sup_{S^2} |\nabla_0^k f| \leq E \quad \text{for all } k = 2, \dots, n-1.$$

There is a constant $C = C(g, E, V, V_0)$ such that for all $q_0 \geq 1$

$$|\Sigma(q_0)|^{n+2} |\nabla^n A|^2 \leq C + C \sup_{S^2} |\nabla_0^n f|^2$$

and

$$\sup_{S^2} |\nabla_0^n f|^2 \leq C + C |\Sigma(q_0)|^{n+2} |\nabla^n A|^2;$$

d) *The mean curvature of $\Sigma(q_0)$ satisfies*

$$H^2 - 4 = 4K(\Sigma) + 2|\dot{A}|^2 - \frac{2\text{tr}_{g_0} h}{\sinh^3 r} + O(\exp(-4r));$$

e)

$$\lim_{q_0 \rightarrow \infty} m_H(\Sigma(q_0)) = \frac{1}{4} \left(\int_{S^2} \exp(2f) d\mu_0 \right)^{1/2} \int_{S^2} \text{tr}_{g_0} h \exp(-f) d\mu_0.$$

f) *There is a coordinate system γ for which*

$$\lim_{q_0 \rightarrow \infty} (\bar{\mathfrak{s}}(q_0) - \underline{\mathfrak{s}}(q_0)) = 0$$

if and only if

$$\widehat{K} = \lim_{q_0 \rightarrow \infty} \widehat{K}(q_0) = 1,$$

where $\widehat{K}(q_0)$ is the Gaussian curvature of $\Sigma(q_0)$ with respect to $\hat{g}(q_0)$.

Note that this proposition also holds, with obvious modifications, if

$$\Sigma(q_0) = \{(q_0 + f_{q_0}(\theta), \theta) \mid \theta \in S^2\},$$

where the functions f_{q_0} converge to a function f on S^2 when q_0 goes to infinity.

Proof. Consider tangent vectors to $\Sigma(q_0)$

$$\partial_i := \frac{\partial f}{\partial \theta_i} \partial_r + \partial_{\theta_i}, \quad i = 1, 2,$$

where (θ_1, θ_2) represent coordinates on S^2 which are orthonormal (with respect to g_0) at a given point p .

The induced metric on $\Sigma(q_0)$ is given by

$$g_{ij} = \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} + \sinh^2(q_0 + f) g_0(\partial_{\theta_i}, \partial_{\theta_j}) + O(\exp(-r))$$

and so

$$\sqrt{\det g_{ij}} = \sinh^2(q_0 + f) \sqrt{\det g_0} + O(1) = \sinh^2(q_0) \exp(2f) \sqrt{\det g_0} + O(1).$$

This implies that

$$(1) \quad \lim_{q_0 \rightarrow \infty} \frac{|\Sigma(q_0)|}{4\pi \sinh^2 q_0} = \int_{S^2} \exp(2f) d\mu_0$$

and the first property follows from the fact that

$$\lim_{q_0 \rightarrow \infty} (\sinh q_0)^{-2} g_{ij} = \exp(2f) g_0(\partial_{\theta_i}, \partial_{\theta_j}).$$

Denoting the connection with respect to the standard hyperbolic metric by \bar{D} , we have

$$\bar{D}_{\partial_r} \partial_r = 0, \quad \bar{D}_{\partial_{\theta_j}} \partial_r = \frac{\cosh r}{\sinh r} \partial_{\theta_j}, \quad \bar{D}_{\partial_{\theta_i}} \partial_{\theta_j} = -\sinh r \cosh r \delta_{ij} \partial_r,$$

and

$$|D - \bar{D}| \leq C \exp(-3r)$$

for some $C = C(g)$.

Thus,

$$\begin{aligned} D_{\partial_i} \partial_j &= \frac{\partial^2 f}{\partial \theta_j \partial \theta_i} \partial_r + \frac{\partial f}{\partial \theta_j} \frac{\partial f}{\partial \theta_i} D_{\partial_r} \partial_r + \frac{\partial f}{\partial \theta_j} D_{\partial_{\theta_i}} \partial_r + \frac{\partial f}{\partial \theta_i} D_{\partial_r} \partial_{\theta_j} + D_{\partial_{\theta_i}} \partial_{\theta_j} \\ &= -\cosh r \sinh r \delta_{ij} \partial_r + \frac{\partial^2 f}{\partial \theta_j \partial \theta_i} \partial_r + O(1) \partial_{\theta_j} + O(\exp(-r)). \end{aligned}$$

An easy computation shows that the exterior unit normal is given by

$$(2) \quad \nu = (1 + O(\exp(-2r))) \partial_r + O(\exp(-2r)) \partial_{\theta_1} + O(\exp(-2r)) \partial_{\theta_2}$$

and thus

$$A_{ij} = 2 \frac{\cosh r}{\sinh r} g_{ij} - \frac{\partial^2 f}{\partial \theta_j \partial \theta_i} + O(1).$$

This implies Property b).

Property c) follows from what was done above plus some tedious computations. We now prove Property d).

It was shown in [13, Lemma 3.1.] that

$$Rc(\nu, \nu) + 2 = -\frac{\text{tr}_{g_0} h}{2 \sinh^3 r} + O(\exp(-4r)) \quad \text{and} \quad R = -6 + O(\exp(-4r)).$$

Combining this with Gauss equations we obtain that

$$\begin{aligned} H^2 - 4 &= 4K(\Sigma(q_0)) + 2|\mathring{A}|^2 + 4(R(\nu, \nu) - R/2 - 1) \\ &= 4K(\Sigma(q_0)) + 2|\mathring{A}|^2 - \frac{2\text{tr}_{g_0} h}{\sinh^3 r} + O(\exp(-4r)). \end{aligned}$$

Combining Property a) with Property d), it follows from the definition of Hawking mass that

$$\begin{aligned} \lim_{q_0 \rightarrow \infty} m_H(\Sigma(q_0)) &= \lim_{q_0 \rightarrow \infty} \frac{|\Sigma(q_0)|^{1/2}}{(16\pi)^{3/2}} \int_{\Sigma(q_0)} \frac{2\text{tr}_{g_0} h}{\sinh^3 r} d\mu \\ &= \lim_{q_0 \rightarrow \infty} \frac{1}{(16\pi)^{3/2}} \int_{\Sigma(q_0)} \frac{2|\Sigma(q_0)|^{3/2}}{\sinh^3 r} \text{tr}_{g_0} h d\hat{\mu} \\ &= \frac{1}{4^{3/2}} \left(\int_{S^2} \exp(2f) d\mu_0 \right)^{3/2} \int_{S^2} 2\text{tr}_{g_0} h \exp(-3f) d\hat{\mu} \\ &= \frac{1}{4} \left(\int_{S^2} \exp(2f) d\mu_0 \right)^{1/2} \int_{S^2} \text{tr}_{g_0} h \exp(-f) d\mu_0. \end{aligned}$$

Finally, we prove Property e). Given a coordinate system induced by an isometry γ of \mathbb{H}^3 , we consider the function on $\Sigma(q_0)$ given by

$$w(x) = s(x) - \hat{q}_0 \quad \text{where} \quad |\Sigma(q_0)| = 4\pi \sinh^2 q_0,$$

where $s(x)$ is the radial function for this coordinate system. For all q_0 sufficiently large, $\Sigma(q_0)$ is graphical over the coordinate spheres for this

new coordinate system and so

$$\limsup_{q_0 \rightarrow \infty} \left(|\partial_s^\top|^2 + (1 - \langle \nu, \partial_s \rangle) \right) \exp(2q_0) < \infty.$$

Due to [13, Propostion 3.3], we know that

$$\begin{aligned} \Delta s &= (4 - 2|\partial_s^\top|^2) \exp(-2s) + 2 - H \\ &\quad + (H - 2)(1 - \langle \partial_s, \nu \rangle) + (1 - \langle \partial_s, \nu \rangle)^2 + O(\exp(-3s)). \end{aligned}$$

Therefore, Property d) implies that w satisfies the following equation with respect to $\hat{g}(q_0)$

$$(3) \quad \hat{\Delta} w = \exp(-2w) - \hat{K}(q_0) + P(q_0),$$

where

$$\lim_{q_0 \rightarrow \infty} \int_{\Sigma(q_0)} |P(q_0)| d\hat{\mu} = 0.$$

Suppose the coordinate system induced by γ is such that

$$\lim_{q_0 \rightarrow \infty} \bar{s}(q_0) - \underline{s}(q_0) = 0.$$

Then

$$\lim_{q_0 \rightarrow \infty} w = 0$$

and so equation (3) implies that

$$\lim_{q_0 \rightarrow \infty} \hat{K}(q_0) = 1.$$

Assume for simplicity that

$$\int_{S^2} \exp(2f) d\mu_0 = 1$$

because, according to (1), this implies that

$$\lim_{q_0 \rightarrow \infty} \hat{q}_0 - q_0 = 0.$$

If $\hat{K} = 1$, then \hat{g} is a round metric on S^2 and hence there is a conformal transformation γ of S^2 for which $\gamma^* \hat{g} = g_0$. From Property a) we know that $\hat{g} = \exp(2f)g_0$ and so $\gamma^* g_0 = \exp(-2f \circ T)g_0$. This conformal transformation induces an isometry of hyperbolic space which we still denote by γ . The relationship between the radial functions $r(x)$ and $s(x)$ is determined by

$$|s(x) + f \circ \gamma(x) - r \circ \gamma(x)| \leq C \exp(-r(x))$$

for some constant C . This implies that for all x in $\Sigma(q_0)$

$$|w(x) + \hat{q}_0 - q_0| \leq C \exp(-q_0),$$

and thus

$$\lim_{q_0 \rightarrow \infty} w = 0.$$

q.e.d.

3. Long time existence for inverse mean curvature flow on asymptotically hyperbolic 3-manifolds

In this section the ambient manifold will be an Anti-de Sitter–Schwarzschild metric $(S^2 \times [s_0, +\infty), g_m)$ with mass $m > 0$.

A sphere Σ_0 satisfies hypothesis (H) if we can find constants $(Q_j)_{j \in \mathbb{N}}$, ε_0 , and δ_0 for which

$$(H) \quad \begin{cases} H \geq \varepsilon_0 & \text{and} & |\Sigma_0| |H^2 - 4| \leq Q_0, \\ \langle \nu, \partial_r \rangle \geq \varepsilon_0 & \text{and} & \langle \nu, \partial_r \rangle \geq 1 - |\Sigma_0|^{-1} Q_1, \\ |\mathring{A}|^2 \leq (1/4 - \delta_0) H^2 & \text{and} & |\Sigma_0|^2 |\mathring{A}|^2 \leq Q_2, \\ \sup_{\Sigma_0} |\nabla^n A|^2 \leq Q_{n+2} |\Sigma_0|^{-(n+2)} & \text{for all } n \geq 1, \\ \Sigma_0 \text{ bounds a compact region containing } S^2 \times \{s_0\}. \end{cases}$$

Recall that \bar{r}_0 and \underline{r}_0 denotes, respectively, the outer radius and the inner radius of Σ_0 .

Theorem 3.1. *Assume that Σ_0 satisfies (H).*

There is a constant $\underline{r} = \underline{r}((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ such that if $\underline{r}_0 \geq \underline{r}$ then the inverse mean curvature flow (Σ_t) with initial condition Σ_0 exists for all time and has the following properties:

- (i) *There is a positive constant $C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ such that the mean curvature of Σ_t satisfies*

$$H \geq C$$

and, for some other constant $C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$,

$$|\Sigma_0| |H^2 - 4| \leq C \exp(-t);$$

- (ii) *For every $n \geq 0$ and $k \geq 1$ there is a constant*

$$C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$$

such that

$$|\Sigma_0|^{n+2} |\partial_t^k \nabla^n A|^2 \leq C \exp(-(n+2)t)$$

and

$$|\Sigma_0|^{n+2} |\nabla^n A|^2 \leq C \exp(-(n+2)t) \quad \text{for } n \geq 1;$$

- (iii) *The surfaces Σ_t can be described as*

$$\Sigma_t = \{(\hat{r}_t + f_t(\theta), \theta) \mid \theta \in S^2\},$$

where \hat{r}_t is such that

$$|\Sigma_t| = 4\pi \sinh^2 \hat{r}_t.$$

Moreover, the functions f_t converge to a smooth function f_∞ defined on S^2 .

(iv) For every $n \geq 0$ and $k \geq 1$ there is a constant

$$C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$$

such that

$$|\Sigma_0|^n |\nabla^n f_t|^2 \leq C \exp(-nt), \quad |\partial_t^k f_t| \leq C \exp(-t),$$

and

$$|\Sigma_0|^n |\partial_t^k \nabla^n f_t|^2 \leq C \exp(-nt) \quad \text{for } n \geq 1.$$

We essentially adapt to our setting some of the ideas used in the work of Huisken–Ilmanen [9] and Claus Gerhardt [7] on smooth solutions to inverse mean curvature flow. We could have been more precise regarding how the constants depend on $(Q_j)_{j \in \mathbb{N}}$ but this version of the theorem suffices for our purposes. The important point is that the estimates do not depend on \underline{r}_0 (only on $\bar{r}_0 - \underline{r}_0$).

Proof. During the first part of this proof, given any geometric quantity T defined on Σ_t we use the notation

$$T = O(\exp(-kr))$$

whenever there is a constant $C = C(m)$ such that

$$|T| \leq C \exp(-kr).$$

Because $H > 0$ we have short-time existence for the flow. Denoting by $\Sigma_t^m := \{|x| = r_t^m\}$ the solution to inverse mean curvature flow with initial condition $\{|x| = t_0\}$ we know that

$$|\Sigma_t^m| = |\Sigma_0^m| \exp(t)$$

and thus we can find a constant $K = K(m)$ such that

$$t/2 - K \leq r_t^m - t_0 \leq t/2 + K.$$

Because two solutions that are initially disjoint must remain disjoint [8, Theorem 2.2], we have that for some constant $K = K(m)$

$$(4) \quad t/2 + \underline{r}_0 - K \leq \underline{r}_t \leq \bar{r}_t \leq t/2 + \bar{r}_0 + K \quad \text{and} \quad \underline{r}_t \geq \underline{r}_0.$$

Therefore, we can find $K = K(m, \bar{r}_0 - \underline{r}_0)$ for which

$$K^{-1} |\Sigma_0| \exp(t) \leq \exp(2r) \leq K |\Sigma_0| \exp(t).$$

We now derive the evolution equations that will be needed later on. We use the notation

$$B_{ij} \approx C_{ij}$$

when B_{ij} and C_{ij} have the same trace-free part.

Set

$$X := \phi(r) \partial_r \quad \text{and} \quad \beta_t := \exp(-t/2) \langle X, \nu \rangle,$$

where the function ϕ is such that $g_m = dr^2 + \phi(r)^2 g_0$ and ν is the exterior normal vector to Σ_t .

Lemma 3.2. *The following evolution equations hold.*

a)

$$\frac{d\beta_t}{dt} = \frac{\Delta\beta_t}{H^2} + \left(\frac{|A|^2}{H^2} - \frac{1}{2} \right) \beta_t + |\partial_r^\top|^2 \left(\frac{3m}{2 \sinh^3 r} + O(\exp(-5r)) \right) \frac{\beta_t}{|H|^2};$$

b)

$$\frac{dH}{dt} = \frac{\Delta H}{H^2} - (|A|^2 + Rc(\nu, \nu)) \frac{1}{H} - \frac{2|\nabla H|^2}{|H|^3};$$

c)

$$\begin{aligned} \frac{d\dot{A}}{dt} &\approx \frac{\Delta\dot{A}}{H^2} - \frac{2\nabla H \otimes \nabla H}{H^3} - \dot{A} - \frac{2\dot{A}^2}{H} \\ &+ \left(\frac{|\dot{A}|^2}{H^2} - \frac{H^2 + 2Rc(\nu, \nu) + O(\exp(-3r))}{2H^2} \right) \dot{A} \\ &+ \left(\frac{1}{H} + \frac{1}{H^2} \right) O(\exp(-3r)); \end{aligned}$$

d)

$$\begin{aligned} \frac{d|\dot{A}|^2}{dt} &\leq \frac{\Delta|\dot{A}|^2}{H^2} + 2 \left(\frac{|\dot{A}|^2}{H^2} - \frac{H^2 + 2Rc(\nu, \nu) + O(\exp(-3r))}{2H^2} \right) |\dot{A}|^2 \\ &- 2|\dot{A}|^2 - 2 \frac{|\nabla\dot{A}|^2}{H^2} - 4 \frac{\langle \nabla H \otimes \nabla H, \dot{A} \rangle}{H^3} \\ &+ \left(\frac{|\dot{A}|}{H^2} + \frac{|\dot{A}|}{H} \right) O(\exp(-3r)). \end{aligned}$$

Proof. For every vector Y we have that

$$D_Y X = \phi'(r)Y$$

and this implies that, using local coordinates (y_1, y_2) for Σ_t ,

$$\langle \nabla\beta_t, \partial_i \rangle = \exp(-t/2)A(\partial_i, X^\top), \quad i = 1, 2$$

and

$$\begin{aligned} \exp(t/2)\Delta\beta_t &= \sum_i (\nabla_{\partial_i} A)(\partial_i, X^\top) + A(\partial_i, \nabla_{\partial_i} X^\top) \\ &= \langle \nabla H, X \rangle + Rc(\nu, X^\top) + \phi' H - \langle X, \nu \rangle |A|^2. \end{aligned}$$

Moreover

$$D_{\partial_t} \nu = \nabla H / H^2, \quad D_{\partial_t} X = \phi' \partial_t,$$

and so

$$\frac{d\beta_t}{dt} = \frac{\phi'}{H} + \frac{\langle \nabla H, X \rangle}{H^2} - \frac{\beta_t}{2}.$$

Therefore

$$\frac{d\beta_t}{dt} = \frac{\Delta\beta_t}{H^2} + \left(\frac{|A|^2}{H^2} - \frac{1}{2} \right) \beta_t - \exp(-t/2) \frac{Rc(\nu, X^\top)}{|H|^2}.$$

Note that denoting by e_1, e_2 a g_m -orthonormal basis for the coordinates spheres

$$\langle \nu, X^\top \rangle = 0 \Rightarrow \langle \nu, \partial_r \rangle \langle \partial_r, X^\top \rangle = - \sum_i \langle \nu, e_i \rangle \langle e_i, X^\top \rangle$$

and hence, we obtain from [12, Lemma 3.1 (iii)] that

$$\begin{aligned} Rc(\nu, X^\top) &= \sum_i \langle \nu, e_i \rangle Rc(e_i, X^\top) + \langle \nu, \partial_r \rangle Rc(\partial_r, X^\top) \\ &= \left(\frac{m}{2 \sinh^3 r} + O(\exp(-5r)) \right) \sum_i \langle \nu, e_i \rangle \langle e_i, X^\top \rangle \\ &\quad - \left(\frac{m}{\sinh^3 r} + O(\exp(-5r)) \right) \langle \nu, \partial_r \rangle \langle \partial_r, X^\top \rangle \\ &= |\partial_r^\top|^2 \left(-\frac{3m}{2 \sinh^3 r} + O(\exp(-5r)) \right) \langle \nu, X \rangle. \end{aligned}$$

The second evolution equation was derived in [8, Section 1].

We now prove the third identity. From [10, Theorem 3.2] it follows that assuming normal coordinates around a point p

$$\frac{d\dot{A}_{ij}}{dt} \approx \frac{dA_{ij}}{dt} - A_{ij} \approx \frac{\nabla_i \nabla_j H}{H^2} - \frac{2\nabla_i H \nabla_j H}{H^3} + \frac{\dot{A}_{ik} \dot{A}_{kj} - R_{\nu i \nu j}}{H}.$$

Arguing like in the proof of Simons' identity for the Laplacian of the second fundamental form A (see for instance [10]), one can see that

$$\begin{aligned} \Delta \dot{A}_{ij} &\approx \nabla_i \nabla_j H + H \dot{A}_{im} \dot{A}_{mj} + \dot{A}_{ij} H^2 / 2 - \dot{A}_{ij} |\dot{A}|^2 + H R_{\nu i \nu j} \\ &\quad - R_{\nu \nu} \dot{A}_{ij} + R_{kikm} \dot{A}_{mj} + R_{kjkm} \dot{A}_{im} + R_{kijm} \dot{A}_{km} + R_{mjik} \dot{A}_{km} \\ &\quad + D_k R_{\nu jik} + D_i R_{\nu kjk}. \end{aligned}$$

Because the metric g_m satisfies

$$\begin{aligned} R_{stuv} &= -(\delta_{su} \delta_{tv} - \delta_{sv} \delta_{tu}) + O(\exp(-3r)) \\ D_q R_{stuv} &= O(\exp(-3r)) \end{aligned}$$

it follows that

$$\begin{aligned} R_{kikm} \dot{A}_{mj} + R_{kjkm} \dot{A}_{im} + R_{kijm} \dot{A}_{km} + R_{mjik} \dot{A}_{km} \\ &= -4 \dot{A}_{ij} + \dot{A}_{ij} O(\exp(-3r)) \\ &= 2Rc(\nu, \nu) \dot{A}_{ij} + \dot{A}_{ij} O(\exp(-3r)), \\ R_{\nu i \nu j} &= -g_{ij} + O(\exp(-3r)), \end{aligned}$$

and therefore

$$\begin{aligned} \frac{d\dot{A}_{ij}}{dt} &\approx \frac{\Delta\dot{A}_{ij}}{H^2} - \frac{2\nabla_i H \nabla_j H}{H^3} \\ &\quad + \left(\frac{|\dot{A}|^2}{H^2} - \frac{H^2 + 2Rc(\nu, \nu) + O(\exp(-3r))}{2H^2} \right) \dot{A}_{ij} \\ &\quad + \left(\frac{1}{H} + \frac{1}{H^2} \right) O(\exp(-3r)). \end{aligned}$$

Using the formula

$$\frac{d\dot{A}}{dt}(\partial_i, \partial_j) = \frac{d\dot{A}_{ij}}{dt} - \langle D_{\partial_t} \partial_i, \partial_k \rangle \dot{A}_{kj} - \langle D_{\partial_t} \partial_j, \partial_k \rangle \dot{A}_{ik}$$

we obtain Lemma 3.2 c). The last identity follows from

$$\frac{d|\dot{A}|^2}{dt} = 2 \left\langle \frac{d\dot{A}}{dt}, \dot{A} \right\rangle$$

and

$$\langle \dot{A}^2, \dot{A} \rangle = 0.$$

q.e.d.

We now argue that we can choose $\underline{r} = \underline{r}(m)$ and a positive constant $C = C(\varepsilon_0, \bar{r}_0 - \underline{r}_0, m)$ such that if $\underline{r}_0 \geq \hat{r}$, then

$$(5) \quad H \geq C \quad \text{and} \quad \langle \nu, \partial_r \rangle \geq C \exp(\underline{r}_0 - \bar{r}_0)$$

while the solution exists.

Choosing \underline{r} large enough so that for all $r \geq \underline{r}$ the term

$$\frac{3m}{2 \sinh^3 r} + O(\exp(-5r))$$

in the equation of Lemma 3.2 a) is positive, we obtain that

$$\frac{d\beta_t}{dt} \geq \frac{\Delta\beta_t}{H^2}$$

while β_t is nonnegative and thus $\beta_t \geq \min \beta_0 > 0$. Note that $\phi(r)$ grows like $\exp(r)$ and so, for some constant $C = C(m)$,

$$\beta_t \leq C \exp(\bar{r}_0) \langle \partial_r, \nu \rangle.$$

This implies the desired bound for $\langle \nu, \partial_r \rangle$.

Set $\alpha_t := \beta_t H$. Because

$$Rc(\nu, \nu) = -2 + O(\exp(-3r)),$$

the previous lemma implies that, provided we choose \underline{r} sufficiently large,

$$\begin{aligned} \frac{d\alpha_t}{dt} &= \frac{\Delta\alpha_t}{H^2} - \frac{2\langle\nabla\alpha_t, \nabla H\rangle}{H^3} + (4 + O(\exp(-3r)) - H^2) \frac{\alpha_t}{2H^2} \\ &\geq \frac{\Delta\alpha_t}{H^2} - \frac{2\langle\nabla\alpha_t, \nabla H\rangle}{H^3} + (3 - H^2) \frac{\alpha_t}{2H^2}. \end{aligned}$$

Because $\alpha_t \leq \sqrt{3}(\min \beta_0)$ implies that $H^2 \leq 3$, it follows from the maximum principle that $\alpha_t \geq \min\{\sqrt{3}(\min \beta_0), \min \alpha_0\}$ for all t and thus we can use the inequalities in (4) in order to obtain the desired bound for the mean curvature.

Lemma 3.3. *We can find constants $\underline{r} = \underline{r}(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ and $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ such that if $\underline{r}_0 \geq \underline{r}$, then*

$$|\Sigma_0||H^2 - 4| \leq C \left(|\Sigma_0| \sup_{\Sigma_0} (|H^2 - 4| + |\mathring{A}|^2) + \exp(-\underline{r}_0) \right) \exp(-t)$$

and

$$|\Sigma_0|^2 |\mathring{A}|^2 \leq C \left(|\Sigma_0|^2 \sup_{\Sigma_0} |\mathring{A}|^2 + \exp(-2\underline{r}_0) \right) \exp(-2t)$$

while the solution exists.

Proof. We assume that the bounds in (5) hold. Let $\alpha_t := |\mathring{A}|^2 H^{-2}$. From Lemma 3.2

$$\frac{dH^{-2}}{dt} = \frac{\Delta H^{-2}}{H^2} + 2 \left(\frac{|\mathring{A}|^2}{H^2} + \frac{H^2 + 2Rc(\nu, \nu)}{2H^2} \right) H^{-2} - \frac{2|\nabla H|^2}{|H|^6}$$

and thus

$$(6) \quad \frac{d\alpha_t}{dt} \leq \frac{\Delta\alpha_t}{H^2} + 4\alpha_t^2 - 2\alpha_t + (\alpha_t + \sqrt{\alpha_t} + H^{-1}\sqrt{\alpha_t}) \frac{O(\exp(-3r))}{H^2} + Q,$$

where

$$Q := 4 \frac{\langle \nabla H, \nabla |\mathring{A}|^2 \rangle}{H^5} - 2 |\mathring{A}|^2 \frac{|\nabla H|^2}{H^6} - 2 \frac{|\nabla \mathring{A}|^2}{H^4} - 4 \frac{\langle \nabla H \otimes \nabla H, \mathring{A} \rangle}{H^3}.$$

We claim that, given $\varepsilon > 0$, we can find a constant $C = C(\varepsilon_0, \varepsilon, \bar{r}_0 - \underline{r}_0, m)$ so that

$$\frac{d\alpha_t}{dt} \leq \frac{\Delta\alpha_t}{H^2} + 4\alpha_t \left(\alpha_t - \frac{1}{2} + \varepsilon \right) + C \exp(-6\underline{r}_0 - 3t) + Q$$

and

$$Q(p) \leq 4 \frac{|\nabla H|^2}{H^4} \left((1 + \varepsilon)\alpha_t(p) - \frac{1}{4} + \varepsilon \right) + C \exp(-6\underline{r}_0 - 3t),$$

whenever p is a critical point of α_t .

The first inequality follows easily from Cauchy's inequalities combined with properties (4) and (5). Denote by $\{v_1, v_2\}$ an eigenbasis for \mathring{A} at p

and assume without loss of generality that $\mathring{A}(v_1, v_1) \geq 0$. Because p is a critical point of α_t the following identities hold at p

$$\nabla|\mathring{A}|^2 = 2|\mathring{A}|^2 H^{-1} \nabla H$$

and

$$|\mathring{A}| H^{-1} \nabla H = \sqrt{2} \nabla \mathring{A}(v_1, v_1) = -\sqrt{2} \nabla \mathring{A}(v_2, v_2).$$

As a result, we obtain that

$$|\nabla \mathring{A}|^2 = |\mathring{A}|^2 \frac{|\nabla H|^2}{H^2} + 2|\nabla \mathring{A}(v_1, v_2)|^2 = \alpha_t^2 |\nabla H|^2 + 2|\nabla \mathring{A}(v_1, v_2)|^2$$

and

$$\begin{aligned} & 2|\nabla \mathring{A}(v_1, v_2)|^2 \\ &= 2|\nabla_{v_2} A(v_1, v_1) + Rc(\nu, v_2)|^2 + 2|\nabla_{v_1} A(v_2, v_2) + Rc(\nu, v_1)|^2 \\ &\leq 2|\nabla_{v_2} A(v_1, v_1)|^2 + 2|\nabla_{v_1} A(v_2, v_2)|^2 \\ &\quad + (\sqrt{\alpha_t} |\nabla H| + |\nabla H|) O(\exp(-3r)) + O(\exp(-6r)) \\ &= 2 \left| \nabla_{v_2} \mathring{A}(v_1, v_1) + \frac{\langle \nabla H, v_2 \rangle}{2} \right|^2 + 2 \left| \nabla_{v_1} \mathring{A}(v_2, v_2) + \frac{\langle \nabla H, v_1 \rangle}{2} \right|^2 \\ &\quad + (\sqrt{\alpha_t} |\nabla H| + |\nabla H|) O(\exp(-3r)) + O(\exp(-6r)) \\ &= 2|\langle \nabla H, v_2 \rangle|^2 \left(\frac{\alpha_t}{\sqrt{2}} + \frac{1}{2} \right)^2 + 2|\langle \nabla H, v_1 \rangle|^2 \left(\frac{\alpha_t}{\sqrt{2}} - \frac{1}{2} \right)^2 \\ &\quad + (\sqrt{\alpha_t} |\nabla H| + |\nabla H|) O(\exp(-3r)) + O(\exp(-6r)) \\ &= \alpha_t^2 |\nabla H|^2 + |\nabla H|^2 / 2 - 2 \frac{\langle \nabla H \otimes \nabla H, \mathring{A} \rangle}{H} \\ &\quad + (\sqrt{\alpha_t} |\nabla H| + |\nabla H|) O(\exp(-3r)) + O(\exp(-6r)). \end{aligned}$$

Moreover, we also have that at the point p

$$4\langle \nabla H, \nabla |\mathring{A}|^2 \rangle = 8\alpha_t \frac{|\nabla H|^2}{H}$$

and thus

$$\begin{aligned} Q(p) &= 4 \frac{|\nabla H|^2}{H^4} \left(\alpha_t(p) - \frac{1}{4} \right) \\ &\quad + (\sqrt{\alpha_t} |\nabla H| + |\nabla H|) \frac{O(\exp(-3r))}{H^4} + \frac{O(\exp(-6r))}{H^4}. \end{aligned}$$

The claim follows from Cauchy's inequalities combined with properties (4) and (5).

As a result, there is a constant $C_1 = C_1(\varepsilon_0, \varepsilon, \bar{r}_0 - \underline{r}_0, m)$ for which if we set

$$\beta_t := \alpha_t + C_1 \exp(-6\underline{r}_0 - 3t),$$

then

$$(7) \quad \frac{d\beta_t}{dt} \leq \frac{\Delta\beta_t}{H^2} + 4\alpha_t \left(\alpha_t - \frac{1}{2} + \varepsilon \right) + Q$$

and

$$Q(p) \leq 4 \frac{|\nabla H|^2}{H^4} \left((1 + \varepsilon)\alpha_t(p) - \frac{1}{4} + \varepsilon \right)$$

whenever p is a critical point of β_t .

Chose $\varepsilon < \delta_0/4$ so that

$$(1 + \varepsilon) \left(\frac{1}{4} - \frac{\delta_0}{4} \right) - \frac{1}{4} + \varepsilon \leq 0$$

and chose \underline{r} so that $C_1 \exp(-6\underline{r}) \leq \delta_0/4$. Thus $\beta_0 \leq 1/4 - 3\delta_0/4$ and

$$\beta_t(x) \leq 1/4 - \delta_0/2 \implies \alpha_t(x) \leq 1/4 - \delta_0/4.$$

Therefore we can apply the maximum principle to β_t and conclude that

$$\alpha_t \leq \sup \alpha_0 + C_1 \exp(-6\underline{r}_0)$$

while the solution exists. This implies that

$$\alpha_t - \frac{1}{2} + \varepsilon \leq -1/4 - \delta_0/8$$

and so we obtain from equation (7) that

$$\alpha_t \leq (\sup \alpha_0 + C \exp(-6\underline{r}_0)) \exp(-(1 + 2\delta_0)t)$$

for some $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. As a result,

$$\begin{aligned} \alpha_t^2 &\leq C(\sup \alpha_0^2 + \exp(-12\underline{r}_0)) \exp(-2t - 4\delta_0 t), \\ \sqrt{\alpha_t} \exp(3r) &\leq C(\sup \alpha_0 + \exp(-6\underline{r}_0)) \exp(-2t - \delta_0 t) \end{aligned}$$

for some $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. Using this bounds in equation (6) we obtain

$$\begin{aligned} \frac{\alpha_t}{dt} &\leq \frac{\Delta\alpha_t}{H^2} - (2 + C \exp(-3t/2))\alpha_t \\ &\quad + C(\sup \alpha_0 + \exp(-6\underline{r}_0)) \exp(-(2 + \delta_0)t) + C \exp(-6\underline{r}_0 - 3t) + Q \end{aligned}$$

for some $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ and hence

$$|\dot{A}|^2 \leq C \left(\sup_{\Sigma_0} |\dot{A}|^2 + \exp(-6\underline{r}_0) \right) \exp(-2t).$$

The evolution equation for H^2 is given by (see [8, Section 1])

$$\frac{dH^2}{dt} = \frac{\Delta H^2}{H^2} - \frac{6|\nabla H|^2}{|H|^2} - 2|\dot{A}|^2 - H^2 - 2Rc(\nu, \nu)$$

and thus, if we set $\phi_t := \exp(t)(H^2 - 4)$, we obtain that

$$\frac{d\phi_t}{dt} = \frac{\Delta\phi_t}{H^2} - \frac{3\langle \nabla\phi_t, \nabla H \rangle}{H^2} - 2|\dot{A}|^2 \exp(t) - \exp(t)(4 + 2Rc(\nu, \nu))$$

From the upper bound derived for $|\dot{A}|$ and the bounds given in (4) and (5) we have that

$$\begin{aligned} \frac{d\phi_t}{dt} &\geq \frac{\Delta\phi_t}{H^2} - \frac{3\langle\nabla\phi_t, \nabla H\rangle}{H^2} \\ &\quad - C \left(\sup_{\Sigma_0} |\dot{A}|^2 + \exp(-6\underline{r}_0) \right) \exp(-t) - C \exp(-t/2 - 3\underline{r}_0) \end{aligned}$$

where $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. The maximum principle implies that

$$H^2 \geq 4 - C \left(\sup_{\Sigma_0} (|H^2 - 4| + |\dot{A}|^2) + \exp(-3\underline{r}_0) \right) \exp(-t).$$

In order to show the existence of some $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ for which

$$H^2 \leq 4 + C \left(\sup_{\Sigma_0} (|H^2 - 4| + |\dot{A}|^2) + \exp(-3\underline{r}_0) \right) \exp(-t)$$

it is enough to note that

$$\frac{dH^2}{dt} \leq \frac{\Delta H^2}{H^2} + 4 - H^2 + C \exp(-3\underline{r}_0 - 3t/2).$$

q.e.d.

Fix some \underline{r} for which Lemma 3.3 holds. Note that in this case we have a uniform bound for $|A|^2$ and so standard estimates can be used to show that the solution $(\Sigma_t)_{t \geq 0}$ exists for all time. Nonetheless, we need shaper estimates on all the derivatives of A and this will occupy most of the rest of the proof. What we have done so far proves Theorem 3.1 (i). The next lemma will be useful in proving Theorem 3.1 (iii).

Lemma 3.4. *There is a constant $C = C(Q_0, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ such that*

$$1 - \langle \partial_r, \nu \rangle \leq C \left(\sup_{\Sigma_0} (1 - \langle \partial_r, \nu \rangle) + \exp(-3\underline{r}_0) \right) \exp(-t)$$

for all t and thus

$$|\Sigma_0| |\nabla r|^2 \leq C \exp(-t)$$

for some other constant $C = C(Q_0, Q_1, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$.

Proof. We denote by Λ any geometric quantity defined on Σ_t for which we can find a constant $C = C(\varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ such that

$$|\Lambda| \leq C(|H - 2| + |\dot{A}| + \exp(-2r))$$

For every vector Y we have that

$$D_Y \partial_r = \phi'(r)/\phi(r) (Y - \langle Y, \partial_r \rangle \partial_r).$$

Therefore

$$\begin{aligned} \frac{d\langle \partial_r, \nu \rangle}{dt} &= \langle D_{\partial_t} \partial_r, \nu \rangle + \langle \partial_r, D_{\partial_t} \nu \rangle \\ &= \frac{\phi'}{\phi} \frac{1}{H} - \frac{\phi'}{\phi} \frac{\langle \partial_r, \nu \rangle^2}{H} + \frac{\langle \partial_r, \nabla H \rangle}{H^2}. \end{aligned}$$

For every tangent vectors Z and W we have

$$\langle \nabla \langle \partial_r, \nu \rangle, Z \rangle = -\phi' / \phi \langle \partial_r, \nu \rangle \langle \partial_r, Z \rangle + A(Z, \partial_r^\top)$$

and

$$\langle \nabla_Z \partial_r^\top, W \rangle = \phi' / \phi \langle Z, W \rangle - \phi' / \phi \langle Z, \partial_r \rangle \langle W, \partial_r \rangle - \langle \partial_r, \nu \rangle A(Z, W),$$

where ∂_r^\top denotes the tangential projection of ∂_r . These identities combined with Lemma 3.3 and with

$$\frac{\phi'}{\phi} = 1 + O(\exp(-2r))$$

imply that

$$\begin{aligned} \operatorname{div} \left(-\frac{\phi'}{\phi} \langle \partial_r, \nu \rangle \partial_r^\top \right) &= 2 \left(\frac{\phi'}{\phi} \right)^2 \langle \partial_r, \nu \rangle (|\partial_r^\top|^2 - 1) + \frac{\phi'}{\phi} \langle \partial_r, \nu \rangle^2 H \\ &\quad - \left(\frac{\phi'}{\phi} \right)' |\partial_r^\top|^2 \langle \partial_r, \nu \rangle - \frac{\phi'}{\phi} \langle \partial_r, \nu \rangle A(\partial_r^\top, \partial_r^\top) \\ &= -2 \left(\frac{\phi'}{\phi} \right)^2 \langle \partial_r, \nu \rangle + \langle \partial_r, \nu \rangle |\partial_r^\top|^2 \\ &\quad + \frac{\phi'}{\phi} \langle \partial_r, \nu \rangle^2 H + |\partial_r^\top|^2 \Lambda \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}(A(\cdot, \partial_r^\top)) &= \langle \nabla H, \partial_r \rangle + Rc(\nu, \partial_r^\top) + \frac{\phi'}{\phi} H - \frac{\phi'}{\phi} A(\partial_r^\top, \partial_r^\top) - |A|^2 \langle \partial_r, \nu \rangle \\ &= \langle \nabla H, \partial_r \rangle + \frac{\phi'}{\phi} H - |\partial_r^\top|^2 - \frac{H^2}{2} \langle \partial_r, \nu \rangle - |A|^2 \langle \partial_r, \nu \rangle \\ &\quad + |\partial_r^\top|^2 \Lambda + O(\exp(-3r)). \end{aligned}$$

As a result we get

$$\frac{d\langle \partial_r, \nu \rangle}{dt} = \frac{\Delta \langle \partial_r, \nu \rangle}{H^2} + Q,$$

where

$$\begin{aligned} H^2 Q &= 2 \left(\frac{\phi'}{\phi} \right)^2 \langle \partial_r, \nu \rangle - \langle \partial_r, \nu \rangle |\partial_r^\top|^2 - 2 \frac{\phi'}{\phi} \langle \partial_r, \nu \rangle^2 H + |\partial_r^\top|^2 + \frac{H^2}{2} \langle \partial_r, \nu \rangle \\ &\quad + |A|^2 \langle \partial_r, \nu \rangle + |\partial_r^\top|^2 \Lambda + O(\exp(-3r)). \end{aligned}$$

Setting $\alpha_t := \langle \partial_r, \nu \rangle - 1$, we obtain from (5) and Lemma 3.3 that

$$\begin{aligned} Q &= 2H^{-2} \left(\frac{H^2}{4} - H \frac{\phi'}{\phi} + \left(\frac{\phi'}{\phi} \right)^2 \right) + \frac{\alpha_t}{4} (\alpha_t^2 - 2\alpha_t - 4) \\ &\quad + |\mathring{A}|^2 \langle \partial_r, \nu \rangle + \alpha_t \Lambda + O(\exp(-3r)) \\ &\geq \frac{\alpha_t}{4} (\alpha_t^2 - 2\alpha_t - 4) + \alpha_t \Lambda + O(\exp(-3r)) \\ &\geq -\alpha_t(1 - \Lambda) + O(\exp(-3r)), \end{aligned}$$

where the last inequality follows from $0 \geq \alpha_t \geq -1$. There is

$$C = C(Q_0, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$$

for which

$$|\Lambda| \leq C \exp(-t)$$

and hence

$$\frac{d\alpha_t}{dt} \geq \frac{\Delta\alpha_t}{H^2} - (1 + C \exp(-t))\alpha_t + C \exp(-3t/2 - 3\underline{r}_0)$$

for some other $C = C(Q_0, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. This equation implies the desired result. q.e.d.

For the rest of the proof, C will denote any constant with dependence

$$C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m).$$

Set \hat{r}_t to be such that $|\Sigma_t| = 4\pi \sinh^2 \hat{r}_t$ and we remark that $\hat{r}_t - t/2$ is uniformly bounded. An immediate consequence of the previous lemma is that Σ_t can be written as the graph of a function f_t over the coordinate sphere $\{|x| = \hat{r}_t\}$ with

$$|f_t| \leq C \quad \text{and} \quad |\Sigma_0| |\nabla f_t|^2 \leq C \exp(-t)$$

for some constant C . Furthermore, Lemma 3.3 and Proposition 2.1 imply the existence of some constant C for which

$$|\Sigma_0|^2 |\nabla^2 f_t|^2 \leq C \exp(-2t).$$

The next lemma is an adaptation of what was done in [7, Section 6].

Given two tensors P and S we denote by $S * T$ any linear combination of tensors formed by contracting over S and T .

Lemma 3.5. *There is $\underline{r} = \underline{r}((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ so that if $\underline{r}_0 \geq \underline{r}$ the following property holds.*

For every $n \geq 0$ there is a constant C such that

$$|\Sigma_0|^n |\nabla^n f_t|^2 \leq C \exp(-nt)$$

for all t . Equivalently, for all $n \geq 1$ there is a constant C for which

$$|\Sigma_0|^{n+2} |\nabla^n A|^2 \leq C \exp(-(n+2)t).$$

Proof. We start by showing that it is enough to bound $\nabla^n \mathring{A}$.

Lemma 3.6. *There exists a constant C for which*

$$|\Sigma_0|^3 |\nabla A| \leq C |\Sigma_0|^3 |\nabla \mathring{A}| + C \exp(-3t/2)$$

and

$$|\Sigma_0|^4 |\nabla^2 A| \leq C |\Sigma_0|^4 |\nabla^2 \mathring{A}| + C \exp(-2t).$$

Moreover, if we can find a constant E for which

$$|\Sigma_0|^{k+2} |\nabla^k \mathring{A}|^2 \leq E \exp(-(k+2)t) \quad \text{for all } k = 1, \dots, n-1,$$

then

$$|\Sigma_0|^{n+3} |\nabla^{n+1} A|^2 \leq C_1 |\Sigma_0|^{n+3} |\nabla^{n+1} \mathring{A}|^2 + C_1 \exp(-(n+3)t).$$

for some constant $C_1 = C_1(E, (Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$.

Proof. On each Σ_t consider the 1-form

$$B(X) = Rc(X, \nu).$$

First we estimate the derivatives of B . In local coordinates (x_1, x_2) , B can be written as

$$B_j = F_j(r, \nabla r), \quad j = 1, 2,$$

where $F_j(r, q_1, q_2)$ is defined on \mathbb{R}^3 and

$$|D^k F_j| \leq S_k \exp(-3r) \quad j = 1, 2$$

for some constant S_k , provided (q_1, q_2) lie on a fixed compact set.

We denote by P any tensor on Σ_t for which $|P| = O(\exp(-3r))$ and by Q any tensor for which $|Q| = O(\exp(-3r))$ and

$$\nabla Q = \nabla r * P + \nabla^2 r * P.$$

Using this notation we have

$$\nabla B = \nabla r * Q + \nabla^2 r * Q$$

and so we can estimate

$$|\Sigma_0|^3 |B|^2 \leq C \exp(-3t), \quad \text{and} \quad |\Sigma_0|^4 |\nabla B|^2 \leq C \exp(-4t)$$

for some constant C .

Let $\{v_1, v_2\}$ be an orthonormal basis for Σ_t . We know that for every integer p

$$\nabla^p A(v_1, v_2) = \nabla^p \mathring{A}(v_1, v_2)$$

and

$$\nabla^p A(v_1, v_1) - \nabla^p A(v_2, v_2) = \nabla^p \mathring{A}(v_1, v_1) - \nabla^p \mathring{A}(v_2, v_2).$$

Moreover, Codazzi equations imply that for $i \neq j$

$$\nabla^p \nabla_{v_i} A(v_j, v_j) = \nabla^p \nabla_{v_j} A(v_1, v_2) - \nabla^p B_i$$

and thus

$$|\nabla^{m+1} A| \leq C |\nabla^{m+1} \mathring{A}| + C |\nabla^m B|$$

for every integer m . This implies the desired result when $n = 0, 1$.

To prove the general result we proceed by induction. The inductive hypothesis implies that

$$|\Sigma_0|^k |\nabla^k r|^2 = |\Sigma_0|^k |\nabla^k f_t|^2 \leq C_1 \exp(-kt) \quad \text{for all } k = 1, \dots, n+1$$

for some $C_1 = C_1(E, (Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$ and thus, using the expression derived for ∇B , we obtain

$$|\Sigma_0|^{k+3} |\nabla^k B|^2 \leq C_1 \exp(-(k+3)t) \quad \text{for all } k = 1, \dots, n$$

for some $C_1 = C_1(E, (Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. Hence, the desired result follows. q.e.d.

In what follows L_1 will denote any tensor that satisfies the following properties. There exists a constant C for which

$$|\Sigma_0| |L_1| \leq C \exp(-t), \quad |\Sigma_0|^3 |\nabla L_1|^2 \leq C |\Sigma_0|^3 |\nabla A|^2 + C \exp(-3t),$$

and if there is a constant E such that

$$|\Sigma_0|^{k+2} |\nabla^k A|^2 \leq E \exp(-(k+2)t) \quad \text{for all } k = 1, \dots, n-1,$$

then

$$|\Sigma_0|^{n+2} |\nabla^n L_1|^2 \leq C_1 |\Sigma_0|^{n+2} |\nabla^n A|^2 + C_1 \exp(-(n+2)t)$$

for some constant $C_1 = C_1(E, (Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. Likewise, L_0 will denote any tensor with the same properties of L_1 except that we just require $|L_0|$ to be uniformly bounded.

We can see from Lemma 3.2 that the evolution equation for \mathring{A} can be written as

$$\frac{d\mathring{A}}{dt} \approx \frac{\Delta \mathring{A}}{H^2} - \mathring{A} + L_1 * \mathring{A} + M + \nabla A * \nabla A * L_0,$$

where the tensor M stands for the term

$$\left(\frac{1}{H} + \frac{1}{H^2} \right) O(\exp(-3r))$$

that appears on Lemma 3.2 c). The relevant property of M is that

$$|\Sigma_0|^2 |M|^2 \leq C \exp(-3t)$$

and

$$|\Sigma_0|^3 |\nabla M|^2 \leq C \exp(-3t) |\Sigma_0|^3 |\nabla A|^2 + C \exp(-4t)$$

for some constant C . If there is a constant E such that for all t

$$|\Sigma_0|^{k+2} |\nabla^k A|^2 \leq E \exp(-(k+2)t) \quad \text{for all } k = 1, \dots, n-1,$$

then

$$|\Sigma_0|^{n+2} |\nabla^n M|^2 \leq C_1 \exp(-3t) |\Sigma_0|^{n+2} |\nabla^n A|^2 + C_1 \exp(-(n+3)t)$$

for some other constant $C_1 = C_1(E, (Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m)$. This follows from the fact that in local coordinates (x_1, x_2)

$$M = \left(\frac{1}{H} + \frac{1}{H^2} \right) F(r, \nabla r),$$

where $F(r, q_1, q_2)$ is a matrix-valued function defined on \mathbb{R}^3 for which there is a constant S_k such that, provided (q_1, q_2) lie on a fixed compact set,

$$|D^k F| \leq S_k \exp(-3r).$$

If K denotes the curvature tensor of Σ_t , then for any tensor T we know that

$$\Delta \nabla T = \nabla \Delta T + K * \nabla T + \nabla K * T$$

and

$$\begin{aligned} \frac{d\nabla T}{dt} &= \nabla \frac{dT}{dt} - \frac{\nabla T}{2} + \nabla T * \dot{A} + T * \nabla A \\ &= \nabla \frac{dT}{dt} - \frac{\nabla T}{2} + \nabla T * L_1 + T * \nabla L_0. \end{aligned}$$

The last identity comes from the fact that, using normal coordinates,

$$\begin{aligned} \frac{d\nabla T}{dt}(\partial_1, \dots, \partial_{n+1}) &= \nabla \frac{dT}{dt}(\partial_1, \dots, \partial_{n+1}) - \nabla T(\partial_1, \dots, \partial_n, D_{\partial_t} \partial_{n+1}) \\ &\quad + (T * \nabla A)(\partial_1, \dots, \partial_{n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d\nabla T}{dt} &= \frac{\Delta \nabla T}{H^2} - \frac{\nabla T}{2} + \nabla \left(\frac{dT}{dt} - \frac{\Delta T}{H^2} \right) + T * \nabla L_1 \\ &\quad + \nabla T * L_1 + \nabla^2 T * \nabla L_1 + T * \nabla L_0. \end{aligned}$$

Proceeding inductively, it can be checked that

$$\begin{aligned} \frac{d\nabla^n \dot{A}}{dt} &\approx \frac{\Delta \nabla^n \dot{A}}{H^2} - \left(\frac{n}{2} + 1 \right) \nabla^n \dot{A} + \sum_{j=0}^n \nabla^j \dot{A} * \nabla^{n-j} L_1 \\ &\quad + \sum_{j=0}^{n-1} \nabla^{j+2} \dot{A} * \nabla^{n-j} L_1 + \nabla^n M + \sum_{j=0}^{n-1} \nabla^j \dot{A} * \nabla^{n-j} L_0 \\ &\quad + \sum_{j,k,l \geq 0, j+k+l=n} \nabla^{j+1} A * \nabla^{k+1} A * \nabla^l L_0 \end{aligned}$$

and thus we can find a constant C for which

$$\begin{aligned} \frac{d|\nabla^n \mathring{A}|^2}{dt} &\leq \frac{\Delta|\nabla^n \mathring{A}|^2}{H^2} - 2\frac{|\nabla^{n+1} \mathring{A}|^2}{H^2} - (n+2)|\nabla^n \mathring{A}|^2 \\ &+ C \sum_{j=0}^n |\nabla^j \mathring{A}| |\nabla^{n-j} L_1| |\nabla^n \mathring{A}| + C \sum_{j=0}^{n-1} |\nabla^{j+2} \mathring{A}| |\nabla^{n-j} L_1| |\nabla^n \mathring{A}| \\ &+ C |\nabla^n M| |\nabla^n \mathring{A}| + C \sum_{j,k,l \geq 0, j+k+l=n} |\nabla^{j+1} A| |\nabla^{k+1} A| |\nabla^l L_0| |\nabla^n \mathring{A}| \\ &+ C \sum_{j=0}^{n-1} |\nabla^j \mathring{A}| |\nabla^{n-j} L_0| |\nabla^n \mathring{A}|. \end{aligned}$$

We now show the desired bound when $n = 1$. Recall that for some constant C we have (see Lemma 3.3 and Lemma 3.6)

$$|\nabla L_0| + |\nabla L_1| + |\nabla A| \leq C(|\nabla \mathring{A}| + \exp(-3t/2)|\Sigma_0|^{-3/2}),$$

$$|\nabla^2 A| \leq C(|\nabla^2 \mathring{A}| + \exp(-2t)|\Sigma_0|^{-2}), \quad \text{and} \quad |\Sigma_0|^2 |\mathring{A}|^2 \leq C \exp(-2t).$$

In this case, we can find $\varepsilon > 0$ such that

$$\begin{aligned} \frac{d|\nabla \mathring{A}|^2}{dt} &\leq \frac{\Delta|\nabla \mathring{A}|^2}{H^2} - (3 - C \exp(-\varepsilon t)) |\nabla \mathring{A}|^2 + C |\nabla \mathring{A}|^4 \\ &+ C \exp(-(3 + \varepsilon)t) |\Sigma_0|^{-3}. \end{aligned}$$

Hence, if we set

$$\alpha_t := |\Sigma_0|^3 |\nabla \mathring{A}|^2 + \exp(-3t),$$

then

$$\frac{d\alpha_t}{dt} \leq \frac{\Delta\alpha_t}{H^2} - (3 - C \exp(-\varepsilon t))\alpha_t + C\alpha_t^2 + C \exp(-(3 + \varepsilon)t)$$

for some other constant C . Moreover, from Lemma 3.2 d) and Lemma 3.3, we can find some positive constant

$$\bar{C} = \bar{C}((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{\mathbf{r}}_0 - \underline{\mathbf{l}}_0, m),$$

so that

$$\begin{aligned} \frac{d|\mathring{A}|^2}{dt} &\leq \frac{\Delta|\mathring{A}|^2}{H^2} + (\bar{C}|\Sigma_0|^{-1} - 1/2)|\nabla \mathring{A}|^2 + (\bar{C}|\Sigma_0|^{-1} - 1)|\mathring{A}|^2 \\ &+ \bar{C} \exp(-3t) |\Sigma_0|^{-3}. \end{aligned}$$

Choose $\underline{\mathbf{r}}$ so that

$$\underline{\mathbf{r}}_0 \geq \underline{\mathbf{r}} \implies \bar{C}|\Sigma_0|^{-1} \leq 1/4.$$

Set

$$\psi_t := \frac{\log \alpha_t^2}{2} + K|\Sigma_0|^3 |\mathring{A}|^2,$$

where the constant K will be chosen later. Note that

$$\begin{aligned} \frac{d\psi_t}{dt} \leq \frac{\Delta\psi_t}{H^2} - (3 - C \exp(-\varepsilon t)) + (C - K/4)|\Sigma_0|^3|\nabla\dot{A}|^2 \\ + \frac{|\nabla \log \alpha_t^2|^2}{4} + C \exp(-3t) \end{aligned}$$

for some constant C . Choose K such that $K > 4C + 4$. If p is a maximum of ψ_t , then at p

$$\frac{|\nabla \log \alpha_t^2|^2}{4} = K^2 |\nabla |\dot{A}|^2|^2 \leq 2K^2 |\nabla \dot{A}|^2 |\dot{A}|^2 \leq CK^2 |\Sigma_0|^{-2} |\nabla \dot{A}|^2.$$

We can now chose \underline{r} so that for al $r_0 \geq \underline{r}$ we have

$$-|\Sigma_0|^3 |\nabla \dot{A}|^2(p) + \frac{|\nabla \log \alpha_t^2|^2(p)}{4} \leq 0.$$

The maximum principle implies that

$$\psi_t \leq -3t + C$$

for some constant C and so

$$|\Sigma_0|^3 |\nabla \dot{A}|^2 \leq C \exp(-3t)$$

for some other constant C .

For $n > 1$ we argue by induction. Thus, assume that

$$|\Sigma_0|^{k+2} |\nabla^k A|^2 \leq C \exp(-(k+2)t) \quad \text{for all } k = 1, \dots, n-1$$

for some constant C . Then, we can find another constant C for which

$$\begin{aligned} |\nabla^j L_0|^2 + |\nabla^j L_1|^2 &\leq C |\Sigma_0|^{-j-2} \exp(-(j+2)t) \quad \text{if } 1 \leq j \leq n-1, \\ |\nabla^n L_0|^2 + |\nabla^n L_1|^2 &\leq C |\nabla^n \dot{A}|^2 + C |\Sigma_0|^{-n-2} \exp(-(n+2)t), \\ |\nabla^{n+1} A|^2 &\leq C |\nabla^{n+1} \dot{A}|^2 + C |\Sigma_0|^{-n-3} \exp(-(n+3)t), \\ |\nabla^n A|^2 &\leq C |\nabla^n \dot{A}|^2 + C |\Sigma_0|^{-n-2} \exp(-(n+2)t), \end{aligned}$$

and

$$|\nabla^n M|^2 \leq C \exp(-3t) |\nabla^n \dot{A}|^2 + C |\Sigma_0|^{-n-2} \exp(-(n+3)t).$$

Looking at the evolution equation of $|\nabla^n \dot{A}|^2$, we see that we can find $\varepsilon > 0$ and a constant C such that

$$\begin{aligned} \frac{d|\nabla^n \dot{A}|^2}{dt} \leq \frac{\Delta|\nabla^n \dot{A}|^2}{H^2} - ((n+2) - C \exp(-\varepsilon t)) |\nabla^n \dot{A}|^2 \\ + C |\Sigma_0|^{-n-2} \exp(-(n+2+\varepsilon)t) \end{aligned}$$

and the maximum principle implies the desired result.

q.e.d.

In what follows, C continues to denote any constant with dependence

$$C = C((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0 - \underline{r}_0, m).$$

One immediate consequence of this lemma is that if we denote by ∇_0 the connection determined by g_0 (the round metric on S^2), then for every $n \geq 0$

$$|\nabla_0^n f_t| \leq C$$

for some constant C . Moreover,

$$\frac{d\hat{r}_t}{dt} = \frac{\sinh \hat{r}_t}{2 \cosh \hat{r}_t}$$

and thus, combining Lemma 3.3 with Lemma 3.4, we have

$$\begin{aligned} \left| \frac{df_t}{dt} \right| &= \left| \frac{1}{H} - \frac{\sinh \hat{r}_t}{2 \cosh \hat{r}_t} + (\langle \partial_r, \nu \rangle - 1)H^{-1} \right| \\ &\leq C(|H - 2| + \exp(-2\underline{r}_0 - 2t) + |\langle \partial_r, \nu \rangle - 1|) \\ &\leq C \left(\sup_{\Sigma_0} (|H^2 - 4| + |\dot{A}|^2 + |\langle \partial_r, \nu \rangle - 1|) + \exp(-2\underline{r}_0) \right) \exp(-2t), \end{aligned}$$

for some other constant C . As a result, we get that the functions f_t converge to a smooth function f_∞ on S^2 and so this proves Theorem 3.1 (iii).

We will now argue that for all integers $k \geq 1$ and $n \geq 0$ there is a constant C such that

$$|\Sigma_0|^n |\partial_t^k \nabla^n f_t|^2 \leq C \exp(-nt) \quad \text{for } n \geq 1, \quad |\partial_t^k f_t| \leq C \exp(-t),$$

and

$$|\Sigma_0|^{n+2} |\partial_t^k \nabla^n A|^2 \leq C \exp(-(n+2)t).$$

This estimates finish the proof of the theorem.

We start with the case $k = 1$. Using normal coordinates, we have that

$$\begin{aligned} \langle \partial_t \nabla f_t, \partial_i \rangle &= \partial_t (\partial_i f_t) - \langle \nabla f_t, D_{\partial_t} \partial_i \rangle \\ &= \partial_i (\langle \partial_r, \nu \rangle) H^{-1} - \langle \partial_r, \nu \rangle \langle \nabla H, \partial_i \rangle H^{-2} - A \langle \nabla f_t, \partial_i \rangle H^{-1} \\ &= -\frac{\phi'}{\phi} \langle \partial_r, \nu \rangle \langle \nabla f_t, \partial_i \rangle H^{-1} - \langle \partial_r, \nu \rangle \langle \nabla H, \partial_i \rangle H^{-2} \end{aligned}$$

and this implies that

$$|\Sigma_0| |\partial_t \nabla f_t|^2 \leq C \exp(-t).$$

The same type of computations shows that for every $n \geq 1$ we can find C such that

$$|\Sigma_0|^n |\partial_t \nabla^n f_t|^2 \leq C \exp(-nt).$$

This implies that, for each $n \geq 1$,

$$|\Sigma_0|^{n+2} |\partial_t \nabla^n A|^2 \leq C \exp(-(n+2)t)$$

for some constant C . Having this estimates one can then show that

$$|\partial_t^2 f_t| \leq C \exp(-t)$$

and, for each $n \geq 1$,

$$|\Sigma_0|^n |\partial_t^2 \nabla^n f_t|^2 \leq C \exp(-nt).$$

Repeating this process gives the desired estimates.

q.e.d.

4. A modified Shi-Tam flow

In this section Σ_0 denotes a sphere satisfying hypothesis (H) and $(\Sigma_t)_{t \geq 0}$ is a solution to inverse mean curvature flow for which Theorem 3.1 holds. Consider the manifold

$$N := \bigcup_{t \geq 0} \Sigma_t$$

where the metric g_m can be written as

$$g_m = \frac{dt^2}{H^2} + g_t.$$

The metric \bar{g} is defined to be

$$\bar{g} := \frac{u^2}{H^2} dt^2 + g_t,$$

where function u satisfies (12).

Lemma 4.1. *The metric \bar{g} has $R(\bar{g}) = -6$.*

Proof. The mean curvature and the exterior normal vector of Σ_t computed with respect to \bar{g} equal

$$\bar{H}(\Sigma_t) = H(\Sigma_t)/u \quad \text{and} \quad \bar{\nu} = \nu/u$$

respectively. Thus

$$\frac{\bar{\nu}}{\bar{H}} = \frac{\nu}{H}$$

and this implies that $(\Sigma_t)_{t \geq 0}$ is indeed a solution to inverse mean curvature flow for the new metric with $\bar{H}(\Sigma_0) = 2$.

We now check that the scalar curvature of \bar{g} is -6 . According to formula (1.10) of [14], given metrics

$$h_0 := dt^2 + g_t \quad \text{and} \quad h_1 = v^2 dt^2 + g_t,$$

the scalar curvature R^0 of h_0 and R^1 of h_1 are related by

$$(8) \quad H^0 \frac{\partial v}{\partial t} = v^2 \Delta_t v + \frac{1}{2}(v - v^3)R_t - \frac{1}{2}uR^0 + \frac{u^3}{2}R^1,$$

where H^0 denotes the mean curvature of Σ_t with respect to h_0 .

Let g_0 be the metric $dt^2 + g_t$. Because the scalar curvature of g_m is -6 , we obtain from combining (8) (setting $v = H^{-1}$) both with Gauss equations and with

$$\frac{dH^{-1}}{dt} = \frac{\Delta H^{-1}}{H^2} + \frac{|A|^2 + Rc(\nu, \nu)}{H^3}$$

that the scalar curvature of g_0 is given by

$$R(g_0) = R_t - 1 - \frac{|A|^2}{H^2}.$$

Consider the function $v := u/H$. Using (8) with $h_0 = g_0$ and $h_1 = \bar{g}$, the condition that $R(\bar{g}) = -6$ is equivalent to

$$\frac{\partial v}{\partial t} = v^2 \Delta_t v + \frac{1}{2}(v - v^3)R_t - \frac{1}{2}vR(g_0) - 3v^3.$$

The evolution equation for u follows from the above equation, Gauss equations, and the evolution equation for H^{-1} .

q.e.d.

Using the identification of Σ_t with S^2 via

$$\Sigma_t = \{(\hat{r}_t + f_t(\theta), \theta) \mid \theta \in S^2\},$$

the function u_t can be identified with a function on S^2 which we still denote by u_t . Recall that the normalized metrics \hat{g}_t (defined on Lemma 5.1) converge to a smooth metric on S^2 . The main purpose of this section is to prove

Theorem 4.2. *Assume that Σ_0 satisfies $H(\Sigma_0) > 0$ and that on Σ_t we have*

$$R_t + 6 - 2H\Delta_t H^{-1} > 0$$

for all t .

Equation (12) admits a smooth solution u with initial condition $u_{\Sigma_0} = H(\Sigma_0)/2$ and satisfying the following properties.

(i) If we denote by u_t the restriction of u to Σ_t , then the functions

$$w_t := 2 \exp(3t/2) |\Sigma_0| (u_t - 1) / (4\pi)$$

converge smoothly to a function w_∞ defined on S^2 .

(ii) For every integer n and k we can find

$$\Lambda = \Lambda((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0, \underline{r}_0, m)$$

such that

$$|\nabla^n w_t|^2 \leq \Lambda \exp(-nt) \quad \text{and} \quad |\partial_t^k \nabla^n w_t| \leq \Lambda \exp(-(n+2)t);$$

(iii) *The metric \bar{g} is asymptotically hyperbolic. More precisely, we can find a coordinate system (s, θ) and a symmetric 2-tensor Q such that*

$$\bar{g} = ds^2 + \sinh^2 s g_0 + \left(\frac{m + (|\Sigma_0|/(4\pi))^{1/2} \exp(3f_\infty) w_\infty}{3 \sinh s} \right) g_0 + Q$$

and

$$|Q| + |DQ| + |D^2Q| + |D^3Q| \leq \Lambda \exp(-4r)$$

for some $\Lambda = \Lambda((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0, \underline{r}_0, m)$.

Except for property (iii), this theorem was essentially proven in [15, Theorem 2.1] when the deformation vector of the foliation $(\Sigma_t)_{t \geq 0}$ equals the unit normal vector. In light of Theorem 3.1 the same techniques apply with no modification (see also [14]). Nonetheless, we need to make sure that some estimates are independent of \underline{r}_0 and so we sketch its proof. During the proof Λ will denote any constant with dependence

$$\Lambda = \Lambda((Q_j)_{j \in \mathbb{N}}, \varepsilon_0, \delta_0, \bar{r}_0, \underline{r}_0, m).$$

Proof. Set

$$h^+(t) = \sup_{\Sigma_t} \left(\frac{R_t + 6 - 2H\Delta_t H^{-1}}{2H^2} \right)$$

and $h^-(t) = h^+(t)$ if $\inf u_0 \leq 1$ or, in case $\inf u_0 > 1$,

$$h^-(t) = \inf_{\Sigma_t} \left(\frac{R_t + 6 - 2H\Delta_t H^{-1}}{2H^2} \right).$$

Moreover, define

$$W^+ = 1 - \left(\sup_{\Sigma_0} u_0 \right)^{-2}, \quad W^- = 1 - \left(\inf_{\Sigma_0} u_0 \right)^{-2},$$

and

$$\gamma^\pm(t) = \left(1 - W^\pm \exp \left(- \int_0^t 2h^\pm(s) ds \right) \right)^{-1/2}.$$

From [14, Lemma 2.2] (see also [15, Section 2.2]) we have that comparison with the ODE

$$\frac{d\gamma}{dt} = h^\pm(t)(\gamma - \gamma^3),$$

implies

$$(9) \quad \gamma^-(t) \leq u_t \leq \gamma^+(t)$$

while the solution exists. Moreover, we know from Theorem 3.1 that

$$|h^\pm(t) - 3/4| \leq \Lambda \exp(-t) \quad \text{and} \quad |w_0| \leq \Lambda.$$

for some constant Λ . Therefore, the inequalities in (9) imply that, while the solution exists,

$$|w_t| \leq \Lambda$$

for some other constant Λ .

Performing the change of variable

$$s = -4\pi|\Sigma_t|^{-1} = -4\pi|\Sigma_0|^{-1} \exp(-t),$$

the evolution equation for w_t becomes (see also [15, Theorem 2.1])

$$(10) \quad \frac{dw_s}{ds} = \frac{u^2}{H^2} \widehat{\Delta}_t w_s + 2u^2 H^{-1} \widehat{g}_t \left(\widehat{\nabla} w_t, \widehat{\nabla} H^{-1} \right) \\ + w_s (4\pi)^{-1} |\Sigma_t| \left(\frac{3}{2} + u(u+1) \left(\frac{\Delta_t H^{-1}}{H} - \frac{R_t + 6}{2H^2} \right) \right),$$

where the operators $\widehat{\Delta}_t$ and $\widehat{\nabla}$ are computed with respect to the normalized metric \widehat{g}_t and the range of s is $-4\pi|\Sigma_0|^{-1} \leq s < 0$.

In order to use the standard theory for quasilinear parabolic equations, we need to make some remarks regarding the last term on the right-hand side of equation (10). Direct computation shows that

$$u(u+1) = 2 + 3\sqrt{\frac{|\Sigma_0|}{16\pi}} (-s)^{3/2} w_s - \frac{|\Sigma_0|}{16\pi} s^3 w_s^2.$$

Thus the term

$$(11) \quad (4\pi)^{-1} |\Sigma_t| \left(\frac{3}{2} + u(u+1) \left(\frac{\Delta_t H^{-1}}{H} - \frac{R_t + 6}{2H^2} \right) \right)$$

can be decomposed as

$$-\frac{3|\Sigma_0|}{164\pi} s^2 w_s^2 - 9\sqrt{\frac{|\Sigma_0|}{164\pi}} \sqrt{-s} w_s + u(u+1) F_t,$$

where

$$F_t = (4\pi)^{-1} |\Sigma_t| \left(\frac{\Delta_t H^{-1}}{H} + \frac{6(H^2 - 4) - 4R_t}{8H^2} \right).$$

Therefore, we obtain from Theorem 3.1 that the term in (11) is bounded by some constant Λ .

Standard theory for quasilinear parabolic equations [11, Section VI, Theorem 6.33] gives a uniform $C^{0,\alpha}$ -bound in space-time for w_s , i.e., for all $\theta, \theta' \in S^2$ and $-4\pi|\Sigma_0|^{-1} \leq s, s' < 0$

$$\frac{|w_s(\theta) - w_s(\theta')|}{\text{dist}(\theta, \theta')^{2\alpha}} + \frac{|w_s(\theta) - w_{s'}(\theta)|}{|s - s'|^\alpha} \leq \Lambda$$

for some constant Λ .

The term in (11) has a uniform $C^{0,\alpha}$ -bound and so standard Schauder estimates imply that $\widehat{\nabla} w_s$ and $\widehat{\nabla}^2 w_s$ are uniformly $C^{0,\alpha}$ -bounded in space-time. Bootstrapping implies the existence of a solution w_s for all s with

$$|\widehat{\nabla}^n w_s| + |\partial_s \widehat{\nabla}^n w_s| \leq \Lambda$$

for every integer n . Rewriting the equation for w_t in terms of the variable t and differentiating it with respect to time we obtain that, for every integer n and k ,

$$|\nabla^n w_t|^2 \leq \Lambda \exp(-nt) \quad \text{and} \quad |\partial_t^k \nabla^n w_t|^2 \leq \Lambda \exp(-(n+2)t).$$

As a result, w_t converges smoothly to a smooth function w_∞ defined on S^2 .

Finally, we show that the metric \bar{g} satisfies the definition of asymptotic hyperbolicity given in the Introduction. The manifold N defined in the beginning of this section is diffeomorphic to $S^2 \times [0, +\infty)$ and thus, besides polar coordinates (r, θ) , admits also coordinates (t, θ) where $r = f_t + \hat{r}_t$. In what follows we will use these coordinate systems, Theorem 3.1, and the previous estimates for the function u without further mention. Let

$$h := w_\infty \exp(3f_\infty) |\Sigma_0|^{1/2} (4\pi)^{-1/2}$$

and denote by Q any 2-tensor that satisfies

$$|Q| + |DQ| + |D^2Q| + |D^3Q| = O(\exp(-4r)).$$

Then

$$\begin{aligned} \bar{g} &= g_m + \frac{u^2 - 1}{H^2} dt^2 = g_m + \frac{u - 1}{2} dt^2 + Q \\ &= g_m + 2(u - 1) dr^2 + Q. \end{aligned}$$

Due to the fact that

$$|\Sigma_t| = 4\pi \sinh^2(r - f_t),$$

we get that

$$\begin{aligned} \bar{g} &= g_m + \frac{2(u - 1) \exp(3t/2) |\Sigma_0|^{3/2}}{(4\pi)^{3/2} \sinh^3(r - f_t)} dr^2 + Q \\ &= g_m + \frac{8h}{\exp(3r)} dr^2 + Q \\ &= (1 + 4h \exp(-3r))^2 dr^2 + (\sinh^2 r + m/(3 \sinh r)) g_0 + Q. \end{aligned}$$

Thus, if we set

$$s := r - 4/3h \exp(-3r),$$

we obtain that

$$\bar{g} = ds^2 + (\sinh^2 s + (h + m)/(3 \sinh s)) g_0 + Q$$

and this implies that \bar{g} is asymptotic hyperbolic if one uses the coordinate system (s, θ) . q.e.d.

5. Proof of the main theorem

We now prove the main theorem.

Proof of Theorem 1.2. Consider the ambient manifold to be Anti-de Sitter–Schwarzschild $(S^2 \times [t_0, +\infty), g_m)$ with positive mass. Set f to be a smooth function on S^2 with

$$\int_{S^2} \exp(2f) d\mu_0 = 1$$

that is invariant under reflection with respect to the three coordinate planes and consider

$$\Sigma(r_0) = \{(r_0 + f(\theta), \theta) \mid \theta \in S^2\} \subset S^2 \times [t_0, +\infty).$$

According to Proposition 2.1 e) we know that

$$\lim_{r_0 \rightarrow \infty} m_H(\Sigma(r_0)) = \frac{m}{2} \int_{S^2} \exp(-f) d\mu_0 > \frac{m}{2},$$

where the last inequality is a consequence of Hölder's inequality.

Choose r_0 sufficiently large such that $\Sigma(r_0)$ satisfies hypothesis (H) of Section 3 and

$$m_H(\Sigma(r_0)) > \frac{m}{2}.$$

This is possible because, due to Proposition 2.1, we know that

$$\lim_{r_0 \rightarrow \infty} H = 2 \quad \text{and} \quad \lim_{r_0 \rightarrow \infty} |\mathring{A}|^2 = 0.$$

Therefore, we can apply Theorem 3.1 and conclude the existence of a smooth solution $(\Sigma_t)_{t \geq 0}$ to inverse mean curvature flow where, by monotonicity of Hawking mass,

$$\frac{m}{2} < m_H(\Sigma(r_0)) \leq \lim_{t \rightarrow \infty} m_H(\Sigma_t).$$

Denote the induced metric on Σ_t by g_t . The above inequality implies

Lemma 5.1. *The Gaussian curvature \widehat{K}_t of Σ_t with respect to the normalized metric*

$$\hat{g}_t := (4\pi)|\Sigma_t|^{-1}g_t$$

does not converge to one when t goes to infinity.

Proof. From Theorem 3.1 (iii) we know that

$$\Sigma_t = \{(\hat{r}_t + f_t(\theta), \theta) \mid \theta \in S^2\},$$

where \hat{r}_t is such that

$$|\Sigma_t| = 4\pi \sinh^2 \hat{r}_t$$

and the functions f_t converge to a smooth function f_∞ defined on S^2 . Moreover, Proposition 2.1 (more precisely, identity (1)) implies that the metric \hat{g}_t converges to $\hat{g} = \exp(2f_\infty)g_0$. Note that the ambient metric is preserved by reflections with respect to the coordinate planes and thus the metric \hat{g} also shares these symmetries.

Suppose that \widehat{K}_t converges to one. Then \hat{g} is a constant scalar curvature metric which is symmetric under reflection on the coordinate planes and so f_∞ must be identically zero. If this were true, it would follow from Proposition 2.1 e) that

$$\lim_{t \rightarrow \infty} m_H(\Sigma_t) = \frac{m}{2}$$

and this is impossible.

q.e.d.

Outside $\Sigma(r_0)$, i.e., on the region

$$N := \bigcup_{t \geq 0} \Sigma_t,$$

the metric g_m can be written as

$$g_m = \frac{dt^2}{H^2} + g_t.$$

We want to find a new asymptotically hyperbolic metric \bar{g} with $R(\bar{g}) = -6$ such that, with respect to this new metric, the mean curvature of Σ_0 is 2, $(\Sigma_t)_{t \geq 0}$ is a solution to inverse mean curvature flow, and the induced metric on Σ_t by \bar{g} coincides with g_t . This would finish the proof for the following two reasons.

First, because $(\Sigma_t)_{t \geq 0}$ is a smooth solution to inverse mean curvature flow for \bar{g} , N is foliated by a family of spheres with positive mean curvature and thus $\Sigma(r_0)$ is outer-minimizing.

Second, the intrinsic geometry of Σ_t is maintained and so we know from Lemma 5.1 that the Gaussian curvature of Σ_t with respect to the normalized metric does not converge to one. Proposition 2.1 f) implies that no matter the coordinate system we choose we will always have

$$\liminf_{t \rightarrow \infty} (\bar{s}_t - \underline{s}_t) > 0.$$

The construction of the metric \bar{g} is inspired by the work of Shi and Tam [14]. Consider smooth positive functions u defined on N such that

$$u_{\Sigma_0} := H(\Sigma_0)/2$$

and

$$(12) \quad 2H^2 \frac{\partial u}{\partial t} = 2u^2 \Delta_t u + 4u^2 H \langle \nabla u, \nabla H^{-1} \rangle + (u - u^3)(R_t + 6 - 2H \Delta_t H^{-1}),$$

where the Laplacian and gradient term are computed with respect to the metric g_t and R_t is the scalar curvature of Σ_t . Having such a function u , the new metric is defined to be

$$\bar{g} := \frac{u^2}{H^2} dt^2 + g_t$$

and it has scalar curvature -6 by Lemma 4.1. Note that the intrinsic geometry of Σ_t is preserved and the mean curvature and exterior normal vector of Σ_t computed with respect to \bar{g} equal

$$\bar{H}(\Sigma_t) = H(\Sigma_t)/u \quad \text{and} \quad \bar{\nu} = \nu/u$$

respectively. Thus

$$\frac{\bar{\nu}}{\bar{H}} = \frac{\nu}{H}$$

and this implies that $(\Sigma_t)_{t \geq 0}$ is indeed a solution to inverse mean curvature flow for the new metric with $\bar{H}(\Sigma_0) = 2$.

We are only left to check that equation (12) has a solution. Note that, provided we choose r_0 sufficiently large, Proposition 2.1 and Theorem 3.1 imply that

$$R_t + 6 - 2H\Delta_t H^{-1} > 0$$

for all t . It is important to remark that this estimate holds because the constants on Theorem 3.1 do not depend on \underline{r}_0 but only on $\bar{r}_0 - \underline{r}_0$. Therefore, Theorem 4.2 implies that equation (12) admits a solution and that the metric \bar{g} is asymptotically hyperbolic.

q.e.d.

Remark 5.2. To extend our argument to the case $\Lambda = -3$, i.e, the case where the initial condition for the flow is a minimal surface, the obvious modification would be to consider a family of metrics

$$g^\varepsilon := \frac{u^2}{H^2} dt^2 + g_t$$

having

$$R = -6 \text{ and } u_{\Sigma_0} := H(\Sigma_0)/\varepsilon.$$

In this case, the sphere Σ_0 would have mean curvature $H = \varepsilon$ with respect to g^ε and $(\Sigma)_{t \geq 0}$ would still be a solution to inverse mean curvature with respect to g^ε . One would then need to show that the metrics g^ε converge when ε goes to zero. We remark that if Σ_0 is a coordinate sphere, then each g^ε is an Anti-De Sitter-Schwarzschild and they indeed converge when ε goes to zero.

5.1. A nontrivial consequence of the Penrose inequality. Assuming that the Penrose inequality holds as conjectured by Xiadong Wang, we will argue that for every smooth function f defined on S^2 with

$$\int_{S^2} \exp(2f) d\mu_0 = 1$$

we have

$$\left(\int_{S^2} K_f \exp(3f) d\mu_0 \right)^2 - \sum_{i=1}^3 \left(\int_{S^2} K_f \exp(3f) x_i d\mu_0 \right)^2 \geq 1,$$

where K_f denotes the Gaussian curvature of $\exp(2f)g_0$. A simple computation shows that an equality is attained if $\exp(2f)g_0$ has constant scalar curvature. Moreover, if we denote

$$I(f) = \left(\int_{S^2} K_f \exp(3f) d\mu_0 \right)^2 - \sum_{i=1}^3 \left(\int_{S^2} K_f \exp(3f) x_i d\mu_0 \right)^2$$

and consider T to be a conformal transformation of S^2 such that

$$T^*(g_0) = \exp(2u)g_0,$$

then $I(f) = I(f \circ T + u)$. One could check this directly or note, as we shall see next, that $I(f)$ can be obtained as the limit of masses of sequence of metrics and so $I(f) = I(f \circ T + u)$ because they are the limit of masses for the same sequence of metrics but written with different coordinate systems.

In what follows we use the same notation as in the proof of Theorem 1.2. Set f to be a smooth function on S^2 with

$$\int_{S^2} \exp(2f) d\mu_0 = 1$$

and consider

$$\Sigma(r_0) = \{(r_0 + f(\theta), \theta) \mid \theta \in S^2\} \subset S^2 \times [t_0, +\infty).$$

Denote by $M(r_0)$ the mass of the metric \bar{g} constructed in the proof of Theorem 1.2. It is not hard to see that, by choosing r_0 sufficiently large, we can have f_∞ (defined in Theorem 3.1 (iii)) and w_∞ (defined in Theorem 4.2 (i)) respectively, as close to f and w_0 as we want. Moreover, from Proposition 2.1 d), we have that

$$2w_0 = |\Sigma_0|(H - 2)/(4\pi) = \widehat{K}(\Sigma(r_0)) + O(\exp(-r_0)),$$

where the Gaussian curvature is computed with respect to the normalized metric $\hat{g}(r_0) := 4\pi|\Sigma(r_0)|^{-1}g_{\Sigma(r_0)}$. Therefore, denoting the mass two tensor of \bar{g} by \bar{h} , we have that

$$(16\pi)^{1/2} \bar{h} |\Sigma(r_0)|^{-1/2}$$

is well approximated by

$$\begin{aligned} & \left(m16\pi^{1/2} |\Sigma(r_0)|^{-1/2} + 2 \exp(3f)w_0 \right) g_0 \\ & = \left(\widehat{K}(\Sigma(r_0)) \exp(3f) + O(\exp(-r_0)) \right) g_0. \end{aligned}$$

Because the metric $\hat{g}(r_0)$ converges to $\exp(2f)g_0$ (Proposition 2.1 a)), we obtain that

$$\begin{aligned} & \lim_{r_0 \rightarrow \infty} M(r_0)^2 \frac{16\pi}{|\Sigma(r_0)|} \\ &= \left(\int_{S^2} K_f \exp(3f) d\mu_0 \right)^2 - \sum_{i=1}^3 \left(\int_{S^2} K_f \exp(3f) x_i d\mu_0 \right)^2. \end{aligned}$$

If we assume the Penrose inequality, we know that

$$M(r_0) \left(\frac{16\pi}{|\Sigma(r_0)|} \right)^{1/2} \geq 1$$

and so the desired inequality follows.

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