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LIMIT LEAVES OF AN H LAMINATION ARE STABLE

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Abstract

Suppose $\mathcal{L}$ is a lamination of a Riemannian manifold by hypersurfaces with the same constant mean curvature H. We prove that every limit leaf of $\mathcal{L}$ is stable for the Jacobi operator. A simple but important consequence of this result is that the set of stable leaves of $\mathcal{L}$ has the structure of a lamination.

1. Introduction.

In this paper we prove that given a codimension one lamination $\mathcal{L}$ in a Riemannian manifold N, whose leaves have a fixed constant mean curvature H (minimality is included), then every limit leaf L of $\mathcal{L}$ is stable with respect to the Jacobi operator. Our result is motivated by a partial result of Meeks and Rosenberg in Lemma A.1 in [**8**], where they proved the stability of L under the constraint that the holonomy representation on any compact subdomain $\Delta \subset L$ has subexponential growth (i.e., the covering space $\widetilde{\Delta}$ of Δ corresponding to the kernel of the holonomy representation has subexponential area growth). In general, if we assume stability for a covering space $\widetilde{M}$ of a constant mean curvature (CMC) hypersurface M in N and for any connected compact domain $\Delta \subset M$ the related restricted covering $\widetilde{\Delta} \to \Delta$ has subexponential area growth, then M is also stable, see Lemma 6.2 in [**5**] for a proof using cutoff functions. However, if the area growth of the covering is exponential over some compact domain in M, then the stability of $\widetilde{M}$ does not imply the stability of M, as can be seen in the example described in the next paragraph, which is due to R. Schoen. The existence of this example makes it clear that the application in [**8**] of cutoff functions used to prove the stability of a limit leaf L with holonomy of subexponential growth cannot be applied to case when the holonomy representation of L has exponential growth.

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Consider a compact surface Σ of genus at least two endowed with a metric g of constant curvature -1, and a smooth function $f\colon \mathbb{R} \to (0,1]$ with $f(0) = 1$ and $-\frac{1}{8} < f''(0) < 0$. Then in the warped product metric $f^2\, g + dt^2$ on $\Sigma \times \mathbb{R}$, each slice $M_c = \Sigma \times \{c\}$ is a CMC surface of mean curvature $-\frac{f'(c)}{f(c)}$ oriented by the unit vector field $\frac{\partial}{\partial t}$, and the stability operator on the totally geodesic (hence minimal) surface $M_0 = \Sigma \times \{0\}$ is $L = \Delta + \mathrm{Ric}(\frac{\partial}{\partial t}) = \Delta - 2f''(0)$, where Δ is the laplacian on M_0 with respect to the induced metric $f(0)^2 g = g$ and Ric denotes the Ricci curvature of $f^2\, g + dt^2$. The first eigenvalue of L in the (compact) surface M_0 is $2f''(0)$, hence M_0 is unstable as a minimal surface. On the other hand, the universal cover $\widetilde{M_0}$ of M_0 is the hyperbolic plane. Since the first eigenvalue of the Dirichlet problem for the laplacian in $\widetilde{M_0}$ is $\frac{1}{4}$, we deduce that the first eigenvalue of the Dirichlet problem for the Jacobi operator on $\widetilde{M_0}$ is $\frac{1}{4} + 2f''(0) > 0$. Thus, $\widetilde{M_0}$ is an immersed stable minimal surface. Similarly, for c sufficiently small, the CMC surface M_c is unstable but its related universal cover is stable.

2. The statement and proof of the main theorem.

In order to help understand the results described in this paper, we make the following definitions.

Definition 1. Let M be a complete, embedded hypersurface in a manifold N. A point $p \in N$ is a *limit point* of M if there exists a sequence $\{p_n\}_n \subset M$ which diverges to infinity on M with respect to the intrinsic Riemannian topology on M but converges in N to p as $n \to \infty$. Let $L(M)$ denote the set of all limit points of M in N. In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where $\overline{M}$ denotes the closure of M.

Definition 2. A *codimension one lamination* of a Riemannian manifold N^{n+1} is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1) $\mathcal{L}$ is a closed subset of N;
2) $\mathcal{A} = \{\varphi_\beta\colon \mathbb{D}^n \times (0,1) \to U_\beta\}_\beta$ is a collection of coordinate charts of N (here $\mathbb{D}^n$ is the open unit ball in $\mathbb{R}^n$, $(0,1)$ the open unit interval and U_β an open subset of N);
3) For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D}^n \times C_\beta$.

We will simply denote laminations by $\mathcal{L}$, omitting the charts φ_β in $\mathcal{A}$. A lamination $\mathcal{L}$ is said to be a *foliation of* N if $\mathcal{L} = N$. Every lamination $\mathcal{L}$ naturally decomposes into a collection of disjoint connected hypersurfaces, called the *leaves* of $\mathcal{L}$. As usual, the regularity of $\mathcal{L}$ requires the corresponding regularity on the change of coordinate charts. Note that if $\Delta \subset \mathcal{L}$ is any collection of leaves of $\mathcal{L}$, then the closure of the union of these leaves has the structure of a lamination within $\mathcal{L}$, which we will call a *sublamination.*

Definition 3. For $H \in \mathbb{R}$, an *H-hypersurface* M in a Riemannian manifold N is a codimension one submanifold of constant mean curvature H. A codimension one H-lamination $\mathcal{L}$ of N is a collection of immersed (not necessarily injectively) H-hypersurfaces $\{L_\alpha\}_{\alpha \in I}$, called the *leaves* of $\mathcal{L}$, satisfying the following properties.

1) $\mathcal{L} = \bigcup_{\alpha \in I}\{L_\alpha\}$ is a closed subset of N.
2) If $H = 0$, then $\mathcal{L}$ is a lamination of N. In this case, we also call $\mathcal{L}$ a *minimal lamination.*
3) If $H \neq 0$, then given a leaf L_α of $\mathcal{L}$ and given a small disk $\Delta \subset L_\alpha$, there exists an $\varepsilon > 0$ such that if (q, t) denote the normal coordinates for $\exp_q(t\eta_q)$ (here exp is the exponential map of N and η is the unit normal vector field to L_α pointing to the mean convex side of L_α), then:
   a) The exponential map $\exp\colon U(\Delta, \varepsilon) = \{(q, t) \mid q \in \mathrm{Int}(\Delta), t \in (-\varepsilon, \varepsilon)\}$ is a submersion.
   b) The inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \mathrm{Int}(\Delta), t \in [0, \varepsilon)\}$ is a lamination of $U(\Delta, \varepsilon)$.

The reader not familiar with the subject of minimal or H-laminations should think about a geodesic γ on a Riemannian surface. If γ is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination $\mathcal{L}$ of the surface. When the geodesic γ has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics in $\mathcal{L}$ through the accumulation points of γ forming the leaves of $\mathcal{L}$. A similar result is true for a complete, embedded H-hypersurface of locally bounded second fundamental form (bounded in compact extrinsic balls) in a Riemannian manifold N, i.e., the closure of a complete, embedded H-hypersurface of locally bounded second fundamental form has the structure of an H-lamination of N. For the sake of completeness, we now give the proof of this elementary fact in the case $H \neq 0$ (see the beginning of Section 1 in [**7**] for the proof in the minimal case).

Consider a complete, embedded H-hypersurface M with locally bounded second fundamental form in a manifold N. Choose a limit point p of M (if there are no such limit points, then M is proper and it is an H-lamination of N by itself), i.e., p is the limit in N of a sequence of divergent points p_n in M. Since M has bounded second fundamental form near p and M is embedded, then for some small $\varepsilon > 0$, a subsequence of the intrinsic ε-balls $B_M(p_n, \varepsilon)$ converges to an embedded

H-ball $B(p,\varepsilon) \subset N$ of intrinsic radius ε centered at p. Since M is embedded, any two such limit balls, say $B(p,\varepsilon)$, $B'(p,\varepsilon)$, do not intersect transversally. By the maximum principle for H-hypersurfaces, we conclude that if a second ball $B'(p,\varepsilon)$ exists, then $B(p,\varepsilon)$, $B'(p,\varepsilon)$ are the only such limit balls and they are oppositely oriented at p.

Now consider any sequence of embedded balls E_n of the form $B(q_n, \frac{\varepsilon}{4})$ such that q_n converges to a point in $B(p, \frac{\varepsilon}{2})$ and such that E_n locally lies on the mean convex side of $B(p,\varepsilon)$. For ε sufficiently small and for n, m large, E_n and E_m must be graphs over domains in $B(p,\varepsilon)$ such that when oriented as graphs, they have the same mean curvature. By the maximum principle, the graphs E_n and E_m are disjoint or equal. It follows that near p and on the mean convex side of $B(p,\varepsilon)$, $\overline{M}$ has the structure of a lamination with leaves of the same constant mean curvature as M. This proves that $\overline{M}$ has the structure of an H-lamination of codimension one.

Definition 4. Let $\mathcal{L}$ be a codimension one H-lamination of a manifold N and L be a leaf of $\mathcal{L}$. We say that L is a *limit leaf* if L is contained in the closure of $\mathcal{L} - L$.

We claim that a leaf L of a codimension one H-lamination $\mathcal{L}$ is a limit leaf if and only if for any point $p \in L$ and any sufficiently small intrinsic ball $B \subset L$ centered at p, there exists a sequence of pairwise disjoint balls B_n in leaves L_n of $\mathcal{L}$ which converges to B in N as $n \to \infty$, such that each B_n is disjoint from B. Furthermore, we also claim that the leaves L_n can be chosen different from L for all n. The implication where one assumes that L is a limit leaf of $\mathcal{L}$ is clear. For the converse, it suffices to pick a point $p \in L$ and prove that p lies in the closure of $\mathcal{L} - L$. By hypothesis, there exists a small intrinsic ball $B \subset L$ centered at p which is the limit in N of pairwise disjoint balls B_n in leaves L_n of $\mathcal{L}$, as $n \to \infty$. If $L_n \neq L$ for all $n \in \mathbb{N}$, then we have done. Arguing by contradiction and after extracting a subsequence, assume $L_n = L$ for all $n \in \mathbb{N}$. Choosing points $p_n \in B_n$ and repeating the argument above with p_n instead of p, one finds pairwise disjoint balls $B_{n,m} \subset L$ which converge in N to B_n as $m \to \infty$. Note that for $(n_1, m_1) \neq (n_2, m_2)$, the related balls B_{n_1,m_1}, B_{n_2,m_2} are disjoint. Iterating this process, we find an uncountable number of such disjoint balls on L, which contradicts that L admits a countable basis for its intrinsic topology.

Definition 5. A minimal hypersurface $M \subset N$ of dimension n is said to be *stable* if for every compactly supported normal variation of M, the second variation of area is non-negative. If M has constant mean curvature H, then M is said to be *stable* if the same variational property holds for the functional $A - nHV$, where A denotes area and V stands for oriented volume. A *Jacobi function* $f\colon M \to \mathbb{R}$ is a solution of the equation $\Delta f + |A|^2 f + \mathrm{Ric}(\eta) f = 0$ on M; if M is two-sided, then the stability of M is equivalent to the existence of a positive Jacobi function on M (see Fischer-Colbrie [**2**]).

The proof of the next theorem is motivated by a well-known application of the divergence theorem to prove that every compact domain in a leaf of an oriented, codimension one minimal foliation in a Riemannian manifold is area-minimizing in its relative $\mathbb{Z}$-homology class. For other related applications of the divergence theorem, see [**11**].

Although we have not mentioned it previously, throughout this paper we are assuming some regularity on an H-lamination; we will need this regularity in the proof of the next theorem. The minimum regularity that we require is Lipschitz regularity from which $C^{1,1}$ regularity can be shown. However, in dimension 3, $C^{1,1}$ regularity of a C^0 lamination follows from the one-sided curvature estimate of Colding and Minicozzi [**1**] when $H = 0$ and for $H > 0$, it follows from the work of Meeks and Tinaglia [**9**]. $C^{1,1}$ regularity for a codimension one minimal foliation holds in any dimension by work of Solomon [**13**].

Theorem 1. *The limit leaves of a codimension one H-lamination of a Riemannian manifold are stable.*

Proof. We will assume that the dimension of the ambient manifold N is three in this proof; the arguments below can be easily adapted to the n-dimensional setting. The first step in the proof is the following result.

Assertion 1. Suppose $\overline{D}(p, r)$ is a compact, embedded H-disk in N with constant mean curvature H (possibly negative), intrinsic diameter $r > 0$ and center p, such that there exist global normal coordinates (q, t) based at points $q \in \overline{D}(p, r)$, with $t \in [0, \varepsilon]$. Suppose that $T \subset [0, \varepsilon]$ is a closed disconnected set with zero as a limit point and for each $t \in T$, there exists a function $f_t : \overline{D}(p, r) \to [0, \varepsilon]$ such that the normal graphs $q \mapsto \exp_q(f_t(q)\eta(q))$ define pairwise disjoint H-surfaces with $f_t(p) = t$, where η stands for the oriented unit normal vector field to $\overline{D}(p, r)$. For each component (t_α, s_α) of $[0, \varepsilon] - T$ with $s_\alpha < \varepsilon$, consider the interpolating graphs $q \mapsto \exp_q(f_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

$$f_t = f_{t_\alpha} + (t - t_\alpha)\frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha}.$$

(See Figure 1). Then, the mean curvature functions H_t of the graphs of f_t satisfy

$$\lim_{t \to 0^+} \frac{H_t(q) - H}{t} = 0 \quad \text{for all } q \in D(p, \varepsilon/2).$$

Proof of Assertion 1. Reasoning by contradiction, suppose there exists a sequence $t_n \in [0, \varepsilon] - T$, $t_n \searrow 0$, and points $q_n \in \overline{D}(p, r/2)$, such that $|H_{t_n}(q_n) - H| > Ct_n$ for some constant $C > 0$. Let $(t_{\alpha_n}, s_{\alpha_n})$ be

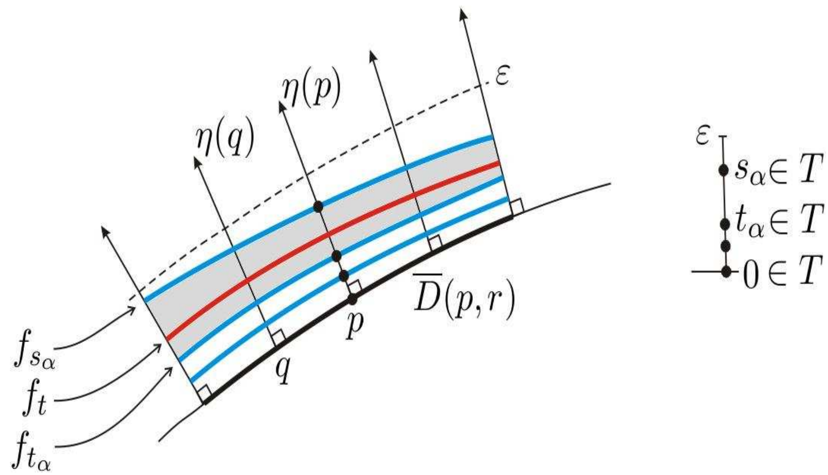

Figure 1. The interpolating graph of f_t between the H-graphs of $f_{t_\alpha}, f_{s_\alpha}$.

the component of $[0,\varepsilon]-T$ which contains t_n. Then, we can rewrite f_{t_n} as

$$f_{t_n} = t_n \left[\frac{t_{\alpha_n}}{t_n} \frac{f_{t_{\alpha_n}}}{t_{\alpha_n}} + \left(1 - \frac{t_{\alpha_n}}{t_n}\right) \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}} \right].$$

After extracting a subsequence, we may assume that as $n \to \infty$, the sequence of numbers $\frac{t_{\alpha_n}}{t_n}$ converges to some $A \in [0,1]$, and the sequences of functions $\frac{f_{t_{\alpha_n}}}{t_{\alpha_n}}, \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}}$ converge smoothly to Jacobi functions F_1, F_2 on $\overline{D}(p,r/2)$, respectively. Now consider the normal variation of $\overline{D}(p,r/2)$ given by

$$\widetilde{\psi}_t(q) = \exp_q\left(t[AF_1 + (1-A)F_2](q)\eta(q)\right),$$

for $t > 0$ small. Since $AF_1 + (1-A)F_2$ is a Jacobi function, the mean curvature $\widetilde{H}_t$ of $\widetilde{\psi}_t$ is $\widetilde{H}_t = H + \mathcal{O}(t^2)$, where $\mathcal{O}(t^2)$ stands for a function satisfying $t\mathcal{O}(t^2) \to 0$ as $t \to 0^+$. On the other hand, the normal graphs of f_{t_n} and of $t_n(AF_1 + (1-A)F_2)$ over $\overline{D}(p,r/2)$ can be taken arbitrarily close in the C^4-norm for n large enough, which implies that their mean curvatures $H_{t_n}, \widetilde{H}_{t_n}$ are C^2-close. This is a contradiction with the assumed decay of H_{t_n} at q_n. q.e.d.

We now continue the proof of the theorem. Let L be a limit leaf of an H-lamination $\mathcal{L}$ of a manifold N by hypersurfaces. If L is one-sided, then we consider the two-sided 2:1 cover $\widetilde{L} \to L$ and pullback the H-lamination $\mathcal{L}$ to a small neighborhood of the zero section $\widetilde{L}_0$ of the normal bundle $\widetilde{L}^\perp$ to $\widetilde{L}$ ($\widetilde{L}_0$ can be identified with $\widetilde{L}$ itself). In this case,

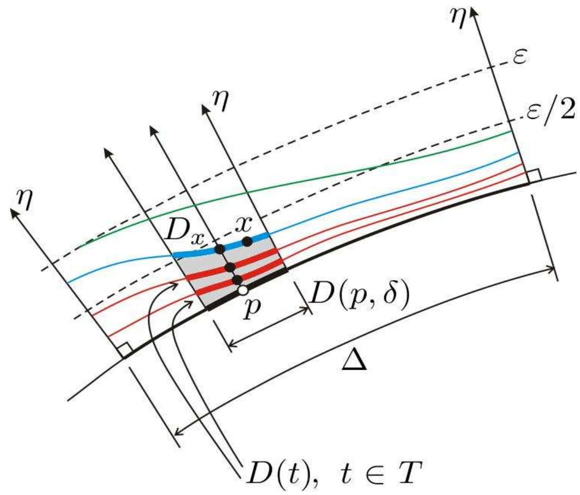

Figure 2. The shaded region between D_x and $D(p,\delta)$ corresponds to $U(p,\delta)$.

we will prove that $\widetilde{L}_0$ is stable, which in particular implies stability for L, see Remark 1. Hence, in the sequel we will assume L is two-sided.

Arguing by contradiction, suppose there exists an unstable compact subdomain $\Delta \subset L$ with non-empty smooth boundary $\partial\Delta$. Given a subset $A \subset \Delta$ and $\varepsilon > 0$ sufficiently small, we define

$$A^{\perp,\varepsilon} = \{\exp_q(t\eta(q)) \mid q \in A,\ t \in [0,\varepsilon]\}$$

to be the one-sided vertical ε-neighborhood of A, written in normal coordinates (q,t) (here we have picked the unit normal η to L such that L is a limit of leaves of $\mathcal{L}$ at the side η points into). Since $\mathcal{L}$ is a lamination and Δ is compact, there exists $\delta \in (0,\varepsilon)$ such that the following property holds:

($\star$) Given an intrinsic disk $D(p,\delta) \subset L$ centered at a point $p \in \Delta$ with radius δ, and given a point $x \in \mathcal{L}$ which lies in $D(p,\delta)^{\perp,\varepsilon/2}$, then there passes a disk $D_x \subset \mathcal{L}$ through x, which is entirely contained in $D(p,\delta)^{\perp,\varepsilon}$, and D_x is a normal graph over $D(p,\delta)$.

Fix a point $p \in \Delta$ and let $x \in \mathcal{L} \cap \{p\}^{\perp,\varepsilon/2}$ be the point above p with greatest t-coordinate. Consider the disk D_x given by property ($\star$), which is the normal graph of a function f_x over $D(p,\delta)$. Since Δ is compact, ε can be assumed to be small enough so that the closed region given in normal coordinates by $U(p,\delta) = \{(q,t) \mid q \in D(p,\delta), 0 \le t \le f_x(q)\}$ intersects $\mathcal{L}$ in a closed collection of disks $\{D(t) \mid t \in T\}$, each of which is the normal graph over $D(p,\delta)$ of a function $f_t \colon D(p,\delta) \to [0,\varepsilon)$ with $f_t(p) = t$, and T is a closed subset of $[0,\varepsilon/2]$, see Figure 2. We now foliate the region $U(p,\delta)-\bigcup_{t\in T} D(t)$ by interpolating the graphing functions as we did in Assertion 1. Consider the union of all these locally defined foliations $\mathcal{F}_p$ with p varying in Δ. Since Δ is compact, we find $\varepsilon_1 \in (0,\varepsilon/2)$ such that the one-sided normal neighborhood $\Delta^{\perp,\varepsilon_1} \subset \bigcup_{p\in\Delta} \mathcal{F}_p$ of Δ is foliated by surfaces which are portions of disks in the locally defined foliations $\mathcal{F}_p$. Let $\mathcal{F}(\varepsilon_1)$ denote this foliation of $\Delta^{\perp,\varepsilon_1}$. By Assertion 1, the mean curvature function of the foliation $\mathcal{F}(\varepsilon_1)$ viewed locally as a function $H(p,t)$ with $p\in\Delta$ and $t\in[0,\varepsilon_1]$, satisfies

$$\lim_{t\to 0^+} \frac{H(p,t)-H}{t} = 0, \quad \text{for all } p\in\Delta. \tag{1}$$

On the other hand since Δ is unstable, the first eigenvalue λ_1 of the Jacobi operator J for the Dirichlet problem on Δ, is negative. Consider a positive eigenfunction h of J on Δ (note that $h=0$ on $\partial\Delta$). For $t\geq 0$ small and $q\in\Delta$, $\exp_q(th(q)\eta(q))$ defines a family of surfaces $\{\Delta(t)\}_t$ with $\Delta(t)\subset\Delta^{\perp,\varepsilon}$ and the mean curvature $\widehat{H}_t$ of $\Delta(t)$ satisfies

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{H}_t = Jh = -\lambda_1 h > 0 \quad \text{on the interior of } \Delta. \tag{2}$$

Let $\Omega(t)$ be the compact region of N bounded by $\Delta\cup\Delta(t)$ and foliated away from $\partial\Delta$ by the surfaces $\Delta(s)$, $0\leq s\leq t$. Consider the smooth unit vector field V defined at any point $x\in\Omega(t)-\partial\Delta$ to be the unit normal vector to the unique leaf $\Delta(s)$ which passes through x, see Figure 3. Since the divergence of V at $x\in\Delta(s)\subset\Omega(t)$ equals $-2\widehat{H}_s$ where $\widehat{H}_s$ is the mean curvature of $\Delta(s)$ at x, then (2) gives

$$\mathrm{div}(V) = -2\widehat{H}_s = -2H + 2\lambda_1 sh + \mathcal{O}(s^2) \quad \text{on } \Delta(s)$$

for $s>0$ small. It follows that there exists a positive constant C such that for t small,

$$\int_{\Omega(t)} \mathrm{div}(V) = -2H\mathrm{Vol}(\Omega(t)) + 2\lambda_1 \int_{\Omega(t)} sh + \mathcal{O}(t^2) < -2H\mathrm{Vol}(\Omega(t)) - Ct. \tag{3}$$

Since the foliation $\mathcal{F}(\varepsilon_1)$ has smooth leaves with uniformly bounded second fundamental form, then the unit normal vector field W to the leaves of $\mathcal{F}(\varepsilon_1)$ is Lipschitz on $\Delta^{\perp,\varepsilon_1}$ and hence, it is Lipschitz on $\Omega(t)$. Since W is Lipschitz, its divergence is defined almost everywhere in $\Omega(t)$ and the divergence theorem holds in this setting. Note that the divergence of W is smooth in the regions of the form $U(p,\delta)-\bigcup_{t\in T} D(t)$ where it is equal to -2 times the mean curvature of the leaves of $\mathcal{F}_p$. Also, the mean curvature function of the foliation is continuous on $\mathcal{F}(\varepsilon_1)$ (see Assertion 1). Hence, the divergence of W can be seen to be a continuous function on $\Omega(t)$ which equals $-2H$ on the leaves $D(t)$, and by Assertion 1, $\mathrm{div}(W)$ converges to the constant $-2H$ as $t\to 0$ to first

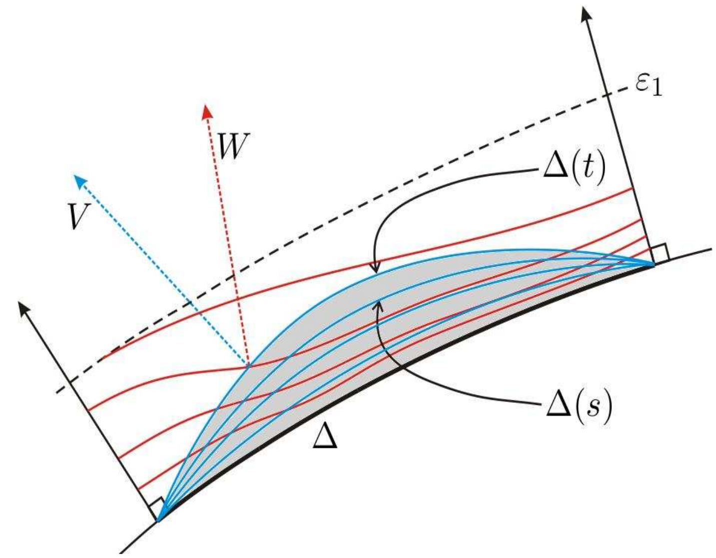

Figure 3. The divergence theorem is applied in the shaded region $\Omega(t)$ between Δ and $\Delta(t)$.

order. Hence,

$$\int_{\Omega(t)} \operatorname{div}(W) > -2H\mathrm{Vol}(\Omega(t)) - Ct, \tag{4}$$

for any $t > 0$ sufficiently small.

Applying the divergence theorem to V and W in $\Omega(t)$ (note that $W = V$ on Δ), we obtain the following two inequalities:

$$\int_{\Omega(t)} \operatorname{div}(V) = \int_{\Delta(t)} \langle V, \eta(t)\rangle - \int_{\Delta} \langle V, \eta\rangle = \mathrm{Area}(\Delta(t)) - \mathrm{Area}(\Delta),$$

$$\int_{\Omega(t)} \operatorname{div}(W) = \int_{\Delta(t)} \langle W, \eta(t)\rangle - \int_{\Delta} \langle V, \eta\rangle < \mathrm{Area}(\Delta(t)) - \mathrm{Area}(\Delta),$$

where $\eta(t)$ is the exterior unit vector field to $\Omega(t)$ on $\Delta(t)$. Hence, $\int_{\Omega(t)} \operatorname{div}(W) < \int_{\Omega(t)} \operatorname{div}(V)$. On the other hand, choosing t sufficiently small such that both inequalities (3) and (4) hold, we have $\int_{\Omega(t)} \operatorname{div}(W) > \int_{\Omega(t)} \operatorname{div}(V)$. This contradiction completes the proof of the theorem.

q.e.d.

Remark 1. The proof of the theorem shows that given any two-sided cover $\widetilde{L}$ of a limit leaf L of $\mathcal{L}$ as described in the statement of the theorem, then $\widetilde{L}$ is stable. This follows by lifting $\mathcal{L}$ to a neighborhood $U(\widetilde{L})$ of $\widetilde{L}$ in its normal bundle, considered to be the zero section in $U(\widetilde{L})$. In the case of non-zero constant mean curvature hypersurfaces, L is already two-sided and then stability is equivalent to the existence of a positive Jacobi function. However, in the minimal case where a hypersurface L may be one-sided, this observation concerning stability of $\widetilde{L}$ is generally a stronger property; for example, the projective plane contained in projective three-space is a totally geodesic surface which is area minimizing in its $\mathbb{Z}_2$-homology class but its oriented two-sided cover is unstable, see Ross [**12**] and also Ritoré and Ros [**10**].

Next we give a useful and immediate consequence of Theorem 1. Let $\mathcal{L}$ be a codimension one H-lamination of a manifold N. We will denote by $\mathrm{Stab}(\mathcal{L})$, $\mathrm{Lim}(\mathcal{L})$ the collections of stable leaves and limit leaves of $\mathcal{L}$, respectively. Note that $\mathrm{Lim}(\mathcal{L})$ is a closed set of leaves and so, it is a sublamination of $\mathcal{L}$.

Corollary 1. *Suppose that N is a not necessarily complete Riemannian manifold and $\mathcal{L}$ is an H-lamination of N with leaves of codimension one. Then, the closure of any collection of its stable leaves has the structure of a sublamination of $\mathcal{L}$, all of whose leaves are stable. Hence,* $\mathrm{Stab}(\mathcal{L})$ *has the structure of a minimal lamination of N and* $\mathrm{Lim}(\mathcal{L}) \subset \mathrm{Stab}(\mathcal{L})$ *is a sublamination.*

Remark 2. Theorem 1 and Corollary 1 have many useful applications to the geometry of embedded minimal and constant mean curvature hypersurfaces in Riemannian manifolds. We refer the interested reader to the survey [**3**] by the first two authors and to our joint papers in [**4**] and [**6**] for some of these applications.

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