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COMPARISON GEOMETRY FOR THE BAKRY-EMERY RICCI TENSOR

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To the memory of Detlef Gromoll

Abstract

For Riemannian manifolds with a measure $(M, g, e^{-f} d\mathrm{vol}_g)$ we prove mean curvature and volume comparison results when the ∞-Bakry-Emery Ricci tensor is bounded from below and f or $|\nabla f|$ is bounded, generalizing the classical ones (i.e. when f is constant). This leads to extensions of many theorems for Ricci curvature bounded below to the Bakry-Emery Ricci tensor. In particular, we give extensions of all of the major comparison theorems when f is bounded. Simple examples show the bound on f is necessary for these results.

1. Introduction

In this paper we study smooth metric measure spaces $(M^n, g, e^{-f} d\mathrm{vol}_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $d\mathrm{vol}_g$ is the Riemannian volume density on M. These objects have been used extensively in geometric analysis and Kähler geometry, they play an essential role in Perelman's work on the Ricci flow, and they arise as smooth collapsed measured Gromov-Hausdorff limits. f is also referred to as the dilaton field in the physics literature. Smooth metric measure spaces are also called manifolds with density. In this paper by the Bakry-Emery Ricci tensor we mean

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f.$$

This is also referred to as the ∞-Bakry-Emery Ricci Tensor. Bakry and Emery [**4**] extensively studied (and generalized) this tensor and its relationship to diffusion processes. The Bakry-Emery tensor also occurs naturally in many different subjects, see e.g. [**24**] and [**31**, 1.3]. The equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci

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flow. Moreover Ric_f has a natural extension to metric measure spaces [**23**, **42**, **43**].

The purpose of this paper is to investigate which geometric and topological results for manifolds with a lower bound on the Ricci tensor extend to smooth metric measure spaces with Bakry-Emery Ricci tensor bounded below. Interestingly, Lichnerowicz [**21**] studied this problem at least ten years before the work of Bakry and Emery. This question has also been investigated by a number of recent authors. We will discuss some of these results below but the interested reader should also see Chapter 18 of [**28**] and the references therein. For another modification of the Ricci tensor involving conformal geometry see [**9**].

The starting point for comparison geometry of Ricci curvature is the Bochner formula. Let $u \in C^3(M)$, then

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla \Delta u, \nabla u).$$

If a Riemannian manifold has a lower bound on Ricci curvature one applies this formula to the distance function to obtain a Ricatti equation which is then used to prove the mean curvature (or Laplacian) comparison. The mean curvature comparison can then be used as a tool to establish classical comparison theorems for Ricci curvature such as Myers' theorem, the Bishop-Gromov volume comparison theorem, the Cheeger-Gromoll splitting theorem, and the Abresch-Gromoll excess estimate. See Zhu's survey paper [**53**] for an excellent account of this approach.

The mean curvature is important because it measures the relative rate of change of the volume element of the geodesic sphere. For the measure $e^{-f}d\mathrm{vol}$, the weighted (or f-)mean curvature is $m_f = m - \partial_r f$, where m is the mean curvature of the geodesic sphere with inward pointing normal vector. The self-adjoint (f-)Laplacian with respect to the weighted measure is $\Delta_f = \Delta - \nabla f \cdot \nabla$. Note that $m_f = \Delta_f(r)$, where r is the distance function. For the f-Laplacian a simple calculation gives the following Bochner formula,

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) + g(\nabla \Delta_f u, \nabla u).$$

At first this looks very similar to the classical Bochner formula. However, since $\mathrm{tr}(\mathrm{Hess} u) \neq \Delta_f(u)$, one does not immediately obtain a Ricatti equation. Indeed, in the next section we give a quick overview with examples where the Myers' theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem, and Abresch-Gromoll's excess estimate are not true when Ric_f is bounded below, so the comparison theory here is more subtle.

One way around this difficulty is to replace Ric_f by the"N"-Bakry-Emery tensor

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{df \otimes df}{N}. \tag{1.1}$$

This extra gradient term then allows one to obtain a Ricatti equation which is identical to the one obtained for a Riemannian manifold of dimension $n + N$. One can then proceed as in the classical case to obtain versions for the N-Bakry-Emery tensor of all the theorems listed above. We sketch out this approach in the Appendix.

In this paper we take a different approach. Instead of modifying the Ricci tensor to simplify the ODE along geodesics, we deal with the ODE directly. We then obtain comparisons that depend on a lower bound on Ric_f and bounds on f or the first derivative of f along geodesics. Using this method we prove three mean curvature comparisons. The first (see Theorem 3.1) is the most general, requiring no assumptions on f, and is quite simple to prove. It appears implicitly in the work of Morgan [**27**], and Naber [**29**] and it recovers some interesting applications for manifolds with positive Bakry-Emery tensor (Corollaries 5.1 and 5.2). The other two are more delicate and require assumptions on f but have stronger applications.

Theorem 1.1 (Mean Curvature Comparison)**.** *Let $p \in M^n$. Assume $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$.*

a) If $\partial_r f \geq -a$ $(a \geq 0)$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H(r) \leq a \tag{1.2}$$

along that minimal geodesic segment from p. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(p) - at$ for all $t < r$.

b) If $|f| \leq k$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/4\sqrt{H}$) then

$$m_f(r) \leq m_H^{n+4k}(r) \tag{1.3}$$

along that minimal geodesic segment from p. In particular when $H = 0$ we have

$$m_f(r) \leq \frac{n + 4k - 1}{r}. \tag{1.4}$$

Here m_H^{n+4k} is the mean curvature of the geodesic sphere in M_H^{n+4k}, the simply connected model space of dimension $n + 4k$ with constant curvature H and m_H is the mean curvature of the model space of dimension n. See (3.18) in Section 3 for the case $H > 0$ and $r \in \left[\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}\right]$ in part b.

As in the classical case, these mean curvature comparisons have many applications. First, we have volume comparison theorems. Let $\mathrm{Vol}_f(B(p,r)) = \int_{B(p,r)} e^{-f} d\mathrm{vol}_g$, the weighted (or f-)volume and $\mathrm{Vol}_H^n(r)$ be the volume of the radius r-ball in the model space M_H^n.

Theorem 1.2 (Volume Comparison.)**.** *Let $(M^n, g, e^{-f}d\mathrm{vol}_g)$ be complete smooth metric measure space with $\mathrm{Ric}_f \geq (n-1)H$. Fix $p \in M^n$.*

a) If $\partial_r f \geq -a$ along all minimal geodesic segments from p then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$),

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq e^{aR}\frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{1.5}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H and $\partial_r f \equiv -a$. In particular if $\partial_r f \geq 0$ and $\mathrm{Ric}_f \geq 0$ then M has f-volume growth of degree at most n.

b) If $|f(x)| \leq k$ then for $R \geq r > 0$ (assume $R \leq \pi/4\sqrt{H}$ if $H > 0$),

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{1.6}$$

In particular, if f is bounded and $\mathrm{Ric}_f \geq 0$ then M has polynomial f-volume growth.

Remark. 1. When $\mathrm{Ric}_f \geq 0$ the condition f is bounded or $\partial_r f \geq 0$ is necessary to show polynomial f-volume growth as shown by Example 2.3. Similar statements are true for the volume of tubular neighborhood of a hypersurface. See Section 4 for another version of volume comparison which holds for all $r > 0$ even when $H > 0$.

Remark. 2. To prove the theorem we only need a lower bound on Ric_f along the radial directions. Given any manifold M^n with Ricci curvature bounded from below one can always choose suitable f to get any lower bound for Ric_f along the radial directions. For example if $\mathrm{Ric} \geq -1$ and $p \in M$, if we choose $f(x) = r^2 = d^2(p,x)$, then $\mathrm{Ric}_f(\partial_r, \partial_r) \geq 1$.

Remark. 3. Volume comparison theorems have been proven for manifolds with N-Bakry Emery Ricci tensor bounded below. See Qian [**39**], Bakry-Qian [**6**], Lott [**24**], and Appendix A. Since $\mathrm{Ric}_f^N \geq 0$ implies $\mathrm{Ric}_f \geq 0$ our result greatly improves the volume comparison result of Qian when N is big and f is bounded, or when $\partial_r f \geq 0$.

The mean curvature and volume comparison theorems have many other applications. We highlight two extensions of theorems of Calabi-Yau [**51**] and Myers' to the case where f is bounded.

Theorem 1.3. *If M is a noncompact, complete manifold with $\mathrm{Ric}_f \geq 0$ for some bounded f then M has at least linear f-volume growth.*

Theorem 1.4 (Myers' Theorem)**.** *If M has $\mathrm{Ric}_f \geq (n-1)H > 0$ and $|f| \leq k$ then M is compact and $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.*

Examples 2.1 and 2.2 show that the assumption of bounded f is necessary in both theorems. Qian [**39**] has proven versions of both theorems for Ric_f^N. For other Myers' theorems see [**14**, **52**, **20**, **27**].

The paper is organized as follows. In the next section we discuss some simple examples and some alternate motivations for the Bakry-Emery tensor. Then in the third section we prove the mean curvature comparisons. In Sections 4 and 5 we prove the volume comparison theorems and discuss their applications, including Theorem 1.3. In Section 6 we apply the mean curvature comparison to prove the splitting theorem for the Bakry-Emery tensor that is originally due to Lichnerowicz. In Section 7 we discuss some other applications of the mean curvature comparison including the Myers' theorem and an extension Abresch-Gromoll's excess estimate to Ric_f. Finally in Appendix A we state the mean curvature comparison for Ric_f^N. This is a special case of an estimate in [**6**], but we have written the result in more Riemannian geometry friendly language. This gives other proofs of the comparison theorems for Ric_f^N mentioned above. See [**45**] for a survey in this direction and [**8**] for the equality case.

After posting the original version of this paper in June 2007 we learned from Fang, Li, and Zhang about their work which is closely related to some of our work here [**12**]. We thank them for sharing their work with us. Motivated from their paper we were able to strengthen the original version of Theorem 1.1 and Theorem 1.2 and give a new proof to Theorem 1.1. This proof of the mean curvature comparison seems to us to be new even in the classical Ricci curvature case. We have moved our original proof using ODE methods to an appendix because we feel it might be useful in other applications.

From the work of [**34**] one expects that the volume comparison and splitting theorem can be extended to the case that Ric_f is bounded from below in the integral sense. We also expect similar versions for metric measure spaces. These will be treated in separate paper.

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2. Some further motivation and examples

In this section we will discuss the simple examples which show that the classical comparison theorems for Ricci curvature do not hold for only a lower bound on the Bakry-Emery tensor. We also discuss a few different viewpoints on the Bakry-Emery tensor which we hope will motivate our results.

One very interesting reason to study smooth metric measure spaces is the fact, observed by Lott in [**24**], that they are examples of collapsed measured Gromov-Hausdorff limits. To see this consider $(M^n \times F^N, g_\epsilon)$, where M and F are compact, with the warped product metric $g_\epsilon = g_M + (\epsilon e^{-f})^2 g_F$. Then, as $\epsilon \to 0$,

$$\left(M^n \times F^N, \widetilde{dvol_{g_\epsilon}}\right) \overset{\text{mGH}}{\longrightarrow} \left(M^n, e^{-Nf} dvol_{g_M}\right).$$

Here $\widetilde{dvol_{g_\epsilon}} = \frac{dvol_{g_\varepsilon}}{\mathrm{Vol}(B_{g_\varepsilon}(\cdot,1))}$ is a renormalized Riemannian measure. Recall that a sequence of compact metric measure spaces $(X_i, \mu_i) \overset{\text{mGH}}{\longrightarrow} (X_\infty, \mu_\infty)$ if the metric spaces converge in the Gromov-Hausdorff sense and for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty.$$

By O'Neill's formula, the Ricci curvature of the warped product metric g_ϵ applied to vectors v, w tangent to the M factor is

$$\mathrm{Ric}_{g_\varepsilon}(v,w) = \mathrm{Ric}_{g_M}(v,w) + \mathrm{Hess}_{g_M} f(v,w) - \frac{df \otimes df}{N}(v,w).$$

This gives another motivation for the definition of Ric_f^N in (1.1).

It is also instructive to consider the relationship between the Bakry-Emery tensor and the characterization of lower bounds on sectional curvature in the spirit of Alexandrov. That is, in terms of concavity properties of distance functions. For example a manifold has nonnegative sectional curvature if and only if

$$\mathrm{Hess}\, r^2 \le 0$$

for all distance functions r. The corresponding characterization for Ricci curvature is in terms of the concavity of the volume form in polar coordinates

$$dvol_g = \mathcal{A}(r,\theta) dr \wedge d\theta^{n-1}.$$

Indeed a standard calculation is that

$$\frac{\partial^2}{\partial r^2}(\mathcal{A}^{\frac{1}{n-1}}) \le -\frac{\mathrm{Ric}(\partial r, \partial r)}{n-1} \mathcal{A}^{\frac{1}{n-1}}.$$

So Ric measures the relative second derivative of the volume form. If we do this calculation for the weighted volume form written in polar coordinates

$$e^{-f} dvol_g = \mathcal{A}_f(r,\theta) dr \wedge d\theta^{n-1} = e^{-f} \mathcal{A}(r,\theta) dr \wedge d\theta^{n-1}$$

we obtain

$$\frac{\partial^2}{\partial r^2}(\mathcal{A}_f^{\frac{1}{n-1}}) \leq \left(-\mathrm{Ric}_f(\partial r, \partial r) - 2\frac{\partial f}{\partial r}\frac{m(r,\theta)}{n-1}\right)\frac{\mathcal{A}_f}{n-1}.$$

Thus from this perspective it seems natural to make assumptions about the boundedness of f or $\frac{\partial f}{\partial r}$, although this equation, in itself, does not easily give us results because of the $m(r,\theta)$ term. Thus we are also motivated to consider m_f instead of m.

We now move on to the examples. The most well known example is the following soliton, often referred to as the Gaussian soliton.

Example 2.1. Let $M = \mathbb{R}^n$ with Euclidean metric g_0, $f(x) = \frac{\lambda}{2}|x|^2$. Then $\mathrm{Hess} f = \lambda g_0$ and $\mathrm{Ric}_f = \lambda g_0$.

This example shows that, unlike the case of Ricci curvature uniformly bounded from below by a positive constant, a metric measure space is not necessarily compact if $\mathrm{Ric}_f \geq \lambda g$ and $\lambda > 0$.

From this we construct the following.

Example 2.2. Let $M = \mathbb{H}^n$ be the hyperbolic space. Fixed any $p \in M$, let $f(x) = (n-1)r^2 = (n-1)d^2(p,x)$. Now $\mathrm{Hess}\, r^2 = 2|\nabla r|^2 + 2r\mathrm{Hess} r \geq 2I$, therefore $\mathrm{Ric}_f \geq (n-1)$.

This example shows that the Cheeger-Gromoll splitting theorem and Abresch-Gromoll's excess estimate do not hold for $\mathrm{Ric}_f \geq 0$, in fact they don't even hold for $\mathrm{Ric}_f \geq \lambda > 0$. Note that the only properties of hyperbolic space used are that $\mathrm{Ric} \geq -(n-1)$ and that Hess $r^2 \geq 2I$. Therefore any Cartan-Hadamard manifold with Ricci curvature bounded below has a metric with $\mathrm{Ric}_f \geq 0$ since $\mathrm{Hess}\, r^2 \geq 2I$ for these spaces. On the other hand these examples are not topologically very interesting. In fact, if $\mathrm{Ric} < 0$ and $\mathrm{Ric}_f \geq \lambda > 0$ then $\mathrm{Hess} f > \lambda g$ which implies M is diffeomorphic to $\mathbb{R}^n$.

A large class of examples are given by gradient Ricci solitons. Compact expanding or steady solitons are Einstein (f is constant) [**31**]. There are nontrivial compact shrinking solitons [**7**, **11**, **13**, **17**]. Some of these examples do not have nonnegative Ricci curvature.

The following example shows that there are manifolds with $\mathrm{Ric}_f \geq 0$ which do not have polynomial f-volume growth.

Example 2.3. Let $M = \mathbb{R}^n$ with Euclidean metric, $f(x_1, \cdots, x_n) = x_1$. Since Hess $f = 0$, $\mathrm{Ric}_f = \mathrm{Ric} = 0$. On the other hand $\mathrm{Vol}_f(B(0,r))$ is of exponential growth. Along the x_1 direction, $m_f - m_H = -1$ which does not goes to zero.

In this example $|\nabla f| \leq 1$, so $\mathrm{Ric}_f \geq 0$ and $|\nabla f|$ bounded does not imply polynomial f-volume growth either.

It is also natural to consider the scalar curvature with measure. As pointed out by Perelman in [**31**, 1.3] the corresponding scalar curvature equation is $S_f = 2\Delta f - |\nabla f|^2 + S$. Note that this is different than taking the trace of Ric_f which is $\Delta f + S$. The Lichnerovicz formula and theorem naturally extend to S_f while any compact manifold has a Riemannian metric and function f with $\Delta f + S > 0$.

On the other hand, if a compact manifold has $\mathrm{Ric}_f > 0$ then the $\hat{A}$-genus must be zero. This was pointed out to us by Aubry. To see it first note that $\mathrm{Ric}_f^N > \epsilon g$ for some sufficiently large N and small ϵ. Now, consider the warped product construction of Lott mentioned above taking F^N to be a Ricci flat manifold which has non-zero $\hat{A}$-genus. (For example, a sufficiently large product of K3-surfaces.) In the warped product metric, the Ricci tensor on vectors tangent to M is Ric_f^N and the Ricci curvature of vectors tangent to F will shrink arbitrarily to zero if we scale the warping function. Therefore, by sufficiently scaling F, we obtain a warped product metric on $M \times F$ which has positive scalar curvature (but not positive Ricci curvature). Then

$$0 = \hat{A}(M \times F) = \hat{A}(M) \cdot \hat{A}(F)$$

so $\hat{A}(M) = 0$.

3. Mean Curvature Comparisons

In this section we prove the mean curvature comparison theorems. First we give a rough estimate on m_f which is useful when $\mathrm{Ric}_f \geq \lambda g$ and $\lambda > 0$.

Theorem 3.1 (Mean Curvature Comparison I.)**.** *If* $\mathrm{Ric}_f(\partial_r, \partial_r) \geq \lambda$ *then given any minimal geodesic segment and* $r_0 > 0$*,*

$$m_f(r) \leq m_f(r_0) - \lambda(r - r_0) \quad \text{for } r \geq r_0. \tag{3.1}$$

Equality holds for some $r > r_0$ *if and only if all the radial sectional curvatures are zero,* $\mathrm{Hess} r \equiv 0$*, and* $\partial_r^2 f \equiv \lambda$ *along the geodesic from* r_0 *to* r*.*

Proof. Applying the Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u) \tag{3.2}$$

to the distance function $r(x) = d(x, p)$, we have

$$0 = |\mathrm{Hess}\, r|^2 + \frac{\partial}{\partial r}(\Delta r) + \mathrm{Ric}(\nabla r, \nabla r). \tag{3.3}$$

Since $\mathrm{Hess}\, r$ is the second fundamental from of the geodesic sphere and Δr is the mean curvature, with the Schwarz inequality, we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \mathrm{Ric}(\partial r, \partial r). \tag{3.4}$$

And equality holds if and only if the radial sectional curvatures are constant. Hence the mean curvature of the model space m_H satisfies

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H. \tag{3.5}$$

Since $m_f' = m' - \mathrm{Hess} f\,(\partial r, \partial r)$, we have

$$m_f' \leq -\frac{m^2}{n-1} - \mathrm{Ric}_f(\partial r, \partial r). \tag{3.6}$$

If $\mathrm{Ric}_f \geq \lambda$, we have

$$m_f' \leq -\lambda.$$

This immediately gives the inequality (3.1).

To see the equality statement, suppose $m_f' \equiv -\lambda$ on an interval $[r_0, r]$, then from (3.6) we have $m \equiv 0$ and

$$m_f' = -\partial_r^2 f = -\mathrm{Ric}_f(\partial_r, \partial_r) = -\lambda. \tag{3.7}$$

So we also have $\mathrm{Ric}(\partial_r, \partial_r) = 0$. Then by (3.3) $\mathrm{Hess}\, r = 0$, which implies the sectional curvatures must be zero. q.e.d.

Proof of Theorem 1.1. Let $\mathrm{sn}_H(r)$ be the solution to

$$\mathrm{sn}_H'' + H\mathrm{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$. Then

$$m_H^n = (n-1)\frac{\mathrm{sn}_H'}{\mathrm{sn}_H}. \tag{3.8}$$

From the Riccati inequality (3.4), equality (3.5), and assumption on Ric_f, we have

$$(m - m_H)' \leq -\frac{m^2 - m_H^2}{n-1} + \partial_r\partial_r f. \tag{3.9}$$

We compute

$$\begin{aligned}
\left(\mathrm{sn}_H^2(m - m_H)\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H(m - m_H) + sn_H^2(m - m_H)' \\
&\leq \mathrm{sn}_H^2\left(\frac{2m_H}{n-1}(m - m_H) - \frac{m^2 - m_H^2}{n-1} + \partial_r\partial_r f\right) \\
&= \mathrm{sn}_H^2\left(-\frac{(m - m_H)^2}{n-1} + \partial_r\partial_r f\right) \\
&\leq \mathrm{sn}_H^2\,\partial_r\partial_r f. \qquad (3.10)
\end{aligned}$$

Here in the 2nd line we have used (3.8) and (3.9).

Integrating (3.10) from 0 to r yields

$$\mathrm{sn}_H^2(r)m(r) \leq \mathrm{sn}_H^2(r)m_H(r) + \int_0^r \mathrm{sn}_H^2(t)\partial_t\partial_t f(t)dt. \tag{3.11}$$

When f is constant (the classical case) this gives the usual mean curvature comparison. This quick proof does not seem to be emphasized in the literature.

Proof of Part a. Using integration by parts on the last term we have

$$\mathrm{sn}_H^2(r)m_f(r) \leq \mathrm{sn}_H^2(r)m_H(r) - \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt. \tag{3.12}$$

Under our assumptions $(\mathrm{sn}_H^2(t))' = 2\mathrm{sn}_H'(t)\mathrm{sn}_H(t) \geq 0$ so if $\partial_t f(t) \geq -a$ we have

$$\begin{aligned}\mathrm{sn}_H^2(r)m_f(r) &\leq \mathrm{sn}_H^2(r)m_H(r) + a\int_0^r (\mathrm{sn}_H^2(t))'dt \\ &= \mathrm{sn}_H^2(r)m_H(r) + \mathrm{sn}_H^2(r)a.\end{aligned} \tag{3.13}$$

This proves the inequality.

To see the rigidity statement suppose that $\partial_t f \geq -a$ and $m_f(r) = m_H(r) + a$ for some r. Then from (3.12) we see

$$a\,\mathrm{sn}_H^2 \leq \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt \leq a\,\mathrm{sn}_H^2. \tag{3.14}$$

So that $\partial_t f \equiv -a$. But then $m(r) = m_f - a = m_H(r)$ so that the rigidity follows from the rigidity for the usual mean curvature comparison.

Proof of Part b. Integrate (3.12) by parts again

$$\mathrm{sn}_H^2(r)m_f(r) \leq \mathrm{sn}_H^2(r)m_H(r) - f(r)(\mathrm{sn}_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \tag{3.15}$$

Now if $|f| \leq k$ and $r \in (0, \frac{\pi}{4\sqrt{H}}]$ when $H > 0$, then $(sn_H^2)''(t) \geq 0$ and we have

$$\mathrm{sn}_H^2(r)m_f(r) \leq \mathrm{sn}_H^2(r)m_H(r) + 2k(\mathrm{sn}_H^2(r))'. \tag{3.16}$$

From (3.8) we can see that

$$(\mathrm{sn}_H^2(r))' = 2\mathrm{sn}_H'\mathrm{sn}_H = \frac{2}{n-1}m_H\mathrm{sn}_H^2.$$

so we have

$$m_f(r) \leq \left(1 + \frac{4k}{n-1}\right)m_H(r) = m_H^{n+4k}(r). \tag{3.17}$$

q.e.d.

Now when $H > 0$ and $r \in \left[\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}\right]$,

$$\begin{aligned}\int_0^r f(t)(sn_H^2)''(t)dt &\leq k\left(\int_0^{\frac{\pi}{4\sqrt{H}}} (sn_H^2)''(t)dt - \int_{\frac{\pi}{4\sqrt{H}}}^r (sn_H^2)''(t)dt\right) \\ &= k\left(\frac{2}{\sqrt{H}} - sn_H(2r)\right).\end{aligned}$$

Hence

$$m_f(r) \le \left(1 + \frac{4k}{n-1} \cdot \frac{1}{\sin(2\sqrt{H}r)}\right) m_H(r). \tag{3.18}$$

This estimate will be used later to prove the Myers' theorem in Section 5.

Remark. In the case $H = 0$, we have $\mathrm{sn}_H(r) = r$ so (3.15) gives the estimate in [**12**] that

$$m_f(r) \le \frac{n-1}{r} - \frac{2}{r} f(r) + \frac{2}{r^2} \int_0^r f(t)dt. \tag{3.19}$$

Remark. The exact same argument gives mean curvature comparison for the mean curvature of distance sphere of hypersurfaces with Ric_f lower bound.

4. Volume Comparisons

In this section we prove the volume comparison theorems. Fix $p \in M^n$, use exponential polar coordinates around p and write the volume element $d\,vol = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$. By the first variation of the area (see [**53**])

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = (\ln(\mathcal{A}(r,\theta)))' = m(r,\theta). \tag{4.1}$$

Therefore

$$\frac{\mathcal{A}_f'}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta). \tag{4.2}$$

And for $r \ge r_0 > 0$

$$\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_f(r_0,\theta)} = e^{\int_{r_0}^r m_f(s,\theta)ds}. \tag{4.3}$$

The volume comparison theorems follow from the mean curvature comparisons through this equation.

First applying the mean curvature estimate Theorem 3.1 we have the following basic volume comparison theorem.

Theorem 4.1 (Volume Comparison I)**.** *Let* $\mathrm{Ric}_f \ge \lambda$ *then for any* r *there are constants* A, B, *and* C *such that*

$$\mathrm{Vol}_f(B(p,R)) \le A + B\int_r^R e^{-\frac{\lambda}{2}t^2 + Ct}dt.$$

The version of Theorem 4.1 for tubular neighborhoods of hypersurfaces is very similar and has been proven by Morgan [**26**], also see [**27**]. As Morgan points out, the theorem is optimal and the constants can not be uniform as the Gaussian soliton shows, see Example 2.1.

Proof. Using the mean curvature estimate (3.1)

$$\int_{r_0}^{r} m_f(r) \leq m_f(r_0)r - \frac{1}{2}\lambda r^2.$$

Hence

$$\mathcal{A}_f(r,\theta) \leq \mathcal{A}_f(r_0,\theta)e^{m_f(r_0,\theta)r - \frac{1}{2}\lambda r^2}.$$

Now let $A(p, r_0, r)$ be the annulus $A(p, r_0, r) = B(p, r) \setminus B(p, r_0)$. Then

$$\begin{aligned}
&(4.4)\quad \mathrm{Vol}_f(A(p,r_0,r)) &&= \int_{r_0}^{r}\int_{S^{n-1}} \mathcal{A}_f(s,\theta)d\theta ds \\
&(4.5) &&\leq \int_{r_0}^{r}\int_{S^{n-1}} \mathcal{A}_f(r_0,\theta)e^{m_f(r_0,\theta)s-\frac{1}{2}\lambda s^2}d\theta ds \\
&(4.6) &&\leq A_f(r_0)\int_{r_0}^{r} e^{Cs - \frac{1}{2}\lambda s^2}ds.
\end{aligned}$$

Where $A_f(r_0)$ is the surface area of the geodesic sphere induced from the f-volume element and C is a constant such that $C \geq m_f(r_0, \theta)$ for all θ where it is defined. Since $\mathrm{Vol}_f(B(p,r)) = \mathrm{Vol}_f(\mathrm{Vol}(B(p,r_0)) + \mathrm{Vol}_f(A(p,r_0,r))$ this proves the theorem. q.e.d.

We also have a rigidity statement for (4.5). That is, if the inequality (4.5) is an equality then we must have equalities in the mean curvature comparison along all the geodesics, this implies that $\mathrm{Hess}\, r \equiv 0$ which implies that

$$A(p, r_0, r) \cong S(p, r_0) \times [r_0, r] \tag{4.7}$$

where $S(p, r_0)$ is the geodesic sphere with radius r_0. Moreover f must also be rigid, namely

$$f(x,t) = f(x) + \partial_r f(x)(r - r_0) + \frac{\lambda}{2}(r - r_0)^2.$$

Now we prove Theorem 1.2 using Theorem 1.1.

Proof of Theorem 1.2: For Part a) we compare with a model space, however, we modify the measure according to a. Namely, the model space will be the pointed metric measure space $M^n_{H,a} = (M^n_H, g_H, e^{-h}d\mathrm{vol}, O)$ where (M^n_H, g_H) is the n-dimensional simply connected space with constant sectional curvature H, $O \in M^n_H$, and $h(x) = -a \cdot d(x, O)$. We make the model a pointed space because the space only has $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$ in the radial directions from O and we only compare volumes of balls centered at O.

Let $\mathcal{A}^a_H$ be the h-volume element in $M^n_{H,a}$. Then $\mathcal{A}^a_H(r) = e^{ar}\mathcal{A}_H(r)$ where $\mathcal{A}_H$ is the Riemannian volume element in M^n_H. By the mean curvature comparison we have $(\ln(\mathcal{A}_f(r,\theta))' \leq a + m_H = (\ln(\mathcal{A}^a_H))'$ so for $r < R$,

$$\frac{\mathcal{A}_f(R,\theta)}{\mathcal{A}_f(r,\theta)} \le \frac{\mathcal{A}_H^a(R,\theta)}{\mathcal{A}_H^a(r,\theta)}. \tag{4.8}$$

Namely $\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_H^a(r,\theta)}$ is nonincreasing in r. Using Lemma 3.2 in [**53**], we get for $0 < r_1 < r$, $0 < R_1 < R$, $r_1 \le R_1$, $r \le R$,

$$\frac{\int_{R_1}^R \mathcal{A}_f(t,\theta)dt}{\int_{r_1}^r \mathcal{A}_f(t,\theta)dt} \le \frac{\int_{R_1}^R \mathcal{A}_H^a(t,\theta)dt}{\int_{r_1}^r \mathcal{A}_H^a(t,\theta)dt}. \tag{4.9}$$

Integrating along the sphere direction gives

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \le \frac{\mathrm{Vol}_H^a(R_1,R)}{\mathrm{Vol}_H^a(r_1,r)}. \tag{4.10}$$

Where $\mathrm{Vol}_H^a(r_1,r)$ is the h-volume of the annulus $B(O,r) \setminus B(O,r_1) \subset M_H^n$. Since $\mathrm{Vol}_H(r_1,r) \le \mathrm{Vol}_H^a(r_1,r) \le e^{ar}\mathrm{Vol}_H(r_1,r)$ this gives (1.5) when $r_1 = R_1 = 0$ and proves Part b).

In the model space the radial function h is not smooth at the origin. However, clearly one can smooth the function to a function with $\partial_r h \ge -a$ and $\partial_r^2 h \ge 0$ such that the h-volume taken with the smoothed h is arbitrary close to that of the model. Therefore, the inequality (4.10) is optimal. Moreover, one can see from the equality case of the mean curvature comparison that if the annular volume is equal to the volume in the model then all the radial sectional curvatures are H and f is exactly a linear function.

Proof of Part b). In this case let $\mathcal{A}_H^{n+4k}$ be the volume element in the simply connected model space with constant curvature H and dimension $n + 4k$.

Then from the mean curvature comparison we have $\ln(\mathcal{A}_f(r,\theta))' \le \ln(\mathcal{A}_H^{n+4k}(r))'$. So again applying Lemma 3.2 in [**53**] we obtain

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \le \frac{Vol_H^{n+4k}(R_1,R)}{Vol_H^{n+4k}(r_1,r)}. \tag{4.11}$$

With $r_1 = R_1 = 0$ this implies the relative volume comparison for balls

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \le \frac{Vol_H^{n+4k}(R)}{Vol_H^{n+4k}(r)}. \tag{4.12}$$

Equivalently

$$\frac{\mathrm{Vol}_f(B(p,R))}{V_H^{n+4k}(R)} \le \frac{\mathrm{Vol}_f(B(p,r))}{V_H^{n+4k}(r)}. \tag{4.13}$$

Since $n + 4k > n$ we note that the right hand side blows up as $r \to 0$ so one does not obtain a uniform upper bound on $\mathrm{Vol}_f(B(p,R))$. Indeed, it is not possible to do so since one can always add a constant to f and not effect the Bakry-Emery tensor.

By taking $r = 1$ we do obtain a volume growth estimate for $R > 1$

$$\mathrm{Vol}_f(B(p,R)) \leq \mathrm{Vol}_f(B(p,1))\mathrm{Vol}_H^{n+4k}(R). \tag{4.14}$$

Note that, from Part a) $\mathrm{Vol}_f(B(p,1)) \leq e^{-f(p)}e^a\omega_n$ if $\partial_r f \geq -a$ on $B(p,1)$. q.e.d.

In the next section we collect the applications of the volume comparison theorems.

5. Applications of the volume comparison theorems.

In the case where $\lambda > 0$ Theorem 4.1 gives two very interesting corollaries. The first is also observed in [**26**].

Corollary 5.1. *If M is complete and $Ric_f \geq \lambda > 0$ then $\mathrm{Vol}_f(M)$ is finite and M has finite fundamental group.*

We note the finiteness of volume is true in the setting of more general diffusion operators [**4**]. Using a different approach the second author has proven that the fundamental group is finite for spaces satisfying $\mathrm{Ric} + \mathcal{L}_X g \geq \lambda > 0$ for some vector field X [**48**]. This had earlier been shown under the additional assumption that the Ricci curvature is bounded by Zhang [**52**]. See also [**29**]. When M is compact the finiteness of fundamental group was first shown by X. Li [**20**, Corollary 3] using a probabilistic method. Also see [**52**, **14**, **39**, **24**]. We would like to thank Prof. David Elworthy for bringing the article [**20**] to our attention.

The second corollary is the following Liouville Theorem, which is a strengthening of a result of Naber [**29**].

Corollary 5.2. *If M is complete with $Ric_f \geq \lambda > 0$, $u \geq 0$, $\Delta_f(u) \geq 0$, and there is $\alpha < \frac{\lambda}{2}$ such that $u(x) \leq e^{\alpha d(p,x)^2}$ for some $p \in M$ then u is constant.*

In particular there are no bounded f-subharmonic functions. Corollary 5.2 follows from Yau's proof that a complete manifold has no positive L^p $(p > 1)$ subharmonic functions [**51**]. The argument only uses integration by parts and a clever choice of test function and so is valid also for the weighted measure and Laplacian, see Theorem 4.2 in [**38**] for a complete proof. In addition to the paper of Naber mentioned above, also see [**36**, **37**, **38**] for applications of Corollary 5.2 to gradient Ricci solitons.

While Theorem 4.1 has applications when $\lambda > 0$ it is not strong enough to extend results for a general lower bound, for these results we apply Theorem 1.2. It is well known that a lower bound on volume growth for manifolds with $\mathrm{Ric} \geq 0$ can be derived from the volume comparison for annulli, see [**53**]. Thus Theorem 1.3 follows from (4.11). We give the proof here for completeness and to motivate Theorem 5.3.

Proof of Theorem 1.3. Let M be a manifold with $\mathrm{Ric}_f \geq 0$ for a bounded function f. Let $p \in M$ and let γ be a geodesic ray based at p in M. Then, applying the annulus relative volume comparison (4.11) to annuli centered at $\gamma(t)$, we obtain

$$\frac{\mathrm{Vol}_f(B(\gamma(t), t-1))}{\mathrm{Vol}_f(A(\gamma(t), t-1, t+1))} \geq \frac{(t-1)^{n+4k}}{(t+1)^{n+4k} - (t-1)^{n+4k}} \geq c(n,k)t$$

$\forall\ t \geq 2$. But $B(\gamma(0), 1) \subset A(\gamma(t), t-1, t+1)$ so we have

$$\mathrm{Vol}_f(B(p, t-1)) \geq c(n,k)\mathrm{Vol}_f(B(p,1))t \qquad \forall t \geq 2.$$

q.e.d.

Using the volume comparison (4.10) in place of (4.11) we can also prove a lower bound on the volume growth for certain convex f.

Theorem 5.3. *If* $\mathrm{Ric}_f \geq 0$ *where* f *is convex function such that the set of critical points of* f *is unbounded, then* M *has at least linear* f*-volume growth.*

The hypothesis on the critical point set is necessary by Examples 2.1 and 2.2.

Proof. Fix $p \in M$. Since the set of critical points of a convex function is connected, for every t there is $x(t)$, a critical point of f, such that $d(p, x(t)) = t$. But $\nabla f(x(t)) = 0$ and f is convex so $\partial r f \geq 0$ in all the radial directions from $x(t)$, therefore we can apply (4.10) and repeat the arguments in the proof of Theorem 1.3 to prove the result. q.e.d.

In [**25**] Milnor observed that polynomial volume growth on the universal cover of a manifold restricts the structure of its fundamental group. Thus Theorem 1.2 also implies the following extension of Milnor's Theorem.

Theorem 5.4. *Let* M *be a complete manifold with* $Ric_f \geq 0$.

1) *If* f *is a convex function that obtains its minimum then any finitely generated subgroup of* $\pi_1(M)$ *has polynomial growth of degree less than or equal* n*. In particular,* $b_1(M) \leq n$.
2) *If* $|f| \leq k$ *then any finitely generated subgroup of* $\pi_1(M)$ *has polynomial growth of degree less than or equal to* $n+4k$*. In particular,* $b_1(M) \leq n + 4k$.

Remark. Part 1) follows because at a pre-image of the minimum point in the universal cover, $\partial_r f \geq 0$.

Remark. Part 2) has been improved recently by Yang [**50**] to the optimal "$\leq n$." For the compact case this can also be proven using the splitting theorem, see the next section.

Gromov [**15**] has shown that a finitely generated group has polynomial growth if and only if it is almost nilpotent. Moreover, the work of the first author and Wilking shows that any finitely generated almost nilpotent group is the fundamental group of a manifold with $\mathrm{Ric} \geq 0$ [**44**, **46**]. Therefore, there is a complete classification of the finitely generated groups that can be realized as the fundamental group of a complete manifold with $\mathrm{Ric} \geq 0$. Combining these results with Theorem 5.4 we expand this classification to a larger class of manifolds.

Corollary 5.5. *A finitely generated group G is the fundamental group of some manifold with*

1) $\mathrm{Ric}_f \geq 0$ *for some bounded f or*
2) $\mathrm{Ric}_f \geq 0$ *for some convex f which obtains its minimum*

if and only if G is almost nilpotent.

It would be interesting to know if Corollary 5.5 holds without any assumption on f. Example 2.3 shows that the Milnor argument can not be applied since the f-volume growth of a manifold with $\mathrm{Ric}_f \geq 0$ may be exponential, so a different method of proof would be needed.

In [**3**] Anderson uses similar covering arguments to show, for example, that if $\mathrm{Ric} \geq 0$ and M has Euclidean volume growth then $\pi_1(M)$ is finite. He also finds interesting relationships between the first betti number, volume growth, and finite generation of fundamental group of manifolds with $\mathrm{Ric} \geq 0$. These relationships also carry over to manifolds satisfying the hypotheses of Theorem 5.4. We leave these statements to the interested reader.

Applying the relative volume comparison Theorem 1.2 we also have the following extensions of theorems of Gromov [**16**] and Anderson [**2**].

Theorem 5.6. *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$), the first Betti number $b_1 \leq C(n,k,HD^2)$ ($C(n,HD^2,aD)$).*

Theorem 5.7. *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{Vol}_f \geq V$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$) there are only finitely many isomorphism types of $\pi_1(M)$.*

Remark. In the case when $|\nabla f|$ is bounded, Ric_f bounded from below implies Ric_f^N is also bounded from below (with different lower bound). Therefore in this case the results can also been proven using the volume comparison in [**39**, **24**, **6**] for the Ric_f^N tensor.

6. The Splitting Theorem.

An important application of the mean curvature comparison is the extension of the Cheeger-Gromoll splitting theorem. After writing the original version of this paper, we learned that Lichnerowicz had proven the splitting theorem, see [**21**, **22**].

Theorem 6.1 (Lichnerowicz-Cheeger-Gromoll Splitting Theorem). *If* $\mathrm{Ric}_f \geq 0$ *for some bounded* f *and* M *contains a line, then* $M = N^{n-1} \times \mathbb{R}$ *and* f *is constant along the line.*

For completeness we retain our complete proof here.

Remark. In [**12**] Fang, Li, and Zhang show that only an upper bound on f is needed in the above theorem. Example 2.2 shows that the upper bound on f is necessary.

Recall that $m_f = \Delta_f(r)$, the f-Laplacian of the distance function. From (1.1), we get a local Laplacian comparison for distance functions

$$\Delta_f(r) \leq \frac{n+4k-1}{r} \quad \text{for all } x \in M \setminus \{p, C_p\}. \tag{6.1}$$

Where C_p is the cut locus of p. To prove the splitting theorem we apply this estimate to the Busemann functions.

Definition 6.2. If γ is a ray then Busemann function associated to γ is the function

$$b^\gamma(x) = \lim_{t\to\infty}(t - d(x, \gamma(t))). \tag{6.2}$$

From the triangle inequality the Busemann function is Lipschitz continuous with Lipschitz constant 1 and thus is differential almost everywhere. At the points where b_γ is not smooth we interpret the f-laplacian in the sense of barriers.

Definition 6.3. For a continuous function h on $M, q \in M$, a function h_q defined in a neighborhood U of q, is a lower barrier of h at q if h_q is $C^2(U)$ and

$$h_q(q) = h(q), \quad h_q(x) \leq h(x) \ (x \in U). \tag{6.3}$$

Definition 6.4. We say that $\Delta_f(h) \geq a$ in the barrier sense if, for every $\varepsilon > 0$, there exists a lower barrier function h_ε such that $\Delta_f(h_\varepsilon) > a - \varepsilon$. An upper bound on Δ_f is defined similarly in terms of upper barriers.

The local Laplacian comparison is applied to give the following key lemma.

Lemma 6.5. *If* M *is a complete, noncompact manifold with* $\mathrm{Ric}_f \geq 0$ *for some bounded function* f *then* $\Delta_f(b^\gamma) \geq 0$ *in the barrier sense.*

Remark. As in [**12**], one can use the inequality (3.19) to prove Lemma 6.5 only assuming an upper bound on f.

Proof. For the Busemann function at a point q we have a family of barrier functions defined as follows. Given $t_i \to \infty$, let σ_i be minimal geodesics from q to $\gamma(t_i)$, parametrized by arc length. The sequence $\sigma_i'(0)$ subconverges to some $v_0 \in T_qM$. We call the geodesic $\overline{\gamma}$ such that $\overline{\gamma}'(0) = v_0$ an asymptotic ray to γ.

Define the function $h_t(x) = t - d(x, \overline{\gamma}(t)) + b^\gamma(q)$. Since $\overline{\gamma}$ is a ray, the points $q = \overline{\gamma}(0)$ and $\overline{\gamma}(t)$ are not cut points to each other, therefore the function $d(x, \overline{\gamma}(t))$ is smooth in a neighborhood of q and thus so is h_t. Clearly $h_t(q) = b^\gamma(q)$, thus to show that h_t is a lower barrier for b^γ we only need to show that $h_t(x) \leq b^\gamma(x)$. To see this, first note that for any s,

$$-d(x, \overline{\gamma}(t)) \leq -d(x, \gamma(s)) + d(\gamma(s), \overline{\gamma}(t)) \tag{6.4}$$

$$= s - d(x, \gamma(s)) - s + d(\gamma(s), \overline{\gamma}(t)).$$

Taking $s \to \infty$ this gives

$$-d(x, \overline{\gamma}(t)) \leq b^\gamma(x) - b^\gamma(\overline{\gamma}(t)). \tag{6.5}$$

Also,

$$\begin{aligned} b^\gamma(q) &= \lim_{i\to\infty} (t_i - d(q, \gamma(t_i))) \\ &= \lim_{i\to\infty} (t_i - d(q, \sigma_i(t)) - d(\sigma_i(t), \gamma(t_i))) \\ &= -d(q, \overline{\gamma}(t)) + \lim_{i\to\infty} (t_i - d(\sigma_i(t), \gamma(t_i))) \\ &= -t + b^\gamma(\overline{\gamma}(t)). \qquad (6.6) \end{aligned}$$

Combining (6.5) and (6.6) gives

$$h_t(x) \leq b^\gamma(x), \tag{6.7}$$

so h_t is a lower barrier function for b^γ. By (6.1), we have that

$$\Delta_f(h_t)(x) = \Delta_f(-d(x, \overline{\gamma}(t))) \geq -\frac{n + 4k - 1}{t}. \tag{6.8}$$

Taking $t \to \infty$ proves the lemma. q.e.d.

Note that since Δ_f is just a perturbation of Δ by a lower order term, the strong maximum principle and elliptic regularity still hold for Δ_f. Namely if h is a continuous function such that $\Delta_f(h) \geq 0$ in the barrier sense and h has an interior maximum then h is constant and if $\Delta_f(h) = 0$ (i.e ≥ 0 and ≤ 0) in the barrier sense then h is smooth. We now apply the lemma and these two theorems to finish the proof of the splitting theorem.

Proof of Theorem 6.1. Denote by γ_+ and γ_- the two rays which form the line γ and let b^+, b^- denote their Busemann functions.

The function $b^+ + b^-$ has a maximum at $\gamma(0)$ and satisfies $\Delta_f(b^+ + b^-) \geq 0$, thus by the strong maximum principle the function is constant and equal to 0. But then $b^+ = -b^-$ so that $0 \leq \Delta_f(b^+) = -\Delta_f(b^-) \leq 0$ which then implies, by elliptic regularity, that the functions b^+ and b^- are smooth.

Moreover, for any point q we can consider asymptotic rays $\overline{\gamma}_+$ and $\overline{\gamma}_-$ to $\gamma_+$and γ_- and denote their Busemann functions by $\overline{b}^+$ and $\overline{b}^-$. From (6.7) it follows that

$$\overline{b}^+(x) + b^+(q) \leq b^+(x). \tag{6.9}$$

The next step is to show that this inequality is, in fact, an equality when γ_+ extends to a line.

First we show that the two asymptotic rays $\overline{\gamma}_+$ and $\overline{\gamma}_-$ form a line. By the triangle inequality, for any t

$$\begin{aligned} d(\overline{\gamma}_+(s_1), \overline{\gamma}_-(s_2)) &\geq d(\overline{\gamma}_-(s_2), \gamma_+(t)) - d(\gamma_+(t), \overline{\gamma}_+(s_1)) \\ &= t - d(\gamma_+(t), \overline{\gamma}_+(s_1)) - (t - d(\overline{\gamma}_-(s_2), \gamma_+(t)), \end{aligned}$$

so by taking $t \to \infty$ we have

$$\begin{aligned} d(\overline{\gamma}_+(s_1), \overline{\gamma}_-(s_2)) &\geq b^+(\overline{\gamma}^+(s_1)) - b^+(\overline{\gamma}^-(s_2)) \\ &= b^+(\overline{\gamma}^+(s_1)) + b^-(\overline{\gamma}^-(s_2)) \\ &\geq \overline{b}^+(\overline{\gamma}^+(s_1)) + b^+(q) + \overline{b}^-(\overline{\gamma}^-(s_2)) + b^-(q) \\ &= s_1 + s_2. \end{aligned}$$

Thus, any asymptotic ray to γ_+ forms a line with any asymptotic ray to γ_-. Applying the same argument given above for b^+ and b^- we see that $\overline{b}^+ = -\overline{b}^-$. By Applying (6.9) to b^-

$$-\overline{b}^-(x) - b^-(q) \geq -b^-(x).$$

Substituting $b^+ = -b^-$ and $\overline{b}^+ = -\overline{b}^-$ we have

$$\overline{b}^+(x) + b^+(q) \geq b^+(x).$$

This along with (6.9), gives

$$\overline{b}^+(x) + b^+(q) = b^+(x).$$

Thus, $\overline{b}^+$ and b^+ differ only by a constant. Clearly, at q the derivative of $\overline{b}^+$ in the direction of $\overline{\gamma}'_+(0)$ is 1. Since $\overline{b}^+$ has Lipschitz constant 1, this implies that $\nabla b^+(q) = \overline{\gamma}'_+(0)$.

From the Bochner formula (3.2) and a direct computation one has the following Bochner formula with measure,

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{6.10}$$

Now apply this to b^+, since $||\nabla b^+|| = 1$, we have

$$0 = ||\mathrm{Hess}\, b^+||^2 + \nabla b^+(\Delta_f(b^+)) + \mathrm{Ric}_f(\nabla b^+, \nabla b^+). \tag{6.11}$$

Since $\Delta_f(b^+) = 0$ and $\mathrm{Ric}_f \geq 0$ we then have that $\mathrm{Hess}\, b^+ = 0$ which, along with the fact that $||\nabla b^+|| = 1$ implies that M splits isometrically in the direction of ∇b^+.

To see that f must be constant in the splitting direction note that from (6.11) we now have $\mathrm{Ric}_f(\nabla b^+, \nabla b^+) = 0$ and ∇b^+ points in the splitting direction so $\mathrm{Ric}(\nabla b^+, \nabla b^+) = 0$. Therefore $\mathrm{Hess} f(\nabla b^+, \nabla b^+) = 0$. Since f is bounded f must be constant in ∇b^+ direction. q.e.d.

As Lichnerowicz points out, the clever covering arguments in [**10**] along with Theorem 6.1 imply the following structure theorem for compact manifolds with $\mathrm{Ric}_f \geq 0$.

Theorem 6.6. *If M is compact and $Ric_f \geq 0$ then M is finitely covered by $N \times T^k$ where N is a compact simply connected manifold and f is constant on the flat torus T^k.*

Theorem 6.6 has the following topological consequences.

Corollary 6.7. *Let M be compact with $Ric_f \geq 0$ then*

1) $b_1(M) \leq n$.
2) $\pi_1(M)$ *has a free abelian subgroup of finite index of rank* $\leq n$.
3) $b_1(M)$ *or* $\pi_1(M)$ *has a free abelian subgroup of rank n if and only if M is a flat torus and f is a constant function.*
4) $\pi_1(M)$ *is finite if* $\mathrm{Ric}_f > 0$ *at one point.*

We also note that the splitting theorem has been used by Oprea [**30**] to derive information about the Lusternik-Schnirelmann category of compact manifolds with non-negative Ricci curvature. These arguments also clearly carry over to the Ric_f case.

For noncompact manifolds with positive Ricci curvature the splitting theorem has also been used by Cheeger and Gromoll [**10**] and Sormani [**41**] to give some other topological obstructions. These results also hold for Ric_f with f bounded.

Theorem 6.8. *Suppose M is a complete manifold with* $\mathrm{Ric}_f > 0$ *for some bounded f then*

1) *M has only one end and*
2) *M has the loops to infinity property.*

In particular, if M is simply connected at infinity then M is simply connected.

7. Other applications of the mean curvature comparison.

Theorem 1.1 can be used to prove an excess estimate. Recall that for $p, q \in M$ the excess function is $e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)$. Let $h(x) = d(x, \gamma)$ where γ is a fixed minimal geodesic from p to q, then (1.4) along with the arguments in [**1**, Proposition 2.3] imply the following version of the Abresch-Gromoll excess estimate.

Theorem 7.1. *Let* $\mathrm{Ric}_f \geq 0$, $|f| \leq k$ *and* $h(x) < \min\{d(p,x), d(q,x)\}$ *then*

$$e_{p,q}(x) \leq 2\left(\frac{n+4k-1}{n+4k-2}\right)\left(\frac{1}{2}Ch^{n+4k}\right)^{\frac{1}{n+4k-1}}$$

where

$$C = 2\left(\frac{n+4k-1}{n+4k}\right)\left(\frac{1}{d(p,x)-h(x)} + \frac{1}{d(q,x)-h(x)}\right).$$

Remark. (1.2) also implies an excess estimate for manifolds with $\mathrm{Ric} \geq (n-1)H$ and $|\nabla f| \leq a$, however the constant C will depend on $H \cdot d(p,q)^2$ and e^{ah}. The mean curvature comparison for Ric_f^N discussed in the appendix also implies an excess estimate.

Theorem 7.1 gives extensions of theorems of Abresch-Gromoll [**1**] and Sormani [**40**] to the case where $\mathrm{Ric}_f \geq 0$ for a bounded f.

Theorem 7.2. *Let be* M *a complete noncompact manifold with* $Ric_f \geq 0$ *for some bounded* f.

1) *If* M *has bounded diameter growth and sectional curvature bounded below then* M *is homeomorphic to the interior of a compact manifold with boundary.*
2) *If* M *has sublinear diameter growth then* M *has finitely generated fundamental group.*

Remark. If we consider $|f| \leq k$, the arguments in [**1**] and [**40**] say slightly more. Namely, the diameter growth in the first part can be of order $\leq \frac{1}{n+4k-1}$ and in the second part one can derive a explicity constant $S_{n,k}$ such that the diameter growth only needs to be $\leq S_{n,k} \cdot r$. Also see [**49**] and [**47**].

We can also apply the mean curvature comparison to the excess function to prove the Myers' theorem. We note that the excess function was also used to prove a Myers' theorem in [**33**]. It is interesting that this proof is exactly suited to our situation, since we only have a uniform bound on mean curvature when $r \leq \frac{\pi}{2\sqrt{H}}$, while other arguments do not seem to easily generalize.

Proof of Theorem 1.4.

Let p_1, p_2 are two points in M with $d(p_1,p_2) \geq \frac{\pi}{\sqrt{H}}$ and set $B = d(p_1,p_2) - \frac{\pi}{\sqrt{H}}$.

Let $r_1(x) = d(p_1,x)$ and $r_2(x) = d(p_2,x)$ and e be the excess function for the points p_1 and p_2. By the triangle inequality, $e(x) \geq 0$ and $e(\gamma(t)) = 0$ where γ is a minimal geodesic from p_1 to p_2. Therefore, $\Delta_f(e)(\gamma(t)) \geq 0$.

Let $y_1 = \gamma\left(\frac{\pi}{2\sqrt{H}}\right)$ and $y_2 = \gamma\left(\frac{\pi}{2\sqrt{H}} + B\right)$. For $i = 1, 2$ $r_i(y_i) = \frac{\pi}{2\sqrt{H}}$ so, by (3.18), we have

$$\Delta_f(r_i(y_i)) \leq 2k\sqrt{H}. \tag{7.1}$$

(1.3) does not give an estimate for $\Delta_f(r_1(y_2))$ since $r_1(y_2) > \frac{\pi}{2\sqrt{H}}$ but by (3.1) and (7.1) we have

$$\Delta_f(r_1(y_2)) \leq 2k\sqrt{H} - B(n-1)H. \tag{7.2}$$

So

$$0 \leq \Delta_f(e)(y_2) = \Delta_f(r_1)(y_2) + \Delta_f(r_2)(y_2) \leq 4k\sqrt{H} - B(n-1)H \tag{7.3}$$

which implies $B \leq \frac{4k}{(n-1)\sqrt{H}}$ and $d(p_1, p_2) \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$. q.e.d.

As we have seen, there is no bound on the distance between two points in a complete manifold with $\mathrm{Ric}_f \geq (n-1)H > 0$. However, by slightly modifying the argument above one can prove a distance bound between two hypersurfaces that depends on the f-mean curvature of the hypersurfaces, here for a hypersurface N the f-mean curvature at a point $x \in N$ with respect to the normal vector n is

$$H^f_n(x) = H_n(x) + \langle n, \nabla f\rangle(x) \tag{7.4}$$

where H_n is the regular mean curvature. m_f is then the f-mean curvature of the geodesic sphere with respect to the inward pointing normal.

Theorem 7.3. *Let* $\mathrm{Ric}_f \geq (n-1)H > 0$ *and let* N_1 *and* N_2 *be two compact hypersurfaces in* M *then*

$$d(N_1, N_2) \leq \frac{\max_{p\in N_1} |H^f_{n_1}(p)| + \max_{q\in N_2} |H^f_{n_2}(q)|}{2(n-1)H}. \tag{7.5}$$

Proof. Let $e_{N_1,N_2}(x) = r_1(x) + r_2(x) - d(N_1, N_2)$ where $r_i(x) = d(x, N_i)$. Then, by applying the Bochner formula to r_i in the same way we applied it to the distance to a point in Section 2, we have

$$\Delta_f(r_i)(x) \leq \max_{p\in N_i} |H^f_{n_i}(x)| - (n-1)Hd(N_i, x).$$

One now can prove the theorem using a similar argument as in the proof of Theorem 1.4. q.e.d.

A similar argument also shows Frankel's Theorem is true for Ric_f.

Theorem 7.4. *Any two compact* f*-minimal hypersurfaces in a manifold with* $\mathrm{Ric}_f > 0$ *intersect.*

One also has a rigidity statement when $\mathrm{Ric}_f \geq 0$ and M has two f-minimal hypersurface which do not intersect, see [**35**] for the statement and proof in the $\mathrm{Ric} \geq 0$ case.

Appendix A. Mean curvature comparison for N-Bakry-Emery Ricci tensor

In [**6**] the volume comparison theorem and Myers' theorem for the N-Bakry-Emery Ricci tensor are proven using what we have called a mean curvature comparison (actually their work is slightly more general than the cases treated in this paper). In this appendix, for clarity, we state this comparison in the language we have used above, which is standard in Riemannian geometry.

Recall the definition of the N-Bakry-Emery tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{1}{N} df \otimes df \quad \text{for } N > 0.$$

The main idea is that the a Bochner formula holds for Ric_f^N that looks like the Bochner formula for the Ricci tensor of an $n+N$ dimensional manifold . This formula seems to have been Bakry and Emery's original motivation for the definition of the Bakry-Emery Ricci tensor and for their more general curvature dimension inequalities for diffusion operators [**4**]. See [**18**, **19**] for elementary proofs of the inequality.

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{N+n} + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u)$$

For the distance function, we have

$$m_f' \leq -\frac{(m_f)^2}{n+N-1} - \mathrm{Ric}_f^N(\partial_r, \partial_r).$$

Thus, using the standard Sturm-Liouville comparison argument one has the mean curvature comparison.

Theorem A.1 (Mean curvature comparison for N-Bakry-Emery)**.** [**6**] *Suppose that* $N > 0$ *and* $\mathrm{Ric}_f^N \geq (n+N-1)H$*, then*

$$m_f(r) \leq m_H^{n+N}(r).$$

This comparison along with the methods used above gives proofs of the comparison theorems for Ric_f^N.

The Bochner formula for Ric_f^N has many other applications to other geometric problems not treated here such as eigenvalue problems and Liouville theorems, see for example [**5**] and [**19**] and the references there in.

Appendix B. ODE proof of mean curvature comparison

In this section we include our original proof of the mean curvature comparison which uses somewhat different methods.

Theorem B.1 (Mean Curvature Comparison)**.** *Assume* $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$*.*

a) If $\partial_r f \geq -a$ $(a \geq 0)$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H(r) \leq a \tag{B-1}$$

along that minimal geodesic segment from p. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(p) - at$ for all $t < r$.
In particular when $a = 0$, we have

$$m_f(r) \leq m_H(r) \tag{B-2}$$

and equality holds if and only if all radial sectional curvatures are H and f is constant.

b) If $|f| \leq k$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H \leq (n-1)e^{\frac{4k}{n-1}} \left(\frac{\sqrt{|H|}\mathrm{sn}_H(2r) + 2|H|r}{\mathrm{sn}_H^2(r)} \right) \tag{B-3}$$

along that minimal geodesic segment from p, where $\mathrm{sn}_H(r)$ is the unique function satisfying

$$\mathrm{sn}''_H(r) + H\mathrm{sn}_H(r) = 0, \quad \mathrm{sn}_H(0) = 0, \quad \mathrm{sn}'_H(0) = 1.$$

In particular when $H = 0$ we have

$$m_f(r) - \frac{n-1}{r} \leq 4(n-1)e^{\frac{4k}{n-1}} \frac{1}{r}. \tag{B-4}$$

Proof. We compare m_f to the mean curvature of the model space. Using $\mathrm{Ric}_f \geq (n-1)H$, and subtracting (3.6) by (3.5) gives

$$\begin{aligned} (m_f - m_H)' &\leq -\frac{1}{n-1}\left[(m_f + \partial_r f)^2 - m_H^2\right] \\ &= -\frac{1}{n-1}\left[(m_f - m_H + \partial_r f)(m_f + m_H + \partial_r f)\right]. \end{aligned} \tag{B-5}$$

Proof of Part a). Write (B-5) as the following
(B-6)

$$(m_f - m_H - a)' \leq -\frac{1}{n-1}\left[(m_f - m_H - a + a + \partial_r f)(m_f + m_H + \partial_r f)\right].$$

Let us define $\psi_{a,H} = \max\{m_f - m_H - a, 0\} = (m_f - m_H - a)_+$, and declare that $\psi_{a,H} = 0$ whenever it becomes undefined. Since $\partial_r f \geq -a$, $a + \partial_r f \geq 0$. When $\psi_{a,H} \geq 0$, $m_f + m_H + \partial_r f \geq a + \partial_r f + 2m_H \geq 2m_H$ which is ≥ 0 if $m_H \geq 0$. Using this and (B-6) gives

$$\psi'_{a,H} \leq -\frac{1}{n-1}(m_f + m_H + \partial_r f)\psi_{a,H} \leq 0. \tag{B-7}$$

Since $\lim_{r\to 0}\psi_{a,H}(r) = (-\partial f_r(0) - a)_+ = 0$, we have $\psi_{a,H}(r) = 0$ for all $r \geq 0$. This finishes the proof of the inequality.

Now suppose that $m_f = m_H + a$, then from (B-5) we have that $m = m_H$ which implies that $\partial_r f = -a$. So $\partial_r^2 f \equiv 0$ which then implies that $\mathrm{Ric}(\partial_r, \partial_r) = \mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$. Now the rigidity for the regular mean curvature comparison implies that all the sectional curvatures are constant and equal to H.

Proof of Part b). By (B-5) we have

(B-8) $$(m_f - m_H)' \leq$$

$$-\frac{1}{n-1}\left[(m_f - m_H)^2 + 2(m_H + \partial_r f)(m_f - m_H) + \partial_r f(2m_H + \partial_r f)\right].$$

Now define $\psi = \max\{m_f - m_H, 0\} = (m_f - m_H)_+$, the error from the mean curvature comparison, and declare that $\psi = 0$ whenever it becomes undefined. Define

(B-9) $$\rho = \left[-\frac{1}{n-1}\partial_r f(2m_H + \partial_r f)\right]_+.$$

Using this notation and inequality (B) we obtain

(B-10) $$\psi' \leq -\frac{1}{n-1}\psi^2 - \frac{2}{n-1}(m_H + \partial_r f)\psi + \rho.$$

We would like to estimate ψ in terms of ρ. It is enough to use the linear differential inequality

(B-11) $$\psi' + \frac{2}{n-1}(m_H + \partial_r f)\psi \leq \rho.$$

When $\partial_r f = 0$ (in the usual case), we have $\rho = 0$ and $\psi = 0$, getting the classical mean curvature comparison. In general, by (B-9), the definition of ρ, when $m_H > 0$

(B-12) $$\rho > 0 \Longleftrightarrow -2m_H < \partial_r f < 0.$$

Also

(B-13) $$\rho \leq \left(-\frac{2}{n-1}(\partial_r f)m_H\right)_+.$$

Therefore we have

$$\rho \leq \frac{4}{n-1}m_H^2.$$

Note that $m_H = (n-1)\frac{\mathrm{sn}_H'(r)}{\mathrm{sn}_H(r)}$. Now (B-11) becomes

$$\psi' + \left(2\frac{\mathrm{sn}_H'(r)}{\mathrm{sn}_H(r)} + \frac{2}{n-1}\partial_r f\right)\psi \leq 4(n-1)\left(\frac{\mathrm{sn}_H'(r)}{\mathrm{sn}_H(r)}\right)^2.$$

Multiply this by the integrating factor $\mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}$ to obtain

$$\left(\mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}\psi(r)\right)' \leq 4(n-1)e^{\frac{2}{n-1}f(r)}(\mathrm{sn}_H'(r))^2.$$

Since $\psi(0)$ is bounded, integrate this from 0 to r gives

$$\text{(B-14)} \qquad \mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}\psi(r) \le 4(n-1)\int_0^r e^{\frac{2}{n-1}f(t)}(\mathrm{sn}_H'(r))^2 dt.$$

Since $|f| \le k$, we have

$$\psi(r) \le (n-1)e^{\frac{4k}{n-1}}\left(\frac{\sqrt{|H|}\mathrm{sn}_H(2r) + 2|H|r}{\mathrm{sn}_H^2(r)}\right).$$

When $H = 0$, $\mathrm{sn}_H(r) = r$, from (B-14) we get

$$\psi(r) \le 4(n-1)e^{\frac{4k}{n-1}}\frac{1}{r}.$$

This completes the proof of Part b). q.e.d.

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