CALABI-YAU COMPONENTS IN GENERAL TYPE HYPERSURFACES

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Abstract

For a one-parameter family $(V, \{\Omega_i\}_{i=1}^{p_g})$ of general type hypersurfaces with bases of holomorphic n-forms, we construct open covers $V = \bigcup_{i=1}^{p_g} U_i$ using tropical geometry. We show that after normalization, each Ω_i is approximately supported on a unique U_i and such a pair approximates a Calabi-Yau hypersurface together with its holomorphic n-form as the parameter becomes large. We also show that the Lagrangian fibers in the fibration constructed by Mikhalkin [**9**] are asymptotically special Lagrangian. As the holomorphic n-form plays an important role in mirror symmetry for Calabi-Yau manifolds, our results is a step toward understanding mirror symmetry for general type manifolds.

1. Introduction

Calabi-Yau manifolds are Kähler manifolds with zero first Chern class. By Yau's theorem [**14**], they admit Ricci flat Kähler metrics. They play important roles in String theory as internal spaces. Up to a scalar multiple, there exists a unique holomorphic volume form $\Omega \in H^{n,0}(Y)$ on any Calabi-Yau manifold Y. In the SYZ proposal [**12**] for the Mirror Symmetry conjecture, Strominger, Yau and Zaslow conjectured that mirror symmetry is a generalization of the Fourier-Mukai transformation along dual special Lagrangian torus fibrations on mirror Calabi-Yau manifolds and it is called the "SYZ transformation". Recall that a Lagrangian submanifold L in Y is called *special* if $\operatorname{Im}\Omega|_L = 0$. It is not easy to construct special Lagrangian fibrations on Calabi-Yau manifolds. Nevertheless, Lagrangian fibrations do exist on Calabi-Yau hypersurfaces in $\mathbb{CP}^{n+1}$, or other toric varieties, by the work of Gross [**7**], Ruan [**11**] and others.

There are generalizations of the Mirror Symmetry conjecture for Fano manifolds (i.e. positive first Chern class) and also recently for general type manifolds (e.g. negative first Chern class). There are many Fano manifolds which are toric varieties and therefore they admit natural

Lagrangian torus fibrations. They have canonical holomorphic volume forms Ω outside singular fibers which make the toric fibrations special. The SYZ transformation along these special Lagrangian fibrations on Fano toric manifolds was studied by Chan and the first author in [**1**].

This paper is an initial step in our studies of the SYZ mirror transformation for general type manifolds. In dimension one, for every $g \geq 2$, there is a family of genus g Riemann surfaces V_t which degenerate to a connected sum of g copies of elliptic curves as t goes to infinity, i.e., $V_\infty = Y_1 \cup \cdots \cup Y_g$ with each Y_i a smooth elliptic curve. Furthermore, we can find a base $\Omega_{1,t}, \ldots, \Omega_{g,t}$ of $H^{1,0}(V_t)$ such that for each $i \in \{1, \ldots, g\}$, $\Omega_{i,t}$ converges to a holomorphic volume form on Y_i as t goes to infinity (see subsection §3.2).

In higher dimensions, we cannot expect to have a connected sum decomposition for general type manifolds V_t into Calabi-Yau manifolds. Instead, we will show in our main theorem that there is a basis $\{\Omega_{1,t} \ldots, \Omega_{p_g,t}\}$ of $H^{n,0}(V_t)$ and a decomposition

$$V_t = \bigcup_{i=1}^{p_g} U_{i,t}$$

such that each $\Omega_{i,t}$ is roughly supported on corresponding $U_{i,t}$ and $(U_{i,t}, \Omega_{i,t})$ approximates a Calabi-Yau manifold $Y_{i,t}$ together with its holomorphic volume form $\Omega_{Y_{i,t}}$ as t goes to infinity. This is not a connected sum decomposition as different $U_{i,t}$'s can have large overlaps. However, it still enables us to have a proper notion of special Lagrangian fibrations on V_t and study the SYZ transformation along them.

If V_t is a family of general type hypersurfaces in $\mathbb{CP}^{n+1}$, i.e. the common degree d of the family of defining polynomials of V_t is bigger than $n + 2$, then its geometric genus

$$p_g(V_t) = \dim H^{n,0}(V_t) = \binom{d-1}{n+1} \geq 2.$$

In fact $p_g(V_t)$ equals to the number of interior lattice points in $\triangle_d$, the standard simplex in $\mathbb{R}^{n+1}$ spanned by $de_1 \ldots de_{n+1}$ and the origin, where $\{e_\alpha\}_{\alpha=1}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. That is, if we denote the set of interior lattice points of $\triangle_d$ by $\triangle_{d,\mathbb{Z}}^0$, then

$$p_g(V_t) = \#\triangle_{d,\mathbb{Z}}^0.$$

The analog formula for p_g holds true for smooth hypersurfaces in toric varieties [**4**]. In this article, we prove the following

Theorem (Main Theorem). *For any positive integers n and d with $d \geq n + 2$, there exists a family of smooth hypersurfaces $V_t \subset \mathbb{CP}^{n+1}$ of*

degree d such that V_t can be written as

$$V_t = \bigcup_{i \in \triangle_{d,\mathbb{Z}}^0} U_{i,t}$$

where $U_{i,t}$ is a family of open subsets $U_{i,t} \subset V_t$ such that after the normalization $H_t : (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^{n+1}$ defined by

$$H_t(z_1, \ldots, z_{n+1}) = \left(|z_1|^{\frac{1}{\log t}} \frac{z_1}{|z_1|}, \ldots, |z_{n+1}|^{\frac{1}{\log t}} \frac{z_{n+1}}{|z_{n+1}|} \right),$$

1) *$U_{i,t}$ is close in Hausdorff distance on $(\mathbb{C}^*)^{n+1}$ to an open subset of a Calabi-Yau hypersurface $Y_{i,t}$ in $\mathbb{CP}^{n+1}$;*
2) *there exists a basis $\{\Omega_{i,t}\}_{i \in \triangle_{d,\mathbb{Z}}^0}$ of $H^{n,0}(V_t)$ such that for each $i \in \triangle_{d,\mathbb{Z}}^0$, $\Omega_{i,t}$ is non-vanishing and close to the holomorphic volume form $\Omega_{Y_{i,t}}$ of $Y_{i,t}$ on $U_{i,t}$ with respect to the pull-back metric $H_t^*(g_0)$ of the invariant toric metric g_0 on $(\mathbb{C}^*)^{n+1}$;*
3) *for any compact subset $B \subset (\mathbb{C}^*)^{n+1} \setminus U_{i,t}$, $\Omega_{i,t}$ tends to zero in $V_t \cap B$ uniformly with respect to $H_t^*(g_0)$.*
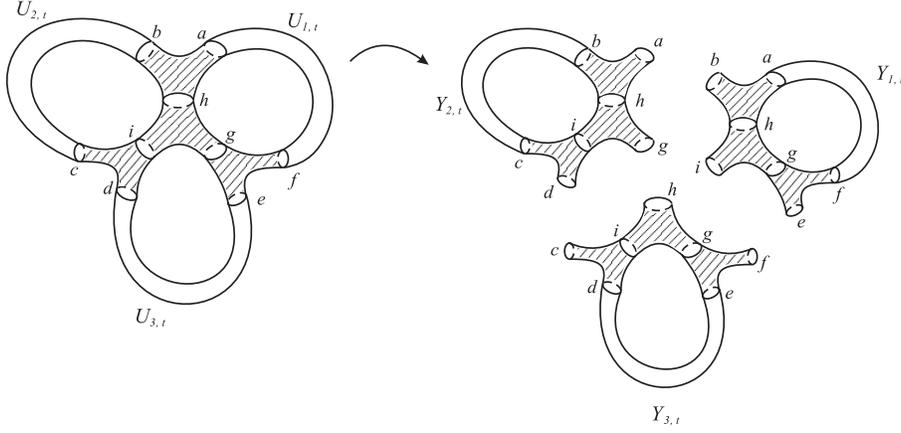

Figure 1

Our proof is based on the results of Mikhalkin [**9**]. In his paper, Mikhalkin constructed torus fibrations on general type hypersurfaces V in $\mathbb{CP}^{n+1}$ and he showed that some of these fibers are Lagrangian. The technique he employed is tropical geometry. He constructed a degenerating family V_t of hypersurfaces to decompose V_t into union of pairs-of-pants and also his fibration can be seen from this tropical degeneration. We are going to make use of his decomposition to construct our open sets $U_{i,t}$ in the main theorem.

Roughly speaking, the main theorem says that as t approaches infinity, V_t decomposes into p_g different Calabi-Yau manifolds $Y_{i,t}$ and each

support a holomorphic n-form $\Omega_{Y_{i,t}}$ on V_t. (In here, we abused the notion of "decomposition" since the open sets $U_{i,t}$ that we obtained in the "decomposition" do overlap even as $t \to +\infty$.) Therefore we can speak of *special Lagrangian submanifolds* in V_t.

Definition 1.1. Let $L_t \subset V_t$ be a smooth family of Lagrangian submanifolds. We call it *asymptotically special Lagrangian* of phase θ with respect to the decomposition if for any $\epsilon > 0$ we have

$$\left| \text{Im} \left(e^{\sqrt{-1}\theta} \Omega_{i,t} \right) \Big|_{L_t \cap U_{i,t}} \right| < \varepsilon$$

for any $i \in \triangle_{d,\mathbb{Z}}^0$ for sufficiently large t.

If $L_t \subset U_{i,t}$ for some $i \in \triangle_{d,\mathbb{Z}}^0$ and $\text{Im} \left(e^{\sqrt{-1}\theta} \Omega_{i,t} \right) |_{L_t} = 0$ then we call L_t a *special Lagrangian* submanifold in V_t.

In section §3.1 we show that the Lagrangian fibers in the torus fibration on V_t constructed by Mikhalkin in [**9**] are asymptotically special Lagrangians.

We start our proof with some preliminaries on the tropical geometry, especially on the theorems of Einsiedler-Kapranov-Lind [**3**] and Mikhalkin [**9**]. The main results and their proofs will be stated in section §3 and §4.

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2. Preliminaries

2.1. Amoebas and Viro's patchworking. Let V^o be a smooth hypersurface in $(\mathbb{C}^*)^{n+1} \subset \mathbb{CP}^{n+1}$ or other toric varieties defined by a Laurent polynomial

$$f(z) = \sum_j a_j z^j,$$

where $j = (j_1, \ldots, j_{n+1}) \in \mathbb{Z}^{n+1}$ are multi-indices. Recall that the *Newton polyhedron* $\triangle \subset \mathbb{R}^{n+1}$ of f, or of V^o, is the convex hull in $\mathbb{R}^{n+1}$ of all $j \in \mathbb{Z}^{n+1}$ such that $a_j \neq 0$. According to [**5**], the *amoeba* of V^o is the image

$$\text{Log}(V^o) \subset \mathbb{R}^{n+1}$$

under the map $\text{Log} : (z_1, \ldots, z_{n+1}) \mapsto (\log|z_1|, \ldots, \log|z_{n+1}|)$.

In this paper, we are looking for deformation of complex structures on V^o together with corresponding basis of holomorphic n-forms satisfying a special limiting property. This leads us to consider deformation of the polynomial f used by the *Viro's patchworking* [**13**] and non-Archimedean amoeba.

Let $v : \triangle_{\mathbb{Z}} \to \mathbb{R}$, where $\triangle_{\mathbb{Z}} = \triangle \cap \mathbb{Z}^{n+1}$, be any function and $f(z) = \sum_{j \in \triangle_{\mathbb{Z}}} a_j z^j$, $a_j \neq 0$ for any $j \in \triangle_{\mathbb{Z}}$, be any polynomial. The *patchworking polynomial* is defined for all $t > 0$ by

$$f_t^v(z) = \sum_{j \in \triangle_{\mathbb{Z}}} a_j t^{-v(j)} z^j.$$

The family f_t^v can be treated as a single polynomial in $(K^*)^{n+1}$, where $K^* = K \setminus \{0\}$ and K is the field of Puiseux series with complex coefficients in t. In order to match the notation in the literatures, for instance [3], we set $\tau = t^{-1}$ and let

$$\mathbb{C}((\tau^q)) = \left\{ g(\tau^q) = \sum_{k=m}^{\infty} g_m (\tau^q)^k \right\}$$

be the field of formal (semi-finite) Laurent series in τ^q. Then the field of Puiseux series is

$$K = \bigcup_{m \geq 1} \mathbb{C}((\tau^{\frac{1}{m}})).$$

The field K is algebraically closed [2] and has a valuation defined by

$$\mathrm{val}_K \left(\sum_{q \in \Lambda_b} b_q \tau^q \right) = \min \Lambda_b.$$

for $b = \sum_{q \in \Lambda_b} b_q \tau^q \in K$, $(b_q \neq 0)$. It is then easy to see that the field K can also be represented by the field of Puiseux series

$$\tilde{b} = \sum_{p \in \tilde{\Lambda}_b} \tilde{b}_p t^p$$

with $\max \tilde{\Lambda}_p < +\infty$ and valuation $\mathrm{val}_K(\tilde{b}) = -\max \tilde{\Lambda}_b$.

Since $e^{-\mathrm{val}_K}$ defines a norm $\| \cdot \|_K$ on K, we can define Log_K on $(K^*)^{n+1}$ analog to Log on $(\mathbb{C}^*)^{n+1}$ by

$$\begin{aligned} \mathrm{Log}_K(a_1, \ldots, a_{n+1}) &= (\log \|a_1\|_K, \ldots, \log \|a_{n+1}\|_K) \\ &= -(\mathrm{val}_K(a_1), \ldots, \mathrm{val}_K(a_{n+1})). \end{aligned}$$

Then for $V_K \subset (K^*)^{n+1}$, the closure $\mathcal{A}_K$ of the image set $\mathrm{Log}_K(V_K)$ in $\mathbb{R}^{n+1}$ is called accordingly the *(non-Archimedean) amoeba* of V_K. Since $\mathbb{C}$ is algebraically close, $\mathcal{A}_K = \mathrm{Log}_K(V_K)$. It is clear that $\mathcal{A}_K = -\mathcal{T}(V_K)$, where $\mathcal{T}(V_K)$ is the tropical variety of V_K which is defined as the closure of $\mathrm{val}_K(V_K)$ in $\mathbb{R}^{n+1}$ [3].

Note that for our family $f_t^v(z) = \sum_{j \in \triangle_{\mathbb{Z}}} a_j t^{-v(j)} z^j$, the coefficient of z^j is $a_j t^{-v(j)} \in K$ which has valuation $\mathrm{val}_K(a_j t^{-v(j)}) = v(j)$. This match the convention in [9].

Following the construction of [**9**], for a finite set A in $\mathbb{Z}^{n+1}$ and a real valued function $v : A \to \mathbb{R}$ on A, one defines Π_v to be the set of non-smooth points (called *corner locus* in [**9**]) of the Legendre transform $L_v : \mathbb{R}^{n+1} \to \mathbb{R}$ of v. Here $L_v(x)$ is defined by

$$L_v(x) = \max_{i \in A} l_{v,i}(x),$$

where $l_{v,i}(x) = \langle x, i \rangle - v(i)$ with $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{n+1}$. In particular, the interior of a top dimensional face of Π_v is given by

$$\mathfrak{F}(j^{(1)}, j^{(2)}) = \left\{ x \in \mathbb{R}^{n+1} : l_{v,j^{(1)}}(x) = l_{v,j^{(2)}}(x) > l_{v,j}(x), \, \forall j \neq j^{(1)}, j^{(2)} \right\}.$$

It was proved in [**9**] that Π_v is a balanced polyhedral complex dual to certain lattice subdivision of the convex hull $\triangle$ of A in $\mathbb{R}^{n+1}$. We refer the reader to [**9**] or the appendix for the definition of a balanced polyhedral complex. We have the following result of Einsiedler-Kapranov-Lind [**3**].

Theorem 2.1. *If $V_K \subset (K^*)^{n+1}$ is a hypersurface given by a polynomial $f = \sum\limits_{j \in \triangle_{\mathbb{Z}}} a_j z^j$, $a_j \in K^*$. Then the (non-Archimedean) amoeba $\mathcal{A}_K$ of V_K is the balanced polyhedral complex Π_v corresponding to the function $v(j) = val_K(a_j)$ defined on the lattice points of the Newton polyhedron $\triangle$.*

Note that the theorem in [**3**] is originally stated for the tropical variety $\mathcal{T}(V_K)$ instead of $\mathcal{A}_K$.

Now we can describe the limiting behavior of the family of varieties $V_t^o = \{f_t^v = 0\}$ in $(\mathbb{C}^*)^{n+1}$ as $t \to +\infty$. For each $t > 0$, we define the amoeba of V_t^o with respect to t by

$$\mathcal{A}_t = \text{Log}_t(V_t^o) \subset \mathbb{R}^{n+1},$$

where $\text{Log}_t(z_1, \dots, z_{n+1}) = (\log_t |z_1|, \dots \log_t |z_{n+1}|)$ on $(\mathbb{C}^*)^{n+1}$, where $\log_t \tau = \log \tau / \log t$ for $\tau > 0$. If we denote accordingly $\mathcal{A}_K = \text{Log}_K(V_K)$ the non-Archimedean amoeba of the family f_t^v regarded as a single polynomial in the field K of Puiseux series. Then, we have the following theorem of Mikhalkin [**9**] which is needed in the proofs of our assertions.

Theorem 2.2. *The amoebas $\mathcal{A}_t$ converge in the Hausdorff distance on $\mathbb{R}^{n+1}$ to the non-Archimedean amoeba $\mathcal{A}_K$ as $t \to +\infty$.*

Recall that the Hausdorff distance between two closed subsets A and B in $\mathbb{R}^{n+1}$ is given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d_{\mathbb{R}^{n+1}}(a, B), \sup_{b \in B} d_{\mathbb{R}^{n+1}}(A, b) \right\}.$$

2.2. Maximal dual complex. As we mentioned, it was proved in [**9**] that Π_v is a balanced polyhedral complex dual to certain lattice subdivision of the convex hull $\triangle$ of A in $\mathbb{R}^{n+1}$. In general, any n-dimensional balanced polyhedral complex Π in $\mathbb{R}^{n+1}$ determines a convex lattice polyhedron $\triangle \subset \mathbb{R}^{n+1}$ and a lattice subdivision of $\triangle$. We call Π a *maximal* polyhedral complex if the elements of the subdivision are simplices of volume $\frac{1}{(n+1)!}$, i.e, the corresponding subdivision is a unimodular lattice triangulation. Note that not all convex lattice polyhedron admit unimodular lattice triangulation. Therefore, not all convex lattice polyhedron admit maximal dual complex. If it does, then we have the following result of [**9**].

Proposition 2.3. *If Π is a maximal dual $\triangle$-complex, then Π is homotopy equivalent to the bouquet of $\#\triangle_{\mathbb{Z}}^0$ copies of $\mathbb{S}^n$, where $\triangle_{\mathbb{Z}}^0 = (\mathrm{Int}\,\triangle) \cap \mathbb{Z}^{n+1}$ is the set of interior lattice points of $\triangle$.*

However, the converse of the proposition is not true. A non-maximal dual $\triangle$-complex may still have the homotopy type stated in the proposition.

It was also shown in [**9**] that on each maximal complex $\Pi \subset \mathbb{R}^{n+1}$, there is a canonical choice of cutting locus Ξ such that each connected component $\mathcal{U}_k$, $(k = 1, \ldots, l)$ called *primitive piece*, of $\Pi \setminus \Xi$ is equivalent to an open neighborhood of the vertex in the *primitive complex* $\Sigma_n \subset \mathbb{R}^{n+1}$ which is the set of non-smooth points of the function $H(x_1, \ldots, x_{n+1}) = \max\{0, x_1, \ldots, x_{n+1}\}$. That is, there exists $M_k \in ASL_{n+1}(\mathbb{Z}) = SL_{n+1}(\mathbb{Z}) \ltimes \mathbb{Z}^n$ such that $M_k(\mathcal{U}_k)$ is an open set of Σ_n containing the vertex. Furthermore, these open sets are parametrized by the vertices of Π. Since Π is dual to the lattice subdivision of $\triangle$ with simplices of volume $\frac{1}{(n+1)!}$, we must have exactly $(n+1)!\mathrm{vol}(\triangle)$ distinct $\mathcal{U}_k$, i.e. $l = (n+1)!\mathrm{vol}(\triangle)$.
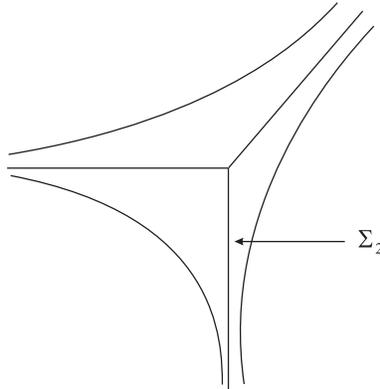

Figure 2: The primitive complex Σ_2 and the amoeba of $1 + z_1 + z_2 = 0$.

2.3. Pairs-of-pants decomposition and stratified fibration. In this subsection, we state the pairs-of-pants decomposition and existence of stratified fibration theorem of Mikhalkin [**9**] which is the main ingredient of the proof of our results. We start with the definition of pair-of-pants and stratified fibration given in [**9**].

As in [**9**], we denote by $\mathcal{H}$ a union of $n + 2$ generic hyperplanes in $\mathbb{CP}^n$ and $\mathcal{U}$ the union of the corresponding ϵ-neighborhoods for a small $\epsilon > 0$. Then $\overline{\mathcal{P}_n} = \mathbb{CP}^n \setminus \mathcal{U}$ is called the *n-dimensional pair-of-pants* while $\mathcal{P}_n = \mathbb{CP}^n \setminus \mathcal{H}$ the *n-dimensional open pair-of-pants*. It is clear that $\mathcal{P}_1$ is diffeomorphic to a 2-sphere with three punctures and $\overline{\mathcal{P}_1}$ is diffeomorphic to a 2-sphere with three holes, or equivalently, a closed disk with two holes. That is, the definition is a generalization of the classical pair-of-pants in one complex dimension.

If V and F are smooth manifolds, and Π is a maximal dual $\triangle$-complex of a lattice polyhedron $\triangle$ of full dimension in $\mathbb{R}^{n+1}$, then a smooth map $\lambda : V \to \Pi$ is a *stratified F-fibration* if it satisfies

1) the restriction of λ over each open n-cell $e \subset \Pi$ is a trivial fibration with fiber F;
2) for each pair of integers (l, k) with $0 \le k \le l \le n$, there exists a smooth "model" map depending only on l and k, $\lambda_{l,k} : V_{l,k} \to \Pi_{l,k}$ with $\Pi_{l,k}$ diffeomorphic to $\mathbb{R}^k \times \Sigma_{l-k} \times [0, +\infty)^{n-l}$ such that any (l, k)-point of Π has a neighborhood U such that

$$\lambda|_U : \lambda^{-1}(U) \to U$$

is diffeomorphic to the model map.

Now, we can state the pairs-of-pants decomposition and existence of stratified fibration theorem of Mikhalkin [**9**].

Theorem 2.4. *Let V be a smooth hypersurface in $\mathbb{CP}^{n+1}$ defined by a polynomial with Newton polyhedron $\triangle_d$. Then for every maximal dual $\triangle_d$-complex Π, there exists a stratified $\mathbb{T}^n$-fibration $\lambda : V \to \Pi$ satisfying*

1) *the induced map $\lambda^* : H^n(\Pi, \mathbb{Z}) \approx \mathbb{Z}^{p_g} \to H^n(V, \mathbb{Z})$ is injective, where $p_g = h^{n,0}(V)$ is the geometric genus of V;*
2) *for each primitive piece $\mathcal{U}_k$ of Π, $\lambda^{-1}(\mathcal{U}_k)$ is diffeomeorphic to an open pair-of-pants $\mathcal{P}_n$;*
3) *for each n-cell e of Π, there exists a point $x \in e$ such that the fiber $\lambda^{-1}(x)$ is a Lagrangian n-torus $\mathbb{T}^n \subset V$;*
4) *there exists Lagrangian embedding $\phi_i : \mathbb{S}^n \to V$, $i = 1, \dots, p_g$ such that the cycles $\lambda \circ \phi_i(\mathbb{S}^n)$ form a basis of $H_n(\Pi)$.*
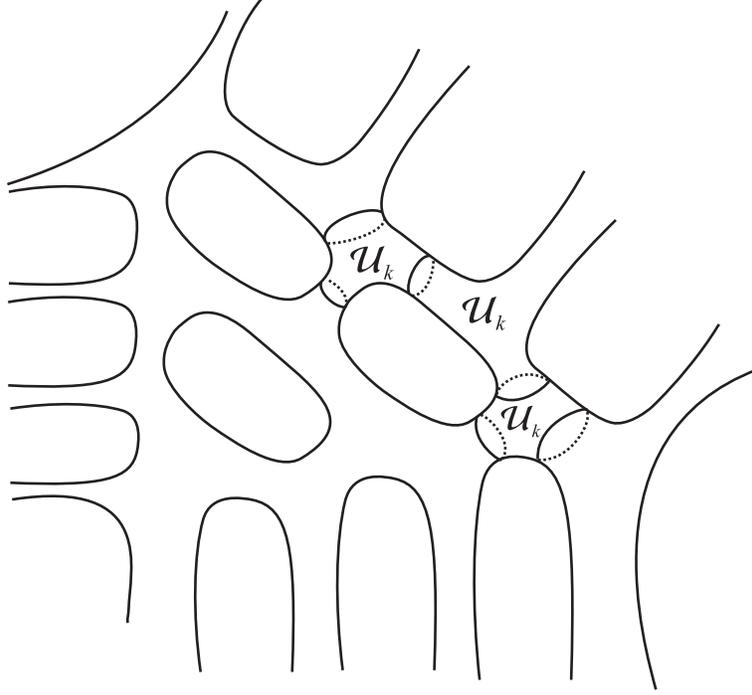
Figure 3: Illustration of some pairs-of-pants $\mathcal{U}_k$.

2.4. Key lemma. To prove the main theorem, we need to show the existence of a real valued function $v : \triangle_{d,\mathbb{Z}} \to \mathbb{R}$, where $\triangle_{d,\mathbb{Z}} = \triangle_d \cap \mathbb{Z}^{n+1}$ and $d \geq n+2$, such that the corresponding lattice subdivision of $\triangle_d$ dual to the balanced polyhedral complex Π_v satisfies some special properties which are needed to obtain the "decomposition" into Calabi-Yau pieces claimed in our main result. The special properties that we need are the property (2) in the following lemma. Existence of function v with property (1) only is well-known.

Lemma 2.5. *Let $\triangle_d$, $d \geq n+2$, be the simplex in $\mathbb{R}^{n+1}$ with vertices $\{0, de_1, \ldots, de_{n+1}\}$, where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$, and the function $v : \triangle_{d,\mathbb{Z}} \to \mathbb{R}$ be defined by*

$$v(j_1, \ldots, j_{n+1}) = \sum_{\alpha=1}^{n+1} j_\alpha^2 + \left(\sum_{\alpha=1}^{n+1} j_\alpha \right)^2.$$

Then

1) *the balanced polyhedral complex Π_v corresponding to v is a maximal dual complex of $\triangle_d$;*
2) *the corresponding subdivision of $\triangle_d$ has the property that for each point $i = (i_1, \ldots, i_{n+1}) \in \triangle_{d,\mathbb{Z}}^0$, the lattice subdivision restricts to a lattice subdivision of the translated simplex $i - 1 + \triangle_{n+2}$, where $1 = (1, \ldots, 1) \in \mathbb{Z}^{n+1}$.*
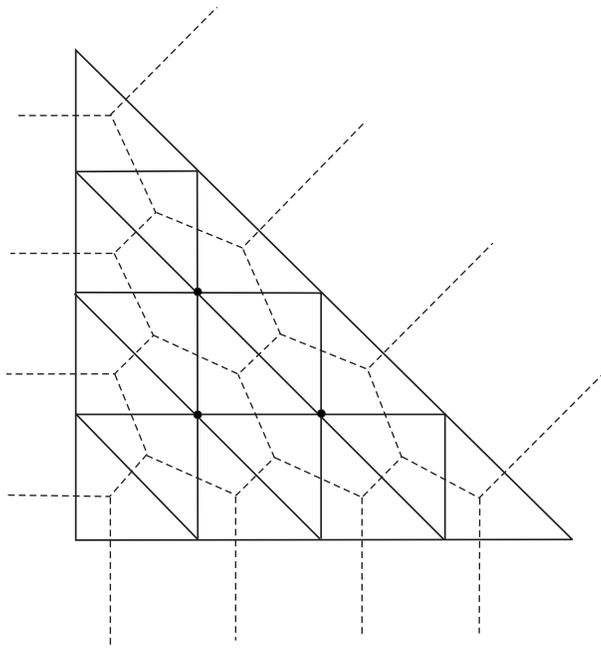
Figure 4: A maximal dual $\triangle_4$ complex with the required property.

Note that we do not claim that the subdivision is translational invariant which probably is not true. We only claim that for each interior lattice point, one can find a union of simplexes which form a standard simplex with exactly one interior lattice point. This clearly corresponds to the Calabi-Yau situation. The proof of this lemma will be given in the last section of this article.

3. Proof of the main theorem

In this section, we give the proof of the main theorem. Recall that we are free to use any hypersurface defined by a homogeneous polynomial of degree d of $n+1$ variables to replace V in order to describe V as a smooth manifold or as a symplectic manifold. The idea of tropical geometry leads us to consider the submanifold $V^o = V \cap (\mathbb{C}^*)^{n+1}$. Since all smooth hypersurfaces with the same Newton polyhedron are isotopic, we are free to choose the non-zero coefficients. In particular, we may take $a_j = 1$ for all $j \in \triangle_{\mathbb{Z}}$. Therefore, a deformation of complex structures on V can be given by a patchworking polynomial $f_t(z) = \displaystyle\sum_{j \in \triangle_{\mathbb{Z}}} t^{-v(j)} z^j$, $t > 0$

in $z \in (\mathbb{C}^*)^{n+1}$. We assume $f_t(z)$ is generic with Newton polyhedron $\triangle = \triangle_d$ and $v(j) < \infty$ for all $j \in \triangle_{d,\mathbb{Z}}$. That is, $f_t(z)$ contains all possible monomials of degree less than or equal to d.

The result of Einsiedler-Kapranov-Lind in the previous section states that the amoebas $\mathcal{A}_t = \mathrm{Log}_t(V_t^o)$ converge in the Hausdorff distance on $\mathbb{R}^{n+1}$ to the non-Archimedean amoeba $\mathcal{A}_K = \Pi_v$ as $t \to +\infty$. By construction, the top dimensional faces are given by two maximal terms in $f_t(z)$ as $t \to +\infty$. To be precise, we note that the highest exponent of t for the term $t^{-v(j)}z^j$ in the polynomial f_t is given by

$$l_{v,j}(x) = \langle j, x \rangle - v(j),$$

where $x = \lim\limits_{t \to +\infty} \mathrm{Log}_t(z) = \lim\limits_{t \to +\infty} (\log_t |z_1|, \ldots, \log_t |z_{n+1}|) \in \mathbb{R}^{n+1}$. Then, the interior of a top dimensional face of $\mathcal{A}_K = \Pi_v$ associated to the terms $t^{-v(j^{(1)})}z^{j^{(1)}}$ and $t^{-v(j^{(2)})}z^{j^{(2)}}$ is

$$\mathfrak{F}(j^{(1)}, j^{(2)}) = \left\{ x \in \mathbb{R}^{n+1} : l_{v,j^{(1)}}(x) = l_{v,j^{(2)}}(x) > l_{v,j}(x), \forall j \neq j^{(1)}, j^{(2)} \right\}.$$

By the results of tropical geometry [9], the generators of the n-dimensional homology of the amoeba are exactly given by the limit of the boundaries of the domains on which a term corresponds to an interior lattice point of the Newton polyhedron is maximal. Recall that the set of interior lattice points of $\triangle_d$ is exactly equal to $p_g = \begin{pmatrix} d - 1 \\ n + 1 \end{pmatrix}$ (see for instance [4]) and note that $\mathfrak{F}(i, j) \neq \emptyset$ only when i and j is connected by an edge in the lattice subdivision of $\triangle_d$ dual to Π_v. Then for each $i \in \triangle_{d,\mathbb{Z}}^0$, $C_i^{\wedge} = \bigcup_j \overline{\mathfrak{F}(i,j)}$ forms an n-cycle representing an element of $H_n(\Pi_v, \mathbb{Z})$. It is clear from Mikhalkin's results mentioned in the previous section that the classes $\{[C_i^{\wedge}]\}_{i \in \triangle_{d,\mathbb{Z}}^0}$ are the generators of $H_n(\Pi_v, \mathbb{Z})$. Since $\Pi_v = \mathcal{A}_K$, this gives the generators of $H_n(\mathcal{A}_K, \mathbb{Z})$ and hence $H_n(\mathcal{A}_t, \mathbb{Z})$ for large t. From this observation, we first prove the following result which gives partial results of the main theorem.

Theorem 3.1. *Let $v : \triangle_{d,\mathbb{Z}} \to \mathbb{R}$ be a real valued function such that the set Π_v of non-smooth points of the Legendre transform of v is a maximal dual $\triangle_d$-complex, $d \geq n + 2$. Let $f_t = \sum\limits_{j \in \triangle_{d,\mathbb{Z}}} t^{-v(j)} z^j$ $(t > 0)$ be the patchworking polynomial of degree d defined by v with non-Archimedean amoeba $\mathcal{A}_K = \Pi_v$. Denote $V_t = \{f_t = 0\} \subset \mathbb{CP}^{n+1}$. Then for all $t > 0$, there exists a basis $\{\Omega_{i,t}\}_{i \in \triangle_{d,\mathbb{Z}}^0}$ of $H^{n,0}(V_t)$, and open subsets $U_{i,t}^{\wedge} \subset V_t \cap (\mathbb{C}^*)^{n+1}$ such that for each $i \in \triangle_{d,\mathbb{Z}}^0$,*

1) *$\mathrm{Log}_t(U_{i,t}^{\wedge})$ tends to an n-cycle $C_i^{\wedge}$ such that $\{[C_i^{\wedge}]\}_{i \in \triangle_{d,\mathbb{Z}}^0}$ forms a basis of $H_n(\mathcal{A}_K(V))$,*
2) *$\Omega_{i,t}$ is nonvanishing on $U_{i,t}^{\wedge}$ for large t, and*
3) *for any compact subset $B \subset \mathbb{CP}^{n+1} \setminus U_{i,t}^{\wedge}$, $\Omega_{i,t}$ tends to zero in $V_t^o \cap B$ uniformly with respect to the metric induced from the pullback metric $H_t^*(g_0)$ of the invariant metric of the torus $(\mathbb{C}^*)^{n+1}$.*
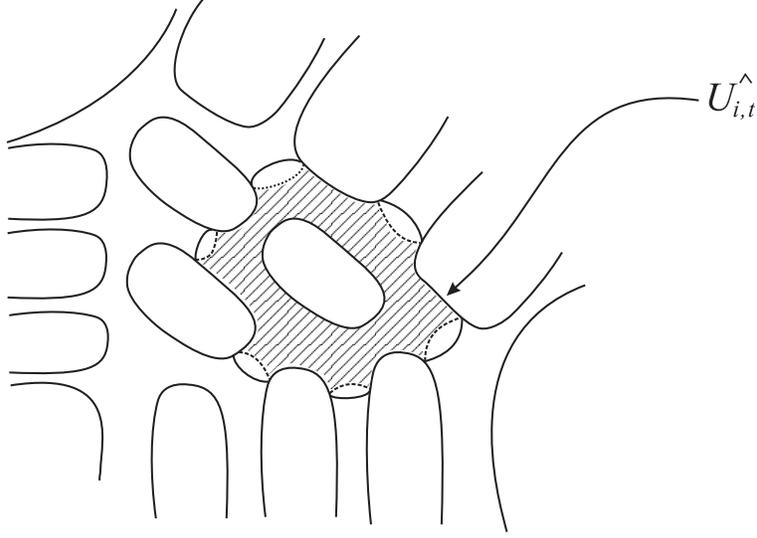
Figure 5: Illustration of an open set $U_{i,t}^{\wedge}$.

Proof. By the above observation, we consider for each $i \in \triangle_{d,\mathbb{Z}}^{0}$, the cycle $C_i^{\wedge} = \bigcup_j \overline{\mathfrak{F}(i,j)}$. Denote the $1/t$-neighborhood of $C_i^{\wedge}$ in Π_v (not in $\mathbb{R}^{n+1}$) by $C_{i,t}^{\wedge}$. (In fact, one can take any function of t instead, as long as it tends to 0 as $t \to +\infty$.) Choose $\delta > 0$ in such a way that Π_v is a deformation retract of the δ-tubular neighborhood T_δ of Π_v. Then for any $t > 0$ such that $\mathrm{Log}_t V_t \subset T_\delta$, we define the open set

$$U_{i,t}^{\wedge} = \lambda^{-1}\left(C_{i,t}^{\wedge}\right),$$

where $\lambda : V_t \to \Pi_v$ is the stratified $\mathbb{T}^n$-fibration given by Mikhalkin [**9**]. Then it is clear that condition (1) is satisfied, namely $\mathrm{Log}_t(U_{i,t}^{\wedge})$ tends to $C_i^{\wedge}$ in Hausdorff distance as $t \to \infty$. This proves the first statement.
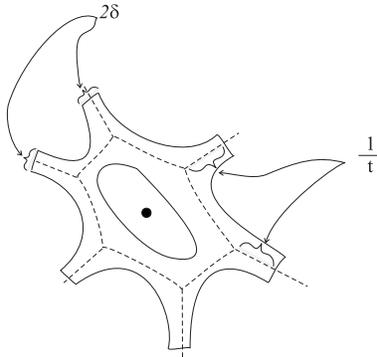
Figure 6: The construction of the neighborhood $C_{i,t}^{\wedge}$.

To see the other two statements, we use the well-known fact [**6**] that on the variety V_t, the Poincaré residues of $f_t^{-1}dz_1 \wedge \cdots \wedge dz_{n+1}$ define a holomorphic n-form on V_t; and all elements in $H^{n,0}(V_t)$ are of the form

$$\Omega = P(z)\operatorname{Res}\frac{dz_1 \wedge \cdots \wedge dz_{n+1}}{f_t(z)},$$

where $P(z)$ is a polynomial of degree at most $d - (n+2)$. To simplify notation, we omit the subscribe t in the following calculations.

For any $\alpha \in \{1,\ldots,n+1\}$, in the region where $f_{z_\alpha} = \frac{\partial f}{\partial z_\alpha} \neq 0$, the residue Ω_o of $f(z)^{-1}dz_1 \wedge \cdots \wedge dz_{n+1}$ is given by

$$\Omega_o = (-1)^{\alpha-1}\frac{dz_1 \wedge \cdots \widehat{dz_\alpha} \cdots \wedge dz_{n+1}}{f_{z_\alpha}}.$$

Now, for each interior lattice point $i \in \triangle_{d,\mathbb{Z}}^0$, we define

$$\Omega_i = (\log t)^{-n}\frac{t^{-v(i)}z^i}{z_1 \cdots z_{n+1}}\Omega_o.$$

Note that i belongs to $\triangle_{d,\mathbb{Z}}^0$ implies $i_\alpha \geq 1$ for all $\alpha = 1,\ldots,n+1$. This shows that $t^{-v(i)}z^i/z_1 \cdots z_{n+1}$ is a polynomial of degree less than or equal to $d - 1 - (n+1) = d - (n+2)$ and hence Ω_i is a holomorphic n-form on V_t. It is also clear from the construction that Ω_i, $i \in \triangle_{d,\mathbb{Z}}^0$, form a basis of $H^{n,0}(V_t)$. Explicitly, in the region with $f_{z_\alpha} \neq 0$,

$$
\begin{aligned}
\Omega_i &= (-1)^{\alpha-1}(\log t)^{-n}\frac{t^{-v(i)}z^i}{z_1 \cdots z_{n+1}}\frac{dz_1 \wedge \cdots \widehat{dz_\alpha} \cdots \wedge dz_{n+1}}{f_{z_\alpha}} \\
&= (-1)^{\alpha-1}(\log t)^{-n}\frac{t^{-v(i)}z^i}{\sum_j j_\alpha t^{-v(j)}z^j}\frac{dz_1}{z_1} \wedge \cdots \left(\widehat{\frac{dz_\alpha}{z_\alpha}}\right) \cdots \wedge \frac{dz_{n+1}}{z_{n+1}}.
\end{aligned}
$$

Now each (non-empty) face $\mathfrak{F}(i,j^{(1)})$ of the n-cycle $C_i^\wedge$ of $\mathcal{A}_K$ corresponding to i is given by

$$l_{v,i}(x) = l_{v,j^{(1)}}(x) > l_{v,j}(x), \quad j \neq i,\ j^{(1)},$$

where $j^{(1)} \in (\triangle_{d,\mathbb{Z}} \setminus \{i\})$. Namely, $t^{-v(i)}z^i$ and $t^{-v(j^{(1)})}z^{j^{(1)}}$ are the two maximal terms of $f_t(z)$ determining the face as $t \to +\infty$. Therefore for any compact subset $R \Subset \operatorname{Int}(\mathfrak{F}(i,j^{(1)}))$ the terms $t^{-v(i)}z^i$ and $t^{-v(j^{(1)})}z^{j^{(1)}}$ dominate other terms of f_t in a neighborhood of $\lambda^{-1}(R) \subset V_t \cap (\mathbb{C}^*)^{n+1}$ in $(\mathbb{C}^*)^{n+1}$ as $t \to +\infty$.

For each $\alpha \in \{1, \ldots, n+1\}$, the definition of $f_t(z)$ gives (omitting the subscribe t)

$$
\begin{aligned}
z_\alpha f_{z_\alpha} &= \sum_j j_\alpha t^{-v(j)} z^j \\
&= i_\alpha t^{-v(i)} z^i + j_\alpha^{(1)} t^{-v(j^{(1)})} z^{j^{(1)}} + \cdots \\
&= i_\alpha \left(t^{-v(i)} z^i + t^{-v(j^{(1)})} z^{j^{(1)}} \right) + \left(j_\alpha^{(1)} - i_\alpha \right) t^{-v(j^{(1)})} z^{j^{(1)}} + \cdots \\
&= i_\alpha \left(f + \cdots \right) + \left(j_\alpha^{(1)} - i_\alpha \right) t^{-v(j^{(1)})} z^{j^{(1)}} + \cdots
\end{aligned}
$$

where "$\cdots$" denotes the terms in f, up to multiple of a constant, other than $t^{-v(i)} z^i$ and $t^{-v(j^{(1)})} z^{j^{(1)}}$, and i_α and $j_\alpha^{(1)}$ are the α-components of i and $j^{(1)}$ respectively.

Since $j^{(1)} \neq i$, there is an index $\alpha \in \{1, \ldots, n+1\}$ such that $j_\alpha^{(1)} \neq i_\alpha$. Therefore, for this α and sufficiently large t, $f_{z_\alpha} \neq 0$ in a neighborhood of $\lambda^{-1}(R)$ for any compact subset $R \Subset \mathrm{Int}(\mathfrak{F}(i, j^{(1)}))$ of the interior of the face $\mathfrak{F}(i, j^{(1)})$. Putting the above expression into the definition of Ω_i and using $f = 0$ on V_t, we have

$$
\Omega_i = (\log t)^{-n} \left[\frac{(-1)^{\alpha-1} t^{-v(i)} z^i}{(j_\alpha^{(1)} - i_\alpha) t^{-v(j^{(1)})} z^{j^{(1)}} + \cdots} \right] \frac{dz_1}{z_1} \wedge \cdots \wedge \left(\widehat{\frac{dz_\alpha}{z_\alpha}} \right) \cdots \wedge \frac{dz_{n+1}}{z_{n+1}}.
$$

Let $\xi = H_t(z) = \left(|z_1|^{\frac{1}{\log t}} \frac{z_1}{|z_1|}, \ldots, |z_{n+1}|^{\frac{1}{\log t}} \frac{z_{n+1}}{|z_{n+1}|} \right)$ be the normalization mentioned in (2) of the main theorem. Then $\mathrm{Log}|\xi| = \mathrm{Log}_t|z|$ and

$$
(\log t)^{-1} \frac{dz_\beta}{z_\beta} = \frac{d|\xi_\beta|}{|\xi_\beta|} + \sqrt{-1} (\log t)^{-1} d \arg z_\beta.
$$

Therefore, by using $t^{-v(j^{(1)})} z^{j^{(1)}} + t^{-v(i)} z^i + \cdots = f(z) = 0$, as $t \to +\infty$

$$
\Omega_i \to \left[\frac{(-1)^{\alpha-1}}{i_\alpha - j_\alpha^{(1)}} \right] \frac{d|\xi_1|}{|\xi_1|} \wedge \cdots \wedge \left(\widehat{\frac{d|\xi_\alpha|}{|\xi_\alpha|}} \right) \wedge \cdots \wedge \frac{d|\xi_{n+1}|}{|\xi_{n+1}|}
$$

locally, with respect to the metric induced from the pull-back $H_t^*(g_0)$ of the invariant toric metric $g_0 = \frac{1}{2\sqrt{-1}} \sum |\xi_\alpha|^{-2} d\xi_\alpha \wedge d\bar{\xi}_\alpha$ on $(\mathbb{C}^*)^{n+1}$, in the neighborhood with image sufficiently near R under the map Log_t.

Next we work on a neighborhood of an vertex $x \in C_i'^{\wedge}$. Near this point, there are $(n + 1)$ terms

$$
t^{-v(j^{(1)})} z^{j^{(1)}}, \ldots, t^{-v(j^{(n+1)})} z^{j^{(n+1)}},
$$

of f that are comparable to $t^{-v(i)} z^i$ and dominating other terms as $t \to +\infty$. Denote $j^{(0)} = i$ and $\zeta_p = t^{-v(j^{(p)})} z^{j^{(p)}}$ for $p = 0, \ldots, n+1$. Then

$$
0 = f(z) = \sum_{p=0}^{n+1} \zeta_p + \cdots,
$$

and

$$
\begin{aligned}
z_\alpha f_{z_\alpha} &= \sum_j j_\alpha t^{-v(j)} z^j \\
&= \sum_{p=0}^{n+1} j_\alpha^{(p)} \zeta_p + \cdots \\
&= j_\alpha^{(0)} \left(\sum_{p=0}^{n+1} \zeta_p \right) + \sum_{p=1}^{n+1} (j_\alpha^{(p)} - j_\alpha^{(0)}) \zeta_p + \cdots \\
&= j_\alpha^{(0)} \left(f + \cdots \right) + \sum_{p=1}^{n+1} (j_\alpha^{(p)} - j_\alpha^{(0)}) \zeta_p + \cdots
\end{aligned}
$$

where "$\cdots$" denotes the terms in f, up to multiple of a constant, other than $t^{-v(j^{(p)})} z^{j^{(p)}}$, $p = 0, \ldots, n+1$. Note that for $\alpha = 1, \ldots, n+1$,

$$
z_\alpha = t^{x_\alpha} b_\alpha + \cdots
$$

with certain leading coefficients $b = (b_1, \ldots, b_{n+1})$, where x_α's are the coordinates of the vertex $x \in C_i^\wedge$. Hence

$$
\zeta_p = t^{l_{v,j^{(p)}}(x)} b^{j^{(p)}} + \cdots,
$$

where $l_{v,j^{(p)}}(x) = \langle x, j^{(p)} \rangle - v(j^{(p)})$. We claim that for any b, there exists $\alpha \in \{1, \ldots, n+1\}$ such that

$$
\lim_{t \to +\infty} \sum_{p=1}^{n+1} (j_\alpha^{(p)} - j_\alpha^{(0)}) \frac{\zeta_p}{\zeta_0} \neq 0.
$$

In fact, if it is not true, then by taking $t \to +\infty$, we have for all α,

$$
\lim_{t \to +\infty} \sum_{p=1}^{n+1} (j_\alpha^{(p)} - j_\alpha^{(0)}) \frac{\zeta_p}{\zeta_0} = \sum_{p=1}^{n+1} (j_\alpha^{(p)} - j_\alpha^{(0)}) \frac{b^{j^{(p)}}}{b^{j^{(0)}}} = 0.
$$

By using $0 = f(z) = \sum \zeta_p + \cdots$, we also have

$$
1 + \sum_{p=1}^{n+1} \frac{b^{j^{(p)}}}{b^{j^{(0)}}} = 0.
$$

Then, by comparing with the polynomial $1 + z_1 + \cdots + z_{n+1}$, it is easy to see that $M = (j_\alpha^{(p)} - j_\alpha^{(0)})_{\alpha,p=1,\ldots,n+1}$ is the matrix of the affine transformation that maps the neighborhood of the vertex x to the primitive complex Σ_n. Since Π_v is a maximal dual complex, M is invertible. This implies $\frac{b^{j^{(p)}}}{b^{j^{(0)}}} = 0$ for all p which is a contradiction.

Using the claim, we see that for large t and each z in the neighborhood, $f_{z_\alpha} \neq 0$ for some α. And

$$\Omega_i \to \frac{(-1)^{\alpha-1}b^{j^{(0)}}}{\sum_{p=1}^{n+1}(j_\alpha^{(p)} - j_\alpha^{(0)})b^{j^{(p)}}} \frac{d|\xi_1|}{|\xi_1|} \wedge \cdots \wedge \left(\widehat{\frac{d|\xi_\alpha|}{|\xi_\alpha|}}\right) \cdots \wedge \frac{d|\xi_{n+1}|}{|\xi_{n+1}|}.$$

This shows that Ω_i is non-vanishing near a vertex for large t. Similarly, we can prove that Ω_i is non-vanishing near any face with dimension between 1 and n. This completes the proof of the second statement.

Finally for the last statement of the theorem, we observe that on any compact subset $B \subset \mathbb{CP}^{n+1} \setminus U_{i,t}^\wedge$, $t^{-v(i)}z^i$ is no longer a dominating term near $V_t \cap B$ and hence $\Omega_i \to 0$ locally in B as $t \to +\infty$ by the above local expression of Ω_i. This completes the proof of the theorem.

q.e.d.

Remark: In his preprint [**10**], Mikhalkin defined a version of "regular 1-form" on tropical curves. From the proof of the theorem 3.1, we see that in this case $(n = 1)$, for each $i \in \triangle_{d,\mathbb{Z}}^0$, the holomorphic 1-form $\Omega_{i,t}$ tends to a limit of the form

$$\frac{(-1)^{\alpha-1}}{i_\alpha - j_\alpha^{(1)}} dx_1 \wedge \cdots \wedge \widehat{dx_\alpha} \wedge \cdots \wedge dx_2,$$

where $x_\alpha = \lim_{t\to+\infty} \text{Log}_t|z| = \lim_{t\to+\infty} \text{Log}|\xi|$. And this limit is in fact a "regular 1-form" on the tropical variety $\mathcal{A}_K(V) = \Pi_v$ in the sense of Mikhalkin and dual to the 1-cycle $C_i^\wedge = \lim_{t\to\infty} \text{Log}_t(U_{i,t}^\wedge)$.

Note that the set of n-cycles $\{C_i^\wedge\}_{i\in\triangle_{d,\mathbb{Z}}^0}$, constructed in the theorem 3.1 does not cover Π_v. Hence $\{U_{i,t}^\wedge\}_{i\in\triangle_{d,\mathbb{Z}}^0}$ cannot cover V_t. In order to obtain an open covering $\{U_{i,t}\}_{i\in\triangle_{d,\mathbb{Z}}^0}$ of V_t, we need to enlarge $U_{i,t}^\wedge$ suitably. In the case of our interest, we have the following

Theorem 3.2. *Let $v : \triangle_{d,\mathbb{Z}} \to \mathbb{R}$ be a function satisfying the condition of lemma 2.5, $f_t = \sum_{j\in\triangle_{d,\mathbb{Z}}} t^{-v(j)}z^j$ $(t > 0)$ be the patchworking polynomial of degree d defined by v with non-Archimedean amoeba $\mathcal{A}_K = \Pi_v$. Denote $V_t = \{f_t = 0\} \subset \mathbb{CP}^{n+1}$. Then for all $t > 0$, there exists a basis $\{\Omega_{i,t}\}_{i\in\triangle_{d,\mathbb{Z}}^0}$ of $H^{n,0}(V_t)$, and open subsets $U_{i,t} \subset V_t$ such that for each $i \in \triangle_{d,\mathbb{Z}}^0$,*

1) *$\text{Log}_t(U_{i,t})$ tends to an n-cycle C_i such that $\Pi_v = \bigcup_{i\in\triangle_{d,\mathbb{Z}}^0} C_i$ and $\{[C_i]\}_{i\in\triangle_{d,\mathbb{Z}}^0}$ forms a basis of $H_n(\mathcal{A}_K(V))$,*

2) *for any compact subset $B \subset \mathbb{CP}^{n+1} \setminus U_{i,t}$, $\Omega_{i,t}$ tends to zero in $V_t^o \cap B$ uniformly with respect to $H_t^*(g_0)$.*

Proof. By the pair-of-pants decomposition theorem 2.4, V_t is a union of pairs-of-pants $\overline{\mathcal{P}}_n$'s. And the set of pair-of-pants $\overline{\mathcal{P}}_n$ are in one-one

correspondence to the set of vertices of Π_v. Hence each $\overline{\mathcal{P}}_n$ corresponds to a simplex in the lattice subdivision of $\triangle_d$ and vice versa. Moreover, if the pair-of-pants $\overline{\mathcal{P}}_n(s)$ corresponds to the simplex σ_s of the lattice subdivision of $\triangle_d$, then $\overline{\mathcal{P}}_n(s)$ is the closure of the preimage $\lambda^{-1}(\mathcal{U}_s)$ of the primitive piece $\mathcal{U}_s$ dual to the simplex σ_s.

Now, for each $i \in \triangle_{d,\mathbb{Z}}^0$, conditions of lemma 2.5 says that the lattice subdivision of $\triangle_d$ corresponding to v restricts to a lattice subdivision of $i - \iota + \triangle_{n+2}$. Let $\{\sigma_s\}$ be the set of simplices of the lattice subdivision of $i - \iota + \triangle_{n+2}$. Let Λ_i be the set of s such that for all $j \in \triangle_{d,\mathbb{Z}}^0 \setminus \{i\}$, j is not a vertex of σ_s. We define

$$C_i = C_i^\wedge \bigcup \left(\bigcup_{s \in \Lambda_i} \overline{\mathcal{U}_s} \right),$$

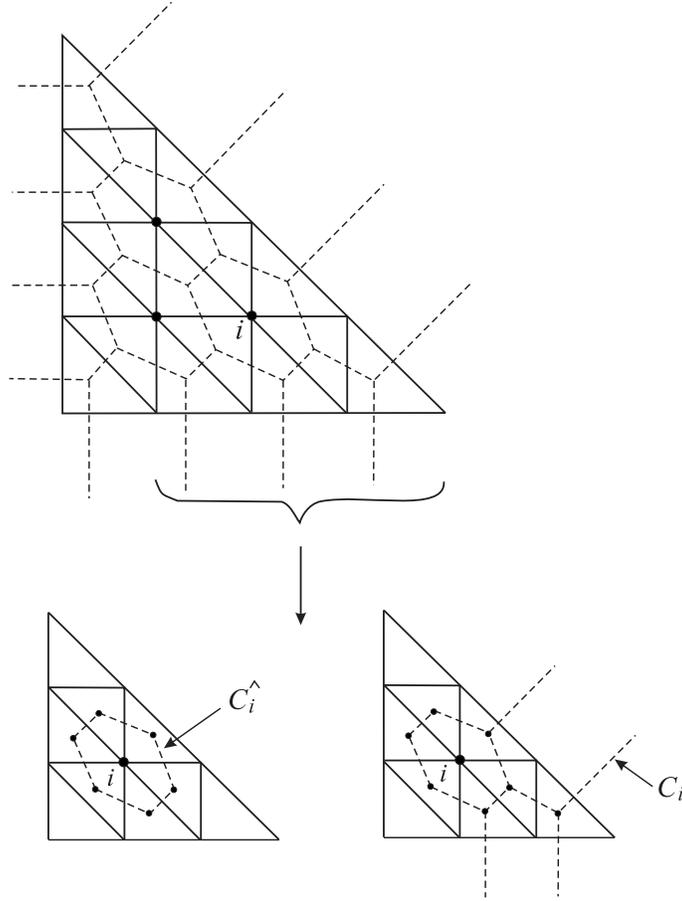where $\mathcal{U}_s$ is the primitive piece dual to the simplex σ_s.

Figure 7: Illustration of the sets $C_i^\wedge$ and C_i.

Then we clearly have

$$\Pi_v = \bigcup_{i \in \triangle_{d,\mathbb{Z}}^0} C_i$$

and $[C_i] = [C_i^\wedge]$ in $H_n(\Pi_v)$. For these C_i, we define their $1/t$-neighborhood $C_{i,t}$ in Π_v similar to those for $C_{i,t}^\wedge$ and define the preimages to be our enlarged open sets

$$U_{i,t} = \lambda^{-1}\left(C_{i,t}\right).$$

Then it is clear that $U_{i,t} \supset U_{i,t}^\wedge$ for all $i \in \triangle_{d,\mathbb{Z}}^0$ and

$$V_t = \bigcup_{i \in \triangle_{d,\mathbb{Z}}^0} U_{i,t}.$$

Finally, as $U_{i,t}^\wedge \subset U_{i,t}$, the second statement of the theorem follows trivially from the corresponding statement in theorem 3.1. The proof is completed.   q.e.d.
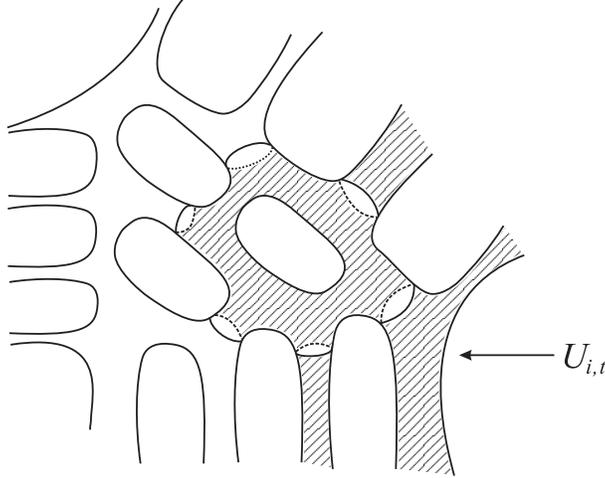

Figure 8: Illustration of an open set $U_{i,t}$.

Proof of the main theorem. By theorems 3.1 and 3.2, we remain to show the following

1) $U_{i,t}$ is close to an open subset of a Calabi-Yau hypersurface $Y_{i,t}$ after normalization H_t,
2) $\Omega_{i,t}$ is close to a scalar multiple of the unique holomorphic volume form $\Omega_{Y_{i,t}}$ of $Y_{i,t}$ on $U_{i,t}$,
3) $\Omega_{i,t}$ is non-vanishing on the whole $U_{i,t}$.

Note that the last item is needed because we have no information of $\Omega_{i,t}$ at the points in $U_{i,t} \cap \left(\mathbb{CP}^{n+1} \setminus (\mathbb{C}^*)^{n+1}\right)$ and the theorem 3.1 shows that $\Omega_{i,t}$ tends to 0 as $t \to +\infty$ in $U_{i,t} \setminus U_{i,t}^\wedge$. We need to show that even

they are tending to 0, $\Omega_{i,t}$ is still non-vanishing on the whole $U_{i,t}$ for large but finite t.

To complete the proof of the main theorem, we take v to be the function given in the lemma 2.5, $f_t = \sum\limits_{j \in \triangle_{d,\mathbb{Z}}} t^{-v(j)} z^j$ be the patchworking polynomial defined by v and $V_t = \{f_t = 0\}$. Recall that the lemma 2.5 implies that the subdivision corresponding to v restricts to a lattice subdivision of $i - \imath + \triangle_{n+2}$. Consider the truncated polynomial $f_{i,t} = \sum\limits_{j \in i - \imath + \triangle_{n+2}} t^{-v(j)} z^j$. This truncated polynomial $f_{i,t}$ factorizes as

$$f_{i,t} = z^{i-\imath} \sum_{j \in \triangle_{n+2}} t^{-v(j+i-\imath)} z^j.$$

So the variety $\{f_{i,t} = 0\} \cap (\mathbb{C}^*)^{n+1}$ can be regarded as an open subset of the Calabi-Yau variety $Y_{i,t}$ defined by the polynomial $f_{i,t}^{CY} = \sum\limits_{j \in \triangle_{n+2}} t^{-v(j+i-\imath)} z^j$.

In [9], it was shown that the normalized varieties $H_t(V_t)$ converge in the Hausdorff metric to the lift $W(\mathcal{A}_K)$ of the corresponding non-Archimedean amoeba $\mathcal{A}_K$, where the lift $W(\mathcal{A}_K)$ is the image of $\mathcal{A}_K$ under the map $W : (K^*)^{n+1} \to (\mathbb{C}^*)^{n+1}$ defined as

$$W(b_1, \ldots, b_{n+1}) = \left(e^{-val_K(b_1) + i \arg(b_1^m)}, \ldots, e^{-val_K(b_{n+1}) + i \arg(b_{n+1}^m)} \right),$$

where $b_\alpha^m \in \mathbb{C}$ is the coefficient of $t^{-val_K(b_\alpha)}$, i.e, the leading coefficient of b_α. For simplicity, we call $\arg(b_1^m), \ldots, \arg(b_{n+1}^m)$ the leading arguments of $b \in K$. It was shown that $W(\mathcal{A}_K)$ depends only on the leading arguments of the coefficients of the defining polynomial f.

We can define similarly the lift $W(C_{i,t})$ for each $i \in \triangle_{d,\mathbb{Z}}^0$. Then the proof in [9] applies directly to show that $W(C_{i,t})$ depends only on the leading arguments of the coefficients of the terms that determines $C_{i,t}$ and the normalized open set $H_t(U_{i,t})$ is close to $W(C_{i,t})$. Since f_t and the truncated polynomial $f_{i,t}$ have the same coefficients corresponding to $C_{i,t}$, we see immediately that $H_t(U_{i,t})$ is close to $H_t(U_{i,t}^{CY})$ in Hausdorff distance, where $U_{i,t}^{CY}$ is the preimage of the neighborhood $C_{i,t}^{CY}$ corresponding to $C_{i,t}$ in the lattice subdivision of $i - \imath + \triangle_{n+2}$. Therefore, $U_{i,t}$ is close to an open set of a Calabi-Yau manifold $Y_{i,t}$ after normalization H_t. This proves the first statement.

To see the other two statements, we observe that the limiting behavior of $\Omega_{i,t}$ shows that $\Omega_{i,t}$ is close to the corresponding holomorphic n-form $\Omega_{Y_{i,t}}$ of the Calabi-Yau hypersurface $\{f_{i,t}^{CY} = 0\} = Y_{i,t} \subset \mathbb{CP}^{n+1}$. In

fact, since $f_{i,t}^{CY} = 0$ on the hypersurface $\{f_{i,t} = 0\}$, we have

$$
\begin{aligned}
\Omega_{Y_{i,t}} &= (\log t)^{-n} \left[\frac{(-1)^{\alpha-1} t^{-v(i)} z^1}{z_\alpha (f_{i,t}^{CY})_{z_\alpha}} \right] \frac{dz_1}{z_1} \wedge \cdots \widehat{\left(\frac{dz_\alpha}{z_\alpha}\right)} \cdots \wedge \frac{dz_{n+1}}{z_{n+1}} \\
&= (\log t)^{-n} \left[\frac{(-1)^{\alpha-1} t^{-v(i)} z^i}{z_\alpha z^{i-1} (f_{i,t}^{CY})_{z_\alpha}} \right] \frac{dz_1}{z_1} \wedge \cdots \widehat{\left(\frac{dz_\alpha}{z_\alpha}\right)} \cdots \wedge \frac{dz_{n+1}}{z_{n+1}} \\
&= (\log t)^{-n} \left[\frac{(-1)^{\alpha-1} t^{-v(i)} z^i}{z_\alpha (z^{i-1} f_{i,t}^{CY})_{z_\alpha}} \right] \frac{dz_1}{z_1} \wedge \cdots \widehat{\left(\frac{dz_\alpha}{z_\alpha}\right)} \cdots \wedge \frac{dz_{n+1}}{z_{n+1}} \\
&= (\log t)^{-n} \left[\frac{(-1)^{\alpha-1} t^{-v(i)} z^i}{z_\alpha (f_{i,t})_{z_\alpha}} \right] \frac{dz_1}{z_1} \wedge \cdots \widehat{\left(\frac{dz_\alpha}{z_\alpha}\right)} \cdots \wedge \frac{dz_{n+1}}{z_{n+1}}.
\end{aligned}
$$

Therefore using the fact that f and $f_{i,t}$ contain the same dominating terms on $U_{i,t}$, we see that $\Omega_{i,t}$ is close to $\Omega_{Y_{i,t}}$ in the sense described in theorem 3.1.

In this local coordinate, the ratio

$$
\frac{\Omega_{i,t}}{\Omega_{Y_{i,t}}} = \frac{z_\alpha (f_{i,t})_{z_\alpha}}{z_\alpha f_{z_\alpha}}
$$

is a holomorphic function. Note that $f_{i,t}$ and f contain the same dominating terms in the neighborhoods corresponding to faces of $C_{i,t}$ and $C_{i,t}^{CY}$ respectively. In a neighborhood of $U_{i,t} \cap \left(\mathbb{CP}^{n+1} \setminus (\mathbb{C}^*)^{n+1} \right)$, we consider those open subsets correspond to top dimensional faces. In these open subsets, the polynomial f and $f_{i,t}$ are dominated by exactly two terms $t^{j^1} z^{j^{(1)}}$ and $t^{j^2} z^{j^{(2)}}$. Then

$$
\begin{aligned}
\frac{\Omega_{i,t}}{\Omega_{Y_{i,t}}} &= \frac{z_\alpha (f_{i,t})_{z_\alpha}}{z_\alpha f_{z_\alpha}} \\
&= \frac{z_\alpha \left(t^{j^1} z^{j^{(1)}} + t^{j^2} z^{j^{(2)}} + \cdots \right)_{z_\alpha}}{z_\alpha \left(t^{j^1} z^{j^{(1)}} + t^{j^2} z^{j^{(2)}} + \cdots \right)_{z_\alpha} + z_\alpha (f - f_{i,t})_{z_\alpha}} \\
&= \frac{j_\alpha^{(1)} t^{j^1} z^{j^{(1)}} + j_\alpha^{(2)} t^{j^2} z^{j^{(2)}} + \cdots}{j_\alpha^{(1)} t^{j^1} z^{j^{(1)}} + j_\alpha^{(2)} t^{j^2} z^{j^{(2)}} + \cdots} \\
&= \frac{\left(j_\alpha^{(1)} - j_\alpha^{(2)} \right) t^{j^1} z^{j^{(1)}} + j_\alpha^{(2)} (f_{i,t} + \cdots) + \cdots}{\left(j_\alpha^{(1)} - j_\alpha^{(2)} \right) t^{j^1} z^{j^{(1)}} + j_\alpha^{(2)} (f + \cdots) + \cdots}.
\end{aligned}
$$

As before, "$\cdots$" means a linear combination of the terms other than $t^{j^1} z^{j^{(1)}}$ and $t^{j^2} z^{j^{(2)}}$. Therefore $\Omega_{i,t}/\Omega_{Y_{i,t}}$ is non-vanishing in these open subsets. Taking closure of these open subsets in the neighborhood, we conclude that $\Omega_{i,t}/\Omega_{Y_{i,t}}$ is non-vanishing on $U_{i,t} \cap \left(\mathbb{CP}^{n+1} \setminus (\mathbb{C}^*)^{n+1} \right)$. Since $\Omega_{Y_{i,t}}$ is the holomorphic volume of the Calabi-Yau hypersurface

$Y_{i,t}$, it is non-vanishing and hence $\Omega_{i,t}$ is also non-vanishing on the whole $U_{i,t}$.

Finally, as $U_{i,t} \supset U_{i,t}^\wedge$, the last statement follows immediately from the last statement of the theorem 3.1 This completes the proof of the main theorem.

3.1. Asymptotically special Lagrangian fibers. From the fibration λ given in [**9**], for each n-cell e of Π_v, there exists a point $x \in e$ such that the fiber $\lambda^{-1}(x)$ is a Lagrangian n-torus $\mathbb{T}^n \subset V$ which is actually given by $\{z \in V_t : \mathrm{Log}_t|z| = x\}$. Therefore, when restricted to this fiber

$$\Omega_i|_{\lambda^{-1}(x)} = \left(\frac{\sqrt{-1}}{\log t}\right)^n \left[\frac{(-1)^{\alpha-1}t^{-v(i)}z^i}{(i_\alpha - j_\alpha^{(1)})t^{-v(i)}z^i + \cdots}\right] d\theta_1 \wedge \cdots \widehat{d\theta_\alpha} \cdots \wedge d\theta_{n+1}.$$

where $\theta_\alpha = \arg z_\alpha$ for $\alpha = 1, \ldots, n+1$. So we have

Theorem 3.3. *Let V_t be the family of smooth hypersurfaces in $\mathbb{CP}^{n+1}$ of degree d, $U_{i,t}$ be the open sets and $\Omega_{i,t}$ be the holomorphic n-forms in the main theorem. Let $\lambda : V_t \to \Pi_v$ be the stratified $\mathbb{T}^n$-fibration given by Mikhalkin [**9**]. Then for any $i \in \triangle_{d,\mathbb{Z}}^0$ and any n-cell e of $C_{i,t}^\wedge$ used in the proof of the main theorem, there exists $x \in e$, independent of t, such that for all $j \in \triangle_{d,\mathbb{Z}}^0$,*

$$\lim_{t \to +\infty} \mathrm{Im}\left(e^{\frac{n\pi\sqrt{-1}}{2}}\Omega_{j,t}\right)\Big|_{\lambda^{-1}(x)} = 0 \quad and$$

$$\lim_{t \to +\infty} \mathrm{Re}\left(e^{\frac{n\pi\sqrt{-1}}{2}}\Omega_{j,t}\right)\Big|_{\lambda^{-1}(x)} \quad is\ non\text{-}vanishing.$$

In particular, the Lagrangian fibers given by Mikhalkin are in fact "asymptotically special Lagrangian of phase $n\pi/2$" with respect to the holomorphic n-form $\Omega_{i,t}$ constructed in the main theorem.

3.2. Hypersurfaces in other toric varieties; the case of curves. It is clear from the works of Mikhalkin [**9**], our result can be modified to include other toric varieties such as $\mathbb{CP}^m \times \mathbb{CP}^n$. In particular, if we apply our method to curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ instead of $\mathbb{CP}^2$, we will obtain stronger results for Riemann surfaces. It applies to Riemann surfaces of all genus, not just for $g = \frac{(d-1)(d-2)}{2}$. And the Calabi-Yau components actually form a connected sum decomposition of the curve. This is a fact which is probably not true in higher dimensions. The key issue is that in this case, one can obtain a subdivision with dual complex Π_v with 1-cycles $\{C_i\}$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$. In summary, we have

Theorem 3.4. *For any integer $g \geq 1$, there is a family of smooth genus g curves V_t of bi-degree $(g+1, 2)$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, such that V_t can be written as*

$$V_t = \bigcup_{i=1}^{g} U_{i,t}$$

where $\{U_{i,t}\}$ is a family of closed subsets $U_{i,t} \subset V_t$ such that topologically $V_t = U_{1,t}\#\cdots\#U_{g,t}$, the connected sum of $U_{i,t}$, $i = 1,\ldots,g$ and after normalization $H_t : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ defined by

$$H_t(z_1, z_2) = \left(|z_1|^{\frac{1}{\log t}} \frac{z_1}{|z_1|}, |z_2|^{\frac{1}{\log t}} \frac{z_2}{|z_2|}\right),$$

1) *$U_{i,t}$ is close in Hausdorff distance on $(\mathbb{C}^*)^2$ to an open subset of an elliptic curves $Y_{i,t}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$;*
2) *there exists a basis $\{\Omega_{i,t}\}_{i=1}^g$ of $H^{1,0}(V_t)$ such that for each $i = 1,\ldots,g$, $\Omega_{i,t}$ is nonvanishing and ϵ-closed to the holomorphic 1-form $\Omega_{Y_{i,t}}$ of $Y_{i,t}$ on $U_{i,t}$ with respect to the metric induced from the pull-back $H_t^*(g_0)$;*
3) *for any compact subset $B \subset (\mathbb{C}^*)^2 \setminus U_{i,t} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$, $\Omega_{i,t}$ tends to zero in $V_t \cap B$ uniformly with respect to $H_t^*(g_0)$.*

Proof. All the steps in the proof of the main theorem apply and give all results concerning the family $U_{i,t}$ except the assertion about the connected-sum. To see this, we need the existence of certain maximal lattice subdivision of the Newton polytope $\triangle = [0, g + 1] \times [0, 2]$ of a generic curves of bi-degree $(g + 1, 2)$ such that the set of vertices $\text{ver}(\sigma)$ of each simplex σ contains at most one interior lattice points of $\triangle$. For this kind of subdivisions, each primitive piece associated to at most one interior lattice point. Then any two cycles $C_i^\wedge$ and $C_j^\wedge$ must have empty intersection for $i \neq j$. As demonstrated in the following figure, the existence of such subdivision is easy to see. In fact, one shows that there exists a function $v : \triangle_{\mathbb{Z}} \to \mathbb{R}$ such that Π_v is a maximal dual complex of $\triangle$ giving the required subdivision.
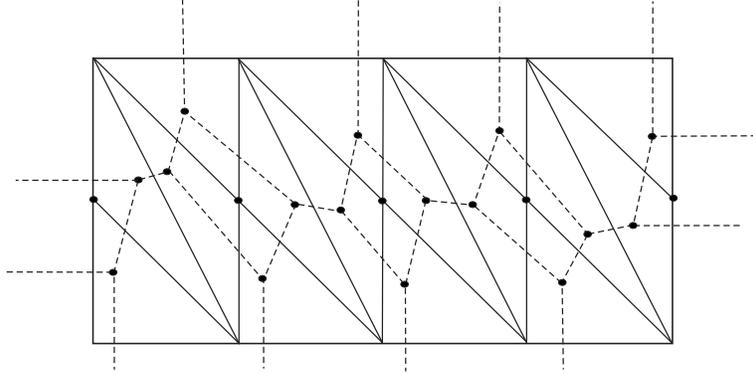

Figure 9: A maximal dual $\triangle$ complex with the required properties.

As in the proof of the main theorem, we can construct open subsets $U_{i,t}$ and holomorphic 1-form $\Omega_{i,t}$ corresponding to each cycles C_i, where $i = 1,\ldots,g$ is in one-one corresponding with the set of interior lattice

points of $\triangle = [0, g+1] \times [0, 2]$. This gives all the assertions except the part of the connected sum.

Note that in this case, two cycles $C_i^{\wedge}$ and $C_j^{\wedge}$ of Π_v corresponding to different interior lattice points of $\triangle$ do not intersect. In fact, they are, at least, separated by an edge e of Π_v. Therefore, either $C_i \cap C_j = \emptyset$ or $C_i \cap C_j = \{x\}$ where $x \in e$ is the common boundary of the corresponding primitive pieces. It is now clear that V_t are just all $U_{i,t}$ gluing along the circles $\lambda^{-1}(x)$ of these x's. So $V_t = U_{1,t} \# \cdots \# U_{g,t}$ and proof is completed. $\hspace{4cm}$ q.e.d.

We would like to remark that in this complex one dimensional case, special Lagrangian submanifolds of $\Omega_{i,t}$ always exists. Namely, they are given by the horizontal or vertical trajectories of the quadratic differential $\Omega_{i,t}^2$. Therefore, we have

Theorem 3.5. *Let V_t be the family of smooth curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of degree $g+1$, $\Omega_{i,t}$ be the holomorphic 1-forms in the theorem 3.4. Then for any $i = 1, \ldots, g$, there exists special Lagrangain fibration of $\Omega_{i,t}$.*

4. Proof of the key lemma

In this section, we prove the lemma 2.5 concerning the dual complex Π_v given by the function $v : \triangle_{d,\mathbb{Z}} \to \mathbb{R}$ defined by

$$v(j) = \sum_{\alpha=1}^{n+1} j_\alpha^2 + \left(\sum_{\alpha=1}^{n+1} j_\alpha \right)^2$$

for $j = (j_1, \ldots, j_{n+1}) \in \triangle_{d,\mathbb{Z}}$. To simplify notation, we will write $j_0 = \sum_{\alpha=1}^{n+1} j_\alpha$. Then $v(j) = \sum_{\alpha=0}^{n+1} j_\alpha^2$. We start with

Lemma 4.1. *Let $i \in \triangle_{d,\mathbb{Z}}$ and $j \in \mathbb{Z}^{n+1} \setminus \left(\{0\} \cup \{\pm e_\alpha\}_{\alpha=1}^{n+1} \cup \{e_\alpha - e_\beta\}_{\alpha \neq \beta} \right)$ such that $i + j \in \triangle_{d,\mathbb{Z}}$. Then there exists no $x \in \mathbb{R}^{n+1}$ such that*

$$
\begin{aligned}
\langle i, x \rangle - v(i) &= \langle i+j, x \rangle - v(i+j) \\
&> \langle r, x \rangle - v(r), \quad r \in \triangle_{d,\mathbb{Z}} \setminus \{i, i+j\}.
\end{aligned}
$$

Proof. We separate the proof into several steps.

Step 1: If i is an interior lattice point of $\triangle_d$, then for all $x \in \mathbb{R}^{n+1}$ satisfying the condition in the lemma, we have

$$
\begin{cases}
|x_\alpha - 2(i_\alpha + i_0)| < 2, & \forall \alpha = 1, \ldots, n+1 \\
|x_\alpha - x_\beta - 2(i_\alpha - i_\beta)| < 2, & \forall \alpha \neq \beta = 1, \ldots, n+1.
\end{cases}
$$

Proof of Step 1: Since $i \in \triangle_{d,\mathbb{Z}}^0$, the points $i \pm e_\alpha$, $i + e_\beta - e_\alpha$ belong to $\triangle_{d,\mathbb{Z}}$, for all α and $\beta \neq \alpha$. By assumption, $j \neq \pm e_\alpha$, $e_i - e_j$, we have

the following strict inequalities from the condition of the lemma

$$\begin{cases} \langle i \pm e_\alpha, x \rangle - v(i \pm e_\alpha) & < \langle i, x \rangle - v(i) \\ \langle i + e_\beta - e_\alpha, x \rangle - v(i + e_\beta - e_\alpha) & < \langle i, x \rangle - v(i). \end{cases}$$

Using the definition of v, we have

$$\begin{cases} \pm x_\alpha - [(\pm 2i_\alpha + 1) + (\pm 2i_0 + 1)] & < 0 \\ x_\beta - x_\alpha - [(2i_\beta + 1) + (-2i_\alpha + 1)] & < 0. \end{cases}$$

Interchanging the β and α gives the required inequalities

$$\begin{cases} |x_\alpha - 2(i_\alpha + i_0)| < 2, & \forall\, \alpha = 1, \ldots, n+1 \\ |x_\beta - x_\alpha - 2(i_\beta - i_\alpha)| < 2, & \forall\, \beta \neq \alpha = 1, \ldots, n+1. \end{cases}$$

Step 2: Either i or $i + j$ belong to boundary of $\triangle_d$.
Proof of Step 2: Suppose not, then both i and $i + j \in \triangle_{d,\mathbb{Z}}^0$. Applying Step 1 to i and $i + j$, we have the following inequalities

$$\begin{cases} |x_\alpha - 2(i_\alpha + i_0)| < 2, & \forall\, \alpha = 1, \ldots, n+1 \\ |x_\beta - x_\alpha - 2(i_\beta - i_\alpha)| < 2, & \forall\, \beta \neq \alpha = 1, \ldots, n+1. \end{cases}$$

and

$$\begin{cases} |x_\alpha - 2(i_\alpha + j_\alpha + i_0 + j_0)| < 2, & \forall\, \alpha = 1, \ldots, n+1 \\ |x_\beta - x_\alpha - 2(i_\beta + j_\beta - i_\alpha - j_\alpha)| < 2, & \forall\, \beta \neq \alpha = 1, \ldots, n+1. \end{cases}$$

Therefore,

$$\begin{cases} |j_\alpha + j_0| < 2, & \forall\, \alpha = 1, \ldots, n+1 \\ |j_\beta - j_\alpha| < 2, & \forall\, \beta \neq \alpha = 1, \ldots, n+1. \end{cases}$$

As j_β are integers, the first inequality above implies for all $\alpha = 1, \ldots, n+1$,

$$-1 \leq j_\alpha + j_0 \leq 1.$$

Recalling $j_0 = \sum_{\beta=1}^{n+1} j_\beta$ and summing over α, we have

$$-(n+1) \leq (n+2)j_0 \leq n+1.$$

Hence $j_0 = \sum_{\beta=1}^{n+1} j_\beta = 0$. On the other hand, the second inequality above implies that either $j_\beta \geq 0$ for all $\beta = 1, \ldots, n+1$ or $j_\beta \leq 0$ for all $\beta = 1, \ldots, n+1$. Therefore, we must have $j = 0$ which is a contradiction. Step 2 is proved.

Step 3: Both i and $i + j$ belong to the boundary of $\triangle_d$.

Proof of Step 3: Suppose this is not true. We may assume i belongs to the interior and $i+j$ belongs to the boundary. Then we can apply Step 1 to i and get

$$\begin{cases} |x_\alpha - 2(i_\alpha + i_0)| < 2, & \forall\, \alpha = 1, \ldots, n+1 \\ |x_\beta - x_\alpha - 2(i_\beta - i_\alpha)| < 2, & \forall\, \beta \neq \alpha = 1, \ldots, n+1. \end{cases}$$

However, we do not have all the inequalities as at least one of the points $i+j \pm e_\alpha$, $i+j+e_\beta - e_\alpha$ lies outside $\triangle_d$.

Case 1: $i_0 + j_0 = \sum_{\beta=1}^{n+1}(i_\beta + j_\beta) \leq d - 1$.

In this case, we still have

$$\begin{cases} |x_\alpha - 2(i_\alpha + j_\alpha + i_0 + j_0)| < 2, & \forall\, \alpha,\ i_\alpha + j_\alpha \geq 1 \\ |x_\beta - x_\alpha - 2(i_\beta + j_\beta - i_\alpha - j_\alpha)| < 2, & \forall\, \beta \neq \alpha,\ i_\beta + j_\beta,\ i_\alpha + j_\alpha \geq 1. \end{cases}$$

And for those α with $i_\alpha + j_\alpha = 0$, we only have the one-sided inequality

$$x_\alpha - 2(i_\alpha + j_\alpha + i_0 + j_0) < 2.$$

Hence for α with $i_\alpha + j_\alpha \geq 1$, we still have

$$|j_\alpha + j_0| < 2.$$

However, for α with $i_\alpha + j_\alpha = 0$, we only have

$$-2 < i_\alpha + j_0.$$

Since $i + j \in \partial\triangle_d$ and we are assuming $i_0 + j_0 = \sum_{\beta=1}^{n+1}(i_\beta + j_\beta) \leq d - 1$

in this case, there exists α_o such that $i_{\alpha_o} + j_{\alpha_o} = 0$. For this α_o, we get

$$-2 < j_{\alpha_o} + j_0 = -i_{\alpha_o} + j_0.$$

As $i \in \triangle_{d,\mathbb{Z}}^0$, we have $i_{\alpha_o} \geq 1$, and hence $j_0 \geq 0$.

If $i + j \neq 0$, then there also exists α such that $i_\alpha + j_\alpha \geq 1$. For all these α, $i + j + e_{\alpha_o} - e_\alpha \in \triangle_{d,\mathbb{Z}}$ and we can apply the condition of the lemma to get the strict inequality

$$\langle i + j + e_{\alpha_o} - e_\alpha, x \rangle - v(i + j + e_{\alpha_o} - e_\alpha) < \langle i + j, x \rangle - v(i + j).$$

Using $i_{\alpha_o} + j_{\alpha_o} = 0$, this gives

$$x_{\alpha_o} - x_\alpha + 2(i_\alpha + j_\alpha) < 2.$$

Together with $|x_{\alpha_o} - x_\alpha - 2(i_{\alpha_o} - i_\alpha)| < 2$, we arrive at

$$-2 < 2\left(1 - (i_\alpha + j_\alpha)\right) - 2(i_{\alpha_o} - i_\alpha),$$

which implies

$$j_\alpha < 2 - i_{\alpha_o} \leq 1.$$

Therefore, $i_\alpha \leq 0$ for those α with $i_\alpha + j_\alpha \geq 1$. Putting these into $0 \leq j_0 = \sum_{\beta=1}^{n+1} j_\beta$, we have

$$0 \leq \sum_{\{\alpha_o : i_{\alpha_o}+j_{\alpha_o}=0\}} j_{\alpha_o} = - \sum_{\{\alpha_o : i_{\alpha_o}+j_{\alpha_o}=0\}} i_{\alpha_o} < 0,$$

which is a contradiction. So we must have $i + j = 0$, that is $i_\alpha + j_\alpha = 0$ for all α. Then for all α, $i + j + e_\alpha = e_\alpha \in \triangle_{d,\mathbb{Z}}$. The condition of the lemma implies

$$\langle i + j + e_\alpha, x \rangle - v(i + j + e_\alpha) < \langle i + j, x \rangle - v(i + j).$$

So $x_\alpha < v(e_\alpha) = 2$ for all α. On the other hand, the condition of the lemma gives the equality

$$\langle i + j, x \rangle - v(i + j) = \langle i, x \rangle - v(i),$$

which is

$$\sum_{\beta=0}^{n+1} i_\beta^2 = \langle i, x \rangle.$$

Using $x_\alpha < 2$ and $i_\alpha \geq 1$, we have

$$\sum_{\beta=0}^{n+1} i_\beta^2 < 2 \sum_{\beta=1}^{n+1} i_\beta = 2i_0.$$

Hence,

$$\sum_{\beta=1}^{n+1} i_\beta^2 + (i_0 - 1)^2 < 1.$$

This is a contradiction as $i_\alpha \geq 1$ for all α. So we have proved that the case 1 with assumption $i_0 + j_0 \leq d - 1$ is impossible and hence we must be in the situation of the following

Case 2: $i_0 + j_0 = d$.

In this case, for any α with $i_\alpha + j_\alpha \geq 1$, we have $i + j - e_\alpha \in \triangle_{d,\mathbb{Z}}$. Therefore, the following strict inequality is satisfied

$$\langle i + j - e_\alpha, x \rangle - v(i + j - e_\alpha) < \langle i + j, x \rangle - v(i + j).$$

Using $i_0 + j_0 = d$, we get

$$-2 < x_\alpha - 2(i_\alpha + j_\alpha + d)$$

provided $i_\alpha + j_\alpha \geq 1$. One also have $i + j + e_\gamma - e_\alpha \in \triangle_{d,\mathbb{Z}}$ for any $\gamma \neq \alpha$. Similar argument implies

$$x_\gamma - x_\alpha - 2(i_\gamma + j_\gamma - i_\alpha - j_\alpha) < 2$$

provided $i_\alpha + j_\alpha \geq 1$ and $\gamma \neq \alpha$.

The second inequality together with

$$|x_\gamma - x_\alpha - 2(i_\gamma - i_\alpha)| < 2$$

imply
$$-2 < x_\gamma - x_\alpha - 2(i_\gamma - i_\alpha) < 2 + 2(j_\gamma - j_\alpha).$$
That is
$$j_\alpha < 2 + j_\gamma$$
for α with $i_\alpha + j_\alpha \geq 1$ and $\gamma \neq \alpha$.

Suppose there is an γ_o such that $i_{\gamma_o} + j_{\gamma_o} = 0$, then $j_{\gamma_o} \neq j$ and hence
$$j_\alpha < 2 + j_{\gamma_o} = 2 - i_{\gamma_o} < 1$$
since $i \in \triangle^0_{d,\mathbb{Z}}$. So $j_\alpha \leq 0$ for any α with $i_\alpha + j_\alpha \geq 1$. It is trivial that $j_\alpha \leq 0$ for those α with $i_\alpha + j_\alpha = 0$, we have $j_\alpha \leq 0$ for all α which in turn implies that $i_0 = d - j_0 = d - \sum_{\beta=1}^{n+1} j_\beta \geq d$. This is impossible as i is in the interior of $\triangle_d$. Therefore, we cannot have γ_o such that $i_{\gamma_o} + j_{\gamma_o} = 0$. That is, $i_\alpha + j_\alpha \geq 1$ for all α.

Now the other two inequalities must be satisfied, i.e.
$$|x_\alpha - 2(i_\alpha + i_0)| < 2;$$
$$-2 < x_\alpha - 2(i_\alpha + j_\alpha + d).$$
Combining these, we have
$$-2 < x_\alpha - 2i_\alpha - 2j_\alpha - 2d < 2 + 2i_0 - 2j_\alpha - 2d.$$
Hence
$$j_\alpha + d < 2 + i_0 = 2 + d - j_0,$$
i.e.
$$j_\alpha + j_0 < 2.$$
Summing over α, we have $(n+2)j_0 < 2(n+1)$ which implies $j_0 \leq 1$. As $i_0 + j_0 = d$ and $i_0 \leq d - 1$, we see that $j_0 = 1$ (and $i_0 = d - 1$). Putting it back into the inequality, we have $j_\alpha < 1$, hence, $j_\alpha \leq 0$ for all α. Summing over to get the contradiction that $j_0 \leq 0$. This proved the Case 2 and the proof of Step 3 is completed.

Step 4: Either $j \in \{y \in \mathbb{R}^{n+1} : y_\beta = 0\} \cap \partial\triangle_d \cap \mathbb{Z}^{n+1}$ for some β and $i + j \in \{y \in \mathbb{R}^{n+1} : y_\alpha = 0\} \cap \partial\triangle_d \cap \mathbb{Z}^{n+1}$ for some α.
Proof of Step 4: Since Step 3 shows that both i and $i + j$ belong to $\partial\triangle_d \cap \mathbb{Z}^{n+1}$, if Step 4 is not true, then either i or $i + j$ belongs to the interior of the face $\{y \in \mathbb{R}^{n+1} : \sum y_\beta = d\} \cap \partial\triangle_d$.

Let first assume that both i and $i+j$ belong to the interior of the face $\{y \in \mathbb{R}^{n+1} : \sum y_\beta = d\} \cap \partial\triangle_d$. Then both $i + e_\beta - e_\alpha$ and $i + j + e_\beta - e_\alpha$ belong to $\triangle_{d,\mathbb{Z}}$, we have the inequalities
$$|x_\beta - x_\alpha - 2(i_\beta - i_\alpha)| < 2,$$
$$|x_\beta - x_\alpha - 2(i_\beta + j_\beta - i_\alpha - j_\alpha)| < 2.$$

Hence $|j_\beta - j_\alpha| < 2$. This implies $j_\alpha \geq 0$ for all α or $j_\alpha \leq 0$ for all α. Together with

$$j_0 = \sum_{\beta=1}^{n+1} j_\beta = \sum_{\beta=1}^{n+1}(i_\beta + j_\beta) - \sum_{\beta=1}^{n+1} i_\beta = d - d = 0,$$

we have $j = 0$ which is a contradiction.

Now we may assume that $i + j$ belongs to the interior of the face $\{y \in \mathbb{R}^{n+1} : \sum y_\beta = d\} \cap \partial\triangle_d$ but $p \in \{y \in \mathbb{R}^{n+1} : y_\beta = 0\} \cap \partial\triangle_d$ for some β. Then we have

$$\begin{cases} i_\beta = 0, \\ i_\alpha + j_\alpha \geq 1, \quad \forall\,\alpha \\ i_0 + j_0 = 0. \end{cases}$$

If $i_0 \leq d - 1$. Then $i + e_\alpha \in \triangle_{d,\mathbb{Z}}$ and hence

$$x_\alpha - 2(i_\alpha + i_0) < 2 \quad \forall\,\alpha.$$

Since we also have $i + j - e_\alpha \in \triangle_{d,\mathbb{Z}}$, we get

$$-2 < x_\alpha - 2(i_\alpha + j_\alpha + i_0 + j_0).$$

These imply

$$j_\alpha + j_0 < 2, \quad \forall\,\alpha.$$

Summing over α implies $j_0 < 2$. So $j_0 \leq 1$. On the other hand, $d = i_0 + j_0 \leq d - 1 + j_0$ implies $j_0 \geq 1$. We must have $j_0 = 1$. But this in turns implies $j_\alpha < 2 - j_0 = 1$. So $j_\alpha \leq 0$ and they cannot sum up to get $\sum_{\beta=1}^{n+1} j_\beta = j_0 = 1$. This contradiction implies $i_0 = d$ and hence $j_0 = 0$.

Consider those γ such that $i_\gamma \leq 1$. (Such γ always exists as $i_0 = d$ implies $i \neq 0$.) For any one of these γ and any $\alpha \neq \gamma$, $i + e_\alpha - e_\gamma \in \triangle_{d,\mathbb{Z}}$. So we have

$$x_\alpha - x_\gamma - 2(i_\alpha - i_\gamma) < 2.$$

Using $i + j + e_\gamma - e_\alpha \in \triangle_{d,\mathbb{Z}}$ for all γ and α, we have

$$|x_\gamma - x_\alpha - 2(i_\gamma - i_\alpha) - 2(j_\gamma - j_\alpha)| < 2.$$

Therefore,

$$j_\alpha - j_\gamma < 2.$$

Applying this inequality to γ_1 and γ_2 with $i_{\gamma_1}, i_{\gamma_2} \geq 1$, we obtain

$$|j_{\gamma_1} - j_{\gamma_2}| < 2.$$

This implies $j_\gamma \geq 0$ for all γ with $i_\gamma \leq 1$ or $j_\gamma \leq 0$ for all γ with $i_\gamma \leq 1$.

If $j_\gamma \geq 0$ for all γ with $i_\gamma \leq 1$, then

$$0 = j_0 = \sum_{\{\gamma\,:\,i_\gamma \geq 1\}} j_\gamma + \sum_{\{\alpha\,:\,i_\alpha=0\}} j_\alpha \geq \sum_{\{\alpha\,:\,i_\alpha=0\}} 1,$$

as $j_\alpha = i_\alpha + j_\alpha \geq 1$ for $i_\alpha = 0$. Hence the set $\{\alpha : i_\alpha = 0\}$ is empty. So for all γ, $i_\gamma \geq 1$ which implies $j_\gamma \geq 0$. Together with $j_0 = 0$, we conclude that $j = 0$ which is a contradiction.

So we must have $j_\gamma \leq 0$ for all γ with $i_\gamma \leq 1$. Then $j_\alpha - j_\gamma < 2$ implies

$$j_\alpha < 2, \quad \forall \alpha \text{ with } i_\alpha = 0.$$

Therefore

$$j_\alpha \leq 1, \quad \forall \alpha \text{ with } i_\alpha = 0.$$

On the other hand, for these α, $j_\alpha = i_\alpha + j_\alpha \geq 1$. Hence $j_\alpha = 1$ for all these α. Putting this back into the inequality, we have

$$j_\gamma > -1, \quad \forall \gamma \text{ with } i_\gamma \geq 1.$$

Hence,

$$j_\gamma = 0, \quad \forall \gamma \text{ with } i_\gamma \geq 1.$$

as $j_\gamma \leq 0$ for these γ. Using $j_0 = 0$, we conclude that the set $\{\alpha : i_\alpha = 0\}$ is empty and obtained a contradiction again. And this completes the proof of Step 4.

Step 5: There exists β such that both i and $i+j$ belong to $\{y \in \mathbb{R}^{n+1} : y_\beta = 0\} \cap \partial \triangle_d \cap \mathbb{Z}^{n+1}$.

Proof of Step 5: By Step 4, there exist β and α such that $i_\beta = 0$ and $i_\alpha + j_\alpha = 0$. If we can choose $\beta = \alpha$ then we are done. If not, then we have

$$i_\beta + j_\beta \geq 1 \quad \text{and} \quad i_\alpha \geq 1.$$

These imply $i + j - e_\beta$, $i + j + e_\alpha - e_\beta$, $i - e_\alpha$, and $i + e_\beta - e_\alpha \in \triangle_{d,\mathbb{Z}}$ and hence we have

$$\begin{cases} -2 < x_\beta - 2(i_\beta + j_\beta + i_0 + j_0) \\ x_\alpha - x_\beta - 2(i_\alpha - i_\beta) - 2(j_\alpha - j_\beta) < 2 \\ -2 < x_\alpha - 2(i_\alpha + i_0) \\ x_\beta - x_\alpha - 2(i_\beta - i_\alpha) < 2. \end{cases}$$

Therefore

$$j_\beta - j_\alpha < 2.$$

Using $i_\beta = 0$, $i_\beta + j_\beta \leq 1$, and $i_\alpha + j_\alpha = 0$, one get

$$1 + i_\alpha \leq j_\beta - j_\alpha < 2,$$

and arrive at the contradiction that $i_\alpha = 0$. This proves the Step 5.

Completion of the proof of the lemma: By Step 5, if there exists $x \in \mathbb{R}^{n+1}$ satisfying the condition of the lemma, then i and $i+j$ belong to $\{y \in \mathbb{R}^{n+1} : y_\beta = 0\} \cap \triangle_{d,\mathbb{Z}}$ for some β. This reduces the argument to one lower dimension. Since the proposition is clearly true for 1-dimension, induction implies the lemma holds.   q.e.d.

Lemma 4.2. *For any $i \in \triangle_{d,\mathbb{Z}}$, there exists at most $n+1$ elements $j_\gamma \in \{\pm e_\beta, e_\beta - e_\alpha\}_{\beta \neq \alpha}$ with $j_{\gamma_1} + j_{\gamma_2} \neq 0$ such that there exists $x \in \mathbb{R}^{n+1}$ satisfying*

$$
\begin{aligned}
\langle i, x \rangle - v(i) &= \langle i + j_\gamma, x \rangle - v(i + j_\gamma) \quad \forall, \gamma \\
&> \langle r, x \rangle - v(r), \quad r \in \triangle_{d,\mathbb{Z}} \setminus \{i, i + j_\gamma\}.
\end{aligned}
$$

Proof. We first claim that for any $i \in \triangle_{d,\mathbb{Z}}$ and $j \neq 0 \in \mathbb{Z}^{n+1}$, there exists no $x \in \mathbb{R}^{n+1}$ such that

$$\langle i, x \rangle - v(i) = \langle i+j, x \rangle - v(i+j) = \langle i-j, x \rangle - v(i-j).$$

In fact, if such x exists, then we have the equality

$$v(i+j) - v(i) = v(i) - v(i-j).$$

This implies $j = 0$ which is a contradiction.

Secondly, we claim that for any $i \in \triangle_{d,\mathbb{Z}}$, there exists no $x \in \mathbb{R}^{n+1}$ such that

$$
\begin{aligned}
\langle i, x \rangle - v(i) &= \langle i + e_\beta, x \rangle - v(i + e_\beta) \\
&= \langle i + e_\alpha, x \rangle - v(i + e_\alpha) \\
&= \langle i + e_\beta - e_\alpha, x \rangle - v(i + e_\beta - e_\alpha),
\end{aligned}
$$

and also no $x \in \mathbb{R}^{n+1}$ such that

$$
\begin{aligned}
\langle i, x \rangle - v(i) &= \langle i + e_\beta, x \rangle - v(i + e_\beta) \\
&= \langle i - e_\alpha, x \rangle - v(i - e_\alpha) \\
&= \langle i + e_\beta - e_\alpha, x \rangle - v(i + e_\beta - e_\alpha),
\end{aligned}
$$

The first set of equalities implies

$$
\begin{cases}
x_\beta &= 2i_\beta + 2i_0 + 2 \\
x_\alpha &= 2i_\alpha + 2i_0 + 2 \\
x_\beta - x_\alpha &= 2(i_\beta - i_\alpha) + 2,
\end{cases}
$$

and the second set implies

$$
\begin{cases}
x_\beta &= 2i_\beta + 2i_0 + 2 \\
x_\alpha &= 2i_\alpha + 2i_0 - 2 \\
x_\beta - x_\alpha &= 2(i_\beta - i_\alpha) + 2,
\end{cases}
$$

Both are impossible.

By the two claims, we see that if $\pm e_\beta$ is one of the j_γ, the $\mp e_\beta$ will not appear in the set $\{j_\gamma\}$; and if $\pm e_\beta$ and $\pm e_\alpha$ belong to the set $\{j_\gamma\}$, then $\pm(e_\beta - e_\alpha)$ will not appear in the set $\{j_\gamma\}$. Therefore, each $\beta = 1, \ldots, n+1$ can appeared once in the set $\{j_\gamma\}$ and this completes the proof of the lemma.

<div align="right">q.e.d.</div>

Proof of the key lemma 2.5: It is clear from the lemmas 4.1 and 4.2, the balanced polyhedral complex Π_v corresponding to $v(j) = \sum_{\beta=0}^{n+1} j_\beta^2$ is a maximal dual complex of $\triangle_d$ which gives (1) of the lemma. To see (2), we observe that lemma 4.1 implies that any simplex of the subdivision with an interior point j of the translated simplex $i - 1 + \triangle_{n+2}$ as a vertex, then all other vertices belong to $i - 1 + \triangle_{n+2}$. Therefore, the subdivision restrict to a subdivision of $i - 1 + \triangle_{n+2}$. This completes the proof of the key lemma.

5. Appendix: Definition of balanced polyhedral complex

In this appendix, we state the Mikhalkin's definition [**9**] of a balanced polyhedral complex for reader's reference.

Definition 5.1. A subset $\Pi \in \mathbb{R}^{n+1}$ is called a *rational polyhedral complex* if it can be represented as a finite union of closed convex polyhedra (possibly semi-infinite) called *cells* in $\mathbb{R}^{n+1}$ satisfying
  1) The slope of the affine span of each cell is rational.
  2) If the dimension of the cell is defined to be the dimension of its affine span and a k-dimensional cell is called a *k-cell*. Then the boundary of a k-cell is a union of $(k-1)$-cells.
  3) Different open cells do not intersect.

Definition 5.2.   1) The maximum of the dimensions of the cells of a polyhedral complex Π is called the *dimension* of Π. And Π is called a polyhedral n-complex if the dimension of Π is n.
  2) A polyhedral n-complex is called *weighted* if there is a *weight* $w(F) \in \mathbb{N}$ assigned to each of its n-cell F.

Definition 5.3. For each n-cell F of a weighted polyhedral n-complex in $\mathbb{R}^{n+1}$ and an co-orientation on F, an integer covector
$$c_F : \mathbb{Z}^{n+1} \to \mathbb{Z}$$
is defined by the following conditions
  1) The kernel of c_F is parallel to F.
  2) The normalized covector $\frac{1}{w(F)} c_F$ is a primitive integer covector.
  3) The covector c_F compatible with the co-orientation of F.

Definition 5.4. A weighted polyhedral n-complex in $\mathbb{R}^{n+1}$ is called *balanced* if for all $(n-1)$-cell $G \subset \Pi$,
$$\sum_s c_{F_s} = 0,$$
where F_s are the n-cells adjacent to G with co-orientation given by a choice of a rotational direction about G.

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