J. DIFFERENTIAL GEOMETRY
82 (2009) 611-628

COMPLETE CONSTANT MEAN CURVATURE SURFACES AND BERNSTEIN TYPE THEOREMS IN $\mathbb{M}^2 \times \mathbb{R}$

José M. Espinar & Harold Rosenberg

Abstract

In this paper we study constant mean curvature surfaces Σ in a product space, $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a complete Riemannian manifold. We assume the angle function $\nu = \langle N, \frac{\partial}{\partial t}\rangle$ does not change sign on Σ. We classify these surfaces according to the infimum $c(\Sigma)$ of the Gaussian curvature of the projection of Σ.

When $H \neq 0$ and $c(\Sigma) \geq 0$, then Σ is a cylinder over a complete curve with curvature $2H$. If $H = 0$ and $c(\Sigma) \geq 0$, then Σ must be a vertical plane or Σ is a slice $\mathbb{M}^2 \times \{t\}$, or $\mathbb{M}^2 \equiv \mathbb{R}^2$ with the flat metric and Σ is a tilted plane (after possibly passing to a covering space).

When $c(\Sigma) < 0$ and $H > \sqrt{-c(\Sigma)}/2$, then Σ is a vertical cylinder over a complete curve of $\mathbb{M}^2$ of constant geodesic curvature $2H$. This result is optimal.

We also prove a non-existence result concerning complete multigraphs in $\mathbb{M}^2 \times \mathbb{R}$, when $c(\mathbb{M}^2) < 0$.

1. Introduction

The image of the Gauss map of a complete minimal surface in $\mathbb{R}^3$ may determine the surface. For example, an entire minimal graph is a plane (Bernsteins' Theorem [**4**]). More generally, if the Gaussian image misses more than four points, then it is a plane ([**14**]). The Gaussian image of Scherks' doubly periodic surface misses exactly four points.

The image of the Gauss map of a non-zero constant mean curvature surface in $\mathbb{R}^3$ does determine the surface under certain circumstances. Hoffman, Osserman and Schoen proved (see [**16**]): *Let $\Sigma \subset \mathbb{R}^3$ be a complete surface of constant mean curvature. If the image of the Gauss map lies in an open hemisphere, then Σ is a plane. If the image is contained in a closed hemisphere, then Σ is a plane or a right cylinder.* Unduloids show that this result is the best possible.

The first author is partially supported by Spanish MEC-FEDER Grant MTM2007-65249, and Regional J. Andalucia Grant P06-FQM-01642.
Received 02/05/2008.

The aim of this paper is to establish results analogous to those of Hoffman-Osserman-Schoen [**16**] for constant mean curvature surfaces in $\mathbb{M}^2 \times \mathbb{R}$. In product spaces, the condition that the image of the Gauss map is contained in a hemisphere, becomes that the angle function, i.e., $\nu = \langle N, \frac{\partial}{\partial t} \rangle$, does not change sign; here N denotes a unit normal vector field along a surface $\Sigma \subset \mathbb{M}^2 \times \mathbb{R}$.

There are many (interesting) complete minimal and constant mean curvature graphs in $\mathbb{M}^2 \times \mathbb{R}$ (e.g., in $\mathbb{H}^2 \times \mathbb{R}$, [**9**]). We will see that conditions on the value of the mean curvature H, and the curvature of $\mathbb{M}^2$, can determine complete surfaces in $\mathbb{M}^2 \times \mathbb{R}$ of constant mean curvature H whose angle function does not change sign.

Let Σ be a complete connected surface of constant mean curvature H immersed in $\mathbb{M}^2 \times \mathbb{R}$. We will also say Σ is an $H-$surface to mean Σ has constant mean curvature H. Let $\pi : \Sigma \longrightarrow \mathbb{M}^2 \equiv \mathbb{M}^2 \times \{0\}$, be the horizontal projection, and define

$$c(\Sigma) = \inf \{\kappa(\pi(p)) \, : \, p \in \Sigma\}$$

where κ is the Gauss (intrinsic) curvature of $\mathbb{M}^2$. $c(\mathbb{M}^2)$ is the infimum of the Gauss curvature of $\mathbb{M}^2$.

The main result of this work is the following. We will assume (up to scaling in the metric of M^2) that $c(\Sigma) = 1$ or $c(\Sigma) = -1$ if $c(\Sigma) \neq 0$.

Theorem 4.1. *Let Σ be a complete immersed $H-$surface in $\mathbb{M}^2 \times \mathbb{R}$, whose angle function ν does not change sign. If $c(\Sigma) < 0$ and $H > \sqrt{-c(\Sigma)}/2$, then Σ is a vertical cylinder over a complete curve of $\mathbb{M}^2$ of constant geodesic curvature $2H$.*

The proof of Theorem 4.1 is inspired by the techniques in [**15**], where it is proved that a complete multi-graph in $\mathbb{H}^2 \times \mathbb{R}$, of constant mean curvature $1/2$, is an entire graph.

Before stating our results and describing the organization of this paper, we discuss some previous work on this subject.

Entire minimal and constant mean curvature graphs $\mathbb{M}^2 \times \mathbb{R}$ have been studied by several authors. When $\mathbb{M}^2$ is a complete surface with nonnegative Gaussian curvature, then an entire minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ is totally geodesic ([**21**]). Hence the graph is a horizontal slice or $\mathbb{M}^2$ is a flat $\mathbb{R}^2$ and the graph is a tilted plane. This result has been generalized to constant mean curvature entire graphs in ([**3**]).

Entire constant mean curvature $1/2$ graphs, in $\mathbb{H}^2 \times \mathbb{R}$ and entire minimal graphs in Heisenberg space have been classified (they are sister surfaces, see [**10**], [**11**] and [**15**]).

S. Fornari and J. Ripoll (see [**13**]) have considered the general problem of constant mean curvature hypersurfaces transverse to an ambient Killing field of a Riemannian manifold, and they obtained several interesting generalizations of the results of Hoffman, Osserman and Schoen

[**16**]. In particular, they prove the Theorem 3.1 stated below under the stronger hypothesis $\mathbb{M}^2$ has non-negative curvature.

Now we describe the organization of the paper.

In Section 2, we present the equations that an immersed $H-$surface in $\mathbb{M}^2 \times \mathbb{R}$ must satisfy.

In Sections 3 we consider the case $c(\Sigma) \geq 0$ and we prove:

Theorem 3.1. *Let $\Sigma \subset \mathbb{M}^2 \times \mathbb{R}$ be a complete $H-$surface whose angle function does not change sign and $c(\Sigma) \geq 0$.*

- *If $H = 0$, then Σ is a vertical plane or Σ is a slice $\mathbb{M}^2 \times \{t\}$, or $\mathbb{M}^2 \equiv \mathbb{R}^2$ with the flat metric and Σ is a tilted plane (after possibly passing to a covering space).*
- *If $H \neq 0$, Σ is a cylinder over a complete curve with curvature $2H$.*

In Section 4, we consider the case $c(\Sigma) < 0$. First, we prove non-existence of certain entire graphs; more precisely:

Lemma 4.1. *Let $\mathbb{M}^2$ be a complete surface with $c(\mathbb{M}^2) < 0$. Then, there are no entire graphs in $\mathbb{M}^2 \times \mathbb{R}$, with constant mean curvature H, with $H > \sqrt{-c(\mathbb{M}^2)}/2$.*

Finally, we prove our main Theorem 4.1 that we previously stated.

2. The geometry of surfaces in $\mathbb{M}^2 \times \mathbb{R}$

Henceforth $\mathbb{M}^2$ denotes a complete Riemannian surface with $\partial\mathbb{M}^2 = \emptyset$. Let g be the metric of $\mathbb{M}^2$ and ∇ the Levi-Civita connection on $\mathbb{M}^2 \times \mathbb{R}$ with the product metric $\langle , \rangle = g + dt^2$.

Let Σ be a complete orientable $H-$surface immersed in $\mathbb{M}^2 \times \mathbb{R}$ and let N be a unit normal to Σ. In terms of a conformal parameter z of Σ, the first and second fundamental forms are given by

$$\begin{aligned} I &= \lambda\,|dz|^2 \\ II &= p\,dz^2 + \lambda\,H\,|dz|^2 + \overline{p}\,d\bar{z}^2, \end{aligned} \tag{1}$$

where $p\,dz^2 = \langle -\nabla_{\frac{\partial}{\partial z}} N, \frac{\partial}{\partial z} \rangle\, dz^2$ is the Hopf differential of Σ.

Let $\pi : \mathbb{M}^2 \times \mathbb{R} \longrightarrow \mathbb{M}^2$ and $h : \mathbb{M}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ be the usual projections. We also call the restriction of h to Σ the *height function*.

First we derive the following necessary equations on Σ, which were obtained in [**2**], but we establish here for the sake of completeness.

Lemma 2.1. *Given an immersed surface $\Sigma \subset \mathbb{M}^2 \times \mathbb{R}$, the following equations are satisfied:*

$$K = K_e + \kappa\,\nu^2 \tag{2}$$

$$|h_z|^2 = \frac{1}{4}\,\lambda\,(1-\nu^2) \tag{3}$$

$$h_{zz} = \frac{\lambda_z}{\lambda}\,h_z + p\,\nu \tag{4}$$

$$h_{z\bar z} = \frac{1}{2}\,\lambda\,H\,\nu \tag{5}$$

$$\nu_z = -H\,h_z - \frac{2}{\lambda}\,p\,h_{\bar z} \tag{6}$$

$$p_{\bar z} = \frac{\lambda}{2}\,(H_z + \kappa\,\nu\,h_z) \tag{7}$$

where $\kappa(z)$ stands for the Gauss curvature of $\mathbb{M}^2$ at $\pi(z)$, K_e the extrinsic curvature and K the Gauss curvature of I.

Proof. Let us write

$$\frac{\partial}{\partial t} = T + \nu\,N$$

where T is a tangent vector field on S. Since $\frac{\partial}{\partial t}$ is the gradient in $\mathbb{M}^2$ of the function t, it follows that T is the gradient of h on S.

Thus, from (1), one gets $T = \frac{2}{\lambda}\left(h_{\bar z}\frac{\partial}{\partial z} + h_z \frac{\partial}{\partial \bar z}\right)$ and so

$$1 = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = \langle T, T\rangle + \nu^2 = \frac{4\,|h_z|^2}{\lambda} + \nu^2,$$

that is, (3) holds.

On the other hand, from (1) we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} &= \frac{\lambda_z}{\lambda}\frac{\partial}{\partial z} + p\,N \\ \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \bar z} &= \frac{1}{2}\,\lambda\,H\,N \\ -\nabla_{\frac{\partial}{\partial z}} N &= H\,\frac{\partial}{\partial z} + \frac{2}{\lambda}\,p\,\frac{\partial}{\partial \bar z}. \end{aligned} \tag{8}$$

The scalar product of these equalities with $\frac{\partial}{\partial t}$ gives us (4), (5) and (6), respectively.

Finally, from (8) we get

$$\left\langle \nabla_{\frac{\partial}{\partial \bar z}} \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} - \nabla_{\frac{\partial}{\partial z}} \nabla_{\frac{\partial}{\partial \bar z}} \frac{\partial}{\partial z}, N \right\rangle \; = \; p_{\bar z} - \frac{1}{2}\,\lambda\,H_z.$$

Hence, using the relationship between the curvature tensors of a product manifold (see, for instance, [**20**, p. 210]), the Codazzi equation becomes

$$\frac{1}{2}\,\lambda\,\kappa\,\nu\,h_z \; = \; p_{\bar z} - \frac{1}{2}\,\lambda\,H_z,$$

that is, (7) holds.

To finish, note that (2) is nothing but the Gauss equation of the immersion. q.e.d.

Now, we will define a quadratic differential depending on $c(\Sigma)$. Denote

$$Q_{c(\Sigma)}\, dz^2 = \left(2H\, p - c(\Sigma) h_z^2\right)\, dz^2.$$

Note that this is either the usual Hopf differential (up to the factor $2H$) when $\mathbb{M}^2 = \mathbb{R}^2$, or the Abresch-Rosenberg differential when $\mathbb{M}^2 = \mathbb{H}^2$ or $\mathbb{M}^2 = \mathbb{S}^2$ (see [**1**]).

Now, we will compute the modulus of the gradient and Laplacian of ν.

Lemma 2.2. *Let Σ be an $H-$surface immersed in $\mathbb{M}^2 \times \mathbb{R}$, with $c(\Sigma) \neq 0$. Then the following equations are satisfied:*

(9)
$$\|\nabla \nu\|^2 = \frac{c(\Sigma)}{4}\left(4H^2 + c(\Sigma)(1-\nu^2) - 2K_e\right)^2 - c(\Sigma)\left(K_e^2 + \frac{4|Q_{c(\Sigma)}|^2}{\lambda^2}\right),$$

(10)
$$\Delta \nu = -\left(4H^2 + \kappa(1-\nu^2) - 2K_e\right)\nu.$$

Proof. From (6)

$$|\nu_z|^2 \;=\; \frac{4\,|p|^2\,|h_z|^2}{\lambda^2} + H^2\,|h_z|^2 + \frac{2\,H}{\lambda}\left(p\, h_{\overline{z}}^2 + \overline{p}\, h_z^2\right),$$

and taking into account that

$$|Q_{c(\Sigma)}|^2 \;=\; 4\,H^2\,|p|^2 + |h_z|^4 - 2\,c(\Sigma)\,H\left(p\, h_{\overline{z}}^2 + \overline{p}\, h_z^2\right),$$

we obtain, using also (3), that

$$\begin{aligned}
|\nu_z|^2 \;&=\; \left(\frac{|p|^2}{\lambda} + \frac{H^2\,\lambda}{4}\right)(1-\nu^2) \\
&\quad + \frac{c(\Sigma)}{\lambda}\left(4\,H^2\,|p|^2 + \frac{\lambda^2}{16}(1-\nu^2)^2 - |Q_{c(\Sigma)}|^2\right) \\
&=\; \frac{\lambda}{4}(2H^2 - K_e)(1-\nu^2) \\
&\quad + c(\Sigma)\frac{\lambda}{4}\left(4\,H^2(H^2 - K_e) + \frac{(1-\nu^2)^2}{4} - \frac{4|Q_{c(\Sigma)}|^2}{\lambda^2}\right)
\end{aligned}$$

where we have used that $4|p|^2 = \lambda^2(H^2 - K_e)$. Thus

$$\begin{aligned}
\|\nabla\nu\|^2 &= \frac{4}{\lambda}|\nu_z|^2 = -c(\Sigma)\left(4H^2 + c(\Sigma)(1-\nu^2)\right)K_e + 2H^2(1-\nu^2) \\
&\quad +4c(\Sigma)H^4 + c(\Sigma)\frac{(1-\nu^2)^2}{4} - c(\Sigma)\frac{4|Q_{c(\Sigma)}|^2}{\lambda^2} \\
&= -c(\Sigma)\left(4H^2 + c(\Sigma)(1-\nu^2)\right)K_e \\
&\quad +\frac{c(\Sigma)}{4}\left(4H^2 + c(\Sigma)(1-\nu^2)\right)^2 - c(\Sigma)\frac{4|Q_{c(\Sigma)}|^2}{\lambda^2} \\
&= \frac{c(\Sigma)}{4}\left(4H^2 + c(\Sigma)(1-\nu^2) - 2K_e\right)^2 \\
&\quad -c(\Sigma)\left(K_e^2 + \frac{4|Q_{c(\Sigma)}|^2}{\lambda^2}\right)
\end{aligned}$$

On the other hand, by differentiating (6) with respect to $\bar z$ and using (4), (5) and (7), one gets

$$\nu_{z\bar z} = -\kappa\,\nu\,|h_z|^2 - \frac{2}{\lambda}\,|p|^2\,\nu - \frac{H^2}{2}\,\lambda\,\nu.$$

Then, from (3),

$$\begin{aligned}
\nu_{z\bar z} &= -\frac{\lambda\,\nu}{4}\left(\kappa(1-\nu^2) + \frac{8\,|p|^2}{\lambda^2} + 2\,H^2\right) \\
&= -\frac{\lambda}{4}\left(4H^2 + \kappa(1-\nu^2) - 2K_e\right)\nu
\end{aligned}$$

thus

$$\Delta\nu = \frac{4}{\lambda}\nu_{z\bar z} = -\left(4H^2 + \kappa(1-\nu^2) - 2K_e\right)\nu$$

q.e.d.

Remark 2.1. Note that (10) is nothing but the Jacobi equation for the Jacobi field ν.

We say Σ is a *vertical plane* when $\Sigma = \gamma \times \mathbb{R}$, $\gamma \subset \mathbb{M}^2$ a complete geodesic.

Lemma 2.3. *Let Σ be a complete $H-$surface immersed in $\mathbb{M}^2 \times \mathbb{R}$ whose angle function ν is constant. Then,*

- *If $H = 0$, Σ is a vertical plane or a slice $\mathbb{M}^2 \times \{t\}$, or $\mathbb{M}^2 \equiv \mathbb{R}^2$ with the flat metric and Σ is a tilted plane (after possibly passing to a covering space).*
- *If $H \neq 0$ when $c(\Sigma) \geq 0$ or $H > 1/2$ when $c(\Sigma) = -1$, then Σ is a cylinder over a complete curve with curvature $2H$ in $\mathbb{M}^2$.*

Proof. We can assume, up to an isometry, that $\nu \leq 0$. We will divide the proof in three cases:

- $\nu = 0$:

  Using (5), h is harmonic and Σ is flat since $\lambda = 4|h_z|^2$ by (3), thus Σ is conformally the plane. So, in this case, Σ must be either a vertical plane if $H = 0$ or Σ is a vertical cylinder over a complete curve of curvature $2H$ in $\mathbb{M}^2$.
- $\nu = -1$:

  In this case, Σ is a slice, and necessarily $H = 0$.
- $-1 < \nu < 0$:

  From (6)

$$Hh_z = -\frac{2p}{\lambda} h_{\bar{z}}$$

then

$$H^2 = \frac{4|p|^2}{\lambda^2} = H^2 - K_e$$

since $|h_z|^2 \neq 0$ from (3), so $K_e = 0$ on Σ.

Thus, from (10), we have

$$4H^2 + \kappa(1 - \nu^2) = 0.$$

So, if $c(\Sigma) = -1$, this is impossible since $0 < 1 - \nu^2 < 1$ and $4H^2 > 1 = -c(\Sigma)$.

If $c(\Sigma) \geq 0$, then $H = 0$ and $\kappa(\pi) \equiv 0$ on Σ, and we will show that

Claim: $\pi(\Sigma) = \mathbb{M}^2$.

It is enough to prove that $\pi(\Sigma)$ has no boundary in $\mathbb{M}^2$. Suppose that there exists $q \in \partial\pi(\Sigma) \subset \mathbb{M}^2$ and $\{p_n\} \subset \Sigma$ a sequence such that $\{\pi(p_n)\} \longrightarrow q$. Fix $p_0 \in \Sigma$ and let γ_n be the complete geodesic in Σ joining p_0 and p_n. Since $q \in \partial\pi(\Sigma)$, γ_n must become almost vertical at p_n for n sufficiently large, which means that $N(p_n)$ must become horizontal, but ν is a constant different from 0, a contradiction.

Thus $\pi(\Sigma) = \mathbb{M}^2$ and $\mathbb{M}^2$ is a complete flat surface since $K = 0$ by the Gauss equation, that is, it is isometrically $\mathbb{R}^2$ (after possibly passing to a covering space), and Σ is a *tilted* plane. This proves the Claim.

q.e.d.

We state some basic facts on the theory of eigenvalue problems on Riemannian manifolds (see [**6**] and [**7**] for details). Given a domain $\Omega \subset \mathbb{M}^2$ such that $\overline{\Omega}$ is compact and its boundary $\partial\Omega$ is C^∞, the real numbers λ, called *eigenvalues*, are those for which there exists a non-trivial solution $\phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to

$$\begin{aligned} \overline{\Delta}\phi + \lambda\phi = 0 \quad &\text{on } \Omega \\ \phi = 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{11}$$

where $\overline{\Delta}$ denotes the usual Laplacian operator associated to the Riemannian metric g on Ω. This problem is called the *Dirichlet eigenvalue problem*.

It is well known that the set of eigenvalues in the Dirichlet problem consists of a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \uparrow +\infty,$$

and we will denote

$$\lambda(\Omega) = \lambda_1 \tag{12}$$

to be the lowest eigenvalue in the Dirichlet eigenvalues problem in Ω.

The last quantity that we will need it is the *Cheeger constant*, that is

$$\jmath(\mathbb{M}^2) = \inf_\Omega \frac{\mathrm{A}(\partial\Omega)}{\mathrm{V}(\Omega)} \tag{13}$$

where Ω varies over open domains on $\mathbb{M}^2$ possesing compact closure and C^∞ boundary.

3. Complete $H-$surfaces Σ in $\mathbb{M}^2 \times \mathbb{R}$ with $c(\Sigma) \geq 0$

In this Section we shall establish:

Theorem 3.1. *Let Σ be a complete $H-$surface immersed in $\mathbb{M}^2 \times \mathbb{R}$ whose angle function ν does not change sign and $c(\Sigma) \geq 0$.*

- *If $H = 0$, then Σ is a vertical plane or Σ is a slice $\mathbb{M}^2 \times \{t\}$, or $\mathbb{M}^2 \equiv \mathbb{R}^2$ with the flat metric and Σ is a tilted plane (after possibly passing to a covering space).*
- *If $H \neq 0$, Σ is a cylinder over a complete curve with curvature $2H$.*

Remark 3.1. As we mentioned in the introduction, Theorem 3.1 generalizes results in [**3**] and [**13**]. Our proof of Theorem 3.1 is inspired by the work of Hoffman, Osserman and Schoen [**16**].

Proof of Theorem 3.1*:*

Without loss of generality, we can assume that Σ is simply-connected and orientable. Otherwise we take its universal cover, if ν is non-positive on the surface, then it is non-positive on its universal cover. Thus, by the Uniformization Theorem, we have three possibilities:

1) Σ is conformally the $2-$sphere:

By (10), ν is a bounded subharmonic function since

$$4H^2 + \kappa(1-\nu^2) - 2K_e \geq 2H^2 + 2(H^2 - K_e) + c(\Sigma)(1-\nu^2) \geq 0,$$

thus ν must be constant since Σ is conformally the $2-$sphere. So, from Lemma 2.3, Σ is a slice and $\mathbb{M}^2$ is necessarily compact.

2) Σ is conformally the plane:

By (10), ν is a bounded subharmonic function, then ν must be constant (Σ is conformally the plane). Thus, again from Lemma 2.3, Σ is either a vertical cylinder over a curve of curvature $2H$ in $\mathbb{M}^2$ or Σ is isometrically $\mathbb{R}^2$ (after possibly passing to a covering space), and Σ is a *tilted* plane.

3) Σ is conformally the disk:

We will show that this case is impossible. Again, by (10), ν is subharmonic. By the Maximum Principle, if $\nu = 0$ at any interior point, then ν must vanish identically on Σ. Thus, using (5), h is harmonic, so Σ is flat since $\lambda = 4|h_z|^2$ by (3), so Σ must be conformally the plane, which is a contradiction.

Therefore, $-1 \leq \nu < 0$ on Σ. Now, from (2) and (10), we have

$$\begin{aligned}\Delta\nu &= -\left(4H^2 + \kappa(1-\nu^2) - 2K_e\right)\nu \\ &= -\left(4H^2 + \kappa(1-\nu^2) - 2(K - \kappa\nu^2)\right)\nu \\ &= +2K\,\nu - \left(4H^2 + \kappa(1+\nu^2)\right)\nu\end{aligned}$$

thus

$$\Delta\nu - 2K\,\nu + \left(4H^2 + \kappa(1+\nu^2)\right)\nu = 0 \tag{14}$$

so ν is a strictly negative solution of (14), but this is impossible by [**12**, Corollary 3 on page 205] since in this paper the authors showed that given a complete metric on the disk $(\mathbb{D}, ds^2)$ there is no positive (or negative) solution to the equation

$$\Delta g - aKg + Pg = 0 \ \text{ on } \mathbb{D},$$

where Δ is the Laplacian operator associated to the Riemannian metric ds^2, K the Gauss curvature of ds^2, P a smooth nonnegative function on $\mathbb{D}$ and $a \geq 1$.

q.e.d.

Remark 3.2. Observe that Case 1, that is, when Σ is conformally the sphere, could be obtained in the following (more geometrical) way. If Σ is conformally $\mathbb{S}^2$, then at the highest point of Σ, one must have H not zero (otherwise Σ is a slice by the Maximum Principle), but then at a lowest point one has the angle function not constant, but ν is constant for (10), so Σ must be a slice.

4. Complete $H-$surfaces Σ with $c(\Sigma) < 0$

When $c(\Sigma) < 0$, this classification is more complicated to obtain. First, we will need the following

Lemma 4.1. *There are no entire $H-$graphs in $\mathbb{M}^2 \times \mathbb{R}$ with $H > 1/2$ and $c(\Sigma) = -1$.*

Proof. Let us suppose that such an entire graph exists. Let

$$\Sigma = \mathrm{Gr}(u) = \left\{ (x, u(x)) \in \mathbb{M}^2 \times \mathbb{R} : \, x \in \mathbb{M}^2 \right\},$$

where $u : \mathbb{M}^2 \longrightarrow \mathbb{R}$ is a solution of

$$\overline{\mathrm{div}} \left(\frac{\overline{\nabla} u}{\sqrt{1 + |\overline{\nabla} u|^2}} \right) = 2H \tag{15}$$

where $\overline{\mathrm{div}}$ and $\overline{\nabla}$ denote the divergence and gradient operators in $\mathbb{M}^2$.

We will obtain a lower bound for the Cheeger constant of $\mathbb{M}^2$ in terms of H, following an argument due to Salavessa [**23**]. Let $\Omega \subset \mathbb{M}^2$ be an open domain with compact closure and smooth boundary $\partial\Omega$, let us denote by η the outwards normal to $\partial\Omega$. Thus, from (15) and the Divergence Theorem, we have

$$\begin{aligned} 2H\mathrm{V}(\Omega) &= \int_\Omega \overline{\mathrm{div}} \left(\frac{\overline{\nabla} u}{\sqrt{1 + |\overline{\nabla u}|^2}} \right) dV = \int_{\partial\Omega} g \left(\frac{\overline{\nabla} u}{\sqrt{1 + |\overline{\nabla} u|^2}}, \eta \right) dA \\ &\leq \mathrm{A}(\partial\Omega). \end{aligned}$$

Then, an immediate consequence of (13) yields

$$2H \leq \jmath(\mathbb{M}^2). \tag{16}$$

Next, we obtain an upper bound for the infimum, $\lambda(\mathbb{M}^2)$, of the spectrum of the Laplacian on $\mathbb{M}^2$. That is, from [**8**], for any $p \in \mathbb{M}^2$ and any $\delta > 0$ we have

$$\lambda(B(p, \delta)) \leq \lambda_{-1}(\delta) \tag{17}$$

where $\lambda(B(p, \delta))$ is the lowest eigenvalue of the Laplacian on the metric ball of radius δ centered at p_0 on $\mathbb{M}^2$ and $\lambda_{-1}(\delta)$ the lowest eigenvalue of the Laplacian on the ball of radius δ on the space form of constant curvature -1.

Now, since $\mathbb{M}^2$ is complete, letting $\delta \longrightarrow +\infty$ in (17), we have

$$\lambda(\mathbb{M}^2) \leq 1/4 \tag{18}$$

where we have used that $\lim_{\delta \longrightarrow +\infty} \lambda_{-1}(\delta) = 1/4$ (see [**6**, Theorem 5, pag 46]).

Finally, Cheeger's inequality (see [**7**, Theorem VI.1.2, pag 161])

$$\jmath(\mathbb{M}^2)^2 \leq 4\lambda(\mathbb{M}^2)$$

combined with (18) give us

$$\jmath(\mathbb{M}^2) \leq 1 = \sqrt{-c(\mathbb{M}^2)}. \tag{19}$$

Thus, from (16) and (19),

$$1 < 2H \leq \jmath(\mathbb{M}^2) \leq 1$$

which is a contradiction and we have proved the Lemma 4.1. q.e.d.

In fact, Lemma 4.1 can be generalized as follows.

Corollary 4.1. *Let $\mathbb{M}^2$ be a surface with $c(\mathbb{M}^2) = -1$. Then, there is no complete entire vertical graph in $\mathbb{M}^2 \times \mathbb{R}$ with*

$$\inf H > 1/2.$$

Proof. With the notation of Lemma 4.1, the only change is

$$\begin{aligned} 2 \inf H \,\mathrm{V}(\Omega) \leq \int_\Omega H \, dV &= \int_\Omega \overline{\mathrm{div}} \left(\frac{\overline{\nabla} u}{\sqrt{1 + |\overline{\nabla} u|^2}} \right) dV \\ &= \int_{\partial\Omega} g \left(\frac{\overline{\nabla} u}{\sqrt{1 + |\overline{\nabla} u|^2}}, \eta \right) dA \leq \mathrm{A}(\partial\Omega) \end{aligned}$$

so,

$$2 \inf H \leq \jmath(\mathbb{M}^2) \tag{20}$$

and we conclude as in the previous result. q.e.d.

Now, we establish our main result.

Theorem 4.1. *Let Σ be a complete immersed $H-$surface in $\mathbb{M}^2 \times \mathbb{R}$, whose angle function ν does not change sign. If $c(\Sigma) < 0$ and $H > \sqrt{-c(\Sigma)}/2$, then Σ is a vertical cylinder over a complete curve of $\mathbb{M}^2$ of constant geodesic curvature $2H$.*

Proof. We divide the proof in two steps. First, we will prove that either the surface is a cylinder over a complete curve or is a multi-graph. Second, we will prove that such a multi-graph can not exist.

Claim A: Σ is either a cylinder over a complete curve with curvature $2H$, or a multi-graph.

Proof of Claim A: Set $\Omega = \{p \in \Sigma \, : \, \nu(p) = 0\}$. We will show that either $\Omega = \emptyset$ or $\Omega = \Sigma$, in the later case, by Lemma 2.3, Σ is a vertical cylinder over a complete curve of curvature $2H$ in $\mathbb{M}^2$.

If $\Omega = \emptyset$ then Σ is a multi-graph. So, we can assume that $\Omega \neq \emptyset$. If we prove that Ω is an open set then, since Ω is closed and Σ is connected, $\Omega = \Sigma$. Let $p \in \Omega$ and $\mathcal{B}(R) \subset \Sigma$ the geodesic ball centered at p of radius R. Such a geodesic ball is relatively compact in Σ. From (10), ν satisfies $\Delta\nu + q\nu = 0$ on $\mathcal{B}(R)$. Thus, Assertion 2.2 in [**17**] asserts that either $\nu < 0$ or $\nu \equiv 0$ in $\mathcal{B}(R)$. Since we are assuming that ν is zero at an interior point of $\mathcal{B}(R)$, then $\mathcal{B}(R) \subset \Omega$, and Ω is an open set. This proves Claim A.

Now, we continue with the second step:

Claim B: Σ can not be a multi-graph.

Proof of Claim B: We know that Σ can not be an entire graph by Lemma 4.1. Thus the proof will be completed when we prove that such a multi-graph is in fact an entire graph. The proof of this will be rather long. The idea originates in the paper [**15**], where it is proved that a complete multi-graph in $\mathbb{H}^2 \times \mathbb{R}$, with $H = 1/2$, is in fact an entire graph.

Let us remark that there is a simple geometrical argument to see that there are no entire vertical graphs with CMC $H > 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$, and this fact is as follows: one could touch such an entire graph by a compact rotational $H-$sphere (touch on the mean convex side of the graph), and the Maximum Principle would say that Σ is equal to the sphere, a contradiction.

Now, we will show that Σ is an entire graph, assuming $\frac{\partial}{\partial t}$ is transverse to Σ:

Since $H > 1/2$, the mean curvature vector of Σ never vanishes, so Σ is orientable. Let N denote a unit normal field to Σ. Since ν is a non-zero Jacobi function on Σ (see Remark 2.1), Σ is strongly stable and thus has bounded curvature. We assume $\nu < 0$ and $\langle N, \overrightarrow{H} \rangle > 0$.

As Σ has bounded geometry, there exists $\delta > 0$ such that for each $p \in \Sigma$, Σ is a graph in exponential coordinates over the disk $D_\delta \subset T_p\Sigma$ of radius δ, centered at the origin of $T_p\Sigma$. This graph, denoted by $G(p)$, has bounded geometry. δ is independent of p and the bound on the geometry of $G(p)$ is uniform as well.

We denote by $F(p)$ the surface $G(p)$ translated to height zero $\mathbb{M}^2 \equiv \mathbb{M}^2 \times \{0\}$, i.e, let ϕ_p be the isometry of $\mathbb{M}^2 \times \mathbb{R}$ which takes p to $\pi(p)$, then $F(p) = \phi_p(G(p))$.

For $q \in \mathbb{M}^2 \times \mathbb{R}$, we will denote by $C_\delta(q)$ an arc of $\mathbb{M}^2$ with geodesic curvature $2H$, of length 2δ and centered at q, i.e, $q \in C_\delta(q)$ is the mid-point.

Claim 1: Let $\{p_n\} \in \Sigma$, satisfying $\nu(p_n) \longrightarrow 0$ as $n \longrightarrow +\infty$, that is, $T_{p_n}\Sigma$ are becoming vertical. Let $\pi(p_n) = q_n$, and assume q_n converges to some point q. Then, there is a subsequence of $\{p_n\}$ (which we also denote by $\{p_n\}$) such that $F(p_n)$ converges to $C_\delta(q) \times [-\delta, \delta]$, for some $2H$ arc $C_\delta(q)$ at q. The convergence is in the C^2-topology.

Proof of Claim 1: The proof is the same as in [**15**, Claim 1] replacing horocycles by arcs of curvature $2H$.

Now, let $p \in \Sigma$ and assume Σ in a neighborhood of p is a vertical graph of a function f defined on B_R, B_R the open geodesic ball of radius R of $\mathbb{M}^2$ centered at $\pi(p) = O \in \mathbb{M}^2$. Denote by $S(R)$ the graph of f over B_R. If Σ is not an entire graph then we let R be the largest such R so that f exists. Since Σ has constant mean curvature, f has bounded gradient on relatively compact subsets of B_R.

Let $q \in \partial B_R$ be such that f does not extend to any neighborhood of q to an $H > 1/2$ graph.

Claim 2: For any sequence $q_n \in B_R$, converging to q, the tangent planes $T_{p_n}S(R)$, where $p_n = (q_n, f(q_n))$, converge to a vertical plane P. P is tangent to ∂B_R at q (after translation of $T_{p_n}S(R)$ to height zero in $\mathbb{M}^2 \times \mathbb{R}$).

Proof of Claim 2: The same proof as in [**15**, Claim 2].

Now, from Claim 1 and Claim 2, we know that for any sequence $q_n \in B_R$ converging to q, the $F(q_n)$ converge to $C_\delta(q) \times [-\delta, \delta]$.

Claim 3: For any $q_n \to q$, $q_n \in B_R$, we have $f(q_n) \longrightarrow +\infty$ or $f(q_n) \longrightarrow -\infty$.

Proof of Claim 3: Let γ be a compact horizontal geodesic of length ε starting at q, entering B_R at q, and orthogonal to ∂B_R at q. Let Γ be the graph of f over γ. Notice that Γ has no horizontal tangents at points near q since the tangent planes of $S(R)$ are converging to P. So assume f is increasing along γ as one converges to q. If f were bounded, then Γ would have a finite limit point (q, c) and Γ would have finite length up till (q, c). Since Σ is complete, $(q, c) \in \Sigma$. But then Σ would have a vertical tangent plane at (q, c), a contradiction. This proves Claim 3.

Now choose $q_n \in \gamma$, $q_n \longrightarrow q$, and $F(p_n)$ converges to $C_\delta(q) \times [-\delta, \delta]$. Let C be the complete curve of $\mathbb{M}^2$ with $q \in C$ and geodesic curvature $2H$, such that C contains $C_\delta(q)$, and parametrize C by arc length; denote $q(s) \in C$ the point at distance s on C from $q(0) = q$, $s \in \mathbb{R}$. Note that C may have self-intersections, and may be compact and smooth. Denote by $\gamma(s)$ a horizontal geodesic arc orthogonal to C at $q(s)$, $q(s)$ is the mid-point of $\gamma(s)$. Assume the length of each $\gamma(s)$ is 2ε and

$$\bigcup_{s \in \mathbb{R}} \gamma(s) = T_\varepsilon(C)$$

is the $\varepsilon-$tubular neighborhood of C.

Let $\gamma^+(s)$ be the part of $\gamma(s)$ on the mean concave side of C; so $\gamma = \gamma^+(0)$. More precisely, the mean curvature vector of Σ points down in $\mathbb{M}^2 \times \mathbb{R}$, and $f \to +\infty$ as one approaches a along γ, so C is concave towards B_R; i.e., B_R is on the concave side of $C_\delta(q)$ at q.

Claim 4: For n large, each $F(q_n)$ is disjoint from $C \times \mathbb{R}$. Also, for $|s| \leq \delta$, $F(q_n) \cap (\gamma(s) \times \mathbb{R})$ is a vertical graph over an interval of $\gamma(s)$.

Proof of Claim 4: Choose n_0 so that for $n \geq n_0$, $\Gamma_n(s) = F(q_n) \cap (\gamma(s) \times \mathbb{R})$ is one connected curve of transverse intersection, for each $s \in [-\delta, \delta]$. Since the $F(q_n)$ are C^2-close to $C_\delta(q) \times [-\delta, \delta]$, $\Gamma_n(s)$ has no horizontal or vertical tangents and is a graph over an interval in $\gamma(s)$.

We now show that this interval is in $\gamma^+(s) - q(s)$. Suppose not, so $\Gamma_n(s)$ goes beyond $C \times \mathbb{R}$ on the convex side. Recall that Γ (see Claim 3) is the graph of f over γ and $f \longrightarrow +\infty$ as one goes up on Γ. We have $p_n = (q_n, f(q_n))$. Fix $n \geq n_0$ and choose new points q_k, $k \geq n$, so that $f(q_{k+1}) - f(q_k) = \delta$; clearly $q_k \longrightarrow q$ as $k \longrightarrow +\infty$. Lift each $\Gamma_k(s)$ to $G(p_k)$ by the vertical translation of $\Gamma_k(s)$ by $f(q_k)$. The curve $\Gamma(s) = \bigcup_{k \geq n} \Gamma_k(s)$ is a vertical graph over an interval in $\gamma(s)$. It has points in the convex side of $C \times \mathbb{R}$ for some $s_0 \in [-\delta, \delta]$. For $s = 0$, $\Gamma(0) = \Gamma$ stays on the concave side of $C \times \mathbb{R}$. So, for some s_1, $0 < s_1 \leq s_0$, $\Gamma(s_1)$ has a point on $C \times \mathbb{R}$ and also inside the convex side of $C \times \mathbb{R}$.

But the $F(q_k)$ converge uniformly to $C_\delta(q) \times [-\delta, \delta]$ as $k \longrightarrow +\infty$, so the curve $\Gamma(s_1)$ converges to $q(s_1) \times \mathbb{R}$ as the height goes to $+\infty$. This obliges $\Gamma(s_1)$ to have a vertical tangent on the convex side of $C \times \mathbb{R}$, a contradiction. This proves Claim 4.

Now we choose $\varepsilon_1 < \varepsilon$ (which we call ε as well) so that $\bigcup_{s \in [-\delta, \delta]} \Gamma(s)$ is a vertical graph of a function g on $\bigcup_{s \in [-\delta, \delta]} (\gamma^+(s) - q(s))$, (the $\gamma^+(s)$ have length ε_1).

Before we continue, note that until here, the proof of Theorem 4.1 is the same proof as in [**15**, Theorem 1.2] with slight modifications. Now the proof continues differently.

The graph of g converges to $C_\delta(q) \times \mathbb{R}$ as the height goes to infinity.

Now we begin this process again, replacing Γ by the curve $\Gamma(\delta)$. This analytically continues the graph g to a graph over $\bigcup_{s \in [-\delta, 2\delta]} (\gamma^+(s) - q(s))$ which converges uniformly to $C(q, [-\delta, 2\delta]) \times \mathbb{R}$ as the height goes to infinity. Here $C(q, [-\delta, 2\delta])$ denotes the arc of C, of length 3δ, between the points $q(-\delta)$ and $q(2\delta)$. We now continue analytically, by extending the graph about $C(2\delta)$. When we refer here to analytic continuation, we mean the unique continuation of the local pieces of the surface.

We want to extend so that the surface we obtain is within the $\varepsilon-$tubular neighborhood, $T_\varepsilon(C \times \mathbb{R}) \equiv T_\varepsilon(C) \times \mathbb{R}$, of $C \times \mathbb{R}$. Note that the graph of g on $\bigcup_{s \in [-\delta, \delta]} (\gamma^+(s) - q(s))$ has this property.

To do this, we go up high enough on $\Gamma(\delta)$ (and $\Gamma(-\delta)$), to height t_1 say, so that all the curves $\Gamma(s)$ starting at height t_1, for $s \in [\delta, 2\delta] \cup [-2\delta, -\delta]$, are $\varepsilon-$close to $C \times \mathbb{R}$.

Now continue this process replacing $\Gamma(\delta)$ and $\Gamma(-\delta)$ by $\Gamma(2\delta)$ and $\Gamma(-2\delta)$; again, going up high enough on these curves so that the graph, possibly immersed if C is not embedded, is within $T_\varepsilon(C \times \mathbb{R})$.

Let M denote the surface obtained by this analytic continuation, M is a union of curves $\Gamma(s)$, starting at different heights, each $\Gamma(s)$ is a graph over $\gamma^+(s)$, converging uniformly to $q(s) \times R^+$.

Let $T_\varepsilon^+(\widetilde{C}\times\mathbb{R})$ be the universal covering space of

$$\bigcup_{s\in\mathbb{R}}(\gamma^+(s)\times\mathbb{R}),$$

and we recall that each $\gamma^+(s)$ is the geodesic of length ε starting at $q(s)$, orthogonal to C, and going to the side of C where M was constructed.

Now, $\widetilde{C}$ is diffeomorphic to $\mathbb{R}$ and $\widetilde{C}\longrightarrow C$ is the immersion of C in $\mathbb{M}^2$. $T_\varepsilon^+(\widetilde{C}\times\mathbb{R})$ is an $\varepsilon-$(one-sided) tubular neighborhood of $\widetilde{C}\times\mathbb{R}$. We give $T_\varepsilon^+(\widetilde{C}\times\mathbb{R})$ the metric induced by that of $\mathbb{M}^2\times\mathbb{R}$. We lift M to an $H-$surface $\widetilde{M}\subset T_\varepsilon^+(\widetilde{C}\times\mathbb{R})$, $\widetilde{M}$ is asymptotic to $\widetilde{C}\times\mathbb{R}$ as the height goes to infinity. Let $\beta=\partial\widetilde{M}$.

Claim 5: The surface

$$Q=\widetilde{C}\times\mathbb{R}$$

is an unstable $H-$surface.

Proof of Claim 5: The stability operator J of Q is

$$\Delta+|A|^2+\mathrm{Ric}(\overrightarrow{n}),$$

where Δ is the Laplacian operator associated to the Riemannian metric induced on Q, $|A|^2$ is the square of the norm of the shape operator associated to Q and $\mathrm{Ric}(\overrightarrow{n})$ is the Ricci curvature in the direction of the unit normal vector field $\overrightarrow{n}$ along Q. Since Q is part of a vertical cylinder, the extrinsic curvature vanishes identically on Q and the unit normal $\overrightarrow{n}$ is horizontal, hence

$$J\equiv\Delta+a,$$

where $a=4H^2+\kappa\geq 4H^2-1>0$.

Consider the operator J on $[0,L]\times[-r,r]$ for $r>0$, where $[0,L]$ is an interval of length L on $\widetilde{C}$.

It is well known that

$$\varphi_1(t)=\cos\left(\frac{\pi t}{L^2}\right)$$

is a first eigenfunction of Δ on $[0,L]$, with eigenvalue $\lambda_1=\frac{\pi^2}{L^2}$. Similarly, a first eigenfunction φ_2 of Δ on $[-r,r]$ is

$$\varphi_2(t)=\cos\left(\frac{\pi t}{2r}\right)$$

with eigenvalue $\lambda_2=\frac{\pi^2}{4r^2}$.

Let $\varphi=\varphi_1\times\varphi_2$, so that

$$\Delta\varphi+(\lambda_1+\lambda_2)\varphi=0\ ,\ \text{on}\ [0,L]\times[-r,r],$$

then,

$$J\varphi+(\lambda_1+\lambda_2-a)\varphi=0\ ,\ \text{on}\ [0,L]\times[-r,r].$$

Hence, if r and L satisfy

$$\lambda_1 + \lambda_2 - a < 0,$$

then the domain is unstable.

This condition is

$$\frac{\pi^2}{L^2} + \frac{\pi^2}{4r^2} < 4H^2 + \kappa, \tag{21}$$

but for L and r large enough (note that we identify $\widetilde{C}$ with $\mathbb{R}$ and we can choose L large), it is clear that

$$\frac{\pi^2}{L^2} + \frac{\pi^2}{4r^2} < 4H^2 - 1,$$

so condition (21) is fulfilled. And the Claim 5 is proved.

Start with a compact stable domain K_0 of Q. Let K_0 expand until one reaches a stable-unstable domain K of Q, K compact. This means there is a smooth function $f : K \longrightarrow \mathbb{R}$, $f = 0$ on ∂K, $f > 0$ on $\mathrm{int}K$, and f satisfies

$$Jf + \lambda f = 0,\ \lambda < 0.$$

Let $K(t)$ be the variation of K given by

$$K(t) = \exp_p(tf(p)N(p)),$$

where $p \in K$ and $N(p)$ is a unit normal to K. $K(t)$ is a smooth surface with $\partial K(t) = \partial K \subset Q$, and for t small, $t \neq 0$, $\mathrm{int}K(t) \cap Q = \emptyset$.

Since the linearized operator J is the first variation of the mean curvature at $t = 0$, and $Jf(p) = -\lambda f(p) > 0$ for $p \in \mathrm{int}K$, we conclude $H(K(t)) > H$ for $t > 0$, and $H(K(t)) < H$ for $t < 0$.

Now on any compact set of Q, β is a positive distance from Q. So for t small enough the surfaces $K(t)$ are disjoint from β and they can be slid up and down Q to remain disjoint from β. But M is asymptotic to Q so for small $t > 0$, the surface $K(t)$ will touch M at a first point, when $K(t)$ is slid up or down Q. But this contradicts the Maximum Principle: If $M(1)$ is on the mean convex side of $M(2)$ near p, then the mean curvature of $M(1)$ at p is greater than or equal to the mean curvature of $M(2)$ at p. So, Claim A is proved.

This completes the proof of Theorem 4.1. q.e.d.

Remark 4.1. In [**22**], the second author proved the following result

Theorem A: Let N^3 be a complete riemannian manifold, and suppose, H and $C > 0$ are constants satisfying

$$3H^2 + S \geq C,$$

where S is the scalar curvature function of N^3. Then, if Σ is a complete stable $H-$surface, one has

$$d_\Sigma(p, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3C}}.$$

When $\partial\Sigma = \emptyset$, Σ is topologically the sphere $\mathbb{S}^2$.

The proof of Theorem A involves studying the metric $d\tilde{s}^2 = uds^2$, where ds^2 is the induced metric on Σ, and u is a positive solution of the linearized operator

$$L = \Delta + |A|^2 + \mathrm{Ric}(n),$$

where Δ is the Laplacian operator associated to ds^2, $|A|^2$ is the square of the norm of the second fundamental form, and Ric is the Ricci curvature in the direction of the normal vector field, n, along Σ. This metric $d\tilde{s}^2$ has positive curvature, so when $\partial\Sigma = \emptyset$, Σ is a compact sphere.

This Theorem A gives some insight into Theorems 3.1 and 4.1 (Theorem 3.1 when $H > 0$ and Claim A of Theorem 4.1 when $H > 1/\sqrt{3}$) since in the case $3H^2 + S \geq C > 0$, Σ is a multi-graph implies Σ is a sphere. This is ruled out by looking at a highest and a lowest point of Σ.

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Institut de Mathmatiques
Université Paris VII
175 Rue du Chevaleret
75013 Paris, France
E-mail address: jespinar@ugr.es

Instituto de Matematica Pura y Aplicada
110 Estrada Dona Castorina
Rio de Janeiro 22460-320, Brazil
E-mail address: rosen@impa.br