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ESTIMATE OF THE CONFORMAL SCALAR CURVATURE EQUATION VIA THE METHOD OF MOVING PLANES. II

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1. Introduction

In this paper, we consider a sequence of positive C^2 solutions u_i of

$$\Delta u_i + K_i(x) u_i^{p_i} = 0 \quad \text{in } B_2 \ , \tag{1.1}$$

where $K_i(x)$ is a sequence of C^1 positive functions defined in $\overline{B}_2$, the ball with center at 0 and radius 2, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplacian of $\mathbb{R}^n$ with $n \geq 3$, and $1 < p_i \uparrow \frac{n+2}{n-2}$. *Throughout this paper, we always assume that K_i is bounded between two fixed positive constants.* One of the motivations in studying equation (1.1) arises from the problem of finding a metric conformal to the standard metric of $\mathbb{R}^n$ such that $K(x)$ is the scalar curvature of the new metric. Recently, there have been many works devoted to this problem. For details please see [2], [3], [6], [11], [15], [16], [23], $\cdots$, and the references therein. It has been shown that for a sequence of solution u_i of (1.1), the blow-up does not occur at a noncritical point of $\{K_i\}$. We refer [15] and [8] for a proof of this statement. Hence in this article, we will assume that 0 is the only critical point of $\{K_i\}$, that is, K_i satisfies the following:

(1.2) For any $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that

$$c(\epsilon) \leq |\nabla K_i(x)| \leq c_1$$

for $|x| \geq \epsilon$, where c_1 is a positive constant independent of i and ϵ.

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Assume that the order of the flatness of K_i at 0 is no less than $n-2$. The authors in [8] have proved that there exists a constant $c>0$ such that the inequality

$$u_i(x+x_i) \le c\ M_i^{-1}|x|^{-n+2} \tag{1.3}$$

holds for $|x|\le 1$, where $M_i = \max_{\overline{B}_1} u_i = u_i(x_i) \to \infty$ for some $x_i \in B_1$. Inequality (1.3) was also derived in [15] and [24] where a global solution of (1.1) on S^n was considered. In the same paper, we also showed by examples that, in order to have (1.3) hold, the assumption on the order of flatness of K at its critical points is optimal. In this paper, we want to consider the situation when the flatness of K_i at its critical points is less than or equal to $n-2$. To state our result, we assume that $K_i \in C^1(\overline{B}_2)$ and satisfies the following conditions:

$$\begin{cases} K_i(x) = K_i(0) + Q_i(x) + R_i(x) \quad \text{in a neighborhood of} \\ 0, \text{ where } Q_i(x) \text{ is a } C^1 \text{ homogeneous function of order} \\ \alpha_i \text{ satisfying} \\ \qquad c_1|x|^{\alpha_i-1} \le |\nabla Q_i(x)| \le c_2|x|^{\alpha_i-1} \\ \text{for some } \alpha_i>1, \text{ and } R_i(x) \text{ satisfies} \\ \qquad \sum_{s=0}^{1} |\nabla^s R_i(x)|\,|x|^{-\alpha_i+s} \to 0 \\ \text{as } |x|\to 0 \text{ uniformly in } i. \text{ Furthermore, we assume} \\ \text{that } K_i(x) \text{ converges uniformly to } K(x) \text{ as } i\to+\infty, \\ \lim_{i\to+\infty} \alpha_i = \alpha > 1 \text{ and } Q_i(x) \text{ converges to } Q(x) \text{ in} \\ C^1(S^{n-1}) \text{ as } i\to+\infty, \text{ where } Q(x) \text{ is a } C^1 \text{ homoge-} \\ \text{neous function of order } \alpha. \text{ For simplicity, we assume} \\ K(0) = n(n-2) \text{ throughout this paper.} \end{cases} \tag{1.4}$$

Let U_0 be the positive smooth solution of

$$\begin{cases} \Delta U_0(y) + n(n-2)U_0^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}^n\ , \\ U_0(0) = \max_{\mathbb{R}^n} U_0(x) = 1\ . \end{cases} \tag{1.5}$$

By a theorem of Caffarelli-Gidas-Spruck (see Corollary 8.2 and Theorem 8.1 in [5]), $U_0(y)$ is radially symmetric with respect to 0. Hence, (1.5) leads to $U_0(y) = \left(1+|y|^2\right)^{-(n-2)/2}$. In addition to (1.4), we also assume that Q satisfies

$$\begin{pmatrix} \int_{\mathbb{R}^n} \nabla Q(\xi+y) U_0^{\frac{2n}{n-2}}(y)\,dy \\ \int_{\mathbb{R}^n} Q(\xi+y) U_0^{\frac{2n}{n-2}}(y)\,dy \end{pmatrix} \neq 0 \quad \text{for all } \xi \in \mathbb{R}^n\ . \tag{1.6}$$

Our first result is

Theorem 1.1. *Suppose u_i is a sequence of positive C^2 solution of (1.1) with $p_i \leq \frac{n+2}{n-2}$ and $\lim\limits_{i\to+\infty} p_i = \frac{n+2}{n-2}$. Assume (1.2), (1.4) and (1.6) are satisfied with $1 < \alpha < n-2$. If we further assume that for any solution ξ of $\int_{\mathbb{R}^n} \nabla Q(x+\xi) U_0^{2n/(n-2)}(y)\,dy = 0$, we have $\int_{\mathbb{R}^n} Q(\xi+x) U_0^{2n/(n-2)}(x)\,dx > 0$. Then u_i is uniformly bounded in $\overline{B}_1$.*

Throughout this paper, $B(x,r)$ always denotes the open ball with center x and radius r. When $x=0$, we simply use B_r for $B(x,r)$. Suppose u_i is a sequence of solutions of (1.1) with $\max\limits_{\overline{B}_1} u_i \to +\infty$ as $i \to +\infty$. Let $S = \{x \,|\, |x| \leq 1$, and there exists $x_i \to x$ such that $\overline{\lim}_{i\to+\infty} u_i(x_i) = +\infty\}$ be the blow-up set of $\{u_i\}$. Assume (1.2) holds. Then, as mentioned above, we have $S = \{0\}$. The blow-up point 0 is called isolated, if there exists a positive constant c such that

$$u_i(x) \leq c\,|x-x_i|^{-\frac{2}{p_i-1}}$$

for $|x| \leq 1$, where $u_i(x_i) = \max\limits_{\overline{B_1}} u_i$. The concept of an isolated blow-up point was first introduced by R. Schoen.

Theorem 1.2. *Assume that (1.2) and (1.4) are satisfied with $1 < \alpha_i$, $\alpha \leq n-2$. Let u_i be a sequence of solutions of (1.1) with $p_i \leq \frac{n+2}{n-2}$, $\lim\limits_{i\to+\infty} p_i = \frac{n+2}{n-2}$ and $\max\limits_{\overline{B}_1} u_i \to +\infty$. Then 0 is an isolated blow-up point.*

In fact, we are going to prove

$$u_i(x)|x|^{\frac{n-2}{2}} \leq c\ , \tag{1.7}$$

a stronger result than Theorem 1.2. In particular, we have

$$|x_i| \leq c\, M_i^{-\frac{p_i-1}{2}}, \tag{1.8}$$

where $u_i(x_i) = \max_{\overline{B}_1} u_i = M_i$. Let $\xi = \lim_{i\to+\infty} M_i^{\frac{p_i-1}{2}} x_i$ and $\tau_i = \frac{n+2}{n-2} - p_i$. In Section 3, we will prove that ξ satisfies

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\, dy = 0 \ , \tag{1.9}$$

and τ_i satisfies

$$\tau_i \le c\, M_i^{-\frac{(p_i-1)\alpha_i}{2}}, \tag{1.10}$$

which, in turns, implies

$$\lim_{i\to+\infty} M_i^{\tau_i} = 1 \ . \tag{1.11}$$

The inequality (1.8) is important when we come to calculate integrals involving the term $u_i^{\frac{2n}{n-2}}$. When $\alpha \ge n-2$, we can show that 0 is a simple blow-up point. For a proof of this statement, we refer the reader to [8], [15] and [24].

Rewrite the equation (1.1) into $\Delta u_i + c_i(x) u_i = 0$, where $c_i(x) = K_i(x) u_i^{p_i-1}(x) \le c|x|^{-2}$ by (1.7). Then, the Harnack inequality can be applied to u_i, i.e., there exists a constant $c > 0$ such that

$$\max_{|x|=r} u_i \le c \min_{|x|=r} u_i \ . \tag{1.12}$$

With the help of the Pohozaev identity, we have

Theorem 1.3. *Suppose that (1.2), (1.4) and (1.6) are satisfied with* $\frac{n-2}{2} \le \alpha_i \le n-2$, *and* u_i *is a sequence of* C^2 *positive solutions of (1.1) with* $p_i = \frac{n+2}{n-2}$. *Suppose* $M_i = \max_{\overline{B}_1} u_i \to +\infty$ *as* $i \to +\infty$. *Let* $m_i = \min_{\overline{B}_1} u_i$. *Then there exists a constant* $c > 0$ *such that the followings hold:*

$$u_i(x + x_i) \le c\, M_i^{-1} |x|^{2-n} \quad \text{for } |x| \le M_i^{-\beta_i}, \tag{1.13}$$

where $u_i(x_i) = M_i$ *and* $\beta_i = \frac{2}{n-2}\left(1 - \frac{\alpha_i}{n-2}\right) \ge 0$.

$$c^{-1} M_i^{1-\frac{2\alpha_i}{n-2}} \le u_i(x) \le c\, M_i^{1-\frac{2\alpha_i}{n-2}} \quad \text{for } |x| \ge \frac{1}{2} M_i^{-\beta_i}. \tag{1.14}$$

In particular,

$$\begin{cases} \lim_{i\to+\infty} m_i = 0 & \text{if } \alpha > \frac{n-2}{2}, \\ c^{-1} \le m_i \le c & \text{if } \alpha_i = \frac{n-2}{2}. \end{cases} \tag{1.15}$$

And for the energy, we have

$$\begin{cases} \lim_{i\to+\infty} \int_{B_1} K_i(x) u_i^{\frac{2n}{n-2}}(x)\,dx = \left(\frac{S_n}{n(n-2)}\right)^{\frac{n}{2}} \\ \qquad\qquad \text{if } \alpha > \frac{n-2}{2}, \\ \lim_{i\to+\infty} \int_{B_r} K_i(x) u_i^{\frac{2n}{n-2}}(x)\,dx = \left(\frac{S_n}{n(n-2)}\right)^{\frac{n}{2}} (1+o(1)) \\ \qquad\qquad \text{if } \alpha = \frac{n-2}{2}, \end{cases} \tag{1.16}$$

where S_n is the best Sobolev constant and $o(1) \to 0$ as $r \to 0$.

For $\alpha < \frac{n-2}{2}$, we have

Theorem 1.4. *Suppose the assumption of Theorem 1.3 holds except that α satisfies $1 < \alpha < \frac{n-2}{2}$. Let u_i be a sequence of solutions of (1.1) with $p_i = \frac{n+2}{n-2}$ and $\max_{\overline{B_1}} u_i \to +\infty$ as $i \to +\infty$. Then*

$$\lim_{i\to+\infty} \int_{B_1} u_i^{\frac{2n}{n-2}}(x)\,dx = +\infty\ .$$

Furthermore, there exists a subsequence of u_i (still denoted by u_i) such that u_i converges to a singular solution u of (1.1) with a nonremovable singularity at 0. The conformal metric $ds^2 = u^{\frac{4}{n-2}}|dx|^2$ is complete in $\overline{B}_1 \setminus \{0\}$ and has unbounded curvature near 0. If we assume 0 is the only zero of

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\,dy = 0\ . \tag{1.17}$$

Then $u(x) = \overline{u}(|x|)(1 + o(1))$ as $x \to 0$, where $\overline{u}(r)$ denotes the integral average of u over the sphere $|x| = r$.

Let u be the singular solution in Theorem 1.4 and

$$\begin{aligned} P(r,u) = \int_{|x|=r} \Bigg(&\frac{n-2}{2} u(x) \frac{\partial u}{\partial \nu} - \frac{|x|}{2} |\nabla u|^2 + |x| \left| \frac{\partial u}{\partial \nu} \right|^2 \\ &+ \frac{n-2}{2n} K(x) |x| u^{\frac{2n}{n-2}}(x) \Bigg)\, d\sigma\ . \end{aligned} \tag{1.18}$$

By the Pohozaev identity, we have for $r \geq s$,

$$P(r;u) - P(s;u) = \int_{s \leq |x| \leq r} (x \cdot \nabla K(x))\, u^{\frac{2n}{n-2}}(x)\, dx \,. \tag{1.19}$$

Since $u(x) \leq c\,|x|^{-\frac{n-2}{2}}$ by Theorem 1.2, $(x \cdot \nabla K(x))\, u^{\frac{2n}{n-2}} \in L^1(B_1)$. Thus, $\lim_{r\to 0} P(r;u) = D$ is always well-defined. Since u is a limit of a sequence of smooth solutions of (1.1), we can prove

$$D = 0 \,. \tag{1.20}$$

This is a new phenomenon different from the case with a constant K. When $K(x) \equiv 1$ and u is a singular solution of

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad B_1 \setminus \{0\} \,,$$

the famous theorem of Caffarelli-Gidas-Spruck says that if 0 is a nonremovable singularity, then there exists an entire singular solution $u_0(x) = u_0(|x|)$ of

$$\begin{cases} \Delta u_0(x) + u_0^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \\ \lim_{|x|\to 0} u_0(x) = +\infty \end{cases} \tag{1.21}$$

satisfying

$$u(x) = u_0(|x|)(1 + o(1)) \,. \tag{1.22}$$

Since the Pohozaev constant $D < 0$ for any solution u_0 of (1.21), as a consequence of (1.20), there exist no entire solutions of (1.21) satisfying (1.22) for this particular u of Theorem 1.4. However, if $\alpha \geq \frac{n-2}{2}$, then the result of Caffarelli-Gidas-Spruck still holds true. We refer the interested readers to [9] for related results.

The estimates of Theorem 1.3 and Theorem 1.4 are important when we want to find an apriori bound for solutions of (1.1) globally defined on S^n. As an application of Theorem 1.3, we proved the following theorem in [10].

Theorem A. *Let K be a positive C^1 function on S^n. Suppose for each critical point P of K, when using the coordinate in $\mathbb{R}^n$ of the stereographic projection from S^n with P as the South pole, K satisfies*

(1.4) and (1.6) with $\frac{n-2}{2} < \alpha < n-2$. *Then there exists a constant* $c > 0$ *such that*

$$u(x) \le c$$

for all $x \in S^n$ *and for all positive solutions of*

$$\frac{4(n-1)}{n-2}\Delta_0 u + n(n-1)u + K(x)u^{\frac{n+2}{n-2}} = 0 \ , \tag{1.23}$$

where Δ_0 *is the Beltrami-Laplacian operator of the standard* S^n.

A special case of Theorem A is

Corollary 1.5. *Suppose* K *is a positive Morse function in* S^n *with* $\Delta K(P) \neq 0$ *for any critical point* P *of* K. *There exists a constant* $c > 0$ *such that for any solution* u *of (1.23), we have*

$$\begin{cases} u(x) \le c & \text{for } n = 5, \\ \int_{S^n} |\nabla u|^2 + \int_{S^n} u^{\frac{2n}{n-2}} \le c & \text{for } n = 6. \end{cases} \tag{1.24}$$

At the first sight, we might apply the degree theory developed by Chang-Yang [11] and Li [15] to find a solution of (1.23). However, a study of radial solutions suggests that the Leray-Schauder degree might be zero in the situation of Theorem A. In a forthcoming paper, we will compute the degree for all solutions of equation (1.23). An immediate consequence of Theorem 1.4 is

Corollary 1.6. *Suppose* K *is a Morse function in* S^n *and satisfies* $\Delta K(P) \neq 0$ *for any critical point* P *of* K. *Let* u_i *be a sequence of solutions of (1.23) with* $\max_{S^n} u_i \to +\infty$ *as* $i \to +\infty$. *Then*

$$\lim_{i \to +\infty} \int_{S^n} K(x) u_i^{\frac{2n}{n-2}}(x)\, dx = +\infty$$

if $n \ge 7$.

The possibility of blowing-up with infinite energy was first mentioned in [21]. It should be an interesting queation whether we can find a blowing-up sequence of solutions in the situation of Corollary 1.6. For the existence of solutions of (1.23) for $n \ge 7$, we refer [11], [1] and [24].

As in [8], there are two main ingradients in our approach. One is the blowing-up anaysis, introduced first by Schoen. Another one is the well-known "method of moving planes", which was first invented by A. D. Alexandrov and has been further developed by Serrin, Gidas-Ni-Nirenberg and Caffarelli-Gidas-Spruck. In this paper, the method of moving planes is used to show that how large of the domain where rescaled solutions can be compared to $U_0(y)$ of (1.5). This is the major step in our approach. See Lemma 3.1 in Section 3.

This paper is organized as follows. In Section 2, we will collect some preliminary results for later uses. Most of them are well-known. However, we will present their proofs here to make the paper self-contained. In Section 3, Theorem 1.1 is proved. Theorem 1.2 will be proved in Section 4. In the final section, both Theorem 1.3 and Theorem 1.4 are proved. In forthcoming papers, we will present some applications of our estimates to equation of (1.1) on S^n.

2. Preliminary results

In this section, we will collect several lemmas which are useful later. First, we formulate a modified version of the well-known methods of moving planes. Let Ω be a smooth open domain in $\mathbb{R}^n$ such that *the complement set* Ω^c *of* Ω *is compact.* Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution of

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega \ ,$$

where $f(x, u)$ is a nonnegative function, Hölder in x, C^1 in $u > 0$ and is defined on $\overline{\Omega} \times [0, \infty)$. For $\lambda < 0$, we denote $T_\lambda = \{x \in \mathbb{R}^n |\, x_1 = \lambda\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^n |\, x_1 > \lambda\}$ and $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$ as the reflection point of x with respect to T_λ. Let

$$\text{(2.1)} \qquad \begin{cases} \lambda^* \equiv \sup\{\lambda |\, \lambda < 0 \quad \text{and} \quad \Omega^c \subset \Sigma_\lambda\} \ , \\ \Sigma'_\lambda = \Sigma_\lambda \cap \Omega \quad \text{for} \quad \lambda < \lambda^* \ , \text{ and} \\ w_\lambda(x) = u(x) - u_\lambda(x) \equiv u(x) - u(x^\lambda) \text{ for } x \in \Sigma'_\lambda \ . \end{cases}$$

For any continuous function $b_\lambda(x)$, we have

$$\text{(2.2)} \qquad \Delta w_\lambda(x) + b_\lambda(x) w_{\lambda(x)} \equiv Q(x, b_\lambda(x)) \text{ in } \Sigma'_\lambda$$

where

$$\text{(2.3)} \qquad Q(x, b_\lambda(x)) = f(x^\lambda, u^\lambda(x)) - f(x, u(x)) + b_\lambda(x) w_\lambda(x) \ .$$

Suppose that $h_\lambda(x)$ and $b_\lambda(x)$ are two families of continuous nonnegative functions defined for $x \in \overline{\Omega}$ and $\lambda_1 \le \lambda \le \lambda_0$ with two constants λ_0 and

$\lambda_1 < \lambda^*$ such that the following conditions are satisfied.

$$0 \le b_\lambda(x) \le C(x)|x|^{-2} \quad \text{for } x \in \Sigma'_\lambda \ , \tag{2.4}$$

where $C(x)$ is independent of λ and tends to zero as $|x| \to +\infty$.

The function $h_\lambda(x)$ is $C^1(\overline{\Sigma}'_\lambda)$ and satisfies

$$\begin{cases} \Delta h_\lambda(x) \ge Q(x, b_\lambda(x)) & \text{in } \Sigma'_\lambda, \\ h_\lambda(x) > 0 & \text{in } \Sigma'_\lambda \end{cases} \tag{2.5}$$

in the distributional sense for $\lambda \in [\lambda_1, \lambda_0]$.

(2.6) $h_\lambda(x) = 0$ on T_λ and $h_\lambda(x) = O(|x|^{-\tau_1})$ as $|x| \to +\infty$ for some constant $\tau_1 > 0$.

$$\begin{cases} h_\lambda(x) < w_\lambda(x) & \text{for } x \in \partial\Omega, \lambda_1 \le \lambda \le \lambda_0 \text{ and}, \\ h_{\lambda_1}(x) \le w_{\lambda_1}(x) & \text{for } x \in \Sigma'_{\lambda_0} \ . \end{cases} \tag{2.7}$$

(2.8) Both $h_\lambda(x)$ and $\nabla_x h_\lambda(x)$ are continuous with respect to both variables x and λ on $\overline{\Sigma}'_\lambda$.

Lemma 2.1. *Let u be a solution of (2.1) satisfying $u(x) = O(|x|^{-\tau_2})$ at ∞ for some $\tau_2 > 0$. Suppose there are two families of continuous nonnegative functions $b_\lambda(x)$ and $h_\lambda(x)$ satisfying (2.4) $\sim$ (2.8) for $\lambda_1 \le \lambda \le \lambda_0$ with $\lambda_0 < \lambda^*$. Then $w_\lambda(x) > 0$ for $x \in \Sigma'_\lambda$ and $\lambda \in [\lambda_0, \lambda_1]$.*

Proof. Lemma 2.1 is a special case of Lemma 2.1 in [8]. For the reader's convenience, we reproduce the proof here.

Step 1. There exists $R_0 > 0$, independent of λ, such that if $(w_\lambda - h_\lambda)(x)$ is negative somewhere in Σ'_λ, and $x_0 \in \Sigma'_\lambda$ is a minimum point of $w_\lambda - h_\lambda$, then $|x_0| < R_0$.

By (2.2) and (2.5), we have

$$\Delta(w_\lambda - h_\lambda) + b_\lambda(w_\lambda - h_\lambda) \le -b_\lambda h_\lambda \le 0 \tag{2.9}$$

in Σ'_λ. Let $0 < \sigma < \min(\tau_1, \tau_2, n-2)$ and $g(x) = |x|^{-\sigma}$. Set $\phi(x) = \frac{w_\lambda(x) - h_\lambda(x)}{g(x)}$. Then ϕ satisfies

$$\Delta\phi + 2\frac{\nabla g}{g} \cdot \nabla\phi + \left(b_\lambda(x) + \frac{\Delta g}{g}\right)\phi \le 0 \ . \tag{2.10}$$

By (2.4), we note that

$$b_\lambda(x) + \frac{\Delta g}{g(x)} = \left(C(x) - \sigma(n-2-\sigma)\right)|x|^{-2} < 0$$

for large $|x|$. Hence, there is a large R_0 with $\Omega^c \subseteq B_{R_0}$ such that

$$b_\lambda(x) + \frac{\Delta g(x)}{g} < 0 \tag{2.11}$$

for $|x| \geq R_0$. Now suppose $w_\lambda - h_\lambda(x_0) = \inf_{\Sigma'_\lambda}(w_\lambda - h_\lambda) < 0$ for some $x_0 \in \Sigma'_\lambda$. Then we want to show $|x_0| < R_0$.

Since $\lim_{|x|\to+\infty} \phi(x) = 0$ and $\phi(x) \geq 0$ on $\partial\Sigma'_\lambda$, there exists $\overline{x}_0$ such that ϕ has its minimum at $\overline{x}_0$. By applying the maximum principle at $\overline{x}_0$, (2.10) implies

$$b_\lambda(\overline{x}_0) + \frac{\Delta g(\overline{x}_0)}{g} \geq 0 \ .$$

By (2.11), we have $|\overline{x}_0| \leq R_0$. Since

$$\begin{aligned}\frac{w_\lambda(x_0) - h_\lambda(x_0)}{g(\overline{x}_0)} \leq \frac{(w_\lambda - h_\lambda)(\overline{x}_0)}{g(\overline{x}_0)} &= \phi(\overline{x}_0) \\ \leq \phi(x_0) &= \frac{w_\lambda(x_0) - h_\lambda(x_0)}{g(x_0)} \ ,\end{aligned}$$

we have $|x_0| \leq |\overline{x}_0| \leq R_0$. Hence Step 1 is proved.

From (2.7) and (2.9), it follows that $w_{\lambda_1} - h_{\lambda_1}$ is a nonegative superharmonic function in Σ'_{λ_1} and is strictly positive on $\partial\Omega$. Hence, by the maximum principle, $w_{\lambda_1} - h_{\lambda_1} > 0$ in Σ'_{λ_1}. Let

$$\tilde{\lambda} = \sup\left\{\lambda \geq \lambda_0 | (w_\mu - h_\mu)(x) > 0 \text{ in } \Sigma'_\mu \text{ for all } \lambda_1 \leq \mu \leq \lambda\right\} \ .$$

It suffices to prove

Step 2. $\tilde{\lambda} = \lambda_0$.

We prove Step 2 by contradiction. Suppose $\tilde{\lambda} < \lambda_0$. Then there exists $\lambda_n \downarrow \tilde{\lambda}$ with $\lambda_n < \lambda_0$, and $\inf_{\Sigma'_{\lambda_n}}(w_{\lambda_n} - h_{\lambda_n}) = (w_{\lambda_n} - h_{\lambda_n})(x_n) < 0$ for some $x_n \in \Sigma'_{\lambda_n}$, because $w_{\lambda_n} - h_{\lambda_n} \geq 0$ on $\partial\Sigma'_\lambda$ and $\lim_{|x|\to\infty}(w_{\lambda_n} - h_{\lambda_n})(x) = 0$. By Step 1, we have $|x_n| \leq R_0$. Without loss of generality, we may assume $\lim_{n\to+\infty} x_n = x_0 \in \overline{\Sigma}'_{\tilde{\lambda}}$. Thus,

$$\nabla(w_{\tilde{\lambda}} - h_{\tilde{\lambda}})(x_0) = 0 \text{ and } (w_{\tilde{\lambda}} - h_{\tilde{\lambda}})(x_0) \leq 0 \ . \tag{2.12}$$

Since $(w_{\tilde{\lambda}} - h_{\tilde{\lambda}})(x) \geq 0$ for $x \in \Sigma'_{\tilde{\lambda}}$, we have

$$\Delta(w_{\tilde{\lambda}} - h_{\tilde{\lambda}}) \leq -b_\lambda(w_{\tilde{\lambda}} - h_{\tilde{\lambda}}) \leq 0$$

in $\Sigma'_{\tilde\lambda}$. From the first part of (2.7) and the maximum principle, it follows that

$$w_{\tilde\lambda} - h_{\tilde\lambda}(x) > 0 \quad \text{for } x \in \Sigma'_{\tilde\lambda} .$$

Therefore, we have $x_0 \in T_{\tilde\lambda}$. However, the first part of (2.12) yields a contradiction to Hopf's boundary point Lemma. Hence, the proof of Lemma 2.1 is finished. q.e.d.

To apply Lemma 2.1 in the proofs of our theorems, we need the following lemma about the Green function $G^\lambda(x,\eta)$ of $-\Delta$ on Σ_λ with the Dirichlet boundary condition. The Green function has the form of

$$G^\lambda(x,\eta) = c_n\left(\frac{1}{|\eta - x|^{n-2}} - \frac{1}{|\eta - x^\lambda|^{n-2}}\right) \tag{2.13}$$

for $x, \eta \in \overline{\Sigma}_\lambda$, where c_n is a positive constant depending on n only.

Lemma 2.2. *There exists positive constants c_1 and c_2, depending on n only, such that the following statements hold:*
(i)

$$G^\lambda(x,0) \geq c_1 \begin{cases} |x|^{2-n} & \text{for } |x| \leq \frac{|\lambda|}{2} , \\ \frac{|\lambda||x_1-\lambda|}{|x|^n} & \text{for } |x| \geq \frac{|\lambda|}{2} . \end{cases}$$

(ii)

$$G^\lambda(x,\eta) \leq c_2 \min\left(|x-\eta|^{2-n}, (x_1-\lambda)|x-\eta|^{1-n}, \frac{(x_1-\lambda)(\eta_1-\lambda)}{|x-\eta|^n}\right) .$$

The proof of Lemma 2.2 is elementary. Please see, for example, [8] for a proof.

Lemma 2.3. *Suppose that u is a positive smooth solution of*

$$\Delta u + K(x)u^p = 0 \quad \text{in } B_{r_0} ,$$

where $0 < a \leq K(x) \leq b$ in B_{r_0} and $1 < p \leq \frac{n+2}{n-2}$. Then there exists a small positive number ϵ_0, depending on a, b and n only such that if $\|u\|_{L^{p^*}} \leq \epsilon_0$ with $p^* = \frac{(p-1)n}{2}$, then the Harnack inequality

$$u(x) \leq c\, u(y)$$

holds for $|x|, |y| \leq \frac{r_0}{4}$, where c is a positive constant depending on a, b and n.

Proof. Let $v(y) = r_0^{\frac{2}{p-1}} u(r_0 y)$ for $|y| < 1$. Then v satisfies

$$\Delta v + \tilde{K}(y) v^p = 0 \quad \text{in } |y| < 1,$$

where $\tilde{K}(y) = K(r_0 y)$. By the assumption, we have

$$\int_{B_1} v^{p^*}(y)\, dy = \int_{B_{r_0}} u^{p^*}\, dy \le \epsilon_0 \ .$$

Then we can apply the standard iteration technique due to Moser, as shown in [14] (see Lemma 6 in [14]), to obtain

$$\int_{|y| \le \frac{1}{2} + \frac{1}{2^k}} |v|^{p^*(\frac{n}{n-2})^k}\, dy \le c_k \int_{|y| \le \frac{1}{2} + \frac{1}{2^{k-1}}} |v|^{p^*(\frac{n}{n-2})^{k-1}}\, dy$$

for $k = 1, 2, \cdots$. Hence, after a finite number of iteration steps, we have $v^p \in L^q(B_{R_0})$ for some $q > \frac{n}{2}$ and some $R_0 > \frac{1}{2}$. By elliptic L^q theory, we have $\max_{B_{\frac{1}{2}}} v \le c$ for some constant. Applying Corollary 8.21 in [13] shows that there exists a constant $c_1 > 0$ such that

$$v(y) \le c_1\, v(y')$$

for $|y|, |y'| \le \frac{1}{4}$. Obviously, Lemma 2.3 follows immediately. q.e.d.

Lemma 2.4. *Suppose $\phi(y)$ satisfies*

$$\Delta\phi(y) + n(n+2) U_0^{\frac{4}{n-2}}(y)\phi(y) = 0 \ \textit{in } \mathbb{R}^n \tag{2.14}$$

with $\phi(y) \to 0$ as $|y| \to +\infty$, where $U_0(y)$ is the solution of (1.5). Then $\phi(y)$ can be written as

$$\phi(y) = c_0 \psi_0(y) + \sum_{j=1}^{n} c_j \psi_j(y)$$

for constants $c_j \in \mathbb{R}$, $j = 0, 1, \cdots, n$, where $\psi_j(y) = \frac{\partial U_0}{\partial y_j}$ for $1 \le j \le n$ and $\psi_0(y) = \frac{n-2}{2} U_0(y) + y \cdot \nabla U_0(y)$.

Proof. Let $\Phi_k(w)$ denote a spherical harmonic of degree k on S^{n-1} and $\phi_k(r) = \int_{|w|=1} \phi(rw)\Phi_k(w)\, ds$. We want to prove $\phi_k(r) \equiv 0$ for $k \ge 2$. Then the conclusion of Lemma 2.4 follows immediately.

It is obvious to see that ϕ_k satisfies

$$\begin{cases} \phi_k'' + \frac{n-2}{r}\phi_k' + \left(n(n+2)U_0^{\frac{4}{n-2}}(r) - \frac{k(n+k-2)}{r^2}\right)\phi_k = 0 \ , \\ \phi_k(0) = 0 \text{ and } \phi_k'(0) = 0 \ . \end{cases} \tag{2.15}$$

Let $\psi(r) = -U'(r)$. Differentiating (1.5) with respect to r, we have

$$\begin{cases} \psi''(r) + \frac{n-1}{r}\psi'(r) + \left(n(n+2)U_0^{\frac{4}{n-2}}(r) - \frac{n-1}{r^2}\right)\psi(r) = 0, \\ \psi(r) > 0 \text{ for } r > 0 \ . \end{cases} \tag{2.16}$$

Since $\psi(r) > 0$ for $r > 0$, by the Sturm-Liouville comparison Theorem, $\phi_k(r)$ does not change its sign for all $r \geq 0$ unless $\phi_k(r) \equiv 0$. We may assume $\phi_k(r) > 0$ for all $r > 0$. For any $R > 0$, we have

$$\begin{aligned} &R^{n-1}\left(\psi(R)\phi_k'(R) - \phi_k\psi'(R)\right) \\ &\quad = \int_0^R (\psi(r)\Delta\phi_k - \phi_k\Delta\psi(r))r^{n-1}\,dr \\ &\quad = [k(n+k-2) - (n-1)]\int_0^R \frac{\phi_k(r)\psi(r)}{r^2}r^{n-1}\,dr > 0 \ . \end{aligned} \tag{2.17}$$

Since $\psi'(R) = O(R^{-n})$ at ∞ and $\phi_k(\infty) = 0$, there exists $R_i \to +\infty$ as $i \to +\infty$ such that $\phi_k'(R_i) \leq 0$ and

$$\overline{\lim}_{i\to+\infty} R_i^{n-1}\left(\psi(R_i)\phi_k'(R_i) - \phi_k(R_i)\psi'(R_i)\right) \leq 0 \ ,$$

which yields a contradiction to (2.17). Hence Lemma 2.4 is proved. q.e.d.

3. Applications of the method of moving planes

In this section, we are mainly concerned with the proof of Theorem 1.1. The proof will be divided into several lemmas. The first one — Lemma 3.1 — is very important in our approach, and will be very useful later. To state it, we consider a sequence solution u_i of (1.1) and let x_i be a *local maximum point* of u_i in $\overline{B}$, with $M_i = u_i(x_i) \to +\infty$ as $i \to +\infty$. We assume K_i satisfies (1.2), (1.4) with $\alpha_i \leq n-2$. Let

$$v_i(y) = M_i^{-1}u_i\left(x_i + M_i^{-\frac{p_i-1}{2}}y\right) \ . \tag{3.1}$$

Obviously, $v_i(y)$ is defined in $|y| \le M_i^{\frac{p_i-1}{2}}$. In Lemma 3.1, v_i is always assumed to satisfy

$$v_i(y) \text{ is uniformly bounded in any bounded set of } \mathbb{R}^n. \tag{3.2}$$

Suppose v_i satisfies (3.2). Without loss of generality, we may assume $v_i(y)$ uniformly converges to $U_0(y)$ in any compact set of $\mathbb{R}^n$. Since v_i satisfies

$$\Delta v_i(y) + \tilde{K}_i(y) v_i^{p_i}(y) = 0 \quad \text{in} \quad |y| \le M_i^{\frac{p_i-1}{2}}, \tag{3.3}$$

where $\tilde{K}_i(y) = K_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right)$, U_0 must satisfy

$$\begin{cases} \Delta U_0 + n(n-2)U_0^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n, \\ U_0(0) = 1, \text{ and } 0 \text{ is a critical point of } U_0. \end{cases} \tag{3.4}$$

By a theorem of Caffarelli-Gidas-Spruck, U_0 is radially symmetric with respect to 0, and

$$U_0(y) = \left(1 + |y|^2\right)^{-\frac{n-2}{2}}. \tag{3.5}$$

In the followings, we let

$$L_i = \min\left\{ \left(M_i^{\frac{p_i-1}{2}} |x_i|^{1-\alpha_i} \right)^{\frac{1}{n-2}}, M_i^{\frac{(p_i-1)\alpha_i}{2(n-2)}} \right\}. \tag{3.6}$$

Obviously, $\lim_{i \to +\infty} L_i = +\infty$. Since

$$\left(M_i^{\frac{p_i-1}{2}} |x_i|^{1-\alpha_i} \right)^{\frac{1}{n-2}} = M_i^{\frac{(p_i-1)(\alpha_i)}{2(n-2)}} \left(M_i^{\frac{(p_i-1)}{2}} |x_i| \right)^{1-\alpha_i},$$

we have $L_i = \left(M_i^{\frac{p_i-1}{2}} |x_i|^{1-\alpha_i} \right)^{\frac{1}{n-2}}$ if $M_i^{\frac{p_i-1}{2}} |x_i| \ge 1$. From (3.6) and $\alpha_i \le n-2$, we always have $M_i^{\frac{p_i-1}{2}} \ge L_i$. Thus, $v_i(y)$ is well-defined for $|y| \le L_i$.

Lemma 3.1. *Assume v_i satisfies (3.2). Then, for any $\epsilon > 0$ there exist $\delta_1 = \delta_1(\epsilon) > 0$ and a positive integer $i_0 = i_0(\epsilon)$ such that for $i \ge i_0$, the inequality*

$$\min_{|y| \le r} v_i(y) \le (1+\epsilon) r^{2-n}$$

holds for all $0 \le r \le \delta_1 L_i$.

Proof. We will prove the lemma by contradiction. Suppose there exists $\epsilon_0 > 0$ such that $\min_{|y| \le r_i} v_i(y) \geqq (1 + 2\epsilon_0) r_i^{2-n}$ for some $r_i \le \delta L_i$, where δ is a small positive number which will be chosen later. Since $v_i(y)$ uniformly converges to $U_0(y)$ in any compact set of $\mathbb{R}^n$, we have $r_i \to +\infty$ as $i \to +\infty$. Let

$$\tilde{v}_i(y) = v_i(y + e_1) \quad \text{with } e_1 = (1, 0, \cdots, 0) \; .$$

Thus,

$$\tilde{v}_i(y) \geqq (1 + \epsilon_0) r_i^{2-n} \tag{3.7}$$

for $|y| \le r_i$. Let $\overline{v}_i(y)$ be the Kelvin transformation of $\tilde{v}_i$, that is,

$$\overline{v}_i(y) = |y|^{2-n} \tilde{v}_i \left(\frac{y}{|y|^2} \right) \; . \tag{3.8}$$

Then $\overline{v}_i$ satisfies

$$\Delta \overline{v}_i + \overline{K}_i(y) \overline{v}_i^{p_i} = 0 \quad \text{for} \quad |y| \ge M_i^{-\frac{p_i - 1}{2}}, \tag{3.9}$$

where $\overline{K}_i(y) = \widetilde{K}_i(y)|y|^{-\tau_i} \equiv K_i \left(x_i + M_i^{-\frac{p_i-1}{2}} \frac{y}{|y|^2} \right) |y|^{-\tau_i}$ and $\tau_i = (n+2) - (n-2)p_i \ge 0$. Since $\tilde{v}_i(y)$ converges to $U_0(y+e)$, $\overline{v}_i(y)$ converges to $\overline{U}_0(y)$ in C^2 in any compact set of $\mathbb{R}^n \cup \{\infty\} \setminus \{0\}$, where $\overline{U}_0(y) = |y|^{2-n} U_0 \left(\frac{y}{|y|^2} + e \right)$. By a straightforward computation, we can prove that $\overline{U}_0(y)$ is radially symmetric with respect to $y_0 = (-\frac{1}{2}, 0, \cdots, 0)$. Therefore, $\overline{v}_i(y)$ has a local maximum y_i near y_0 for large i.

Let $-\frac{1}{2} < \lambda_0 \le -\frac{1}{4}$, where λ_0 will be chosen to be sufficiently close to $-\frac{1}{2}$. For $\lambda \le \lambda_0$, as in Section 2, let $T_\lambda = \{x \mid x_1 = \lambda\}$, $\Sigma'_\lambda = \{x \mid x_1 > \lambda,\ |x| \ge r_i^{-1}\}$ and $x^\lambda = (2\lambda - x_1, \cdots, x_n)$ denote the reflection point of x with respect to T_λ. We claim for large i,

$$\overline{v}_i(y^\lambda) < \overline{v}_i(y) \tag{3.10}$$

holds for $y \in \Sigma'_\lambda$ and $\lambda \le \lambda_0$. Obviously, (3.10) yields a contradiction to the fact that $\overline{v}_i(y)$ has a local maximum at y_i.

Let $w_\lambda(y) = \overline{v}_i(y) - \overline{v}_i(y^\lambda)$. (The index i is omitted for the sake of simplicity.) Then w_λ satisfies

$$\Delta w_\lambda + b_\lambda(y) w_\lambda(y) = Q_\lambda(y) \quad \text{in} \quad \Sigma'_\lambda, \tag{3.11}$$

where $b_\lambda(y) = \overline{K}_i(y)\left(\overline{v}_i(y)^{p_i} - \overline{v}_i(y^\lambda)^{p_i}\right)\left(\overline{v}_i(y) - \overline{v}_i(y^\lambda)\right)^{-1}$, and $Q_\lambda(y) = \left(\overline{K}_i(y^\lambda) - \overline{K}_i(y)\right)\left(\overline{v}_i(y^\lambda)\right)^{p_i}$.

By (3.7) and (3.8), for $|y| = r_i^{-1}$ we have

$$\overline{v}_i(y) \geq r_i^{n-2} \min_{|y|\leq r_i} \tilde{v}_i \geq 1 + \epsilon_0. \tag{3.12}$$

On the other hand, $\overline{v}_i\left(y^{-\frac{1}{2}}\right)$ uniformly converges to $\overline{U}_0(0^{-\frac{1}{2}}) = \overline{U}_0(0) = 1$ for $|y| = r_i^{-1}$, where $y^{-\frac{1}{2}}$ and $0^{-\frac{1}{2}}$ are the reflection point of y and 0 with respect to the hyperplane $T_{-\frac{1}{2}}$ respectively. Hence, there exists $-\frac{1}{4} \geq \lambda_0 > -\frac{1}{2}$ such that

$$\overline{v}_i(y^\lambda) \leq 1 + \frac{\epsilon_0}{2} \tag{3.13}$$

for $|y| = r_i^{-1}$, $\lambda \leq \lambda_0$ and large i. Together with (3.12), it implies that when $|y| = r_i^{-1}$, we have

$$w_\lambda(y) \geq \frac{\epsilon_0}{2} \tag{3.14}$$

for $\lambda \leq \lambda_0$ and large i. In the followings, $\lambda_0 > -\frac{1}{2}$ is chosen so that the inequality

$$w_\lambda(y) \geq \frac{\epsilon_0}{2} \geq c_0 r_i^{-(n-2)} G^\lambda(y,0) \tag{3.15}$$

holds for $|y| = r_i^{-1}$, $\lambda \leq \lambda_0$ and large i, where c_0 is a constant depending on ϵ_0 and n only.

Since $\overline{v}_i$ has a harmonic asymptotic expansion at ∞, we have

$$\begin{cases} \overline{v}_i(y) = |y|^{2-n}\left(\overline{c}_{0,i} + \sum_{j=1}^n \overline{c}_{j,i} \frac{y_i}{|y|^2}\right) + O\left(\frac{1}{|y|^n}\right), \\ \frac{\partial \overline{v}_i}{\partial y_1}(y) = -(n-2)\frac{c_{0,i}\, y_1}{|y|^n} + O\left(\frac{1}{|y|^n}\right), \end{cases} \tag{3.16}$$

where constants $\overline{c}_{0,i}$ and $\overline{c}_{j,i}$ converge to some $\overline{c}_0 > 0$ and $\overline{c}_j$ as $i \to +\infty$. By elementary calculations and Lemma 2.2, there are constants c_1 and $c_2 > 0$ such that

$$\begin{aligned} w_\lambda(y) &= \overline{v}_i(y) - \overline{v}_i(y^\lambda) \\ &\geq c_1 \begin{cases} \frac{(y_1-\lambda)|\lambda|}{|y|^n} & \text{if } |y^\lambda| \leq 2|y| \\ \frac{1}{|y|^{n-2}} & \text{if } |y^\lambda| \geq 2|y| \end{cases} \\ &\geq c_2\, G^\lambda(y,0) \end{aligned} \tag{3.17}$$

for $y \in \Sigma_\lambda$, $\lambda \le \lambda_1 < 0$ and $|y| \ge R$ if both $|\lambda_1|$ and R are sufficiently large, but independent of i. (For a proof, see Lemma 2.3 in [5].) Since $\overline{v}_i$ is superharmonic in Σ'_λ and $\overline{v}_i \ge 1$ on $|y| = r_i^{-1}$, for $r_i^{-1} \le |y| \le R$ and $y \in \Sigma'_\lambda$ we have

$$\overline{v}_i(y) \ge \inf_{|y|=R} \overline{v}_i \ge c_3 > 0 \ ,$$

where c_3 is a constant independent of i. Hence, if $|\lambda_1|$ is sufficiently large, then

$$w_\lambda(y) \ge \frac{c_3}{2}$$

for $r_i^{-1} \le |y| \le R$ and $\lambda \le \lambda_1 < 0$. Since w_λ is superharmonic in Σ'_λ for $\lambda \le \lambda_1$, by (3.15), we have for large i

$$w_\lambda(y) \ge c_0\, r_i^{-(n-2)} G^\lambda(y, 0) \tag{3.18}$$

for $y \in \Sigma'_\lambda$ and $\lambda \le \lambda_1$.

Let $Q_\lambda^+ = \max(0, Q_\lambda)$, and set

$$h_\lambda(y) = A L_i^{n-2} G^\lambda(y, 0) - \int_{\Sigma_\lambda} G^\lambda(y, \eta) Q_\lambda^+(\eta)\, d\eta, \tag{3.19}$$

where $G^\lambda(y, \eta)$ is the Green's function in Section 2, and A is a positive constant to be chosen later. Obviously, h_λ satisfies

$$\Delta h_\lambda = Q_\lambda^+(y) \ge Q_\lambda(y) \quad \text{in} \quad \Sigma'_\lambda. \tag{3.20}$$

Since $|\eta^\lambda| \ge |\eta|$ for $\eta \in \Sigma_\lambda$ and $\lambda \le \lambda_0 \le -\frac{1}{4}$, we have

$$\begin{aligned} Q_\lambda(y) &= \left(\widetilde{K}_i(\frac{\eta^\lambda}{|\eta^\lambda|^2}) |\eta^\lambda|^{-\tau_i} - \widetilde{K}_i(\frac{\eta}{|\eta|^2}) |\eta|^{-\tau_i} \right) \overline{v}_i^{p_i}(\eta^\lambda) \\ &\le \left(\widetilde{K}_i(\frac{\eta^\lambda}{|\eta^\lambda|^2}) - \widetilde{K}_i(\frac{\eta}{|\eta|^2}) \right) |\eta^\lambda|^{-\tau_i} \overline{v}_i^{p_i}(\eta^\lambda) \ . \end{aligned}$$

Hence,

$$\begin{aligned} Q_\lambda^+(y) &\le 4^{\tau_i} \left| \widetilde{K}_i(\frac{\eta^\lambda}{|\eta^\lambda|^2}) - \widetilde{K}_i(\frac{\eta}{|\eta|^2}) \right| \overline{v}_i^{p_i}(\eta^\lambda) \\ &\le 2 \left| \widetilde{K}(\frac{\eta^\lambda}{|\eta^\lambda|^2}) - \widetilde{K}_i(\frac{\eta}{|\eta|^2}) \right| \left(1 + |\eta^\lambda|\right)^{-(n-2)p_i} \ . \end{aligned} \tag{3.21}$$

From (1.4) it follows that, for $|y| \geq r_i^{-1}$,

$$\begin{aligned}
&\left|\widetilde{K}_i(\frac{y}{|y|^2}) - K_i(x_i)\right| \\
&\leq c_1 \left\{|x_i|^{\alpha_i - 1} + M_i^{-\frac{(p_i-1)(\alpha_i-1)}{2}} \left(1 + |y|^{1-\alpha_i}\right)\right\} \left\{M_i^{-\frac{p_i-1}{2}} \left(1 + |y|^{-1}\right)\right\} \\
&\leq c_2 \left\{|x_i|^{\alpha_i-1} M_i^{-\frac{p_i-1}{2}} \left(1 + |y|^{-1}\right) + M_i^{-\frac{(p_i-1)\alpha_i}{2}} \left(1 + |y|^{-\alpha_i}\right)\right\} \ .
\end{aligned}$$

If $M_i^{\frac{p_i-1}{2}} |x_i| \geq 1$, then

$$M_i^{-\frac{(p_i-1)\alpha_i}{2}} = M_i^{-\frac{p_i-1}{2}} |x_i|^{\alpha_i - 1} \left(M_i^{\frac{p_i-1}{2}} |x_i|\right)^{1-\alpha_i} \leq L_i^{-(n-2)} \ .$$

If $M_i^{\frac{p_i-1}{2}} |x_i| \leq 1$, then

$$M_i^{-\frac{p_i-1}{2}} |x_i|^{\alpha_i-1} \leq M_i^{-\frac{\alpha_i(p_i-1)}{2}} = L_i^{-(n-2)} \ .$$

In any case,

$$\left|\widetilde{K}_i(\frac{y}{|y|^2}) - K_i(x_i)\right| \leq c_2 \, L_i^{-(n-2)} \left(1 + |y|^{-\alpha_i}\right) \ . \tag{3.22}$$

Thus, by (3.21) and (3.22), we have

$$Q_\lambda^+(\eta) \leq c_3 \, L_i^{-(n-2)} \left(1 + |\eta|^{-\alpha_i}\right) \left(1 + |\eta^\lambda|\right)^{-(n-2)p_i} \ . \tag{3.23}$$

For $0 < \beta < n$, we want to estimate

$$S_\beta(y) = \int_{\Sigma_\lambda} G^\lambda(y, \eta) |\eta|^{-\beta} \left(1 + |\eta^\lambda|\right)^{-(n-2)p_i} \, d\eta \ .$$

Case 1: $|y| \leq \frac{|\lambda|}{2}$.

By Lemma 2.2, we obtain $G^\lambda(y, \eta) \leq c \, |y - \eta|^{2-n}$. Hence

$$S_\beta(y) \leq \int_{\Sigma_\lambda} |y - \eta|^{2-n} |\eta|^{-\beta} \left(1 + |\eta^\lambda|\right)^{-(n-2)p_i} \, d\eta \ .$$

Decompose

$$\mathbb{R}^n = \{\eta |\, |y-\eta| \le \frac{|y|}{2}\} \cup \{\eta |\, |y-\eta| \ge \frac{|y|}{2},\, |\eta| \le 2|y|\} \cup \{\eta |\, |y-\eta| \ge \frac{|y|}{2},\, |\eta| \ge 2|y|\} \equiv A_1 \cup A_2 \cup A_3.$$

Elementary calculations give

$$\int_{A_1} |y-\eta|^{2-n}|\eta|^{-\beta}\left(1+|\eta^\lambda|\right)^{-(n-2)p_i}\, d\eta \le c_1|y|^{2-\beta}(1+|\lambda|)^{-(n-2)p_i}\ ,$$

$$\int_{A_2} |\eta-y|^{2-n}|\eta|^{-\beta}(1+|\eta^\lambda|)^{-(n-2)p_i}\, d\eta \le c_1|y|^{2-\beta}(1+|\lambda|)^{-(n-2)p_i}\ .$$

For $|y| \le 1$,

$$\int_{A_3} |y-\eta|^{2-n}|\eta|^{-\beta}(1+|\eta^\lambda|)^{-(n-2)p_i}\, d\eta \ \le\ c_3 \begin{cases} |y|^{2-\beta} & \text{if } \beta > 2 \\ \log\frac{2}{|y|} & \text{if } \beta = 2 \\ 1 & \text{if } \beta < 2 \end{cases}$$

For $|y| \ge 1$,

$$\int_{A_3} |y-\eta|^{2-n}|\eta|^{-\beta}(1+|\eta^\lambda|)^{-(n-2)p_i}\, d\eta \le c_4\int_{A_3}|\eta|^{-2n-\beta+\tau_i}\, d\eta \le c_5|y|^{-n-\beta+\tau_i}\ .$$

We also note that, for $1 \le |y| \le \frac{|\lambda|}{2}$,

$$|y|^{2-\beta}(1+|\lambda|)^{-(n-2)p_i} = |y|^{-n}|y|^{n+2-\beta}(1+|\lambda|)^{-(n-2)p_i} \le c_4|y|^{-n+\tau_i}\ .$$

In conclusion, we have for $|y| \le 1$,

$$S_\beta(y) \le c_3 \begin{cases} |y|^{2-\beta} & \text{if } \beta > 2\ , \\ \log\frac{2}{|y|} & \text{if } \beta = 2\ , \\ 1 & \text{if } \beta < 2\ , \end{cases} \tag{3.24}$$

and for $|y| \ge 1$,

$$S_\beta(y) \le c_4|y|^{-n+\tau_i}\ . \tag{3.25}$$

Case 2. $|y| \geq \frac{|\lambda|}{2}$

As before, let $A_1 = \{\eta|\,|y-\eta| \leq \frac{|y|}{2}\}$ and $A_2 = \{\eta|\,|y-\eta| \geq \frac{|y|}{2}\}$. For $\eta \in A_1$, by Lemma 2.2, we have

$$G^\lambda(y,\eta) \leq c\,(y_1-\lambda)|y-\eta|^{1-n} .$$

Thus,

$$\begin{aligned}\int_{A_1} & G^\lambda(y,\eta)|\eta|^{-\beta}(1+|\eta|)^{-(n-2)p_i}\,d\eta \\ &\leq c\,(y_1-\lambda)(1+|y|)^{-(n-2)p_i}\int_{A_1}|y-\eta|^{1-n}\,d\eta \\ &\leq c_1(y_1-\lambda)|y|^{-n+\tau_i} .\end{aligned}$$

For $\eta \in A_2$, we apply $G^\lambda(y,\eta) \leq c\,(y_1-\lambda)(\eta_1-\lambda)|y-\eta|^{-n}$. Then,

$$\begin{aligned}\int_{A_2} & G^\lambda(y,\eta)|\eta|^{-\beta}(1+|\eta^\lambda|)^{-(n-2)p_i} \\ &\leq c_1\,(y_1-\lambda)|y|^{-n}\int_{\mathbb{R}^n}|\eta|^{-\beta}(1+|\eta|)^{1-(n-2)p_i}\,d\eta \\ &= c_2\,(y_1-\lambda)|y|^{-n} .\end{aligned}$$

Combining these two estimates together yields

$$S_\beta(y) \leq c_2(y_1-\lambda)|y|^{-n+\tau_i} . \tag{3.26}$$

By (3.23)∼(3.26) and Lemma 2.2, we obtain

$$\int_{\Sigma'_\lambda} G^\lambda(y,\eta)Q^+_\lambda(\eta)\,d\eta \leq c_6 L_i^{-n+2}G^\lambda(y,0) \tag{3.27}$$

for some constant $c_6 > 0$. Set $A = 2\,c_6$ in (3.19). By (3.27), we have

$$0 < c_6 L_i^{2-n}G^\lambda(y,0) \leq h_\lambda(y) \leq 2\,c_6 L_i^{2-n}G^\lambda(y,0) . \tag{3.28}$$

Recall $r_i \leq \delta L_i$. Choose δ to be sufficiently small such that $\delta^{-(n-2)} \geq \frac{3\,c_6}{c_0}$, where c_0 is the constant in (3.18). Then, when i is large,

$$w_\lambda(y) > h_\lambda(y)$$

holds for $|y| = r_i^{-1}$ and $\lambda \leq \lambda_0$, and holds for $y \in \Sigma'_{\lambda_1}$. It is obvious that $h_\lambda(y)$ satisfies the assumption of Lemma 2.1 for $\lambda_1 \leq \lambda \leq \lambda_0$ and large i. Applying Lemma 2.1 gives (3.10). Thus, the proof of Lemma 3.1 is finished. q.e.d.

Lemma 3.2. *Suppose $v_i(y)$ satisfies (3.2) and $v_i(y) \leq 2$ for $|y| \leq c_0 L_i$. Then there exist positive constants δ_2 and c such that*

$$v_i(y) \leq c\, U_0(y)$$

for $|y| \leq \delta_2 L_i$, where c is a constant depending on n only.

Proof. Let $G_i(y, \eta)$ be the Green's function of the Laplacian operator in the ball $B_i = \{\eta|\, |\eta| \leq L_i\}$ with zero boundary value. For any $\epsilon > 0$, let δ_1 be the positive number stated in Lemma 3.1. Let $\overline{\delta}$ be sufficiently small (independent of i) such that

$$G_i(y, \eta) \geq \frac{1-\epsilon}{\sigma_n(n-2)}|y-\eta|^{2-n}$$

for $|y| = \delta_1 L_i$ and $|\eta| \leq \overline{\delta} L_i$, where σ_n denotes the area of the unit sphere S^{n-1}.

Let $|y_i| = \delta_1 L_i$ satisfy $v_i(y_i) = \min_{|y| \leq \delta L_i} v_i(y)$. Then, by Lemma 3.1, we have

$$\begin{aligned} \frac{1+\epsilon}{\delta_1^{n-2} L_i^{n-2}} &\geq v_i(y_i) \geq \int_{B_i} G_i(y_i, \eta)\overline{K}_i(\eta) v_i^{p_i}(\eta)\, d\eta \\ &\geq \frac{n(n-2)(1-2\epsilon)}{\sigma_n(n-2)(\delta_1+\overline{\delta})^{n-2} L_i^{n-2}} \int_{|\eta| \leq \overline{\delta}_i L_i} v_i^{p_i}(\eta)\, d\eta. \end{aligned}$$

Let $\overline{\delta} \ll \delta_1$. Then

$$\int_{|\eta| \leq \overline{\delta} L_i} v_i^{p_i}(\eta)\, d\eta \leq \frac{\sigma_n}{n}(1+4\epsilon)\ . \tag{3.29}$$

Since v_i uniformly converges to $U_0(y)$ in any compact set of $\mathbb{R}^n$ and $U_0(y)$ satisfies

$$\int_{\mathbb{R}^n} U_0^{\frac{n+2}{n-2}}(y)\, dy = \frac{\sigma_n}{n}\ ,$$

there exists a large R such that

$$\int_{R \leq |\eta| \leq \overline{\delta} L_i} v_i^{p_i}(\eta)\, d\eta \leq \frac{5\,\sigma_n \epsilon}{n} \tag{3.30}$$

holds for large i. Since $v_i(y) \le 2$, we have

$$\int_{R\le|\eta|\le\bar{\delta}L_i} v_i^{p_i^*}(\eta)\,d\eta \le \frac{10\,\sigma_n\epsilon}{n}\ . \tag{3.31}$$

Let ϵ be sufficiently small such that $\frac{10\,\sigma_n\epsilon}{n} \le \epsilon_0$, where ϵ_0 is the small number in Lemma 2.3. An applying of Lemma 2.3 shows that there exists a constant $c > 0$ such that

$$\max_{|y|=r} v_i(y) \le c \min_{|y|=r} v_i(y) \tag{3.32}$$

holds for $2R \le r \le \frac{\bar{\delta}}{2}L_i$. By Lemma 3.1, we have

$$v_i(y) \le c\,U_0(y) \tag{3.33}$$

for $2R \le |y| \le \frac{\bar{\delta}}{2}L_i$. Obviously, (3.33) holds true for $|y| \le 2R$ also. Hence we have finished the proof of Lemma 3.2. q.e.d.

Let $l_i = \delta_2 L_i$, where δ_2 is the positive constant stated in Lemma 3.2.

Lemma 3.3. *Suppose v_i satisfies the assumptions of Lemma 3.2. Then there exists a constant $c > 0$ such that*

$$\max_{|y|\le l_i} |v_i(y) - U_i(y)| \le c\,l_i^{-(n-2)}\ ,$$

where $U_i(y)$ is the C^2 positive solution of

$$\begin{cases} \Delta U_i + K_i(x_i)U_i^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n\ , \\ U_i(0) = 1 = \max\limits_{\mathbb{R}^n} U_i(y)\ . \end{cases}$$

Proof. Rewrite equation (3.3) into

$$\Delta v_i + c_i(y)v_i(y) = 0 \quad \text{for } |y| \le l_i$$

with $c_i(y) = \tilde{K}_i(y)v_i^{p_i-1}(y) \le c\,(1+|y|)^{-(p_i-1)(n-2)}$ by Lemma 3.2. Note that $(p_i-1)(n-2) > 2$ for large i. Hence, by applying the gradient estimates for the linear elliptic equations, we obtain

$$|\nabla v_i(y)| \le c_1\,v_i(y)(1+|y|)^{-1} \tag{3.34}$$

for $|y| \le \frac{l_i}{2}$. In particular, we have

$$|\nabla v_i(y)| \le c_1\,l_i^{-n+1} \tag{3.35}$$

for $|y| = \frac{l_i}{2}$.

By (3.3) and the Pohozaev identity, from (3.35) we conclude

$$\begin{aligned}\left(\frac{n}{p_i+1} - \frac{n-2}{2}\right) \int_{|y|\le \frac{l_i}{2}} \widetilde{K}_i(y) v_i^{p_i+1}(y)\, dy \\ + \frac{1}{p_i+1}\int_{|y|\le\frac{l_i}{2}} \left(y\cdot\nabla\widetilde{K}_i(y)\right) v_i^{p_i+1}\, dy \\ = \int_{|y|=\frac{l_i}{2}} \frac{n-2}{2} v_i \frac{\partial v_i}{\partial r} + \left|\frac{\partial v_i}{\partial r}\right|^2 |y| \\ - \frac{1}{2}|\nabla v_i|^2|y| + \frac{|y|}{p_i+1}\widetilde{K}_i(y) v_i^{p_i+1}\, d\sigma \\ \le c\, l_i^{-n+2}\ .\end{aligned}$$

Since

$$\begin{aligned}&\left|y\cdot\nabla\widetilde{K}_i(y)\right| \\ &\le M_i^{-\frac{p_i-1}{2}}\left(|x_i|^{\alpha_i-1} + M_i^{-\frac{p_i-1}{2}(\alpha_i-1)}|y|^{\alpha_i-1}\right)|y| \\ &\le c\, l_i^{-(n-2)}\left(1+|y|^{\alpha_i}\right)\ ,\end{aligned} \tag{3.36}$$

we have

$$\begin{aligned}&\int_{|y|\le\frac{l_i}{2}} \left|y\cdot\nabla\widetilde{K}_i(y)\right| v_i^{p_i+1}(y)\, dy \\ &\le c\, l_i^{-(n-2)} \int_{\mathbb{R}^n} \left(1+|y|^{\alpha_i}\right)\left(1+|y|\right)^{-(n-2)(p_i+1)}\, dy \\ &\le c\, l_i^{-(n-2)}\ .\end{aligned}$$

Thus

$$\tau_i = (n+2) - (n-2)p_i \le c\, l_i^{-(n-2)}\ , \tag{3.37}$$

which implies $\lim_{i\to+\infty} l_i^{\tau_i} = 1$.

Let $\Lambda_i = \max_{|y|\le l_i} |v_i - U_i| = v_i(y_i) - U_i(y_i)$ for some $|y_i| \le l_i$. Suppose the conclusion of Lemma 3.3 does not hold true, i.e., $\Lambda_i l_i^{n-2} \to +\infty$ as $i \to +\infty$. Let $w_i(y) = \Lambda_i^{-1}\left(v_i(y) - U_i(y)\right)$. By (3.3), w_i satisfies

$$\Delta w_i + b_i w_i = \widetilde{Q}_i(y), \tag{3.38}$$

where $b_i(y) = \tilde{K}_i(y)\left(\frac{v_i^{p_i}-U_i^{p_i}}{v_i-U_i}\right)$ and

$$
\widetilde{Q}_i(y) = \Lambda_i^{-1}\left\{\left(K_i(x_i) - K_i(x_i + M_i^{-\frac{p_i-1}{2}}y)\right)U_i^{p_i}(y) + K_i(x_i)\left(U_i^{\frac{n+2}{n-2}} - U_i^{p_i}\right)\right\} . \tag{3.39}
$$

By Lemma 3.2 and (3.37), we have

$$
b_i(y) \le c\,(1+|y|)^{-4} \quad \text{for} \quad |y| \le l_i . \tag{3.40}
$$

By a straightforward calculations,

$$
\begin{aligned}
|\widetilde{Q}_i(y)| &\le c\,\Lambda_i^{-1}\left\{L_i^{-(n-2)}(1+|y|)^{-(n+2-\alpha_i)} + \tau_i\,(1+|y|)^{-(n+2)}|\log U_i|\right\} \\
&\le c\,\Lambda_i^{-1} l_i^{2-n}(1+|y|)^{-4} ,
\end{aligned} \tag{3.41}
$$

for $|y| \le l_i$.

Applying the Green representation's Theorem leads to

$$
w_i(y) = \int_{B_i} G_i(y,\eta)\left(b_i(\eta)w_i(\eta) + \widetilde{Q}_i(\eta)\right)d\eta - \int_{\partial B_i}\frac{\partial G_i}{\partial \nu}(y,\eta)w_i(\eta)\,ds ,
$$

where $B_i = B(0,l_i)$, and G_i is the Green function of Δ in B_i. Thus, by (3.40) and (3.41), we obtain

$$
\begin{aligned}
|w_i(y)| &\le c_1\left\{\int_{B_i}|y-\eta|^{2-n}(1+|\eta|)^{-4}\,d\eta + \Lambda_i^{-1}l_i^{-(n-2)}\right\} \\
&\le c_2\left\{(1+|y|)^{-2} + \Lambda_i^{-1}l_i^{-(n-2)}\right\} ,
\end{aligned} \tag{3.42}
$$

where we note that $|w_i(\eta)| \le \Lambda_i^{-1}l_i^{-(n-2)}$ for $|\eta| = l_i$ by Lemma 3.2.

Since w_i is bounded in C^2_{loc}, there exists a subsequence of w_i (still denoted by w_i) such that w_i converge to w in C^2_{loc} by elliptic estimates, where w satisfies

$$
\begin{cases}
\Delta w + n(n+2)U_0^{\frac{4}{n-2}}(y)w(y) = 0 \quad \text{in } \mathbb{R}^n , \\
|w(y)| \le c(1+|y|)^{-2} .
\end{cases}
$$

By Lemma 2.4, we get $w(y) = \sum_{j=1}^{n} c_j \frac{\partial U_0}{\partial y_j} + c_0\left(|y|U_0'(|y|) + \frac{n-2}{2}U_0(|y|)\right)$. Since $w(0) = \frac{\partial w}{\partial y_j}(0) = 0$, we must have $c_j = 0$ for $0 \le j \le n$, namely, $w(y) \equiv 0$. Hence $\lim_{i\to+\infty}|y_i| = +\infty$.

Applying (3.42) at $y = y_i$ gives

$$1 = |w_i(y_i)| \le \left\{(1+|y_i|)^{-2} + \Lambda_i^{-1} l_i^{-(n-2)}\right\} ,$$

which obviously yields a contradiction. Thus, $\Lambda_i l_i^{n-2}$ must be bounded.
q.e.d.

Let $x_i \in \overline{B}_1$ satisfy $u_i(x_i) = \max_{\overline{B}_1} u_i(x_i) = M_i$. Suppose $M_i \to +\infty$. For this sequence of maximum points x_i of u_i, the rescaled function $v_i(y)$, defined in (3.1), obviously satisfies (3.2) and $v_i(y) \le 1$ for $|y| \le M_i^{\frac{p_i-1}{2}}$. We have

Lemma 3.4. *Let x_i satisfy $u_i(x_i) = \max_{\overline{B}_1} u_i(x_i) = M_i$. Then $M_i^{\frac{p_i-1}{2}}|x_i|$ is bounded.*

Proof. Suppose $\lim_{i\to+\infty} M_i^{\frac{p_i-1}{2}}|x_i| = +\infty$. By (3.6), we have $L_i = (M_i^{\frac{p_i-1}{2}}|x_i|^{1-\alpha_i})^{\frac{1}{n-2}}$. By Lemma 3.3, $w_i(y) = l_i^{n-2}\,(v_i(y) - U_i(y)\,)$ is uniformly bounded in $|y| \le l_i$. Thus, we may assume $w_i(y)$ uniformly converges to $w(y)$. By (3.35), we have

$$|\nabla w_i(y)| \le c_1\, l_i^{-1} \tag{3.43}$$

for $|y| = \frac{1}{2} l_i$.

Let $e_i = |\nabla K_i(x_i)|^{-1}\nabla K_i(x_i)$. Without loss of generality, we may assume $\lim_{i\to+\infty} e_i = (1,0,\cdots,0)$. For any $R > 0$, from (3.39) it follows that

$$\begin{aligned}\widetilde{Q}_i(y) =& l_i^{n-2} M_i^{-\frac{p_i-1}{2}} |\nabla K_i(x_i)|\, \{(e_i,y) + o(1)\}\, U_i^{\frac{n+2}{n-2}}(y) \\ &+ l_i^{n-2} K_i(x_i)\left(U_i^{\frac{n+2}{n-2}} - U_i^{p_i}\right) .\end{aligned} \tag{3.44}$$

for $|y| \le R$ and large i. For $|y| \ge R$, by (3.41) we have

$$|\widetilde{Q}_i(y)| \le c\,(1+|y|)^{-4} \tag{3.45}$$

for a constant c independent of i.

Thus, by (3.44) and (3.45) it is easy to see that

$$\lim_{i\to+\infty} \int_{|y|\le \frac{l_i}{2}} \widetilde{Q}_i(y)\psi_1(y)\,dy = c_1 \int_{\mathbb{R}^n} \psi_1(y) y_1 U_0^{\frac{n+2}{n-2}}(y)\,dy \tag{3.46}$$

for some constant $c_1 > 0$, where

$$c_1 = \lim_{i\to+\infty} l_i^{n-2} M_i^{-\frac{p_i-1}{2}} |\nabla K(x_i)| = \delta_2^{n-2} \lim_{i\to+\infty} |x_i|^{1-\alpha_i} |\nabla K_i(x_i)|$$

and $\psi_1 = \frac{\partial U_0}{\partial y_1}$.

On the other hand, multiplying ψ_1 on both sides of (3.38) gives

$$\begin{aligned} &\int_{|y|\le\frac{l_i}{2}} w_i(y)\left(\Delta\psi_1 + b_i(y)\psi_1(y)\right) dy \\ &\qquad + \int_{|y|=\frac{l_i}{2}} \left(\psi_1 \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial \psi_1}{\partial \nu}\right) ds \\ &\qquad = \int_{|y|\le\frac{l_i}{2}} \widetilde{Q}_i(y)\psi_1(y)\, dy. \end{aligned} \tag{3.47}$$

By (3.43), the boundary term of the above tends to 0 as $i \to +\infty$. Since $|w_i(y)| \le c$, we can easily prove

$$\begin{aligned} &\lim_{i\to+\infty} \int_{|y|\le\frac{l_i}{2}} w_i(y)\left(\Delta\psi_1(y) + b_i(y)\psi_1(y)\right) dy \\ &= \int_{\mathbb{R}^n} w(y)\left(\Delta\psi_1 + n(n+2)U_0^{\frac{4}{n-2}}\psi_1\right) dy \\ &= 0\ , \end{aligned} \tag{3.48}$$

which obviously yields a contradiction to (3.47). Thus, the proof of Lemma 3.4 is finished. q.e.d.

Remark 3.5. Since $M_i^{\frac{p_i-1}{2}}|x_i|$ is bounded,

$$c\, M_i^{\frac{(p_i-1)\alpha_i}{2(n-2)}} \le L_i \le M_i^{\frac{(p_i-1)\alpha_i}{2(n-2)}}$$

for some positive constant c. By (3.37), we have

$$\tau_i = O(1)\left(\max_{\overline{B}_1} u_i\right)^{-\frac{(p_i-1)\alpha_i}{2}} . \tag{3.49}$$

By Lemma 3.4, without loss of generality, we may assume

$$\xi = \lim_{i\to+\infty} M_i^{\frac{p_i-1}{2}} x_i\ . \tag{3.50}$$

Lemma 3.6. *Let* x_i *satisfy* $u_i(x_i) = \max_{B_1} u_i(x) \to +\infty$ *as* $i \to +\infty$ *and* ξ *be the vector in* $\mathbb{R}^n$*, given by (3.50). Then* ξ *satisfies*

$$\int_{\mathbb{R}^n} \nabla Q(y+\xi) U_0^{\frac{2n}{n-2}}(y)\, dy = 0\ . \tag{3.51}$$

Proof. Following the notation of Lemma 3.3 and Lemma 3.4, let $w_i(y) = l_i^{n-2}(v_i(y) - U_i(y)\,)$, where $l_i = \delta_2\, L_i$. Then w_i satisfies

$$\Delta w_i + b_i(y) w_i = \widetilde{Q}_i(y)\ , \tag{3.52}$$

where

$$\begin{aligned}\widetilde{Q}_i(y) =& l_i^{n-2}\left(K_i(x_i) - K_i\left(x_i + M_i^{-\frac{p-1}{2}} y\right)\right) U_i^{p_i}(y) \\ &+ K_i(x_i)\left(U_i^{\frac{n+2}{n-2}} - U_i^{p_i}\right).\end{aligned}$$

By (3.49) and (1.4), we have

$$\begin{aligned} &K_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right) - K_i(0) \\ &= Q_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right) + R_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right) \\ &= M_i^{-\frac{(p_i-1)\alpha_i}{2}}\left(Q_i(\xi_i + y) + o(1)(1+|y|^{\alpha_i})\right)\end{aligned} \tag{3.53}$$

for $|y| \le l_i$ with $\xi_i = M_i^{\frac{p-1}{2}} x_i$.

By Lemma 3.3 and Remark 3.5, $M_i^{\frac{(p_i-1)\alpha_i}{2}} l_i^{2-n}$ is bounded and $w_i(y)$ is uniformly bounded in $|y| \le \frac{1}{2} l_i$. Without loss of generality, we may assume $c = \lim_{i\to+\infty} M_i^{\frac{(p_i-1)\alpha_i}{2}} l_i^{2-n} > 0$ and w_i converges to w uniformly in any compact set of $\mathbb{R}^n$. Let $\psi_j(y) = \frac{\partial U_0}{\partial y_j}$ for $1 \le j \le n$. Since

$$\begin{aligned}\int_{|y|\le \frac{l_i}{2}} \psi_j(y) \Big[(K_i(0) - K_i(x_i)\,) U_i^{p_i}(y) \Big] \\ + K_i(x_i)\left[U_i^{\frac{n+2}{n-2}}(y) - U_i^{p_i}(y)\right] dy = 0\ ,\end{aligned}$$

by (3.53) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} Q(\xi+y)U_0^{\frac{n+2}{n-2}}(y)\psi_j(y)\,dy \\
&= \lim_{i\to+\infty} M_i^{\frac{(p_i-1)\alpha_i}{2}} l_i^{2-n} \int_{|y|\le\frac{l_i}{2}} Q_i(y)\psi_j(y)\,dy \\
&= c\left(\lim_{i\to+\infty}\int_{|y|\le\frac{l_i}{2}} \psi_j(\Delta w_i + b_i w_i)\,dy\right) \\
&= c\left(\lim_{i\to+\infty}\int_{|y|\le\frac{l_i}{2}} w_i(\Delta\psi_j + b_i(y)\psi_j)\,dy + \text{ boundary term}\right) \\
&= c\int_{\mathbb{R}^n} w(\Delta\psi_j + n(n+2)U_0^{\frac{4}{n-2}}\psi_j(y)\,)\,dy = 0\ .
\end{aligned}$$

Applying the integration by part gives

$$\frac{n-2}{2n}\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}Q(\xi+y)U_0^{\frac{2n}{n-2}}(y)\,dy = \int_{\mathbb{R}^n} Q(\xi+y)U_0^{\frac{n+2}{n-2}}(y)\psi_j(y)\,dy = 0\ .$$

Hence, Lemma 3.6 is proved. q.e.d.

Lemma 3.7. *Let x_i satisfy $u_i(x_i) = \max_{\overline{B}_1} u_i(x)$. Suppose $\alpha < n-2$. Then for any $R>0$, there exists a constant $c>0$ such that $u_i(x_i+y)|y|^{\frac{n-2}{2}} \le c$ for $|y|\le RM_i^{-\beta_i}$, where $\beta_i = \frac{p_i-1}{2}(1-\frac{\alpha_i}{n-2})$.*

Proof. By Lemma 3.1 and Lemma 3.2, there exist δ_2 and c_1 such that

$$v_i(y) \le c_1 U_0(y) \tag{3.54}$$

holds for $|y|\le \delta_2 L_i$. Since $v_i(y)$ is superharmonic, it is easy to show

$$v_i(y) \ge c_2 U_0(y) \tag{3.55}$$

for some constant $c_2>0$ and $|y|\le\delta_2 L_i$. Therefore

$$c_2 M_i^{1-\frac{(p_i-1)\alpha_i}{2}} \le u_i(x_i+y) \le c_1 M_i^{1-\frac{(p_i-1)\alpha_i}{2}} \tag{3.56}$$

for $|y| = \delta_2 M_i^{-\beta_i}$ and for two constants c_1 and c_2 which is independent of i, and also (by 3.54),

$$u_i(x_i+y)|y|^{\frac{2}{p_i-1}} = v_i\left(M_i^{\frac{p_i-1}{2}}y\right)\left(M_i^{\frac{p_i-1}{2}}|y|\right)^{\frac{2}{p_i-1}} \le c_1 \tag{3.57}$$

for $|y| \le \delta_2 M_i^{-\beta_i}$.

Now suppose the conclusion of Lemma 3.7 does not hold. Then we can apply a blow-up argument due to R. Schoen (see [17] or the proof of Lemma 4.1 in §4) to show that there exists a sequence y_i such that the followings hold:

1. $u_i(x_i + y_i)|y_i|^{\frac{2}{p_i-1}} \to +\infty$ as $i \to +\infty$,
2. $u_i(x_i + y)$ has a local maximum at y_i,
3. The function $\widetilde{v}_i(z) = \widetilde{M}_i^{-1} u_i\left(x_i + y_i + \widetilde{M}_i^{-\frac{p_i-1}{2}} z\right)$ uniformly converges to $U_0(z)$ in $C^2_{loc}(\mathbb{R}^n)$, where $\widetilde{M}_i = u_i(x_i + y_i)$, and
4. $\delta_0 M_i^{-\beta_i} \le |y_i| \le 2RM_i^{-\beta_i}$.

Since $\widetilde{v}_i$ is superharmonic, by the maximum principle, we have

$$\widetilde{v}_i(z) \ge c_3|z|^{2-n} \tag{3.58}$$

for some constant c_3 when $1 \le |z| \le \frac{1}{2}\widetilde{M}_i^{\frac{p_i-1}{2}}$.

Let $S_i = \left\{y \,|\, |y| = \frac{\delta_0}{2} M_i^{-\beta_i}\right\}$ and $\overline{y}_i \in S_i$ satisfy $|y_i - \overline{y}_i| = d(y_i, S_i)$. Set $z_i = \widetilde{M}_i^{\frac{p_i-1}{2}}(\overline{y}_i - y_i)$. By (3.56) and (3.58), we have

$$c_3|z_i|^{2-n}\widetilde{M}_i \le u_i(x_i + \overline{y}_i) \le c_1 M_i^{1-\frac{(p_i-1)\alpha_i}{2}} .$$

Then

$$\widetilde{M}_i^{1-\frac{(p_i-1)(n-2)}{2}} \le c_4 M_i^{1-\frac{(p_i-1)\alpha_i}{2}} |y_i - \overline{y}_i|^{n-2} \le c_5 M_i^{1-\frac{(p_i-1)(n-2)}{2}},$$

where $|y_i - \overline{y}_i| \le |y_i| + |\overline{y}_i| \le (R + \delta_0)M_i^{-\beta_i}$. Since $1 - \frac{(p_i-1)(n-2)}{2} < 0$, we have

$$M_i \le c_5 \widetilde{M}_i \ , \tag{3.59}$$

which implies $\widetilde{v}_i(z) \le c_5$ for $|z| \le \widetilde{M}_i^{\frac{p_i-1}{2}}$. Following the proof of Lemma 3.4 with x_i replaced by $x_i + y_i$, we can show the identity

$$\int_{\mathbb{R}^n} \psi_1(y) y_1 U_0^{\frac{n+2}{n-2}}(y)\, dy = 0$$

holds, where we assume $\lim_{i\to+\infty} \nabla K(x_i+y_i)|\nabla K(x_i+y_i)|^{-1} = (1,0,\cdots,0)$. Obviously, it yields a contradiction. Hence the proof of Lemma 3.7 is finished. q.e.d.

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $M_i = \max_{\overline{B}_1} u_i = u_i(x_i) \to +\infty$ as $i \to +\infty$. Let $r_i = M_i^{-\beta_i}$ and $u_i^*(y) = r_i^{\frac{2}{p_i-1}} u(x_i + r_i y)$, where we recall $\beta_i = \frac{p_i-1}{2}\left(1 - \frac{\alpha_i}{n-2}\right)$. Then $u_i^*(0) = M_i r_i^{\frac{2}{p_i-1}} = M_i^{\frac{\alpha_i}{n-2}} \to +\infty$ as $i \to +\infty$. By Lemma 3.2, we have

$$u_i^*(y) \le c u_i^*(0)^{-1}|y|^{-n+2} \tag{3.60}$$

for $|y| \le \delta_0$. By Lemma 3.7, $u_i^*(y)|y|^{\frac{n-2}{2}}$ is uniformly bounded in any compact set of $\mathbb{R}^n$. Applying the Harnack inequality and (3.60), $u_i^*(0)u_i^*(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n \setminus \{0\}$. Therefore, there exists a subsequence $u_i^*(0)u_i^*(y)$ (still denoted by $u_i^*(0)u_i^*(y)$) such that $u_i^*(0)u_i^*(y)$ converges to $h(y)$ in C^2 topology in any compact set of $\mathbb{R}^n \setminus \{0\}$. It is not difficult to see $h(y)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$; thus,

$$h(y) = \frac{a}{|y|^{n-2}} + b$$

with both a and $b \ge 0$.

Applying the Pohozaev identity, we have

$$\begin{aligned} &\frac{1}{p_i+1} r_i \int_{B_1} (y \cdot \nabla K_i(x_i + r_i y))\, u_i^*(y)^{p_i+1}\, dy \\ &= P(1; u_i^*) - \left(\frac{n}{p_i+1} - \frac{n-2}{2}\right) \int K_i(x_i + r_i y) u_i^*(y)^{p_i+1}\, dy \\ &\le p(1; u_i^*)\ , \end{aligned} \tag{3.61}$$

where

$$\begin{aligned} P(1; u_i^*) = \int_{\partial B_1} \Bigg(&\frac{n-2}{2} u_i^* \frac{\partial u_i^*}{\partial \nu} - \frac{1}{2}|\nabla u_i^*|^2 \\ &+ \left|\frac{\partial u_i^*}{\partial \nu}\right|^2 + \frac{1}{p_i+1} K_i(x_i + r_i y) {u_i^*}^{p_i+1} \Bigg)\, dy \end{aligned}$$

Since $u_i^*(0)u_i^*(y)$ converges to $h(y)$, a simple calculation leads to

$$\lim_{i\to+\infty} {u_i^*}^2(0) P(1; u_i^*) = -(n-2)\sigma_n ab \le 0\ , \tag{3.62}$$

where σ_n denotes the area of the unit sphere S^{n-1}. On the other hand, the left hand side of (3.61) tends to

$$\begin{aligned} &\lim_{i\to+\infty} u_i^{*^2}(0) r_i \int_{B_1} (y\cdot\nabla K_i(x_i+r_i y))\, u_i^*(y)^{p_i+1}\, dy \\ (3.63)\qquad &= \lim_{i\to+\infty} M^{\frac{2(\alpha_i-1)}{n-2}} \int_{|y|\le L_i} y\cdot\nabla K_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right) v_i^{p_i+1}\, dy \\ &= \int_{\mathbb{R}^n} (y\cdot\nabla Q(\xi+y))\, U_0^{p_i+1}(y)\, dy\ , \end{aligned}$$

where $\lim\limits_{i\to+\infty} M_i^{\tau_i} = 1$ is utilized.

Applying Lemma 3.6, (3.62) and (3.63), we have

$$0 < \int_{\mathbb{R}^n} Q(\xi+y) U_0^{p_i+1}(y)\, dy = \frac{1}{\alpha}\int_{\mathbb{R}^n} (y+\xi)\cdot\nabla Q(y+\xi) U_0^{p_i+1}(y)\, dy \le 0\ ,$$

which yields a contradiction. Therefore, the proof of Theorem 1.1 is completely finished. q.e.d.

4. Isolated Blowing-UP

Suppose that Theorem 1.2 does not hold, that is,

$$\lim_{i\to+\infty} \sup_{\overline{B}_1} \left(u_i(x)|x|^{\frac{p_i-1}{2}}\right) = +\infty\ . \tag{4.1}$$

Let x_i be a local maximum point of u_i. Following the notation in previous sections, we set

$$\begin{cases} v_i(y) = M_i^{-1} u_i\left(x_i + M_i^{-\frac{p_i-1}{2}} y\right)\ , \\ \tilde{v}_i(y) = v_i(y+e_1)\ ,\quad \text{and}\ , \\ \overline{v}_i(y) = |y|^{2-n}\tilde{v}_i\left(\frac{y}{|y|^2}\right)\ , \end{cases} \tag{4.2}$$

where $M_i = u_i(x_i)$ and $e_1 = (1,0,0,\cdots)$. Similarly, we define

$$\overline{U}_0(y) = |y|^{2-n} U_0\left(\frac{y}{|y|^2} + e_1\right)\ . \tag{4.3}$$

It is easy to see that $\overline{U}_0(y) = \left(\frac{2}{1+4|y-\overline{y}_0|^2}\right)^{\frac{n-2}{2}}$ and $\overline{U}_0(0) = \overline{U}_0(-e_1) = 1$ where $\overline{y}_0 = (-\frac{1}{2}, 0, \cdots, 0)$.

Given $\epsilon_0 > 0$ with $\epsilon_0 \ll 1$, there exists $\lambda_0 = \lambda_0(\epsilon_0) < 0$ and $c_n > 0$ such that

$$\begin{cases} -\frac{1}{2} < \lambda_0(\epsilon_0) \leq -\frac{1}{4} \ , \quad \text{and} \\ \overline{U}_0(y^\lambda) \leq 1 + \frac{\epsilon_0}{2} \end{cases} \tag{4.4}$$

for $|y| \leq c_n \epsilon_0$ and $\lambda \leq \lambda_0(\epsilon_0)$, where c_n depends on n only.

In the followings, $\delta_0 < \frac{1}{2}$ is a fixed positive number, but small enough such that (4.13), (4.15) and (4.16) below are satisfied.

Lemma 4.1. *Given ϵ_0, R_0 where $\epsilon_0 \ll 1 \ll R_0$ and $R_0^{-1} \leq c_n \epsilon_0$, there exists a positive constant $C_0 > 0$ such that the following statements hold true.*

(i) *If $u_i(x)|x|^{\frac{2}{p_i-1}} \geq C_0$, then there exists a local maximum point $x_i \in B(x, \delta_0|x|)$ of u_i with $u_i(x_i) \geq u_i(x)$ such that the rescaled function v_i of (4.2) satisfies (4.5)—(4.7).*

(4.5) *The origin 0 is the only local maximum of v_i in $B(0, 4R_0)$.*

(4.6) $|v_i(y) - U_0(y)|_{C^2(B(0,4R_0))} \leq \epsilon_0(4R_0)^{2-n}$,

(4.7) *$\overline{v}_i(y)$ has a local maximum point $\overline{y}_i$ near $\overline{y}_0$ such that $\overline{y}_{i,1} \leq \frac{1}{2}\left(\lambda_0 - \frac{1}{2}\right) < \lambda_0$ for all i where $\overline{y}_{i,1}$ denotes the x_1-coordinate of $\overline{y}_i$ and λ_0 is the constant in (4.4).*

(ii) *Let $\left\{x_j^i\right\}_{j=1}^{m_i}$ denote all local maximum points of u_i with $u_i(x_j^i)|x_j^i|^{\frac{p_i-1}{2}} \geq C_0$ such that (4.5), (4.6) and (4.7) hold. Then*

$$u_i(x) \leq 2\,C_0|x|^{-\frac{2}{p_i-1}} \qquad \textit{for all} \quad x \notin \Omega_i, \tag{4.8}$$

where $\Omega_i = \cup_j B\left(x_j^i, 2\delta_0\left|x_j^i\right|\right)$. Furthermore,

$$\left|x_j^i - x_k^i\right| \geq 4R_0 u_i\left(x_j^i\right)^{-\frac{p_i-1}{2}} \tag{4.9}$$

for $j \neq k$.

Proof of part(i). We will prove (i) by a blow-up argument, which was originally due to R. Schoen. Suppose the conclusion of (i) of Lemma 4.1 does not hold true. Then there exists a subsequence of u_i (still denoted by u_i) and x_i with $u_i(x_i)|x_i|^{\frac{n-2}{2}} \to +\infty$ such that u_i has no local maximum which is no less than $u_i(x_i)$ in $B(x_i, |x_i|\delta_0)$ and satisfies (4.5), (4.6) and (4.7).

Let $l_i = \delta_0 |x_i|$, and

$$S_i(x) = u_i(x)(l_i - |x - x_i|)^{\frac{2}{p_i - 1}} \ . \tag{4.10}$$

Let $\overline{x}_i$ satisfy

$$S_i(\overline{x}_i) = \sup_{|x - x_i| \le l_i} S_i(x) \ .$$

Set

$$\begin{aligned} v_i(y) &= \overline{M}_i^{-1} u_i(\overline{x}_i + \overline{M}_i^{-\frac{p_i-1}{2}} y) \\ &= \frac{S_i(x)}{S_i(\overline{x}_i)} \left(\frac{l_i - |\overline{x}_i - x_i|}{l_i - |x - x_i|} \right)^{\frac{2}{p_i - 1}} \quad , \end{aligned} \tag{4.11}$$

where $\overline{M}_i = u_i(\overline{x}_i)$ and $x = \overline{x}_i + \overline{M}_i^{-\frac{p_i-1}{2}} y$. For

$$|y| \le \frac{1}{2} \overline{M}_i^{\frac{p_i-1}{2}} (l_i - |\overline{x}_i - x_i|),$$

$$\begin{aligned} l_i - |x - x_i| &\ge l_i - |\overline{x}_i - x_i| - \overline{M}_i^{-\frac{p_i-1}{2}} |y| \\ &\ge \frac{1}{2}(l_i - |\overline{x}_i - x_i|) \ . \end{aligned} \tag{4.12}$$

Since $\overline{M}_i^{\frac{p_i-1}{2}} (l_i - |\overline{x}_i - x_i|) \ge u_i^{\frac{p_i-1}{2}}(x_i) l_i \to +\infty$ as $i \to +\infty$, $v_i(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n$. Therefore, there exists a subsequence of v_i (still denoted by v_i) which converges to $V_0(y)$ in $C^2_{loc}(\mathbb{R}^n)$, where $V_0(y)$ is a positive entire smooth solution of

$$\Delta V_0(y) + n(n-2) V_0^{\frac{n+2}{n-2}} = 0 \qquad \text{in} \quad \mathbb{R}^n \ .$$

Applying a theorem of Caffarelli-Gidas-Spruck, $V_0(y)$ is radially symmetric about some point y_0 in $\mathbb{R}^n$, and $V_0(y)$ has a nondegenerate maximum at y_0. Thus, for large i, $v_i(y)$ has a local maximum at y_i near y_0. Going back to u_i, we have found a local maximum point x_i^* of u_i with $|x_i^* - \overline{x}_i| \le c \overline{M}_i^{-\frac{p_i-1}{2}}$ for some constant $c > 0$, and

$$u_i(x_i^*) \ge u_i(\overline{x}_i) \ge u_i(x_i).$$

Obviously, $|x_i^* - \overline{x}_i| \le c \overline{M}_i^{-\frac{p_i-1}{2}} = o(1)(l_i - |\overline{x}_i - x_i|)$. It is easy to see that x_i^* satisfies all conditions in (i) when i is large. Hence we have a contradiction, and (i) is proved.

Proof of part (ii). Recall $\Omega_i = \cup_j B\left(x^i_j, 2\delta_0 \left|x^i_j\right|\right)$ where $\left\{x^i_j\right\}_{j=1}^{m_i}$ is the set of local maximum points of u_i which satisfy the conditions in part (i). Suppose that x satisfies $u_i(x)|x|^{\frac{2}{p_i-1}} \geq 2C_0$. By (i), there exists a local maximum point $x_i \in B(x, \delta_0|x|)$ with $u_i(x_i) \geq u_i(x)$ such that (4.5)—(4.7) are satisfied. Since $|x_i| \geq (1-\delta_0)|x|$, we have

$$\begin{aligned} u_i(x_i)|x_i|^{\frac{2}{p_i-1}} &\geq (1-\delta_0)^{\frac{2}{p_i-1}} u_i(x)|x|^{\frac{2}{p_i-1}} \\ &\geq 2(1-\delta_0)^{\frac{2}{p_i-1}} C_0 \geq C_0, \end{aligned}$$

if δ_0 is small such that

$$2(1-\delta_0)^{\frac{2}{p_i-1}} > 1 \ . \tag{4.13}$$

Hence $x_i = x^i_j$ for some j. Since $|x^i_j| \geq (1-\delta_0)|x|$ and $\delta_0 < \frac{1}{2}$, we have

$$|x^i_j - x| \leq \delta_0|x| \leq \frac{\delta_0}{1-\delta_0}|x^i_j| < 2\delta_0|x^i_j| \ .$$

Thus $x \in \Omega_i$, and (4.8) is proved. The inequality (4.9) is an immediate consequence of (4.5). q.e.d.

Let $\{x^i_j\}_{j=1}^{m_i}$ be the set of local maximum points of u_i in Lemma 4.1. Points x^i_j can be ordered by $u_i(x^i_1) \geq u_i(x^i_2) \geq \cdots \geq u_i(x^i_{m_i})$. Assume (4.1). Then there is a subsequence of u_i (still denoted by u_i) and $x^i_{j_i}$ such that $u_i(x^i_{j_i})|x^i_{j_i}|^{\frac{2}{p_i-1}} \geq i$ and $u_i(x^i_j)|x^i_j|^{\frac{2}{p_i-1}} < i$ for $1 \leq j < j_i$. It is obvious that $u_i(x^i_j) \to +\infty$ as $i \to +\infty$ for $j \leq j_i$. Hence $|x^i_j| \to 0$ for $j \leq j_i$.

Lemma 4.2. *There exists a positive integer i_0 such that, for $i \geq i_0$, $u_i(x) \leq 2u_i(x^i_j)$ for $x \in B(x^i_j, 2\delta_0|x^i_j|)$ with $j \leq j_i$ and for $i \geq i_0$.*

Proof. Suppose the conclusion of Lemma 4.2 does not hold true. Then we claim that there is a subsequence of u_i (still denoted by u_i) and $k_i < l_i \leq j_i$ such that (i) $|x^i_{l_i}| \leq 2|x^i_{k_i}|$, and (ii) $u_i(x) \leq 2u_i(x^i_{k_i})$ for all $x \in B(x^i_{k_i}, 2\delta_0|x^i_{k_i}|)$.

To see this, suppose $u_i(x) = \max_{\overline{B}_i} u_i \geq 2u(x^i_j)$ for some i and $j \leq j_i$ and for some $x \in \overline{B}_i$ where $B_i = B(x^i_j, 2\delta_0|x^i_j|)$. Then, by Lemma 4.1, there exists $x^i_k \in B(x, \delta_0|x|)$ such that $u(x^i_k) \geq u_i(x) \geq 2u(x^i_j)$. By the ordering on $\{x^i_j\}$, we have $k < j \leq j_i$. Since

$$|x^i_k| \geq (1-\delta_0)|x| \geq (1-\delta_0)(1-2\delta_0)|x^i_j|,$$

we have

$$
\begin{aligned}
u_i(x_k^i)|x_k^i|^{\frac{2}{p_i-1}} &\geq 2\left((1-\delta_0)(1-2\delta_0)\right)^{\frac{2}{p_i-1}} u(x_j^i)|x_j^i|^{\frac{2}{p_i-1}} \\
&\geq \left(\frac{3}{2}\right) u(x_j^i)|x_j^i|^{\frac{2}{p_i-1}} ,
\end{aligned} \tag{4.14}
$$

if δ_0 satisfies

$$
[(1-\delta_0)(1-2\delta_0)]^{\frac{2}{p_i-1}} \geq \frac{3}{4} . \tag{4.15}
$$

If $u_i(x) \leq 2u_i(x_k^i)$ for all $x \in B(x_k^i, 2\delta_0|x_k^i|)$, then we let $k_i = k$ and $l_i = j$. Thus, the claim is proved. If there exists $x \in B(x_k^i, 2\delta_0|x_k^i|)$ such that $u_i(x) \geq 2u_i(x_k^i)$, then we can repeat the argument above to have $k_m < k_{m-1} < \cdots < k_1 < j$ such that

$$
|x_{k_m}^i| \geq (1-\delta_0)(1-2\delta_0)|x_{k_{m-1}}^i| \geq [(1-\delta_0)(1-2\delta_0)]^m |x_j^i|,
$$

and by (4.14),

$$
\begin{aligned}
i \geq u_i(x_{k_m}^i)|x_{k_m}^i|^{\frac{2}{p_i-1}} &\geq \left(\frac{3}{2}\right)^m u(x_j^i)|x_j^i|^{\frac{2}{p_i-1}} \\
&\geq \left(\frac{3}{2}\right)^m C_0 .
\end{aligned}
$$

Thus, after finite steps, we can find $k_i \in N$, such that

$$
|x_{k_i}^i| \geq (1-\delta_0)(1-2\delta_0)|x_{k_{i-1}}^i| ,
$$

and,

$$
u_i(x) \leq 2u_i(x_{k_i}^i)
$$

for $x \in B(x_{k_i}^i, 2\delta_0|x_{k_i}^i|)$. Let δ_0 satisfy

$$
(1-\delta_0)(1-2\delta_0) \geq \frac{1}{2} . \tag{4.16}
$$

Then our claim is proved.

However, by Lemma 4.4 below, we have $|x_{k_i}^i| = o(1)|x_{l_i}^i|$, which yields a contradiction to the claim above. Hence the proof of Lemma 4.2 is finished. q.e.d.

To complete the proof of Lemma 4.2, we need the following two lemmas.

Lemma 4.3. *Let $k_i \le j_i$ be a sequence of positive integers, and suppose that $u_i(x) \le 2u_i(x^i_{k_i})$ for $x \in B(x^i_{k_i}, 2\delta_0|x^i_{k_i}|)$. Then*

$$\lim_{i\to+\infty} L_i(M_i^{\frac{p_i-1}{2}}|x^i_{k_i}|)^{-1} = +\infty,$$

where $L_i = (M_i^{\frac{p_i-1}{2}}|x^i_{k_i}|^{1-\alpha_i})^{\frac{1}{n-2}}$ and $M_i = u_i(x^i_{k_i})$.

Proof. Suppose $\underline{\lim}_{i\to+\infty} L_i \left(M_i^{\frac{p_i-1}{2}}|x^i_{k_i}|\right)^{-1} < +\infty$. Without loss of generality, we may assume

$$L_i \le c_1\, M_i^{\frac{p_i-1}{2}}|x^i_{k_i}| \tag{4.17}$$

for all i and some constant c_1 independent of i. Since

$$u(x^i_{k_i}) \ge u(x^i_{j_i}) \to +\infty$$

as $i \to +\infty$, we have $\lim\limits_{i\to+\infty} x^i_{k_i} = 0$ and

$$\lim_{i\to+\infty} M_i^{\frac{p_i-1}{2}}|x^i_{k_i}| \ge c_1^{-1} \lim_{i\to+\infty} L_i = +\infty\ .$$

Hence, the scaled function $v_i(y) = M_i^{-1}u_i\left(x^i_{k_i} + M_i^{-\frac{p_i-1}{2}}y\right)$ uniformly converges to $U_0(y)$ in any compact set of $\mathbb{R}^n$ as $i \to +\infty$. Therefore, by Lemma 3.1 we have for any $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$\min_{|y|=r} v_i(y) \le (1+\epsilon)U_0(r)$$

holds for all $0 \le r \le \delta_1 L_i$. As in the proof of Lemma 3.2 (See (3.30)), there exists a $\delta_2 > 0$ such that

$$\int\limits_{R\le|y|\le\delta_2 L_i} v_i^{p_i}(y)\,dy \le \frac{5\sigma_n}{n}\epsilon \tag{4.18}$$

for some $R = R(\epsilon) > 0$, which is independent of i. By (4.17) δ_2 may be choosen small such that $\delta_2 L_i \le 2\delta_0 M_i^{\frac{p_i-1}{2}}|x^i_{k_i}|$. Hence $v_i(y) \le 2$ for $|y| \le \delta_2 L_i$. Recall $p_i^* = \frac{n}{2}(p_i - 1) > p_i$ and $p_i^* - p_i \le 1$. By (4.18),

$$\int\limits_{R\le|y|\le\delta_2 L_i} v_i^{p_i^*}(y)\,dy \le \frac{10\sigma_n}{n}\epsilon\ . \tag{4.19}$$

If ϵ is choosen small, then, by Lemma 2.3 and the Harnack inequality, we have

$$v_i(y) \le c_2\, U_0(y) \tag{4.20}$$

for all $|y| \le \delta_2 L_i$ and for some constant c_2 independent of i. By (4.20), Lemma 3.3 holds for v_i also. Repeating the proofs of (3.44), (3.46) and (3.47) in Lemma 3.4, we can obtain

$$\int_{\mathbb{R}^n} \psi_1(y) y_1 U_0^{\frac{2n}{n-2}}(y)\, dy = 0 \ ,$$

which yields a contradiction. Hence Lemma 4.3 is proved. q.e.d.

Lemma 4.4. *Let $k_i \le l_i \le m_i$ be two sequences of positive integers. Suppose $u_i(x) \le 2u_i(x^i_{k_i})$ for $x \in B(x^i_{k_i}, 2\delta_0 |x^i_{k_i}|)$. Then, for any $\epsilon > 0$, there exists a positive integer $i_0 = i_0(\epsilon)$ such that*

$$|x^i_{k_i}| \le \epsilon |x^i_{l_i}|$$

for $i \ge i_0$.

Proof. Suppose the claim of Lemma 4.4 does not hold. Without loss of generality, we may assume

$$|x^i_{l_i}| \le c_1\, |x^i_{k_i}| \tag{4.21}$$

for all i and some $c_1 > 0$ independent of i.

Let ϵ_0 and R_0 be the constants in Lemma 4.1. Let $v_i(y) = M_i^{-1} u_i(x^i_{k_i} + M_i^{-\frac{p_i-1}{2}} y)$ with $M_i = u_i(x^i_{k_i})$. First, we note that, by (4.5)—(4.7), Lemma 3.1 holds for $v_i(y)$ also, that is, there exist $\delta_1 = \delta_1(\epsilon_0)$ and $i = i_0(\epsilon_0)$ such that

$$\min_{|y|=r} v_i(y) \le (1 + 2\epsilon_0) U_0(r) \tag{4.22}$$

for $0 \le r \leqq \delta_1 L_i$ and $i \ge i_0$, where $L_i = \left(M_i^{\frac{p_i-1}{2}} |x^i_{k_i}|^{1-\alpha_i} \right)^{\frac{1}{n-2}}$.

Since L_i is not tending to $+\infty$ in general as $i \to +\infty$, the claim of (4.22) is viewed as a "finite" version of Lemma 3.1. Under conditions of (4.5)—(4.7), however, the proof of (4.22) can go through as in Lemma 3.1 without too much modification. In the followings, we would like to sketch its proof briefly.

Suppose (4.22) does not hold true for a subsequence of v_i (still denoted by v_i), i.e., there exists a sequence of r_i such that

$$\min_{|y|=r_i} v_i(y) \geq (1+2\epsilon_0)U_0(r_i)$$

for some $r_i \leq \delta_1 L_i$, where δ_1 will be chosen later. By Lemma 4.1, we have $r_i \geq 4R_0$. Let $\tilde{v}_i$ and $\overline{v}_i$ be defined as in (4.1). Thus, we have $\min_{|y|=r_i-1} \tilde{v}_i(y) \geq (1+2\epsilon_0)U_0(r_i) \geq (1+\epsilon_0)U_0(r_i-1)$, if $R_0^{-1} \leq c_n\,\epsilon_0$ where c_n is independent of i. For simplicity of notation, we replace $r_i - 1$ by r_i, i.e., we have

$$\min_{|y|=r_i} \tilde{v}_i(y) \geq (1+\epsilon_0)U_0(r_i) \ , \tag{4.23}$$

and r_i satisfies

$$2R_0 \leq r_i \leq \delta_1 L_i \ . \tag{4.24}$$

By (4.23), we have

$$\overline{v}_i(y) \geq r_i^{n-2} \min_{|y|\leq r_i} \tilde{v}_i \geq (1+\epsilon_0) \quad \text{for} \quad |y| = r_i^{-1} \ . \tag{4.25}$$

Let $\lambda_0 = \lambda_0(\epsilon_0)$ be the number defined in (4.4). For $|y| \geq \frac{1}{4}$, by (4.6) we have

$$\left|\overline{v}_i(y) - \overline{U}_0(y)\right| \leq \epsilon_0 |y|^{2-n}(4R_0)^{2-n}$$

which implies

$$\overline{v}_i(y) \leq \overline{U}_0(y) + \epsilon_0 R_0^{2-n} \ .$$

By (4.4), for $|y| = r_i^{-1}$ and $\lambda \leq \lambda_0$ we have

$$\begin{aligned} \overline{v}_i(y^\lambda) &\leq \overline{U}_0(y^\lambda) + \epsilon_0 R_0^{2-n} \\ &\leq 1 + \frac{\epsilon_0}{2} + \epsilon_0 R_0^{2-n} \leq 1 + \frac{3}{4}\epsilon_0 \ . \end{aligned} \tag{4.26}$$

Let $w_\lambda(y) = \overline{v}_i(y) - v_i(y^\lambda)$. Applying (4.25) and (4.26) together gives

$$\begin{aligned} w_\lambda(y) &\geq \frac{\epsilon_0}{4} \geq c_0\, r_i^{2-n} G^\lambda(y,0) \\ &= c_0\, \delta_1^{2-n} L_i^{2-n} G^\lambda(y,0) \end{aligned} \tag{4.27}$$

for $|y| = r_i^{-1}$ and $\lambda \leq \lambda_0$, where c_0 depends on n and ϵ_0 only.

As in the proof of Lemma 3.1, $\overline{v}_i$ has a harmonic asymptotic expansion (3.16) at ∞,

$$\begin{cases} \overline{v}_i(y) = |y|^{2-n}\left(\overline{c}_{0,i} + \sum \overline{c}_{j,i}\frac{y_i}{|y|^2}\right) + O_i\left(\frac{1}{|y|^n}\right) \,, \\ \overline{v}_{i_{y_i}} = -(n-2)\frac{\overline{c}_{0,i}\, y_1}{|y|^n} + O_i\left(\frac{1}{|y|^n}\right) \,, \end{cases}$$

where $\overline{c}_{0,i} \to \overline{c}_0$, $\overline{c}_{j,i}$ are uniformly bounded as $i \to +\infty$, and $O_i(|y|^{-n}) \leq c\,|y|^{-n}$ for some constant $c > 0$ independent of i, by (4.6). Therefore, as in (3.17), there exists $\lambda_1 < 0$, independent of i, such that

$$w_\lambda(y) \geq c_1\, G^\lambda(y,0) \tag{4.28}$$

for all $\lambda \leq \lambda_1$ and $y \in \Sigma'_\lambda = \{y|\, y_1 > \lambda \text{ and } |y| \geq r_i^{-1}\}$.

As in Lemma 3.1, we let

$$h_\lambda(y) = AL_i^{2-n}G^\lambda(y,0) - \int_{\Sigma'_\lambda} G^\lambda(y,\eta)Q^+_\lambda(y)\, d\eta \,. \tag{4.29}$$

By the same estimates in Lemma 3.1, we can find a constant A, independent of i, such that $h_\lambda(y) > 0$ in Σ'_λ. Furthermore, we have

$$c_2\, L_i^{2-n}G^\lambda(y,0) \leq h_\lambda(y) \leq c_3\, L_i^{2-n}G^\lambda(y,0) \,,$$

for $y \in \Sigma'_\lambda$, $\lambda \leq \lambda_0$ and two constants c_2 and c_3, independent of i. Hence, if δ_1 satisfies $\delta_1^{2-n} \geq \frac{2\,c_3}{c_0}$, then, by (4.27), (4.28) and Lemma 2.1, we have

$$w_\lambda(y) > 0$$

for $y \in \Sigma'_\lambda$ and $\lambda \leq \lambda_0(\epsilon_0)$. However, it yields a contradiction to the fact that $\overline{v}_i$ has a local maximum point $\overline{y}_i$ with $\overline{y}_{i,1} \leq \frac{1}{2}(\lambda_0 - \frac{1}{2}) < \lambda_0$. Hence, (4.22) is proved.

As in (3.29), (4.22) implies that there exists $\delta_2 = \delta_2(\epsilon_0) < \delta_1$ such that

$$\int_{|y|\leq \delta_2 L_i} v_i^{p_i}(y)\, dy \leq \frac{\sigma_n}{n}(1+4\epsilon_0) \,. \tag{4.30}$$

Let

$$B_i = \left\{x \;\middle|\; |x - x^i_{l_i}| \leq 2R_0 u(x^i_{l_i})^{-\frac{p_i-1}{2}}\right\}$$

and

$$\widetilde{B}_i = \{y|\; x = x^i_{k_i} + M_i^{-\frac{p_i-1}{2}} y \in B_i\}.$$

For $y \in \widetilde{B}_i$, by (4.21) we have

$$\begin{aligned} M_i^{-\frac{p_i-1}{2}}|y| &\le \left|x - x^i_{l_i}\right| + \left|x^i_{l_i} - x^i_{k_i}\right| \\ &\le 2R_0 u(x^i_{l_i})^{-\frac{p_i-1}{2}} + 2\,c_1\,|x^i_{k_i}| \\ &= 2R_0\left(u(x^i_{l_i})^{-\frac{p_i-1}{2}})|x^i_{l_i}|^{-1}\right)|x^i_{l_i}| + 2\,c_1\,|x^i_{k_i}| \\ &\le c_4\,|x^i_{k_i}|\ , \end{aligned}$$

where $c_4 = 2(1 + R_0\,C_0^{-\frac{p_i-1}{2}})\,c_1$. Thus, by Lemma 4.3,

$$|y| \le c_4\,M_i^{\frac{p_i-1}{2}}|x^i_{k_i}| \le \frac{\delta_2}{2}L_i \tag{4.31}$$

for large i. On the other hand, we have

$$\begin{aligned} M_i^{-\frac{p_i-1}{2}}|y| &\ge \left|x^i_{k_i} - x^i_{l_i}\right| - \left|x^i_{l_i} - x\right| \\ &\ge \left|x^i_{k_i} - x^i_{l_i}\right| - 2R_0 u(x^i_{l_i})^{-\frac{p_i-1}{2}}\ . \end{aligned}$$

Moreover, by Lemma 4.1 and $M_i \ge u_i(x^i_{l_i})$,

$$\begin{aligned} |y| &\ge u_i^{\frac{p_i-1}{2}}(x^i_{l_i})\left|x^i_{k_i} - x^i_{l_i}\right| - 2R_0 \\ &\ge 2R_0\ , \end{aligned} \tag{4.32}$$

which combined together with (4.31) gives $\widetilde{B}_i \subset \left\{y \mid 2R_0 \le |y| \le \frac{\delta_1}{2}L_i\right\}$. From (4.5) and (4.6) it follows that $u_i(x) \le u_i(x^i_{l_i})$ for $x \in \overline{B}_i$. Since $u_i(x^i_{l_i}) \le u_i(x^i_{k_i})$, we have $v_i(y) \le 1$ on $\widetilde{B}_i$, and therefore

$$\begin{aligned} \int_{B_i} u_i^{p_i^*}\,dy &= \int_{\widetilde{B}_i} v_i^{p_i^*}\,dy \\ &\le \int\limits_{2R_0 \le |y| \le \delta_2 L_i} v_i^{p_i}\,dy\ . \end{aligned} \tag{4.33}$$

Let R_0 be sufficiently large such that

$$\int_{|y| \le 2R_0} U_0^{p_i}(y)\,dy \ge \frac{\sigma_n}{n}(1 - \epsilon_0)\ .$$

Then, by (4.6) and (4.30), we obtain

$$\int\limits_{2R_0 \le |y| \le \delta_2 L_i} v_i^{p_i}\,dy \le \overline{c}_n\,\epsilon_0$$

for some constant $\overline{c}_n$ depending on n only. Together with (4.33), the inequality above implies

$$\frac{1}{2}\int_{\mathbb{R}^n} U_0^{\frac{2n}{n-2}}(y)\,dy \le \int_{B_i} u_i^{p_i^*}(y)\,dy \le \overline{c}_n\,\epsilon_0,$$

which obviously yields a contradiction if ϵ_0 is sufficiently small. Hence, Lemma 4.4 is proved. q.e.d.

Proof of Theorem 1.2. Suppose the conclusion of Theorem 1.2 does not hold true. Let $\epsilon_0 \ll 1 \ll R_0$ be true positive constants satisfying $R_0^{-1} \le c_n\,\epsilon_0$ for some small constant c_n. By Lemma 4.1 and Lemma 4.2, there exists a constant C_0 and the set of local maximum points $\{x_j^i\}_{j=1}^{m_i}$ of u_i satisfying $u_i(x_j^i)|x_j^i|^{\frac{2}{p_i-1}} \ge C_0$, (4.5), (4.6) and (4.7). The set $\{x_j^i\}_{j=1}^{m_i}$ can be ordered by $u_i(x_1^i) \ge u_i(x_2^i) \ge \cdots \ge u_i(x_{m_i}^i)$. Without loss of generality, we may assume that, for each i, there exists a positive integer j_i such that $u_i(x_{j_i}^i)|x_{j_i}^i|^{\frac{2}{p_i-1}} \ge i$ and $u_i(x_j^i)|x_j^i|^{\frac{2}{p_i-1}} < i$. Let $\Omega_i = \cup_{j=1}^{m_i} B(x_j^i, 2\delta_0|x_j^i|)$. Then

$$u_i(x) \le 2C_0|x|^{-\frac{2}{p_i-1}} \tag{4.34}$$

for $x \notin \Omega_i$, and

$$u_i(x) \le 2u_i(x_j^i) \tag{4.35}$$

for $x \in B(x_j^i, 2\delta_0|x_j^i|)$ where $1 \le j \le j_i$.

By Lemma 4.3, we have

$$\lim_{i\to+\infty}\inf_{j\le j_i} L_{i,j}\left(M_{i,j}^{\frac{p_i-1}{2}}|x_j^i|\right)^{-1} = +\infty\ , \tag{4.36}$$

where $M_{i,j} = u_i(x_j^i)$ and $L_{i,j} = \left(u_i(x_j^i)^{\frac{p_i-1}{2}}|x_j^i|^{1-\alpha_i}\right)^{\frac{1}{n-2}}$. Moreover, by Lemma 4.4, we can show that for any δ with $0 < \delta \ll 1$, there exists $i_0 = i_0(\delta)$ such that for $i \ge i_0$,

$$|x_{j-1}^i| \le \frac{\delta}{2}|x_j^i| \tag{4.37}$$

holds for $2 \le j \le j_i + 1$, and

$$|x_{j_i}^i| \le \frac{\delta}{2}|x_j^i| \tag{4.38}$$

for $j_i + 1 \le j \le m_i$. From (4.37), (4.38) and Lemma 4.1 it follows that

$$u_i(x) \le u_i(x^i_{j_i}) \quad \text{for } |x| \ge \delta |x^i_{j_i}| \ . \tag{4.39}$$

for $i \ge i_1 = i_1(\delta) \ge i_0$. Obviously, (4.37) implies

$$|x^i_j| \le \left(\frac{\delta}{2}\right)^k |x^i_{j_i}| \tag{4.40}$$

for $j < j_i$ and $k = j_i - j$. By (4.22), (4.30) and (4.36), we obtain

$$\int\limits_{B(x^i_j, 2\delta_0 |x^i_j|)} u_i^{p_i^*}(y)\, dy \le 2 \int\limits_{|y| \le \delta_2 L_{i,j}} v_{i,j}^{p_i}(y)\, dy \le 2 \left(\frac{\sigma_n}{n}(1 + 3\epsilon_0)\right) \ , \tag{4.41}$$

for large i where $v_{i,j}(y) = M_{i,j}^{-1} u_i \left(x^i_j + M_{i,j}^{-\frac{p_i - 1}{2}} y \right)$.

In the followings, both ϵ_0 and R_0 will be fixed. For the simplicity of notation, we let $x_i = x^i_{j_i}$. Note that $\lim\limits_{i \to +\infty} u_i(x_i)|x_i|^{\frac{2}{p_i - 1}} = +\infty$. As in (4.2), we let $v_i(y) = M_i^{-1} u_i(x_i + M_i^{-\frac{p_i - 1}{2}} y)$ with $M_i = u_i(x_i)$. By Lemma 3.1 and Lemma 3.2, for any $\epsilon > 0$ there exist $\delta_2 = \delta_2(\epsilon) > 0$ and a positive integer $i_3 = i_3(\epsilon)$ such that for $i \ge i_3$,

$$\min v_i(y) \le (1 + \epsilon) U_0(r)$$

holds for $0 \le r \le \delta_2 L_i$ and, by (3.29) we obtain

$$\int_{|y| \le \delta_2 L_i} v_i^{p_i}(y)\, dy \le \frac{\sigma_n}{n}(1 + 4\epsilon) \ , \tag{4.42}$$

where $L_i = \left(M_i^{\frac{p_i - 1}{2}} |x_i|^{1 - \alpha_i} \right)^{\frac{1}{n-2}}$. In particular, there exists $R = R(\epsilon) > 0$ such that for $i \ge i_3$,

$$\int_{R \le |y| \le \delta_2 L_i} v_i^{p_i}(y)\, dy \le \frac{5 \sigma_n \epsilon}{n} \ . \tag{4.43}$$

Therefore, by Lemma 2.3 and (4.39), there exists a constant $c_1 > 0$ such that

$$v_i(y) \le c_1 U_0(y) \tag{4.44}$$

for $|y| \geq 2M_i^{\frac{p_i-1}{2}}|x_i|$ and large i.

Let $e_i = |\nabla K_i(x_i)|^{-1}\nabla K_i(x_i)$ and let y_i satisfy $x_i - y_i = |x_i|e_i$. Applying the Pohozaev identity, we obtain

$$\begin{aligned}
&\frac{n-2}{2n}\int_{|x|\leq l_i}(x-y_i)\cdot\nabla K_i(x)u_i^{p_i+1}(x)\,dx \\
&+\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)\int_{|x|\leq l_i}K_i\cdot u_i^{p_i+1}\,dx \\
&=\int_{|x|=l_i}\left[(x-y_i,\nabla u_i)\frac{\partial u_i}{\partial\nu}-(x-y_i,\nu)\frac{|\nabla u_i|^2}{2}+\frac{n-2}{2}u_i\frac{\partial u_i}{\partial\nu}\right.\\
&\qquad\left.+\frac{(x-y_i,\nu)}{p_i+1}K_i(x)u_i^{p_i+1}\right]d\sigma\ ,
\end{aligned}\tag{4.45}$$

where $l_i = \frac{\delta_2}{2}L_iM_i^{-\frac{p_i-1}{2}}$. By (4.44) and the gradient estimates, we have for $|y| = \frac{\delta_2}{2}L_i$,

$$|\nabla v_i(y)| \leq c_1\, v_i(y)|y|^{-1}\ ,$$

which implies for $|x| = l_i$,

$$\begin{cases} u_i(x) \leq c_2\, M_iL_i^{-n+2}\ , \\ |\nabla u_i(x)| \leq c_2\, M_i^{1+\frac{p_i-1}{2}}L_i^{-n+1}\ . \end{cases}\tag{4.46}$$

By (3.49), we have

$$\lim_{i\to+\infty} M_i^{\tau_i} = 1\ ,\tag{4.47}$$

which and (4.46) lead to

$$\begin{aligned}
\text{the right-hand side of (4.45)}\ &\leq c_3\, L_i^{-n+2} \\
&= c_3\, M_i^{-\frac{p_i-1}{2}}|x_i|^{\alpha_i-1} \\
&= o(1)|x_i|^{\alpha_i}\ .
\end{aligned}\tag{4.48}$$

To estimate the left-hand side of (4.41), we decompose

$$B(0,l_i) = B(0,\delta|x_i|)\cup A_1\cup A_2\cup A_3,$$

where

$$A_1 = \{x|\ |x-x_i| \leq M_i^{-\frac{p_i-1}{2}}R\},\quad A_2 = \{x|\ |x-x_i| \geq M_i^{-\frac{p_i-1}{2}}R$$

and

$$\delta|x_i| \le |x| \le 3|x_i|\}, \quad A_3 = \{x|\ 3|x_i| \le |x| \le l_i\},$$

and $R = R(\epsilon)$ in (4.43). It is easy to calculate

$$\begin{aligned}\int_{A_1} (x - y_i) \cdot \nabla K_i(x_i) u_i^{p_i+1}(x)\, dx &\ge c_4\, |x_i|^{\alpha_i} \int_{|y|\le 1} v_i^{p_i^*}\, dy \\ &\ge c_5\, |x_i|^{\alpha_i}\ ,\end{aligned} \tag{4.49}$$

where c_5 depends on n and the lower bound of $|\nabla K_i(x)|\ |x|^{-\alpha_i+1}$.

Let $\widetilde{\Omega}_i = \cup_{j=1}^{j_i-1} B(x_j^i, 2\delta_0|x_j^i|)$. Then from (4.37) it follows that

$$\widetilde{\Omega}_i \subset B(0,\, \delta|x_i|)$$

for $i \ge i_0(\delta)$. Since $u_i(x) \le 2C_0|x|^{-\frac{2}{p_i-1}}$ for $x \in B(0, \delta|x_i|) \setminus \widetilde{\Omega}_i$, by (4.47) we obtain

$$\begin{aligned}&\int_{B(0,\delta|x_i|)\setminus\tilde{\Omega}_i} |x - y_i|\, |\nabla K_i(x)| u_i^{p_i+1}(x)\, dx \\ &\le c_6\, |x_i| \int_{B(0,\delta|x_i|)} |x|^{\alpha_i - 1 - \frac{2(p_i+1)}{p_i-1}}\, dx \\ &\le c_7\, \delta^{\alpha_i - 1} |x_i|^{\alpha_i}\end{aligned} \tag{4.50}$$

for $i \ge i_0$. Let $B_j = B(x_j^i, 2\delta_0|x_j^i|)$ and $k = j_i - j$. Then by (4.40) and (4.41) we have

$$\begin{aligned}&\int_{B_j} |x - y_i|\, |\nabla K_i(x)| u_i^{p_i+1}(x)\, dx \\ &\le c_8\, |x_i|\, |x_j^i|^{\alpha_i - 1} \int_{B_j} u_i^{p_i+1}\, dx \\ &\le c_9\, |x_i|\, |x_j^i|^{\alpha_i-1} \le c_9\, |x_i|^{\alpha_i} \delta^k\ ,\end{aligned}$$

Therefore,

$$\int_{\widetilde{\Omega}_j} |x - y_j|\, |\nabla K_i(x)| u_i^{p_i+1}(x)\, dx \le 2\, c_9\, |x_i|^{\alpha_i} \delta\ . \tag{4.51}$$

Let δ be sufficiently small such that

$$\int_{B(0,\delta|x_i|)} |x - y_i|\, |\nabla K_i(x)| u_i^{p_i+1}\, dx \le \frac{c_5}{2} |x_i|^{\alpha_i} \tag{4.52}$$

holds for $i \geq i_0$. For the rest of the proof, δ will be fixed.

By (4.39), (4.43) and (4.47), for $i \geq \max(i_2(\delta) i_3(\epsilon))$ we have

$$\begin{aligned}
&\int_{A_2} |x - y_i|\, |\nabla K_i(x)| u_i^{p_i+1}\, dx \\
&\quad \leq c_{10}\, |x_i|^{\alpha_i} \int_{R \leq |y| \leq \delta_2 Li} v_i^{p_i^*}\, dy \\
&\quad \leq c_{10} |x_i|^{\alpha_i} \int_{R \leq |y| \leq \delta_2 L_i} v_i^{p_i}\, dy \\
&\quad \leq \frac{1}{4} c_5\, |x_i|^{\alpha_i}\ ,
\end{aligned} \tag{4.53}$$

if ϵ is sufficiently small.

For $x \in A_3$, let $x = x_i + M_i^{-\frac{p_i - 1}{2}} y$. Then

$$|y| \geq M_i^{\frac{p_i-1}{2}} |x - x_i| \geq \frac{1}{2} M_i^{\frac{p_i - 1}{2}} |x|\ ,$$

which implies $|x| \leq 2 M_i^{-\frac{p_i-1}{2}} |y|$. Together with (4.44) and (4.47), we have

$$\begin{aligned}
&\int_{A_3} |x - y_i|\, |\nabla K(x)| u_i^{p_i+1}\, dx \\
&\quad \leq c_{10}\, M_i^{-\frac{(p_i-1)\alpha_i}{2}} \int_{R \leq |y| \leq \delta_2 L_i} |y|^{\alpha_i} v_i^{\frac{2n}{n-2}}(y)\, dy \\
&\quad \leq c_{11}\, M_i^{-\frac{(p_i-1)\alpha_i}{2}} \int_{R \leq |y| \leq \delta_2 L_i} |y|^{\alpha_i - 2n}\, dy \\
&\quad \leq c_{11}\, M_i^{-\frac{(p_i-1)\alpha_i}{2}} \\
&\quad = c_{11}\, |x_i|^{\alpha_i} \left(M_i^{-\frac{p_i-1}{2}} |x_i| \right)^{-\alpha_i} \\
&\quad = o(1) |x_i|^{\alpha_i}\ .
\end{aligned} \tag{4.54}$$

Combining (4.48), (4.49) and (4.52)—(4.54) gives

$$\frac{1}{4} c_5\, |x_i|^{\alpha_i} \leq o(1) |x_i|^{\alpha_i},$$

which obviously yields a contradiction. Hence, the proof of Theorem 1.2 is completely finished. q.e.d.

5.

In this section, we are going to prove both Theorem 1.3 and Theorem 1.4. The key step for the proof of both theorems is the following lemma — Lemma 5.1. To state Lemma 5.1, we rewrite equation (1.1) into $\Delta u_i + c_i(x)u_i = 0$ with $c_i(x) = K_i(x)u_i^{\frac{4}{n-2}}$. By Theorem 1.2, we have $c_i(x) \le c|x|^{-2}$ for some constant $c > 0$. Applying the Harnack inequality and the gradient estimates of linear elliptic equations, we have

$$\sup_{|x|=r} u_i(x) \le c_1 \inf_{|x|=r} u_i(x) \tag{5.1}$$

and

$$|\nabla u_i(x)| \le c_1 u_i(x)|x|^{-1} \tag{5.2}$$

hold for $|x| \le 1$.

Let $w_i(t) = \overline{u}_i(r)r^{\frac{n-2}{2}}$ and $r = e^t$, where

$$\overline{u}_i(r) = \frac{1}{|\partial B_r|}\int_{|x|=r} u_i(x_i + x)\, d\sigma$$

is the integral average of $u_i(x_i+x)$ over the sphere $|x| = r$. By (5.1) and (5.2), both $w_i(t)$ and $w_i'(t)$ are uniformly bounded for all $t \le 0$, where w_i' denotes the first derivative of w_i with respect to t. By elementary calculations, w_i satisfies

$$\left(\frac{n-2}{2}\right)^2 w_i - c_1 w_i^{\frac{n+2}{n-2}} \le w_i'' \le \left(\frac{n-2}{2}\right)^2 w_i - c_2 w_i^{\frac{n+2}{n-2}}(t) \tag{5.3}$$

for all $t \le 0$ and two positive constants c_1 and c_2. From (5.3), there exists a small positive number $\epsilon_1 > 0$ such that $w_i''(t) > 0$ whenever $w_i(t) \le \epsilon_1$. For simplicity, we replace w_i by $w(t)$ in the following lemma.

Lemma 5.1. *There is a small positive number $\epsilon_0 < \epsilon_1$ such that the followings hold:*

(i) Suppose that $w(t)$ is nonincreasing in (t_0, t_1) with $w(t_0) \le \epsilon_0$. Then the inequality

$$t_1 - t_0 \le \frac{2}{n-2}\log\frac{w(t_0)}{w(t_1)} + c \tag{5.4}$$

holds, where c is a constant. Futhermore, if t_1 is a local minimum point of w, then the inequality

$$t_1 - t_0 \ge \frac{2}{n-2}\log\frac{w(t_0)}{w(t_1)} \tag{5.5}$$

holds.

(ii) Suppose that $w(t)$ is nondecreasing in (t_1, t_2) with $w(t_2) \le \epsilon_0$. Then

$$t_2 - t_1 \le \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} + c \tag{5.6}$$

for some constant $c > 0$. Furthermore if t_1 is a local minimum point of w, then

$$t_2 - t_1 \ge \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} \tag{5.7}$$

holds.

Proof. Suppose w is nonincreasing in (t_0, t_1). By the first half of inequality (5.3), $w_t^2 - \left(\frac{n-2}{2}\right)^2 w^2 + cw^{\frac{2n}{n-2}}(t)$ is nonincreasing in (t_0, t_1) where $c = \frac{n-2}{n} c_1$. Hence

$$w_t^2 - g(w) \ge -g(w(t_1)) \tag{5.8}$$

for $t \in [t_0, t_1)$ where $g(w) = \left(\frac{n-2}{2}\right)^2 w^2 - cw^{\frac{2n}{n-2}}$. Integrating (5.8) gives

$$t_1 - t_0 \le \int_{w(t_1)}^{w(t_0)} \frac{dw}{\sqrt{g(w) - g(w(t_1))}} \ . \tag{5.9}$$

By scaling,

$$\int_{w(t_1)}^{w(t_0)} \frac{dw}{\sqrt{g(w) - g(w(t_1))}} = \int_1^{\frac{w(t_0)}{w(t_1)}} \frac{d\eta}{\sqrt{\overline{g}(\eta) - \overline{g}(1)}} \tag{5.10}$$

where $\overline{g}(\eta) = \left(\frac{n-2}{2}\right)^2 \eta^2 - cw(t_1)^{\frac{4}{n-2}} \eta^{\frac{2n}{n-2}}$. For $1 \le \eta \le \frac{w(t_0)}{w(t_1)} \le \frac{\epsilon_0}{w(t_1)}$, we have

$$w^{\frac{4}{n-2}}(t_1) \left(\frac{\eta^{\frac{2n}{n-2}} - 1}{\eta^2 - 1} \right) \le c_2 w(t_1)^{\frac{4}{n-2}} \eta^{\frac{4}{n-2}} \le c_3 \epsilon_0^{\frac{4}{n-2}} \ .$$

Hence, if ϵ_0 is sufficiently small, then

$$\begin{aligned} &\int_1^{\frac{w(t_0)}{w(t_1)}} \frac{d\eta}{\sqrt{\overline{g}(\eta) - \overline{g}(1)}} \\ &\le \frac{2}{n-2} \int_1^{\frac{w(t_0)}{w(t_1)}} \frac{d\eta}{\sqrt{\eta^2 - 1}} + c_3 w^{\frac{4}{n-2}}(t_1) \int_1^{\frac{w(t_0)}{w(t_1)}} \frac{\eta^{\frac{4}{n-2}}}{\sqrt{\eta^2 - 1}} \, d\eta \\ &\le \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} + c_4 \end{aligned}$$

for some constant c_4. Here, we have used

$$w^{\frac{4}{n-2}}(t_1)\int_1^{\frac{w(t_0)}{w(t_1)}} \frac{\eta^{\frac{4}{n-2}}}{\sqrt{\eta^2-1}}\,d\eta \le c_5 w^{\frac{4}{n-2}}(t_1)\left(\frac{w(t_0)}{w(t_1)}\right)^{\frac{4}{n-2}} \le c_5\epsilon_0 \ .$$

Therefore, the first part of (i) is proved.

For the proof of the second part of (i), we use

$$w_{tt} \le \left(\frac{n-2}{2}\right)^2 w \ .$$

Hence $w_t^2 - \left(\frac{n-2}{2}\right)^2 w$ is nondecreasing in (t_0, t_1). In particular, we have

$$w_t^2 - \left(\frac{n-2}{2}\right)^2 w^2(t) \le -\left(\frac{n-2}{2}\right)^2 w^2(t_1) \ , \tag{5.11}$$

because $w'(t_1) = 0$. Integrating (5.11) gives

$$t_1 - t_0 \ge \frac{2}{n-2}\int_{w(t_1)}^{w(t_0)} \frac{dw}{\sqrt{w^2(t_0) - w^2(t_1)}} \ge \frac{2}{n-2}\log\frac{w(t_0)}{w(t_1)} \ .$$

Hence, the second part of (i) is proved.

If we let $\tilde{w}(t) = w(2t_1 - t)$ for $t \in (2t_1 - t_2, t_1)$, then (ii) immediately follows by similar arguments to (i). q.e.d.

Proof of Theorem 1.3. Obviously, (1.13) is a consequence of Lemma 3.2 and Theorem 1.2. Since $u_i(x) \sim M_i^{1-\frac{2\alpha_i}{n+2}}$ for $|x| = M_i^{-\beta_i}$ where $a_i \sim b_i$ denotes that a_i/b_i are bounded below and above by two constants independent of i, it suffices to prove the lower bound of (1.14).

Let x_i satisfy $u_i(x_i) = \max\limits_{\overline{B}_1} u_i(x) = M_i$. By Lemma 3.4, we may assume $\lim\limits_{i\to+\infty} M_i^{\frac{2}{n-2}} x_i = \xi$. By Lemma 3.6, ξ satisfies

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\,dy = 0 \ .$$

Let $u_i^*(y) = r_i^{\frac{n-2}{2}} u_i(x_i + r_i y)$ with $r_i = M_i^{-\beta_i}$, where

$$\beta_i = \frac{2}{n-2}\left(1 - \frac{\alpha_i}{n-2}\right).$$

In Section 3, we have proved $u_i^*(0)u_i^*(y)$ converges to $h(y) = a|y|^{2-n} + b$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ where $a \geq 0$ and $b \geq 0$. Moreover, from (3.62) and (3.63), we have

$$\begin{aligned}
&\lim_{i\to+\infty} u_i^{*^2}(0)P(1;u_i^*) \\
&\quad = \lim_{i\to+\infty} u_i^{*^2}(0) r_i \int_{B_1} y\cdot\nabla K_i(x_i + r_i y) u_i^*(y)^{\frac{2n}{n-2}}\,dy \\
&\quad = \int_{\mathbb{R}^n} y\cdot\nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\,dy \\
&\quad = \int_{\mathbb{R}^n} Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\,dy\ ,
\end{aligned}$$

where

$$\begin{aligned}
P(1;u_i^*) = \int_{\partial B_1} \Bigg(&\frac{n-2}{2} u_i^* \frac{\partial u^*}{\partial\nu} - \frac{1}{2}|\nabla u_i^*|^2 + |\frac{\partial u_i^*}{\partial\nu}|^2 \\
&+ \frac{n-2}{2n} K_i(x_i + r_i y) u_i^{*\frac{2n}{n-2}} \Bigg)\, d\sigma_y\ .
\end{aligned}$$

Since $u_i^*(0)u_i^*$ converges to $h(y)$, a simple calculation leads to

$$\lim_{i\to+\infty} u_i^{*^2}(0)P(1;u_i^*) = -(n-2)\sigma_n ab \leq 0\ ,$$

where σ_n is the area of unit sphere S^{n-1}. Therefore, by the assumption of Theorem 1.3, we have

$$\int_{\mathbb{R}^n} Q(\xi + y) U_0^{\frac{2n}{n-2}}(y)\,dy < 0\ , \tag{5.12}$$

from which both a and $b > 0$. Hence it implies $w_i(t)$ has its first local minimum at $t_i = -\beta_i \log M_i + c + o(1)$, where c is a constant. We also have $w(t_i) = \text{const.}\, M_i^{\frac{-\alpha_i}{n-2}}$. We want to prove $w(t) \leq \epsilon_0$ for $t \in (t_i, 0)$, where ϵ_0 is the positive number stated in Lemma 5.1.

Suppose the claim is not true. Let $t_i^* < t_i < \bar{t}_i$ satisfy $w_i(t_i^*) = w_i(\bar{t}_i) = \epsilon_0$ and $w_i(t) \leq \epsilon_0$ for $t \in (t_i^*, \bar{t}_i)$. Since $u_i^*(0)u_i^*(y)$ converges to $h(y) = h(|y|)$, we have $u_i(x_i + x) = \overline{u}_i(|x|)(1 + o(1))$ and $|\nabla u_i(x_i + x)| = -\overline{u}_i'(|x|)(1 + o(1))$ at $|x| = e^{t_i}$. By a simple computation, we have for

$r_i = e^{t_i}$,

(5.13)

$$\begin{aligned} &P(r_i;u_i)\\ &= \sigma_n\left\{\frac{1}{2}w_i'^2(t_i) - \frac{1}{2}\left(\frac{n-2}{2}\right)^2 w_i^2(t_i) + \frac{n-2}{2n}\overline{K}_i(r_i)w_i^{\frac{2n}{n-2}}(t_i)\right\}\\ &\quad + \left(w_i'^2(t_i) + w_i^2(t_i)\right)o(1)\ , \end{aligned}$$

where $\overline{K}_i(r) = \frac{1}{|\partial B_r|}\int\limits_{|x-x_i|=r} K\,d\sigma$ and

$$\begin{aligned} P(r_i;u_i) = \int_{|x-x_i|=r_i} &\frac{n-2}{2}u_i\frac{\partial u_i}{\partial\nu} - \frac{r_i}{2}|\nabla u_i|^2 + |\frac{\partial u_i}{\partial\nu}|^2\, r_i\\ &+ \frac{n-2}{2n}K_i(y)u_i^{\frac{2n}{n-2}}(y)\, r_i\, d\sigma_y\ . \end{aligned}$$

Since $w'(t_i) = 0$, (5.13) implies

$$\begin{aligned} w_i^2(t_i) &\le c_n|P(r_i)|\\ &= c_n\left(\int_{B_{r_i}\setminus B_{r_i^*}} |x\cdot\nabla K_i(x)|u_i^{\frac{2n}{n-2}}\,dx\right.\\ &\qquad\left. + \int_{B_{r_i^*}} |x\cdot\nabla K_i(x)|u_i^{\frac{2n}{n-2}}(x)\,dx\right)\\ &\equiv I_1 + I_2\ , \end{aligned} \tag{5.14}$$

where $r_i^* = e^{t_i^*}$. Since $|x\cdot\nabla K_i(x)| \le c|x|^{\alpha_i}$,

$$|I_2| \le c_2(r_i^*)^{\alpha_i} = c_2\exp(\alpha_i t_i^*)\ . \tag{5.15}$$

To estimate I_1, by (5.5), we have for $t_i^* \le t \le t_i$,

$$w(t) \le c_3 w(t_i)\exp\left[\frac{n-2}{2}(t_i - t)\right]\ .$$

Thus,

$$\begin{aligned} |I_1| \le& c_3 w^{\frac{2n}{n-2}}(t_i)\exp(nt_i)\int_{t_i^*}^{t_i}\exp -(n-\alpha_i)t\,dt\\ \le& c_4 w^{\frac{2n}{n-2}}(t_i)\exp(nt_i)\exp(\alpha_i - n)t_i^*\ . \end{aligned}$$

From (5.4) it follows that

$$w(t_i) \leq c_5 w_i(t_i^*) \exp\left[\left(\frac{n-2}{2}\right)(t_i^* - t_i)\right].$$

Putting these two estimates together gives

$$|I_1| \leq c_6 \epsilon_0^{\frac{2n}{n-2}} \exp(\alpha_i t_i^*) \ . \tag{5.16}$$

Therefore,

$$w(t_i) \leq c_7 \exp(\frac{\alpha_i}{2} t_i^*) \ . \tag{5.17}$$

Applying (5.5) and (5.6), we have

$$t_i - t_i^* \geq \frac{2}{n-2} \log \frac{w(t_i^*)}{w(t_i)} = \frac{2}{n-2} \log \frac{\epsilon_0}{w(t_i)} \ ,$$

and

$$\bar{t}_i - t_i \geq \frac{2}{n-2} \log \frac{w(\bar{t}_i)}{w(t_i)} = \frac{2}{n-2} \log \frac{\epsilon_0}{w(t_i)}.$$

Putting these two inequalities and (5.17) together yields

$$\bar{t}_i - t_i^* \geq \frac{4}{n-2} \log \frac{\epsilon_0}{w(t_i)} \geq -\frac{2\alpha_i}{n-2} t_i^* - c_8 \ .$$

Hence

$$\bar{t}_i + \left(\frac{2\alpha_i}{n-2} - 1\right) t_i^* \geq -c_8 \ .$$

Suppose $\alpha = \lim_{i \to +\infty} \alpha_i > \frac{n-2}{2}$. Then

$$t_i^* \geq -c_9 \ ,$$

which yields a contradiction, because $\lim_{i \to +\infty} t_i^* \leq \lim_{i \to +\infty} t_i = -\infty$. Hence $w_i(t)$ is increasing in $(t_i, 0]$ with $w_i(0) \leq \epsilon_0$. By (ii) of Lemma 5.1,

$$\begin{aligned} \overline{u}_i(1) = w_i(0) &\geq c_{10} w_i(t_i) e^{-\frac{n-2}{2} t_i} \\ &\geq c_{11} \, M_i^{1 - \frac{2\alpha_i}{n-2}} \ . \end{aligned}$$

Applying the Harnack inequality gives the lower bound of (1.14) for $|x| \geq M_i^{-\beta_i}$.

If $\alpha = \frac{n-2}{2}$, then $\overline{t}_i \geq -c_8$ and $\left(\frac{2\alpha_i}{n-2} - 1\right) t_i^* \geq -c_8$. Since $t_i^* \leq t_i$, we have

$$M_i^{\frac{2\alpha_i}{n-2}-1} \leq c_{12}$$

for some constant c_{12}, and there exists a t_0, which is independent of i, such that w_i is increasing in $[t_i, t_0]$ with $w_i(t_0) \leq \epsilon_0$. Let $r_0 = e^{t_0}$. By (ii) of Lemma 5.1,

$$\begin{aligned}\overline{u}_i(r_0) =& w_i(r_0)e^{-\frac{n-2}{2}t_0} \geq c_{10}\, w_i(t_i)e^{-\frac{n-2}{2}t_i} \\ =& c_{10}\, \overline{u}_i(e^{t_i}) \geq c_{11}\, M_i^{1-\frac{2\alpha_i}{n-2}} \ .\end{aligned}$$

Applying the Harnack inequality, we have the lower bound of (1.15) for the case of $\alpha = \frac{n-2}{2}$. Obviously, (1.16) is an immediate consequence of (1.13)—(1.15). Thus, the proof of Theorem 1.3 is considered completely finished. q.e.d.

Proof of Theorem 1.4. By Theorem 1.2, we have

$$u_i(x) \leq c_1\, |x|^{-\frac{n-2}{2}} \quad \text{for} \ \ |x| \leq 1 \ . \tag{5.18}$$

Applying estimates of linear elliptic equations, $u_i(x)$ is bounded in $C^2_{\mathrm{loc}}(\overline{B}_1 \setminus \{0\})$. Without loss of generality, we may assume u_i converges to some positive function u in $C^2_{\mathrm{loc}}(\overline{B}_1 \setminus \{0\})$, where u is a postive smooth function of

$$\Delta u + K(x)\, u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \ \ B_1 \setminus \{0\} \ , \tag{5.19}$$

and $K(x) = \lim_{i\to+\infty} K_i(x)$. In the following, we want to prove u has a nonremovable singularity at 0. In fact, we claim that

(5.20) For any $u_0 > 0$, there exists a positive $r_0 > 0$ and i_0 such that $\overline{u}_i(r_0) \geq u_0$ for $i \geq i_0$, where

$$\overline{u}_i(r) = \frac{1}{|\partial B_r|} \int_{|x|=r} u_i\, d\sigma \ .$$

Now suppose (5.20) is not true. Then there exists $u_0 > 0$ and $\overline{u}_i(r_i) = u_0$ for some $r_i > 0$ such that $\lim_{i\to+\infty} r_i = 0$. Let $w_i(t) = \overline{u}_i(r) r^{\frac{n-2}{2}}$ and $t = \log r$. Denote $t_i = \log r_i$. Then we have $w_i(t_i) = u_0 e^{\frac{(n-2)}{2}t_i} \to 0$ as $i \to +\infty$. Hence we may assume $w_i(t_i) < \epsilon_0$ for all i where ϵ_0 is the constant in Lemma 5.1.

Let $t_i^* \equiv \sup\{t < t_i \mid w_i(t) = \epsilon_0\}$. Without loss of generality, we may assume there are no local minimum of w_i in (t_i^*, t_i). To see this, we assume there is a local minimum $\overline{t}_i \in (t_i^*, t_i)$. Then, by (5.6), we have

$$u_0 = \overline{u}_i(r_i) \le \overline{u}(e^{\overline{t}_i}) \le c\,\overline{u}_i(r_i) = c\,u_0 \ ,$$

for some constant $c > 0$. Let t_i and u_0 be replaced by $\overline{t}_i$ and $c\,u_0$ respectively and then we may assume there are no local minimal points of w_i in (t_i^*, t_i). Thus, we have $w_i'(t) < 0$ for $t \in (t_i^*, t_i)$.

Let $r_i^* = e^{t_i^*}$ and let

$$\tilde{u}_i(y) = u_i(r_i^* y)(r_i^*)^{\frac{n-2}{2}} \ . \tag{5.21}$$

Since $\tilde{u}_i(y)$ satisfies

$$\Delta \tilde{u}_i + K_i\,(r_i^* y)\,\tilde{u}_i^{\frac{n+2}{n-2}} = 0 \ ,$$

and is uniformly bounded in any compact set of $\mathbb{R}^n \backslash \{0\}$, $\tilde{u}_i(y)$ converges in $C^2_{\rm loc}\,(\mathbb{R}^n \setminus \{0\})$ to $\tilde{u}_0$, where $\tilde{u}_0$ satisfies

$$\Delta \tilde{u}_0 + n(n-2)\tilde{u}_0^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\} \ . \tag{5.22}$$

Applying the Pohozaev identity leads to

$$P(1; \tilde{u}_i) = \frac{(n-2)r_i^*}{2n} \int_{|y| \le 1} y \cdot \nabla K_i(r_i^* y)\tilde{u}_i^{\frac{2n}{n-2}}(y)\,dy \ , \tag{5.23}$$

where $P(r, \tilde{u}_i)$ is defined in (1.18). Since

$$|y \cdot \nabla K_i(r_i^* y)|\tilde{u}_i^{\frac{2n}{n-2}}(y) \le c\,r_i^{*\alpha_i - 1}|y|^{\alpha_i - n} \in L^1(B_1)$$

by Theorem 1.2, we have for any $r > 0$,

$$P(r, \tilde{u}_0) = \lim_{i \to +\infty} P_i(r; \tilde{u}_i) = 0 \ .$$

If $\tilde{u}_0$ has a singularity at 0, then $\tilde{u}_0(x) = \tilde{u}_0(|x|)$ and $P(r; \tilde{u}_0) \equiv$ constant < 0 by an elementary calculation. Hence $\tilde{u}_0$ is smooth at 0. By a theorem of Caffarelli-Gidas-Spruck, $\tilde{u}_0$ can be written as

$$\tilde{u}_0(y) = \left(\frac{\lambda}{1 + \lambda^2|y - \eta_0|^2}\right)^{\frac{n-2}{2}} \tag{5.24}$$

for some $\lambda > 0$ and $\eta_0 \in \mathbb{R}^n$. We have from (5.18),

$$\lambda|\eta_0| \le c_1 \ .$$

Step 1. We claim $\eta_0 = 0$.

First, let us assume $\eta_0 \neq 0$. Hence, $\tilde{u}_i$ has a local maximum at η_i and, by (5.21), u_i has a local maximum at y_i, where

$$y_i = r_i^* \eta_i, \quad \text{and,} \quad \lim_{i \to +\infty} \eta_i = \eta_0 \ . \tag{5.25}$$

Let $\xi_i = u_i(y_i)^{\frac{2}{n-2}} y_i$. Then

$$\begin{aligned} \lim_{i\to+\infty} \xi_i &= \lim_{i\to+\infty} \tilde{u}_i(\eta_i)^{\frac{2}{n-2}} (r_i^*)^{-1} y_i \\ &= \lim_{i\to+\infty} \tilde{u}_i(\eta_i)^{\frac{2}{n-2}} \eta_i \\ &= \lambda \eta_0 \equiv \xi_0 \ . \end{aligned} \tag{5.26}$$

Thus,

$$0 < c_2^{-1} \le u_i(y_i)^{\frac{2}{n-2}} |y_i| \le c_2 \ . \tag{5.27}$$

Since (5.18) holds for all $|x| \le 1$, we have for large $R > 0$, by (5.27)

$$\begin{aligned} u_i(y) &\le c_1 \, |y|^{-\frac{n-2}{2}} \\ &\le c_1 \, R^{-\frac{n-2}{2}} |y_i|^{-\frac{n-2}{2}} \\ &\le u_i(y_i) \ , \end{aligned}$$

when $|y| \ge R|y_i|$. From the uniform convergence of $\tilde{u}_i$ in any compact set of $\mathbb{R}^n \setminus \{0\}$ and $|y_i| =$ const. r_i^*, it follows that

$$u_i(y_i) = \max_{|x| \ge \delta |y_i|} u_i(x) \tag{5.28}$$

for any fixed but small positive δ.

Let

$$v_i(y) = M_i^{-1} u_i \left(y_i + M_i^{-\frac{2}{n-2}} y \right) \ ,$$

where $M_i = u_i(y_i)$. Obviously, $v_i(y)$ converges to $U_0(y)$ uniformly in any compact set of $\mathbb{R}^n \setminus \{-\xi_0\}$, where ξ_0 is the vector in (5.26). By the same arguments in Lemma 3.1, we can prove Lemma 3.1 still holds for

$v_i(y)$ outside of a small neighborhood of $\{-\xi_0\}$, i.e., for any $\epsilon > 0$, there exists $\delta_1 = \delta(\epsilon)$ and $i_0 = i_0(\epsilon)$ such that

$$\min_{|y|=r} v_i(y) \leq (1+\epsilon)U(r) \tag{5.28}$$

for $2|\xi_0| \leq r \leq \delta_1 L_i$ with $L_i = M_i^{\frac{2\alpha_i}{(n-2)^2}}$.

To see this, we suppose (5.28) is not true. Then there exist an ϵ_0 and a sequence of $r_i \to +\infty$ as $i \to +\infty$ such that

$$\min_{|y|=r_i} v_i(y) \geq (1+2\epsilon_0)U_0(r_i) \ ,$$

where $r_i \leq \delta_1 L_i$ for some small $\delta_1 > 0$ to be chosen later. Without loss of generality, we may assume $-\xi_0 = 2\tau_0 e_1$ for some $\tau_0 > 0$. Let

$$\begin{cases} \tilde{v}_i(y) = v_i(y+\tau_0 e_1) \ , \\ \overline{v}_i(y) = \left(\frac{\tau_0}{|y|}\right)^{n-2} \tilde{v}_i\left(\frac{\tau_0^2 y}{|y|^2}\right) \ , \\ \overline{U}_0(y) = \left(\frac{\tau_0}{|y|}\right)^{n-2} U_0\left(\frac{\tau_0^2 y}{|y|^2} + \tau_0 e_1\right) \ . \end{cases}$$

By a straighforward calculation, we have

$$\overline{U}_0(y) = \left(\frac{\lambda}{1+\lambda^2|y+y_0|^2}\right)^{\frac{n-2}{2}} \ ,$$

and

$$\overline{U}_0(0) = \tau_0^{-n+2} \ ,$$

where $\lambda = \frac{1+\tau_0^2}{\tau_0^2}$ and $y_0 = \frac{\tau_0^3}{1+\tau_0^2}e_1$. It is easy to see that there exists a small $\delta > 0$ such that the image of the neighborhood $\overline{B(-\xi_0,\delta)}$ of $-\xi_0$ under the map $y \to \frac{\tau_0^2 y}{|y|^2} + \tau_0 e_1$ is contained in the half-plane $\{(y_1,\cdots,y_n)|\, y_1 > 0\}$. In Lemma 3.1, what we have to need about $\overline{v}_i$ is the estimates of $\overline{v}_i(y^\lambda)$ for $\lambda \leq \lambda_0$ and $y_1 \geq \lambda_0$, where $\lambda_0 = -\frac{1}{2}\frac{\tau_0^3}{1+\tau_0^2}$. Since y^λ is not contained in the image of $\overline{B(-\xi_0,\delta)}$ under the inversion, $\frac{\tau_0^2 y^\lambda}{|y^\lambda|^2} + \tau_0 e_1 \notin B(-\xi_0,\delta)$ and we have

$$\overline{v}_i(y^\lambda) = \left(\frac{\tau_0}{|y^\lambda|}\right)^{n-2} \tilde{v}_i\left(\frac{\tau_0^2 y^\lambda}{|y^\lambda|^2}\right) \leq c\,|y^\lambda|^{2-n}$$

for some constant $c > 0$ and for $\lambda \leq \lambda_0$ and $y_1 \geq \lambda$. Then we can obtain all the estimates in Lemma 3.1 without any modification, and apply the method of moving planes to obtain a contradition.

Applying Lemma 3.2, there exists $R = R(\epsilon) > 0$ such that

$$\int_{R(\epsilon)\le|y|\le\delta_2 L_i} v_i^{\frac{n+2}{n-2}}(y)\,dy \le \frac{4\sigma_n}{n}\epsilon\ .$$

Choose ϵ so small such that Lemma 2.3 can be applyed. Thus,

$$v_i(y) \le c_4\, U_0(y) \tag{5.29}$$

for $2|\xi_0| \le y \le l_i = \delta_2 L_i$ where c_4 and δ_2 are two constant independent of i. In particular,

$$\begin{cases} v_i(y) \le c_4\, l_i^{-n+2}, \\ |\nabla v_i(y)| \le c_5\, l_i^{-n+1} \end{cases} \tag{5.30}$$

for $|y| = l_i$.

Multiplying $\frac{\partial v_i}{\partial y_i}$ on the equation for v_i, we have

$$\begin{aligned} &\frac{n-2}{2n} M_i^{\frac{-2}{n-2}} \int_{|y|\le l_i} \frac{\partial K_i}{\partial x_j}\left(y_i + M_i^{\frac{-2}{n-2}} y\right) v_i^{\frac{2n}{n-2}}(y)\,dy \\ &\qquad = \int_{|y|=l_i} \left[\left(\frac{\partial v_i}{\partial y_j}\frac{\partial v_i}{\partial \nu}\right) - \frac{1}{2}|\nabla v_i|^2 \nu_j \right. \\ &\qquad\qquad \left. + \frac{n-2}{2n} K_i\left(y_i + M_i^{\frac{-2}{n-2}} y\right) v_i^{\frac{2n}{n-2}}\right] d\sigma\ . \end{aligned} \tag{5.31}$$

By (5.30), the absolute value of the boundary term is bounded by $c_6\, l_i^{-n+1}$. Hence,

$$\lim_{i\to+\infty} \left(L_i^{n-2}\,|\text{ the boundary term }|\right) = 0\ .$$

On the other hand, we have

$$\begin{aligned} &\lim_{i\to+\infty} L_i^{n-2} M_i^{\frac{-2}{n-2}} \int_{|y|\le l_i} \frac{\partial K_i}{\partial x_j}\left(y_i + M_i^{\frac{-2}{n-2}}(y)\right) v_i^{\frac{2n}{n-2}}(y)\,dy \\ &\quad = \lim_{i\to+\infty} \int_{|y|\le l_i} \frac{\partial K_i}{\partial x_j}\left(M_i^{\frac{2}{n-2}} y_i + y\right) v_i^{\frac{2n}{n-2}}(y)\,dy \\ &\quad = \int_{\mathbb{R}^n} \frac{\partial Q}{\partial x_j}(\xi_0 + y)\, U_0^{\frac{2n}{n-2}}(y)\,dy\ , \end{aligned}$$

where we ultilize for any $\delta > 0$,

$$\begin{aligned} & M_i^{-\frac{2}{n-2}} L_i^{n-2} \int_{B(-\xi_0,\delta)} \left|\frac{\partial K_i}{\partial x_j}\right| \left(y_i + M_i^{-\frac{2}{n-2}} y\right) v_i^{\frac{2n}{n-2}}(y)\, dy \\ & \le M_i^{-\frac{2}{n-2}} L_i^{n-2} \int_{|y|\le \frac{2\delta}{|\xi_0|}|y_i|} \left|\frac{\partial K_i}{\partial x_j}(y)\right| u_i^{\frac{2n}{n-2}}(y)\, dy \\ & \le c_7\, M_i^{-\frac{2}{n-2}} L_i^{n-2} \int_{|y|\le \frac{2\delta}{|\xi_0|}|y_i|} |y|^{\alpha_i - 1 - n}\, dy \\ & \le c_8\, \delta^{\alpha_i - 1} |y_i|^{\alpha_i - 1} L_i^{n-2} M_i^{-\frac{2}{n-2}} \\ & \le c_9\, \delta^{\alpha_i - 1} \ . \end{aligned}$$

Therefore, ξ_0 satisfies

$$\int_{\mathbb{R}^n} \nabla Q(\xi_0 + y) U_0^{\frac{2n}{n-2}}(y)\, dy = 0 \ . \tag{5.32}$$

By (5.18), we have

$$u_i(y_i + y)|y|^{\frac{n-2}{2}} \le c_1 \quad \text{for} \ \ 2|y_i| \le |y| \le 1 \ . \tag{5.33}$$

Let $\tilde{r}_i = M_i^{-\frac{2}{n-2}} L_i = M_i^{-\beta_i}$ where $\beta_i = \frac{2}{n-2}\left(1 - \frac{\alpha_i}{n-2}\right)$, and $u_i^*(y) = \tilde{r}_i^{\frac{n-2}{2}} u_i(y_i + \tilde{r}_i y)$. Then $u_i^*(0) = \tilde{r}_i^{\frac{n-2}{2}} u_i(y_i) = M_i^{\frac{\alpha_i}{n-2}} \to +\infty$ as $i \to +\infty$. By (5.33), $u_i^*(y)$ is uniformly bounded in $\mathbb{R}^n \setminus \{0\}$. By (5.29) and the Harnack inequality,

$$u_i^*(0) u_i^*(y) = L_i^{-(n-2)} v_i(L_i y)$$

is uniformly bounded in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. Without loss of generality, we may assume $u_i^*(0)u_i^*(y)$ converges to $h(y)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, where $h(y)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Thus, by Liouville's Theorem,

$$h(y) = a|y|^{2-n} + b$$

with $a, b \ge 0$. By Pohozaev's identity, we have

$$\frac{n-2}{2n} \tilde{r}_i \int_{B_1} y \cdot \nabla K_i(y_i + \tilde{r}_i y) u_i^*(y)^{\frac{2n}{n-2}}\, dy = P(1; u_i^*) \ ,$$

where $P(1; u_i^*)$ is given in (1.18).

By elementary calculations, we have

$$\lim_{i\to+\infty} u_i^{*^2}(0)P_i(1;u_i^*) = -(n-2)\sigma_n ab\ , \tag{5.34}$$

where σ_n is the area of S^{n-1}.

On the other hand,

$$\begin{aligned} u_i^{*^2}(0)\tilde{r}_i \int_{B_1} y\cdot\nabla K_i(y_i+\tilde{r}_i y)u_i^*(y)^{\frac{2n}{n-2}}\,dy \\ = \int_{|y|\le L_i} y\cdot\nabla Q_i(\xi_i+y)v_i^{\frac{2n}{n-2}}(y)\,dy \\ + o(1)\int_{|y|\le L_i} |y|\,|\xi_i+y|^{\alpha_1-1}v_i^{\frac{2n}{n-2}}\,dy\ . \end{aligned} \tag{5.35}$$

For any $\delta>0$, we have the estimate

$$\begin{aligned} &\left|\int_{B(-\xi;\delta)} y\cdot\nabla K_i(\xi_i+y)v_i^{\frac{2n}{n-2}}(y)\,dy\right| \\ &= M_i^{\frac{2\alpha_i}{n-2}}\left|\int_{B(-y_i,M_i^{\frac{-2}{n-2}}\delta)} y\cdot\nabla K_i(y_i+y)u_i^{\frac{2n}{n-2}}(y_i+y)\,dy\right| \\ &\le M_i^{\frac{2\alpha_i}{n-2}}\left|\int_{|y+y_i|\le c_2\,\delta|y_i|} (y\cdot\nabla K_i(y_i+y))\,u_i^{\frac{2n}{n-2}}(y_i+y)\,dy\right| \\ &\le c_3\,M_i^{\frac{2\alpha_i}{n-2}}|y_i|\int_{|y|\le c_2\,\delta|y_i|}|y|^{\alpha_i-1-n}\,dy \\ &= c_4\,M_i^{\frac{2\alpha_i}{n-2}}|y_i|^{\alpha_i}\delta^{\alpha_i-1} \\ &\le c_5\,\delta^{\alpha_i-1}\ , \end{aligned} \tag{5.36}$$

where c_5 is a constant independent of i. Since v_i uniformly converges to $U_0(y)$ in $\overline{B}_R\setminus B(-\xi_o,\delta)$ for any large $R>0$, we have by (5.29), (5.32) and (5.34)—(5.36),

$$\begin{aligned} -(n-2)\sigma_n ab &= \left(\frac{n-2}{2n}\right)\lim_{i\to+\infty}\int_{|y|\le L_i} y\cdot\nabla Q_i(\xi_i+y)v_i^{\frac{2n}{n-2}}\,dy \\ &= \frac{n-2}{2n}\int_{\mathbb{R}^n} y\cdot\nabla Q(\xi+y)U_0^{\frac{2n}{n-2}}(y) \\ &= \frac{\alpha(n-2)}{2n}\int_{\mathbb{R}^n} Q(\xi+y)U_0^{\frac{2n}{n-2}}(y)\,dy \le 0\ . \end{aligned}$$

From the assumption, it follows that

$$\int_{\mathbb{R}^n} Q(\xi_0 + y) U_0^{\frac{2n}{n-2}}(y)\, dy < 0 \ , \tag{5.37}$$

so that both a and $b > 0$.

Let $\hat{w}_i(t) = \hat{u}_i(r) r^{\frac{n-2}{2}}$ and $r = e^t$ where $\hat{u}_i(r)$ is the integral average of $u_i(y_i + y)$ over the sphere $|y| = r$. Since $u_i^*(0) u_i^*(y) \to a|y|^{2-n} + b$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ with both $a, b > 0$, $\hat{w}_i$ has a first local minimum at $T_i = -\beta_i \log M_i + c + o(1)$. Recall $w(t_i^*) = \epsilon_0$ and $\lim_{i\to+\infty} w_i(t_i) = 0$. Thus, we have $r_i^* = o(1) \min(e^{\frac{n-2}{2}T_i}, r_i)$ as $i \to +\infty$. Meanwhile, by the Harnack inequality, we have

$$c_6^{-1}\, \overline{u}_i(r) \le \hat{u}_i(r) \le c_6\, \overline{u}_i(r)$$

for $r \ge 2|y_i|$, where c_6 is a constant independent of u_0 and i.

If $t_i \ge T_i$, then, $\hat{w}_i(t)$ uniformly tends to 0 for $T_i \le t \le t_i$ as $i \to +\infty$. Therefore, $\hat{w}_i$ has no local minimum point in $(T_i, t_i]$ for large i. By (ii) of Lemma 5.1, we have

$$c_7\, M_i^{1-\frac{2\alpha_i}{n-2}} \le \hat{u}_i(e^{T_i}) \le c\, u_i(e^{t_i}) \le c_8\, u_0 \ .$$

Since $\lim_{i\to+\infty} 1 - \frac{2\alpha_i}{n-2} = 1 - \frac{2\alpha}{n-2} > 0$, M_i is bounded, which yields a contradition.

If $t_i \le T_i$, Then

$$c_9\, M_i^{1-\frac{\alpha_i}{n-2}} \le \hat{u}_i(T_i) \le \hat{u}_i(t_i) = u_0 \ ,$$

which again leads to a contradiction. Therefore, we have proved $\eta_0 = 0$.

Step 2.

Applying a variant of the Pohozaev identity (see (5.31)), we have

$$\begin{aligned} &\left(\frac{n-2}{2n}\right) r_i^* \int_{|y|\le\lambda_i} \frac{\partial K_i}{\partial x_j}(r_i^* y)\, \tilde{u}_i^{\frac{2n}{n-2}}(y)\, dy \\ &= \int_{|y|=\lambda_i} \left[\frac{\partial \tilde{u}_i}{\partial y_j}\frac{\partial \tilde{u}_i}{\partial \nu} - \frac{1}{2}|\nabla \tilde{u}_i|^2 \nu_j + \frac{n-2}{2n} K_i(r_i^* y) \tilde{u}_i^{\frac{2n}{n-2}}(y)\right] dy \ , \end{aligned} \tag{5.38}$$

where $\lambda_i = (r_i^*)^{\frac{-\alpha_i}{n-2}}$. In the followings, we discuss two cases seperately.

Case 1. Suppose w_i has no local minimum after t_i. Then (5.4) and the Harnack inequality give

$$\begin{aligned} \tilde{u}_i(y)|y|^{n-2} &= u_i(r_i^* y)\,(r_i^*|y|)^{n-2}\,(r_i^*)^{-\frac{n-2}{2}} \\ &\le c\, u_i(r_i^*)\,(r_i^*)^{\frac{n-2}{2}} \\ &= c\,\epsilon_0 \end{aligned} \tag{5.39}$$

for $1 \le |y| \le (r_i^*)^{-1}$. By gradient estimates, we have

$$|\nabla \tilde{u}_i(y)| \le c_1\, \tilde{u}_i(y)|y|^{-1} \le c_1\,|y|^{-n+1}$$

for $|y| \ge 2$. Hence, the absolute value of the right-hand side of (5.38) $\le c_3\,\lambda_i^{-n+1}$. Multiplying $\lambda_i^{n-2} = (r_i^*)^{-\alpha_i}$ on both sides of (5.38) leads to

$$\begin{aligned} 0 &= \left(\frac{n-2}{2n}\right) \lim_{i\to+\infty} (r_i^*)^{-\alpha_i+1} \int_{|y|\le\lambda_j} \frac{\partial K_i}{\partial x_j}(r_i^* y)\tilde{u}_i^{\frac{2n}{n-2}}(y)\,dy \\ &= \frac{n-2}{2n}\int_{\mathbb{R}^n} \frac{\partial Q}{\partial x_j}(y)\tilde{u}_0^{\frac{2n}{n-2}}(y)\,dy \\ &= \frac{n-2}{2n\lambda^{\alpha-1}}\int_{\mathbb{R}^n} \frac{\partial Q}{\partial x_j}(y)U_0^{\frac{2n}{n-2}}(y)\,dy\ , \end{aligned} \tag{5.40}$$

where we have untilized (5.39) and the following estimate: For any $\delta > 0$, by Theorem 1.2,

$$\begin{aligned} \int_{|y|\le\delta} \left|\frac{\partial K_i}{\partial x_j}\right| (r_i^* y)\tilde{u}_i^{\frac{2n}{n-2}}(y)\,dy &\le c_4\,(r_i^*)^{\alpha_i-1}\int_{|y|\le\delta} |y|^{\alpha_i-1-n}\,dy \\ &= c_5\,(r_i^*)^{\alpha_i-1}\delta^{\alpha_i-1}\ . \end{aligned}$$

Case 2. Suppose w_i has a local minimum after t_i, then, by (5.4) and (5.5), we have

$$c_1\,u_i(r_i^*)(r_i^*)^{n-2} \le u_i(r_i)r_i^{n-2} = u_0 r_i^{n-2} \le c_2\,u_i(r_i^*)(r_i^*)^{n-2}\ .$$

Recall $u_i(r_i^*)\,(r_i^*)^{\frac{n-2}{2}} = \epsilon_0$. Hence,

$$c_3\,(r_i^*)^{\frac{1}{2}} \le r_i \le c_4\,(r_i^*)^{\frac{1}{2}} \tag{5.41}$$

where both c_3 and c_4 are independent of i. Thus, as $i \to +\infty$,

$$(r_i^*)^{1-\frac{\alpha_i}{n-2}} = o(1)r_i\ , \tag{5.42}$$

which and (5.4) give (5.39) again, that is,

$$\tilde{u}_i(y) \le \epsilon_0 |y|^{2-n} \tag{5.39$'$}$$

for $1 \le |y| \le (r_i^*)^{\frac{-\alpha_i}{n-2}} = \lambda_i$. Hence, by (5.38), we have the same conclusion as (5.40).

Let $u_i^*(y) = \tilde{u}_i(\lambda_i y)\lambda_i^{\frac{n-2}{2}}$. By Theorem 1.2, $u_i^*(y) \le c\,|y|^{-\frac{n-2}{2}}$. Therefore $u_i^*(y)$ is uniformly bounded in $C^2(\mathbb{R}^n \setminus \{0\})$. Since $\lambda_i^{\frac{n-2}{2}} u_i^*(y)$ satisfies

$$\Delta\left(\lambda_i^{\frac{n-2}{2}} u_i^*(y)\right) + K_i(\lambda_i r_i^* y)(u_i^*)^{\frac{4}{n-2}}\left(\lambda_i^{\frac{n-2}{2}} u_i^*(y)\right) = 0 ,$$

and, by (5.39) and (5.39$'$), $\lambda_i^{\frac{n-2}{2}} u_i^*(y) = \lambda_i^{n-2}\tilde{u}_i(\lambda_i y)$ is uniformly bounded in any compact of $\mathbb{R}^n \setminus \{0\}$, $\lambda_i^{\frac{n-2}{2}} u_i^*(y)$ converges to a harmonic function $h(y)$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Using Liouville's Theorem, we have $h(y) = a|y|^{2-n} + b$ for $a, b \ge 0$. By a similar argument as in Step 1, we have

$$\begin{aligned}
0 \ge & -(n-2)\sigma_n ab \\
= & \frac{n-2}{2n} \lim_{i\to+\infty} \lambda_i^{n-2}(\lambda_i r_i^*) \int_{B_1} y \cdot \nabla K_i(\lambda_i r_i^* y)(u_i^*)^{\frac{2n}{n-2}}(y)\, dy \\
= & \frac{n-2}{2n} \lim_{i\to+\infty} \lambda_i^{n-2} r_i^* \int_{|y|\le\lambda_i} y \cdot \nabla K_i(r_i^* y)\tilde{u}_i^{\frac{2n}{n-2}}(y)\, dy \\
= & \frac{(n-2)\alpha}{2n\lambda} \int_{\mathbb{R}^n} Q(y) U_0^{\frac{2n}{n-2}}(y)\, dy .
\end{aligned}$$

Thus, by (5.40) the assumption (1.6),

$$\int_{\mathbb{R}^n} Q(y) U_0^{\frac{2n}{n-2}}(y)\, dy < 0 , \tag{5.43}$$

which implies that both a and $b > 0$. Therefore, we conclude that w_i has at least one local minimum at $T_i = \left(1 - \frac{\alpha_i}{n-2}\right) t_i^* + c + o(1)$ after t_i^*. Since $1 - \frac{\alpha_i}{n-2} > \frac{1}{2}$, we have by (5.41),

$$t_i^* < T_i = \left(1 - \frac{\alpha_i}{n-2}\right) t_i^* + c < \frac{1}{2} t_i^* \le t_i$$

for large i, which yields a contradiction to the assumption that there exists no local minimum point of w_i between t_i^* and t_i. Thus, (5.20) is proved. Since u has a nonremovable singularity at 0, we have $\int_{B_1} u^{\frac{2n}{n-2}} = +\infty$, and therefore $\lim_{i\to+\infty} \int_{B_1} u_i^{\frac{2n}{n-2}}(x)\,dx = +\infty$.

By (1.7) and the Harnack inequalty,

$$+\infty = \int_{B_1} u^{\frac{2n}{n-2}}(x)\,dx \le c_1 \int_{B_1} u^{\frac{2}{n-2}}(x)|x|^{-n+1}\,dx \le c_2 \int_0^1 \left(\inf_{|x|=r} u^{\frac{2}{n-2}}(x)\right) dr \ ,$$

from which the completeness of $u^{\frac{4}{n-2}}|dx|^2$ follows immediately.

Suppose $Q(x)$ satisfies that 0 is the unique zero of

$$\int_{\mathbb{R}^n} \nabla Q(\xi+y) U_0^{\frac{2n}{n-2}}(y)\,dy = 0 \ .$$

We want to prove $u(x)$ is asymptotically symmetric. Suppose the contrary. Then there exists a sequence of $x_i \to 0$ as $i \to +\infty$ such that

$$u(x_i) \ge (1+\epsilon_0)\overline{u}(|x_i|) \tag{5.39}$$

for some positive ϵ_0, where $\overline{u}(r)$ denotes the integral average of u over $|x| = r$. Let $v_i(y) = u(|x_i|y)|x_i|^{\frac{n-2}{2}}$. By Theorem 1.2, $v_i(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n \setminus \{0\}$. If $\overline{u}(|x_i|)|x_i|^{\frac{n-2}{2}} \to 0$ as $i \to +\infty$, then there is a subsequence of v_i (still denoted by v_i) such that $\frac{v_i(y)}{v_i(e_1)}$ converges to a positive harmonic function $h(y)$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. By Liouville's Theorem, $h(y) = a|y|^{2-n} + b$ with $a, b \ge 0$ and $a + b > 0$. Obviously, it is a contradiction to (5.39). Suppose $\overline{u}(|x_i|)|x_i|^{\frac{n-2}{2}} \ge c > 0$ for some constant c. Then $v_i(y)$ converges to $\tilde{U}_0(y)$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. As the argument in Step 1, we see that $\tilde{U}_0(y)$ is smooth at 0. Hence

$$\widetilde{U}_0(y) = \left(\frac{\lambda}{1+\lambda^2|y-\eta_0|^2}\right)^{\frac{n-2}{2}} \ .$$

Suppose $\eta_0 \neq 0$. Then u has a local maximum at x_i where x_i satisfies

$$\lim_{i\to+\infty} \left(u(x_i)^{\frac{2}{n-2}} x_i\right) = \lambda\eta_0 \equiv \xi_0 \ .$$

Since u_i converges to u in $C^2_{\text{loc}}(\overline{B}_1 \setminus \{0\})$, there is a subsequence of u_i (still denoted by u_i) and a sequence of local maximum points y_i of u_i such that

$$\lim_{i\to+\infty} u_i^{\frac{2}{n-2}}(y_i)|y_i| = \xi_0 \ .$$

Thus, we can repeat the same argument as in Step 1 to prove that ξ_0 satisfies

$$\int_{\mathbb{R}^n} \nabla Q(\xi_0 + y) U_0^{\frac{2n}{n-2}}(y)\, dy = 0 \ .$$

By the assumption, we have $\xi_0 = 0$, which obviously yields a contradiction. Hence we have proved $\eta_0 = 0$. However, it also yields a contradiction to (5.39). The completeness of the comformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ is the consequence of the fact that u has a nonremovable singularity at 0 and the Harnack inequality (1.12) holds. The unboundedness of curvatures of g is an immediate consequence of Proposition 2.6 in [22]. Therefore, the proof of Theorem 1.4 is completely finished. q.e.d.

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