

P-ADIC UNIFORMIZATION OF UNITARY SHIMURA VARIETIES. II

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Abstract

In this paper we show that certain Shimura varieties, uniformized by the product of complex unit balls, can be p -adically uniformized by the product of Drinfeld upper half-spaces and their equivariant coverings. We also extend a p -adic uniformization to automorphic vector bundles. It is a continuation of our previous work [38] and contains all cases (up to a central modification) of a uniformization by known p -adic symmetric spaces. The idea of the proof is to show that an arithmetic quotient of the product of Drinfeld upper half-spaces cannot be anything else than a certain unitary Shimura variety. Moreover, we show that difficult theorems of Yau and Kottwitz appearing in [38] may be avoided.

1. Introduction

Let M be a Hermitian symmetric domain (=Hermitian symmetric space of non-compact type), and let Δ be a torsion-free cocompact lattice in $\text{Aut}(M)$. Then the quotient $\Delta \backslash M$ is a complex manifold, which has a unique structure of a complex projective variety Y_Δ (see [34, Ch. IX, §3]). A well-known theorem says that when Δ is an arithmetic congruence subgroup, the Shimura variety Y_Δ has a canonical structure over some number field E (see for example [22, II, Thm. 5.5]).

Let v be a prime of E . We are interested in a question whether Y_Δ can be p -adically (or more precisely v -adically) uniformized. By this we mean that the E_v -analytic space $(Y_\Delta \otimes_E E_v)^{an}$ is isomorphic to $\Delta \backslash \Omega$ for some E_v -analytic symmetric space Ω and some arithmetic group Δ , acting on Ω discretely. In the cases where a p -adic uniformization exists we are interested in the relation between M and Ω , Δ and Γ .

The main obstacle for attacking such a general problem is that there is no general definition of a p -adic symmetric space. The only p -adic

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analytic spaces which are generally called “symmetric” can be described as follows. For a p -adic field L and a natural number $d \geq 2$ let Ω_L^d be the open analytic subspace of \mathbb{P}_L^{d-1} obtained by removing the union of all L -rational hyperplanes.

Ω_L^d is called $(d-1)$ -dimensional Drinfeld upper half-space over L , and has a lot of good properties. For example, it is a generic fiber of some explicitly constructed regular formal scheme $\widehat{\Omega}_L^d$, flat over the ring of integers \mathcal{O}_L of L . Moreover the natural L -rational action of $\mathbf{PGL}_d(L)$ on Ω_L^d extends to an \mathcal{O}_L -rational action on $\widehat{\Omega}_L^d$ (see [27] or [19]). Both $\widehat{\Omega}_L^d$ and Ω_L^d are closely connected with the Bruhat-Tits building of $\mathbf{SL}_d(L)$ (see [27] and [2]), and $\mathbf{PGL}_d(L)$ is the group of all automorphisms of $\widehat{\Omega}_L^d$ over \mathcal{O}_L (see [27, Prop. 4.2]) and of Ω_L^d over L (see [2]). Furthermore, Drinfeld constructed a projective system $\{\Sigma_L^{d,n}\}_{n \in \mathbb{N}}$ of $\mathbf{GL}_d(L)$ -equivariant finite étale Galois coverings of $\Omega_L^d \widehat{\otimes}_L \widehat{L}^{nr}$, where \widehat{L}^{nr} is the completion of the maximal unramified extension of L (see [11] or [38, 1.4.1]). (For simplicity of notation we introduce a pro-analytic space $\Sigma_L^d := \varprojlim_n \Sigma_L^{d,n}$, which was written as $\{\Sigma_L^{d,n}\}_n$ in the notation of [38, Def. 1.3.3-1.3.5]).

But the most important for us property of Ω_L^d is that for each torsion-free cocompact lattice $\Gamma \subset \mathbf{PGL}_d(L)$ the quotient $\Gamma \backslash \Omega_L^d$ exists and has a unique structure of a smooth projective variety X_Γ over L . Moreover, X_Γ satisfies the Hirzebruch proportionality principle, and its canonical divisor is ample (see Lemma 2.9 and Proposition A.1 for a generalization for products). These and some other properties of the Ω_L^d 's (and of the Σ_L^d 's) enable us to call them “symmetric spaces”.

Since there are essentially no other known examples of p -adic analytic spaces having analogous properties, especially the algebraization property of the quotients (compare [30, Thm 4.8]), we have to restrict ourself to Shimura varieties, p -adically uniformized by the products of (the equivariant coverings of) Drinfeld upper half-spaces. This enables us to obtain some arithmetic information about these Shimura varieties such as a description of their reduction modulo p . (To be more precise, the symmetric spaces of [30, Thm 4.8] include also some equivariant projective bundles over Drinfeld upper half-spaces. However our Second Main Theorem implies that these spaces p -adically uniformize the corresponding projective bundles over Shimura varieties.)

For the proof of a p -adic uniformization there are two completely different methods. The first one, due to Cherednik [6], is based on Ihara's

method of elliptic elements. Cherednik treated the case of a uniformization by the Ω^2 's of Shimura curves associated to adjoint groups.

The second method, due to Drinfeld [11] (explained in more detail in [1]), is based on constructing moduli problems whose solution are simultaneously Shimura variety and a p -adically uniformized variety. Drinfeld treated the case of a uniformization by the Σ^2 's, but only for Shimura curves defined over \mathbb{Q} . Later Rapoport and Zink developed Drinfeld's method (see [32, 31]), and their recent work [33] treats the case of a uniformization by the products of the Ω_L^d 's (and of the Σ_L^d 's). We also would like to mention the work of Boutot and Zink [4], where they deduce the uniformization for curves from [33] (compare Section 5).

Generalizing Cherednik's method, the author treated in his thesis [38] the case of Shimura varieties, uniformized by (the equivariant coverings of) Drinfeld upper half-spaces. Moreover, the result was extended to the case of automorphic vector bundles and standard principal bundles.

As was indicated in the introduction of [38], our method carries to the case of products. We do this here, and we also show that the use in [38] of the classification of algebraic groups and of difficult theorems of Kottwitz and Yau was unnecessary. The main new idea of the proof in comparison to that of [38] is to work not just with a Shimura variety but with a triple consisting of a Shimura variety, a standard principal bundle with a flat connection over it, and an equivariant map from the standard principal bundle to the corresponding Grassmann variety.

For the convenience of the reader we now compare the results of this paper with those of [33]. Note first that our Shimura varieties differ from those of [33] by some abelian twist (see Remark 3.13). The Shimura varieties treated by [33] are moduli varieties of abelian varieties with some additional structure, whereas our choice enabled us to write the p -adic uniformization in a simpler form (without the twist appearing in [33, Prop. 6.49]). Moreover, our Shimura varieties have weight defined over \mathbb{Q} (the trivial one), therefore they are moduli varieties of abelian motives (see [25, Thm. 3.3.1]) and have some other advantages (see for example [22, III, §8]).

Our result is almost identical to (the twist of) that of [33], but a more technically involved modular approach of Rapoport and Zink requires them to make some unnecessary assumptions (see [33, 6.38]), while our group-theoretic method applies without any difference to the general case. We should mention that our proof also makes an indirect

use of the moduli theory of p -divisible groups, since this is the only known approach allowing to construct the Σ_L^d 's. However, for most of the applications one is only interested in Shimura varieties, which are maximal at p , and these Shimura varieties are p -adically uniformized by the maximal unramified extension of scalars of the product of the Ω_L^d 's, whose construction is completely elementary.

The paper is organized as follows. In the second and the third sections we formulate and prove our main results respectively. As an application we find an algebraic connection between Drinfeld upper half-spaces and the complex unit balls. In the fourth section we give an alternative more direct approach to the differential-geometric part of the proof, which uses Yau's theorem on the existence and the uniqueness of the Kähler-Einstein metric. In the fifth section we deduce the analog of our Main Theorems for quaternion Shimura varieties from the unitary case. In the last section we show that the Shimura varieties treated in this paper are the most general ones (up to a central modification) which can be p -adically uniformized by the product of Drinfeld upper half-spaces. The appendix treats the Hirzebruch proportionality principle for quotients of the products of Drinfeld upper half-spaces. It is a generalization of [19, Thm. 2.2.8].

Notation and conventions

- 1) For a group G let $Z(G)$ be the center of G , and let $PG := G/Z(G)$ be the adjoint group of G .
- 2) For a Lie group G let G^0 be its connected component of the identity.
- 3) For a totally disconnected topological group E let $\mathcal{F}(E)$ be the set of all compact and open subgroups of E , and let E^{disc} be the group E with the discrete topology.
- 4) For a subgroup Δ of a topological group G let $\overline{\Delta}$ be the closure of Δ in G .
- 5) For an element g of a group G let $\text{Inn}(g)$ be the inner automorphism of G by g .
- 6) For a natural number n let I_n be the $n \times n$ identity matrix, and let $B^n \subset \mathbb{C}^n$ be the n -dimensional complex unit ball.
- 7) For an analytic space or a scheme X let $T(X)$ be the tangent bundle on X .
- 8) For a vector bundle V on X and a point $x \in X$ let V_x be the fiber

of V over x .

9) For an algebra D let D^{opp} be the opposite algebra of D .

10) For a number field F and a finite set N of finite primes of F let \mathbb{A}_F^f be the ring of finite adeles of F , and let $\mathbb{A}_F^{f;N}$ be the ring of finite adeles of F without the components from N .

11) For a field extension K/F let $\mathbf{R}_{K/F}$ be the functor of the restriction of scalars from K to F .

12) For a scheme X over a field K and a field extension L of K we will write X_L or $X \otimes_K L$ instead of $X \times_{\mathrm{Spec}K} \mathrm{Spec}L$.

13) For a complex analytic space M let $\mathrm{Aut}(M)$ be the topological group of holomorphic automorphisms equipped with the compact-open topology.

14) For an analytic space X over a complete non-archimedean field K and a complete non-archimedean field extension L of K let $X \widehat{\otimes}_K L$ be the image of X under the ground field extension functor from K to L . (The completion sign will be omitted in the case of a finite extension).

15) By a p -adic field we mean a finite field extension of \mathbb{Q}_p for some prime number p .

16) By a p -adic analytic space we mean an analytic space over a p -adic field in the sense of Berkovich.

17) For a variety X over the field of complex numbers or over a p -adic field let X^{an} be the corresponding complex or p -adic analytic space.

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2. Statement of the results

2.1. Let F be a totally real number field of degree g over \mathbb{Q} , let K be a totally imaginary quadratic extension F , and let $\infty_1, \dots, \infty_g$ be the archimedean completions of F . Let $d \geq 2$ and $1 \leq r \leq g$ be natural numbers. Let D^{int} be a central simple algebra over K of dimension d^2 with an involution of the second kind α^{int} over F . We will make the following assumptions:

- i) α^{int} has signature $(d-1, 1)$ at $\infty_1, \dots, \infty_r$;
- ii) α^{int} is positive definite at $\infty_{r+1}, \dots, \infty_g$.

Let $\mathbf{G}^{int} := \mathbf{GU}(D^{int}, \alpha^{int})$ be the algebraic group over F of unitary similitudes characterized by

$$\mathbf{G}^{int}(R) = \{d \in (D^{int} \otimes_F R)^\times \mid d \cdot \alpha^{int}(d) \in R^\times\}$$

for each F -algebra R .

2.2. Set $\mathbf{H}^{int} := \mathbf{R}_{F/\mathbb{Q}} \mathbf{G}^{int}$. For each $i = 1, \dots, g$ fix an embedding $\tilde{\infty}_i : K \hookrightarrow \mathbb{C}$, extending $\infty_i : F \hookrightarrow \mathbb{R}$. Then $\tilde{\infty}_1, \dots, \tilde{\infty}_g$ define an identification of $\mathbf{H}^{int}(\mathbb{R})$ with $\mathbf{GU}_{d-1,1}(\mathbb{R})^r \times \mathbf{GU}_d(\mathbb{R})^{g-r} \subset \mathbf{GL}_d(\mathbb{C})^g$, which is unique up to an inner automorphism.

Define a homomorphism $h : \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \rightarrow \mathbf{H}^{int}_{\mathbb{R}}$ by requiring for each $z \in \mathbb{C}^\times$

$$h(z) = \underbrace{(\text{diag}(1, \dots, 1, z/\bar{z})^{-1}; \dots; \text{diag}(1, \dots, 1, z/\bar{z})^{-1})}_r; \underbrace{I_d; \dots; I_d}_{g-r}$$

$$\in \mathbf{GU}_{d-1,1}(\mathbb{R})^r \times \mathbf{GU}_d(\mathbb{R})^{g-r}$$

(compare Remark 3.13). Let M^{int} be the $\mathbf{H}^{int}(\mathbb{R})$ -conjugacy class of h . Then M^{int} is isomorphic to $(B^{d-1})^r := \underbrace{B^{d-1} \times \dots \times B^{d-1}}_r$ if $d > 2$

and to $\mathfrak{h}^r := \underbrace{\mathfrak{h} \times \dots \times \mathfrak{h}}_r$, where $\mathfrak{h} = \mathbb{C} \setminus \mathbb{R}$, if $d = 2$. In particular, each

connected component of M^{int} is isomorphic to $(B^{d-1})^r$. Notice that the pair $(\mathbf{H}^{int}, M^{int})$ satisfies Deligne's axioms (see [8, 1.5 and 2.1] or [22, II, 2.1]).

2.3. Let \tilde{X}^{int} be the canonical model of the Shimura variety corresponding to $(\mathbf{H}^{int}, M^{int})$. (For the definition of the canonical model we take that of [24], which has a different sign convention from those of [9] and [22] (see the discussion in [24, 1.10], [38, Rem. 3.1.13] and Step 4 of Section 3). Then \tilde{X}^{int} is a scheme over the reflex field $E \subset \mathbb{C}$

of $(\mathbf{H}^{\text{int}}, M^{\text{int}})$. Moreover, the group $\mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$ acts on \tilde{X}^{int} in such a way that for each $S \in \mathcal{F}(\mathbf{G}^{\text{int}}(\mathbb{A}_F^f))$ the quotient $S \backslash \tilde{X}^{\text{int}}$ is a projective scheme over E and $\tilde{X}^{\text{int}} \cong \varprojlim_S S \backslash \tilde{X}^{\text{int}}$.

Set $\mathbf{PG}^{\text{int}}(F)_+ := \mathbf{PG}^{\text{int}}(F) \cap \mathbf{PG}^{\text{int}}(F \otimes_{\mathbb{Q}} \mathbb{R})^0$ and write $\mathbf{PGU}_{d-1,1}(\mathbb{R})_+$ instead of $\mathbf{PGU}_{d-1,1}(\mathbb{R})^0$. Then $(\tilde{X}_{\mathbb{C}}^{\text{int}})^{\text{an}}$, defined as $\varprojlim_S (S \backslash \tilde{X}_{\mathbb{C}}^{\text{int}})^{\text{an}}$, is isomorphic to

$$[(B^{d-1})^r \times (\mathbf{G}^{\text{int}}(\mathbb{A}_F^f) / \overline{\mathbf{Z}(\mathbf{G}^{\text{int}})(F)}^{\text{disc}})] / \mathbf{PG}^{\text{int}}(F)_+,$$

where $(x, g)\gamma = (\gamma^{-1}x, g\gamma)$ for all $x \in (B^{d-1})^r$, $g \in \mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$ and $\gamma \in \mathbf{PG}^{\text{int}}(F)_+$. Furthermore, the action of $\mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$ on \tilde{X}^{int} corresponds by this isomorphism to left multiplication on the second factor.

Observe that for each $S \in \mathcal{F}(\mathbf{G}^{\text{int}}(\mathbb{A}_F^f))$ the analytic space $(S \backslash \tilde{X}_{\mathbb{C}}^{\text{int}})^{\text{an}}$ is a finite disjoint union of quotients of the form $\Delta_{aSa^{-1}} \backslash (B^{d-1})^r$ for some $a \in \mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$, where by

$$\Delta_{aSa^{-1}} \subset \text{Aut}((B^{d-1})^r)^0 \cong \mathbf{PGU}_{d-1,1}(\mathbb{R})_+^r$$

we denote the projection of $\mathbf{G}^{\text{int}}(F)_+ \cap (aSa^{-1})$. Then each $\Delta_{aSa^{-1}}$ is a cocompact lattice (torsion-free if S is sufficiently small), hence each geometrically connected component of $S \backslash \tilde{X}^{\text{int}}$ is of the form described in the introduction.

2.4. The number field $E \subset \mathbb{C}$ can be described as follows (compare [38, 3.1.1 and Prop. 3.1.3]). Let $K_0 \subset \mathbb{C}$ be the composite of the fields $\tilde{\omega}_1(K), \dots, \tilde{\omega}_r(K)$. Set

$$\begin{aligned} \Sigma := \{ \sigma \in \text{Aut}(K_0/\mathbb{Q}) \mid \forall i = 1, \dots, r \exists \sigma(i) \in \{1, \dots, r\} : \\ \sigma(\tilde{\omega}_i(k)) = \tilde{\omega}_{\sigma(i)}(k) \forall k \in K \}. \end{aligned}$$

Then E is the subfield of K_0 , fixed elementwise by Σ . In particular, the extension K_0/E is Galois with a Galois group Σ .

2.5. Let v be a finite prime of E . Set $X^{\text{int}} := \tilde{X}^{\text{int}} \otimes_E E_v$. In this paper we are going to show that under certain assumptions X^{int} admits a p -adic uniformization.

Let p be the restriction of v to \mathbb{Q} . Then the completion of the algebraic closure of E_v is \mathbb{C}_p . Choose a field isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$, extending the natural embedding $E \hookrightarrow E_v \hookrightarrow \mathbb{C}_p$. From now on we

identify \mathbb{C} with \mathbb{C}_p by means of this isomorphism. In particular, we will view E_v as a subfield of \mathbb{C} .

For each $i = 1, \dots, r$ the embedding $\widetilde{\infty}_i : K \hookrightarrow \mathbb{C}$ corresponds to an embedding $\alpha_i : K \hookrightarrow \mathbb{C}_p$, which extends to a continuous embedding $\alpha_i : K_{w_i} \hookrightarrow \mathbb{C}_p$ for some prime w_i of K , lying over p . Since the group $\text{Gal}(K_0/E)$ preserves the set $\{\widetilde{\infty}_1, \dots, \widetilde{\infty}_r\}$, the set $\{w_1, \dots, w_r\}$ does not depend on the isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$. Let v_i be the restriction of w_i to F . To the assumptions made in 2.1 we add the following:

iii) the v_i 's are distinct;

iv) the v_i 's split in K ;

(Notice that these two conditions are satisfied automatically if p splits completely in K_0)

v) the Brauer invariant of each $D^{int} \otimes_K K_{w_i}$ is $1/d$.

Lemma 2.6. $E_v \subset \mathbb{C}_p$ is the composite field of the $\alpha_i(K_{w_i})$'s, $i = 1, \dots, r$.

Proof. By the definitions, the closure of the image $K_0 \hookrightarrow \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ coincides with the composite field of all the $\alpha_i(K_{w_i})$'s. Therefore we have to show that v splits completely in K_0 . For this it is sufficient to show that if $\sigma \in \Sigma = \text{Gal}(K_0/E)$ acts continuously on $K_0 \subset \mathbb{C}_p$, then $\sigma = 1$. Such a σ induces a continuous isomorphism $\alpha_i(K) \xrightarrow{\sim} \alpha_{\sigma(i)}(K)$ for each $i = 1, \dots, r$. By the definition this implies that $w_{\sigma(i)} = w_i$. Hence, by our assumption, $\sigma(i) = i$, so that σ induces the identity on each $\widetilde{\infty}_i(K)$. Therefore σ acts trivially on the composite field K_0 .

q.e.d.

Now we are going to describe the p -adic uniformization of X^{int} .

2.7. Let D be a central simple algebra over K of dimension d^2 with an involution of the second kind α over F satisfying the following conditions:

vi) the pairs (D, α) and (D^{int}, α^{int}) are locally isomorphic at all finite places of F , except at v_1, \dots, v_r ;

vii) D splits at w_1, \dots, w_r ;

viii) α is positive definite at all the archimedean places of F .

The existence of such D and α follows from the results of Kottwitz and Clozel (see [7, §2]) as in [7, Prop. 2.3].

Set $\mathbf{G} := \mathbf{GU}(D, \alpha)$, and for each $i = 1, \dots, r$ fix a central skew field \tilde{D}_{w_i} over K_{w_i} of Brauer invariant $1/d$. Set also

$$\mathcal{G}' := \prod_{i=1}^r F_{v_i}^\times \times \mathbf{G}(\mathbb{A}_F^{f;v_1, \dots, v_r})$$

and

$$\tilde{\mathcal{G}} := \prod_{i=1}^r \tilde{D}_{w_i}^\times \times \mathcal{G}'.$$

Then for each $i = 1, \dots, r$ the pair consisting of an algebra isomorphism $D \otimes_K K_{w_i} \xrightarrow{\sim} \text{Mat}_d(K_{w_i})$ (resp. $D^{int} \otimes_K K_{w_i} \xrightarrow{\sim} \tilde{D}_{w_i}$) and the similitude homomorphism $\mathbf{G}(F_{v_i}) \rightarrow F_{v_i}^\times$ (resp. $\mathbf{G}^{int}(F_{v_i}) \rightarrow F_{v_i}^\times$) gives us an isomorphism $\mathbf{G}(F_{v_i}) \xrightarrow{\sim} \mathbf{GL}_d(K_{w_i}) \times F_{v_i}^\times$ (resp. $\mathbf{G}^{int}(F_{v_i}) \xrightarrow{\sim} \tilde{D}_{w_i}^\times \times F_{v_i}^\times$). Using in addition an algebra isomorphism

$$(D, \alpha) \otimes_F \mathbb{A}_F^{f;v_1, \dots, v_r} \xrightarrow{\sim} (D^{int}, \alpha^{int}) \otimes_F \mathbb{A}_F^{f;v_1, \dots, v_r}$$

we obtain isomorphisms

$$\mathbf{G}(\mathbb{A}_F^f) \xrightarrow{\sim} \prod_{i=1}^r \mathbf{GL}_d(K_{w_i}) \times \mathcal{G}'$$

and

$$\mathbf{G}^{int}(\mathbb{A}_F^f) \xrightarrow{\sim} \tilde{\mathcal{G}}.$$

Abusing notation, we will sometimes write these isomorphisms as equalities.

For each $S \in \mathcal{F}(\mathcal{G}')$ and each $n \in \mathbb{N} \cup \{0\}$ consider a double quotient

$$\tilde{X}_{S,n} = S \backslash \left[\prod_{i=1}^r \Sigma_{K_{w_i}}^{d,n} \otimes_{K_{w_i}} E_v \times \mathcal{G}' \right] / \mathbf{G}(F)$$

(see [38, 1.3.1 and 1.4.1] for our sign convention, which is different from that of Drinfeld).

Proposition 2.8. *Each $\tilde{X}_{S,n}$ has a canonical structure of an E_v -analytic space and of a projective scheme $X_{S,n}$ over E_v .*

Proof. The proof will follow closely that of [38, Prop. 1.5.2]. Set $Z_S = \mathbf{Z}(\mathbf{G})(F) \cap S$. Then the group Z_S is a subgroup of F^\times , which projects to a subgroup of finite index in $\prod_{i=1}^r (E_v^\times / \mathcal{O}_{E_v}^\times) \cong \mathbb{Z}^r$. Since each $k \in K_{w_i}^\times = Z(\mathbf{GL}_d(K_{w_i}))$ acts on $\Omega_{K_{w_i}}^d \widehat{\otimes}_{K_{w_i}} \widehat{K}_{w_i}^{nr}$ by the action of the Frobenius automorphism in the power $d \cdot \text{val}(k)$ on the second factor, the quotient $Z_S \backslash \prod_{i=1}^r (\Omega_{K_{w_i}}^d \widehat{\otimes}_{K_{w_i}} \widehat{K}_{w_i}^{nr}) \otimes_{K_{w_i}} E_v$ is of the form $\prod_{i=1}^r (\Omega_{K_{w_i}}^d \otimes_{K_{w_i}} \widetilde{K}_{w_i}) \otimes_{K_{w_i}} E_v$ for some finite unramified extensions \widetilde{K}_{w_i} 's of K_{w_i} 's. Hence this quotient is finite over the E_v -analytic space $\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v$. By the formal arguments of [38, Prop. 1.5.2] the following lemma completes the proof q.e.d.

Lemma 2.9. *Let L be a p -adic field, let L_1, \dots, L_r be p -adic subfields of L , let $d \geq 2$ be a natural number, and let Γ be a torsion-free cocompact lattice in $\prod_{i=1}^r \mathbf{PGL}_d(L_i)$. Then*

- a) *the action of Γ on $\prod_{i=1}^r \Omega_{L_i}^d \otimes_{L_i} L$ is discrete and free;*
- b) *the quotient $\Gamma \backslash \prod_{i=1}^r \Omega_{L_i}^d \otimes_{L_i} L$ exists and has a unique structure of a smooth projective variety over L , whose canonical bundle is ample.*

Proof. GAGA results imply the uniqueness in b). Rest of the proof follows from the arguments of [2, Lem. 6] and [27, App. 1], working without change in our situation. q.e.d.

2.10. Set $X = \varprojlim_{S,n} X_{S,n}$. Then, as in [38, Con. 1.5.1], the group

$\widetilde{\mathcal{G}}$ acts on X in such a way that for each $T \in \mathcal{F}(\widetilde{\mathcal{G}})$ the quotient $T \backslash X$ is a projective scheme over E_v and $X = \varprojlim_T T \backslash X$. Moreover, for each

sufficiently small $T \in \mathcal{F}(\widetilde{\mathcal{G}})$ the projection $X \rightarrow T \backslash X$ is étale, and the quotient $T \backslash X$ is smooth.

Remark 2.11. Since the natural projections $(X_{S_1, n_1})^{an} \rightarrow (X_{S_2, n_2})^{an}$ are étale for all $S_1 \subset S_2$ sufficiently small and all $n_1 \geq n_2$, we can define, as in [38, Prop. 1.5.3 f)], a pro-analytic space $X^{an} := \varprojlim_{S,n} X_{S,n}^{an}$. Then

the natural $\widetilde{\mathcal{G}}$ -equivariant map

$$\eta : \left[\prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times (\mathcal{G}')^{disc} \right] / \mathbf{G}(F) \rightarrow X^{an}$$

of pro-analytic spaces over E_v is étale and surjective.

Remark 2.12. The proof of Proposition 2.8 (and that of [38, Prop. 1.5.2]) implies that for each sufficiently small $T \in \mathcal{F}(\tilde{\mathcal{G}})$, each connected component of $(T \backslash X)^{an}$ is a finite étale cover of an analytic space of the form $\Gamma \backslash \prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v$ for some irreducible arithmetic torsion-free cocompact lattice $\Gamma \subset \prod_{i=1}^r \mathbf{PGL}_d(K_{w_i})$. In particular, the canonical bundle of such an $T \backslash X$ is ample.

First Main Theorem 2.13. *There exists an isomorphism*

$$\varphi : X \xrightarrow{\sim} X^{int}$$

commuting with the action of the group $\tilde{\mathcal{G}} = \mathbf{G}^{int}(\mathbb{A}_F^f)$.

Now we describe the p -adic uniformization of certain automorphic vector bundles and a certain principal bundle.

2.14. Let P^{int} be the canonical model of the standard \mathbf{PH}^{int} -principal bundle over X^{int} (see [22, Thm. 4.3]). Then $(P_{\mathbb{C}}^{int})^{an}$ is isomorphic to

$$[(B^{d-1})^r \times (\mathbf{PH}_{\mathbb{C}}^{int})^{an} \times (\mathbf{G}^{int}(\mathbb{A}_F^f) / \overline{\mathbf{Z}(\mathbf{G}^{int})(F)}^{disc}) / \mathbf{PG}^{int}(F)_+,$$

where $\mathbf{PG}^{int}(F)_+$ acts on $(\mathbf{PH}_{\mathbb{C}}^{int})^{an} = (\mathbf{PG}_{F \otimes_{\mathbb{Q}} \mathbb{C}}^{int})^{an}$ by right multiplication. Therefore $P_{\mathbb{C}}^{int}$ has a natural $\tilde{\mathcal{G}}$ -invariant flat connection $\tilde{\mathcal{H}}^{int}$ (see [38, 1.9.15-1.9.19] for the definitions) such that the restriction of $(\tilde{\mathcal{H}}^{int})^{an}$ to $(B^{d-1})^r \times (\mathbf{PH}_{\mathbb{C}}^{int})^{an} \times \{1\}$ is trivial, that is consists of vectors, tangent to $(B^{d-1})^r$ (compare the proof of [38, Prop. 4.4.2]). Furthermore, P^{int} is the only $\mathbf{PH}_{\mathbf{E}_v}^{int} \times \tilde{\mathcal{G}}$ -equivariant model of $P_{\mathbb{C}}^{int}$, to which $\tilde{\mathcal{H}}^{int}$ descends (see for example [38, 4.7]). Denote the descent of $\tilde{\mathcal{H}}^{int}$ to P^{int} by \mathcal{H}^{int} .

Let \check{M}^{int} be the Grassmann variety corresponding to the pair $(\mathbf{H}^{int}, M^{int})$ (see [22, III, §1]). It is a homogeneous space for the group $\mathbf{PH}_{\mathbf{E}}^{int}$, satisfying $\check{M}_{\mathbb{C}}^{int} \cong (\mathbb{P}_{\mathbb{C}}^{d-1})^r$.

Let $\beta_{\mathbb{R}} : (B^{d-1})^r \hookrightarrow ((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$ be the natural (Borel) embedding, and let $\rho^{int} : P_{E_v}^{int} \rightarrow \check{M}_{E_v}^{int}$ be the natural $\mathbf{PH}_{\mathbf{E}_v}^{int} \times \tilde{\mathcal{G}}$ -equivariant map (for the trivial action of $\tilde{\mathcal{G}}$ on $\check{M}_{E_v}^{int}$) defined by $(\rho_{\mathbb{C}}^{int})^{an}[x, h, g] = h(\beta_{\mathbb{R}}(x))$ for all $x \in (B^{d-1})^r$, $h \in (\mathbf{PH}_{\mathbb{C}}^{int})^{an}$ and $g \in \tilde{\mathcal{G}}$ (see [38, 4.3.2 and 4.3.4]).

Let W^{int} be a $\mathbf{PH}_{\mathbf{E}_v}^{int}$ -equivariant vector bundle on $\check{M}_{E_v}^{int}$, and let $V^{int} = V^{int}(W^{int})$ be the canonical model of the automorphic vector bundle on X^{int} corresponding to W^{int} . Then

$$(V_{\mathbb{C}}^{int})^{an} \cong [\beta_{\mathbb{R}}^*(W_{\mathbb{C}}^{int})^{an} \times (\mathbf{G}^{int}(\mathbb{A}_F^f) / \overline{\mathbf{Z}(\mathbf{G}^{int})(F)}^{disc}) / \mathbf{PG}^{int}(F)_+$$

and

$$V^{int} \cong \mathbf{PH}_{\mathbf{E}_v}^{int} \backslash (\rho^{int})^*(W^{int})$$

(see [22, III, Prop. 3.5 and Thm. 5.1]).

We now describe the corresponding objects on the p -adic side.

2.15. Set $\mathbf{H} := \mathbf{R}_{F/\mathbb{Q}}\mathbf{G}$. Then for some group $\tilde{\mathbf{H}}$ over E_v we have natural isomorphisms

$$\mathbf{PH}_{\mathbf{E}_v} \cong (\mathbf{PGL}_d)^r \times \tilde{\mathbf{H}}$$

and

$$\mathbf{PH}_{\mathbf{E}_v}^{int} \cong \prod_{i=1}^r \mathbf{PGL}_1(\tilde{D}_{w_i})_{\mathbf{E}_v} \times \tilde{\mathbf{H}},$$

where the first r factors correspond to the natural embeddings $F \hookrightarrow F_{v_i} \xrightarrow{\alpha_i} E_v$. Let $\mathbf{PH}_{\mathbf{E}_v}$ acts on $(\mathbb{P}_{E_v}^{d-1})^r$ by the natural action of the first r factors and the trivial action of the last one.

Let $\pi \in E_v$ be a uniformizer, and let $\tilde{\Pi}$ be an element of $\mathbf{GL}_d(E_v)$ satisfying $\tilde{\Pi}^d = \pi$. For each $i = 1, \dots, r$ set $d_i := [E_v : \alpha_i(K_{w_i})]$. Finally, denote the projection of $(\tilde{\Pi}^{d_1}, \dots, \tilde{\Pi}^{d_r})$ to $\mathbf{PGL}_d(E_v)^r$ by Π' , and set $\Pi := (\Pi', 1) \in \mathbf{PGL}_d(E_v)^r \times \tilde{\mathbf{H}}(E_v) \cong \mathbf{PH}(E_v)$. Let $E_v^{(d)}$ be the unramified field extension of E_v of degree d . Since the Brauer invariant of each $\tilde{D}_{w_i} \otimes_{K_{w_i}} E_v$ is d_i/d (see [5, Ch. VI, Sec. 1, Thm. 3]), the group $\mathbf{PH}_{\mathbf{E}_v}^{int}$ is isomorphic to the quotient of $\mathbf{PH}_{\mathbf{E}_v} \otimes_{E_v} E_v^{(d)}$ by the equivalence relation $\text{Fr}(x) \sim \Pi^{-1}x\Pi$, where $\text{Fr} \in \text{Gal}(E_v^{(d)}/E_v)$ is the Frobenius automorphism.

For an E_v -scheme Y with an E_v -rational action of the group $\mathbf{PH}_{\mathbf{E}_v}$ define a twist $Y^{tw} := (\text{Fr}(x) \sim \Pi^{-1}x) \backslash Y \otimes_{E_v} E_v^{(d)}$. Then $Y \otimes_{E_v} E_v^{(d)} \cong Y^{tw} \otimes_{E_v} E_v^{(d)}$, and the natural action of $\mathbf{PH}_{\mathbf{E}_v}^{int}$ on Y^{tw} is E_v -rational. Using the definition of the twist, we see that $M_{E_v}^{int} \cong ((\mathbb{P}_{E_v}^{d-1})^r)^{tw}$ and that there exists a unique $\mathbf{PH}_{\mathbf{E}_v}$ -equivariant vector bundle W on $(\mathbb{P}_{E_v}^{d-1})^r$ such that $W^{tw} \cong W^{int}$.

Let

$$\beta_v : \prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \rightarrow \prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \hookrightarrow ((\mathbb{P}_{E_v}^{d-1})^r)^{an}$$

be the natural map. As in [38, 4.1.2-4.1.3 and 4.2.2-4.2.3], we construct a $\tilde{\mathcal{G}}$ -equivariant vector bundle $V = V(W)$ on X and a $\tilde{\mathcal{G}}$ -equivariant

$\mathbf{PH}_{\mathbf{E}_v}$ -principal bundle P over X such that

$$(S \setminus V)^{an} \cong S \setminus [\beta_v^*(W) \times \mathcal{G}'] / \mathbf{G}(F)$$

and

$$(S \setminus P)^{an} \cong S \setminus \left[\prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times (\mathbf{PH}_{\mathbf{E}_v})^{an} \times \mathcal{G}' \right] / \mathbf{G}(F)$$

for each $S \in \mathcal{F}(\tilde{\mathcal{G}})$. Of course V and P are unique. P also has (as in the proof of [38, Prop. 4.4.2]) a natural $\tilde{\mathcal{G}}$ -invariant flat connection \mathcal{H} . As in [38, Prop. 4.3.2], there exists a $\mathbf{PH}_{\mathbf{E}_v} \times \tilde{\mathcal{G}}$ -equivariant morphism $\rho : P \rightarrow (\mathbb{P}_{E_v}^{d-1})^r$, defined by $\rho^{an}[x, h, g] = h(\beta_w(x))$ for each

$$x \in \prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v, h \in (\mathbf{PH}_{\mathbf{E}_v})^{an} \quad \text{and} \quad g \in \mathcal{G}'.$$

By the construction, V is naturally isomorphic to the quotient $\mathbf{PH}_{\mathbf{E}_v} \setminus \rho^*(W)$.

Second Main Theorem 2.16. *a) Any isomorphism*

$$\varphi : X \xrightarrow{\sim} X^{int}$$

from the First Main Theorem lifts to a $\tilde{\mathcal{G}}$ -equivariant isomorphism

$$\varphi_V : V \xrightarrow{\sim} V^{int}$$

of automorphic vector bundles.

b) Any φ as in a) lifts to a $\tilde{\mathcal{G}}$ -equivariant isomorphism

$$\varphi_P : P^{tw} \xrightarrow{\sim} P^{int}$$

of $\mathbf{PH}_{\mathbf{E}_v}^{int}$ -principal bundles, which maps \mathcal{H}^{tw} and ρ^{tw} to \mathcal{H}^{int} and ρ^{int} respectively.

Remark 2.17. The observations made above show that a) is an immediate consequence of b).

The analogs of the Main Theorems for quaternion Shimura varieties will be given in Section 5.

3. Proof of the Main Theorems

Step 1. Exactly by the same arguments as in the proof of [38, Prop. 1.5.3 and 1.6.1], we show that there exists an inverse limit of $\{(S \setminus X_{\mathbb{C}})^{an}\}_{S \in \mathcal{F}(\tilde{\mathcal{G}})}$ in the category of complex analytic spaces, which we denote by $(X_{\mathbb{C}})^{an}$. Moreover, the group $\tilde{\mathcal{G}}$ acts transitively on the set of connected components of $(X_{\mathbb{C}})^{an}$, and $\mathcal{G}_0 := \overline{\mathbf{Z}(\mathbf{G}^{int})(F)}$ is the kernel of the action of the group $\tilde{\mathcal{G}} = \mathbf{G}^{int}(\mathbb{A}_F^f)$ on X . Set $\mathcal{G} := \tilde{\mathcal{G}}/\mathcal{G}_0$.

Let M be a connected component of $(X_{\mathbb{C}})^{an}$, and let Δ be the stabilizer of M in $\tilde{\mathcal{G}}$. Then Δ is naturally embedded into $\text{Aut}(M) \times \tilde{\mathcal{G}}$, and $(X_{\mathbb{C}})^{an} \cong [M \times \mathcal{G}^{disc}]/\Delta$. Let $\pi : \tilde{M} \rightarrow M$ be the universal cover of M , let $\tilde{\Delta} \subset \text{Aut}(\tilde{M})$ be the set of all automorphisms of \tilde{M} which lift some automorphism from $\Delta \subset \text{Aut}(M)$, and let $\tilde{\pi} : \tilde{\Delta} \rightarrow \Delta$ be the natural projection. Then $(X_{\mathbb{C}})^{an} \cong [\tilde{M} \times \mathcal{G}^{disc}]/\tilde{\Delta}$, where $\tilde{\Delta}$ acts on \mathcal{G} through $\tilde{\pi}$.

As in [38, 4.4.1], the \mathcal{G} -equivariant $\mathbf{PH}_{\mathbf{E}_v}$ -principal bundle P over X with a flat connection \mathcal{H} define a $(\mathbf{PH}_{\mathbb{C}})^{an}$ -principal bundle $(P_{\mathbb{C}})^{an}$ over $(X_{\mathbb{C}})^{an} \cong [\tilde{M} \times \mathcal{G}^{disc}]/\tilde{\Delta}$ with a flat connection $(\mathcal{H}_{\mathbb{C}})^{an}$. Therefore $\rho : P \rightarrow (\mathbb{P}_{E_v}^{d-1})^r$ defines the $\mathbf{PH}(\mathbb{C}) \times \mathcal{G}$ -equivariant map $(\rho_{\mathbb{C}})^{an} : (P_{\mathbb{C}})^{an} \rightarrow ((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$. Since \tilde{M} is simply connected, we show, as in [38, Prop.4.4.2], that there exists a homomorphism $j : \tilde{\Delta} \rightarrow \mathbf{PH}(\mathbb{C})$ and a $\mathbf{PH}(\mathbb{C}) \times \mathcal{G}$ -equivariant isomorphism $(P_{\mathbb{C}})^{an} \cong [\tilde{M} \times (\mathbf{PH}_{\mathbb{C}})^{an} \times \mathcal{G}^{disc}]/\tilde{\Delta}$ over X , where $\delta \in \tilde{\Delta}$ acts on the second factor by right multiplication by $j(\delta)$. Furthermore, $(\mathcal{H}_{\mathbb{C}})^{an}$ corresponds then to the natural \mathcal{G} -invariant flat connection on the right hand side.

Let ρ_0 be the pull-back of $(\rho_{\mathbb{C}})^{an}$ to $\tilde{M} \times \{1\} \times \{1\} \cong \tilde{M}$, and let $i : \tilde{\Delta} \rightarrow \text{Aut}(\tilde{M}) \times \mathbf{PH}(\mathbb{C}) \times \mathcal{G}$ be the product of the natural embedding, of j and of $\tilde{\pi} : \tilde{\Delta} \rightarrow \Delta \subset \mathcal{G}$. Then to prove the complex versions of our Main Theorems it will suffice to show (see [38, Lem. 2.2.6 and Rem. 4.4.4]) that $M \cong \tilde{M} \cong (B^{d-1})^r$, that ρ_0 is the Borel embedding, that $\Delta \cong \tilde{\Delta} \cong \mathbf{PG}^{int}(F)_+$, and that i is conjugate to the diagonal embedding of $\mathbf{PG}^{int}(F)_+$ into $\prod_{i=1}^r \mathbf{PG}^{int}(F_{\infty_i})^0 \times \mathbf{PH}^{int}(\mathbb{C}) \times (\mathbf{G}^{int}(\mathbb{A}_F^f)/\overline{\mathbf{Z}(\mathbf{G}^{int})(F)})$ (for the last property we identify \tilde{M} with $(B^{d-1})^r$ and $\tilde{\Delta}$ with $\mathbf{PG}^{int}(F)_+$).

Step 2. This differential-geometric part of the proof significantly differs from the case of one factor, treated in [38]. The method we use here is very similar to (but was obtained mostly independently of) that of [28]. Another method will be shown in the next section.

By the definition, ρ induces isomorphisms $(\mathcal{H}_{\mathbb{C}_p})_x \xrightarrow{\sim} T_{\rho(x)}(\mathbb{P}_{\mathbb{C}_p}^{d-1})^r$ for all $x \in P$. Therefore ρ_0 induces isomorphisms between $T_x(\widetilde{M})$ and $T_{\rho_0(x)}((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$ for all $x \in \widetilde{M}$. It follows that ρ_0 is a local isomorphism and that $\rho_0(\widetilde{M})$ is an open analytic subspace of $((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$. Furthermore, since $(\rho_{\mathbb{C}})^{an}$ is $\mathbf{PH}(\mathbb{C}) \times \mathcal{G}$ -equivariant, we have $\rho_0(\delta x) = j(\delta)\rho_0(x)$ for all $\delta \in \widetilde{\Delta}$ and $x \in \widetilde{M}$. In particular, each $j(\delta)$ induces an automorphism of $\rho_0(\widetilde{M})$.

Let j' be the composition of $j : \widetilde{\Delta} \rightarrow \mathbf{PH}(\mathbb{C}) \cong \mathbf{PGL}_d(\mathbb{C})^g$ with the projection to the product of the first r factors (corresponding to the embeddings $\infty_1, \dots, \infty_r$ of F into \mathbb{R}). Then the local isomorphism ρ_0 is j' -equivariant. Denote by \widetilde{J} the closure of $\widetilde{\Delta}$ in $\text{Aut}(\widetilde{M})$.

Lemma 3.1. *j' can be extended to a continuous homomorphism $\phi : \widetilde{J} \rightarrow \mathbf{PGL}_d(\mathbb{C})^r$, and the map ρ_0 is ϕ -equivariant.*

Proof. Choose a $g \in \widetilde{J}$ and a subsequence $\{\delta_n\}_n \subset \widetilde{\Delta}$, converging to g . Then for all $x_1, x_2 \in \widetilde{M}$ the equality $\rho_0(x_1) = \rho_0(x_2)$ implies that

$$\begin{aligned} \rho_0(g(x_1)) &= \lim_n \rho_0(\delta_n(x_1)) = \lim_n j'(\delta_n)\rho_0(x_1) = \lim_n j'(\delta_n)\rho_0(x_2) \\ &= \lim_n \rho_0(\delta_n(x_2)) = \rho_0(g(x_2)). \end{aligned}$$

Hence the map $\phi_0(g) : \rho_0(x) \mapsto \rho_0(g(x))$ is a well-defined map of $\rho_0(\widetilde{M})$ into itself. Moreover, since ρ_0 is a local isomorphism, the constructed map $g \mapsto \phi_0(g)$ is a continuous homomorphism from \widetilde{J} to $\text{Aut}(\rho_0(\widetilde{M}))$, extending j' .

Denote the stabilizer of $\rho_0(\widetilde{M})$ in $\mathbf{PGL}_d(\mathbb{C})^r$ by G . Then G , being the stabilizer of the closed subset $((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an} \setminus \rho_0(\widetilde{M})$ of $((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$, is a closed subgroup of $\mathbf{PGL}_d(\mathbb{C})^r = \text{Aut}((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$. By the identity theorem (see [13, Vol. I, A, Thm.3]), the restriction homomorphism $res : G \rightarrow \text{Aut}(\rho_0(\widetilde{M}))$ is injective. Moreover, it induces a homeomorphism between G and $res(G) \subset \text{Aut}(\rho_0(\widetilde{M}))$. In particular, the induced topology on $res(G)$ is locally compact. Hence $res(G)$ is a closed subgroup of $\text{Aut}(\rho_0(\widetilde{M}))$ (see [35, Prop. 1.4]). Therefore $\phi_0(\widetilde{J}) \subset res(G)$, and $\phi := res^{-1} \circ \phi_0 : \widetilde{J} \rightarrow G \subset \mathbf{PGL}_d(\mathbb{C})^r$ is the required homomorphism.

q.e.d.

Our aim is to obtain more information about \widetilde{J} and ϕ . First we show, as in the proof of [38, Prop. 1.5.3 d), f) and 1.3.8 a)], that the

quotient $X' := (\prod_{i=1}^r \tilde{D}_{w_i}^\times \cdot Z(\tilde{\mathcal{G}})) \backslash X$ exists, that the projection $X \rightarrow X'$ is étale, and that

$$(X')^{an} \cong \left[\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times (P\mathcal{G}')^{disc} \right] / \mathbf{PG}(F).$$

As in [38, 4.2.2-4.2.3], there exists a unique $\mathbf{PH}_{\mathbf{E}_v}$ -principal bundle P' over X' , satisfying

$$(P')^{an} \cong \left[\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times (\mathbf{PH}_{\mathbf{E}_v})^{an} \times (P\mathcal{G}')^{disc} \right] / \mathbf{PG}(F).$$

Then $P' \cong (\prod_{i=1}^r \tilde{D}_{w_i}^\times \cdot Z(\tilde{\mathcal{G}})) \backslash P$, hence

$$(P'_\mathbb{C})^{an} \cong [\tilde{M} \times (\mathbf{PH}_\mathbb{C})^{an} \times (P\mathcal{G}')^{disc}] / \tilde{\Delta}$$

and

$$(X'_\mathbb{C})^{an} \cong [\tilde{M} \times (P\mathcal{G}')^{disc}] / \tilde{\Delta}.$$

For each $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ and $y \in \tilde{M}$ set

$$\Gamma_x := \{\gamma \in \mathbf{PG}(F) \mid \gamma(x) = x\}$$

and

$$\Delta_y := \{\delta \in \tilde{\Delta} \mid \delta(y) = y\}.$$

Since the projection $X \rightarrow X'$ is étale, \tilde{M} is the universal cover of each connected component of $(X'_\mathbb{C})^{an} \cong [\tilde{M} \times (P\mathcal{G}')^{disc}] / \tilde{\Delta}$. Hence each Δ_y projects injectively to $P\mathcal{G}'$. In particular, Γ_x 's and Δ_y 's are naturally embedded into

$$\mathbf{PH}(\mathbb{C}_p) \times P\mathcal{G}' = \mathbf{PH}(\mathbb{C}) \times P\mathcal{G}'.$$

The following proposition is a technical heart of the proof (compare [38, Prop. 2.2.8]).

Proposition 3.2. *For each $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ there exists a (non-unique) $y \in \tilde{M}$ and for each $y \in \tilde{M}$ there exists a (non-unique) $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ such that the subgroups Γ_x and Δ_y are conjugate in $\mathbf{PH}(\mathbb{C}) \times P\mathcal{G}'$. In particular, the closure of each $j(\Delta_y) \subset \mathbf{PH}(\mathbb{C})$ is compact, and there exists $y \in \tilde{M}$ such that the closure $\overline{j(\Delta_y)}$ is a maximal compact torus of $\mathbf{PH}(\mathbb{C})$.*

Proof. For each $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ consider a point

$$[x, 1, 1] \in \left[\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} \mathbb{C}_p \times (\mathbf{PH}_{\mathbb{C}_p})^{an} \times (P\mathcal{G}')^{disc} \right] / \mathbf{PG}(F) \cong (P'_{\mathbb{C}_p})^{an}.$$

It gives us a point of $P'(\mathbb{C}_p) = P'(\mathbb{C})$. Therefore it corresponds to a certain point

$$[y, h, g] \in [\widetilde{M} \times (\mathbf{PH}_{\mathbb{C}})^{an} \times (P\mathcal{G}')^{disc}] / \widetilde{\Delta} \cong (P'_{\mathbb{C}})^{an}.$$

Each $\gamma \in \Gamma_x \subset \mathbf{PH}(\mathbb{C}) \times P\mathcal{G}'$ fixes $[x, 1, 1] \in (P'_{\mathbb{C}_p})^{an}$. Therefore its conjugate

$$(h, g)\gamma(h, g)^{-1} \in \mathbf{PH}(\mathbb{C}) \times P\mathcal{G}'$$

fixes

$$[y, 1, 1] \in [\widetilde{M} \times (\mathbf{PH}_{\mathbb{C}})^{an} \times (P\mathcal{G}')^{disc}] / \widetilde{\Delta}.$$

It follows that $(h, g)\gamma(h, g)^{-1}$ belongs to Δ_y as claimed. The proof of the opposite direction is exactly the same.

By the proven above it will suffice to show the remaining statements for the closures of Γ_x 's in $\mathbf{PH}(\mathbb{C}) = \mathbf{PG}(F \otimes_{\mathbb{Q}} \mathbb{C})$.

As each Γ_x is contained in $\mathbf{PH}(\mathbb{R}) \cong \mathbf{PGU}_d(\mathbb{R})^g$, its closure is compact. Choose an elliptic element $\gamma \in \mathbf{G}(F) \subset D$ with an elliptic point $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ (see [38, Def. 1.7.1]). (Such an element exists, since by the weak approximation theorem the closure of $\mathbf{PG}(F) \subset \prod_{i=1}^r \mathbf{PGL}_d(K_{w_i})$ contains $\prod_{i=1}^r \mathbf{PSL}_d(K_{w_i})$, and since the set of elliptic elements of $\prod_{i=1}^r \mathbf{PSL}_d(K_{w_i})$ is open and non-empty (see [38, Prop. 1.7.3]). By [38, Lem 1.7.4], g generates a maximal commutative subfield $L \subset D$, invariant under α . Then

$$\mathbf{T} := \mathbf{H} \cap \mathbf{R}_{L/\mathbb{Q}}\mathbf{G}_m \subset \mathbf{R}_{D/\mathbb{Q}}\mathbf{G}_m$$

is a maximal \mathbb{Q} -rational torus of \mathbf{H} , and each $\gamma \in \mathbf{T}(\mathbb{Q}) = \mathbf{G}(F) \cap L^\times$ fixes x . Moreover, one check by direct computation that $\mathbf{T}(\mathbb{Q})$ is the stabilizer of x in $\mathbf{G}(F)$. Therefore the projection $\mathbf{T}' \subset \mathbf{PH}$ of \mathbf{T} is also a maximal \mathbb{Q} -rational torus, and $\Gamma_x = \mathbf{T}'(\mathbb{Q})$. The theorem on the real approximation (see [8, (0.4)]) now implies that the closure $\overline{\Gamma_x}$ is a maximal compact torus $\mathbf{T}'(\mathbb{R})$ of $\mathbf{PH}(\mathbb{C})$. q.e.d.

Using the proposition, fix a point $y_0 \in \widetilde{M}$ such that the closure of $j'(\Delta_{y_0})$ is a maximal compact torus T in $\mathbf{PGL}_d(\mathbb{C})^r$.

3.3. Let J be the closure of $j'(\Delta) \subset \mathbf{PGL}_d(\mathbb{C})^r$. Then $\mathrm{Lie} J$ is an $\mathrm{Ad}(J)$ -invariant real Lie subalgebra of $\mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})^r$. Denote by $\mathrm{Lie} J_{\mathbb{C}}$ the \mathbb{C} -span of $\mathrm{Lie} J$. As in [38, Prop. 4.5.2], the subgroup $j(\Delta)$ is Zariski dense in $\mathbf{PH}_{\mathbb{C}}$, and therefore $\mathrm{Lie} J_{\mathbb{C}}$ is an ideal in $\mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})^r$. Since $\mathrm{Lie} J_{\mathbb{C}}$ contains the Cartan subalgebra $\mathcal{T} := (\mathrm{Lie} T)_{\mathbb{C}}$ of $\mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})^r$, it then has to be all of $\mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})^r$. Lie algebra $\mathrm{Lie} J$ is invariant under the compact group $\mathrm{Ad}(T)$, hence it decomposes as $\prod_{i=1}^r \mathcal{J}_i \subset \mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})^r$, and each \mathcal{J}_i is either all of $\mathrm{Lie} \mathbf{PGL}_d(\mathbb{C})$ or one of its real form. Thus J^0 decomposes as $\prod_{i=1}^r J_i$, where each J_i is either all of $\mathbf{PGL}_d(\mathbb{C})$ or $\mathbf{J}_i(\mathbb{R})^0$ for some real form \mathbf{J}_i of \mathbf{PGL}_d . In both cases $J^0 \cong \mathbf{J}(\mathbb{R})^0$ for some semi-simple adjoint real group \mathbf{J} .

3.4. Choose a maximal compact subgroup $J^{\mathrm{comp}} \supset T$ of $\mathbf{PGL}_d(\mathbb{C})^r$. Equip $(\mathbb{P}_{\mathbb{C}}^{d-1})^{an}$ with a standard J^{comp} -invariant Riemannian metric (making it a Hermitian symmetric space). Then the pull-back of this metric to \widetilde{M} is $\phi^{-1}(T)$ -invariant. Recall that $\overline{\Delta}_{y_0} \subset \mathrm{Aut}(\widetilde{M})$ fixes $y_0 \in \widetilde{M}$, that $\phi(\widetilde{J})$ is dense in J , and that $\phi(\overline{\Delta}_{y_0})$ is dense in $j'(\Delta_{y_0}) = T$. Since the isotropy group at any point of a connected Riemannian manifold is compact (see for example [17, II, Thm. 1.2]), we conclude that $\overline{\Delta}_{y_0}$ is compact. Hence $\phi(\overline{\Delta}_{y_0}) = T$. As $\mathrm{Lie} T$ spans all of $\mathrm{Lie} J$ as an $\mathrm{Ad}(j'(\widetilde{\Delta}))$ -module over \mathbb{R} , the image of ϕ contains an open neighbourhood of the identity in J . Therefore $\phi : \widetilde{J} \rightarrow J$ is surjective. Using the fact that ρ_0 is a local isomorphism, we see that \widetilde{J} is a Lie group and that ϕ is a topological covering.

Proposition 3.5. *The stabilizer $J_y \subset J$ of each point $\rho_0(y) \in \rho_0(\widetilde{M})$ is compact.*

Proof. We start from the following lemma.

Lemma 3.6. *Every Cartan subgroup of J , contained in J_y , is compact.*

Proof. Suppose that the lemma is false, and let $C \subset J_y$ be a non-compact Cartan subgroup of J . Then the set $C' := \{c \in C : \text{the group generated by } c \text{ is not relatively compact in } C\}$ is a non-empty open subset of C . Set $\widetilde{C} := \phi^{-1}(C)^0$. Then \widetilde{C} stabilizes $y \in \widetilde{M}$, and the subset $\widetilde{C}' := \{c \in \widetilde{C} : c \text{ is regular in } \widetilde{J} \text{ and } \phi(c) \in C'\}$ is a non-empty open subset of \widetilde{C} . Hence $\widetilde{J}' := \cup_{g \in \widetilde{J}} g \widetilde{C}' g^{-1}$ is an open non-empty subset of \widetilde{J} (see for example the proof of [16, Thm 7.101]). It follows that there exists a δ belonging to $\widetilde{\Delta} \cap \widetilde{J}'$. Then by the very definition of \widetilde{J}' , δ stabilizes some point on \widetilde{M} , and the subgroup generated by

$j'(\delta)$ is not relatively compact in J . This contradicts to Proposition 3.2. q.e.d.

Corollary 3.7. J is a real form of $(\mathbf{PGL}_d)^r$.

Proof. By our choice of y_0 and by 3.3, the group J_{y_0} always contains a Cartan subgroup of J . Hence the lemma shows that J can not contain a factor of $\mathbf{PGL}_d(\mathbb{C})$. This shows the statement. q.e.d.

Now we claim that every J_y contains a Cartan subgroup of J . Notice that $J_y = J \cap \mathbf{P}_y(\mathbb{C}) \subset \mathbf{J}(\mathbb{C})$ for some parabolic subgroup $\mathbf{P}_y \subset \mathbf{J}_{\mathbb{C}}$. Hence $J_y = J \cap \mathbf{P}_y(\mathbb{C}) \cap \overline{\mathbf{P}_y}(\mathbb{C})$, where $\overline{\mathbf{P}_y}$ is a parabolic of $\mathbf{J}_{\mathbb{C}}$, complex conjugate to \mathbf{P}_y . The subgroup $\mathbf{P}_y \cap \overline{\mathbf{P}_y} \subset \mathbf{J}_{\mathbb{C}}$ is rational over \mathbb{R} , and, being an intersection of two parabolics, contains a maximal torus of \mathbf{J} (see [29, Ch. II, Cor of Thm. 5]). Therefore it contains a maximal \mathbb{R} -rational torus, proving the claim.

Using the lemma, it now remains to show that J_y is reductive. We have seen that $(\text{Lie } J_y)_{\mathbb{C}}$ contains a certain Cartan subalgebra $\mathcal{T}_y := (\text{Lie } T_y)_{\mathbb{C}}$ of $(\text{Lie } J)_{\mathbb{C}}$. Therefore it decomposes as a direct sum $\mathcal{T}_y \oplus \bigoplus_{\beta \in \Phi} \mathcal{J}^{\beta}$, where Φ is a subset of root system of $(\text{Lie } J)_{\mathbb{C}}$ with respect to \mathcal{T}_y and \mathcal{J}^{β} 's are the corresponding root spaces. The Lie algebra $\text{Lie } J$ is stable under the maximal compact subalgebra $\text{Lie } T_y$ of \mathcal{T}_y , and therefore the complex conjugation of $(\text{Lie } J)_{\mathbb{C}}$ defined by its real form $\text{Lie } J$ maps \mathcal{J}^{β} to $\mathcal{J}^{-\beta}$. As $(\text{Lie } J_y)_{\mathbb{C}}$ is stable under this conjugation, we have $\beta \in \Phi$ if and only if $-\beta \in \Phi$, so that J_y is reductive. q.e.d.

Take any $y \in \widetilde{M}$. Since J_y is compact, we may assume that $J_y \subset J^{comp}$. Then J_y is contained in the stabilizer J_y^{comp} of $\rho_0(y)$ in J^{comp} . We know that $\dim \rho_0(\widetilde{M}) = \dim ((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$, that $\dim J^0 = \dim J^{comp}$, and that J^{comp} acts transitively on $((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$. Therefore $J_y = J_y^{comp} = \mathbf{U}_{d-1}(\mathbb{R})$, and the orbit $J(\rho_0(y))$ contains an open neighbourhood of $\rho_0(y)$. Since a point y was chosen arbitrary and since $\rho_0(\widetilde{M})$ is connected, we see that J_0 acts transitively on $\rho_0(\widetilde{M})$. Hence \widetilde{J} acts transitively on \widetilde{M} , and $\rho_0 : \widetilde{M} \rightarrow \rho_0(\widetilde{M})$ is a topological covering. As J_y and $\rho_0(\widetilde{M})$ are connected, J is connected as well.

Let $J_{y,i} \subset J_i$ be the i -th factor of J_y . As $\mathbf{U}_{d-1}(\mathbb{R})$ is a maximal connected proper subgroup of $\mathbf{PGU}_d(\mathbb{R})$, we see by [14, Ch. VIII, Thm. 6.1] (considering separately compact and non-compact cases) that $\rho_0(\widetilde{M}) \cong J^0/J_y = \prod_{i=1}^r J_i/J_{y,i}$ is a Hermitian symmetric space, not containing Euclidean factors. In particular, (see [14, Ch. VIII,

Thm. 4.6]) it is simply connected. Therefore $\rho_0 : \widetilde{M} \rightarrow \rho_0(\widetilde{M})$ and $\phi : \widetilde{J} \rightarrow J$ are isomorphisms.

The kernel of $\widetilde{\pi} : \widetilde{\Delta} \rightarrow \Delta$ is isomorphic to the fundamental group of M . Hence it is normal in $\widetilde{\Delta}$ and discrete in $\text{Aut}(\widetilde{M})$. Therefore it is central in $\widetilde{J} \cong J$, so that it is trivial. This shows that $\widetilde{\Delta} \cong \Delta$ and $\widetilde{M} \cong M$.

Recall that $M_S := (\Delta \cap S) \backslash M$ is a connected component of $(S \backslash X_{\mathbb{C}})^{an}$ for each $S \in \mathcal{F}(\mathcal{G})$, hence for a sufficiently small S the projection $M \rightarrow M_S$ is a topological covering, M_S is smooth, and its canonical bundle is ample (see 2.10 and Remark 2.12). Therefore (see [20, IV, Thm. 5.3]) M_S and M are measure-hyperbolic. In particular, M has no compact factors. Then [14, Ch. VIII, Thm. 7.13] implies that $((\mathbb{P}_{\mathbb{C}}^{d-1})^r)^{an}$ is a compact dual of M and that ρ_0 is the Borel embedding. It follows that $M \cong (B^{d-1})^r$ and that $J \cong \text{Aut}((B^{d-1})^r)^0 \cong \mathbf{PGU}_{d-1,1}(\mathbb{R})_+^r$.

Step 3. Now we are going to determine Δ . Recall that Δ is naturally embedded into $\mathbf{PGU}_{d-1,1}(\mathbb{R})_+^r \times \mathcal{G}$ and that its projection to $\mathbf{PGU}_{d-1,1}(\mathbb{R})_+^r \subset \mathbf{PGL}_d(\mathbb{C})^r$ coincides with j' .

The following immediate generalization of [38, Cor. 2.2.9] plays the central role in this step.

Lemma 3.8. *For each $\delta \in \Delta$ with an elliptic projection $\delta_{\infty} \in \mathbf{PGU}_{d-1,1}(\mathbb{R})_+^r$ there exists a representative*

$$\widetilde{\delta} = (\widetilde{\delta}_{\infty}, \widetilde{\delta}_v, \widetilde{f}_v, \widetilde{\delta}^{f;v}) \in \mathbf{GU}_{d-1,1}(\mathbb{R})_+^r \times \prod_{i=1}^r \widetilde{D}_{w_i}^{\times} \times \prod_{i=1}^r F_{v_i}^{\times} \times \mathbf{G}(\mathbb{A}_F^{f;v_1, \dots, v_r})$$

such that the following hold:

a) Let us view K as a subset of $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^g$, of $\prod_{i=1}^r K_{w_i}$ and of $K \otimes_F \mathbb{A}_F^{f;v_1, \dots, v_r}$ respectively. Then the characteristic polynomials of $\widetilde{\delta}_{\infty}$, $\widetilde{\delta}_v$ and $\widetilde{\delta}^{f;v}$ have their coefficients in K and coincide.

b) \widetilde{f}_v and the similitudes factor of $\widetilde{\delta}^{f;v}$ belong to F^{\times} , viewed as a subset of $\prod_{i=1}^r F_{v_i}^{\times}$ and of $(\mathbb{A}_F^{f;v_1, \dots, v_r})^{\times}$ respectively, and coincide.

Proof. Since the arguments of the proof of [38, Prop. 2.2.8 and Cor. 2.2.9] work without changes in our case, we omit the proof. Alternatively, the statement can be proved by very similar considerations to those of Proposition 3.2, working with P instead of P' . q.e.d.

As in [38, Prop. 1.6.1], Δ is a cocompact lattice of $\mathbf{J}(\mathbb{R})^0 \times \mathcal{G}$. Recall that the projection of Δ to $\mathbf{J}(\mathbb{R})^0$ is injective and that the center $Z(\mathcal{G}) \cong (\mathbb{A}_K^f)^{\times} / \overline{K}^{\times}$ is compact. Hence the projection Δ' of Δ to $\mathbf{J}(\mathbb{R})^0 \times P\mathcal{G}$ is

also a cocompact lattice, isomorphic to Δ . Now we proceed as in the proof of [38, Thm. 2.2.5]. For each finite set N of finite places of F and for each $S \in \mathcal{F}(\mathbb{A}_F^{f;N})$ we define a cocompact lattice

$$\Delta^S \subset \prod_{i=1}^r \mathbf{J}_i(\mathbb{R}) \times \prod_{u \in N} \mathbf{PG}^{\text{int}}(F_u)$$

consisting of the projections of elements of Δ' , whose components outside N belongs to S (see [38, 2.4.1]).

Lemma 3.9. *The lattice Δ^S is irreducible (in the sense of [21, Ch. III, Def. 5.9]).*

Proof. We will prove a stronger assertion saying that for each $u \in \{\infty_1, \dots, \infty_r\} \cup N$ the projection pr_u of Δ' to the u 'th component is injective. If not, set $\Delta^u := \text{Ker pr}_u$. Then Δ^u is a non-trivial normal subgroup of Δ' . Since the projection $\Delta' \rightarrow \mathbf{J}(\mathbb{R})^0$ is dense and injective, the closure $\overline{\Delta_\infty^u}$ of $\Delta_\infty^u \subset \mathbf{J}(\mathbb{R})^0$ is a normal non-trivial closed subgroup of $\mathbf{J}(\mathbb{R})^0$. Therefore it is equal to $\prod_{i \in I} \mathbf{J}_i(\mathbb{R})^0$ for some non-empty subset $I \subset \{1, \dots, r\}$. Recall that the set of elliptic elements of $\mathbf{PGU}_{d-1,1}(\mathbb{R})_+$ is open and non-empty (see [38, Prop. 1.7.3]). Therefore there exists an element $\delta \in \Delta^u$, whose projection $\delta_{\infty_i} \in \mathbf{J}_i(\mathbb{R})^0$ is elliptic for all $i \in I$ and trivial for all $i \notin I$. In particular, δ_∞ fixes some point on $(B^{d-1})^r$. Proposition 3.2 and Lemma 3.8 now show that δ_u can not be equal to 1, contradicting to our assumption. (Observe that the above proof is simpler than that of [38, Prop. 2.4.5].) q.e.d.

3.10. From this point all the arguments from [38, 2.3-2.6] work without changes. First we show, as in [38, 2.4.5-2.4.6], that by Margulis' theorem [21, Thm. (B), p. 298] the subgroup Δ^S is arithmetic. As in [38, 2.3 and 2.4.6] (using Proposition 3.2), we see that an absolutely simple adjoint group $\tilde{\mathbf{G}}$, defining Δ^S , is defined over F and does not depend on N and S . We also get that $\Delta' \cong \Delta$ is naturally embedded into $\tilde{\mathbf{G}}(F)$. We conclude from Lemma 3.8, as in [38, 2.5.2], that $\tilde{\mathbf{G}}$ is locally isomorphic to \mathbf{PG}^{int} at all places of F .

3.11. Next we show directly that $\tilde{\mathbf{G}}$ is an inner form of \mathbf{PG}^{int} . In fact, let c be the element of $H^1(\text{Gal}(\bar{F}/F), \text{Aut}_{\bar{F}}(\mathbf{PG}^{\text{int}}))$ corresponding to $\tilde{\mathbf{G}}$, and let \mathcal{D} be the Dynkin diagram of $\mathbf{PG}^{\text{int}}/F$. Thus our aim is to show that the image \bar{c} of c in $H^1(\text{Gal}(\bar{F}/F), \text{Aut}(\mathcal{D})) = \text{Hom}(\text{Gal}(\bar{F}/F), \mathbb{Z}/2\mathbb{Z})$ is trivial. Set L be the subfield of \bar{F} of fixed elements by $\text{Ker}(\bar{c})$. Then L is a quadratic or trivial extension of F ,

and $\bar{c} \in H^1(\text{Gal}(L/F), \text{Aut}(\mathcal{D}))$. Let u be a finite prime of F , inert in L . Then $\text{Gal}(L/F) \cong \text{Gal}(L_u/F_u)$. Hence it remains to show the triviality of the image $\bar{c}_u \in H^1(\text{Gal}(L_u/F_u), \text{Aut}(\mathcal{D}))$ of \bar{c} . Since \mathbf{PG}^{int} and $\tilde{\mathbf{G}}$ are locally isomorphic at all places of F , the image $c_u \in H^1(\text{Gal}(\bar{F}_u/F_u), \text{Aut}_{\bar{F}_u}(\mathbf{PG}^{\text{int}}))$ of c is trivial. But \bar{c}_u is the image of c_u , and we are done.

It follows that $\tilde{\mathbf{G}} \cong \mathbf{PGU}(D', \alpha')$, where (D', α') is a form of $(D^{\text{int}}, \alpha^{\text{int}})$, that is D' is a central simple algebra over K of dimension d^2 , and α' is an involution of D' of the second kind over F . Thus we have avoided the use of the classification in [38, 2.5.1].

3.12. As in [38, 2.5.4-2.5.6], we see that $\tilde{\mathbf{G}} \cong \mathbf{PG}^{\text{int}}$ and that the image of the natural embedding $\Delta \hookrightarrow \mathcal{G} \cong \mathbf{G}^{\text{int}}(\mathbb{A}_F^f)/\overline{\mathbf{Z}(\mathbf{G}^{\text{int}})(F)}$ lies in $\mathbf{PG}^{\text{int}}(F)_+$. Since Δ is cocompact in \mathcal{G} , and since $\mathbf{PG}^{\text{int}}(F)_+ \subset \mathcal{G}$ is discrete, the image of Δ has a finite index in $\mathbf{PG}^{\text{int}}(F)_+$.

To show that these groups are isomorphic we can compare the volumes, as in [38, 2.6], using Kottwitz' results on Tamagawa measures [18, Thm. 1] and the Hirzebruch proportionality principle (Proposition A.1). Alternatively one can argue as follows.

Let $h \in \mathbf{PG}^{\text{int}}(F)_+$ be an elliptic element, let $y \in (B^{d-1})^r$ be an elliptic point of h , and let \mathbf{T} be the maximal (F -rational) torus of \mathbf{PG}^{int} such that $\mathbf{T}(F)$ fixes y and $h \in \mathbf{T}(F)$. Then by Proposition 3.2 there exist $x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p)$ and $g \in P\mathcal{G}'$ such that $g(\Delta \cap \mathbf{T}(F))g^{-1} \subset \Gamma_x \subset \mathbf{PG}(F) \subset P\mathcal{G}'$. Since \mathbf{T} is connected, the subgroup $\mathbf{T}(F) \cap \Delta$ is Zariski dense in \mathbf{T} (see [3, Ch. V, Cor. 13.3]). Therefore the map $t \mapsto gtg^{-1}$ defines an algebraic F -rational map from \mathbf{T} to \mathbf{PG} . In particular, ghg^{-1} belongs to $\mathbf{PG}(F)$. Moreover, since the stabilizer of a point is an algebraic subgroup, $ghg^{-1} \in \Gamma_x$. Proposition 3.2 now shows that $h \in \Delta$. Since the set of elliptic elements of $\mathbf{PG}^{\text{int}}(F)_+$ generates the whole group (see [21, Ch. IX, Lem. 3.3] and [38, Prop. 1.7.3]), $\Delta \cong \mathbf{PG}^{\text{int}}(F)_+$.

Step 4. The proven above implies the existence of a \mathcal{G} -equivariant isomorphism $\tilde{\varphi} : (X_{\mathbb{C}})^{\text{an}} \xrightarrow{\sim} (X_{\mathbb{C}}^{\text{int}})^{\text{an}}$. By GAGA results (compare [38, Lem. 2.2.6]) $\tilde{\varphi}$ gives us an algebraic isomorphism $\varphi : X_{\mathbb{C}} \xrightarrow{\sim} X_{\mathbb{C}}^{\text{int}}$.

To see that φ is E_v -rational we will compare the action of the Galois group on the set of special points on the two sides. Fix some special point $x \in X^{\text{int}}(\mathbb{C})$. Then x corresponds to some maximal torus $\mathbf{T} \subset \mathbf{H}^{\text{int}}$ and to some element h' in the conjugacy class M^{int} of h . As in [38, Lem. 3.1.5], \mathbf{T} is equal to the intersection of \mathbf{H}^{int} with $\mathbf{R}_{L/\mathbb{Q}}\mathbb{G}_m$ for some maximal commutative subfield L of D^{int} , stable under α^{int} .

Moreover, L is a CM -field, and the restriction of α^{int} to L is the complex conjugation. For each $i = 1, \dots, r$ the point $h' \in M^{int}$ determines, as in [38, 3.1.6], an embedding $\tilde{\omega}_i : L \hookrightarrow \mathbb{C}$, extending $\tilde{\omega}_i : K \hookrightarrow \mathbb{C}$.

The reflex field $\tilde{E} \subset \mathbb{C}$ of (\mathbf{T}, h) is the subfield of the composite L_0 of the fields $\tilde{\omega}_1(L), \dots, \tilde{\omega}_r(L)$ fixed elementwise by those automorphisms of L_0/\mathbb{Q} which permute the $\tilde{\omega}_i$'s. Moreover, the canonical morphism $r' : \mathbf{R}_{\tilde{E}/\mathbb{Q}}\mathbb{G}_m \rightarrow \mathbf{T}$ (in the notation of [38, 3.1.4]) is characterized as the unique morphism such that the composition map $r' \circ N_{L_0/\tilde{E}} : \mathbf{R}_{L_0/\mathbb{Q}}\mathbb{G}_m \rightarrow \mathbf{T}$ is given by

$$r'(N_{L_0/\tilde{E}}(l)) = \prod_{i=1}^r \tilde{\omega}_i^{-1}(N_{L_0/\tilde{\omega}_i(L)}(\bar{l}/l)) \in \mathbf{T}(\mathbb{Q}) \subset L^\times$$

for each $l \in L_0^\times$, where by \bar{l} we denote the complex conjugate of l (compare [38, 3.1.7]).

For each $i = 1, \dots, r$ set $L_{w_i} := L \otimes_K K_{w_i}$. Then L_{w_i} is a maximal commutative subfield of the skew field $D^{int} \otimes_K K_{w_i} \cong D_{w_i}$. As in 2.5, each $\tilde{\omega}_i$ defines a continuous embedding $\tilde{\alpha}_i : L_{w_i} \hookrightarrow \mathbb{C}_p$. Moreover, the composite field of the $\tilde{\alpha}_i(L)$'s is the completion \tilde{E}_v of $\tilde{E} \subset \mathbb{C}_p$ (compare Lemma 2.6).

For each $l \in \tilde{E}_v^\times$ define an element

$$\tilde{r}(l) := (\tilde{\alpha}_1^{-1}(N_{\tilde{E}_v/\tilde{\alpha}_1(L_{w_1})}(l^{-1})), \dots, \tilde{\alpha}_r^{-1}(N_{\tilde{E}_v/\tilde{\alpha}_r(L_{w_r})}(l^{-1})); 1)$$

in

$$\prod_{i=1}^r L_{w_i}^\times \times \mathcal{G}' \subset \prod_{i=1}^r \tilde{D}_{w_i}^\times \times \mathcal{G}' = \tilde{\mathcal{G}}.$$

Let $\theta_{\tilde{E}_v} : \tilde{E}_v^\times \rightarrow \text{Gal}(\tilde{E}_v^{ab}/\tilde{E}_v)$ be the Artin homomorphism, sending the uniformizer to the arithmetic Frobenius automorphism. By the definition of X^{int} (see for example [38, 3.1.4, 3.1.12]), x is rational over \tilde{E}_v^{ab} , and for each $l \in \tilde{E}_v^\times$ the element $\theta_{\tilde{E}_v}(l) \in \text{Gal}(\tilde{E}_v^{ab}/\tilde{E}_v)$ maps x to $\tilde{r}(l)(x)$ (compare [38, Prop. 3.1.9]). Since these properties (for all special points) characterize the weakly-canonical models, it will suffice to show that

- i) the point $y := \varphi^{-1}(x) \in X(\mathbb{C}_p)$ is rational over \tilde{E}_v^{ab} ;
- ii) $\theta_{\tilde{E}_v}(l)(y) = \tilde{r}(l)(y)$ for each $l \in \tilde{E}_v^\times$.

Let

$$[\tilde{y}, g] \in \prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times \mathcal{G}'$$

be a representative of y . Then $[\sigma(\tilde{y}), g]$ is a representative of $\sigma(y)$ for each (not necessary continuous) $\sigma \in \text{Aut}(\mathbb{C}_p/\tilde{E}_v)$. As in [38, Prop. 3.2.3], we show that for each $i = 1, \dots, r$ there exists an embedding $L_{w_i} \hookrightarrow \text{Mat}_d(K_{w_i})$ such that \tilde{y} belongs to the image of the corresponding embedding

$$\prod_{i=1}^r \Sigma_{L_{w_i}}^1 \otimes_{K_{w_i}} E_v \hookrightarrow \prod_{i=1}^r \Sigma_{K_{w_i}}^d \otimes_{K_{w_i}} E_v$$

(see [38, 1.4.4]). Then the statement follows from [38, Lem. 1.4.3]. This completes the proof of the First Main Theorem.

Remark 3.13. a) Let $(X^{int})'$ be the weakly-canonical model over E_v of the Shimura variety, corresponding to the same group \mathbf{H}^{int} as X^{int} and to a homomorphism $h' : \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow \mathbf{H}_{\mathbb{R}}^{\text{int}}$ defined by requirement that for each $z \in \mathbb{C}^\times$ we have

$$h(z) = \underbrace{(\text{diag}(\bar{z}, \dots, \bar{z}, z)^{-1}; \dots; \text{diag}(\bar{z}, \dots, \bar{z}, z)^{-1})}_r; \underbrace{I_d; \dots; I_d}_{g-r} \\ \in \mathbf{GU}_{d-1,1}(\mathbb{R})^r \times \mathbf{GU}_d(\mathbb{R})^{g-r}.$$

Then $(X^{int})'$ is a certain abelian unramified twist of X^{int} . Moreover, comparing the action of the Galois group on the set of special points, one can check that the difference of the Galois actions on X^{int} and on $(X^{int})'$ is given by some explicit homomorphism $\text{Aut}(\mathbb{C}/E_v) \rightarrow \mathcal{G}_1 := \{1\} \times \prod_{i=1}^r F_{v_i}^\times \times \{1\} \subset \tilde{\mathcal{G}} = \mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$. In particular, the quotients $\mathcal{G}_1 \backslash X^{int}$ and $\mathcal{G}_1 \backslash (X^{int})'$ (see Lemma 5.4) are naturally isomorphic (over E_v).

b) Rapoport and Zink work with $(X^{int})'$ instead of X^{int} , therefore their Shimura varieties are uniformized by the corresponding twist (see [33, Prop. 6.49]) of the product of Drinfeld upper half-spaces, and [33, Thm. 6.50] follows from our First Main Theorem by twisting.

Step 5. We have already shown in the second and the third steps that the homomorphism

$$j' : \mathbf{PG}^{\text{int}}(F)_+ \rightarrow \mathbf{PGL}_d(\mathbb{C})^r \cong \prod_{i=1}^r \mathbf{PG}^{\text{int}}(K_{\infty_i})$$

is a diagonal embedding. Arguing as in [38, 4.6.2-4.6.4 and 4.5.5] we show that this holds for the whole j . For the descent to E_v we use the uniqueness arguments of [38, 4.7]. This completes the proof of the Second Main Theorem.

Corollary 3.14. *Let X_Γ be a projective scheme over a p -adic field L such that $X_\Gamma^{an} \cong \Gamma \backslash \prod_{i=1}^r \Omega_{L_i}^d \otimes_{L_i} L$ for p -adic subfields L_1, \dots, L_r of L , a natural number $d \geq 2$, and a torsion-free cocompact lattice $\Gamma \subset \prod_{i=1}^r \mathbf{PGL}_d(L_i)$ (see Lemma 2.9). Then for each field embedding $L \hookrightarrow \mathbb{C}$ the universal cover of $(X_{\Gamma, \mathbb{C}})^{an}$ is $(B^{d-1})^r$.*

Proof. For curves ($r = 1$ and $d = 2$) the statement follows from the fact that X_Γ is a smooth curve of genus greater than 1 (see [26, Thm. 3.3]).

In the remaining cases, the rank of $\prod_{i=1}^r \mathbf{PGL}_d(K_{w_i})$ is $r(d-1) \geq 2$. If Γ is irreducible, then it is arithmetic by [21, Thm. (A), p. 298]. Therefore there exists a number field F , an adjoint absolutely simple group \mathbf{G}' over F , and finite primes v_1, \dots, v_r of F such that $\mathbf{G}'(F_{v_i}) \cong \mathbf{PGL}_d(L_i)$ for each $i = 1, \dots, r$, the group $\mathbf{G}'(F_{\infty_i})$ is compact for each archimedean completion F_{∞_i} of F , and Γ is commensurable with

$$\mathbf{G}'(F) \cap S \subset \prod_{i=1}^r \mathbf{G}'(F_{v_i}) \cong \prod_{i=1}^r \mathbf{PGL}_d(K_{w_i})$$

for every $S \in \mathcal{F}(\mathbf{G}'(\mathbb{A}_F^{f; v_1, \dots, v_r}))$. Since the universal cover of $(X_{\Gamma, \mathbb{C}})^{an}$ does not change after replacement of Γ by another torsion-free subgroup of $\prod_{i=1}^r \mathbf{PGL}_d(K_{w_i})$, commensurable with Γ , we may assume that $\Gamma = \mathbf{G}'(F) \cap S$ for a sufficiently small S . Now we can apply the first two steps of the proof of the Main Theorems (using \mathbf{G}' instead of \mathbf{G}) to conclude that $(X_{\Gamma, \mathbb{C}})^{an} \cong \Delta_S \backslash (B^{d-1})^r$ for some torsion-free cocompact lattice $\Delta_S \subset \mathbf{PGU}_{d-1, 1}(\mathbb{R})_+^r$. This proves the statement for irreducible Γ 's.

If Γ is reducible, then possibly replacing Γ by a subgroup of finite index we may assume that there exists a non-trivial proper subset $I \subset \{1, \dots, r\}$ and torsion-free cocompact lattices

$$\Gamma_1 \subset \prod_{i \in I} \mathbf{PGL}_d(K_{w_i})$$

and

$$\Gamma_2 \subset \prod_{i \notin I} \mathbf{PGL}_d(K_{w_i})$$

such that $\Gamma = \Gamma_1 \times \Gamma_2$. Then

$$\Gamma \backslash \prod_{i=1}^r \Omega_{L_i}^d \otimes_{L_i} L \cong (\Gamma_1 \backslash \prod_{i \in I} \Omega_{L_i}^d \otimes_{L_i} L) \times (\Gamma_2 \backslash \prod_{i \notin I} \Omega_{L_i}^d \otimes_{L_i} L),$$

and the statement follows from the previous cases by induction on the number of factors. q.e.d.

Remark 3.15. Observe that the proof uses neither the classification of algebraic groups nor of Hermitian symmetric domains.

Remark 3.16. We know that the canonical bundle of X_Γ is ample (Lemma 2.9) and that the Chern numbers of X_Γ are proportional to those of $(\mathbb{P}^{d-1})^r$ (see Proposition A.1). Therefore in the case $r = 1$ the corollary follows immediately from Yau's theorem ([39]).

Remark 3.17. The proof shows that for arithmetic Γ 's the corollary is a direct consequence of the results of the Step 2. Conversely, the arguments of [38, proof of Prop. 1.6.1] show that the corollary implies that $M \cong (B^{d-1})^r$. In particular, in the case $r = 1$ the differential-geometric part of the First Main Theorem immediately follows from Yau's Theorem.

4. Another approach

In this section we give another approach to the differential-geometric part of the proof of the First Main Theorem. Using Yau's theorem on the existence and the uniqueness of the Kähler-Einstein metric, (but not appealing to principal bundles and to the structure theory of reductive groups) we will show directly that (in the notation of Step 1 of Section 3) M is a Hermitian symmetric domain. Here we use the ideas from Milne's unpublished manuscript [23].

Lemma 4.1. *There exists a Δ -invariant Kähler metric g on M .*

Proof. Recall the following theorem of Yau (see [40, Thm. 5]). Let V be a smooth complex projective variety whose canonical bundle is ample. Then there exists a unique Kähler metric g_V on V^{an} , called the Kähler-Einstein metric, whose Ricci tensor satisfies $\text{Ric}(g_V) = -g_V$ (notice that the sign of Yau's Ricci tensor differs from the standard one).

We are going to apply this theorem to $M_S := (\Delta \cap S) \backslash M$ for sufficiently small $S \in \mathcal{F}(\mathcal{G})$ (see the last paragraph of Step 2 of Section 3).

Let g_S be the Kähler-Einstein metric on M_S , and let $g_{(S)}$ be its inverse image on M . From the uniqueness of the metric we conclude that $g := g_{(S)}$ does not depend on S . For every $\delta \in \Delta$ we have a

commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\delta} & M \\ \text{proj} \downarrow & & \downarrow \text{proj} \\ M_S & \xrightarrow{\delta} & M_{\delta S \delta^{-1}} \end{array}$$

Using again the uniqueness of the metric we conclude that g is Δ -invariant. q.e.d.

Since (M, g) is a Kähler manifold, the group $\text{Is}(M, g)$ of its isometries is a Lie group (see [17, II, Thm. 1.2]). Since g is Δ -invariant, Δ is contained in $\text{Is}(M, g)$, therefore its closure G is a Lie subgroup of $\text{Is}(M, g)$. Notice also that G is also contained in $\text{Aut}(M)$.

For each $y \in M$ we denote the stabilizers of y in Δ and in G by Δ_y and G_y respectively.

Lemma 4.2. *There exist a point $y \in M$ and an element $\delta \in G_y$ of order 4, inducing the multiplication by $\sqrt{-1}$ on $T_y(M)$.*

Proof. (Compare the proof of Proposition 3.2). Let

$$x \in \prod_{i=1}^r \Omega_{K_{w_i}}^d(\mathbb{C}_p) \subset \mathbb{P}^{d-1}(\mathbb{C}_p)^r$$

be an elliptic point for the action of $\mathbf{PG}(F)$. Then the point

$$[x, 1] \in \left[\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} E_v \times (P\mathcal{G}')^{disc} \right] / \mathbf{PG}(F) \cong (X')^{an}$$

corresponds to a certain point $z \in X'(\mathbb{C}_p) = X(\mathbb{C})$ and therefore to a point

$$[y, g] \in [M \times (P\mathcal{G}')^{disc}] / \Delta \cong (X'_\mathbb{C})^{an}.$$

Hence the map $\delta \mapsto g\delta g^{-1}$ defines an isomorphism between Δ_y and the stabilizer $\Gamma_x \subset \mathbf{PG}(F) \subset P\mathcal{G}'$ of x . Now we identify $\mathbb{C} = \mathbb{C}_p$ -vector spaces $T_y(M)$ and $T_x(\mathbb{P}_{\mathbb{C}_p}^{d-1})^r = T_x(\mathbb{P}_{\mathbb{C}}^{d-1})^r$ via the sequence of canonical isomorphisms

$$\begin{aligned} T_y(M) &\cong T_{[y,1]}(X'_\mathbb{C})^{an} \cong T_{[y,g]}(X'_\mathbb{C})^{an} \\ &\cong T_z(X'_\mathbb{C}) - T_z(X'_{\mathbb{C}_p}) \cong T_{[x,1]}(X'_{\mathbb{C}_p})^{an} \\ &\cong T_x\left(\prod_{i=1}^r \Omega_{K_{w_i}}^d \otimes_{K_{w_i}} \mathbb{C}_p\right) \cong T_x(\mathbb{P}_{\mathbb{C}_p}^{d-1})^r. \end{aligned}$$

Then the above isomorphism $\Delta_y \xrightarrow{\sim} \Gamma_x$ commutes with embeddings $\Delta_y \hookrightarrow \text{Aut}(T_y(M))$ and $\Gamma_x \hookrightarrow \text{Aut}(T_x(\mathbb{P}_{\mathbb{C}}^{d-1})^r)$, hence it extends by continuity to the isomorphism between the closures $\overline{\Delta}_y \subset \text{Aut}(T_y(M))$ and $\overline{\Gamma}_x \subset \text{Aut}(T_x(\mathbb{P}_{\mathbb{C}}^{d-1})^r)$. Since these closures are naturally isomorphic to the closures in G and in $\mathbf{PH}(\mathbb{R})$ respectively, it remains to show the existence of an element in $\overline{\Gamma}_x \subset \mathbf{PH}(\mathbb{R}) \cong \mathbf{PGU}_d(\mathbb{R})^r$, which has order 4 and induces the multiplication by $\sqrt{-1}$ on $T_x(\mathbb{P}_{\mathbb{C}}^{d-1})^r$. As $\overline{\Gamma}_x$ is a maximal compact torus of $\mathbf{PGU}_d(\mathbb{R})^r$ (see the last paragraph of the proof of Proposition 3.2), this is clear. q.e.d.

Proposition 4.3. *M is a Hermitian symmetric domain.*

Proof. First we show that G acts transitively on M . Let $y \in M$ be as in the lemma, and let N be the orbit Gy . Consider the map $\varphi_y : \text{Is}(M, g) \rightarrow M; \delta \mapsto \delta(y)$. By [17, p.41], it factors as a composition of a closed embedding of $\text{Is}(M, g)$ into the bundle $O(M)$ of orthogonal frames over M and the projection of $O(M)$ to M . This shows that φ_y is proper, therefore N is closed in M . By [14, II, Thm. 3.2 and 4.2], $N \cong G/G_y$ then has a canonical structure of a G -equivariant closed real submanifold of M . In particular, its tangent space $T_y(N) \subset T_y(M)$ is invariant under the action of G_y . By Lemma 4.2 it then invariant under the multiplication by $\sqrt{-1}$. Since the group G acts on N transitively, N is an almost complex submanifold of the complex manifold M . Therefore N has a unique G -invariant structure of a closed complex submanifold of M (see [14, Ch. VIII, Thm. 1.2 and p.284]).

It follows that $\tilde{N} := [N \times \mathcal{G}^{disc}]/\Delta$ is a \mathcal{G} -invariant closed (complex) analytic subspace of $(X_{\mathbb{C}})^{an} \cong [M \times \mathcal{G}^{disc}]/\Delta$.

Lemma 4.4. *There exists a closed \mathcal{G} -invariant subscheme Y of $X_{\mathbb{C}}$ such that $Y^{an} \cong \tilde{N}$.*

Proof. Take an $S \in \mathcal{F}(\mathcal{G})$. Then the quotient map

$$\pi_S : (X_{\mathbb{C}})^{an} \rightarrow (S \backslash X_{\mathbb{C}})^{an}$$

is open, and \tilde{N} is S -invariant. Therefore the quotient

$$\pi_S(\tilde{N}) \cong S \backslash [N \times \mathcal{G}^{disc}]/\Delta$$

is a closed analytic subspace of $(S \backslash X_{\mathbb{C}})^{an}$. The scheme $S \backslash X_{\mathbb{C}}$ is projective, hence by GAGA (see for example [38, Cor. 1.2.2]) there exists a projective subscheme Y_S of $S \backslash X_{\mathbb{C}}$ such that

$$Y_S^{an} \cong S \backslash [N \times \mathcal{G}^{disc}]/\Delta.$$

The inverse limit of the Y_S 's satisfies the conditions of the lemma.

q.e.d.

Since every \mathcal{G} -orbit in $X_{\mathbb{C}}$ is Zariski dense (as in [38, Prop. 1.5.3 e))), we obtain that $Y = X_{\mathbb{C}}$. Hence $N = M$, so that G acts transitively on M , as claimed.

Let $\delta \in G$ be as in Lemma 4.2. Then for each $z = g(y) \in M$ with $g \in G$ we have an involutive holomorphic isometry $g\delta^2g^{-1}$ of M with an isolated fixed point z . This means that M is a Hermitian symmetric space. Moreover, as in Step 2 of Section 3 we see that M is actually a Hermitian symmetric domain. q.e.d.

Remark 4.5. Knowing the proposition one can obtain an alternative proof of the results of Step 2 of Section 3. For showing this one can simply proceed along the lines of the above mentioned step, which would be much easier assuming the proposition. Alternatively, one can apply Margulis' superrigidity theorem (as in [38, 4.6]).

5. The case of quaternion Shimura varieties

In this section we deduce a p -adic uniformization of quaternion Shimura varieties from the unitary case (compare [4], where the moduli approach (see [8, §6]) is used).

5.1. Let F be a totally real number field of degree g over \mathbb{Q} , let $\infty_1, \dots, \infty_g$ be the archimedean completions of F , and let $1 \leq r \leq g$ be a natural number. Let D^{int} be a quaternion algebra over F , split at $\infty_1, \dots, \infty_r$ and ramified at $\infty_{r+1}, \dots, \infty_g$. Let $\mathbf{G}^{int} = \mathbf{GL}_1(D^{int})$ be the algebraic group over F , characterized by $\mathbf{G}^{int}(R) = (D^{int} \otimes_F R)^\times$ for each F -algebra R , and set $\mathbf{H}^{int} := \mathbf{R}_{F/\mathbb{Q}}\mathbf{G}^{int}$.

Define a homomorphism $h : \mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbf{G}_m \rightarrow \mathbf{H}_{\mathbb{R}}^{int}$ by requiring for each $z = x + iy \in \mathbb{C}^\times$

$$h(z) = \left(\underbrace{\left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} ; \dots ; \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \right)}_r ; \underbrace{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \dots ; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)}_{g-r} \right) \\ \in \mathbf{GL}_2(\mathbb{R})^r \times (\mathbb{H}^\times)^{g-r} \cong \mathbf{H}^{int}(\mathbb{R}),$$

where \mathbb{H} denotes the field of Hamilton's quaternions. Then M^{int} , the $\mathbf{H}^{int}(\mathbb{R})$ -conjugacy class of h , is isomorphic to \mathfrak{h}^r . The reflex field

$E \subset \mathbb{C}$ of $(\mathbf{H}^{\text{int}}, M^{\text{int}})$ is the subfield of the composite F_0 of the fields $\infty_1(F), \dots, \infty_r(F)$ fixed elementwise by those automorphisms of F_0/\mathbb{Q} , which permute the ∞_i 's ($i = 1, \dots, 2$).

Let v be a finite prime of E , and let v_1, \dots, v_r be the finite primes of F , corresponding to $\infty_1, \dots, \infty_r$ as in 2.5. Assume that the v_i 's are distinct and that D^{int} is ramified at v_1, \dots, v_r . Then, as in Lemma 2.6, we see that for each $i = 1, \dots, r$ we have a natural embedding $\alpha_i : F_{v_i} \hookrightarrow E_v$ and that E_v is the composite field of the $\alpha_i(F_{v_i})$'s. Assume that the v_i 's are distinct and that D^{int} is ramified at v_1, \dots, v_r .

Let X^{int} be the weakly-canonical model over E_v of the Shimura variety corresponding to $(\mathbf{H}^{\text{int}}, M^{\text{int}})$. Let \check{M}^{int} be the Grassmann variety corresponding to the pair $(\mathbf{H}^{\text{int}}, M^{\text{int}})$, and let W^{int} be a $\mathbf{PH}_{\mathbf{E}_v}^{\text{int}}$ -equivariant vector bundle on $\check{M}_{E_v}^{\text{int}}$. Let P^{int} (resp. $V^{\text{int}} = V^{\text{int}}(W^{\text{int}})$) be the canonical model of the standard $\mathbf{PH}_{\mathbf{E}_v}^{\text{int}}$ -principal bundle over X^{int} (resp. the automorphic vector bundle on X^{int} corresponding to W^{int}).

5.2. Let D be the quaternion algebra over F ramified at $\infty_1, \dots, \infty_r$, split at v_1, \dots, v_r , and locally isomorphic to D^{int} at all other places of F . Notice that D is definite.

Define algebraic groups \mathbf{G} and \mathbf{H} by $\mathbf{G} := \mathbf{GL}_1(D)$ and $\mathbf{H} := \mathbf{R}_{F/\mathbb{Q}}\mathbf{G}$. For each $i = 1, \dots, r$ fix a quaternion division algebra \tilde{D}_{v_i} over F_{v_i} . Set $\mathcal{G}' := \mathbf{G}(\mathbb{A}_F^{f;v_1, \dots, v_r})$ and $\tilde{\mathcal{G}} := \prod_{i=1}^r \tilde{D}_{v_i}^\times \times \mathcal{G}'$. Fix algebra isomorphisms $D \otimes_F F_{v_i} \cong \text{Mat}_2(F_{v_i})$, $D^{\text{int}} \otimes_F F_{v_i} \cong \tilde{D}_{v_i}$ and $D \otimes_F \mathbb{A}_F^{f;v_1, \dots, v_r} \cong D^{\text{int}} \otimes_F \mathbb{A}_F^{f;v_1, \dots, v_r}$. They induce isomorphisms

$$\mathbf{G}(\mathbb{A}_F^f) \cong \prod_{i=1}^r \mathbf{GL}_2(F_{v_i}) \times \mathcal{G}'$$

and $\mathbf{G}^{\text{int}}(\mathbb{A}_F^f) \cong \tilde{\mathcal{G}}$. We will often identify the groups $\mathbf{G}^{\text{int}}(\mathbb{A}_F^f)$ and $\tilde{\mathcal{G}}$ by means of the last isomorphism.

Let X be a scheme over E_v with a $\tilde{\mathcal{G}}$ -action such that for each $S \in \mathcal{F}(\tilde{\mathcal{G}})$ the quotient $S \backslash X$ is projective and satisfies

$$(S \backslash X)^{an} \cong S \backslash \left[\prod_{i=1}^r \Sigma_{F_{v_i}}^2 \otimes_{F_{v_i}} E_v \times \mathcal{G}' \right] / \mathbf{G}(F)$$

(compare Proposition 2.8 and [38, Prop. 1.5.2]). Let P (resp. V) be the $\mathbf{PH}_{\mathbf{E}_v}$ -principal (resp. the automorphic vector) bundle over X defined as in 2.15.

Theorem 5.3. *There exists a $\tilde{\mathcal{G}}$ -equivariant isomorphism $\varphi : X \xrightarrow{\sim} X^{int}$. Moreover, φ lifts to $\tilde{\mathcal{G}}$ -equivariant isomorphisms*

- a) $\varphi_V : V \xrightarrow{\sim} V^{int}$ of automorphic vector bundles and
- b) $\varphi_P : P^{tw} \xrightarrow{\sim} P^{int}$ of $\mathbf{PH}_{\mathbf{E}_v}^{int}$ -principal bundles.

Proof. The theorem can be proved exactly by the same argument as the Main Theorems. In fact, the proof would be much easier technically. Instead we will embed our schemes into certain quotients of the corresponding schemes in the unitary case and will deduce the theorem from the Main Theorems.

First we define the corresponding objects in the unitary case. Let K be a totally imaginary quadratic extension of F in which all v_1, \dots, v_r split. Then for each $i = 1, \dots, r$ we can and do identify F_{v_i} (resp. \tilde{D}_{v_i}) with K_{w_i} (resp. \tilde{D}_{w_i}), defined as in Section 2. As a consequence, the field E_v has the same meaning, as in Section 2. Define the involution of the second kind α (resp. α^{int}) on $D \otimes_F K$ (resp. $D^{int} \otimes_F K$), as the tensor product of the main involution of D (resp. D^{int}) and the conjugation of K over F .

Without further mention we will write all objects in the unitary case, defined from the above data as in Section 2, with an upper subscript un , meaning "unitary". For example, we write \mathbf{G}^{un} for $\mathbf{GU}(D \otimes_F K, \alpha)$ and $(\mathbf{G}^{int})^{un}$ for $\mathbf{GU}(D^{int} \otimes_F K, \alpha^{int})$. Then the above algebra isomorphisms identify, as in 2.7, $\tilde{\mathcal{G}}^{un}$ with $(\mathbf{G}^{int})^{un}(\mathbb{A}_F^f)$. Define a subgroup $\mathcal{G}_1 \subset \tilde{\mathcal{G}}^{un} = (\mathbf{G}^{int})^{un}(\mathbb{A}_F^f)$ as in Remark 3.13.

The diagonal embeddings $\mathbf{G} \hookrightarrow \mathbf{G}^{un}$ and $\mathbf{G}^{int} \hookrightarrow (\mathbf{G}^{int})^{un}$ define us embeddings

$$\tilde{\mathcal{G}} = \prod_{i=1}^r \tilde{D}_{v_i}^\times \times \mathbf{G}(\mathbb{A}_F^{f;v_1, \dots, v_r}) \hookrightarrow \prod_{i=1}^r \tilde{D}_{w_i}^\times \times \mathbf{G}^{un}(\mathbb{A}_F^{f;v_1, \dots, v_r}) = \mathcal{G}_1 \backslash \tilde{\mathcal{G}}^{un}$$

and

$$\mathbf{G}^{int}(\mathbb{A}_F^f) \hookrightarrow (\mathbf{G}^{int})^{un}(\mathbb{A}_F^f) \rightarrow \mathcal{G}_1 \backslash (\mathbf{G}^{int})^{un}(\mathbb{A}_F^f),$$

commuting with the above identifications.

The following lemma and its obvious analogs will assure the existence of the quotients, used below.

Lemma 5.4. *For each closed subgroup $H \subset Z(\tilde{\mathcal{G}}^{un})$ the quotient scheme $H \backslash X^{un}$ exists.*

Proof. (Compare [38, Lem. 1.3.11 c) and Prop. 1.5.3 c]). In the notation of Step 1 of Section 3, the group $Z(\mathcal{G}^{un}) = Z(\tilde{\mathcal{G}}^{un})/\mathcal{G}_0^{un}$ is

compact. Hence for each $S \in \mathcal{F}(\tilde{\mathcal{G}}^{un})$ the subgroup $H \cdot S$ of $\tilde{\mathcal{G}}^{un}$ is open and compact modulo \mathcal{G}_0^{un} . Since \mathcal{G}_0^{un} acts trivially on X^{un} , each quotient $(H \cdot S) \backslash X^{un}$ exists, and their inverse limit is $H \backslash X^{un}$. q.e.d.

Next we notice that the natural equivariant embeddings

$$\left[\prod_{i=1}^r \Sigma_{F_{v_i}}^2 \times \mathbf{G}(\mathbb{A}_F^{f;v_1, \dots, v_r}) \right] / \mathbf{G}(F) \hookrightarrow \left[\prod_{i=1}^r \Sigma_{K_{w_i}}^2 \times \mathbf{G}^{un}(\mathbb{A}_F^{f;v_1, \dots, v_r}) \right] / \mathbf{G}^{un}(F)$$

and

$$[\mathfrak{h}^r \times \mathbf{G}^{int}(\mathbb{A}_F^f)] / \mathbf{G}^{int}(F) \hookrightarrow [\mathfrak{h}^r \times (\mathcal{G}_1 \backslash (\mathbf{G}^{int})^{un}(\mathbb{A}_F^f))] / (\mathbf{G}^{int})^{un}(F)$$

define by GAGA equivariant embeddings $X \hookrightarrow \mathcal{G}_1 \backslash X^{un}$ and $i : X_{\mathbb{C}}^{int} \hookrightarrow \mathcal{G}_1 \backslash (X^{int})_{\mathbb{C}}^{un}$.

To show that i is E_v -rational we observe that the natural embedding $\mathbf{H}^{int} \hookrightarrow (\mathbf{H}^{int})^{un}$ maps homomorphism h to the conjugate of $(h^{un})'$ (in the notation of Remark 3.13). Therefore the natural embedding $X^{int} \hookrightarrow ((X^{int})^{un})'$ is E_v -rational (see [8, Cor. 5.4]). Dividing by \mathcal{G}_1 we get our statement from Remark 3.13. (Alternatively one can check it directly by calculating the action of the Galois group on the set of special points (as in Step 4 of Section 3, where the unitary case is treated)).

Next we observe that after dividing by the center, we have $\mathbf{P}\mathbf{G} \cong \mathbf{P}\mathbf{G}^{un}$ and $\mathbf{P}\mathbf{G}^{int} \cong \mathbf{P}(\mathbf{G}^{int})^{un}$, implying that the above embeddings of schemes induce isomorphisms

$$Z(\tilde{\mathcal{G}}) \backslash X \xrightarrow{\sim} Z(\tilde{\mathcal{G}}^{un}) \backslash X^{un}$$

and

$$Z(\mathbf{G}^{int}(\mathbb{A}_F^f)) \backslash X^{int} \xrightarrow{\sim} Z((\mathbf{G}^{int})^{un}(\mathbb{A}_F^f)) \backslash (X^{int})^{un}.$$

Let $\varphi^{un} : X^{un} \xrightarrow{\sim} (X^{int})^{un}$ be a $\tilde{\mathcal{G}}^{un}$ -equivariant isomorphism as in the First Main Theorem. It gives us a $\mathcal{G}_1 \backslash \tilde{\mathcal{G}}^{un}$ -equivariant isomorphism

$$\bar{\varphi} := \mathcal{G}_1 \backslash \varphi^{un} : \mathcal{G}_1 \backslash X^{un} \xrightarrow{\sim} \mathcal{G}_1 \backslash (X^{int})^{un}.$$

Fix an $x \in X \subset \mathcal{G}_1 \backslash X^{un}$. As it was explained above, there exists $g \in Z(\tilde{\mathcal{G}}^{un})$ such that

$$g\bar{\varphi}(x) \in X^{int} \subset \mathcal{G}_1 \backslash (X^{int})^{un}.$$

Hence $g\bar{\varphi}$ maps the $\tilde{\mathcal{G}}$ -orbit of x into X^{int} . Since each $\mathbf{G}^{int}(\mathbb{A}_F^f) (= \tilde{\mathcal{G}})$ -orbit is Zariski dense in both X and X^{int} (see [38, Prop. 1.3.8 and 1.5.3]), $g\bar{\varphi}$ restricts to a $\tilde{\mathcal{G}}$ -equivariant isomorphism $\varphi : X \xrightarrow{\sim} X^{int}$. The existence of the liftings φ_V and φ_P is now an immediate consequence of the Second Main Theorem. q.e.d.

6. On the generality of our results

In this section we will show that the Shimura varieties treated in this paper are the most general one (up to a central modification) which can be p -adically uniformized by the product of Drinfeld upper half-spaces.

Let $\tilde{\mathbf{H}}$ be a reductive group over \mathbb{Q} , and let \tilde{M} be a conjugacy class of homomorphisms $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow \tilde{\mathbf{H}}_{\mathbb{R}}$ satisfying Deligne's axioms. Let $E' \subset \mathbb{C}$ be the reflex field of $(\tilde{\mathbf{H}}, \tilde{M})$, and let v' be a finite prime of E' . Let X' be the canonical model of the Shimura variety corresponding to $(\tilde{\mathbf{H}}, \tilde{M})$. Suppose that there exists $T \in \mathcal{F}(\tilde{\mathbf{H}}(\mathbb{A}^f))$ such that the projection $X' \rightarrow T \backslash X'$ is étale and that some connected component of $(T \backslash X' \otimes_{E'} E'_{v'})^{an}$ is a finite étale cover of an analytic space of the form $\Gamma \backslash \prod_{i=1}^r \Omega_{E_i}^d \otimes_{E_i} E'_{v'}$ for some p -adic subfields E_1, \dots, E_r of $E'_{v'}$, some natural number $d \geq 2$, and some irreducible arithmetic torsion-free cocompact lattice $\Gamma \subset \prod_{i=1}^r \mathbf{PGL}_d(E_i)$ (compare Remark 2.12). Recall that when $r(d-1) \geq 2$, that is in all cases except for curves, the arithmeticity follows from the irreducibility.

Claim 6.1. $\mathbf{P}\tilde{\mathbf{H}}$ is isomorphic to the group $\mathbf{R}_{F/\mathbb{Q}}\mathbf{PGU}(D^{int}, \alpha^{int})$ for some F , D^{int} and α^{int} satisfying the assumptions of Section 2.

Proof. Since Γ is arithmetic, there exist F , \mathbf{G}' and v_1, \dots, v_r as in the proof of Corollary 3.14. In particular, \mathbf{G}' is a form of \mathbf{PGL}_d , F is totally real, and each $\mathbf{G}'_{\mathbf{F}_{\infty_i}}$ is isomorphic to \mathbf{PGU}_d , the unique \mathbb{R} -anisotropic form of \mathbf{PGL}_d .

Moreover, by the classification of simple algebraic groups (see [37]), $\mathbf{G}' \cong \mathbf{PGU}(D, \alpha)$ for some central simple algebra D of dimension d^2 over a totally imaginary quadratic extension K over F and some involution of the second kind α of D over F . Since $\mathbf{G}'_{\mathbf{F}_{v_i}} \cong \mathbf{PGL}_d$ for each $i = 1, \dots, r$, the v_i 's split in K , and D splits at each prime w_i of K lying over v_i . Hence D and α satisfy the assumptions of Section 2.

Let D^{int} and α^{int} correspond to D and α , as in Section 2, and set $(\mathbf{G}')^{int} := \mathbf{PGU}(D^{int}, \alpha^{int})$. Our assumptions together with First Main Theorem imply that $(B^{d-1})^r$ is the universal cover of some connected component \tilde{X}_T^0 of $(T \backslash X' \otimes_{E'} \mathbb{C})^{an}$. Furthermore, the fundamental group Γ_1 of \tilde{X}_T^0 is commensurable with

$$\Gamma_2 := (\mathbf{G}')^{int}(F)_+ \cap S \subset \mathbf{PGU}_{d-1,1}(\mathbb{R})^r$$

for every $S \in \mathcal{F}((\mathbf{G}')^{int}(\mathbb{A}_F^f))$. In particular, the arithmetic lattice $\Gamma_1 \subset \mathbf{P}\tilde{\mathbf{H}}(\mathbb{R})$ is irreducible. By Deligne's assumption, $\mathbf{P}\tilde{\mathbf{H}}$ has no \mathbb{Q} -rational \mathbb{R} -anisotropic factors, so it is \mathbb{Q} -simple. Hence there exists a

number field F' and an adjoint absolutely simple group $\tilde{\mathbf{G}}$ over F' such that $\mathbf{P}\tilde{\mathbf{H}} \cong \mathbf{R}_{F'/\mathbb{Q}}\tilde{\mathbf{G}}$.

Notice now that Γ_1 and Γ_2 are Zariski dense in $\tilde{\mathbf{G}}$ and $(\mathbf{G}')^{\text{int}}$ respectively (see [21, I, Prop. 3.2.10]). Since Γ_1 and Γ_2 are commensurable, we therefore see that $F' = F$ (using [21, VIII, Prop. 3.22]) and that $\tilde{\mathbf{G}} \cong (\mathbf{G}')^{\text{int}}$ (using [21, I, 0.11]). Hence $\mathbf{P}\tilde{\mathbf{H}} \cong \mathbf{R}_{F/\mathbb{Q}}(\mathbf{G}')^{\text{int}}$, as claimed. q.e.d.

Appendix A. The Hirzebruch proportionality principle

The following proposition generalizes [19, Thm. 2.2.8].

Proposition A.1. *Let X_Γ be a projective scheme over a p -adic field L such that $X_\Gamma^{\text{an}} \cong \Gamma \backslash \prod_{i=1}^r \Omega_{L_i}^d \otimes_{L_i} L$ for p -adic subfields L_1, \dots, L_r of L , a natural number $d \geq 2$, and a torsion-free cocompact lattice $\Gamma \subset \prod_{i=1}^r \mathbf{PGL}_d(L_i)$ (see Lemma 2.9). Then for any positive integers i_1, \dots, i_l such that $i_1 + \dots + i_l = r(d-1)$ we have*

$$c_{i_1} \dots c_{i_l}(T(X_\Gamma)) = \chi_E(\Gamma) \cdot c_{i_1} \dots c_{i_l}(T(\mathbb{P}^{d-1})^r),$$

where by $c_{i_1} \dots c_{i_l}$ we denote the corresponding Chern number, and by $\chi_E(\Gamma)$ the Euler-Poincaré characteristic of Γ .

Proof. The proof is essentially the same as that of Kurihara, so we will content ourselves with a brief sketch. In addition we give a proof of a key lemma which Kurihara merely states.

Let R and $\kappa \cong \mathbb{F}_q$ be the ring of integers and the residue field of L respectively.

Lemma A.2 ([19, Lem. 4.6.1]). *Let Y be a flat projective scheme over R which is locally a complete intersection over R of relative dimension n . Then for any positive integers i_1, \dots, i_l such that $i_1 + \dots + i_l = n$ we have*

$$(A.1) \quad c_{i_1} \dots c_{i_l}(T_{Y_\eta/L}) = \sum_E \text{length}(\mathcal{O}_{Y_0, e}) c_{i_1} \dots c_{i_l}(\iota_E^*(T_{Y_0/\kappa})),$$

where Y_η (resp. Y_0) is the generic (resp. the special) fiber of Y , E runs over irreducible components of Y_0 , e denotes the generic point of E , ι_E denotes the closed immersion $E \hookrightarrow Y_0$, and $T_{Y_\eta/L}$ (resp. $T_{Y_0/\kappa}$) denotes the virtual tangent bundle of Y_η/L (resp. of Y_0/κ).

The virtual tangent bundle of a locally complete intersection morphism $Z_1 \rightarrow Z_2$ is a certain element in the Grothendieck group $K^0(Z_1)$ of vector bundles on Z_1 , defined for example in [12, App. B.7.6].

Proof. Let $i : \text{Spec } \kappa \hookrightarrow \text{Spec } R$ and $j : \text{Spec } L \hookrightarrow \text{Spec } R$ be the natural closed regular and open embeddings respectively. Let $T_{Y/R}$ be the virtual tangent bundle of Y/R . Then $T_{Y_\eta/L} = j^*T_{Y/R}$ and $T_{Y_0/\kappa} = i^*T_{Y/R}$. Therefore for the proof of the lemma it will suffice to show that for any l classes $F_1, \dots, F_l \in K^0(Y)$ we have

$$c_{i_1}(j^*F_1) \dots c_{i_l}(j^*F_l) = \sum_E \text{length}(\mathcal{O}_{Y_0, \epsilon}) c_{i_1}(\iota_E^*(i^*F_1)) \dots c_{i_l}(\iota_E^*(i^*F_l)).$$

Since Grothendieck group is generated by vector bundles, we may suppose that all the F_i 's are vector bundles. We will use the notation and results from [12].

Recall that the map j^* from cycles on Y to cycles on Y_η is surjective and that the specialization map σ from cycles on Y_η to cycles on Y_0 defined by $\sigma(j^*\beta) := i^!(\beta)$ is well-defined (see [12, 20.3]). Then $\sigma[Y_\eta] = [Y_0] = \sum_E \text{length}(\mathcal{O}_{Y_0, \epsilon}) (\iota_E)_*[E]$ and $\deg(\sigma(\alpha)) = \deg(\alpha)$ for each 0-cycle α on Y_η . Moreover, by the commutativity of Chern classes with Gysin maps ([12, Prop. 6.3]) and with a flat pull-back ([12, Thm. 3.2.(d)]), we obtain that

$$\begin{aligned} \sigma(c_m(j^*F) \cap \alpha) &= \sigma(c_m(j^*F) \cap j^*\beta) = \sigma(j^*(c_m(F) \cap \beta)) \\ &= i^!(c_m(F) \cap \beta) = c_m(i^*F \cap i^!\beta) = c_m(i^*F \cap \sigma(\alpha)) \end{aligned}$$

for each vector bundle F on Y , each cycle $\alpha = j^*(\beta)$ on Y_η , and each $m \in \mathbb{N}$. By induction this implies that for any l vector bundles F_1, \dots, F_l on Y we have

$$\begin{aligned} &\sigma(c_{i_1}(j^*F_1) \cap \dots \cap c_{i_l}(j^*F_l) \cap [Y_\eta]) \\ &= \sum_E \text{length}(\mathcal{O}_{Y_0, \epsilon}) c_{i_1}(i^*F_1) \cap \dots \cap c_{i_l}(i^*F_l) \cap (\iota_E)_*[E]. \end{aligned}$$

Using the projection formula [12, Thm. 3.2(c)] and taking degrees of both sides, we obtain the required equality. \square q.e.d.

In our case, X_Γ is a generic fiber of a scheme \tilde{X}_Γ satisfying the condition of the lemma (see [27, App. 1]). Each two irreducible components of the special fiber $\tilde{X}_{\Gamma, 0}$ have isomorphic neighborhoods, which

are moreover isomorphic to the product of r neighborhoods of components in the case of one factor. Therefore all the summands from the right-hand side of (A.1) are equal. Using [36, Thm. 7] we see that the number of irreducible components in $\tilde{X}_{\Gamma,0}$ is

$$\left(\frac{d}{(1-q)(1-q^2) \cdots (1-q^{d-1})} \right)^r \chi_E(\Gamma).$$

Hence to prove the proposition it remains to show that

$$\begin{aligned} & c_{i_1} \cdots c_{i_r}(T(\mathbb{P}^{d-1})^r) \\ &= \left(\frac{d}{(1-q)(1-q^2) \cdots (1-q^{d-1})} \right)^r c_{i_1} \cdots c_{i_r}(i_E^*(T_{\tilde{X}_{\Gamma,0}/\mathbb{F}_q})). \end{aligned}$$

By the multiplicativity of Chern classes, this follows from the corresponding formula [19, (4.6.3)] in the case of one factor. q.e.d.

Remark A.3. The proposition can be considered as a numerical evidence for First Main Theorem in two different ways:

1) It gives evidence (using the classical Hirzebruch proportionality [15]) for Corollary 3.14 and thus to the results of Step 2 of Section 3 (see Remark 3.17). Moreover in the case $r = 1$ it actually implies them (see Remark 3.16).

2) Using the proposition one can show (by the arguments of [38, 2.6]) that the volumes (=top Chern classes) of Shimura variety and of the corresponding p -adically uniformized variety are equal.

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