J. DIFFERENTIAL GEOMETRY
48 (1998) 61-90

THE QUANTUM COHOMOLOGY OF BLOW-UPS OF $\mathbb{P}^2$ AND ENUMERATIVE GEOMETRY

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1. Introduction

The enumerative geometry of curves in algebraic varieties has taken a new direction with the appearance of Gromov-Witten invariants and quantum cohomology. Gromov-Witten invariants originate in symplectic geometry and were first defined in terms of pseudo-holomorphic curves. In algebraic geometry, these invariants are defined using moduli spaces of stable maps.

Let X be a nonsingular projective variety over $\mathbb{C}$. Let $\beta \in H_2(X, \mathbb{Z})$. In [13], the moduli space $\overline{M}_{0,n}(X, \beta)$ of stable n-pointed genus 0 maps is defined. This moduli space parametrizes the data $[\mu : C \to X, p_1, \ldots, p_n]$ where C is a connected, reduced, (at worst) nodal curve of genus 0, $p_1, \ldots, p_n$ are nonsingular points of C, and μ is a morphism. $\overline{M}_{0,n}(X, \beta)$ is equipped with n morphisms $\rho_1, \ldots, \rho_n$ to X where

$$\rho_i([\mu : C \to X, p_1, \ldots, p_n]) = \mu(p_i).$$

X is a convex variety if $H^1(\mathbb{P}^1, f^*(T_X)) = 0$ for all maps $f : \mathbb{P}^1 \to X$. In this case, $\overline{M}_{0,n}(X, \beta)$ is a projective scheme of pure expected dimension equal to

$$dim(X) + n - 3 + \int_\beta c_1(T_X)$$

Received November 18, 1996. The second author was partially supported by an NSF post-doctoral fellowship.

Key words and phrases. Quantum cohomology, Gromov-Witten invariants.

with only finite quotient singularities. Given classes $\gamma_1, \ldots, \gamma_n$ in $H^*(X, \mathbb{Z})$, the Gromov-Witten invariants $I_\beta(\gamma_1 \ldots \gamma_n)$ are defined by:

$$I_\beta(\gamma_1 \ldots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \ldots \cup \rho_n^*(\gamma_n).$$

The intuition behind these invariants is as follows. If the γ_i are the cohomology classes of subvarieties $Y_i \subset X$ in general position, then $I_\beta(\gamma_1 \ldots \gamma_n)$ should count the (possibly virtual) number of irreducible rational curves C in X of homology class β which intersect all the Y_i. In case X is a homogeneous space, a correspondence between the Gromov-Witten invariants and the enumerative geometry of rational curves in X can be proven by transversality arguments (see [9]).

One can use the Gromov-Witten invariants to define the big quantum cohomology ring $QH^*(X)$ of X. The associativity of this ring yields relations among the invariants $I_\beta(\gamma_1 \ldots \gamma_n)$ which often are sufficient to determine them all recursively from a few basic ones. The model case for this approach is the recursive determination of the numbers N_d of nodal rational curves of degree d in the projective plane [13], [17].

If X is not convex, the moduli space $\overline{M}_{0,n}(X, \beta)$ in general will not have the expected dimension. Recently, Gromov-Witten invariants have been defined and proven to satisfy basic geometric properties via the construction of virtual fundamental classes of the expected dimension [2], [1], [15] and, in the symplectic context, [16], [8], [18]. In particular, these Gromov-Witten invariants have been proven to satisfy the axioms of [13], [3]. Therefore, they again define an associative quantum cohomology ring $QH^*(X)$.

The aim of this paper is to study the Gromov-Witten invariants of the blow-up X_r of $\mathbb{P}^2$ in a finite set $x_1, \ldots x_r$ of points and to give enumerative applications. X_r is a particularly simple example of a nonconvex variety, so this study (at least in the context of algebraic geometry) necessitates the use of the above constructions. Let S be a nonsingular, rational, projective surface. S is either deformation equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $X_{r(S)}$ where $r(S)+1 = rank(A^1(S))$. Together with the invariants of $\mathbb{P}^1 \times \mathbb{P}^1$, the Gromov-Witten invariants of X_r therefore determine the invariants of all these rational surfaces (the invariants are constant in flat families of nonsingular varieties). For enumerative applications, it is necessary to consider the blow-up X_r of $\mathbb{P}^2$ in a finite set of *general* points.

Let H be the pull-back to X_r of the hyperplane class in $\mathbb{P}^2$, and let $E_1, \ldots, E_r$ be the exceptional divisors. Our aim is to count the number of irreducible rational curves C in X_r of class $dH - \sum_{i=1}^{r} a_i E_i$ passing through $3d - \sum_{i=1}^{r} a_i - 1$ general points. By associating to a curve in $\mathbb{P}^2$ its strict transform in X_r, this number can also be interpreted as the number of irreducible rational curves in $\mathbb{P}^2$ having singularities of order a_i at the (fixed) general points x_i and passing through $3d - \sum_{i=1}^{r} a_i - 1$ other general points.

The paper is naturally divided into two parts. First, we use the associativity of the quantum product to show that the Gromov-Witten invariants of X_r can be computed from simple initial values by means of explicit recursion relations. There are $r+1$ initial values required for X_r:

(i) The number of lines in the plane passing through 2 points, $N_{1,(0,\dots,0)} = 1$.

(ii) The number of curves in the exceptional class E_i, $N_{0,-[i]} = 1$.

The relations are then used to prove properties of these invariants.

In the second half of the paper, the enumerative significance of the invariants is investigated. Our main tool is a degeneration argument in which the points x_i are specialized to lie on a nonsingular cubic in $\mathbb{P}^2$. The idea of using such degenerations is due independently to J. Kollár and, in joint work, to L. Caporaso and J. Harris [4]. For a general blow-up X_r, the Gromov-Witten invariants are proven to be a count (with possible multiplicities) of the finite number of solutions to the corresponding enumerative problem on X_r. Let $\beta = dH - \sum_{i=1}^{r} a_i E_i$ be a class in $H_2(X_r, \mathbb{Z})$. If the expected dimension of the moduli space $\overline{M}_{0,0}(X_r, \beta)$ is strictly positive or if there exists a multiplicity $a_i \in \{1, 2\}$, then the corresponding Gromov-Witten invariant is proven to be an actual count of the number of irreducible, degree d, rational plane curves of multiplicity a_i at the (fixed) general points x_i which pass through $3d - \sum_{i=1}^{r} a_i - 1$ other general points. In the Del Pezzo case ($r \leq 8$), all invariants are shown to be enumerative (see also [17]). A basic symmetry of the Gromov-Witten invariants of the spaces X_r obtained from the classical Cremona transformation is discussed in Section 5.1. These considerations show that for $d \leq 10$, the Gromov-Witten invariants always coincide with enumerative geometry. Tables of these invariants in low degrees are given in Section 5.2.

In [13], an associativity equation for Del Pezzo surfaces (corresponding to our relation $R(m)$) is derived. The small quantum cohomology ring of Del Pezzo surfaces is studied in [6]. In Section 11 of [6], the associativity of the small quantum product on X_r is used to derive some relations among the Gromov-Witten invariants of these surfaces. The invariants of $\mathbb{P}^2$ blown-up in a point are computed in [5], [10], and [11]. In [10], A. Gathmann computes more generally the invariants of the blow-up of $\mathbb{P}^n$ in a point and studies their enumerative significance. In [7], the Gromov-Witten invariants of X_6 are computed via associativity. Our recursive strategy for X_6 differs.

The first author would like to thank K. Hulek for suggesting to him the possibility of studying the quantum cohomology of blow-ups, and also the University of Chicago and in particular W. Fulton for an invitation which made this collaboration possible. The second author thanks J. Kollár for many remarkable insights on these questions. In particular, the main immersion result (Lemma 4.9) was first observed by him. Thanks are due to J. Harris for conversations in which the degeneration argument was explained. We also thank B. Fantechi and T. Graber for helpful discussions.

Both authors thank the Mittag-Leffler-Institut for support.

2. Notation and background material

Let X be a nonsingular projective variety. Assume for simplicity that the Chow and homology rings of X coincide. Let $dim(X)$ be the complex dimension. Denote by $\alpha \cup \beta$ the cup product of classes $\alpha, \beta \in H^*(X, \mathbb{Z})$ and let $(\alpha \cdot \beta) = \int_X \alpha \cup \beta$. By definition, $(\alpha \cdot \beta)$ is zero if $\alpha \in H^{2i}(X, \mathbb{Z})$, $\beta \in H^{2j}(X, \mathbb{Z})$, and $i + j \neq dim(X)$.

We recall the definition of quantum cohomology from [13] in a slightly modified form for nonconvex varieties. Let $B \subset H_2(X, \mathbb{Z})$ be the semigroup of non-negative linear combinations of classes of algebraic curves. Let $\beta \in H_2(X, \mathbb{Z})$. Let $n_\beta = dim(X) + \int_\beta c_1(T_X) - 3$. Let $n \geq 0$. For classes $\gamma_i \in H^{2j_i}(X, \mathbb{Z})$ with $\sum_{i=1}^n j_i = n_\beta + n$, let $I_\beta(\gamma_1 \ldots \gamma_n)$ be the corresponding Gromov-Witten invariant:

$$I_\beta(\gamma_1 \ldots \gamma_n) = \int_{[\overline{M}_{0,n}(X,\beta)]} \rho_1^*(\gamma_1) \cup \ldots \cup \rho_n^*(\gamma_n),$$

where $[\overline{M}_{0,n}(X, \beta)]$ is the virtual fundamental class. Note that if $n_\beta = 0$ and $n = 0$, then I_β is just the degree of the fundamental class. Kontsevich and Manin introduced a set of axioms for the Gromov-Witten invariants which have now been established for nonsingular projective varieties (see Section 1). If $\overline{M}_{0,n}(X,\beta)$ is empty, then $I_\beta(\gamma_1 \dots \gamma_n) = 0$; in particular, all invariants vanish for $\beta \notin B$. Let $T_0 = 1, T_1, \dots, T_m$ be a homogeneous $\mathbb{Z}$-basis for $H^*(X,\mathbb{Z})$. We assume that $T_1, \dots, T_p$ form a basis of $H^2(X,\mathbb{Z}) = Pic(X)$. We denote by $T_i^\vee$ the corresponding elements of the dual basis: $(T_i \cdot T_j^\vee) = \delta_{ij}$. Denote by (g_{ij}) the matrix of intersection numbers $(T_i \cdot T_j)$ and by (g^{ij}) the inverse matrix. For variables $y_0, q_1, \dots, q_p, y_{p+1}, \dots, y_m$ (abbreviated to q, y), define the formal power series

$$\Gamma(q,y) = \sum_{n_{p+1}+\dots+n_m \geq 0} \sum_{\beta \in B \setminus \{0\}} I_\beta(T_{p+1}^{n_{p+1}} \cdots T_m^{n_m}) \cdot q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p} \frac{y_{p+1}^{n_{p+1}} \cdots y_m^{n_m}}{n_{p+1}! \cdots n_m!} \tag{2.0.1}$$

in the ring

$$\mathbb{Q}[[q,q^{-1},y]] = \mathbb{Q}[[y_0, q_1, \dots, q_p, q_1^{-1}, \dots, q_p^{-1}, y_{p+1}, \dots, y_m]].$$

In case X is a homogeneous space, the substitution $q_i = e^{y_i}$ in (2.0.1) yields a formal power series which equals the quantum part of the potential function of [13] modulo a quadratic polynomial in the variables $y_1, \dots, y_m$. The form (2.0.1) of the potential function is chosen to avoid convergence issues in the nonconvex case. Let

$$\partial_i = \begin{cases} q_i \frac{\partial}{\partial q_i} & i = 1, \dots, p, \\ \frac{\partial}{\partial y_i} & i = 0, p+1, \dots, m, \end{cases}$$

and denote $f_{ijk} = \partial_i \partial_j \partial_k f$ for $f \in \mathbb{Q}[[q,q^{-1},y]]$. Define a $\mathbb{Q}[[q,q^{-1},y]]$-algebra structure on the free $\mathbb{Q}[[q,q^{-1},y]]$-module generated by $T_0, \dots, T_m$ by:

$$T_i * T_j = T_i \cup T_j + \sum_{e,f=0}^{m} \Gamma_{ije} g^{ef} T_f.$$

By definition, this is the quantum cohomology ring of X, $QH^*(X)$.

We sketch the proof of the associativity of this quantum product following [13] and [9]. First, a formal calculation (using the axiom of divisor) yields:

$$\Gamma_{ijk} = \sum_{n \geq 0} \sum_{\beta \in B \setminus \{0\}} \frac{1}{n!} I_\beta(\gamma^n \cdot T_i T_j T_k) q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p}, \tag{2.0.2}$$

where $\gamma = y_{p+1}T_{p+1} + \ldots + y_m T_m$, and the $\mathbb{Q}[[y_0, y_{p+1}, \ldots, y_m]]$-linear extension of I_β is used. Define the symbol Φ_{ijk} by

$$\Phi_{ijk} = I_0(T_iT_jT_k) + \Gamma_{ijk}.$$

In case X is homogeneous, Φ_{ijk} is the partial derivative of the full potential function. The $*$-product can be expressed by:

$$T_i * T_j = \sum_{e,f=0}^{m} \Phi_{ije} g^{ef} T_f.$$

Let

$$F(i,j|k,l) = \sum_{e,f=0}^{m} \Phi_{ije} g^{ef} \Phi_{fkl}.$$

Associativity is now equivalent to $F(i,j|k,l) = F(j,k|i,l)$. Following [9], we let

$$\begin{aligned} &G(i,j|k,l)_{\beta,n} \\ (2.0.3)\qquad &= \sum \binom{n}{n_1} g^{ef} I_{\beta_1}(\gamma^{n_1} \cdot T_iT_jT_e) I_{\beta_2}(\gamma^{n_2} \cdot T_kT_lT_f), \end{aligned}$$

where the sum runs over all $n_1, n_2 \geq 0$ with $n_1 + n_2 = n$ and all $\beta_1, \beta_2 \in B$ with $\beta_1 + \beta_2 = \beta$. As before, $\gamma = y_{p+1}T_{p+1} + \ldots + y_m T_m$. A calculation using equations (2.0.2) and (2.0.3) yields:

$$F(i,j|k,l) = \sum_{\beta \in B} q_1^{\int_\beta T_1} \ldots q_p^{\int_\beta T_p} \sum_{n \geq 0} \frac{1}{n!} G(i,j|k,l)_{\beta,n}.$$

On the other hand, we can use the splitting axiom and linear equivalence on $\overline{M}_{0,4} = \mathbb{P}^1$ to see that $G(i,j|k,l)_{\beta,n} = G(j,k|i,l)_{\beta,n}$, and thus the associativity follows.

3. Quantum cohomology of blow-ups of $\mathbb{P}^2$

Notation 3.1. Let $r \geq 0$. Let X_r be the blowup of $\mathbb{P}^2$ in r general points $x_1, \ldots, x_r$. Denote by $H \in H_2(X, \mathbb{Z})$ the hyperplane class, and by E_i, for $i = 1, \ldots, r$, the exceptional divisors. Let $m = r + 2$, and $T_0 = 1$. Let T_1, T_{i+1} (for $i = 1, \ldots, r$), and T_m be the Poincaré dual cohomology classes of H, E_i and the class of a point respectively. Let

$\epsilon_1 = 1$ and $\epsilon_i = -1$ for $i = 2, \dots, r+1$. Then, $T_0^\vee = T_m$ and $T_i^\vee = \epsilon_i T_i$ for $i = 1, \dots, r+1$. For an r-tuple $\alpha = (a_1, \dots, a_r)$ of integers, denote by (d, α) the class $dH - \sum_{i=1}^r a_i E_i$. Let $|\alpha| = \sum_i a_i$, and let $n_{d,\alpha} = 3d - |\alpha| - 1$ be the expected dimension of the moduli space $\overline{M}_{0,0}(X_r, (d, \alpha))$. If $n_{d,\alpha} \geq 0$, let

$$N_{d,\alpha} = I_{(d,\alpha)}(T_m^{n_{d,\alpha}})$$

be the corresponding Gromov-Witten invariant. When writing $N_{d,\alpha}$ for a sequence α of length r, we will always mean the Gromov-Witten invariant on X_r.

The components of the finite sequences α, β, γ are denoted by the corresponding roman letters a_i, b_i, c_i. For any r, we write $[i]_r$ for the sequence $(j_1, \dots, j_r)$ with $j_k = \delta_{ik}$. We just write $[i]$ if r is understood. For a sequence $\beta = (b_1, \dots, b_{r-1})$, we denote by (β, k) the sequence obtained by adding $b_r = k$. For a permutation σ of $\{1, \dots, r\}$, denote by α_σ the sequence $(a_{\sigma(1)}, \dots, a_{\sigma(r)})$. For an integer k, we write $\alpha \geq k$ to mean that $a_i \geq k$ for all i.

The invariants $N_{1,(0,\dots,0)}$ and $N_{0,-[i]_r}$ are first determined. A result relating virtual and actual fundmental classes is needed. Let $\overline{M}^*_{0,0}(X, \beta)$ denote the open locus of automorphism-free maps ($\overline{M}^*_{0,0}(X, \beta)$ is a fine moduli space).

Proposition 3.2. *If $\overline{M}_{0,0}(X, \beta) = \overline{M}^*_{0,0}(X, \beta)$ and the moduli space is of pure expected dimension, then the virtual fundamental class is the ordinary scheme theoretic fundamental class $[\overline{M}_{0,0}(X, \beta)]$.*

If, in addition, the expected dimension is 0, then the Gromov-Witten invariant N_β equals the (scheme-theoretic) length of $\overline{M}_{0,0}(X, \beta)$. This result is a direct consequence of the construction in [2].

Lemma 3.3. $N_{1,(0,\dots,0)} = 1$ *and* $N_{0,-[i]_r} = 1$.

Proof. A simple check shows that $\overline{M}_{0,2}(X_r, H) = \overline{M}^*_{0,2}(X_r, H)$. Also, the moduli space is irreducible of dimension 4 and (at least) generically nonsingular. For two general points $p_1, p_2 \in X_r$, $\rho_1^{-1}(p_1) \cap \rho_2^{-2}(p_2)$ consists of one reduced point corresponding to preimage of the unique line connecting the images of p_1 and p_2 in $\mathbb{P}^2$. Hence, $N_{1,(0,\dots,0)} = 1$ by Proposition 3.2.

The moduli space $\overline{M}_{0,0}(X_r, (0, -[i])$ consists of one automorphism-free map $\mu : \mathbb{P}^1 \xrightarrow{\sim} E_i \subset X_r$. The Zariski tangent space to $\overline{M}_{0,0}(X_r, (0, -[i])$ at $[\mu]$ is $H^0(\mathbb{P}^1, N_{X_r}) = 0$ where $N_{X_r} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$

is the normal bundle of the map μ. Hence, $\overline{M}_{0,0}(X_r, (0, -[i])$ is nonsingular and $N_{0,-[i]} = 1$ by Proposition 3.2. q.e.d.

The invariants $N_{d,\alpha}$ will be determined by explicit recursions. In addition, these Gromov-Witten invariants will be shown to satisfy the following geometric properties.

(P1) $N_{0,\alpha} = 0$ unless $\alpha = -[i]$ for some i.

(P2) $N_{d,\alpha} = 0$ if $d > 0$ and any of the a_i is negative.

(P3) $N_{d,\alpha} = N_{d,\alpha_\sigma}$ for any permutation σ of $\{1, \ldots, r\}$.

(P4) $N_{d,\alpha} = N_{d,(\alpha,0)}$. In particular $N_{d,(0,\ldots,0)}$ is the number of rational curves on $\mathbb{P}^2$ passing through $3d-1$ general points computed by recursion in [13].

(P5) If $n_{d,\alpha} > 0$, then $N_{d,\alpha} = N_{d,(\alpha,1)}$.

Remark 3.4. Let Y be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point with exceptional divisor E, and let F, G be the pullbacks of the classes of the fibres of the two projections to $\mathbb{P}^1$. There is an isomorphism $\phi : X_2 \to Y$ with $\phi_*(H) = F + G - E$, $\phi_*(E_1) = F - E$, $\phi_*(E_2) = G - E$. Let (d, α) be given with $r \geq 2$. If $d - a_1 - a_2 \geq 0$, then pushing down first to X_2 and then further to $\mathbb{P}^1 \times \mathbb{P}^1$ gives a bijection between the irreducible rational curves in $|(d, \alpha)|$ on X_r passing through $n_{d,\alpha}$ general points and the irreducible rational curves of bidegree $(d - a_1, d - a_2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, with points of multiplicities $d - a_1 - a_2, a_3, \ldots, a_r$ at $r-1$ general points and passing through $n_{d,\alpha}$ other general points.

We obtain recursion formulas determining the $N_{d,\alpha}$ from the associativity of the quantum product. All effective classes (d, α) on X_r satisfy $\alpha \leq d$. Therefore, we can write

$$\Gamma(q, y) = \sum_{(d,\alpha)} N_{d,\alpha} q_1^d q_2^{a_1} \ldots q_{r+1}^{a_r} \frac{y_m^{n_{d,\alpha}}}{n_{d,\alpha}!},$$

where the sum runs over all $(d, \alpha) \neq 0$ satisfying $n_{d,\alpha} \geq 0$, $d \geq 0$, and $\alpha \leq d$. Let $\Gamma_{ijk} = \partial_i \partial_j \partial_k \Gamma$ (following the notation of Section 2) . The quantum product of T_i and T_j is given by

$$T_i * T_j = (T_i \cdot T_j) T_m + \sum_{k=1}^{r+1} \epsilon_k \Gamma_{ijk} T_k + \Gamma_{ijm} T_0.$$

Lemma 3.5. *For $i, j, k, l \in \{1, \ldots, m\}$, there is a relation:*

$$(R_{i,j,k,l}) \quad (T_i \cdot T_j)\Gamma_{klm} - (T_k \cdot T_j)\Gamma_{ilm} + (T_k \cdot T_l)\Gamma_{ijm} - (T_i \cdot T_l)\Gamma_{kjm} = \sum_{s=1}^{m-1} \epsilon_s(\Gamma_{jks}\Gamma_{isl} - \Gamma_{ijs}\Gamma_{ksl}).$$

Proof. We write

$$(T_i * T_j) * T_k - (T_k * T_j) * T_i = \sum_{l=0}^{m} r_{i,j,k,l} T_l^\vee.$$

By associativity, we obtain the relation $r_{i,j,k,l} = 0$. We show this relation is equivalent to $(R_{i,j,k,l})$. We compute directly

$$\begin{aligned}(T_i * T_j) * T_k =& (T_i \cdot T_j)T_m * T_k + \sum_{s=1}^{m-1} \epsilon_s \Gamma_{ijs} T_s * T_k + \Gamma_{ijm} T_k \\ =& \sum_{l=1}^{m} (T_i \cdot T_j)\Gamma_{klm} T_l^\vee \\ &+ \sum_{s=1}^{m-1} \left(\epsilon_s \Gamma_{ijs}(T_s \cdot T_k)T_m + \sum_{l=1}^{m} \epsilon_s \Gamma_{ijs}\Gamma_{ksl} T_l^\vee \right) \\ &+ \Gamma_{ijm} T_k.\end{aligned}$$

It is easy to see that

$$\Gamma_{ijm} T_k = \sum_{l=1}^{m} \Gamma_{ijm}(T_k \cdot T_l)T_l^\vee + \Gamma_{ijm}\delta_{km} T_0^\vee,$$

$$\sum_{s=1}^{m-1} \epsilon_s \Gamma_{ijs}(T_s \cdot T_k)T_m = \Gamma_{ijk}(1 - \delta_{km})T_0^\vee.$$

Therefore, the sum of these two terms is just

$$\sum_{l=1}^{m} \Gamma_{ijm}\big((T_k \cdot T_l)T_l^\vee + \Gamma_{ijk} T_0^\vee.$$

Thus

$$\begin{aligned}(T_i * T_j) * T_k =& \sum_{l=1}^{m} \left((T_i \cdot T_j)\Gamma_{klm} + (T_k \cdot T_l)\Gamma_{ijm} + \sum_{s=1}^{m-1} \epsilon_s \Gamma_{ijs}\Gamma_{ksl} \right) T_l^\vee \\ &+ \Gamma_{ijk} T_0^\vee,\end{aligned}$$

and the result follows by exchanging the role of i and k and subtracting. q.e.d.

For the recursive determination of the $N_{d,\alpha}$, only the following relations are needed:

$$(R_{1,1,m,m}) \qquad \Gamma_{mmm} = \sum_{s=1}^{m-1} \epsilon_s (\Gamma_{1sm}^2 - \Gamma_{11s}\Gamma_{smm}),$$

and for all $i = 2 \dots r+1$

$$(R_{1,1,i,i}) \qquad \Gamma_{iim} - \Gamma_{11m} = \sum_{s=1}^{m-1} \epsilon_s (\Gamma_{1is}^2 - \Gamma_{11s}\Gamma_{iis}).$$

Note that in case $r = 0$, only the relation $(R_{1,1,m,m})$ occurs and coincides with that of [13]. In the summations below, the following notation is used. Let the symbol $\vdash(d,\alpha)$ denote the set of pairs $((d_1,\beta),(d_2,\gamma))$ satisfying:

(i) $(d_1,\beta),(d_2,\gamma) \neq 0$,

(ii) $(d_1,\beta) + (d_2,\gamma) = (d,\alpha)$,

(iii) $n_{d_1,\beta}, n_{d_2,\gamma} \geq 0$, $d_1, d_2 \geq 0$, $\beta \leq d_1$, and $\gamma \leq d_2$.

The notation $\vdash(d,\alpha), d_i > 0$ will be used to denote the subset of $\vdash(d,\alpha)$ satisfying $d_1, d_2 > 0$. The binomial coefficient $\binom{p}{q}$ is defined to be zero if $q < 0$ or $p < q$.

Theorem 3.6. *The $N_{d,\alpha}$ are determined by the initial values:*

(i) $N_{1,\underbrace{(0,\dots,0)}_{r}} = 1$, *for all* r,

(ii) $N_{0,-[i]_r} = 1$, *for* $i \in \{1,\dots,r\}$,

and the following recursion relations.

If $n_{d,\alpha} \geq 3$, *then relation* $R(m)$ *holds:*

$$N_{d,\alpha} = \sum_{\vdash(d,\alpha),d_i>0} N_{d_1,\beta} N_{d_2,\gamma} \Big(d_1 d_2 - \sum_{k=1}^{r} b_k c_k \Big) \cdot \left(d_1 d_2 \binom{n_{d,\alpha}-3}{n_{d_1,\beta}-1} - d_1^2 \binom{n_{d,\alpha}-3}{n_{d_1,\beta}} \right).$$

If $n_{d,\alpha} \geq 0$, then for any $i \in \{1, \dots, r\}$ relation $R(i)$ holds:

$$\begin{aligned} d^2 a_i N_{d,\alpha} =&(d^2 - (a_i - 1)^2) N_{d,\alpha-[i]} \\ &+ \sum_{\vdash(d,\alpha-[i]), d_i>0} N_{d_1,\beta} N_{d_2,\gamma} \Big(d_1 d_2 - \sum_{k=1}^{r} b_k c_k \Big) \\ &\qquad \cdot \left(d_1 d_2 b_i c_i - d_1^2 c_i^2 \right) \binom{n_{d,\alpha}}{n_{d_1,\beta}}. \end{aligned}$$

Furthermore, the properties (P1)-(P5) hold.

Proof. From the relation $(R_{1,1,i+1,i+1})$ above, we get immediately (for $n_{d,\alpha} \geq 1$) the recursion formula $R(i)^*$:

$$\begin{aligned} &(a_i^2 - d^2) N_{d,\alpha} \\ &= \sum_{\vdash(d,\alpha)} N_{d_1,\beta} N_{d_2,\gamma} \Big(d_1 d_2 - \sum_{k=1}^{r} b_k c_k \Big) \left(d_1 d_2 b_i c_i - d_1^2 c_i^2 \right) \binom{n_{d,\alpha} - 1}{n_{d_1,\beta}}. \end{aligned}$$

We now show property (P1). If $N_{0,\alpha} \neq 0$, then $(0, \alpha)$ is effective and therefore $\alpha \leq 0$. If $n_{0,\alpha} = 0$ we get $\alpha = -[i]$ for some $i \in \{1, \dots, r\}$. If $n_{0,\alpha} > 0$, we apply $R(i)^*$ for an i with $a_i \neq 0$. We see that all summands on the right side are divisible by $d_1 = 0$, and thus (P1) follows.

The relation $R(m)$ is obtained from $R_{1,1,m,m}$ in two steps. The relation $R_{1,1,m,m}$ immediately yields a recursion relation identical to $R(m)$ except for the fact that the sum is over $\vdash (d, \alpha)$ instead of $\vdash(d, \alpha), d_i > 0$. It will be shown that the terms with $d_1 = 0$ or $d_2 = 0$ vanish. Since all summands are divisible by d_1, only the case $d_2 = 0$ needs be considered. By (P1), either $N_{0,\gamma} = 0$ or $\gamma = -[i]$. In the second case, both binomal coefficients vanish. Thus, relation $R(m)$ follows.

Now we show relation $R(i)$ holds. We apply relation $R(i)^*$ to $N_{d,\alpha-[i]}$. All summands on the right side of $R(i)^*$ are divisible by d_1, thus all nonvanishing summands have $d_1 > 0$. By (P1), $N_{0,\gamma}$ can only be nonzero if $\gamma = -[j]$ for some $j \in \{1, \dots, r\}$. Since the right side of $R(i)^*$ is divisible by c_i, the only nonzero summand on the right side with $d_2 = 0$ occurs for $(d_2, \gamma) = (0, -[i])$ and is $-d^2 a_i N_{d,\alpha}$. Bringing this term to the left side and bringing $((a_i - 1)^2 - d^2) N_{d,\alpha-[i]}$ to the right side, we obtain the relation $R(i)$. Note that $n_{d,\alpha} \geq 0$ implies $n_{d,\alpha-[i]} \geq 1$.

We now show that the invariants $N_{d,\alpha}$ are determined recursively by the relations $R(1), \dots, R(r)$, $R(m)$ and the intial values. By (P1), all

$d = 0$ invariants are determined. Let $d > 0$. If $n_{d,\alpha} \geq 3$, then relation $R(m)$ determines $N_{d,\alpha}$ in terms of $N_{e,\lambda}$ with $e < d$. Assume now that $0 \leq n_{d,\alpha} < 3$. Either $(d, \alpha) = (1, (0, \ldots, 0))$ (and $N_{d,\alpha} = 1$) or there exists an i_0 with $a_{i_0} \neq 0$. By relation $R(i_0)$, we can determine $N_{d,\alpha}$ in terms of $N_{e,\lambda}$ satisfying either $e < d$ or $e = d$ and $n_{d,\lambda} > n_{d,\alpha}$. After at most 3 applications of a suitable $R(i)$, $R(m)$ may be applied. $N_{d,\alpha}$ is then expressed in terms of the intial values and $N_{e,\lambda}$ with $e < d$. This completes the recursion.

Finally, we verify (P2)–(P5). First, (P2) is proven. For $d = 0$, the statement of (P2) is void. Let $d > 0$, and assume by induction that (P2) holds for all $d_0 < d$. Let (d, α) be given with $d > 0$, $a_j < 0$. If $n_{d,\alpha} \geq 3$, we can apply $R(m)$ to express $N_{d,\alpha}$ as a linear combination of products $N_{d_1,\beta} N_{d-d_1,\alpha-\beta}$ with $d_1, d - d_1 > 0$. Furthermore $a_j < 0$ implies $b_j < 0$ or $a_j - b_j < 0$. Therefore, $N_{d,\alpha} = 0$ by induction. If $0 \leq n_{d,\alpha} < 3$, we apply $R(j)$ to express $N_{d,\alpha}$ as a linear combination of $N_{d,\alpha-[j]}$ and terms of the form $N_{d_1,\beta} N_{d-d_1,\alpha-[j]-\beta}$ with $d_1, d - d_1 > 0$. These last terms vanish by induction. Thus $N_{d,\alpha}$ is just a multiple of $N_{d,\alpha-[j]}$. As $n_{d,\alpha-[j]} = n_{d,\alpha} + 1$, we can repeat this process to reduce to the case $n_{d,\alpha} \geq 3$.

(P3) is obvious, as the initial values and the set $R(1), \ldots R(r), R(m)$ of relations are symmetric.

(P4) Let (d, α) be given. We will show that $N_{d,\alpha} = N_{d,(\alpha,0)}$. By (P1) and the intial values, the result holds for $d = 0$. Let $d > 0$ and assume by induction that the result holds for all $d_1 < d$. Case 1: $n_{d,\alpha} \geq 3$. Apply $R(m)$ to express $N_{d,\alpha}$ as a linear combination of terms $N_{d_1,\beta} N_{d-d_1,\alpha-\beta}$ and to express $N_{d,(\alpha,0)}$ as a linear combination of terms $N_{d_1,\beta_0} N_{d-d_1,(\alpha,0)-\beta_0}$ with $d_1, d - d_1 > 0$. (P2) implies, for nonzero terms, that β_0 must be of the form $(\beta, 0)$. Furthermore the coefficient of $N_{d_1,(\beta,0)} N_{d_2,(\gamma,0)}$ in the expression for $N_{d,(\alpha,0)}$ is the same as that of $N_{d_1,\beta} N_{d_2,\gamma}$ in the expression for $N_{d,\alpha}$. Thus the result follows by induction on d.

Case 2: $0 \leq n_{d,\alpha} < 3$. If $\alpha \leq 0$, then (d, α) must be $(1, (0 \ldots, 0))$ and $N_{d,\alpha} = N_{d,(\alpha,0)} = 1$. If there exists an i with $a_i < 0$, then $N_{d,\alpha} = N_{d,(\alpha,0)} = 0$ by (P2). Assume there exists a j with $a_j > 0$. We apply $R(j)$ both to $N_{d,\alpha}$ and $N_{d,(\alpha,0)}$. Then $N_{d,\alpha}$ is expressed as a linear combination of $N_{d,\alpha-[j]}$ and the $N_{d_1,\beta} N_{d-d_1,\alpha-[j]-\beta}$ with $d_1, d - d_1 > 0$. Using (P2), the expression for $N_{d,(\alpha,0)}$ is obtained by replacing $N_{d_1,\beta} N_{d_2,\gamma}$ by $N_{d_1,(\beta,0)} N_{d_2,(\gamma,0)}$ and $N_{d,\alpha-[i]}$ by $N_{d,(\alpha,0)-[i]}$. By induction on d, it is enough to show the result for $N_{d,\alpha-[i]}$. Iterating the argument we reduce to $n_{d,\alpha} \geq 3$ or to $\alpha \leq 0$, where we already showed the result.

(P5) Let (d, α) be given with $n_{d,\alpha} \geq 0$ and $a_j = 1$ for some j. We show that $N_{d,\alpha} = N_{d,\alpha-[j]}$. By (P1), we can assume $d > 0$. We apply relation $R(j)$ to express $N_{d,\alpha}$ as a linear combination of $N_{d,\alpha-[j]}$ and terms $N_{d_1,\beta}N_{d-d_1,\alpha-[j]-\beta}$ with $d_1, d-d_1 > 0$. Furthermore, by (P2), all nonzero terms have $b_j = c_j = 0$. The coefficient of these terms is divisible by c_j. Therefore, $R(j)$ just reads $d^2 N_{d,\alpha} = d^2 N_{d,\alpha-[j]}$. q.e.d.

4. Moduli analysis

4.1. Results

As before, let X_r be the blow-up of $\mathbb{P}^2$ at r general points $x_1, \ldots, x_r$. In this section, the connection between Gromov-Witten invariants and the enumerative geometry of curves in X_r is examined. Let $\alpha = (a_1, \ldots, a_r)$. Let (d, α) denote the class $dH - \sum_{i=1}^{r} a_i E_i$ in $H_2(X_r, \mathbb{Z})$. Let $n_{d,\alpha} = 3d - |\alpha| - 1$ be the expected dimension of the moduli space of maps $\overline{M}_{0,0}(X_r, (d, \alpha))$. If $n_{d,\alpha} \geq 0$, let $N_{d,\alpha}$ be the corresponding Gromov-Witten invariant. In this case, the number of genus 0 stable maps of class (d, α) passing through $n_{d,\alpha}$ general points of X_r is proven to be *finite*. $N_{d,\alpha}$ is then shown to be a count with (possible) multiplicities of the finite solutions to this enumerative problem. Hence, the Gromov-Witten invariant $N_{d,\alpha}$ is always non-negative. An analysis of the moduli space of maps yields a more precise enumerative result.

Theorem 4.1. *Let $n_{d,\alpha} \geq 0$, $d > 0$, and $\alpha \geq 0$. Let (at least) one of the following two conditions hold for the class (d, α):*

(i) $n_{d,\alpha} > 0$.

(ii) $a_i \in \{1, 2\}$ for some i.

Then, $N_{d,\alpha}$ equals the number of genus 0 stable maps of class (d, α) passing through $n_{d,\alpha}$ general points in X_r. Moreover, in this case, each solution map is an immersion of $\mathbb{P}^1$ in X_r.

4.2. Dimension 0 moduli

Three coarse moduli spaces will be considered:

$$M^{\#}_{0,0}(X_r, (d, \alpha)) \subset M_{0,0}(X_r, (d, \alpha)) \subset \overline{M}_{0,0}(X_r, (d, \alpha)).$$

$M_{0,0}(X_r,(d,\alpha))$ is the open set of maps with domain $\mathbb{P}^1$. $M^{\#}_{0,0}(X_r,(d,\alpha))$ is the open set of maps with domain $\mathbb{P}^1$ that are *birational* onto their image. As a first step, these unpointed moduli spaces are shown to be empty when their expected dimensions are negative. As always, X_r is general.

Lemma 4.2. *Let $(d,\alpha)\neq 0$ satisfy $n_{d,\alpha}<0$. Then, $\overline{M}_{0,0}(X_r,(d,\alpha))$ is empty.*

Proof. If $d<0$, $\overline{M}_{0,0}(X_r,(d,\alpha))$ is clearly empty. Next, the case $d=0$ is considered. The only classes $(0,\alpha)\neq 0$ that can be represented by a connected curve are the classes $(0,-k[i])$ for $k\geq 1$. Since $3\cdot 0+k-1\geq 0$, these classes are ruled out by the assumption $n_{d,\alpha}<0$. It can now be assumed that $d>0$.

Let $\mathcal{B}_r$ be the open configuration space of r distinct ordered points on $\mathbb{P}^2$. $\mathcal{B}_r$ is an open set of $\mathbb{P}^2\times\cdots\times\mathbb{P}^2$ (with r factors). Let $\pi:\mathcal{X}_r\to\mathcal{B}_r$ be the universal family of blown-up $\mathbb{P}^2$'s. The fiber of π over the point $b=(b_1,\ldots,b_r)\in\mathcal{B}_r$ is simply $\mathbb{P}^2$ blown-up at $b_1,\ldots,b_r$. The morphism π is projective. Let $\tau:\overline{M}_{0,0}(\pi,(d,\alpha))\to\mathcal{B}_r$ be the relative coarse moduli space of stable maps associated to the family π. The morphism τ is projective. The fiber $\tau^{-1}(b)$ is the corresponding moduli space of maps $\overline{M}_{0,0}(\pi^{-1}(b),(d,\alpha))$ to the fiber $\pi^{-1}(b)$.

Assume that $\overline{M}_{0,0}(X_r,(d,\alpha))$ is nonempty for general X_r. It follows that τ is a dominant projective morphism and thus surjective onto $\mathcal{B}_r$. Let $b=(b_1,\ldots,b_r)\in\mathcal{B}_r$ be r general points on a nonsingular plane cubic $E\subset\mathbb{P}^2$. Let $X_b=\pi^{-1}(b)$. Since τ is surjective, there exists a stable map $\mu:C\to X_b$. By the numerical assumption,

$$C\cdot\mu^*(c_1(T_{X_b}))=3d-|\alpha|=n_{d,\alpha}+1\leq 0.$$

Since the points $b_1,\ldots,b_r$ lie on E, the strict transform of E is a representative of the divisor class $c_1(T_{X_b})$ on X_b. Moreover, since E is elliptic, no component of C surjects upon E. Let $C=\bigcup C_j$ be the decompositon of C into irreducible components. For each C_j, $\mu(C_j)$ is either a point or an irreducible curve in X_b not equal to E. Hence, $C_j\cdot\mu^*(E)\geq 0$. Since

$$\sum_j C_j\cdot\mu^*(E)=C\cdot\mu^*(c_1(T_{X_b}))\leq 0,$$

$C_j\cdot\mu^*(E)=0$ for all components C_j. Since $d>0$, there exists a component C_l such that $\mu(C_l)$ is of class (d_l,α_l) with $d_l>0$. Then,

$\mu(C_l)$ is curve and $\mu(C_l) \cap E = \emptyset$. Now consider the image of $\mu(C_l)$ in $\mathbb{P}^2$ (using the natural blow-down map $X_b \to \mathbb{P}^2$). The image of $\mu(C_l)$ is a degree $d_l > 0$ plane curve meeting E only at the points $b_1, \ldots, b_r$. Hence, there is an equality in the Picard group of E:

$$\mathcal{O}_{\mathbb{P}^2}(d_l)|_E \cong \mathcal{O}_E(\sum_{i=1}^{r} m_i b_i)$$

for some non-negative integers $m_1, \ldots, m_r$. Since $b_1, \ldots, b_r$ were chosen to be general points on E, no such equality can hold. A contradiction is reached and the Lemma is proven q.e.d.

A map $\mu : \mathbb{P}^1 \to X_r$ is simply incident to a point $y \in X_r$ if $\mu^{-1}(y)$ is scheme theoretically a single point in $\mathbb{P}^1$.

Lemma 4.3. *Let (d, α) satisfy $n_{d,\alpha} \geq 0$. Then every map*

$$[\mu] \in \overline{M}_{0,0}(X_r, (d, \alpha))$$

incident to $n_{d,\alpha}$ general points in X_r is a birational map with domain $\mathbb{P}^1$. Morever, every such map is simply incident to the $n_{d,\alpha}$ points.

Proof. Let C be a reducible curve. Assume there exists a genus 0 (unpointed) stable map $\mu : C \to X_r$ representing the class (d, α) incident to $n_{d,\alpha}$ general points. It is first claimed that at least two irreducible components are mapped nontrivially by μ. If no component is mapped to a point, the claim is trivial, otherwise, let K be a maximal connected component of C that is mapped to a point. K must meet the union of the irreducible components mapped nontrivially in at least 3 points. Since C is a tree, these 3 points lie on *distinct* components of C. Let $C_1, \ldots, C_s$ be the irreducible components mapped nontrivially by μ. Let $(d_1, \alpha_1), \ldots, (d_s, \alpha_s)$ be the classes represented by these components. Let p_i be the number of the $n_{d,\alpha}$ general points contained in $\mu(C_i)$. Since

$$n_{d,\alpha} = s - 1 + \sum_{i=1}^{s} n_{d_i,\alpha_i} > \sum_{i=1}^{s} n_{d_i,\alpha_i},$$

and $\sum_{i=1}^{s} p_i \geq n_{d,\alpha}$, it follows that for some j, $p_j > n_{d_j,\alpha_j}$. Let $y_1, \ldots, y_{p_j}$ be the general points contained in $\mu(C_j)$. Let X_{r+p_j} be the blow-up of X_r at these points. Consider the strict transform of the map μ to the map $\mu' : C_j \to X_{r+p_j}$. The class represented by μ' is $\beta = (d_j, (\alpha_j, m_1, \ldots, m_{p_j}))$ where $m_i \geq 1$ for all $1 \leq i \leq p_j$. Therefore

$n_\beta \leq n_{d_j,\alpha_j} - p_j < 0$. By Lemma 4.2, $\overline{M}_{0,0}(X_{r+p_j}, \beta)$ is empty. A contradiction is reached. Hence, no stable maps in $\overline{M}_{0,0}(X_r, (d, \alpha))$ with reducible domains pass through $n_{d,\alpha}$ general points of X_r.

Next, assume there exists a stable map $\mu : \mathbb{P}^1 \to X_r$ passing through $n_{d,\alpha}$ general points, which is not birational onto its image. Let $\mu : \mathbb{P}^1 \to Im(\mu)$ be a generically k-sheeted cover for $k \geq 2$. Let $\gamma : \mathbb{P}^1 \to Im(\mu)$ be a desingularization of the image. The map γ represents the class $(d/k, \alpha/k) \neq (0,0)$ and is incident to the $n_{d,\alpha}$ general points. Note that

$$n_{d/k,\alpha/k} = 3 \cdot \frac{d}{k} - \frac{1}{k}|\alpha| - 1 < n_{d,\alpha}.$$

As before, a contradiction is reached. Hence, the stable maps in $\overline{M}_{0,0}(X_r, (d, \alpha))$ passing through $n_{d,\alpha}$ general points of X_r are birational.

Finally, assume there exists a stable map $\mu : \mathbb{P}^1 \to X_r$ passing through $n_{d,\alpha}$ general points $y_1, \ldots, y_{n_{d,\alpha}}$ which is not simply incident to the point y_1. Let $X_{r+n_{d,\alpha}}$ be the blow-up of X_r at the general points. Then, the strict transform of μ to $X_{r+n_{d,\alpha}}$ represents the class $\beta = (d, (\alpha, m_1, \ldots, m_{n_{d,\alpha}}))$ where $m_i \geq 1$ for all $1 \leq i \leq n_{d,\alpha}$ and $m_1 \geq 2$. Again, $n_\beta \leq n_{d,\alpha} - n_{d,\alpha} - 1 < 0$ and a contradiction is reached. q.e.d.

Corollary 4.4. *Let (d, α) satisfy $n_{d,\alpha} = 0$. Then*

$$\overline{M}_{0,0}(X_r, (d, \alpha)) = M^{\#}_{0,0}(X_r, (d, \alpha)).$$

A scheme Z is of *pure dimension 0* if every irreducible component is a (possibly non-reduced) point. Z may be empty.

Lemma 4.5. *Let (d, α) satisfy $n_{d,\alpha} = 0$. Then, $\overline{M}_{0,0}(X_r, (d, \alpha))$ is of pure dimension 0.*

Proof. By Corollary 4.4, $\overline{M}_{0,0}(X_r, (d, \alpha)) = M^{\#}_{0,0}(X_r, (d, \alpha))$. Let $\mu : \mathbb{P}^1 \to X_r$ correspond to a point $[\mu] \in M^{\#}_{0,0}(X_r, (d, \alpha))$. Consider the normal (sheaf) sequence on $\mathbb{P}^1$ determined by μ:

$$0 \to T_{\mathbb{P}^1} \to \mu^* T_{X_r} \to N_{X_r} \to 0.$$

The sheaf N_{X_r} has generic rank 1 and degree equal to

$$3d - |\alpha| - 2 = n_{d,\alpha} - 1 = -1.$$

There is a canonical torsion sequence:

$$0 \to \tau \to N_{X_r} \to F \to 0.$$

The torsion subsheaf, τ, is supported on the locus where μ fails to be an immersion. F is a line bundle of degree equal to $-1 - dim(\tau)$. It follows that

$$H^0(\mathbb{P}^1, N_{X_r}) = H^0(\mathbb{P}^1, \tau). \tag{4.5.1}$$

Let $\lambda : \mathcal{C} \to M^{\#}_{0,0}(X_r, (d, \alpha))$ be any morphism of an irreducible curve to the moduli space. It will be shown that the image of λ is a point. It can be assumed that $\mathcal{C}$ is nonsingular. Since $M^{\#}_{0,0}(X_r, (d, \alpha))$ is contained in the automorphism-free locus, there exist a universal curve $\pi : \mathcal{P} \to M^{\#}_{0,0}(X_r, (d, \alpha))$ and a universal morphism $\mu : \mathcal{P} \to X_r$ (see [9]). Moreover, π is a $\mathbb{P}^1$-fibration. Let $\pi : S \to \mathcal{C}$ be the pull-back of $\mathcal{P}$ via λ and let $\mu : S \to X_r$ be the induced map. S is a nonsingular surface. Let $d\mu : T_S \to \mu^* T_{X_r}$ be the differential of μ. Let $T_V \subset T_S$ be the line bundle of π-vertical tangent vectors, and let $U \subset S$ be the open set where $d\mu : T_V \to T_{X_r}$ is a bundle injection. The torsion result (4.5.1) directly implies that the bundle map $d\mu : T_S \to T_{X_r}$ is of constant rank 1 on U. Hence, by the complex algebraic version of Sard's theorem, $\mu(S)$ is irreducible of dimension 1. The μ-image of S must equal the μ-image of each fiber of π. It now follows easily that the image of λ is a point. q.e.d.

4.3. The map μ over E_i

The results of the previous section do not show that $\overline{M}_{0,0}(X_r, (d, \alpha))$ is a nonsingular collection of points when $n_{d,\alpha} = 0$. Conditions for nonsingularity will be established in Section 4.4. Preliminary results concerning the the map μ over the exceptional divisors are required. First, the injectivity of the differential over E_i is established.

Lemma 4.6. *Let (d, α) satisfy $n_{d,\alpha} = 0$. Let $\mu : \mathbb{P}^1 \to X_r$ correspond to a point $[\mu] \in \overline{M}_{0,0}(X_r, (d, \alpha))$. Then $d\mu$ is injective at all points in $\mu^{-1}(E_i)$ for all i.*

Proof. Consider again the relative coarse moduli space

$$\tau : \overline{M}_{0,0}(\pi, (d, \alpha)) \to \mathcal{B}_r$$

and the universal family of blown-up $\mathbb{P}^2$'s, $\pi : \mathcal{X}_r \to \mathcal{B}_r$. Let $\mathcal{U}_r \subset \mathcal{B}_r$ denote the open subset to which the conclusions of Corollary 4.4 and

Lemma 4.5 apply. For $b = (b_1, \ldots, b_r) \in \mathcal{B}_r$, let E_i in $\pi^{-1}(b)$ denote the exceptional divisor corresponding to the point b_i. Assume, for a general point $b \in \mathcal{U}_r$, there exists a map $\mu : \mathbb{P}^1 \to \pi^{-1}(b)$ satisfying:

(i) $[\mu] \in \overline{M}_{0,0}(\pi^{-1}(b), (d, \alpha))$.

(ii) There exists a point $p \in \mathbb{P}^1$ such that $d\mu(p) = 0$ and $\mu(p) \in E_i$ for some i.

In this case, there must exist a fixed index j such that for general $b \in \mathcal{U}_r$ the moduli space $\overline{M}_{0,0}(\pi^{-1}(b), (d, \alpha))$ contains a map with vanishing differential at some point over E_j. Let $Y \subset \tau^{-1}(\mathcal{U}_r)$ denote the locus of maps with vanishing differential at some point over E_j. Y is closed in $\tau^{-1}(\mathcal{U}_r)$. Let $\overline{Y}$ denote the closure of Y in $\overline{M}_{0,0}(\pi, (d, \alpha))$. Let $[\mu] \in \overline{Y}$ where $\mu : C \to \pi^{-1}(\tau([\mu]))$. It is easily seen that one of the following two cases hold:

(i) There exists a point $p \in C_{nonsing}$ satisfying $d\mu(p) = 0$ and $\mu(p) \in E_j$.

(ii) There is a node of C mapped to E_j.

These are the two possible degenerations of the singular point of the morphism μ over E_j. Since Y dominates $\mathcal{B}_r$, the map $\overline{Y} \to \mathcal{B}_r$ is surjective.

Define a complete curve $\mathcal{F} \subset \mathcal{B}_r$ as follows. Let the points $e_1, \ldots, e_r$ be distinct points on a nonsingular cubic plane curve $F \subset \mathbb{P}^2$. Choose a zero for the group law on F. Let the curve $\mathcal{F} \subset \mathcal{B}_r$ be determined by elliptic translates of the tuple $(e_1, \ldots, e_r)$. There is a natural map $\epsilon_j : \mathcal{F} \to F$ given by $\epsilon_j(f = (f_1, \ldots, f_r)) = f_j$. Consider the fibration of blown-up $\mathbb{P}^2$'s over $\mathcal{F}$, $\pi^{-1}(\mathcal{F}) \to \mathcal{F}$. Let $S \subset \pi^{-1}(\mathcal{F})$ be the subfibration of $\mathbb{P}^1$'s determined by the exceptional divisor E_j:

$$S \subset \pi^{-1}(\mathcal{F}) \to \mathcal{F}.$$

Via composition with ϵ_j, there is a natural projection $S \to F$. There is a canonical isomorphism $S \stackrel{\sim}{=} \mathbb{P}(T_{\mathbb{P}^2}|_F) \to F$ of varieties over F.

Let $\gamma : \mathcal{D} \to \overline{Y}$ be an irreducible curve that surjects onto $\mathcal{F}$ via τ. After a possible base change, a flat family of stable maps which induces the morphism γ exists over $\mathcal{D}$. (In [9], the moduli space of maps is constructed locally as finite quotient of a fine moduli space of rigidified maps, so a base change with a universal family exists on an open set of $\mathcal{D}$. The properness of the functor of stable maps implies, after further base changes, that this family can be completed over $\mathcal{D}$.) Denote this family of stable maps over $\mathcal{D}$ by $\eta : \mathcal{C} \to \mathcal{D}$ and $\mu : \mathcal{C} \to \pi^{-1}(\mathcal{F})$. Let $Z \subset \mathcal{C}$ be the locus of nodes of the fibers of η union the locus of nonsingular points of the fibers where $d\mu$ vanishes on the tangent space to the fiber. Z is a closed subvariety.

Let $Z' \subset \mathcal{C}$ denote the (closed) intersection $Z \cap \mu^{-1}(S)$. The subvariety $T = \mu(Z') \subset S = \mathbb{P}(T_{\mathbb{P}^2}|_F)$ dominates F by the properties of $\overline{Y}$. There is a natural section $F \to \mathbb{P}(T_{\mathbb{P}^2}|_F)$ given by the differential of F. By Lemma 4.7 below, $F \cap T$ is nonempty. Let $\zeta \in F \cap T$.

There are now two cases. First, let $d \in \mathcal{D}$ be such that there exists a nonsingular point $p \in \mathcal{C}_d$ at which the differential of μ_d vanishes satisfing $\zeta = \mu_d(p)$. Consider the map μ_d from $\mathcal{C}_d$ to $\mathbb{P}^2$ blown-up at the points $f = (f_1, \dots, f_r)$. Since $\zeta \in F \subset \mathbb{P}(T_{\mathbb{P}^2}|_F)$, the strict transform of F in this blow-up passes through $\zeta = \mu_d(p) \in E_j$. If p lies on a component of $\mathcal{C}_d$ not mapped to a point, then $\mathcal{C}_d \cdot \mu^*(F) \geq 2$ because of the vanishing differential at p. However, since $n_{d,\alpha} = 0$ and F represents the first Chern class of the surface, $\mathcal{C}_d \cdot \mu^*(F) = 1$. A contradiction is reached. If p lies on a component mapped to a point, let K be the maximal connected subcurve of $\mathcal{C}_d$ which contains p and is mapped to a point. By stability of the map, K must intersect the other components of $\mathcal{C}_d$ at least at 3 points. By maximality, these intersection points lie on components not mapped to a point by μ_d. Hence, in this case, $\mathcal{C}_d \cdot \mu^*(F) \geq 3$. Again a contradiction is reached.

Second, let $d \in \mathcal{D}$ be such that a node $p \in \mathcal{C}_d$ maps to ζ. Again consider the map μ_d from $\mathcal{C}_d$ to $\mathbb{P}^2$ blown up at the points $f = (f_1, \dots, f_r)$. The strict transform of F in this blow-up passes through $\zeta = \mu_d(p) \in E_j$. If the node p is an intersection of 2 components of $\mathcal{C}_d$ neither of which is mapped to a point by μ_d, then $\mathcal{C}_d \cdot \mu^*(F) \geq 2$ and a contradiction is reached. If the node is on a component that is mapped to a point, then $\mathcal{C}_d \cdot \mu^*(F) \geq 3$ as before and a contradiction is again reached. q.e.d.

Lemma 4.7. *Let $\iota : F \hookrightarrow \mathbb{P}^2$ be a nonsingular plane cubic. Let $F \to \mathbb{P}(T_{\mathbb{P}^2}|_F)$ be the canonical section induced by the differential. Then $F \cap V$ is nonempty for any curve $V \subset \mathbb{P}(T_{\mathbb{P}^2}|_F)$.*

Proof. First the divisor class of the section F is calculated. Consider the tangent sequence on the plane cubic F:

$$0 \to \mathcal{O}_F = T_F \to T_{\mathbb{P}^2}|_F \to \mathcal{O}_{\mathbb{P}^2}(3)|_F = \mathcal{O}_F(3) \to 0. \tag{4.7.1}$$

Let $S = \mathbb{P}(T_{\mathbb{P}^2}|_F)$ and let $\rho : S \to F$ denote the projection. Let L denote the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on S. Via a degeneracy locus computation, sequence (4.7.1) implies that the section F is a divisor in the linear series of the line bundle $L \otimes \rho^* \mathcal{O}_F(3)$. Note that:

$$H^0(S, L \otimes \rho^* \mathcal{O}_F(3)) = H^0(F, T^*_{\mathbb{P}^2}|_F(3)).$$

The dual of the Euler sequence tensored with $\mathcal{O}_{\mathbb{P}^2}(3)$ restricted to F yields:

$$0 \to T^*_{\mathbb{P}^2}|_F(3) \to \oplus_1^3 \mathcal{O}_F(2) \to \mathcal{O}_F(3) \to 0.$$

It is easy to see the corresponding sequence on global sections is exact. Hence $H^0(S, L \otimes \rho^* \mathcal{O}_F(3)) = 9$. Therefore, for any $s \in S$, there exists a divisor linearly equivalent to F passing through s. Also, it is easy to calculate $F \cdot F = 9$.

Let V be an irreducible curve in S and assume $V \cap F$ is empty. Hence, $V \cdot F = 0$ and V is not a fiber of ρ. Let G be a divisor equivalent to F meeting V. By the equation $V \cdot G = 0$, V must be a component of G. Write $G = c_V V + \sum_i c_i W_i$. Let f be a general fiber of ρ.

$$c_v V \cdot f + \sum_i c_i W_i \cdot f = G \cdot f = 1.$$

$V \cdot f \geq 1$ since V is not a fiber. Therefore, $V \cdot f = 1$, $c_V = 1$, and $W_i \cdot f = 0$. This implies each W_i is a fiber. Then,

$$9 = F \cdot F = F \cdot G = \sum_i F \cdot c_i W_i = \sum_i c_i.$$

V is therefore a section of $\mathcal{O}_S(F) \otimes \rho^* N$ where N is degree -9 line bundle on F. Again $H^0(S, \mathcal{O}_S(F) \otimes \rho^* N) = H^0(F, T^*_{\mathbb{P}^2}|_F \otimes \mathcal{O}_F(3) \otimes N)$. The latter is seen to be zero by the dual Euler sequence argument. No such V exists. q.e.d.

The Lemma 4.6 showed the branches of the image curve $\mu(\mathbb{P}^1)$ are nonsingular at their intersections with the E_i. Next, it is shown that distinct branches of the image curve do not intersect in the exceptional divisors.

Lemma 4.8. *Let (d, α) satisfy $n_{d,\alpha} = 0$. Let $\mu : \mathbb{P}^1 \to X_r$ correspond to a point $[\mu] \in \overline{M}_{0,0}(X_r, (d, \alpha))$. Let I be the image curve $\mu(\mathbb{P}^1)$. Then the set $I \cap E_i$ is contained in the nonsingular locus of I (for all i).*

Proof. The proof of this lemma exactly follows the proof of Lemma 4.6. If the assertion is false, a quasi-projective subvariety

$$W \subset \overline{M}_{0,0}(\pi, (d, \alpha))$$

can be found where the image curve has distinct branches meeting in E_j (for a fixed index j). The closure $\overline{W}$ of W then surjects upon $\mathcal{B}_r$. Let $\mu : C \to X_b$ be a limit map $[\mu] \in \overline{W}$. At least one of the following properties must be satisfied:

(i) Distinct points of C are mapped by μ to the same point of E_j.

(ii) There exists a point $p \in C_{nonsing}$ satisfying $d\mu(p) = 0$ and $\mu(p) \in E_j$.

(iii) There is a node of C mapped to E_j.

The same curve $\mathcal{F} \subset \mathcal{B}_r$ is considered. Let $\gamma : \mathcal{D} \to \overline{W}$ be an irreducible curve that surjects onto $\mathcal{F}$ via τ. As before, a curve in $T \subset S = \mathbb{P}(T_{\mathbb{P}^2}|_F)$ can be found representing the points on E_j where the singularities occur. Using Lemma 4.7, $F \cap T$ is non-empty. It is then deduced that stable maps exist satisfying $\mu^* c_1(T_{X_b}) \geq 2$ as before. A contradiction is reached. q.e.d.

4.4. Nonsingularity conditions

The main nonsingularity result needed for the proof of Theorem 4.1 can now be proven.

Lemma 4.9. *Let (d, α) satisfy $d > 0$, $\alpha \geq 0$, and $n_{d,\alpha} = 0$. If there exists an index i for which $a_i \in \{1, 2\}$, then $\overline{M}_{0,0}(X_r, (d, \alpha))$ is nonsingular of pure dimension 0. Moreover, the points of $\overline{M}_{0,0}(X_r, (d, \alpha))$ correspond to immersions of $\mathbb{P}^1$ in X_r.*

Proof. If $\overline{M}_{0,0}(X_r, (d, \alpha))$ is empty for generic X_r, the Lemma is trivially true. Let $\mu : \mathbb{P}^1 \to X_r$ be a map in $\overline{M}_{0,0}(X_r, (d, \alpha))$. By the genericity assumption, the natural map:

$$d\tau : T_{\overline{M}_{0,0}(\pi,(d,\alpha)),[\mu]} \to \tau^* T_{\mathcal{B}_r, \tau([\mu])} \tag{4.9.1}$$

must be surjective. The Lemma is proved in two steps. First, the surjectivity of (4.9.1) is translated into a condition on the global sections map of a normal sheaf sequence associated to μ. The map μ is then shown to be an *immersion.* N_{X_r} is therefore locally free of rank 1 and degree $3d - |\alpha| - 2 = n_{d,\alpha} - 1 < 0$. The Zariski tangent space to $\overline{M}_{0,0}(X_r, (d, \alpha))$ at $[\mu]$ is $H^0(\mathbb{P}^1, N_{X_r}) = 0$. Hence, $[\mu]$ is a nonsingular point of $\overline{M}_{0,0}(X_r, (d, \alpha))$.

Let X_r be the blow-up of $\mathbb{P}^2$ at the points $x_1, \ldots, x_r$. The deformation problem as the blown-up points $x_1, \ldots, x_r$ vary is considered. There is a projection $X_r \to \mathbb{P}^2$ which yields a sequence on X_r:

$$0 \to T_{X_r} \to T_{\mathbb{P}^2} \to Q \to 0. \tag{4.9.2}$$

Q is a sheaf supported on the exceptional curves E_i. $Q|_{E_i}$ is a line bundle on E_i. More precisely, if the point $e \in E_i$ corresponds to the tangent direction $T_e \subset T_{\mathbb{P}^2,x_i}$, then the fiber of Q at e is $T_{\mathbb{P}^2,x_i}/T_e$. The space of deformations of the points $x_1, \ldots, x_r$ is $\oplus_{i=1}^r T_{\mathbb{P}^2,x_i} = H^0(X_r, Q)$. $\oplus_{i=1}^r T_{\mathbb{P}^2,x_i}$ is also canonically the tangent space to $\mathcal{B}_r$ at the point $x = (x_1, \ldots, x_r)$. Therefore a vector $0 \neq v \in \oplus_{i=1}^r T_{\mathbb{P}^2,x_i}$ defines a first order deformation of X_r in the family $\mathcal{X}_r$. Let $\lambda : \triangle \to \mathcal{B}_r$ be a nonsingular curve in $\mathcal{B}_r$ passing through x with tangent direction $\mathbb{C}v$. Let $\mathcal{X}_\triangle = \lambda^{-1}\mathcal{X}_r \to \triangle$. This deformation naturally yields a differential sequence on X_r:

$$0 \to T_{X_r} \to T_{\mathcal{X}_\triangle} \to \mathcal{O}_{X_r} \to 0. \tag{4.9.3}$$

Sequences (4.9.2) and (4.9.3) are related by a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T_{X_r} \arrow[r] \arrow[d, "="] & T_{\mathcal{X}_\triangle} \arrow[r, "a"] \arrow[d, "b"] & \mathcal{O}_{X_r} \arrow[r] \arrow[d, "c"] & 0 \arrow[d] \\
0 \arrow[r] & T_{X_r} \arrow[r] & T_{\mathbb{P}^2} \arrow[r, "d"] & Q \arrow[r] & 0.
\end{tikzcd} \tag{4.9.4}$$

Moreover, it is easy to check that the image of

$$c : H^0(X_r, \mathcal{O}_{X_r}) \to H^0(X_r, Q)$$

is simply $\mathbb{C}v$.

Since $d \geq 1$, $Im(\mu)$ is not contained in any E_i. Therefore the above commutatitive diagram *stays exact* when pulled back to $\mathbb{P}^1$. Let $N_{\mathbb{P}^2}$ and $N_{\mathcal{X}_\triangle}$ denote the normal sheaves on $\mathbb{P}^1$ of the maps to $\mathbb{P}^2$ and $\mathcal{X}_\triangle$ induced by μ. Consider the commutative diagram of exact sequences obtained by pulling back (4.9.4) to $\mathbb{P}^1$ and quotienting by the inclusion of sheaves induced by the differential $d\mu : T_{\mathbb{P}^1} \to \mu^* T_{X_r}$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & N_{X_r} \arrow[r] \arrow[d, "="] & N_{\mathcal{X}_\triangle} \arrow[r, "a"] \arrow[d, "b"] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d, "c"] & 0 \arrow[d] \\
0 \arrow[r] & N_{X_r} \arrow[r] & N_{\mathbb{P}^2} \arrow[r, "d"] & \mu^* Q \arrow[r] & 0.
\end{tikzcd}$$

$H^0(\mathbb{P}^1, N_{\mathcal{X}_\triangle})$ is the space of first order deformations of the map μ considered as a map to $\mathcal{X}_\triangle$. By the surjectivity of (4.9.1), there must exist a first order deformation of $[\mu]$ not contained in X_r. Therefore, the image of $a : H^0(\mathbb{P}^1, N_{\mathcal{X}_\triangle}) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ must be non-zero. This condition is equivalent to the splitting of the top sequence. Using this splitting and the morphism b, it is seen that the section $v \in H^0(\mathbb{P}^1, Q)$ must be in the image of $d : H^0(\mathbb{P}^1, N_{\mathbb{P}^2}) \to H^0(\mathbb{P}^1, \mu^*Q)$.

The conclusion of the above considerations is the following. For every element $v \in \oplus_{i=1}^r T_{\mathbb{P}^2, x_i}$, there exists a section of $H^0(\mathbb{P}^1, N_{\mathbb{P}^2})$ which has image $v \in H^0(\mathbb{P}^1, \mu^*Q)$. The map μ will now be shown to be an immersion.

Suppose $p \in \mathbb{P}^1$ satisfies $\mu(p) \in E_i$. By Lemma 4.6, $d\mu(p)$ is injective. Let m be the multiplicity of μ^*E_i at p. Local calculations show that the following hold in a neighborhood $U \subset \mathbb{P}^1$ of p with local parameter t:

(i) $N_{\mathbb{P}^2}$ has torsion part $\mathbb{C}[t]/(t^{m-1})$, where t is a local parameter at p.

(ii) $\mu^*(Q)$ is the torsion sheaf $\mathbb{C}[t]/(t^m)$.

(iii) The map on torsion parts from $N_{\mathbb{P}^2}$ to $\mu^*(Q)$ is multiplication by t.

Let τ be the torsion part of $N_{\mathbb{P}^2}$. By (iii), the natural map of sheaves on U:

$$N_{\mathbb{P}^2}/\tau \to \mu^*(Q) \otimes \mathcal{O}_p = \mathbb{C}$$

is surjective. Therefore, a section $\overline{s}$ of the line bundle $N_{\mathbb{P}^2}/\tau$ is zero at p if and only if the image of $\overline{s}$ in $\mu^*(Q) \otimes \mathcal{O}_p$ is zero.

Decompose $\tau = A \oplus B$ where A is the torsion part supported at the points $\bigcup_i \mu^{-1}(E_i)$, and B is the torsion part supported elsewhere. Let n equal the set theoretic cardinality $|\bigcup_i \mu^{-1}(E_i)|$. For each point $z \in \mathbb{P}^1$ lying over an exceptional divisor E, let m_z be the multiplicity of μ^*E at z. The following equations are obtained:

$$\sum_{z \in \bigcup_i \mu^{-1}(E_i)} m_z = \sum_i a_i,$$

$$degree(A) = \sum_{z \in \bigcup_i \mu^{-1}(E_i)} (m_z - 1) = -n + \sum_i a_i.$$

The degree of $N_{\mathbb{P}^2}$ is $3d - 2$. The degree of

$$N_{\mathbb{P}^2}/A = 3d - 2 + n - \sum_i a_i = n - 1.$$

Let $b = degree(B)$. Then, the degree of $N_{\mathbb{P}^2}/\tau$ is $n-1-b$. Note that μ is an immersion if and only if $b = 0$.

Without loss of generality, let $a_1 \in \{1, 2\}$. First consider the case $a_1 = 1$. There is a unique point z_1 in $\mu^{-1}(E_1)$. Let $v = \oplus_i v_i$ where $v_i \in T_{\mathbb{P}^2, x_i}$ satisfy:

(i) $v_1 \neq 0$ in $\mu^* Q \otimes \mathcal{O}_{z_1}$.

(ii) $v_i = 0$ for $i \geq 2$.

Since there exists a section s of $H^0(\mathbb{P}^1, N_{\mathbb{P}^2})$ with image

$$v \in H^0(\mathbb{P}^1, \mu^*(Q)),$$

there must exist a nonzero section $\overline{s}$ of $H^0(\mathbb{P}^1, N_{\mathbb{P}^2}/\tau)$ vanishing at (at least) $n-1$ points (all the z's except z_1) by (iii). Therefore,

$$degree(N_{\mathbb{P}^2}/\tau) \geq n-1.$$

It follows that $b = 0$.

Next, consider the case $a_1 = 2$. There are two possiblities. Either $\mu^{-1}(E_1)$ consists of two points or one point. If there is a unique point in $\mu^{-1}(E_1)$, the argument proceeds exactly as in the $a_1 = 1$ case and $b = 0$. Now suppose $\mu^{-1}(E_1) = \{z_1, z_2\}$. By Lemma 4.8, $\mu(z_1) \neq \mu(z_2)$. Let $v = \oplus_i v_i$ satisfy:

(i) $v_1 \neq 0$ in $\mu^* Q \otimes \mathcal{O}_{z_1}$,

(ii) $v_1 = 0$ in $\mu^* Q \otimes \mathcal{O}_{z_2}$,

(iii) $v_i = 0$ for $i \geq 2$.

Such a selection of v_1 is possible since $T_{\mathbb{P}^2, x_1}$ surjects upon $\mu^* Q \otimes \mathcal{O}_{z_1} \oplus \mu^* Q \otimes \mathcal{O}_{z_2}$ for $\mu(z_1) \neq \mu(z_2)$. As before, there must exist a nonzero section $\overline{s}$ of $H^0(\mathbb{P}^1, N_{\mathbb{P}^2}/\tau)$ vanishing at least $n-1$ points (all the z's except z_1) by (iv). Therefore, $degree(N_{\mathbb{P}^2}/\tau) \geq n-1$. It follows that $b = 0$. q.e.d.

Lemma 4.10. *Let $d > 0$, $\alpha \geq 0$, $r \leq 8$, and $n_{d,\alpha} = 0$. Then, $\overline{M}_{0,0}(X_r, (d, \alpha))$ is nonsingular of pure dimension 0. Moreover, the points of $\overline{M}_{0,0}(X_r, (d, \alpha))$ correspond to immersions of $\mathbb{P}^1$ in X_r.*

Proof. Let $\mu : \mathbb{P}^1 \to X_r$ be a map in $\overline{M}_{0,0}(X_r, (d, \alpha))$. By Lemma 4.6, μ is an immersion at the points of $\mathbb{P}^1$ mapping to the exceptional curves E_i. Suppose $p \in \mathbb{P}^1$ is a point where μ is not an immersion ($\mu(p) \notin E_i$). Since the number of blown-up points $x_1, \ldots, x_r$ is at most 8, there is curve in the linear series $3H - \sum_{i=1}^{8} E_i$ passing through $\mu(p)$. Let F denote this cubic (which may be reducible). There are now two cases. If $\mu(\mathbb{P}^1)$ is not contained in any component of F, then $\mathbb{P}^1 \cdot \mu^*(F) \geq 2$ because μ is not an immersion at p. This is a contradiction since the numerical assumption implies $\mathbb{P}^1 \cdot \mu^*(F) = 1$. If $\mu(\mathbb{P}^1)$ is contained in a component of F, then d must equal 1,2, or 3 (since μ is birational). For these low degree cases, $\overline{M}_{0,0}(X_r, (d, \alpha))$ is empty unless $a_i = 1$ for some i. Then, Lemma 4.9 yields a contradiction. We conclude μ is an immersion and $\overline{M}_{0,0}(X_r, (d, \alpha))$ is nonsingular. q.e.d.

4.5. Proof of Theorem 4.1

First, the case $n_{d,\alpha} = 0$ is considered. Since $d > 0$, $\alpha \geq 0$, and $a_i \in \{1, 2\}$ (for some i), Lemma 4.9 shows that $\overline{M}_{0,0}(X_r, (d, \alpha))$ is a nonsingular set of points. By Proposition 3.2, $N_{d,\alpha}$ equals the number of points in $\overline{M}_{0,0}(X_r, (d, \alpha))$. Moreover, by Lemma 4.9, the points of $\overline{M}_{0,0}(X_r, (d, \alpha))$ represent immersions of $\mathbb{P}^1$. Theorem 4.1 is established for classes (d, α) satisfying $n_{d,\alpha} = 0$.

Proceed now by induction on $n = n_{d,\alpha}$. If $n_{d,\alpha} > 0$, consider the class $(d, (\alpha, 1))$ on $\mathbb{P}^2$ blown-up at $r + 1$ points $x_1, \ldots, x_{r+1}$. Certainly, $n_{d,(\alpha,1)} = n - 1$. By property (P5) of Section 3,

$$N_{d,\alpha} = N_{d,(\alpha,1)}.$$

The class $(d, (\alpha, 1))$ satisfies condition (ii) in the hypotheses of Theorem 4.1. By induction, $N_{d,(\alpha,1)}$ equals the number of genus 0 stable maps of class $(d, (\alpha, 1))$ passing through $n_{d,\alpha} - 1$ points $p_1, \ldots, p_{n-1}$ in X_{r+1}. This is precisely equal to the number of stable maps of class (d, α) passing through the $n_{d,\alpha}$ points $p_1, \ldots, p_{n-1}, x_{r+1}$ in X_r by Lemma 4.3. Since the solution curves are immersions in X_{r+1}, it follows easily that the corresponding curves in X_r are also immersions. The proof of Theorem 4.1 is complete.

5. Symmetries and computations

5.1. The cremona transformation

Let p_1, p_2, p_3 be 3 non-collinear points in $\mathbb{P}^2$. Let L_1, L_2, L_3 be the 3 lines determined by pairs of points where $p_i, p_j \in L_k$ for distinct indices i, j, k. Let S be the blow-up of $\mathbb{P}^2$ at the points p_1, p_2, p_3. Let E_1, E_2, E_3 be the exceptional divisors of this blow-up. Let F_1, F_2, F_3 be the strict transforms of the lines L_1, L_2, L_3. The F_k are disjoint (-1)-curves on S and can be blown-down. The resulting surface is another projective plane $\overline{\mathbb{P}}^2$. The blow-down maps are:

$$\mathbb{P}^2 \xleftarrow{e} S \xrightarrow{f} \overline{\mathbb{P}}^2. \tag{5.0.1}$$

This is the classical Cremona transformation of the plane. Let $q_1, q_2, q_3 \in \overline{\mathbb{P}}^2$ be the points $f(F_1), f(F_2), f(F_3)$. Let H and $\overline{H}$ denote the hyperplane classes in $A_1(\mathbb{P}^2)$ and $A_1(\overline{\mathbb{P}}^2)$ respectively. There are now 2 bases of $A_1(S)$ corresponding to the two blow-downs: H, E_1, E_2, E_3 and $\overline{H}, F_1, F_2, F_3$. The relationship between these bases is:

$$\begin{aligned} dH - a_1E_1 - a_2E_2 - a_3E_3 & \\ = (2d - a_1 - a_2 - a_3)\overline{H} &- (d - a_2 - a_3)F_1 \\ &- (d - a_1 - a_3)F_2 - (d - a_1 - a_2)F_3. \end{aligned}$$

Let $x_4, \dots, x_r \in \mathbb{P}^2$ be additional general points on $\mathbb{P}^2$ which correspond via the maps (5.0.1) to general points $s_4, \dots, s_r \in S$ and $y_4, \dots, y_r \in \overline{\mathbb{P}}^2$. The blow-up of S at the points $s_4, \dots, s_r$ may be viewed as a general blow-up of $\mathbb{P}^2$ at $p_1, p_2, p_3, x_4, \dots, x_r$ or as a general blow-up of $\overline{\mathbb{P}}^2$ at $q_1, q_2, q_3, y_4, \dots, y_r$. Let $G_4, \dots, G_r$ denote the exceptional divisors of the blow-up of S.

Since the class $dH - a_1E_1 - a_2E_2 - a_3E_3 - \sum_{i=4}^r a_iG_i$ equals the class

$$\begin{aligned} (2d - a_1 - a_2 - a_3)\overline{H} - (d - a_2 - a_3)F_1 - (d - a_1 - a_3)F_2 & \\ - (d - a_1 - a_2)F_3 - \sum_{i=4}^r a_iG_i&, \end{aligned}$$

the Gromov-Witten invariant $N_{d,\alpha}$ on the blow-up of $\mathbb{P}^2$ equals the invariant $N_{d',\alpha'}$ on the blow-up of $\overline{\mathbb{P}}^2$ where

$$(d', \alpha') = (2d - a_1 - a_2 - a_3, (d - a_2 - a_3, d - a_1 - a_3, d - a_1 - a_2, a_4, \dots, a_r)).$$

It follows that $\overline{M}_{0,0}(X_r, (d, \alpha))$ is nonsingular if and only if $\overline{M}_{0,0}(X_r, (d', \alpha'))$ is nonsingular. Therefore, $N_{d,\alpha}$ is enumerative if and only if $N_{d',\alpha'}$ is enumerative. The Cremona symmetry of the Gromov-Witten invariants of X_r is discussed in [6] from a slightly different perspective.

For example, let $(d, \alpha) = (10, (4, 4, 3, 3, 3, 3, 3, 3, 3)) = (10, (4^2, 3^7))$ where the last equality is just notational convenience. Then, $n_{10,(4^2,3^7)} = 30 - 29 - 1 = 0$. The class $(10, (4^2, 3^7))$ does not satisfy either condition (i) or (ii) of Theorem 4.1. Applying the Cremona transformation, $(d', \alpha') = (9, (3, 3, 2, 3^6))$. Theorem 4.1 applies to (d', α'). Therefore, the moduli space $\overline{M}_{0,0}(X_r, (10, (4^2, 3^7))$ is nonsingular (and all points correspond to immersions). $N_{10,(4^2,3^7)} = 520$ is enumerative in this case.

5.2. Tables

The arithmetic genus of the class (d, α) on X_r is determined by:

$$g_a(d, \alpha) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \frac{a_i(a_i - 1)}{2}.$$

The arithmetic genus of a reduced, irreducible curve is non-negative. By Corollary 4.4, $\overline{M}_{0,0}(X_r, (d, \alpha))$ is empty when $g_a(d, \alpha) < 0$ and $n_{d,\alpha} = 0$. A simple reduction to the case of expected dimension zero shows that $N_{d,\alpha} = 0$ if $g_a(d, \alpha) < 0$.

If $a_i + a_j > d$ for indices $i \neq j$, then $N_{d,\alpha} = 0$ unless $(d, \alpha) = (1, (1, 1))$. This follows again by a reduction to the expected dimension zero case. Then, Corollary 4.4 shows that $\overline{M}_{0,0}(X_r, (d, \alpha))$ is empty (unless $(d, \alpha) = (1, (1, 1))$) by considering the intersection of a map with the line in $\mathbb{P}^2$ connecting the points x_i and x_j.

In the first table below, Gromov-Witten invariants $N_{d,\alpha}$ for $d \leq 5$ and $\alpha \geq 0$ are listed. By properties (P3), (P4), and (P5), it suffices to list the invariants for ordered sequences α satisfying $\alpha \geq 2$. Moreover, if $g_a(d, \alpha) < 0$ or if $a_i + a_j > d$, the invariant vanishes and is omitted from the table. The invariants were computed by a Maple program via the recursive algorithm of the proof of Theorem 3.6.

$d=1$	2	3	4	5	5
$N_1=1$	$N_2=1$	$N_3=12$	$N_4=620$	$N_5=87304$	$N_{5,(2^6)}=1$
		$N_{3,(2)}=1$	$N_{4,(2)}=96$	$N_{5,(2)}=18132$	$N_{5,(3)}=640$
			$N_{4,(2^2)}=12$	$N_{5,(2^2)}=3510$	$N_{5,(3,2)}=96$
			$N_{4,(2^3)}=1$	$N_{5,(2^3)}=620$	$N_{5,(3,2^2)}=12$
			$N_{4,(3)}=1$	$N_{5,(2^4)}=96$	$N_{5,(3,2^3)}=1$
				$N_{5,(2^5)}=12$	$N_{5,(4)}=1$

The Cremona transformation applied to the class $(5,(2,2,2))$ yields $N_{5,(2,2,2)} = N_{4,(1,1,1)}$. By Property (P5), $N_{4,(1,1,1)} = N_4 = 620$. The following table lists all the Gromov-Witten invariants for degrees 6 and 7 which are not obtained from lower degree numbers by the Cremona transformation.

$d=6$	7	7
$N_6=26312976$	$N_7=14616808192$	$N_{7,(3,2)}=90777600$
$N_{6,(2)}=6506400$	$N_{7,(2)}=4059366000$	$N_{7,(3,2^2)}=23133696$
$N_{6,(2^2)}=1558272$	$N_{7,(2^2)}=1108152240$	$N_{7,(3,2^3)}=5739856$
$N_{6,(2^3)}=359640$	$N_{7,(2^3)}=296849546$	$N_{7,(3,2^4)}=1380648$
$N_{6,(2^4)}=79416$	$N_{7,(2^4)}=77866800$	$N_{7,(3,2^5)}=320160$
$N_{6,(2^5)}=16608$	$N_{7,(2^5)}=19948176$	$N_{7,(3,2^6)}=71040$
$N_{6,(2^6)}=3240$	$N_{7,(2^6)}=4974460$	$N_{7,(3,2^7)}=14928$
$N_{6,(2^7)}=576$	$N_{7,(2^7)}=1202355$	$N_{7,(3,2^8)}=2928$
$N_{6,(2^8)}=90$	$N_{7,(2^8)}=280128$	$N_{7,(3^2)}=6508640$
$N_{6,(3)}=401172$	$N_{7,(2^9)}=62450$	$N_{7,(4)}=7492040$
$N_{6,(3,2)}=87544$	$N_{7,(2^{10})}=13188$	$N_{7,(4,2)}=1763415$
$N_{6,(4)}=3840$	$N_{7,(3)}=347987200$	$N_{7,(5)}=21504$

In [7], the Gromov-Witten invariants of X_6 are computed. Our computation $N_{6,(2^6)} = 3240$ disagrees with [7]. We have checked our number using different recursive strategies.

Let (d,α) be a class for which all the hypotheses of Theorem 4.1 and Lemma 4.10 fail. Then, $r \geq 9$, $3d = |\alpha| + 1$, and $\alpha \geq 3$. Hence, $d \geq 10$. If $d = 10$, then there are only two possible values (up to reordering) for α: $(4^2, 3^7)$ or $(5, 3^8)$. The invariant $N_{10,(4^2,3^7)}$ was shown to be enumerative by the Cremona transformation in Section 5.1. Applying the transformation to $(10,(5,3^8))$ yields $(9,(4,2^2,3^6))$. Hence, $N_{10,(5,3^8)} = N_{9,(4,2^2,3^6)} = 90$ is enumerative by Theorem 4.1. We have shown all invariants of degree $d \leq 10$ are enumerative. The only invariants of degree 11 not proven to be enumerative by the methods of this paper correspond to the classes $(11,(5,3^9))$ and $(11,(4^2,3^8))$. $N_{11,(5,3^9)} = 707328$ and $N_{11,(4^2,3^8)} = 2350228$. It is not known to the authors whether non-trivial multiplicities arise.

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