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HOMOLOGICAL REDUCTION OF CONSTRAINED POISSON ALGEBRAS

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Reduction of a Hamiltonian system with symmetry and/or constraints has a long history. There are several reduction procedures, all of which agree in "nice" cases [1]. Some have a geometric emphasis - reducing a (symplectic) space of states [39], while others are algebraic - reducing a (Poisson) algebra of observables [43]. Some start with a momentum map whose components are constraint functions [15]; some start with a gauge (symmetry) algebra whose generators, regarded as vector fields, correspond via the symplectic structure to constraints [10]. The relation between symmetry and constraints is particularly tight in the case Dirac calls "first class". The present paper is concerned entirely with this first class case and deals with the reduction of a Poisson algebra via homological methods, although there is considerable motivation from topology, particularly via the models central to rational homotopy theory.

Homological methods have become increasingly important in mathematical physics, especially field theory, over the last decade. In regard to constrained Hamiltonians, they came into focus with Henneaux's Report [22] on the work of Batalin, Fradkin and Vilkovisky [2], [3]-[5], emphasizing the acyclicity of a certain complex, later identified by Browning and McMullan as the Koszul complex of a regular ideal of constraints. I was able to put the BFV construction into the context of homological

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perturbation theory [44] and, together with Henneaux et al [13], extend the construction to the case of non-regular geometric constraints of first class. Independently, using a mixture of homological and C^1 -patching techniques, Dubois-Violette extended the construction to regular but not-necessarily-first-class constraints [11].

I am grateful to all of the above for their input and inspiration, whether in their papers or in conversation. The present version has also profitted from conversations at the MSRI Workshop on Symplectic Topology. Finally, I would like to express my thanks to the referee who has read several versions with extreme care, suggesting extensive improvements, both factual and stylistic. While early revision was in progress, Kimura sent me a copy of [35] which has also had a significant influence on the present exposition, as has his continued interaction while with me at UNC as an NSF Post-Doc.

1. Preliminaries

This research touches on questions which it is hoped will be of interest to mathematical physicists, symplectic and algebraic geometers and homotopy theorists. The techniques used here are primarily those of differential commutative algebra and rational homotopy theory. We write with a dual vision and hopefully a dual audience; for example, the constraints are functions on a symplectic manifold and the physics literature speaks almost entirely in terms of the constraints whereas the algebra can be expressed more invariantly in terms of the ideal generated by the constraints. We work entirely over the reals $\mathbb{R}$ as our ground field, although any field of characterisitic 0 would do and the complex numbers $\mathbb{C}$ are more common in certain physical applications. The major Theorem 4.2 is expressed in algebraic terms, followed by remarks specifically in terms of the constraints themselves.

We begin therefore with a brief (very!) review of the motivating background: a tiny bit of symplectic geometry, slightly more of Poisson algebra and the essentials of constraint varieties and their symmetries in the first class case. The reader who desires more extensive background or a more leisurely exposition may consult a variety of sources listed in the bibliography. The relations between the algebra and the motivating geometry are exposed particularly clearly in [35].

1.1. The Hamiltonian Formalism. The motivating physical systems are described as differential equations of motion or evolution involving smooth functions on a manifold. The underlying manifold W is assumed to be symplectic. This means there is a 2-form ω such that $d\omega = 0$ and $\omega^{dimW} \neq 0$. Equivalently, ω induces an isomorphism

$$TW \to T^*W.$$

(With an eye to future applications, we would like to allow W to be infinite dimensional, in which case the appropriate definition is that the induced map $TW \to T^*W$ be one-to-one.) In local coordinates $q^1, ..., q^n, p_1, ..., p_n$, the form ω looks like $dq^i \wedge dp_i$ (the summation convention will be assumed throughout this paper).

From an algebra point of view, the crucial point is two-fold: For any function $f \in C^\infty(W)$, there is a Hamiltonian vector field X_f defined by $\omega(X_f, \ ) = df$. For two functions $f, g \in C^\infty(W)$, their Poisson bracket $\{f, g\} \in C^\infty(W)$ is defined by

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f).$$

This bracket makes $C^\infty(W)$ into a Poisson algebra, that is, a commutative algebra P (with product denoted fg) together with a bracket $\{\ ,\ \} : P \otimes P \to P$ forming a Lie algebra such that $\{f,\ \}$ is a derivation of P as a commutative algebra: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

A typical Hamiltonian system is one of the form $\{f, H\} = df/dt$ for fixed H. Symmetries of such a system are given by functions g which Poisson commute with H; they form a sub-Lie algebra of $C^\infty(W)$. Symmetries arise also in connection with "constraints". Regarded as in a dynamical system, solutions can be constrained to lie in a sub-manifold $V \subset W$ (more generally, V is just a sub-space), hereafter called the **constraint locus**, also known in the literature as a constraint surface. As in algebraic geometry, we can think of V as the zero set of some functions $\phi_\alpha : W \to R$, called **constraints**. The algebra of functions C^∞-in-the-sense-of-Whitney on V can be identified with $C^\infty(W)/I$ where I is the ideal of functions which vanish on V. If $V \subset W$ is a closed and embedded submanifold, this agrees with the usual notion of smooth functions on V.

Now if W is symplectic (or just given a Poisson bracket on $C^\infty(W)$), Dirac calls the constraints **first class** if I is closed under the Poisson bracket. (If the $\mathbb{R}$-linear span of the ϕ_α is closed under the bracket, physicists say the ϕ_α **close** on a Lie algebra; this is a very nice case, but the more general first class case is where homological techniques are really important.) When the constraints are first class, we have that the Hamiltonian vector fields X_{ϕ_α} determined by the constraints are tangent to V (where V is smooth) and give a foliation $\mathcal{F}$ of V. Similarly, $C^\infty(W)/I$ is a Lie module over I with respect to the Poisson bracket. In symplectic geometry, when V is smooth, it is usually called a **coisotropic** submanifold (see [49] for generalizations when V is not smooth). For the general case, we will call the constraint locus **coisotropic** if the ideal is first class.

In many cases of interest, I does not arise from the Lie algebra of some Lie group of transformations of W or even V, but the corresponding Hamiltonian vector fields X_{ϕ_α} are still referred to as (infinitesimal) symmetries. In the nicest case, e.g. where the foliation $\mathcal{F}$ is given by a principal G-bundle structure on a smooth V, the algebra $C^\infty(V/\mathcal{F})$ can be identified with the I-invariant sub-algebra of $C^\infty(W)/I$. In great (if not complete) generality, this I-invariant sub-algebra represents the true observables of the constrained system.

In this context, the "classical BRST construction", at least as developed by Batalin-Fradkin-Vilkovisky and phrased in terms of constraints, is a homological construction for performing the reduction of the Poisson algebra $C^\infty(W)$ of smooth functions on a Poisson manifold W by the ideal I of functions which vanish on a coisotropic constraint locus. But the construction produces cohomology in other degrees than zero, which at least in some cases, admits a geometric interpretation.

Instead of considering just the "observable" functions, one can consider the deRham complex of longitudinal or vertical forms of the foliation $\mathcal{F}$, that is, the complex $\Omega(V,\mathcal{F})$ consisting of forms on vertical vector fields, those tangent to the leaves. If we think of $\mathcal{F}$ as an involutive sub-bundle of the tangent bundle to V, then $\Omega(V,\mathcal{F})$ consists of sections of $\Lambda^*\mathcal{F}$. In adapted local coordinates $(x^1,...,x^{r+s})$ with $(x^1,...,x^r)$ being coordinates on a leaf, a typical longitudinal form is

$$f_J(x)dx^J$$

where $J=(j_1,...,j_q)$ with $1\le j_1 < ...j_q \le r$, the leaf dimension.
The usual exterior derivative of differential forms restricts to determine the vertical exterior derivative because $\mathcal{F}$ is involutive. This complex is familiar in foliation theory, c.f. [21]. The classical BRST-BFV construction has, in the nice cases, the same cohomology as this complex of longitudinal forms.

A major motivating example for the BFV construction was provided by gauge theory. Here W is $T^*\mathcal{A}$ where $\mathcal{A}$ is the space of connections for a fixed principal G-bundle $G \to P \to B$. The reduced phase space is $T^*(\mathcal{A}/\mathcal{G})$ where $\mathcal{G}$ is the group of "gauge transformations", the vertical automorphisms of P.

In considering what the physicists [2],[3]-[5],[12], [14],[22],[8] did in some special cases, I recognized a homological "model" for $\Omega(V, \mathcal{F})$ in roughly the sense of rational homotopy theory [46]. This is the same sense in which the Cartan-Chevalley-Eilenberg complex [9] for the cohomology of a Lie algebra $\mathfrak{g}$ is a "model" for $\Omega^*(G)$ where G is a compact Lie group with Lie algebra $\mathfrak{g}$. The physicists' model is itself crucially a Poisson algebra extension of a Poisson algebra P, and its differential contains a piece which reinvented the Koszul complex for the ideal I. The differential also contains a piece which looks like the Cartan-Chevalley-Eilenberg differential. Generalizations of the Cartan-Chevalley-Eilenberg differential as they occur in physics are usually referred to as BRST operators. This honors seminal work of Becchi, Rouet and Stora [6] and, independently, Tyutin [48]. Apparently it was the search for such an operator in aid of quantization which motivated the work of Batalin, Fradkin and Vilkovisky.

It was Browning and McMullan [8] who first identified the Koszul complex within the construction in the regular case, (Henneaux had already called attention to the relevance of that acyclicity) leading both Dubois-Violette [11] and myself [44] independently to adopt a more fully homological approach, although with somewhat different emphases. Dubois-Violette retains some of the symplectic geometry and is able to handle regular general (not necessarily first class) constraints. On the other hand, by restricting to first class constraints, in joint work with Henneaux et al [13], I was able to handle non-regular ideals in suitable geometric circumstances.

In the present paper, I start at the level of the purely (Poisson) algebraic structures. In particular, I adapt the notion of "model" from rational homotopy theory and use the techniques of homological perturbation theory. Although the treatment of BFV is basis dependent (individual constraints) and nominally finite dimensional, I attempt to work more invariantly in terms of the ideal generated by the constraints and take care to avoid assumptions of finite dimensionality. Although originally invented in the context of quantization, both BRST cohomology as they described it and the BFV-generalization are mathematically interesting in the 'classical' setting. The present paper is concerned only with the clasical setting but in the full generality of a first class ideal, in contrast to the paper of Kostant and Sternberg [36] whose main interest is in quantization issues for the case of an equivariant moment map and hence do not deal with the BFV-generalization nor with homological perturbation methods.

2. Reduction

We have presented a geometric picture of reduction as referring to $W \hookleftarrow V \to V/\mathcal{F}$. There is a variety (pun intended) of difficulties with this approach. The constraint locus V can fail to be a submanifold. Even if it is a submanifold, the quotient $\hat{V} := V/\mathcal{F}$ may not be a manifold, in fact, may not even be Hausdorff. (An intermediate situation of considerable interest occurs with the quotient $V/\mathcal{F}$ being a stratified symplectic space [38].)

When (W, ω) is a symplectic manifold with a smooth coisotropic submanifold, one of the nicest cases is called 'regular', namely when the quotient $V/\mathcal{F}$ is a manifold and the projection $V \to \hat{V}$ is a submersion. This implies further that $\omega|V$ has constant rank on TV (so that $\omega|V$ is a presymplectic form on V), and $\mathcal{F}$ is an involutive distribution given by $ker\ \omega|V$ which is fibrating. Then a standard argument, due essentially to E. Cartan [39] or [20, Thm. 25.2], shows that there exists a unique symplectic form $\hat{\omega}$ on $\hat{V}$ satisfying $\pi^*\hat{\omega} = \omega|V$. The reduction of (W, ω) is then the symplectic manifold $(\hat{V}, \hat{\omega})$, and the corresponding reduced Poisson algebra is $C^\infty(\hat{V})$ with the Poisson bracket that is associated to $\hat{\omega}$.

In the "singular" case, when these conditions fail to hold, reduction in the above sense will not be well defined. Various definitions of reduction are possible, depending upon which aspects of the theory are considered primary. (Of course, each such definition should agree with regular reduction when both apply.) Below we present two such definitions (following [1]), although there are undoubtedly others.

The first type of reduction we shall consider is based upon the notion of an "observable". Following Bergman, we call a function on W an **observable** iff its Poisson bracket with each first class constraint is again a constraint, i.e., $h \in C^\infty(W)$ is an observable if and only if $\{h, I\} \subset I$. Bergman emphasized observables (rather than the points in V which are **states**) because observables represent measurable quantities. (The condition $\{h, I\} \equiv 0$ on V is a gauge invariance condition.) The set $\mathcal{O}(\mathcal{V})$ of observables forms a subalgebra of the associative algebra $C^\infty(W)$.

" Dirac reduction" takes two states $x, y \in V$ to be physically equivalent iff they cannot be distinguished by observables. This amounts to defining an equivalence relation $\sim$ on V by $x \sim y$ iff $h(x) = h(y)$ for all observables h. The corresponding reduced space is $\hat{V} = V/\sim$. The observables after reduction are identified with the elements of $\mathcal{O}(\mathcal{V})$ which are fixed under the adjoint action of I (with respect to Poisson bracket). Since we are dealing with first class constraints, these observables inherit a Poisson bracket.

Example: Zero angular momentum in two dimensions.

Here $W = T^*\mathbb{R}^2 \approx \mathbb{R}^2 \times \mathbb{R}^2 = \{(q,p)\}$ and the angular momentum is $q \times p = q_1p_2 - q_2p_1$ with constraint set $V = \{(q,p) | q_1p_2 - q_2p_1 = 0\}$. The foliation $\mathcal{F}$ is in fact given by the orbits of the standard circle action on $\mathbb{R}^2$ lifted to $T^*\mathbb{R}^2$. The Dirac reduction can be identified with the symplectic orbifold $\mathbb{C}/Z_2$.

Sniatycki and Weinstein [43] have defined an algebraic reduction in the context of group actions and momentum maps which is guaranteed to produce a reduced Poisson algebra but not necessarily a reduced space of states (cf. [50]). (In contrast, Kostant and Sternberg use the Marsden-Weinstein reduction [39].) The S-W (Sniatycki and Weinstein) reduced Poisson algebra is $(C^\infty(W)/I)^G$ where $V = J^{-1}(0)$ for some equivariant Poisson map $J : W \to \mathfrak{g}^*$ (called a **moment** map), equivariant with respect to a given G-action on W, with $\mathfrak{g}$ being the Lie algebra of G. (If G is compact and connected, $(C^\infty(W)/I)^G$ is isomorphic to the Dirac reduction $C^\infty(W)^G/I^G$.) With hindsight, the generalization of S-W reduction to a general first class constraint ideal I is obvious. The issue of its suitability is not one of geometry necessarily, but rather one of physics.

The present paper grew out of the realization that the BFV construction could be regarded as a homological model which in degree zero models the I-invariants of $C^\infty(W)/I$. The whole construction turned out in many cases to be a model for the complex of longitudinal forms $\Omega^*(V,\mathcal{F})$. From an algebraic geometric point of view, it is indeed natural to define the observables on V by restriction of observables on W, that is, to consider the quotient algebra $C^\infty(W)/I$, which corresponds to the algebra of smooth (in-the-sense-of Whitney) functions on V. In physics, this is expressed by saying two functions on W are weakly equal ($f \approx g$) if their difference vanishes on V.

Now let us recast the problem in purely algebraic terms. Consider an arbitrary Poisson algebra P with an ideal I which is closed under the Poisson bracket. Reduction is then achieved by passing to the

I-invariant subalgebra of P/I. Note that a class $[g]$ is I-invariant if $\{I, g\} \subset I$, equivalently, if $\{\phi, g\} \approx 0$ for all constraints $\phi \in I$. This subalgebra inherits a Poisson bracket even though P/I does not: For $f, g \in P$ and $\phi \in I$, we have $\{f + \phi, g\} = \{f, g\} + \{\phi, g\}$ where $\{\phi, g\}$ need not belong to I, but will if the class of g is I-invariant.

The Poisson algebra of invariants amounts to the quotient $N_P(I)/I$ where $N_P(I)$ denotes the normalizer of I in P in the sense of Lie algebras; the ideal I is a Poisson ideal in $N_P(I)$.

In this context, the analog of longitudinal forms are the alternating multilinear-over-P/I functions $h : I/I^2 \otimes \cdots \otimes I/I^2 \to P/I$ which again form a graded commutative algebra, which we denote

$$Alt_{P/I}(I/I^2, P/I).$$

We use I/I^2 because the corresponding Hamiltonian vector fields are restricted to V in providing the foliation $\mathcal{F}$.

The fact that I is a sub-Lie algebra of P but is not a Lie algebra over P (the bracket is $\mathbb{R}$-linear but not P-linear) is a significant subtlety. One way to handle this is to observe that I/I^2 inherits the structure of what Rinehart called an $(\mathbb{R}, P/I)$-**Lie algebra**. This corresponds to what Herz [34] called a quasi-Lie algebra and what Palais [40] called a d-Lie ring. Since it is Rinehart's paper that establishes the relation to the geometry and was his major contribution in a tragically short career, we prefer to refer to the Lie-Rinehart pair $(I/I^2, P/I)$.

Definition 2.1 [42],[40]. A Lie-Rinehart pair (L, A) over a ground ring k consists of a commutative k-algebra A and a Lie ring L over k which is a module over A together with an A-morphism $\rho : L \to Der\ A$ such that

$$[\phi, f\psi] = (\rho(\phi)f)\psi + f[\phi, \psi] \text{ for } \phi, \psi \in L, f \in A.$$

Notice this is the condition satisfied by $L = I/I^2$ and $A = P/I$ with $\rho(\phi)f = \{\phi, f\}$. Hence we can consider the **Rinehart complex** $Alt_{P/I}(I/I^2, P/I)$ with differential d given by

$$\begin{aligned} (dh)(\phi_0, ..., \phi_q) = &\sum_{i<j} (-1)^{i+j} h([\phi_i, \phi_j], ..., \hat{\phi}_i, ..., \hat{\phi}_j, ...) \\ &+ \sum_i (-1)^i \rho(\phi_i) h(..., \hat{\phi}_i, ...). \end{aligned} \tag{2.1}$$

Realizing that d is a derivation with respect to the usual product of alternating functions, it is sufficient to know the above definition for $q = 0$ and 1. This differential given by Rinehart [42] is an obvious generalization of that of Cartan-Chevalley-Eilenberg.

When P/I is replaced by $P = C^\infty(W)$ and I/I^2 by the Lie algebra corresponding to vector fields on W, the Rinehart complex becomes the de Rham complex of W. As remarked by Stephen Halperin, the Rinehart complex $Alt_{P/I}(I/I^2, P/I)$ is the complex $\Omega^*(V, F)$ of longitudinal forms, when $P = C^\infty(W)$ and I is a first class ideal. (See Huebschmann [24], [25], [28] for further applications of Rinehart's complex to Poisson algebras.)

This is the complex we wish to "model". We will do this using just the Poisson algebra structure of P and the sub-Lie algebra and P-ideal I, in contrast to the treatments of [13] and [11] which retain some of the local manifold properties of W.

3. Differential graded commutative algebras

One of the hallmarks of homological algebra is the use of resolutions; for differential homological algebra, "models", in the sense to be described, are more useful for many purposes. For our approach to constrained Hamiltonian systems, one of the basic objects is the deRham complex $(\Omega^*(M), d)$ of differential forms on a smooth manifold regarded as a DGCA (differential graded commutative algebra):

$\Omega^*(M) = \{\Omega^p(M)\}$ where $\Omega^p(M)$ denotes the (real) vector space of differential p-forms,

the wedge product $\omega \wedge \eta$ of forms gives $\Omega^*(M)$ the structure of a graded commutative algebra (over $\mathbb{R}$) : $\Omega^p \wedge \Omega^q \subset \Omega^{p+q}$ with $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$,

the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ is a graded derivation: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ and $d^2 = 0$.

Another DGCA that plays an important role in mathematical physics is the Cartan-Chevalley-Eilenberg complex $(\Lambda\mathfrak{g}^*, d)$ for the cohomology of a Lie algebra $\mathfrak{g}$. Here, if $\mathfrak{g}$ is finite dimensional, $\Lambda\mathfrak{g}^*$ is usually interpreted as the exterior algebra $E(\mathfrak{g}^*)$ on the $\mathbb{R}$-dual of $\mathfrak{g}$, but, in general, $\Lambda\mathfrak{g}^*$ should be interpreted as $Alt_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$, the algebra of alternating multilinear functions on $\mathfrak{g}$. The coboundary d is given by (2.1) with $\phi_i \in \mathfrak{g}$.

Rational homotopy theory is much simpler than ordinary homotopy theory because, for a large class of spaces, it is completely equivalent to the homotopy theory of DGCAs over the rationals [41]. Moreover, computations as well as theoretical analysis can be carried out effectively in terms of the models of Sullivan [46].

Definition 3.1. In the category of DGCAs over any k-algebra P, a **model** of a DGCA (A, d) is a morphism $\pi : (\mathcal{A}, \partial) \to (A, d)$ of DGCAs such that $\mathcal{A}$ is free as a graded commutative algebra over P and $\pi^* : H(\mathcal{A}) \approx H(A)$.

Here, free as a graded commutative algebra over P means $\mathcal{A}$ is of the form $P \otimes E(Z^{odd}) \otimes k[Z^{even}]$ where $E =$ exterior algebra and $k[\ ] =$ polynomial algebra and Z is some free graded vector space of finite type. Following the tradition in rational homotopy theory, the free graded commutative algebra on a graded vector space Z will be denoted ΛZ.

A major point of the Cartan-Chevalley-Eilenberg construction in the case of a compact Lie group G is a natural map $(\Lambda\mathfrak{g}^*, d) \to \Omega^*(G)$ inducing a homology isomorphism, i.e., a model for $\Omega^*(G)$.

The main thrust of this paper is the construction of a differential graded Poisson algebra which is, in many cases, a model for the forms along the leaves of the constraint variety of a first class system, in particular, H^0 will be isomorphic to the algebra of observables in the reduced sense: $(P/I)^I$.

4. Models for P/I and $Alt_{P/I}(I/I^2, P/I)$ and the BRST generator

Now we reverse the procedure of BFV and first provide a model for P/I as a P-module. This model is a DGCA $(P \otimes \Lambda\Psi, \delta)$ where Ψ is a graded vector space (in fact, negatively graded) and Λ continues to denote the free graded commutative algebra. (This grading is the opposite of the usual convention in homological algebra, but is chosen to correspond to the (anti-) ghost grading in the physics literature and because we are modelling a DGCA of differential forms.) This model is constructed as follows: Let Φ be the space of P-indecomposables of I , i.e., $\Phi = I/\bar{P}I$ where $\bar{P}$ is a complement to the constants $\mathbb{R} \subset P$. Let $s\Phi$ denote a copy of Φ but regarded as having degree -1. Let δ be the derivation of $P \otimes \Lambda s\Phi$ determined by choosing a splitting $\Phi \to I$ and factoring it as $\delta s : \Phi \to s\Phi \to I$. (In terms of representatives $\rho \in \Phi, \quad \delta\rho$ is $s^{-1}\rho$.) In other words, $P \otimes \Lambda s\Phi$ is the Koszul complex for the ideal I in the commutative algebra P [37], [7]. If I is what is now known as a regular (at one time: Borel) ideal (an algebraic condition, but implied by

I being the defining ideal in $C^\infty(W)$ for $V = J^{-1}(0)$ when 0 is a regular value of $J : W \to \mathbb{R}^N$), the Koszul complex $(P \otimes \Lambda s\Phi, \delta)$ is a model for P/I. For more general ideals, this fails, i.e., $H^i(P \otimes \Lambda s\Phi, \delta) \neq 0$ for some $i \neq 0$. The Tate resolution [47] kills this homology by systematically enlarging $s\Phi$ to a graded vector space Ψ and gives a model $(P \otimes \Lambda\Psi, \delta)$ as desired. We refer to this model as K_I for brevity. It is graded by the grading on Ψ extended multiplicatively, δ being still of degree 1.

Now we wish to replace P/I by K_I in $Alt_{P/I}(I/I^2, P/I)$ with the Rinehart generalization of the Cartan-Chevalley-Eilenberg differential d and further alter it to a model which is itself a (graded) Poisson algebra. The construction can be carried out quite generally, but we succeed in showing it is a model in our sense most easily in the case of a regular ideal, which obtains under reasonable geometric conditions. Following the major theorem, we describe a few other cases in which the model property also holds. (Lars Kjeseth is continuing the purely algebraic version of this class of examples. Kimura [35] has shown that for constraints which are not first class, the corresponding complex is NOT in general a model for the complex of forms along the leaves.)

Theorem 4.1. *If I is a first class ideal, there is a structure of differential graded Poisson algebra on $(\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi$ and a map of differential graded Poisson algebras*

$$\pi : ((\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi), \partial) \to Alt_{P/I}(I/I^2, P/I)$$

which induces an isomorphism on cohomology in degree zero. Here ∂ is $\delta + d+$ "terms of higher order" in a sense to be made precise below.

The algebra structure on $(\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi$ is that of the algebra of graded symmetric multilinear functions. The map π is fairly straightforward. Map $P \otimes \Lambda\Psi \to P/I$ by projection onto P and then by the quotient onto P/I. Similarly project $(\Lambda\Psi)^*$ onto $(\Lambda s\Phi)^*$ (recall $s\Phi$ is a summand of Ψ) and then, identifying $(\Lambda s\Phi)^*$ with $Alt(\Phi, R)$, map this to $Alt_P(I, \mathbb{R})$ by pulling back over the quotient $I \to I/\bar{P}I = \Phi$. Finally, note the isomorphism of algebras $Alt_P(I, P/I) \approx Alt_{P/I}(I/I^2, P/I)$.

We will construct the differential ∂ without any assumption on the ideal I other than that it is first class. The entire construction $((\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi), \partial)$ we will denote by X. In the full generality of a first class ideal, we will show $H^i(X) = 0$ for $i < 0$ and $H^0(X) \approx (P/I)^I$ and moreover the isomorphism is given by the inclusion $(P/I)^I \hookrightarrow P/I \hookrightarrow P \otimes \Lambda\Psi$ via the chosen splitting $P/I \hookrightarrow P$. This then gives a "no-ghost theorem": $H^0(X)$ is represented completely by elements of P without any ghost (or antighost) contributions from $\Lambda\Psi^*$ (or $\Lambda\Psi$).

For $i > 0$, $H^i(X)$ must be represented with ghosts. When this involves only ghosts corresponding directly to constraints (i.e., elements of $(s\Phi)^*$) but no ghosts-of-ghosts, "geometrically" we are looking at longitudinal forms. It is only from the transverse ("gauge-fixed") point of view that the ghosts inherit their name.

The key to the main theorem comes from the Hamiltonian and BRST formalisms. Let $(\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi$ be given a bigrading (r, s). Assuming P ungraded (see §6 for the graded or super case), $P \otimes \Lambda\Psi$ is already (negatively) graded and this grading is s, called the **resolution** degree. Then $(\Lambda\Psi)^*$ inherits the dual (positive) grading r, called the **ghost** degree, adopting the term from the physics literature (where the negative of the resolution degree is called the anti-ghost degree). The total degree is the sum $r + s$ of the ghost degree and the resolution degrees. Batalin, Fradkin, and Vilkovisky make X into a Poisson algebra by extending the Poisson bracket on P to one on X by defining

$$\{h, \psi\} = h(\psi) \quad \text{for} \quad h \in \Psi^*, \quad \psi \in \Psi,$$

all other brackets not determined by the derivation property being set equal to zero. This extended bracket is of total degree zero, but mixed bidegrees.

4.1. The BRST generator. The sought-for differential ∂ is constructed to be of the form $\partial = \{Q, \quad \}$ where Q is a formal sum of terms Q_n defined by induction (on n). In physics, Q is referred to as a BRST generator or operator, in keeping with the philosophy mentioned in §2 with particular emphasis on the facts that 1) $\partial^2 = 0$ or equivalently, $\{Q, Q\} = 0$ and 2) Q contains a piece corresponding to the Cartan-Chevalley-Eilenberg differential.

The proof of the existence of Q can be handled effectively by the "step-by-step obstruction" methods of homological perturbation theory [16], [18], [19], [17], [29]-[32], [33]. We adapt the details to this case, rather than appealing to the general theory. We make crucial use of the filtration of X by the form or monomial degree, i.e., $(\Lambda^i\Psi)^* \otimes P \otimes \Lambda\Psi$ is the part of X of form degree i, or equivalently, "form degree i" refers to an i-multilinear graded symmetric function from Ψ to $P \otimes \Lambda\Psi$. The filtration is defined by: $\mathcal{F}^n = \mathcal{F}^n X$ is the space of forms of degree $> n$. We use the strict inequality so that this filtration is multiplicative with respect to both parts of the Poisson algebra structure:

$$\mathcal{F}^p\mathcal{F}^q \subset \mathcal{F}^{p+q+1} \subset \mathcal{F}^{p+q} \text{ and } \{\mathcal{F}^p, \mathcal{F}^q\} \subset \mathcal{F}^{p+q}.$$

Start with $Q_0 : \Psi \to P \otimes \Lambda\Psi$ as the Koszul-Tate differential δ restricted to Ψ. As an element of X, this Q_0 is of total degree 1 and form degree 1, but $\{Q_0, \quad\}$ is a sum of two pieces, of form degree 0 (namely $1 \otimes \delta$) and of form degree 1. Since the bracket restricts to the pairing (by evaluation) of $(\Lambda\Psi)^*$ and $\Lambda\Psi$, the term of form degree 1 includes the adjoint of δ taking $Hom_P(\Lambda\Psi \otimes P, P)$ to itself. The remainder of $\{Q_0, \quad\}$ is given by the original bracket (in P) of the coefficients of Q_0 with elements of P.

Since all our objects are at least vector spaces, the model property of $P \otimes \Lambda\Psi$ can be evidenced by a "contracting homotopy" $s : P \otimes \Lambda\Psi \to P \otimes \Lambda\Psi$ of degree -1 such that $s\delta + \delta s = 1 - \bar{\pi}$ where $\bar{\pi} : P \otimes \Lambda\Psi \to P \to P/I \hookrightarrow P \otimes \Lambda\Psi$ is given by π composed with an $\mathbb{R}$-linear splitting $P/I \hookrightarrow P$.

For any element $R \in X$, let R^2 denote $\frac{1}{2}\{R, R\}$. Now construct $R_n = \sum_{i \le n} Q_i$ by induction so that

$$\{R_n, R_n\} \in \mathcal{F}^{n+2} \quad \text{and} \quad \delta\{R_n, R_n\} \in \mathcal{F}^{n+3}.$$

Define $Q_{n+1} = -s/2\{R_n, R_n\} = -sR_n^2$.

The following slightly complicated computation shows R_{n+1} satisfies the inductive assumption.

Both δ and s preserve the filtration, and from the way Q_0 is defined, $\{Q_0, \quad\} - 1 \otimes \delta$ increases filtration. Start with

$$R_{n+1}^2 = (R_n + Q_{n+1}^2)^2 = R_n^2 - \{R_n, sR_n^2\} + (sR_n^2)^2.$$

The last term $(sR_n^2)^2 \in \mathcal{F}^{2n+4}$ since $sR_n^2 \in \mathcal{F}^{n+2}$ and $2n + 4 \ge n + 4$. On the other hand,

$$\{R_n, sR_n^2\} \equiv (1 \otimes \delta)(sR_n^2) \quad mod \quad \mathcal{F}^{n+3},$$

since $R_n = Q_0 + Q_1 + \ldots$ and the $\{Q_i, \quad\}$ for $i > 0$ increase filtration. Thus

$$\{R_n, sR_n^2\} \equiv -(1 \otimes s\delta)R_n^2 + R_n^2 \quad mod \quad \mathcal{F}^{n+3},$$

so

$$R_{n+1}^2 \equiv -(1 \otimes s\delta)R_n^2 + R_n^2 \quad mod \quad \mathcal{F}^{n+3} \tag{4.1}$$

$$\equiv 0 \quad mod \quad \mathcal{F}^{n+3} \tag{4.2}$$

by the assumption on δR_n^2 .

Similarly

$$\delta R_{n+1}^2 \equiv \delta R_n^2 - \delta\{R_n, sR_n^2\} + \delta(sR_n^2)^2 \tag{4.3}$$

$$\equiv \delta R_n^2 \quad mod \quad \mathcal{F}^{n+4}. \tag{4.4}$$

Now we need to commute δ with $\{R_n, \quad\}$. Since $\{R_n, \quad\} - 1 \otimes \delta$ increases filtration by at least one, its square does so by at least two. Thus

$$\{R_n, \{R_n, \quad\}\} - \{R_n, 1 \otimes \delta\} - 1 \otimes \delta\{R_n, \quad\}$$

applied to sR_n^2 is of filtration at least $n+4$. Now the graded Jacobi identity gives

$$2\{R_n, \{R_n, \quad\}\} = \{\{R_n, R_n\}, \quad\},$$

which increases filtration by $n+2$, thus

(4.5)

$$\delta R_{n+1}^2 \equiv \delta R_n^2 + \{R_n, \delta s R_n^2\} \quad mod \quad \mathcal{F}^{n+4}$$

$$\equiv \delta R_n^2 + \{R_n, R_n^2\} - \{R_n, s\delta R_n^2\} \quad mod \quad \mathcal{F}^{n+4} \tag{4.6}$$

$$\equiv \delta R_n^2 - \delta R_n^2 \quad mod \quad \mathcal{F}^{n+4} \tag{4.7}$$

since $\{R_n, s\delta R_n^2\} \equiv \delta s \delta R_n^2 \quad mod \mathcal{F}^{n+4}$ and $\{x, \{x, x\}\} = 0$ for x of any degree (over a field of characteristic not equal to 3).

Thus we have constructed a differential graded Poisson algebra for any coisotropic ideal. Where possible, we will show that we have a model for $Alt_{P/I}(I/I^2, P/I)$ by the usual techniques of comparison in homological perturbation theory, namely comparison of spectral sequences. In one final case, we can do this locally but appeal to a geometric arguement to patch the local results. After establishing that, we will look at issues involving choices (possibly non-minimal) of generators (constraints) for the ideal I.

From the definition of the filtration $\mathcal{F}^p$, the associated graded $E_0(X)$ is isomorphic to $(\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi$. To analyze d_0, notice that since s preserves the form degree, $Q_{i+1} = -sR_i^2 \in \mathcal{F}^{i+2}$ and hence $\{Q_i, \ \}$ increases filtration by at least i. As mentionned earlier, $\{Q_o, \ \} - 1 \otimes \delta$ also increases filtration so d_0 is (induced by) the Koszul differential δ. Thus

$$E_1(X) \approx (\Lambda\Psi)^* \otimes P/I \approx Alt_{\mathbb{R}}(\Psi, P/I),$$

and $E_1(X)$ is concentrated in anti-ghost degree 0; the spectral sequence necessarily collapses from E_2. To determine $H^0(X) \approx E_2^{0,0}$, we need only analyze d_1 on Φ. For $h \in P/I$, consider $\{Q_0, h\} : I \to P/I$. It is given by $\{Q_0, h\}(\phi) = \{\phi, h\}$ for $\phi \in I$. Thus $H^0(X)$ is isomorphic to the I- invariants of P/I.

When the ideal I is regular, $\Psi = s\Phi$ and we can analyze d_1 on $\Lambda s\Phi$ similarly. For example, for $h : I \to P/I$, consider $\{Q_0, h\} : I \wedge I \to P/I$. It is given by $\{Q_0, h\}(\phi_1, \phi_2) = \{\phi_1, h(\phi_2)\} - \{\phi_2, h(\phi_1)\}$ while $\{Q_1, h\}(\phi_1, \phi_2) = -\frac{1}{2}\{s\{Q_0, Q_0\}, h\}(\phi_1, \phi_2) = -h\{\phi_1, \phi_2\}$. (At this point, one appreciates the facility of non-invariant description in terms of a generating set of constraints $\{\phi_\alpha\}$ for I and a dual set $\{\eta^\alpha : I \to P\}$.)

Thus we see d_1 (up to sign) looks like the Rinehart generalization of the Cartan-Chevalley-Eilenberg differential. It is this identification of (E_1, d_1) which motivates the name **BRST generator** for Q.

Now to make the comparison with the complex of longitudinal forms, since Φ is defined as a quotient of I, there is the induced chain map

$$\begin{aligned} \pi : X \to Alt_k(\Psi, P/I) \to Alt_k(s\Phi, P/I) &\to Alt_P(I, P/I) \\ &\approx Alt_{P/I}(I/I^2, P/I) \end{aligned}$$

as described above. In the regular case all maps except π are isomorphisms. For the constrained Hamiltonian setting with which we began, in which P is $C^\infty(W)$, we have identified $Alt_{P/I}(I/I^2, P/I)$ with the longitudinal forms of the foliation $\mathcal{F}$ of V and d_1 with the exterior derivative "along the leaves".

Theorem 4.2. *If I is a regular first class ideal in $C^\infty(W)$, the map π induces an isomorphism $H(X) \approx H(\Omega(V, \mathcal{F}))$.*

When I is not regular, we still have the map but in general lack sufficient information to conclude an isomorphism in cohomology.

Now the physicists do not work with the ideal explicitly but rather with a set of constraints, which is a set (not necessarily minimal) of generators for the ideal. The corresponding BFV construction starts with Φ as the vector space spanned by the constraints, rather than with $I/\overline{P}I$. In certain cases, even though the constraints do not form a regular sequence, we can still make the identification of $H(X)$ with $H(\Omega(V, \mathcal{F}))$.

The redundant case: The set of constraints may be reducible in a trivial way; a proper subset may consist of a regular sequence of generators. Then we can split Φ as $\Xi \oplus \Upsilon$ where Ξ is the span of the minimal subset and Υ is spanned by the complementary subset. The

Koszul-Tate resolution of P/I splits as the Koszul resolution determined by Ξ tensored with a contractible DCGA. Then $Alt(\Psi, \ )$ splits similarly and the BRST generator can be constructed first in the Ξ part and then extended so the results will be the same as when using $\Phi = I/\overline{P}I$.

In particular, if the constraints are given by an equivariant moment map $J : W \to \mathfrak{g}^*$ where G acts by symplectomorphisms but with kernel H, then $I/\overline{P}I$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$ but the span of the constraints would be isomorphic to $\mathfrak{g}$. Here choose a splitting $\Xi \oplus \Upsilon$ such that $\Upsilon = \mathfrak{h}$ and $\Xi \approx \mathfrak{g}/\mathfrak{h}$, then proceed as in the redundant case.

In [13] and [23], the setting is specifically that of a symplectic manifold (phase space) with a constraint submanifold ("surface") and moreover the assumption is made that locally the constraints can be separated into "independent constraint functions" and dependent ones which can be expressed as functional linear combinations of the independent ones with coefficients which are regular in a neighborhood of the constraint submanifold. Thus locally we are in the redundant case so identities involving the globally defined BRST generator and comparisons with the complex of forms along the leaves can be verified locally; we again have $H(X) \approx H(\Omega(V, \mathcal{F}))$.

Finally, the construction of ∂ and of Q involves a choice of contracting homotopy s and implicitly of a choice of splitting $P/I \hookrightarrow P$. A change in s produces changes in ∂ but not in the homotopy type of (X, ∂) as DGCA. Moreover the change in s can be realized by an automorphism of $\Lambda\Psi$ and the induced one on $\Lambda\Psi^*$. This is an example of what is known as a canonical transformation, a basic automorphism of any Hamiltonian system.

5. Generalizations: Infinite dimensions and super-algebras

If I is regular and finitely generated over P (so Φ is finite dimensional over $\mathbb{R}$), $Alt_P(\Phi, P \otimes s\Phi)$ is finitely generated as a P-module and $Q_n = 0$ for sufficiently large n. If I is finitely generated but not regular, Ψ may easily be infinite dimensional, though finite in each grading, and so all Q_n may be non-zero.

More importantly, there are many examples occurring in physics (field theory) in which Φ is itself infinite dimensional. That is why we have been careful to emphasize Alt or to take the dual of $\Lambda\Psi$ rather than $\Lambda(\Psi^*)$. Actually both physical and mathematical considerations (cf. Gel'fand-Fuks cohomology) suggest that the alternating functions might better be restricted to being continuous in an appropriate topology.

Early in the development of Batalin, Fradkin and Vilkovisky's approach, attention was called to the generalization to a super-Poisson algebra $P = P_0 \oplus P_1$ with super-constraints. This means that P is a GCA (graded by $Z/2 = \{0,1\}$) with a graded bracket $\{\ \ ,\ \ \}$:

$$P_0 \otimes P_0 \to P_0, \tag{5.1}$$

$$P_0 \otimes P_1 \to P_1, \tag{5.2}$$

$$P_1 \otimes P_1 \to P_0 \tag{5.3}$$

with graded anticommutativity, graded Jacobi identity, and graded derivation property (Leibnitz rule):

$$\{f, gh\} = \{f, g\}h + (-1)^{|f||g|} g\{f, h\},$$

where $f \in P_{|f|},\ g \in P_{|g|}$.

It has long been known in algebraic topology how to generalize the construction of models such as the Koszul-Tate complex or the Chevalley-Eilenberg complex to the graded setting, e.g.,Ψ is now a graded vector space and $s\Phi$ is an isomorphic copy of a Φ regraded down by 1 so that δ is still of degree 1. The use of Λ to denote the free graded commutative algebra on a graded vector space means that the only necessary change in our treatment is to specify the resolution degree as the one implied by the degree on $s\Phi$ with δ being of resolution degree 1. Notice this is not the same as ignoring the internal grading on $s\Phi$ and just counting the algebraic degree. (It is spelled out in [17] for example.) From there on, the signs take care of themselves if we follow the usual conventions, introducing a sign $(-1)^{pq}$ whenever a term of total degree p is pushed past a term of total degree q.

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