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ON THE TOPOLOGICAL ENTROPY OF GEODESIC FLOWS

RICARDO MAÑÉ

1. Introduction

Let M^n be a closed connected C^∞ manifold and let SM be its unit tangent bundle, defined as usual as $SM = \{\theta = (x, v) : \ x \in M, \ v \in T_xM, \ ||v|| = 1\}$. The geodesic flow $\varphi_t : SM \to SM$ is defined by $\varphi_t(x, v) = (\gamma(t), \dot\gamma(t))$, where $\gamma : \mathbf{R} \to M$ is the geodesic with initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = v$.

Given x and y in M, define $n_T(x, y)$ as the number of geodesics of length $\leq T$ (parametrized by arc length) joining x and y. A standard application of Sard's Theorem to the exponential maps of M shows that $n_T(x, y)$ is finite and locally constant on an open full measure subset of $M \times M$.

Our aim is to relate *the exponential growth rate of* $n_T(x, y)$, as a function of T, with the *topological entropy of the geodesic flow* $h_{top}(\varphi)$. In that direction, among other results, we shall prove that

$$h_{top}(\varphi) = \lim_{T\to+\infty} \frac{1}{T} \log \int_{M\times M} n_T(x, y)\, dxdy.$$

While proving this result, we shall also prove that *Przytycki's upper estimate* for the topological entropy of general C^2 flows [8], *is always an equality for* C^∞ *geodesic flows.* Since Przytycki's inequality will be a key tool in our proofs we begin by recalling its statement. Given a

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linear map $L : E \to F$ between finite dimensional Hilbert spaces, we define its *expansion* $ex(L)$ by

$$ex(L) = \max_S |\det(L|_S)|,$$

where the maximum is taken over all subspaces $S \subset E$. Przytycki's inequality states that *for a C^2 flow $\psi_t : N \to N$ on a closed manifold N,*

$$h_{top}(\psi) \leq \liminf_{t\to\infty} \frac{1}{t} \log \int_N ex(d_x\psi_t)\, dx.$$

The main result of this paper is Theorem 1.1 below. Half of its proof relies on a combination of Yomdin's theorem [9] and a formula due to Berger and Bott [2] that, although not difficult to prove, provides the link between the numbers $n_T(x, y)$ and the dynamics of the geodesic flow. We took this combination from [6], but for our purposes an improved version of Yomdin's theorem will be needed. The other half relies on Przytycki's inequality, Berger and Bott's formula, and a careful change of variables, where the novelty of this work resides.

Theorem 1.1.

$$\begin{aligned} h_{top}(\varphi) &= \lim_{T\to+\infty} \frac{1}{T} \log \int_{M\times M} n_T(x, y)\, dxdy \\ &= \lim_{T\to+\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta\varphi_T)\, d\theta. \end{aligned}$$

When the manifold has no conjugate points, it is easy to check, using the fact that the exponential map $exp_x : T_xM \to M$ is a covering map admitting a fundamental domain of diameter $\leq diam(M) \stackrel{\text{def}}{=} c$, that for any $x_1, y_1, x_2, y_2 \in M$,

$$n_T(x_1, y_1) \leq n_{T+2c}(x_2, y_2).$$

This property and Theorem 1.1 imply:

Corollary 1.2. *If M has no conjugate points, then*

$$\lim_{T\to+\infty} \frac{1}{T} \log n_T(x, y) = h_{top}(\psi),$$

for all $x, y \in M$.

From this corollary we can recover the results of Freire and Mañé [3] on the equality of the topological entropy of the geodesic flow and the volume growth rate of the manifold when there are no conjugate points. Recall that the volume growth rate $\lambda(M)$ of M is defined by

$$\lambda(M) = \lim_{r \to +\infty} \frac{1}{r} \log \mathrm{Vol}(B_r(x)),$$

where $B_r(x)$ denotes the ball of radius r and center x in the universal covering $\tilde{M}$ of M, and $\mathrm{Vol}(B_r(x))$ denotes its volume. Manning proved [5] that this limit exists and is independent of x. Moreover he proved that $\lambda(M) \leq h_{top}(\varphi)$ for *every* closed Riemannian manifold and $\lambda(M) = h_{top}(\varphi)$ when M has sectional curvatures ≤ 0. In [3], Freire and Mañé extended this result to manifolds *without conjugate points* through a different technique (see Remark 1.5 below). Here we can obtain it from Corollary 1.2.

Corollary 1.3. *If M has no conjugate points, then*

$$\lambda(M) = h_{top}(\varphi).$$

To deduce Corollary 1.3 from Corollary 1.2, it suffices to show that $\lambda(M) \geq h_{top}(\varphi)$, because, as we explained above, the inequality $\lambda(M) \leq h_{top}(\varphi)$ always holds. If $p : \tilde{M} \to M$ is the covering map and $x \in \tilde{M}$, then the number $n_T(p(x), p(x))$ is just the number of points in the set

$$G_T \stackrel{\text{def}}{=} \{z \in M : \ p(z) = p(x), \ d(z, x) \leq T\}.$$

Observe that there exists $r_0 > 0$ such that any two distinct points z' and z'' with $p(z') = p(z'')$, satisfy $d(z', z'') \geq r_0$. Clearly

$$B_{T+r_0}(x) \supset \bigcup_{z \in G_T} B_{r_0}(z),$$

and the sets $B_{r_0/2}(z)$, $z \in G_T$ are disjoint. Hence if

$$k \stackrel{\text{def}}{=} \min\{\mathrm{Vol}(B_{r_0/2}(a)) : \ a \in \tilde{M}\},$$

we have

$$\mathrm{Vol}(B_{T+r_0}(x)) \geq k \# G_T = k \, n_T(p(x), p(x)).$$

Therefore

$$\begin{aligned}\lambda(M) &= \lim_{T\to+\infty} \frac{1}{T+r_0} \log \mathrm{Vol}(B_{T+r_0}(x)) \\ &\geq \lim_{T\to+\infty} \frac{1}{T+r_0} \log k\, n_T(p(x), p(x)) = h_{top}(\varphi),\end{aligned}$$

thus completing the proof of the corollary.

In the next statement, $V(\theta)$ will denote the vertical fibre at $\theta = (x, v) \in SM$, defined by

$$V(\theta) = d_\theta \pi^{-1}(\{0\}),$$

where π is the projection map $\pi : SM \to M$.

Theorem 1.4.

$$\begin{aligned}h_{top}(\varphi) &= \lim_{T\to+\infty} \frac{1}{T} \log \int_{SM} |\det(d_\theta \varphi_T|_{V(\theta)})|\, d\theta \\ &= \lim_{T\to+\infty} \frac{1}{T} \log \int_M \mathrm{Vol}(\varphi_T(S_x M))\, dx.\end{aligned}$$

The second equality is trivial because

$$\mathrm{Vol}(\varphi_T(S_x M)) = \int_{S_x M} |\det(d_{(x,v)} \varphi_T|_{V(x,v)})|\, dv.$$

Remark 1.5. Manning's proof of Corollary 1.3 for manifolds of non-positive curvature relies on the fact that for such manifolds, any two geodesics $\gamma_i : \mathbf{R} \to \tilde{M}$ in the universal covering satisfy:

$$d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma_1(0), \gamma_2(0)) + d(\gamma_1(T), \gamma_2(T)),$$

for all $0 \leq t \leq T$. In [3] the authors observed that for manifolds where this property holds with the right term multiplied by a constant independent of the geodesics, Manning's proof can be applied with insignificant changes, thus providing a much simpler proof of their result *if* the existence of such a constant could be established for manifolds without conjugate points. However the example of Ballmann, Brin and Burns [1], that appeared four years later, proved that such a constant *does not exist*.

Remark 1.6. On the question of the exponential growth rate of $n_T(x,y)$, without assumptions like the absence of conjugate points we have little to say; only that (see Appendix) as a corollary of a result of G. Paternain [6], it follows that for every $x \in M$,

$$\limsup_{T\to+\infty} \frac{1}{T} \log n_T(x,y) \leq h_{top}(\varphi),$$

for a.e. $y \in M$. This naturally poses the following question:

Problem I. Is it true that for a.e. $(x,y) \in M \times M$,

$$\lim_{T\to+\infty} \frac{1}{T} \log n_T(x,y) = h_{top}(\varphi)? \tag{0.1}$$

Since an affirmative answer to this problem may sound too good to be true, a humbler, and more feasible question is the following one:

Problem II. Is it true that (0.1) holds for generic Riemannian metrics when $\dim M = 2$?

Remark 1.7. Theorem 1.1 above was announced, and its proof sketched, in a preprint of the author circulated in January 93. The proof that we shall use here is essentially shorter than that outlined there.

The proof of Theorem 1.1 relies on three inequalities. Two of them (inequalities (A) and (A′) below) are where the novelty of this paper resides. The other inequality (inequality (B) below) is a variation of an inequality due to G. Paternain [6], which in turn comes from Yomdin's theorem [9]. One way to prove this inequality is to combine G. Paternain's method with a certain uniformity in Yomdin's theorem, visible in Gromov's exposition of this celebrated result in the Bourbaki Seminaire [4]. Such was the method employed in our announcement. Afterwards, G. Paternain and M. Paternain [7] proved that inequality using directly Yomdin's theorem without having to appeal to the uniformity mentioned above.

2. Proof of Theorem 1.1

We begin by recalling the basic formalism of geodesic flows, stressing its symplectic properties, which will play a key role in our proofs.

Given $\theta = (x, v) \in SM$ define

$$E(\theta) = \{w \in T_xM : \langle w, v\rangle = 0\}.$$

Denote by $\pi : SM \to M$ the canonical projection and set

$$N(\theta) = d_\theta\pi^{-1}(E(\theta)),$$

$$V(\theta) = d_\theta\pi^{-1}(\{0\}).$$

If X is the geodesic vector field on SM (i.e., the vector field generated by the geodesic flow), then $T_\theta SM$ is the direct sum of $N(\theta)$ and the one-dimensional subspace spanned by $X(\theta)$. Moreover, N is φ_t-invariant, i.e.,

$$d_\theta\varphi_t(N(\theta)) = N(\varphi_t(\theta)),$$

for all $\theta \in SM$ and $t \in \mathbf{R}$. On each N_θ there exist an inner product $< , >_\theta$ and an isometry $J_\theta : N(\theta) \to N(\theta)$ of this inner product, such that for all θ:

a) $J_\theta^2 = -I$;
b) $V(\theta)$ and $J_\theta V(\theta)$ are orthogonal;
c) $\langle J_{\varphi_t(\theta)} d_\theta\varphi_t(\zeta), d_\theta\varphi_t(\eta)\rangle_{\varphi_t(\theta)} = \langle J_\theta\zeta, \eta\rangle_\theta$;
d) $\langle\ ,\ \rangle_\theta$ and J_θ are C^∞ functions of θ.

We shall say that a subspace $S \subset N(\theta)$ is Lagrangian if its orthogonal complement $S^\perp$ is $J_\theta S$.

These properties can be translated into symplectic terms by defining, for each $\theta \in SM$, a 2-form $\omega_\theta : N(\theta) \times N(\theta) \to \mathbf{R}$ by

$$\omega_\theta(\zeta, \eta) = \langle J_\theta\zeta, \eta\rangle_\theta.$$

Then ω_θ is non-degenerate (by (a)) and $d\varphi_t$-invariant (by (c)). A subspace S is now Lagrangian if and only if $\dim S = \dim N(\theta)/2$ and $\omega_\theta|_{S\times S} = 0$. By property (b), $V(\theta)$ is a Lagrangian subspace for all θ.

The above discussion presents the properties of $\langle\ ,\ \rangle_\theta$ and J_θ that will be used here. We now give their definitions. First we define the so-called connector map $K_\theta : T_\theta SM \to T_xM$. Given $\zeta \in TT_\theta SM$, let $\theta(s) = (x(s), v(s))$ be a curve in SM such that $\theta(0) = \theta$ and $\dot\theta(0) = \zeta$. Then $K_\theta(\zeta) = Dv/ds(0)$, i.e., $K_\theta(\zeta)$ is the covariant derivative at $s = 0$ of the vector field $v(s)$ along the curve $x(s)$. There is a linear isomorphism $i_\theta : N(\theta) \to E(\theta) \oplus E(\theta)$ defined by

$$i_\theta(\zeta) = (d_\theta\pi\zeta, K_\theta\zeta).$$

The map J_θ is the pullback by i_θ of the rotation $(v, v') \to (v', -v)$ on $E(\theta) \oplus E(\theta)$, and

$$\langle \zeta, \eta \rangle_\theta = \langle d_\theta \pi \zeta, d_\theta \pi \eta \rangle + \langle K_\theta \zeta, K_\theta \eta \rangle.$$

The proof of Theorem 1.1 consists of proving the following inequalities:

$$A) \quad \liminf_{T \to +\infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y)\, dx dy \geq \liminf_{T \to +\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta.$$

$$A') \quad \limsup_{T \to +\infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y)\, dx dy \geq \limsup_{T \to +\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta.$$

$$B) \quad \limsup_{T \to +\infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y)\, dx dy \leq h_{top}(\varphi).$$

These inequalities, plus Przytycki's inequality imply Theorem 1.1.

First we shall prove (A) and (A$'$). For this purpose, and also for the proof of inequality (B), we shall introduce, following Berger and Bott [2], a number $A(\theta, t)$ associated to each $\theta = (x, v) \in SM$ and $t > 0$, defined by

$$A(\theta, t) = |\det(d_\theta(\pi \circ \varphi_t)|_{V(\theta)})|.$$

Berger and Bott proved ([2])

$$\int_0^T \int_{S_x M} A((x, v), t)\, dv dt = \int_M n_T(x, y)\, dy.$$

Integrating this equality over M we obtain

$$\int_0^T \int_{SM} A(\theta, t)\, d\theta dt = \int_{M \times M} n_T(x, y)\, dx dy.$$

Next observe that:

$$\liminf_{T \to +\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta = \liminf_{T \to +\infty} \frac{1}{T} \log \int_0^T \int_{SM} ex(d_\theta \varphi_t)\, d\theta dt,$$

and

$$\limsup_{T\to+\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta$$
$$= \limsup_{T\to+\infty} \frac{1}{T} \log \int_0^T \int_{SM} ex(d_\theta \varphi_t)\, d\theta dt.$$

Therefore the proof of (A) and (A′) is reduced to showing:

$$\liminf_{T\to+\infty} \frac{1}{T} \log \int_0^T \int_{SM} A(\theta,t)\, d\theta dt$$
$$\geq \liminf_{T\to+\infty} \frac{1}{T} \log \int_0^T \int_{SM} ex(d_\theta \varphi_t)\, d\theta dt, \tag{0.1}$$

and

$$\limsup_{T\to+\infty} \frac{1}{T} \log \int_0^T \int_{SM} A(\theta,t)\, d\theta dt \tag{0.2}$$
$$\geq \limsup_{T\to+\infty} \frac{1}{T} \log \int_0^T \int_{SM} ex(d_\theta \varphi_t)\, d\theta dt. \tag{0.3}$$

Before going into the formal proof of these inequalities, let us informally describe the strategy we shall follow. Obviously the proof will rely on the analysis of the maps $d_\theta\varphi_t : N(\theta) \to N(\varphi_t(\theta))$. The key idea is that the analysis becomes more accurate for our objective if we shift the initial point θ backwards to a certain point $\varphi_{-\tau_1}(\theta)$, $\tau_1 > 0$, and shift the final point $\varphi_t(\theta)$ forward to $\varphi_{t+\tau_2}(\theta)$, $\tau_2 > 0$. It will be important that these numbers τ_1 and τ_2 can be chosen so that they are measurable functions of the pair (θ, t), and their possible values lie in a finite set of the form $\{0, 1/m, 2/m, ..., 1\}$, where $m \geq 1$ is an integer. Clearly inequalities (0.1) and (0.3) follow from

Lemma 2.1. *There exists $C > 0$ such that for all $T > 0$,*

$$\int_0^{T+2} \int_{SM} A(\theta,t)\, d\theta dt \geq C \int_0^T \int_{SM} ex(d_\theta \varphi_t)\, d\theta dt.$$

To prove this lemma we first introduce a definition. For subspaces S_1, S_2 of $N(\theta)$ with $\dim S_1 = \dim S_2 = \dim N(\theta)/2$, we define

$$\alpha(S_1, S_2) = |\det(P|_{S_1})|,$$

where $P : N(\theta) \to S_2^{\perp}$ is the orthogonal projection. Clearly α depends continuously on the subspaces and $\alpha(S_1, S_2) = 0$ if and only if $S_1 \cap S_2 \neq \{0\}$.

The next two lemmas will be proved later.

Lemma 2.2. *For each $\theta \in SM$ and $t \in \mathbf{R}$, there is a Lagrangian subspace $R_t(\theta) \subset N(\theta)$, which depends measurably on t and θ, and satisfies:*

a) $|\det(d_\theta \varphi_t|_{R_t(\theta)})| = ex(d_\theta \varphi_t)$*;*

b) if S is a subspace of $N(\theta)$ with $\dim S = \dim N(\theta)/2$, then

$$|\det(d_\theta \varphi_t|_S)| \geq \alpha(S, R_t^{\perp}(\theta)) ex(d_\theta \varphi_t).$$

Lemma 2.3. *There exist $\delta > 0$, an integer $m \geq 1$ and measurable functions $\tau_i : SM \times \mathbf{R} \to \{0, 1/m, 2/m, ..., 1\}$, $i = 1, 2$, such that, after abbreviating $\tau_i(\theta, t)$ to τ_i for $i = 1, 2$ and setting $\tau = \tau(\theta, t) = \tau_1 + \tau_2$,*

$$\theta_1 = \varphi_{-\tau_1}(\theta), \quad \theta_2 = \varphi_{t+\tau_2}(\theta), \quad \textit{and} \quad V_i = V(\theta_i) \textit{ for } i = 1, 2,$$

we have for all θ and t:

a) $\alpha((d_{\theta_1}\varphi_{\tau_1})V_1, R_t^{\perp}(\theta)) \geq \delta$

and

b) $\alpha((d_{\theta_1}\varphi_{t+\tau})V_1, V_2) \geq \delta$*.*

From these two lemmas we shall deduce the following:

Corollary 2.4. *There exists $C > 0$ such that for all $t > 0$ and $\theta \in SM$, the functions τ_1 and τ given by Lemma 2.3 satisfy*

$$A(\varphi_{-\tau_1(\theta,t)}(\theta), t + \tau(\theta, t)) \geq C\, ex(d_\theta \varphi_t).$$

Proof of Corollary 2.4. Set $S = (d_{\theta_1}\varphi_{\tau_1})V_1$. By Lemma 2.2 and property (a) of Lemma 2.3 we have

$$|\det(d_\theta \varphi_t|_S)| \geq \alpha(S, R_t^{\perp}(\theta))\, ex(d_\theta \varphi_t) \geq \delta ex(d_\theta \varphi_t). \tag{0.4}$$

Take $C_1 > 0$ such that

$$|\det(d_\zeta \varphi_s|_L)| \geq C_1, \tag{0.5}$$

for every $\zeta \in SM$, $s \in [0,1]$, and every non-trivial linear subspace $L \subset N(\zeta)$. Set $\widehat{\theta} = \varphi_t(\theta)$ and $\widehat{S} = d_\theta \varphi_t S$. Then equations (0.4) and (0.5) imply

$$\begin{aligned} &|\det(d_{\theta_1}\varphi_{t+\tau}|_{V_1})| \\ (0.6)\qquad &= |\det(d_{\widehat{\theta}}\varphi_{\tau_2}|_{\widehat{S}})| \cdot |\det(d_\theta \varphi_t|_S)| \cdot |\det(d_{\theta_1}\varphi_{\tau_1}|_{V_1})| \\ &\geq C_1^2 \delta ex(d_\theta \varphi_t). \end{aligned}$$

Now set $S_2 = (d_{\theta_1}\varphi_{t+\tau})V_1$. By property (b) of Lemma 2.3 and the definition of α we have respectively

$$\alpha(S_2, V_2) \geq \delta,$$

$$|\det(d_{\theta_2}\pi|_{S_2})| > \delta, \tag{0.7}$$

which together with inequality (0.6) implies that

$$\begin{aligned} A(\varphi_{-\tau_1(\theta,t)}(\theta), t + \tau(\theta,t)) &= |\det(d_{\theta_1}(\pi \circ \varphi_{t+\tau})|_{V_1})| \\ &= |\det(d_{\theta_2}\pi|_{S_2})| \cdot |\det(d_{\theta_1}\varphi_{t+\tau}|_{V_1})| \\ &\geq \delta |\det(d_{\theta_1}\varphi_{t+\tau}|_{V_1}| \geq \delta^2 C_1^2 ex(d_\theta \varphi_t). \end{aligned}$$

Hence the proof of the corollary with $C = \delta^2 C_1^2$ is completed.

Before proving Lemmas 2.2 and 2.3, let us see how Lemma 2.1 follows from the corollary.

Define $F : SM \times [0,T] \to SM \times [0, T+2]$ by

$$F(\theta, t) = (\varphi_{-\tau_1(\theta,t)}(\theta), t + \tau(\theta,t))),$$

where $\tau_1(\theta,t)$ and $\tau(\theta,t)$ are defined as in Lemma 2.3. Given integers $0 \leq i \leq m$, $0 \leq j \leq m$, we define

$$A(i,j) = \{(\theta,t) \in SM \times [0,T] : \ \tau_1(\theta,t) = i/m, \ \tau_2(\theta,t) = j/m\}.$$

On each $A(i,j)$, F is injective, and if we set $d\mu = d\theta dt$, then $\mu(F(S)) = \mu(S)$ for every Borel set $S \subset A(i,j)$. Hence for any integrable function $\Phi : SM \times [0, T+2] \to \mathbf{R}$ we have

$$\int_{F(A(i,j))} \Phi \, d\mu = \int_{A(i,j)} (\Phi \circ F) \, d\mu.$$

Suppose now that $\Phi > 0$. Then

$$\begin{aligned}\int_{SM\times[0,T]}(\Phi\circ F)\,d\mu &= \int_{\bigcup_{i,j}A(i,j)}(\Phi\circ F)\,d\mu = \sum_{i,j}\int_{A(i,j)}(\Phi\circ F)\,d\mu\\ &= \sum_{i,j}\int_{F(A(i,j))}\Phi\,d\mu \le \sum_{i,j}\int_{SM\times[0,T+2]}\Phi\,d\mu\\ &= (m+1)^2\int_{SM\times[0,T+2]}\Phi\,d\mu.\end{aligned}$$

From this inequality and Corollary 2.4, it follows that

$$\begin{aligned}&\int_0^{T+2}\int_{SM}A(\theta,t)d\theta dt\\ &\qquad\ge\frac{1}{(m+1)^2}\int_{SM\times[0,T]}A(F(\theta,t))\,d\mu\\ &\qquad=\frac{1}{(m+1)^2}\int_0^T\int_{SM}A(\varphi_{-\tau_1(\theta,t)}(\theta),t+\tau(\theta,t))\,d\theta dt\\ &\qquad\ge\frac{C}{(m+1)^2}\int_0^T\int_{SM}ex(d_\theta\varphi_t)\,d\theta dt,\end{aligned}$$

thus completing the proof of Lemma 2.1.

Proof of Lemma 2.2. Consider the polar decomposition

$$d_\theta\varphi_t = L_t(\theta)O_t(\theta),$$

where $L_t(\theta) : N(\theta) \to N(\theta)$ is symmetric and positive, and $O_t(\theta) : N(\theta) \to N(\varphi_t(\theta))$ is an isometry, both being C^∞ functions of θ. Since $L_t(\theta) = ((d_\theta\varphi_t)^*(d_\theta\varphi_t))^{1/2}$ and $(d_\theta\varphi_t)^*$ is symplectic (because so is $d_\theta\varphi_t$), $L_t(\theta)$ is *symplectic and symmetric*. Thus, if ζ is an eigenvector of $L_t(\theta)$ associated to an eigenvalue λ, then $J_\theta\zeta$ is an eigenvector associated to the eigenvalue λ^{-1} because $L_t(\theta)J_\theta = J_\theta L_t(\theta)^{-1}$ (by the symmetry and the symplecticity of $L_t(\theta)$) and hence

$$L_t(\theta)J_\theta\zeta = J_\theta L_t(\theta)^{-1}\zeta = \lambda^{-1}J_\theta\zeta.$$

Using this property it is possible to construct for each t an orthonormal basis of $N(\theta)$ of the form $\{\zeta_1, ..., \zeta_{n-1}, J_\theta\zeta_1, ..., J_\theta\zeta_{n-1}\}$, where ζ_i is an eigenvector of $L_t(\theta)$ associated to an eigenvalue $\lambda_i \ge 1$. Let $R_t(\theta)$ be the subspace spanned by $\{\zeta_1, ..., \zeta_{n-1}\}$. Clearly $R_t(\theta)$ is Lagrangian and satifies property (a).

To prove property (b), observe first that

$$|\det(d_\theta\varphi_t|_S)| = |\det(L_t(\theta)|_S)|,$$

because $O_t(\theta)$ is an isometry. Notice also that $L_t(\theta)$ leaves $R_t(\theta)$ and $R_t^\perp(\theta)$ invariant, because both of these spaces are spanned by the eigenvectors of $L_t(\theta)$. Hence $L_t(\theta)$ commutes with the orthogonal projection $P : N(\theta) \to R_t(\theta)$, i.e., $L_t(\theta) \circ P = P \circ L_t(\theta)$. Let us suppose that $S \cap R_t^\perp(\theta) = \{0\}$, otherwise there is nothing to prove. Then $P(S) = R_t(\theta)$ and thus

$$\begin{aligned}|\det(L_t(\theta)|_{R_t(\theta)})||\det(P|_S)| &= |\det(P|_{L_t(\theta)(S)})||\det(L_t(\theta)|_S)| \\ &\leq |\det(L_t(\theta)|_S)|.\end{aligned}$$

Hence

$$ex(d_\theta\varphi_t)\alpha(S, R_t^\perp(\theta)) \leq |\det(d_\theta\varphi_t|_S)|.$$

Finally we show the measurability of $R_t(\theta)$ as a function of t and θ. Let $\mathcal{F}$ denote the vector bundle over SM consisting of pairs (θ, g), in which $\theta \in SM$ and $g : N(\theta) \to N(\theta)$ is a symmetric linear map. Given positive integers p and l_i, $1 \leq i \leq p$, let $\mathcal{F}(p, l_1, ...l_p)$ be the set of pairs $(\theta, g) \in \mathcal{F}$, where g has p eigenvalues $\lambda_1 < \cdots < \lambda_p$ with multiplicities $l_1, ..., l_p$. Then $\mathcal{F}(p, l_1, ...l_p)$ is a Borel set (check it) and so is the subset $\mathcal{P}(p, l_1, ..., l_p)$ of $SM \times \mathbf{R}$ defined by $\{(\theta, t) : \ L_t(\theta) \in \mathcal{F}(p, l_1, ..., l_p)\}$. Now observe that $R_t(\theta)$ can be chosen to be continuous on each set $\mathcal{P}(p, l_1, ..., l_p)$. Since these sets are Borel and there are finitely many of them, the measurability is proved.

Proof of Lemma 2.3. It suffices to prove that we can find $\delta_1, \delta_2 > 0$, integers $m_1, m_2 \geq 1$ and measurable functions $\tau_i : SM \times \mathbf{R} \to \{0, 1/m, 2/m, ..., 1\}$ such that properties (a) and (b) of Lemma 2.3 hold with δ changed to δ_1 in (a) and to δ_2 in (b). Then we can easily obtain Lemma 2.3 with $m = m_1 m_2$ and $\delta = \min(\delta_1, \delta_2)$.

We shall prove first the existence of τ_1 and τ_2. The measurability will be discussed after that. We shall use the following well known property of the vertical subbundle: if $\theta \in SM$ and $S \subset N(\theta)$ is a Lagrangian subspace, then the set of values $s \in \mathbf{R}$ such that

$$d_\theta\varphi_s(S) \cap V(\varphi_s(\theta)) \neq \{0\}$$

is discrete.

We begin by proving the existence of τ_1. Suppose by contradiction that it does not exist. Then for every integer $m \geq 1$, there exists $(\theta_m, t_m) \in SM \times \mathbf{R}$ such that for all $s \in \{0, 1/m, 2/m, ..., 1\}$ we have

$$\alpha(d_{\varphi_{-s}(\theta_m)}\varphi_s(V(\varphi_{-s}(\theta_m))),\, R^{\perp}_{t_m}(\theta_m)) \leq 1/m. \tag{0.8}$$

Since M is compact, the sequence $(\theta_m, R^{\perp}_{t_m}(\theta_m))$ has a subsequence $(\theta_{m_k}, R^{\perp}_{t_{m_k}}(\theta_{m_k}))$ that converges to (θ, S), where S is a Lagrangian subspace. From equation (0.8) and the continuity of α we deduce

$$\alpha(d_{\varphi_{-s}(\theta)}\varphi_s(V(\varphi_{-s}(\theta))), S) = 0,$$

for all $s \in [0, 1]$. This is equivalent to

$$d_{\varphi_{-s}(\theta)}\varphi_s(V(\varphi_{-s}(\theta))) \cap S \neq \{0\},$$

for all $s \in [0, 1]$. Hence

$$d_\theta\varphi_{-s}(S) \cap V(\varphi_{-s}(\theta)) \neq \{0\},$$

for all $s \in [0, 1]$. This contradicts the property of Lagrangian subspaces mentioned above.

Now we shall prove the existence of τ_2. Suppose that it does not exist. Then there exists a sequence $(\theta_m, t_m) \in SM \times \mathbf{R}$, such that after setting $\tau_1(m) = \tau_1(\theta_m, t_m)$ and $\beta_m = \varphi_{-\tau_1(m)}(\theta_m)$, we have

$$\alpha(d_{\beta_m}\varphi_{\tau_1(m)+s+t_m}(V(\beta_m)),\, V(\varphi_{t_m+s}(\theta_m))) \leq 1/m, \tag{0.9}$$

for all $s \in \{0, 1/m, 2/m, \ldots, 1\}$. By compactness, there is a subsequence

$$\left(\varphi_{t_{m_k}}(\theta_{m_k}), d_{\beta_{m_k}}\varphi_{\tau_1(m_k)+t_{m_k}}(V(\beta_{m_k}))\right)$$

which converges to (θ, S), where S is a Lagrangian subspace. From equation (0.9) and the continuity of α, we deduce

$$\alpha(d_\theta\varphi_s(S), V(\varphi_s(\theta))) = 0,$$

for all $s \in [0, 1]$. Therefore

$$d_\theta\varphi_s(S) \cap V(\varphi_s(\theta)) \neq \{0\},$$

for all $s \in [0, 1]$. Since S is Lagrangian, this again contradicts the above property and completes the proof of the existence of the functions τ_1 and τ_2.

The measurability follows easily from observing that they can be taken locally constant on each of the subsets $\mathcal{P}(p, l_1, ..., l_p)$, which we defined while proving Lemma 2.2. This finishes the proof of Inequalities (A) and (A′).

Now let us prove Inequality (B). We shall use a slightly strengthened version of Yomdin's theorem combined with the method employed by G. Paternain [6] in his proof that for every $x \in M$,

$$\limsup_{T \to +\infty} \frac{1}{T} \log \int_M n_T(x, y)\, dy \leq h_{top}(\varphi).$$

Let N be a closed manifold and $f : N \to N$ a C^r diffeomorphism. We shall consider N embedded in a Euclidean space $\mathbf{R}^m$ and f extended to a C^r map $f : U \to U$, where U is an open neighborhood of N. Given a C^r map $g : U \to \mathbf{R}^m$, we define

$$||d_r g|| = \sup\{||d_x^k g|| : \quad x \in U, \quad 1 \leq k \leq r\}.$$

Fix an integer $l \geq 1$. If $Y \subset N$, we can define the C^r-size of Y as an l-dimensional set to be the infimum of $s > 0$ such that there exists a C^r map $h : [0,1]^l \to \mathbf{R}^m$ satisfying

$$h([0,1]^l) \supset Y \qquad \text{and} \qquad ||d_r h|| \leq s.$$

If no such s exists, the C^r size of Y is ∞. If $Y \subset N$ is a submanifold, the C^r size of Y will mean the C^r size of Y as a $\dim Y$-dimensional set. These definitions are taken from Gromov [4]. We can assume that $dist(f(U), \partial U) \geq 1/\sqrt{l}$ (as required by Gromov in [4]) by rescaling the embedding and thus increasing $dist(f(U), \partial U)$.

The next result is the slight improvement of Yomdin's theorem that we shall need. For simplicity, we shall state it and prove it for the case of diffeomorphisms; however we shall use it for flows.

For the function f considered above, we define

$$K = K(f) = 1 + \limsup_{k \to +\infty} \frac{1}{k} \log \|df^k\|. \tag{0.10}$$

Theorem 2.5. *Let $f : N \to N$ and $K = K(f)$ be as above. For any $S, \varepsilon > 0$ and any integers $r, l \geq 1$, there exist $C > 0$ and an integer $n_0 \geq 1$ such that, for every l-dimensional submanifold $Y \subset N$ with C^r size $\leq S$, we have*

$$\mathrm{Vol}(f^n(Y)) \leq C \exp\left\{ \left(h_{top}(f|_N) + \varepsilon + \frac{lK}{r} \right) n \right\}$$

for all $n \geq n_0$.

Proof. As observed by Gromov [4, 3.2 p.231], every set of C^r-size $\leq S$ can be divided into j^l subsets of C^r-size $\leq S/j$ for all $j = 1, 2, ...$ Thus it suffices to prove the theorem in the case where Y has C^r-size ≤ 1. We shall need the following lemma proved by Gromov [4, 3.6 p.233].

Lemma 2.6. *There exists $C = C(l, m, r)$ independent of f, such that if $Y_0 \subset N$ is a C^r l-dimensional submanifold of C^r-size ≤ 1 and Q_β, $\beta = 1, ..., i$, are unit cubes contained in space $\mathbf{R}^m$ in which we embedded N, then*

$$\mathrm{Vol}(f^i(Y_0 \cap (\bigcap_{b=1}^{i} f^{-\beta}(Q_\beta)))) \leq (C||d_r f||^{l/r} + 1)^i.$$

Given a continuous map $g : X \to X$ of a compact metric space, denote by $n(\delta, i, g)$ the minimal cardinality of a (δ, i, g)-spanning set.

Assume that the points x_s, $s = 1, ..., n(1/2, i, f|_N)$, form a $(1/2, i, f|_N)$-spanning set. Then the manifold N can be covered by sets of the form

$$A_s \stackrel{\text{def}}{=} \bigcap_{\beta=1}^{i} f^{-\beta}(Q_\beta), \qquad 1 \leq s \leq n(1/2, i, f|_N),$$

where Q_β is the unit cube centered at $f^\beta(x_s)$. If Y_0 is as in Lemma 2.6, we have

$$\begin{aligned} \mathrm{Vol}(f^i(Y_0)) &\leq \sum_s \mathrm{Vol}(f^i(Y_0 \cap A_s)) \\ &\leq n(1/2, i, f|_N)(C||d_r f||^{l/r} + 1)^i. \end{aligned} \tag{0.11}$$

Now define $f_j : jU \to jU$, for $j \geq 1$, by $f_j(x) = jf(j^{-1}x)$. Then $f_j(jN) = jN$. Let Y_0 be a submanifold of C^r-size ≤ 1. Observe that jY_0 can be covered by j^l-sets, Y_0^k, $k = 1, ..., j^l$, with C^r-size ≤ 1 [4]. Therefore we can apply equation (0.11) to each Y_0^k, obtaining

$$\begin{aligned} \mathrm{Vol}(f_j^i(jY_0)) &\leq \sum_k \mathrm{Vol}(f_j^i(Y_0^k)) \\ &\leq j^l n(1/2, i, f_j|_{jN})(C||d_r f_j||^{l/r} + 1)^i. \end{aligned} \tag{0.12}$$

But

$$n(1/2, i, f_j|_{jN}) = n(\frac{1}{2j}, i, f|_N).$$

By the definition of topological entropy, given $\delta > 0$, there exist $i_0 \geq 1$ and $j_0 \geq 1$ such that

$$n(\frac{1}{2j}, i, f|_N) \leq \exp\left\{(h_{top}(f|_N) + \delta)i\right\},$$

for all $i \geq i_0$, $j \geq j_0$. Then for $i \geq i_0$, $j \geq j_0$, inequality (0.12) implies

$$\begin{aligned} \mathrm{Vol}(f^i(Y_0)) &= \mathrm{Vol}(j^{-1} f_j^i(jY_0)) = j^{-l}\mathrm{Vol}(f_j^i(jY_0)) \\ &\leq n(\frac{1}{2j}, i, f|_N)(C||d_r f_j||^{l/r} + 1)^i \\ &\leq (\exp\left\{(h_{top}(f|_N) + \delta)i\right\}) \cdot (C||d_r f_j||^{l/r} + 1)^i. \end{aligned} \tag{0.13}$$

The definition of K in equation (0.10) allows us to choose k so that

$$\frac{1}{k}\log(C(2||df^k||)^{l/r} + 1) \leq \frac{lK}{r}.$$

Observe that $\|d^s f_j\| = j^{1-s}\|d^s f\|$ for $s = 1, 2, \ldots$. We see from this that we can choose $j \geq j_0$ such that

$$||d_r f_j^k|| \leq 2||df^k||.$$

If $i \geq i_0$, then inequality (0.13) applied to f^k with $\delta = k\varepsilon$ gives

$$\begin{aligned} \mathrm{Vol}(f^{ki}(Y_0)) &\leq (\exp\{(h_{top}(f^k|_N) + k\varepsilon)i\}) \cdot (C||d_r f_j^k||^{l/r} + 1)^i \\ &\leq \exp\left\{\left(k h_{top}(f|_N) + k\varepsilon + \log(C(2||df^k||)^{l/r} + 1)\right) i\right\} \\ &\leq \exp\left\{\left(h_{top}(f|_N) + \varepsilon + \frac{lK}{r}\right) ki\right\}. \end{aligned}$$

If $n \geq ki_0$, we can choose $i \geq i_0$ such that $0 \leq n - ki < k$. For $n \geq ki_0$, we have

$$\begin{aligned} \mathrm{Vol}((f^n(Y_0)) &= \mathrm{Vol}(f^{n-ki}(f^{ki}(Y_0))) \\ &\leq \|df\|^{l(n-ki)} \exp\left\{\left(h_{top}(f|_N) + \varepsilon + \frac{lK}{r}\right) ki\right\} \\ &\leq C \exp\left\{\left(h_{top}(f|_N) + \varepsilon + \frac{lK}{r}\right) n\right\}, \end{aligned}$$

where $C = \|df\|^{lk}$. This completes the proof of the theorem.

Now let us complete the proof of Inequality (B). Observe that

$$\int_{S_xM} A((x,v),t)\,dv \leq \mathrm{Vol}(\varphi_t(S_xM)).$$

By the theorem above, given $\varepsilon > 0$ and an integer $r \geq 1$, there exist $t_0 \geq 0$ and $C_r > 0$ such that

$$\mathrm{Vol}(\varphi_t(S_xM)) \leq C_r \exp\left\{\left(h_{top}(\varphi) + \varepsilon + \frac{(n-1)K}{r}\right) t\right\},$$

for all $t \geq t_0$ and all $x \in M$, since it is easy to see (using the compactness of M) that there exists $S > 0$ such that the $(n-1)$-dimensional manifold S_xM has C^r-size $\leq S$ for all $x \in M$. Hence there is a constant C'_r such that, for all large enough T, we have

$$\begin{aligned}\int_{M\times M} n_T(x,y)\,dxdy &= \int_0^T \int_M \int_{S_xM} A(x,v,t)\,dvdxdt \\ &\leq C'_r \exp\left\{\left(h_{top}(\varphi) + \varepsilon + \frac{(n-1)K}{r}\right) T\right\}.\end{aligned}$$

Therefore

$$\limsup_{T\to+\infty} \frac{1}{T} \log \int_{M\times M} n_T(x,y)\,dxdy \leq h_{top}(\varphi) + \varepsilon + \frac{(n-1)K}{r},$$

concluding the proof of Inequality (B), since the above inequality holds for all $\varepsilon > 0$ and all $r \geq 1$.

3. Proof of Theorem 1.4

We shall use here the following lemma.

Lemma 3.1. *There exist a constant $K > 0$, an integer $m \geq 1$ and a measurable function $\tau : SM \times \mathbf{R}^+ \to \{0, 1/m, 2/m, \dots, 1\}$ such that, after setting*

$$G_T(\theta) = \varphi_{-\tau(\theta,T)}\theta,$$

we have

$$|\det(d_{G_T(\theta)}\varphi_T|_{V(G_T(\theta))})| \geq K\, ex(d_\theta\varphi_T)$$

for any $T > 0$.

We shall not prove this lemma because it is obtained by the same methods as the lemmas of Section 2.

Now, given $T > 0$, we prove as in Section 2, that there exists $K_1 > 0$ (independent of T) such that

$$\int_{SM} (f \circ G_T)\, d\theta \le K_1 \int_{SM} f\, d\theta,$$

for every integrable function $f : SM \to (0, +\infty)$. Hence

$$\begin{aligned}\int_{SM} ex(d_\theta \varphi_T)\, d\theta &\le K^{-1} \int_{SM} |\det(d_{G_T(\theta)} \varphi_T|_{V(G_T(\theta))})|\, d\theta \\ &\le K_1 K^{-1} \int_{SM} |\det(d_\theta \varphi_T|_{V(\theta)})|\, d\theta,\end{aligned}$$

and in consequence of Theorem 1.1,

$$\begin{aligned}h_{top}(\varphi) &= \lim_{T\to+\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta \\ &\le \liminf_{T\to+\infty} \frac{1}{T} \log \int_{SM} |\det(d_\theta \varphi_T|_{V(\theta)})|\, d\theta.\end{aligned}$$

Since $ex(d_\theta \varphi_T) \ge |\det(d_\theta \varphi_T|_{V(\theta)})|$, by definition of $ex(.)$, we also have:

$$\begin{aligned}\limsup_{T\to+\infty} \frac{1}{T} \log \int_{SM} |\det(d_\theta \varphi_T|_{V(\theta)})|\, d\theta &\le \lim_{T\to+\infty} \frac{1}{T} \log \int_{SM} ex(d_\theta \varphi_T)\, d\theta \\ &= h_{top}(\varphi).\end{aligned}$$

4. Appendix: An upper bound for the growth rate of $n_T(x, y)$.

Here we shall prove the following property:

Proposition 4.1. *For every $x \in M$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log n_T(x, y) \le h_{top}(\varphi),$$

for a.e. $x \in M$.

This is an immediate corollary of the following inequality due to G. Paternain [6]:

Proposition 4.2. *For every $x \in M$, we have*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \int_M n_T(x,y)\,dy \le h_{top}(\varphi),$$

and the following application of the Borel-Cantelli Lemma:

Lemma 4.3. *Let $(X, \mathcal{A}, \mu)$ be a probability space, and $f_n : X \to (0, +\infty)$ a sequence of integrable functions. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n(x) \le \limsup_{n\to\infty} \frac{1}{n} \log \int_X f_n\,d\mu,$$

for μ-a.e. $x \in X$.

Proof. Set

$$\sigma = \limsup_{n\to\infty} \frac{1}{n} \log \int_X f_n\,d\mu.$$

Define $S(n,\epsilon) = \{x :\ f_n(x) \ge \exp(\sigma+\epsilon)n\}$. Then

$$\begin{aligned} \sigma &= \limsup_{n\to\infty} \frac{1}{n} \log \int_X f_n\,d\mu \ge \limsup_{n\to\infty} \frac{1}{n} \log \int_{S(n,\epsilon)} f_n\,d\mu \\ &\ge \limsup_{n\to\infty} \frac{1}{n} \log[\mu(S(n,\epsilon)) \exp(\sigma+\epsilon)n] \\ &= \sigma + \epsilon + \limsup_{n\to\infty} \frac{1}{n} \log \mu(S(n,\epsilon)). \end{aligned}$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(S(n,\epsilon)) \le -\epsilon,$$

which implies

$$\sum_n \mu(S(n,\epsilon)) < +\infty.$$

By the Borel-Cantelli Lemma, for a.e. x, there exists $m(x)$ such that $x \notin S(n,\epsilon)$ for all $n \ge m(x)$. This means that $f_n(x) \le \exp(\sigma+\epsilon)n$ for all $n \ge m(x)$ and then

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n(x) \le \sigma + \epsilon.$$

Since this holds for every $\epsilon > 0$, the lemma is proved.

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