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SYMPLECTIC SUBMANIFOLDS AND ALMOST-COMPLEX GEOMETRY

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1. Introduction

In this paper we develop a general procedure for constructing symplectic submanifolds. Recall that if (V, ω) is a symplectic manifold, a submanifold $W \subset V$ is called symplectic if the restriction of ω to W is non-degenerate. Paradigms are complex submanifolds of complex Kähler manifolds. In general questions about complex submanifolds of high codimension can be intractable, but one has a rather good grip on submanifolds of complex codimension 1, which can be studied through the familar aparatus of line bundles, linear systems and cohomology—effectively linearising the problem. The idea of this paper is to extend these techniques in complex geometry to general symplectic manifolds. The main result we prove is the following existence theorem.

Theorem 1. *Let (V, ω) be a compact symplectic manifold of dimension $2n$, and suppose that the de Rham cohomology class $[\omega/2\pi] \in H^2(V; \mathbf{R})$ lies in the integral lattice $H^2(V; \mathbf{Z})/Torsion$. Let $h \in H^2(V; \mathbf{Z})$ be a lift of $[\omega/2\pi]$ to an integral class. Then for sufficiently large integers k the Poincaré dual of kh, in $H_{2n-2}(V; \mathbf{Z})$, can be realised by a symplectic submanifold $W \subset V$.*

In the case when V is a Kähler manifold and ω is the Kähler form, this reduces to a standard, but central, result in complex geometry. In that case one would argue that h is the first Chern class of a *positive line bundle* L over V, having a connection with curvature $-i\omega$, and then show that for large k the tensor power L^k has many holomorphic sections. The zero set of a generic section would provide the desired

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submanifold W, a complex codimension-1 submanifold of V. The result is closely related to the famous Kodaira Embedding Theorem: for large k the holomorphic sections of L^k define an embedding of V in a projective space $\mathbf{CP}^N$ and the submanifold W is obtained as a hyperplane section $V \cap \mathbf{CP}^{N-1}$. Our approach, for a general symplectic manifold (V, ω), is to choose a compatible *almost-complex* structure J on V, and then to extend familiar results about positive line bundles, suitably formulated, to the almost-complex case. There are similarities in style between our work and the ideas developed by Gromov and many other authors, studying *pseudo-holomorphic curves* in general symplectic manifolds: both approaches extend techniques from complex geometry to almost-complex manifolds. A simple corollary of our main result is a general existence theorem for pseudo-holomorphic curves, see Cor. 7 below. However the two theories move in some respects in opposite directions. In Gromov's theory one studies submanifolds of complex dimension 1, obtained as the images of maps $f : \Sigma \to V$, where Σ is a Riemann surface. The dimension of Σ is rather fundamental, since the relevant "Cauchy-Riemann" equations become over-determined in higher dimensions, and one does not expect to find any solutions for generic almost-complex structures on V. In this paper, by contrast, we study submanifolds "cut out" in V as the zeros of suitable line bundle sections, and the theory is specific to complex *codimension* one. The two points of view illustrate the elementary principle that one can present a submanifold either as the image of a map or as the set of solutions of a system of equations.

One situation in which the interaction between thse two points of view is particularly interesting is that in which the symplectic manifold V has real dimension 4, so complex dimension one and codimension one co-incide. It is easy to see that in this case a submanifold $W^2 \subset V^4$ is symplectic if and only if it is a pseudo-holomorphic curve for some compatible almost-complex structure. This situation, leading to a vista of interactions between 4-dimensional topology and symplectic geometry, was the principle motivation for the present work, and the earlier article [1] contains a discussion of some of these ideas (together with a preliminary attack on the central problem solved in this paper). The pregnancy of these interactions was heightened, some while after [1], by the work of Kronheimer and Mrowka on the genus of embedded surfaces in 4-manifolds. Kronheimer and Mrowka showed in [5] that for a large class of smooth 4-manifolds X, with $b^+(X) > 1$, there is a set of preferred "basic" classes in $H^2(X)$. These were obtained from the instanton invariants of X. If κ is a basic class and $\Sigma \subset X$ is an embedded surface of positive self-intersection and genus g, then

$$2g - 2 \geq \Sigma.\Sigma + |\kappa.\Sigma|.$$

On the other hand it is a rather elementary fact that if X is symplectic and Σ is a symplectic submanifold, then

$$2g - 2 = \Sigma.\Sigma + K_X.\Sigma,$$

where $K_X \in H^2(X)$ is the "canonical class": minus the first Chern class of the tangent bundle of X, with any compatible almost-complex structure. Thus the existence of symplectic submanifolds realising $k\omega/2\pi$ for large k leads to the constraint:

$$|\kappa.\omega| \leq K_X.\omega. \tag{2}$$

This can be used to make various deductions. Consider triples (X, K, ϕ) where X is a compact oriented 4-manifold, K and ϕ are classes in $H^2(X)$ such that $K = w_2(X) \mod 2, K^2 = 2\chi(X) + 3\sigma(X), \phi^2 > 0$. These are the elementary conditions required for K, ϕ to be realised as $K_X, [\omega]$ respectively, for some symplectic structure on Σ. Using (2), together with calculations of the basic classes for specific manifolds, one could write down for the first time triples (X, K, ϕ) which cannot be realised by any symplectic structure, for example any case where $K.\phi < 0$ and X has non-trivial instanton invariants. However these applications have, to a large extent, been overtaken by subsequent events. A few months after the proof of Theorem 1 was completed the Seiberg-Witten invariants of 4-manifolds were introduced in [10], and these were applied by Taubes [6] to prove a stronger version of the inequality (2) (in this case for the Seiberg- Witten basic classes, but there is overwhelming evidence that these are the same as those of Kronheimer and Mrowka). Moreover, Taubes has established a direct link between the Seiberg-Witten basic classes and the pseudo-holomorphic curves of Gromov's theory [7], [8]; which leads in particular to a different existence theorem for symplectic submanifolds: if X is a symplectic 4-manifold with $b^+(X) > 1$, then the Poincaré dual of K_X is realised by a symplectic submanifold.

We will now explain some of the ideas involved in the proof in more detail, and formulate our results more precisely. We begin with a little linear algebra. Let $\mathbf{C}^n$ have its standard metric and symplectic form ω, and let G be the Grassmannian of oriented real $2n - 2$-planes in

$\mathbf{C}^n$. Write $G^+ \subset G$ for the open set of "symplectic" $(2n-2)$-planes Π, those for which the restriction of ω^{n-1} to Π is positive, relative to the orientation on Π. Clearly G^+ depends only on the symplectic structure on $\mathbf{C}^n$. Given the metric, and hence a volume form Ω_Π on each subspace, we can define a map—the "Kähler angle"— $\theta : G \to [0, \pi]$ by

$$\theta(\Pi) = \cos^{-1}\left(\frac{1}{(n-1)!}\frac{\omega^{n-1}|_\Pi}{\Omega_\Pi}\right).$$

One can show, although we do not need this, that θ completely classifies the orbits of $U(n)$ acting on G. The complex-linear subspaces are just those with $\theta(\Pi) = 0$, so θ measures the amount by which a subspace fails to be complex-linear. Clearly the set G^+ is $\theta^{-1}[0, \pi/2)$.

Now suppose that a linear subspace Π in $\mathbf{C}^n$ is obtained as the kernel of an $\mathbf{R}$-linear map $A : \mathbf{C}^n \to \mathbf{C}$. We can write A as the sum $a' + a''$ where a' is complex linear and a'' is antilinear. We let $|a'|, |a''|$ be the standard norms defined by the Hermitian metric. A little calculation shows that

1. A has (real) rank 2 unless $\overline{a''} = e^{i\alpha} a'$ for some real α,

2. if A has rank 2 and $\Pi = ker(A)$, then

$$\tan(\theta(\Pi)) = \frac{2\sqrt{|a'|^2|a''|^2 - |\langle a', \overline{a''}\rangle|^2}}{|a'|^2 - |a''|^2}.$$

One sees from this that
$$\theta(\Pi) \leq 2\frac{|a''|}{|a'|},$$

so the ratio $|a''|/|a'|$ controls the deviation of the kernel from being a complex linear subspace. For key observation we need for our main result is the following:

Proposition 3. *If $a', a'' : \mathbf{C}^n \to \mathbf{C}$ are respectively complex linear and anti-linear maps and if $|a''| < |a'|$, then the subspace $ker(a' + a'') \subset \mathbf{C}^n$ is symplectic.*

Of course it is easy to verify this directly, without introducing the function θ. Now consider a symplectic manifold (V, ω) with a compatible almost complex structure. If $W \subset V$ is a C^∞ submanifold of real codimension 2, we can define at each point p of W a number $\theta_p(W)$, by applying the above discussion to the tangent space of W in V. This measures the extent to which W fails to be a pseudo-holomorphic submanifold. Suppose that $L \to V$ is a complex line bundle, and s is a smooth section of ξ. The derivative ∇s is well defined on the zero-set of s and can be split into the complex linear and antilinear parts $\partial s, \overline{\partial} s$. We see then that if

$$|\overline{\partial} s| < |\partial s| \tag{4}$$

everywhere on the zero-set, then this zero set is a symplectic codimension 2 submanifold of V, with orientation compatible with the symplectic structure. The homology class of the zero set is of course the Poincaré dual of the first Chern class of L. This is the way in which we will construct symplectic submanifolds. However we will actually be able to manage rather more. Recall first that, given the hypotheses of Theorem 1, there is a line bundle L over V with $c_1(L) = h$, an integral lift of $[\omega/2\pi]$. We can endow L with a unitary connection having curvature form $-i\omega$, which will play a fundamental role in the proof, although it is not actualy involved in the statement of the main result, as follows:

Theorem 5. *Let $L \to V$ be a complex line bundle over a compact symplectic manifold V with compatible almost-complex structure, and with $c_1(L) = [\omega/2\pi]$. Then there is a constant C such that for all large k there is a section s of $L^{\otimes k}$ with*

$$|\overline{\partial} s| < \frac{C}{\sqrt{k}} |\partial s|$$

on the zero set of s.

This Theorem, together with Proposition 3, implies Theorem 1, but it shows further that there exist submanifolds W_k realising kh which are very close to being pseudo-holomorphic submanifolds, in that

$$\theta_p(W_k) \leq \frac{2C}{\sqrt{k}}$$

for all $p \in W_k$. The existence of these "approximately pseudo-holomorphic" submanifolds is not obvious even locally in V, and is perhaps counter to ones intuition about almost-complex structures, given the over-determined nature of the Cauchy-Riemann equations in higher dimensions. The point is that these submanifolds become extremely complicated everywhere, "filling out" all of V. In fact we will show in Section

6 that the sequence of currents $k^{-1}W_k$ converges to the symplectic form ω as $k \to \infty$.

Using Theorem 5 we can prove a version of Theorem 1 without the integrality hypothesis. If (V, ω) is any compact symplectic manifold there is symplectic structure ω' arbitrarily close to ω such that a multiple of ω' satisfies the integrality hypothesis. Fix almost-complex structures J, J' compatible with ω, ω' respectively. Then one can apply Theorem 5 to get submanifolds which are everywhere close to being J'-pseudo-holomorphic, and hence also symplectic with respect to ω. Another simple extension is obtained by repeating the constuction, replacing V by its symplectic submanifold W, to get submanifolds of arbitrary codimension (the analogues of "complete intersections"in complex geometry). Putting everything together we have

Corollary 6. *If (V, ω) is any compact symplectic manifold, then the following hold:*

1. *V contains symplectic submanifolds of any even codimension*

2. *If J is a compatible almost-complex structure on V, then there are almost-complex structures J' arbitrarily close in C^0 to J such that V contains J'-pseudo-holomorphic curves.*

The construction of sections s satisfying the inequality in Theorem 4 will involve two parts. The first, and easier, part is to construct a suitable family of "approximately holomorphic" sections s with $\overline{\partial} s$ everywhere small, where $\overline{\partial} s$ is defined over V using the connection on L^k. This is done in Section 2. The other task is to select a section where the other part ∂s of the derivative is not small on the zero set. This task forms the core of the paper. The lack of integrability of the almost complex structure is not particularly relevant here: the essential problem is already present in the classical case of complex geometry, and in Section 6 we will modify our set-up slightly to obtain new results on the geometry of complex submanifolds of high degree inside a Kähler manifold. The selection of the good sections is essentially a question of "quantative transverality". There is a "local-to-global" aspect which is covered in Section 3, the crucial local result being proved in Section 4. This local result in turn depends on considerations from real algebraic geometry, following the method of Yomdin [11], which we review for completeness in Section 5.

Looking beyond the existence question, one can hope to use similar techniques to mimic various familar topics in complex geometry in the symplectic, almost-complex case. One that we consider here, in Section 6 below, is the analogue of the Lefschetz hyperplane theorem. In a sequel to this paper we will discuss other topics involving *families* of symplectic submanifolds.

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2. Local theory

In this section we study the interaction between "almost-complex" geometry and the curvature of line bundles. Our goal is to construct approximately holomorphic sections of our line bundle $L^k \to V$, when k is large. We begin by reviewing some background in these two areas.

We start with linear algebra. A complex structure on a real vector space W may be viewed as a decomposition of the complexified space $W^* \otimes \mathbf{C}$ as a sum of complex conjugate subspaces, the complex linear and anti-linear functionals. If we fix one reference complex structure, so $W = \mathbf{C}^n$, these are the familiar 1-forms of types $(1,0)$ and $(0,1)$:

$$Hom_{\mathbf{R}}(\mathbf{C}^n, \mathbf{C}) = \Lambda^{1,0} \oplus \Lambda^{0,1}.$$

Any other complex structure J on $\mathbf{C}^n$ may be specified in terms of this reference by a complex linear map

$$\mu : \Lambda^{1,0} \to \Lambda^{0,1},$$

such that the J-complex linear forms $\Lambda_J^{1,0}$ are those of the shape $\phi + \mu(\phi)$; i.e., $\Lambda_J^{1,0}$ is the *graph* of the linear map μ. Taking conjugates, the subspace $\Lambda_J^{0,1}$ is the graph of $\overline{\mu} : \Lambda^{0,1} \to \Lambda^{1,0}$, and the condition that $\Lambda_J^{1,0}, \Lambda_J^{0,1}$ be transverse is that $(1 - \mu\overline{\mu})$ be invertible. For any $\alpha^{1,0} \in \Lambda^{1,0}, \alpha^{0,1} \in \Lambda^{0,1}$ the $\Lambda_J^{0,1}$ component of $\alpha^{1,0} + \alpha^{0,1}$ is $\psi + \overline{\mu}(\psi)$ where

$$\psi = (1 - \mu\overline{\mu})^{-1}(\alpha^{0,1} - \mu(\alpha^{1,0})).$$

Now consider an almost-complex structure J on a neighbourhood Ω of the origin in $\mathbf{C}^n$. By the above discussion this is specified, in terms of the reference structure, by a map $\mu_z : \Lambda^{1,0} \to \Lambda^{0,1}$ which varies smoothly with $z \in \Omega$. In other words μ is now a bundle map. The

$\overline{\partial}$-operator $\overline{\partial}_J$ of this almost-complex structure maps a function f to the $\Lambda^{0,1}_J$ component of its derivative df: thus if we identify $\Lambda^{0,1}_J$ with $\Lambda^{0,1}$ by the map $\varpi_J : \Lambda^{0,1}_J \to \Lambda^{0,1}$ given by projection followed by the endomorphism $(1-\mu\overline{\mu})^{-1}$, we can write

$$\overline{\partial}_J(f) = \overline{\partial} f - \mu(\partial f),$$

where $\overline{\partial}, \partial$ are the ordinary operators defined by the standard complex structure on $\mathbf{C}^n$.

It is clear that we can change co-ordinates to make the two structures agree at $0 \in \mathbf{C}^n$: that is, to make $\mu_0 = 0$. The *Nijenhius tensor* N of the almost complex structure J appears as the obstruction to removing the first derivative of μ; thus we cannot in general make μ vanish to higher order. The fundamental, and very elementary, mechanism that we will exploit is the effect of *scaling*. For $\rho < 1$ let $\delta_\rho : \mathbf{C}^n \to \mathbf{C}^n$ be the dilation $\delta_\rho(z) = \rho z$, and let $\tilde{J}$ be the almost-complex structure over $\rho^{-1}\Omega$ defined by pulling back J using δ_ρ. This structure is defined by a bundle map $\tilde{\mu}$ which is just

$$\tilde{\mu}_z = \mu_{\rho z}.$$

So it is clear that, if we restrict J to an interior ball in Ω and hence $\tilde{J}$ to a ball of radius $O(\rho^{-1})$, we have:

$$|\tilde{\mu}_z| \leq C\rho|z| \ , \ |\nabla\tilde{\mu}| \leq C\rho, \tag{8}$$

for some constant C, related to the norm of the Nijenhius tensor and its derivative. (Throughout the paper, we use the convention that C represents a positive constant which is allowed to vary from line to line.) These inequalities just reflect the fact that examined through a microscope the almost-complex structure J appears close to the flat structure.

We now turn to the second topic, the relation between curvature and complex geometry. Consider the standard Kähler form

$$\omega_0 = \frac{i}{2}\sum_{\alpha=1}^{n} dz_\alpha \overline{dz}_\alpha$$

on $\mathbf{C}^n$. We can write $\omega_0 = idA$ where

$$A = \tfrac{1}{4}\left(\sum_{\alpha=1}^{n} z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha\right),$$

so $-i\omega_0$ is the curvature form of a $U(1)$ connection on the trivial complex line bundle over $\mathbf{C}^n$ with connection matrix A. Expressed rather more invariantly, if $\xi \to \mathbf{C}^n$ is a line bundle with connection having curvature ω_0 there is a preferred trivialisation of ξ, up to an overall scalar, defined by radial parallel transport from the origin, and the connection matrix in this trivialisation is A. The connection A defines a coupled $\overline{\partial}$-operator $\overline{\partial}_A$,

$$\overline{\partial}_A(f) = \overline{\partial} f + A^{0,1} f,$$

where $A^{0,1}$ is the (0,1) component of A, and we observe that

$$\overline{\partial}_A(e^{-|z|^2/4}) = \tfrac{1}{4}(\sum z_\alpha d\overline{z}_\alpha - z_\alpha d\overline{z}_\alpha) e^{-|z|^2/4} = 0.$$

The other component of the covariant derivative is

$$(\partial + A^{1,0}) e^{-|z|^2/4} = \tfrac{1}{2}(\sum \overline{z}_\alpha dz_\alpha) e^{-|z|^2/4}. \tag{9}$$

In other words, the line bundle ξ—which is endowed with a holomorphic structure by its connection—has a holomorphic section σ with norm $|\sigma|^2 = e^{-|z|^2/2}$. (Another way of going about things is to use the formula $\overline{\partial}\partial(\log(|\sigma|^2))$ for the curvature of a Hermitian holomorphic line bundle with a holomorphic section σ.) Thus the effect of the *positive* curvature tensor ω_0 is to give a holomorphic section which *decays* rapidly at infinity in $\mathbf{C}^n$. Replacing ω_0 by $k\omega_0$, for an integer $k > 0$, we get a holomorphic section of ξ^k with norm $e^{-k|z|^2/4} = e^{-|\sqrt{k}z|^2/4}$: so the effect of replacing ξ by ξ^k is the same as applying a dilation with scale factor $k^{-1/2}$ to $\mathbf{C}^n$.

We now bring these two discussions together. Let (V, ω) be the compact symplectic manifold of Section 1, with a fixed compatible almost-complex structure J, and a line bundle $L \to V$ with a $U(1)$ connection having curvature $-i\omega$. We let g be the Riemannian metric determined by ω and J, and $g_k = kg$ be that determined by $k\omega$ and J. For any point $p \in V$ we can, by Darboux's Theorem, choose a chart $\chi = \chi_p : B^{2n} \to V$, where B^{2n} is the unit ball in $\mathbf{C}^n$, with $\chi(0) = p$ and such that $\chi^*(\omega)$ is the standard form ω_0 on $\mathbf{C}^n$. We may suppose that all derivatives of χ—measured with respect to the metric g and its Levi-Civita connection—are bounded, independent of p. (For this we need to apply a version of Darboux's Theorem which incorporates smooth dependence on parameters.) We may also suppose that the derivative of χ at 0 is complex-linear with respect to the complex structure J on TV_p. Thus if we pull back the almost-complex structure J using χ we get a structure represented by a bundle map μ over B^{2n}, as envisaged above, and all derivatives of μ satisfy bounds, independent of the point p. Given k we now compose the chart χ with the dilation map with scale factor $k^{-1/2}$; thus we put

$$\tilde{\chi} = \chi \circ \delta_{k^{-1/2}} : k^{1/2} B^{2n} \to V.$$

On the one hand, the almost-complex structure is represented, in the chart $\tilde{\chi}$, by a bundle map $\tilde{\mu}$, which satisfies bounds coming from (8):

$$\begin{aligned} |\tilde{\mu}_z| &\leq Ck^{-1/2}|z|, \\ |\nabla\tilde{\mu}| &\leq Ck^{-1/2}, \end{aligned}$$

where the constant C does not depend on p. On the other hand, the pull-back $\tilde{\chi}^*(-ik\omega)$ of the curvature of the line bundle L^k over V is the standard form $-i\omega_0$ on $\mathbf{C}^n$, so we may lift $\tilde{\chi}$ to a connection-preserving bundle map, which we will also denote by $\tilde{\chi}$, from the standard line bundle ξ, with connection matrix A, to the line bundle L^k (restricted to a neighbourhood of p). Therefore we may regard the section σ of ξ as a local section of L^k.

Let $\overline{\partial}_{A,\tilde{J}}$ denote the $\overline{\partial}$-operator defined by the almost complex structure $\tilde{J}$ over the large ball $k^{1/2}B^{2n}$ in $\mathbf{C}^n$ and the connection matrix A. Thus, using the map $\varpi_{\tilde{J}}$ we can write

$$\overline{\partial}_{A,\tilde{J}}(f) = (\overline{\partial} f + A^{0,1} f) + \tilde{\mu}(\partial f + A^{1,0} f).$$

If $f = e^{-|z|^2/4}$, we get

$$\overline{\partial}_{A,\tilde{J}} f = \tfrac{1}{2}\mu(\textstyle\sum \overline{z}_\alpha dz_\alpha) e^{-|z|^2/4},$$

and, in consequence of (9),

$$|\overline{\partial}_{A,\tilde{J}} f_z| \leq |\tilde{\mu}_z| \, |\partial f + A^{1,0} f| \leq Ck^{-1/2}|z|^2 e^{-|z|^2/4}. \tag{10}$$

This inequality (10) is the crucial estimate which makes precise the sense in which the Gaussian section in the flat model becomes an approximately holomorphic section of the line bundle L^k. We can also bound the derivative of $\overline{\partial}_{A,\tilde{J}} f$, in these co-ordinates and bundle trivialisation:

$$\begin{aligned} |\nabla(\overline{\partial}_{A,\tilde{J}} f)| \leq & C(|\nabla\mu| \, |z| e^{-|z|^2/4} + |\mu| \, |\nabla(\textstyle\sum \overline{z}_\alpha dz_\alpha e^{-|z|^2/4}) \\ \leq & Ck^{-1/2}(|z| + |z|^3) e^{-|z|^2/4}. \end{aligned}$$

We now introduce a cut-off function, $\beta_k : \mathbf{C}^n \to \mathbf{R}$ depending on the (large) parameter k. Let β be a standard cut-off function of a single variable, with $\beta(x) = 1$ when $x \leq \frac{1}{2}$ and $\beta(x) = 0$ when $x \geq 1$. Then set

$$\beta_k(z) = \beta(k^{-1/6}z).$$

Thus β_k is supported in the large ball of radius $k^{1/6}$ in $\mathbf{C}^n$, which is nevertheless much smaller than the ball of radius $k^{1/2}$ on which the co-ordinate chart $\tilde{\chi}$ is defined. The derivative $\nabla\beta_k$ is $O(k^{-1/6})$, supported on an annulus where $|z|$ is $O(k^{1/6})$, so

$$|\nabla\beta_k| \leq Ck^{-1/2}|z|^2,$$

and we have

$$|\overline{\partial}_{A,\bar{j}}(\beta_k f)| \leq |\overline{\partial}_{A,\bar{j}} f| + |\nabla\beta_k| f \leq Ck^{-1/2}|z|^2.$$

Similarly,

$$\begin{aligned} |\nabla(\overline{\partial}_{A,\bar{j}}\beta_k f)| \leq & |\nabla\nabla\beta_k| + |\nabla\beta_k||\nabla_A f| + |\nabla(\overline{\partial}_{A,\bar{j}} f)| \\ \leq & Ck^{-1/2}(|z| + |z|^3)e^{-|z|^2/4}, \end{aligned}$$

since

$$|\nabla\nabla\beta_k| \leq C|z|k^{-1/2}.$$

In short, the cut-off function does not affect the estimates.

We can now define a smooth section σ_p of the line bundle L^k over V to be equal to $\tilde{\chi}(\beta_k\sigma)$ in the image of the co-ordinate chart $\tilde{\chi}$, extended by 0 over V. Thus σ_p is supported in a ball of radius $O(k^{-1/2}k^{1/6}) = O(k^{-1/3})$ about p, measured in the fixed metric g on V. It will however be more convenient to state the estimates in terms of the rescaled metric g_k. We write d_k for the distance function in this metric (so $d_k = k^{1/2}d$), and set

$$\begin{aligned} e_k(p,q) &= e^{-d_k(p,q)^2/5} \text{ if } d_k(p,q) \leq k^{1/4} \\ &= 0 \qquad \text{if } d_k(p,q) > k^{1/4}. \end{aligned}$$

We have then

Proposition 11. *For each point p in V and any sufficiently large integer k there is a smooth section σ_p of L^k over V such that at a point q in V:*

1. *for fixed R, $|\sigma_p(q)| \geq C^{-1}$ if $d_k(p,q) \leq R$,*

2. $|\sigma_p(q)| \leq e_k(p,q)$,

3. $|\nabla_V \sigma_p| \leq C(1 + d_k(p,q))e_k(p,q)$,

4. $|\overline{\partial}\sigma_p(q)| \leq Ck^{-1/2} d_k(p,q)^2 e_k(p,q)$,

5. $|\nabla_V \overline{\partial}_L \sigma_p(q)| \leq Ck^{-1/2}(d_k(p,q) + d_k(p,q)^3)e_k(p,q)$,

where $\overline{\partial}_L$ is the $\overline{\partial}$-operator on L^k, ∇_V denotes the covariant derivative defined by the Levi-Civita connection on V and the connection on L^k, and the constants are independent of k.

This result follows easily from the discussion above. The main point is that the chart $\tilde{\chi}$ is an approximate isometry with respect to the metric g_k on V. All that is required is to compare the estimates in the co-ordinate chart, used above, with the invariant estimates on V. So, for example, if $q \in V$ is the point corresponding to $z \in \mathbf{C}^n$ under the co-ordinate chart $\tilde{\chi}_p$ we have

$$|d_k(p,q)^2 - |z|^2 \;| \leq Ck^{-1/2}|z|^3,$$

which means that $|z|^2/4 \geq d_k(p,q)^2/5$, for q in the support of σ_p, once k is sufficiently large.

The most complicated case is (5). In the co-ordinate chart $\tilde{\chi}$,

$$\nabla_V(\overline{\partial}_L\sigma) = (\nabla + A + \Gamma)(\overline{\partial}_{A,\overline{J}}\sigma),$$

where Γ is the connection matrix for the connection on $T_V^{0,1} \subset TV^{\mathbf{C}}$ induced by the Levi-Civita connection and the bundle map $\varpi_{\overline{J}}$. Thus the essential point is that A and Γ are bounded on the ball of radius $k^{1/6}$ in $\mathbf{C}^n$—the support of σ_p. For A this is clear from the formulae above. For Γ one only needs to use the fact that the derivative of the chart $\tilde{\chi}$, and hence the ordinary Christoffel symbols, are bounded, along with $\tilde{\mu}$ and its first derivatives. In fact it is not at all essential to introduce the covariant derivative ∇_V here: all we really need is that the derivative estimate in one co-ordinate chart $\tilde{\chi}_p$ gives a corresponding estimate in any other such chart $\tilde{\chi}_{p'}$.

3. The global construction

In this section we present the main construction of the paper, leading to a section of our line bundle $L^k \to V$ satisfying the condition of

Theorem 5. The building blocks are the sections σ_p of the previous section. We now choose a collection of points p_i in V such that the balls of g_k-radius 1 about the p_i cover V. We emphasise again that the whole procedure depends on the parameter k, which appears now via the metric g_k on V. So we have to choose a separate collection p_i for each k. The points p_i must be chosen in a sensible way, so that the cover by unit balls is reasonably economical. Nothing very subtle is required however.

Lemma 12. *There is a constant C such that for each k we can cover V by a collection of g_k-unit balls with centres $p_i, i = 1, \dots M$ such that for any point q in V*

$$\sum_{i=1}^{M} d_k(p_i, q)^r e_k(p_i, q) \le C, \qquad \text{for } r = 0, 1, 2, 3.$$

To prove this Lemma we reduce to the Euclidean case. The essential point is that if Λ is a lattice in $\mathbf{C}^n$, then for any $a, r > 0$ and $w \in \mathbf{C}^n$ the infinite sums

$$\sum_{\mu \in \Lambda} |\mu - w|^r e^{-a|\mu - w|^2}$$

are bounded uniformly in w. Now fix a finite cover, independent of k, of V by charts $\phi_s : O_s \to V$, $s = 1, \dots, S$, where $O_s \subset \mathbf{C}^n$ is bounded, such that for all $x, y \in O_s$

$$\tfrac{1}{2}|x - y| \le d(\phi x, \phi y) \le 2|x - y|.$$

We may choose slightly smaller open sets $O_s'' \subset\subset O_s' \subset\subset O_s$, such that V is covered by the images of the O_s''. Let Λ be the lattice $\alpha(\mathbf{Z}^n \oplus i\mathbf{Z}^n)$ in $\mathbf{C}^n$ where $\alpha = \frac{1}{2}\left(\frac{n}{2}\right)^{1/2} k^{-1/2}$. This is chosen so that the Euclidean balls of radius $\frac{1}{2}k^{-1/2}$ centred on lattice points cover $\mathbf{C}^n$. Let $\Lambda_s \subset V$ be the image under ϕ_s of $\Lambda \cap O_s'$. Then when k is large, the balls of g radius $k^{-1/2}$—i.e., of g_k-radius 1—centred on the points of Λ_s cover $\phi_s(O_s)$. We define the set $\{p_i\} \subset V$ to be the union of the Λ_s, as s runs from 1 to S. So the g_k-unit balls centred on the p_i cover V, by construction. To bound the sums appearing in (12) it suffices to bound the individual contributions

$$R_s(q) = \sum_{p \in \Lambda_s} d_k(p, q)^r e_k(p, q),$$

since S does not depend on k. Now recall that $e_k(p,q)$ vanishes if $d_k(p,q) \geq k^{1/4}$, i.e., if $d(p,q) \geq k^{-1/4}$. So, when k is sufficiently large, $R_s(q)$ vanishes if q is not in $\phi_s(O_s)$. On the other hand if $q = \phi_s(z)$ is in $\phi_s(O_s)$ we have a bound

$$R_s(q) \leq \sum_{\lambda \in \Lambda} 2^r k^{r/2} |z-\lambda|^r e^{-k|z-\lambda|^2/20} = \sum_{\mu \in \Lambda_0} 2^r |w-\mu|^r e^{-|\mu - w|^2/20},$$

where Λ_0 is the fixed lattice $\frac{1}{2}\left(\frac{n}{2}\right)^{1/2}(\mathbf{Z}^n \oplus i\mathbf{Z}^n)$, and $w = k^{1/2}z$.

From now on we suppose that we have, for each k, fixed a collection $p_i, i = 1, \ldots, M$, satisfying the conditions of Lemma 12. We change our notation to write

$$\sigma_i = \sigma_{p_i}.$$

To generate the desired section we simply consider an appropriate linear combination of the σ_i. Thus for complex numbers $w_i, i = 1, \ldots, M$, making up a vector $\underline{w}$, we have a section

$$s = s_{\underline{w}} = \sum_{i=1}^{M} w_i \sigma_i \tag{13}$$

of L^k. We will always consider co-efficients w_i with $|w_i| \leq 1$.

Lemma 14. *For any choice of co-efficients $\underline{w}$ with $|w_i| \leq 1$, the section $s = s_{\underline{w}}$ satisfies*

$$\begin{aligned} |s| &\leq C, \\ |\overline{\partial}_L s| &\leq C k^{-1/2}, \\ |\nabla_V \overline{\partial}_L s| &\leq C k^{-1/2} \end{aligned}$$

everywhere on V.

The proof of this lemma is an immediate consequence of (11), (12).

We now reach the nub of the problem, whose solution will take up the remainder of this section.

Proposition 15. *There is an $\epsilon > 0$ such that for all large k we can choose $\underline{w}$, with $|w_i| \leq 1$, so that $s = s_{\underline{w}}$ satisfies the transversality condition*

$$|\partial s| > \epsilon$$

on the zero-set $W(s)$.

Combining with the second inequality of Lemma 14, this Proposition plainly completes the proof of the main result, Theorem 5.

We will construct the desired co-efficient vector in a series of stages. At each stage we will adjust some of the co-efficients w_i and leave others fixed. The problem we have to overcome is that the total number M of co-efficients *grows* with the parameter k (at least as fast as k^{2n}). The key to our approach is that the number of adjustment stages will be independent of k, so we need to divide up the set of points p_i in a suitable way. In turn this requires us to retrace ours steps slightly, to the choice of the set of centres p_i.

Lemma 16. *Given any $D > 0$ there is a number $N = N(D)$, independent of k, such that for any large k we can choose a collection of centres p_i satisfying the conditions of Lemma 12 in addition with the property that there is a partition of the set $I = \{1, 2, \dots, M\}$ into N disjoint subsets $I = I_1 \cup I_2 \cup \cdots \cup I_N$ such that for each α*

$$d(p_i, p_j) \geq D \textit{ if } p_i, p_j \in I_\alpha.$$

More precisely we can take $N(D) = CD^{2n}$.

To prove this we follow the same scheme as for Lemma 12. For integral D, the standard lattice $\Lambda = \mathbf{Z}^n \oplus i\mathbf{Z}^n$ can be partioned into the D^{2n} cosets $\Lambda / D\Lambda$, and the distance between any two members of the same coset is at least D. Now, following the construction of Lemma 12, it suffices to partition each collection Λ_s of centres, since the number S of these collections is independent of k. Then an appropriate coset partition of the corresponding lattice does the trick.

From now on we assume that D is fixed, that we have chosen the centres p_i as in the Lemma above, and that we have fixed a corresponding partition $\{I_\alpha\}$ of the index set. Of course we may also regard this as a partition of the set of co-efficients w_i, and of the set of balls B_i covering V. (It may help, in keeping track of the construction, to think of the partition as a "colouring" of the balls, using $N(D)$ colours.) The parameter D will be chosen at the end of the proof.

Our main strategy will now be to adjust the co-efficients w_i "belonging" to the same I_α (i.e., with $i \in I_\alpha$) at the same stage in the construction scheme. Thus we start with any co-efficient vector $\underline{w}^0$ (for example $\underline{w}^0 = 0$) and corresponding section $s^0 = s_{\underline{w}^0}$. At the first stage in the construction we will change the co-efficients w_i^0 belonging to I_1, leaving the others unchanged, to get a vector $\underline{w}_1$ and section $s^1 = s_{\underline{w}^1}$. At the second stage we change the co-efficients of $\underline{w}_1$ belonging to I_2 to get a section s^2, and so on. Thus the total number N of stages in the construction depends on D but not on k. (Of course at each stage we will maintain the bound $|w_i^\alpha| \leq 1$.)

With this framework in place, we can begin the discussion of the criteria we will use in adjusting the co-efficients. The essential idea is that at stage α we will achieve a controlled amount of transversality over the balls belonging to I_α (the balls "coloured by α"). We write

$$V_\alpha = \bigcup_{i \in I_\beta, \beta \leq \alpha} B_i,$$

so $\emptyset = V_0 \subset V_1 \subset V_2 \cdots \subset V_N = V$. What we want to achieve is that, for a suitable ϵ, the section s^α satisfies $|\partial s_\alpha| > \epsilon$ on $W(s_\alpha) \cap V_\alpha$ where $W(s_\alpha)$ is the zero-set, as before. Broadly speaking, this leads to two requirements at each stage:

1. The change in the co-efficients belonging to I_α must achieve controlled transversality over the balls belonging to I_α.
2. The change in the co-efficients belonging to I_α should not destroy the control of the transversality over the balls belonging to I_β for $\beta < \alpha$.

To take the discussion further we need to pin down precisely a notion of "controlled transversality".

Definition 17. Let $f : U \to \mathbf{C}$ be a smooth map on an open set $U \subset \mathbf{C}$ and let $w \in \mathbf{C}$. For $\eta > 0$ we say that f is η-transverse to w over U if for any $z \in U$ such that $|f(z) - w| \leq \eta$ the derivative satisfies $|(\partial f)_z| \geq \eta$.

This definition is most clearly relevant to holomorphic functions, but we will need to apply it also in the case where f is approximately holomorphic. An obvious point to note about this definition is stability under C^1 perturbations. If f is η transverse to w, and $g : U \to \mathbf{C}$ is another map with $\|f - g\|_{C^1} \leq \delta$, then g is $\eta - \delta$-tansverse to w. Of course this statement is a vacuous if $\eta < \delta$.

We pause now to consider any section s of our line bundle $L^k \to V$. Write $\tilde{\chi}_i = \tilde{\chi}_{p_i}$ for the co-ordinate chart centred on p_i chosen in Section 2. This chart is defined on a large ball of radius $k^{1/6}$ in $\mathbf{C}^n$, but from now on we can fix attention on bounded regions of $\mathbf{C}^n$. We may suppose that $\tilde{\chi}_i^{-1}(B_i)$ is contained in the ball $\Delta = \frac{11}{10}B^{2n}$, and we fix a larger set $\Delta^+ \subset \mathbf{C}^n$, say the polydisc $\{|z_\alpha| \leq 22/10\}$. Over $\tilde{\chi}_i(\Delta)$ we have a standard trivialisation of L^k furnished by the section σ_i, and we write $s = f_i\sigma_i$ for a function f_i, which we regard as a function on Δ^+. We say that the section s is η-transverse over B_i if the map f_i is η-transverse to 0 over the corresponding set Δ in $\mathbf{C}^n$.

The next two lemmas express the salient properties of the sections $s_{\underline{w}}$ in terms of their representation by the functions f_i.

Lemma 18. *If $s = s_{\underline{w}}$ is a section of L^k with $|w_i| \leq 1$, and f_i are the corresponding functions on $\Delta^+ \subset \mathbf{C}^n$, then the following hold:*

1. $\|f_i\|_{C^1(\Delta^+)} \leq C$.
2. $\|\overline{\partial} f_i\|_{C^1(\Delta^+)} \leq Ck^{-1/2}$.
3. *If $|\partial f_i| > \epsilon$ on $f_i^{-1}(0) \cap \Delta$, then (once k is sufficiently large) $|\partial_L s| > C^{-1}\epsilon$ on $W(s) \cap B_i$.*

Lemma 19. *Let $s = s_{\underline{w}}$ for a co-efficient vector $\underline{w}$ with $|w_i| \leq 1$. For any α, let $\underline{w}'$ be another vector which agrees with $\underline{w}$ except for the co-efficients belonging to I_α, i.e.,*

$$w'_j = w_j \text{ if } j \notin I_\alpha,$$

and suppose that $|w'_j - w_j| \leq \delta$ for $j \in I_\alpha$. Write $s' = s_{\underline{w}'}$, and let f_i, f'_i be the functions representing these sections, as above. Then the following hold:

1. *For any i,*
$$\|f'_i - f_i\|_{C^1(\Delta^+)} \leq C\delta.$$
2. *If $i \in I_\alpha$ and $w'_i = w_i + \theta_i$ then*
$$\|f'_i - f_i - \theta_i\|_{C^1(\Delta^+)} \leq C\exp(-D^2/5)\delta.$$

The proofs of these lemmas are straightforward. The main point is that $|\sigma_i|$ is bounded below on $\phi(\Delta^+)$; then we have for example:

$$\overline{\partial}_L(s) = \overline{\partial}_L(f_i\sigma_i) = (\overline{\partial}_J f_i)\sigma_i + f_i(\overline{\partial}_L\sigma_i),$$

so, dividing by σ_i, bounds on $\overline{\partial}_L s, \overline{\partial}_L\sigma_i$ give corresponding bounds on $\overline{\partial}_J f_i$. Similarly for the full covariant derivative. Thus we go from the $\overline{\partial}$-operator $\overline{\partial}_J$ of the almost complex structure to the standard $\overline{\partial}$-operator using the fact that $\overline{\partial}_J = \overline{\partial} + \tilde{\mu}\partial$ where μ is of order $k^{-1/2}$ over Δ^+.

We will now bring in the local result, for functions on a ball, which will be the fuel for our global construction. The proof will be given in Sections 4 and 5. As a piece of notation, for an integer p we will write Q_p for the function

$$Q_p(\delta) = \log(\delta^{-1})^{-p}, \quad \delta > 0.$$

Theorem 20. *For $\sigma > 0$, let $\mathcal{H}_\sigma$ denote the set of functions f on Δ^+ such that*

1. $\|f\|_{C^0(\Delta^+)} \leq 1$,
2. $\|\overline{\partial} f\|_{C^1(\Delta^+)} \leq \sigma$.

Then there is an integer p, depending only on the dimension n, such that for any δ with $0 < \delta < \frac{1}{2}$, if $\sigma \leq Q_p(\delta)\delta$, then for any $f \in \mathcal{H}_\sigma$ there is a $w \in \mathbf{C}$ with $|w| \leq \delta$ such that f is $Q(\delta)\delta$-transverse to w over the interior region $\Delta \subset \Delta^+$. Moreover, if ζ is any non-zero complex number, we can suppose that w lies in the half-plane $\{w \in \mathbf{C} : Re\ (w\zeta) > 0\}$.

All the ingredients are now in place for the main global construction. We suppose that, at the beginning of stage α in the construction, we have chosen $\underline{w}^{\alpha-1}$ so that $s_{\alpha-1}$ is $\eta_{\alpha-1}$-transverse over the set $V_{\alpha-1} \subset V$, (the balls coloured with all the earlier colours) for some small positive $\eta_{\alpha-1}$, say with $0 < \eta_{\alpha-1} < \rho$. We want to choose the coefficients w_i^α, for $i \in I_\alpha$. We will require that $|w_i^\alpha - w_i^{\alpha-1}| \leq \delta_\alpha$ for some δ_α to be specified shortly. For any choice of $\underline{w}^\alpha$ satisfying this condition, (1) of Lemma 19 implies that s^α is $\eta_{\alpha-1} - C\delta_\alpha$-transverse over $V_{\alpha-1}$, for some fixed C. Thus if

$$\delta_\alpha = C^{-1}\eta_{\alpha-1},$$

we can assume that s^α is still $\frac{1}{2}\eta_{\alpha-1}$- transverse over $V_{\alpha-1}$. Now consider the situation over a single ball B_i, for $i \in I_\alpha$. Here the section $s^{\alpha-1}$ is represented by a function $f_i = f_i^{\alpha-1} = s^{\alpha-1}/\sigma_i$. The function f_i is bounded by a fixed constant C over the larger set Δ^+, and the estimates (18) show that $C^{-1}f_i$ lies in $\mathcal{H}_\sigma$ for $\sigma = C^{-1}k^{-1/2}$. For suitably small ρ, we can apply (20) to $C^{-1}f_i$, with $\delta = C^{-1}\delta_\alpha$, so long as

$$k^{-1/2} \leq C\delta_\alpha Q_p(\delta_\alpha). \tag{21}$$

Assuming this, we find a $v = v_i$ with $|v| \leq \delta_\alpha$ such that f_i is $Q_p(\delta_\alpha)\delta_\alpha$-transverse to v_i over the unit ball. (We can absorb constants into the factor Q_p, by changing p suitably, since we may as well suppose that δ_α is small.) Equivalently $f_i - v_i$ is $Q_p(\delta_\alpha)\delta_\alpha$-transverse to 0. But $f_i - v_i$ is the function representing the section $s_{\underline{w}'}$ where

$$\begin{aligned} w'_j &= w_j^{\alpha-1} && \text{if } j \neq i, \\ w'_i &= w_i^{\alpha-1} - v_i. \end{aligned}$$

So we conclude that, assuming the side condition (21), we can, by changing a *single* co-efficient w_i by at most δ_α, make a new section $s^{\underline{w}'}$ which is $Q_p(\delta_\alpha)\delta_\alpha$-transverse over the ball B_i. Furthermore, by choosing v_i in a suitable half-plane, we can assume that $|w'_i| \leq 1$.

We now define $\underline{w}^\alpha$, and hence the new section s^α, by making all these changes simultaneously, for all $i \in I_\alpha$. Thus

$$\begin{aligned} w_j^\alpha &= w_j^{\alpha-1} && \text{if } j \notin I_\alpha, \\ w_j^\alpha &= w_j^{\alpha-1} - v_j && \text{if } j \in I_\alpha. \end{aligned}$$

To control the transversality of s^α over a typical ball B_i, for $i \in I_\alpha$, we want to regard it as a small perturbation of the section $s^{w'}$ considered above. By (2) of Lemma 19 the C^1-norm of the difference of these two sections over B_i is bounded by $C\exp(-D^2/5)\delta_\alpha$. So s^α is $\frac{1}{2}Q_p(\delta_\alpha)\delta_\alpha$-transverse over B_i, say, so long as

$$C\exp(-D^2/5)\delta_\alpha \leq \tfrac{1}{2}Q_p(\delta_\alpha)\delta_\alpha.$$

We can cancel a factor δ_α from each side of the equation and again absorb the constant by redefining p suitably. Then the condition becomes

$$\exp(-D^2/5) \leq Q_p(\delta_\alpha). \tag{22}$$

Suppose this condition is met. We may assume—by choosing ρ small—that $Q_p(\delta_\alpha)\delta_\alpha < \eta_{\alpha-1}$; then the new section s^α is η_α-transverse everywhere on V_α, where $\eta_\alpha = \frac{1}{2}Q_p(\delta_\alpha)\delta_\alpha$. Finally, we have $\delta_\alpha Q_p(\delta_\alpha) \geq C^{-1}\eta_{\alpha-1}Q_p(\eta_{\alpha-1})$, so by redefining p again we may suppose that

$$\eta_\alpha = Q_p(\eta_{\alpha-1})\eta_{\alpha-1}.$$

We can summarise this discussion in the following statement:

Proposition 23. *There are constants $\rho < 1$ and p such that if $s^{\alpha-1} = s_{\underline{w}^{\alpha-1}}$ is a section of L^k which is $\eta_{\alpha-1}$-transverse over $V_{\alpha-1}$ with $\eta_{\alpha-1} \leq \rho$, and if*

1. $k^{-1/2} \leq Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$,

2. $\exp(-D^2/5) \leq Q_p(\eta_{\alpha-1})$,

then there is another section $s^\alpha = s_{\underline{w}^\alpha}$ *which is* η_α*-transverse over* V_α, *where*

$$\eta_\alpha = \eta_{\alpha-1}Q_p(\eta_{\alpha-1}).$$

Notice that $\eta_\alpha < \eta_{\alpha-1}$, so the condition $\eta_\alpha < \rho$ holds rather trivially. Notice also that the argument needs to be reshaped slightly at the first stage, when $\alpha = 1$. In this case the construction produces a section s^1 which is η_1-transverse over V_1, for some small η_1.

We claim now that this iterative construction will produce the desired section s^N, starting from any initial s^0, provided D is chosen suitably and k is sufficiently large. To verify this claim we have to work through the book-keeping of the iterative definition in (23) of the η_α, and the constraints involving k and D.

Lemma 24. *Let* $p > 0$ *and let* $x_\alpha, \alpha = 1, 2, \ldots,$ *be a sequence of real numbers such that*

$$x_\alpha = x_{\alpha-1} + p\log x_{\alpha-1}.$$

Then for any $q > p$ *there is an integer* α_1, *depending only on* q *and* x_1, *such that*

$$x_\alpha \leq q(\alpha + \alpha_1)\log(\alpha + \alpha_1).$$

To prove the Lemma we define a sequence y_α by

$$y_\alpha = q\alpha\log\alpha;$$

then

$$\begin{aligned} y_\alpha - y_{\alpha-1} &= q(\alpha\log\alpha - (\alpha-1)\log(\alpha-1)) \\ &= q((\alpha-(\alpha-1))\log(\alpha-1) + \alpha(\log\alpha - \log(\alpha-1)) \\ &\geq q(\log(\alpha-1)+1). \end{aligned}$$

While

$$p\log y_{\alpha-1} = p(\log(\alpha-1) + \log\log(\alpha-1) + \log q),$$

so $y_\alpha - y_{\alpha-1} \geq p\log y_\alpha$ once α is large, say if $\alpha > \alpha_0$. Now if we choose α_1 so that $\alpha_1 \geq \alpha_0$ and that $x_\alpha \leq q(\alpha+\alpha_1)\log(\alpha = \alpha_1)$ when $\alpha = 1$, then the same inequality holds for all $\alpha \geq 1$ by comparison of the two sequences $(x_\alpha), (y_{\alpha+\alpha_1})$.

We apply this Lemma to the sequence $x_\alpha = -\log \eta_\alpha$ where the η_α are defined inductively by $\eta_\alpha = Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$ as in Proposition 23, with $\eta_0 \leq \rho$. We get the rough bound

$$Q_p(\eta_{\alpha-1}) \geq \frac{C}{(\alpha \log \alpha)^p},$$

for some constant C depending on ρ. Hence

$$Q_p(\eta_{\alpha-1}) \geq \frac{C}{(N \log N)^p},$$

where N is the number of "colours", as above. Now we know that $N \leq const.D^{2n}$ where D is the separation parameter. So

$$Q_p(\eta_{\alpha-1}) \geq \frac{C}{D^{2np+1}},$$

say, once D is large. Since

$$e^{-D^2/5} \leq \frac{C}{D^{2np+1}},$$

for large D, we conclude that if we make the parameter D large enough, then condition (2) of Proposition 23 will be satisfied at each stage. Now the other condition (1) of Proposition 23 has a different status, it only involves k and not D, and the parameter k does not affect the value of η_N. Thus having fixed D we can choose k large enough for (1) to hold at all stages and we get a section s^N which is η_N transverse over the whole of V. This conpletes the proof of the existence result, Theorem 5.

4. Transversality with estimates

In this Section we will prove Theorem 20: a refinement of Sard's theorem for approximately holomorphic functions. The input for this proof, following the method of Yomdin [9], [3], is a result on the complexity of *real* algebraic sets which we state first, and which in turn will be proved in Section 5. Let $P : \mathbf{R}^m :\to \mathbf{R}$ be a polynomial function of degree d. Let $S \subset \mathbf{R}^n$ be the subset

$$S = \{x \in \mathbf{R}^n : |x| \leq 1, P(x) \leq 1\}.$$

and let $S(\theta) = \{x : |x| \leq 1, P(x) \leq 1 + \theta\}$. The result we will use is

Proposition 25. *There are constants C, ν, depending only on the dimension m, such that for any polynomial P there are arbitrarily small, positive θ so that S may be decomposed into pieces*

$$S = S_1 \cup S_2 \cdots \cup S_A,$$

where $A \leq Cd^{\nu}$, in such a way that any pair of points in the same piece S_r can be joined by a path in $S(\theta)$ of length less than Cd^{ν}.

The statement of this Proposition is slightly complicated: a simpler statement, which is also true, is that the number of components of S, and the diameter of each component in the induced path-length metric are bounded by Cd^{ν}. We have chosen the more complicated statement because the proof is easier and suffices for our application.

We begin the proof of Theorem 20 by considering the case of holomorphic functions. Thus we wish to prove

Proposition 26. *There is a constant p such that if f is any holomorphic function on $\Delta^+ \subset \mathbf{C}^n$ with $|f(z)| \leq 1$ everywhere, and if $\eta \leq \frac{1}{2}$, then there is a w in $\mathbf{C}$ with $|w| \leq \eta$ and Re $(w) \geq 0$ such that f is $\eta Q_p(\eta)$-transverse to w over the interior region Δ.*

Here, as in Section 3, $Q_p(\eta) = \log(\eta^{-1})^{-p}$. The first step is the following Lemma.

Lemma 27. *Let $f : \Delta^+ \to \mathbf{C}$ be a holomorphic map with $|f(z)| \leq 1$. Then for any ϵ with $0 < \epsilon < 1/2$ there is a complex polynomial function g on $\mathbf{C}^n$ of degree less than $C \log \epsilon^{-1}$ such that $|f(z) - g(z)|, |\partial f - \partial g| \leq \epsilon$ on the interior region Δ.*

To prove this Lemma we truncate the ordinary Taylor series of f. Recall that Δ^+ is the polydisc $\{|z_\alpha| \leq 22/10\}$. A holomorphic function f on Δ^+ has a Taylor expansion

$$f(z) = \sum a_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n},$$

and the co-efficients are given by the Cauchy formula:

$$a_I = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w_1, \dots w_n)}{w_1^{i_1+1} \dots w_n^{i_n+1}} dw_1 \dots dw_n,$$

where the contour Γ is the distinguished boundary $|z_\alpha| = 22/10$. For any integer $s > 0$ let g be the polynomial given by ignoring all terms in the series which contain any power z_α^r with $r > s$, so g has degree at most ns. The interior region Δ is the ball $\{|z| \leq 11/10\}$. If z lies in Δ we write

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{(w_1 - z_1)\dots(w_n - z_n)} dw_1 \dots dw_n.$$

Then we expand the denominator in the familiar way; just as in the ordinary proof of the Taylor theorem. Using the fact that for $z \in \Delta$ and $w \in \Gamma$, $|z_\alpha / w_\alpha| \leq 1/2$, one obtains that

$$f(z) - g(z) = \frac{1}{(2\pi i)^n} \int_\Gamma f(w) E(w) dw_1 \dots dw_n,$$

where $|E(w)| \leq n 2^{-s}$, and it follows that

$$|f(z) - g(z)| \leq n 2^{-s} \text{ on } \Delta.$$

By a similar, straightforward, argument one finds that $|\partial f - \partial g| \leq \sqrt{n}(s+n)2^{-s}$. Thus in sum we see that

$$|f - g|, |\partial f - \partial g| \leq C e^{-\lambda d},$$

for some λ, where $d = ns$ is the degree of g. Inverting this, to approximate f in C^1 with error $\epsilon < 1/2$ we can take a polynomial of degree d where we need $Ce^{-\lambda d} < \epsilon$ and d to be an integer divisible by n. We can do this with

$$d \leq \frac{\log \epsilon^{-1} - \log C^{-1}}{\lambda} + n,$$

which is bounded by a multiple of $\log \epsilon^{-1}$, once $\epsilon < 1/2$, as required.

We now come to the central argument of this Section, which will prove Proposition 26. Let f be, as above, a holomorphic function on Δ^+ with $|f| \leq 1$. Given a small $\epsilon > 0$ we use Lemma 27 to approximate f in C^1 to within ϵ over the interior ball Δ by a polynomial g of degree $d \leq C \log \epsilon^{-1}$. Now consider the set

$$S^f = \{z \in B^{2n} \subset \mathbf{C}^n : |\partial f| \leq \epsilon\}.$$

Clearly S^f is a subset of S^g where

$$S^g = \{z \in B^{2n} : |\partial g| \leq 2\epsilon\}.$$

Thus the image $f(S^f) \subset \mathbf{C}$ is contained in $f(S^g)$, which is in turn contained in the ϵ-neighbourhood of $g(S^g)$. Now the set S^g is a semi-algebraic set of the kind considered in Proposition 25, where we identify

$\mathbf{C}^n$ with $\mathbf{R}^{2n}$ and take $P = (2\epsilon)^{-2}|\partial g|^2$, which is a real polynomial of degree $2(d-1)$. We takes $\theta \leq 3$, say, and decompose S^g into A pieces as in Proposition 25. If z_1, z_2 are in the same piece of S^g then, integrating the derivative of g over a path of length less than $L = Cd^\nu$ in $P(\theta)$ joining z_1 to z_2, we have $|g(z_1) - g(z_2)| \leq 4\epsilon L$. Thus $g(S^g)$ is contained in the union of A discs in $\mathbf{C}$ each of radius $4L\epsilon$. Hence $f(S^g)$, and so also $f(S^f)$, is contained in the union of A slightly larger discs of radius $(2L+1)\epsilon$. The condition on w in $\mathbf{C}$ that f is ϵ-transverse to w is that w lie ouside the ϵ-neighbourhood of $f(S^f)$, and this neighborhood is contained in the union of A discs of radius $(4L+2)\epsilon$. The total area in $\mathbf{C}$ of these discs is at most $A\pi(4L+2)^2\epsilon^2$. If we choose ρ such that the area of the half-disc

$$\Omega = \{w \in \mathbf{C} : \ |w| \leq \rho, \ \ \text{Re }(\rho) \geq 0\}$$

is bigger than the total covered by these discs, then there is w in Ω not contained in the ϵ-neighbourhood of $f(S^f)$. The condition on ρ is

$$\tfrac{1}{2}\pi\rho^2 > A\pi(4L+2)^2\epsilon^2,$$

that is

$$\rho > \sqrt{(}2A)(4L+2)\epsilon.$$

Now we know that A and L are bounded by powers of the degree of P, hence of the degree d of g which is bounded by a power of $\log(\epsilon^{-1})$. Putting everything together, we see that for any sufficiently small ϵ there is a w in $\mathbf{C}$ such that f is ϵ-transverse to w, w lies in the half plane $\Re w \geq 0$ and

$$|w| \leq C\epsilon \log(\epsilon^{-1})^p$$

for some p. To obtain the result stated in Proposition 26 we only need to re-organise the parameters. Observe that, if C, p are fixed, the function h given by $h(\epsilon) = C\epsilon\log(\epsilon^{-1})^p$ is increasing for small ϵ and tends to zero as $\epsilon \to 0$. If $\eta = h(\epsilon)$ then

$$\eta \log(\eta^{-1})^{-p} = C\epsilon \left(\frac{\log \epsilon^{-1}}{\log\epsilon^{-1} - p\log\log\epsilon^{-1} - \log C} \right)^p,$$

which is less than $2C\epsilon$, say, once ϵ is sufficiently small. Inverting the function h, we see that for small η there is a w with $|w| \leq \eta$ such that f is $\frac{1}{2C}\eta(\log\eta^{-1})^{-p}$ transverse to w, and finally by increasing p, and assuming η to be sufficiently small, we can replace the factor $1/2C$ by 1, to get the result stated in Proposition 26.

We now move on to consider approximately holomorphic functions, and to prove Proposition 25. We need the following result from linear analysis:

Lemma 28. *For each $r < 1$ there is a constant $K = K(r)$ such that if f is any smooth complex-valued function on $\Delta^+ \subset \mathbf{C}^n$, then there is a holomorphic function $\tilde{f}$ on the interior region $r\Delta^+ \subset \Delta^+$ such that*

$$\|f - \tilde{f}\|_{C^1(r\Delta^+)} \leq K(\|\overline{\partial} f\|_{C^0(\Delta^+)} + \|\nabla \overline{\partial} f\|_{C^0(\Delta^+)}).$$

This is a standard fact. One approach is to use the Hormander theory of weighted L^2 spaces [4]. Using a suitable weighted L^2 norm on Δ^+, which compares uniformly with the standard norm on an interior region $r'\Delta^+$, where we choose $r < r' < 1$, one obtains a bounded solution to the $\overline{\partial}$-problem: i.e., for any $\overline{\partial}$-closed $(0,1)$ form ρ on the ball there is a function $T(\rho)$ with $\overline{\partial} T(\rho) = \rho$ and

$$\|T(\rho)\|_{L^2(r'\Delta^+)} \leq C\|\rho\|_{L^2(\Delta^+)}.$$

We take $\rho = \overline{\partial} f$, so $\tilde{f} = f - T(\rho)$ is holomorphic. Write h for $T(\rho)$, i.e., $f - \tilde{f}$. Then the L^2-norm of h and the C^1-norm of $\overline{\partial} h = \overline{\partial} f$ over the region $r'\Delta^+$ are bounded by multiples of $\|\overline{\partial} f\|_{C^1(\Delta^+)}$, and so by standard elliptic theory (in Hölder or Sobolev spaces) the same is true for the C^1-norm of h over the interior ball $r\Delta^+$.

The proof of Theorem 20 follows easily from this. If f is a function in $\mathcal{H}_\sigma$ we approximate f by a holomorphic function $\tilde{f}$ on a slightly smaller region, with $\|f - \tilde{f}\|_{C^1} \leq C\sigma$. Then we apply Proposition 26 to $\tilde{f}$. Here we need to modify the statement of Proposition 26 slightly to take account of the smaller region on which $\tilde{f}$ is defined, but it is clear that this will just make a small change to the constants involved. Hence we find a w with $|w| \leq \eta$ such that $\tilde{f}$ is $\eta Q_p(\eta)$-transverse to w: if $\sigma \leq \frac{1}{2}\eta Q_p(\eta)$, then $\tilde{f}$ is also $\frac{1}{2}\eta Q_p(\eta)$-transverse to w. Again the factor of $1/2$ is absorbed by a change of p. Finally the requirement that w lie in a given half-plane is obtained from the corresponding condition in Proposition 26, multiplying f by a complex number ζ of unit modulus.

5. Complexity of real algebraic sets

In this Section we prove Proposition 25. As a piece of terminology, we will say that a quantity defined by a polynomial function $P : \mathbf{R}^m \to$

$\mathbf{R}$ is *p-bounded* if it is bounded by a function Cd^{ν}, where the constants C, ν depend only on the dimension m. We will use the word *diameter*, for a subset of $\mathbf{R}^n$, to mean the diameter in the induced path-length metric. Now our main goal is to prove:

Proposition 29. *Let $P : \mathbf{R}^m \to \mathbf{R}$ be a polynomial function such that 0 is a regular value of P on the unit ball B^m and of the restriction of P to the boundary sphere S^{m-1}. Then the number of components of $\{x \in B^m : P(x) \leq 1\}$ and the diameters of the components are p-bounded.*

We make a number of simple remarks:

1. Proposition 29 implies Proposition 26. For any P we choose regular values $1 + \epsilon$ of $P|_{B^m}$ and $P|_{S^{m-1}}$, for arbitrarily small positive ϵ, and apply Proposition 29 to $P - \epsilon$.

2. It is the the same to prove Proposition 29 as to prove the corresponding bounds on the number and diameter of the components of the hypersurface
$$\Sigma = \{x \in B^m : P(x) = 1\}.$$

3. Under the hypotheses of Proposition 29, Σ is an $(m-1)$-manifold with boundary, whose topology is not changed by small pertubations of P. Since the diameter of a compact Riemannian manifold with boundary varies continously with the Riemannian metric (in the C^{∞} topology), it suffices to prove Proposition 29 for any dense set of polynomials P.

4. To prove Proposition 29 it suffices to decompose Σ into any p-bounded collection of sets of p-bounded diameter.

Now our proof follows the argument of Gromov in [3, p.124]. Before diving into the main proof we bring to the centre of the stage the principle geometric ingredient in the argument, which is the Crofton formula from integral geometry.

Let H be the set of affine hyperplanes in $\mathbf{R}^m$. There is a smooth measure on H, unique up to scale, which is invariant under the transitive action of the affine Euclidean group. If C is any compact, smooth, arc in $\mathbf{R}^m$, then almost all hyperplanes $\Pi \in H$ meet C transversely in a finite number $i(\Pi, C)$ of points. The Crofton formula states that, with a suitable normalisation of the measure, the length of C is

$$l(C) = \int_H i(\Pi, C) d\Pi.$$

The set of hyperplanes which meet the ball B^m is compact, so has finite volume V say. If C is the intersection of B^m with a real algebraic curve of degree δ, then the intersection number $i(\Pi, C)$ is at most δ (almost everywhere) and it follows that the length of C is at most $V\delta$.

We now begin the main proof, which goes by induction on the dimension m. We choose co-ordinates $(t, y_1, \dots y_{m-2}, z)$ on $\mathbf{R}^m$. We will regard the first co-ordinate t as a parameter and the last co-ordinate z as a "height" function, in the manner of Morse theory. Write π for the projection from $\mathbf{R}^m$ onto the first co-ordinate and decompose Σ into "slices"

$$\Sigma_t = \pi^{-1}(t) \subset \Sigma.$$

For fixed t the critical points of the height function z on Σ_t are the solutions of the polynomial equations $\frac{\partial P}{\partial y_i} = 0$. Let $Q_0 : B^m \to \mathbf{R}^{m-1}$ be the map with components $(P - 1, \frac{\partial P}{\partial y_i})$. Let C be $Q_0^{-1}(0)$—the union of the critical sets, as the parameter t varies over $[-1, 1]$. As we shall show below, it suffices to treat the case where the following general position conditions are met:

General position conditions (30).

1. *The points* $(\pm 1, 0, 0)$ *are not in* Σ.

2. *The set* C *is a smooth curve in* B^m *which meets the boundary sphere transversally in a finite number of points.*

3. *The slices* Σ_t *are smooth manifolds, transverse to the boundary sphere for all but finitely many parameter values* t.

4. *For all but finitely many* t, *the height function* z *is a Morse function on* Σ_t.

If these conditions hold we get a finite set $E \subset [-1, 1]$ of exceptional parameter values: the union of the finite sets in (3),(4) and the projection of the set in (2). We also include the end points ± 1 in E. Then the complement of E is a finite union of open intervals $J_\beta \subset [-1, 1]$ and the restriction of π to each $\pi^{-1}(J_\beta) \subset \Sigma$ is a fibration of manifolds with boundary. Moreover the fibration is compatible with the family of Morse functions, and the index of the critical points is constant along each component of $C_\beta = C \cap \pi^{-1}(J_\beta)$.

Lemma 31. *For an open dense set of polynomials of a given degree d the general position conditions (30) hold, and the number of points in the set E is p-bounded.*

We assume this lemma for the moment, and complete the proof. Remark (3) following Proposition 29 shows that it suffices to prove the result for polynomials satisfying (30). For such P the set C is a portion of a real algebraic curve of degree at most d^{m-1}, given by the equations $P = 1, \frac{\partial P}{\partial y_i} = 0$, so its length is p-bounded by the Crofton formula.

We now apply induction on the dimension m. First, the boundary sphere S^{m-1} is a rational variety, and can be covered by the images of two balls under maps $f : \mathbf{R}^{m-1} \to S^{m-1}$ whose components are rational functions. It follows that the main result (29) for the ball B^{m-1} implies that the number and diameter of the components of $\Sigma \cap S^{m-1}$ are p-bounded. We now fix attention on a single interval J_β. Let us say that a point x in $\pi^{-1}(J_\beta)$ is *accessible from the boundary* if x lies in a slice Σ_t, and the component of Σ_t containing x also meets the boundary. Applying the main result in dimension $m-1$ to the slice Σ_t we see that in this case x can be joined to a boundary point by a path of p-bounded length. So the set of points which are accessible from the boundary can be covered by a p-bounded number of sets (just the number of components of $\Sigma \cap S^{m-1}$) each of p-bounded diameter. Now let $\Sigma^*(\beta) \subset \pi^{-1}(J_\beta)$ be the set of points which are *not* accessible from the boundary, and let x_1 be in $\Sigma^*(\beta)$, lying on a slice Σ_{t_1}. We minimise the height function z on the component of Σ_{t_1} which contains x_1. The minimum is not a boundary point since x_1 is not accessible from the boundary, so it must be a critical point of index 0, a point, c_1 say, of the curve C_β. Again by induction, applying the main result to the slice Σ_{t_1}, x_1 can be joined to c_1 by a path of p-bounded length. If x_2 is another point in the same portion $\Sigma^*(\beta)$ which lies in a slice Σ_{t_2}, we can similarly join x_2 to a point c_2 in C_β by a p-bounded path. If c_1, c_2 are in the same component of the curve C_β, then we would succeed in joining x_1 to x_2 by a p-bounded path, since the total length of C is p-bounded as we have seen. Thus we can decompose $\Sigma^*(\beta)$ into pieces, labelled by the components of $C(\beta)$ of index 0, each of p-bounded diameter. Now the number of components of C_β is the same as the number of intersection points $C \cap \pi^{-1}(t_1)$, hence bounded by the degree of C, and so p-bounded.

So we conclude that $\Sigma^*(\beta)$ is covered by a p-bounded family of sets of p-bounded diameter. Adding the points accessible from the boundary and summing over the p-bounded collection of intervals J_β we deduce that the same is true for the set $\Sigma_0 = \bigcup_\beta \pi^{-1}(J_\beta)$. Finally, any point in Σ lies in the closure of Σ_0 (because the transversality conditions prevent Σ containing an open set in a hyperplane $\pi^{-1}(t)$, and the end-points $(\pm 1, 0, 0)$ are not in Σ), and since Σ is locally path-connected we obtain the corresponding result for Σ.

We will now prove Lemma 31. This follows rather standard lines; it is clear that each condition is open, so it suffices to show that each one individually is satisfied by a dense set of polynomials of fixed degree. First condition (1) of (30) is clear. Now consider condition (2), that C be a curve, transverse to the boundary sphere. Recall that C is the zero-set of the map $Q_0 = (P-1, \frac{\partial P}{\partial y_i}) : B^m \to \mathbf{R}^{m-1}$. We consider an $m-2$ dimensional family of variations of the polynomial P, with a parameter $\underline{s} = (s_1, \dots, s_{m-2})$

$$P_{\underline{s}} = P + \sum s_i y_i,$$

and define a map $F : B^m \times \mathbf{R}^{m-2} \to \mathbf{R}^{m-1}$ by $F(x, \underline{s}) = (P_{\underline{s}}, \frac{\partial P_{\underline{s}}}{\partial y_i})$. Then the derivative of F at any point $(x, 0)$, for x in Σ, is surjective by construction, so $F^{-1}(0)$ is cut out transversally in a neighbourhood of $\Sigma \times \{0\}$. Now apply Sard's theorem to choose an arbitrarily small regular value $\underline{s}$ of the projection from $F^{-1}(0)$ to the $\underline{s}$ co- ordinate; then the corresponding map $P_{\underline{s}}$ has a critical set $C_{\underline{s}}$ which is a smooth curve. Finally choose a regular value $\rho < 1$, arbitrarily close to 1, of the radius function on $C_{\underline{s}}$. Then $C_{\underline{s}}$ meets the ρ-sphere transversally, and composing $P_{\underline{s}}$ with the dilation with factor ρ we get a small variation which satisfies (2). Clearly, in the transverse situation, the contribution to the exceptional set E from condition (2): the number of intersection points of C with the boundary sphere is p-bounded: it is at most twice the degree of C since the boundary sphere is an algebraic surface of degree 2.

The arguments for the other conditions are similar. For condition (3) we consider the map $Q_1 = (P-1, \frac{\partial P}{\partial y_i}, \frac{\partial P}{\partial t}) : B^m \to \mathbf{R}^m$. The zeros of Q_1 are the singular points of slices Σ_t. One constructs a family of variations just as in the case of Q_0 above to arrange that, after arbitrarily small perturbation, Q_1 is transverse to 0. In that case the zero-set of Q_1 is finite, and the number of points is bounded by the degree $d(d-1)^{m-1}$ of Q_1. Condition (3) also requires that, with a finite number of exceptions, the slices Σ_t are transverse to the boundary. To arrange this it suffices to arrange that the function on $\Sigma \cap S^{m-1}$ given by the spherical distance from $(1, 0, 0)$ is a Morse function. One can achieve this by a small rotation of the axes (i.e., the complement of the set of "focal points" of a submanifold of the sphere is open and dense: the proof is just the same as in the Euclidean case). To bound the number of points where Σ_t fails to be transverse to the boundary sphere we can suppose, by making a small rotation of the (y_i, z) co-ordinates if necessary, that none of these points lies on the hyperplane section $z = 0$. Then the non-transverse points are solutions of the $m - 1$ equations

$$P = 1, \; z\frac{\partial P}{\partial y_i} - y_i\frac{\partial P}{\partial z} = 0$$

on S^{m-1}, and the number of such solutions is at most $2d^{m-1}$, by considering the degrees of the equations involved.

Next we consider condition (3). Given a polynomial P we let H be the polynomial function given by the Hessian $H(t, y_i, z) = \det\left(\frac{\partial^2 P}{\partial y_i \partial y_j}\right)$, and let $Q_2 : B^m \to \mathbf{R}^m$ be the map $(P-1, \frac{\partial P}{\partial y_i}, H)$. The degenerate critical points of the height function z are the zeros of Q_2. We consider the variation of P by quadratic terms $P_{\underline{u}} = P + \sum u_{ij} y_i yj$, where $\underline{u} = (u_{ij})$, and put

$$G(x, \underline{u}) = (P, \frac{\partial P_{uu}}{\partial y_i}, \det\left(\frac{\partial^2 P_{\underline{u}}}{\partial y_i \partial y_j}\right)).$$

The derivative of the last component of G in the $\underline{u}$ variable is given by

$$\underline{u} \mapsto \operatorname{Tr}\,(\underline{u}\ \mathrm{adj}\ (A)),$$

where $A = \left(\frac{\partial^2 P_{\underline{u}}}{\partial y_i \partial y_j}\right)$, and adj (A) is the "adjugate" matrix of co-factors. It follows that the derivative of G is surjective so long as adj (A) is not zero, that is, so long as the rank of A is at least $m - 3$. However if we have chosen P as above so that the derivative of Q_0 is surjective, a short calculation of the partial derivative shows that the rank of A is indeed at least $m - 3$. (More geometrically, the kernel of A is at most 1-dimensional, consisting of the tangent space to the curve C.) Thus we can proceed as before, perturbing P to make the corresponding map Q_2 transverse to 0. In this transversal situation, the number of degenerate critical points is bounded by the degree $d(d-1)^{m-1}(d-2)^{m-2}$ of the map Q_2.

6. Further results

In this Section we consider three topics beyond the main result of the paper. As the referee has pointed out, some of the results are very similar to those proved by Tian in [9].

The integrable case

We will here consider the special case where our symplectic manifold admits a compatible *complex* structure. In this case L is a holomorphic line bundle over V, and large powers $L^{\otimes k}$ have many global holomorphic sections: this is the essence of the standard Kodaira embedding theorem. As we explained in Section 1, the existence theorem (5) is a standard result in this case, but we can use the techniques of our proof to obtain result—perhpas new—on the geometry of hypersurfaces of high degree.

Proposition 32. *Let (V,ω) be a compact Kähler manifold and $L \to V$ be a Hermitian holomorphic line bundle having a connection with curvature $-i\omega$. Then there is a constant $\eta > 0$ such that for large enough k, L^k has a holomorphic section s such that $|s| \leq 1$ everywhere on V, and $|\partial s| \geq \eta\sqrt{k}$ on the zero set of s.*

A more vivid statement is:

Corollary 33. *There is a constant c such that for large k there is a complex hypersurface W_k in V representing the Poincaré dual of $k[\omega]$ whose first fundamental form β_W and Riemann curvature tensor R_W (for induced metric induced from the fixed metic on V) satisfy*

$$|\beta_W| \leq c\sqrt{k}, \quad |R_W| \leq ck.$$

To derive Corollary we let W_k be the zero-set of the section s given in Proposition 32. As we shall see below, over any ball in V of radius $k^{-1/2}$ there is a holomorphic section σ of L^k with $A^{-1} < |\sigma| \leq A$. Over such a ball we let $f = s/\sigma$, so f is holomorphic function bounded by A. Choose a local holomorphic co-ordinate chart $\tilde{\theta}$ (similar to the charts $\tilde{\chi}$ used in Section 2) which dilates by a factor $k^{1/2}$, so maps the unit ball B^{2n} in $\mathbf{C}^n$ to a neighbourhood of diameter $O(k^{-1/2})$ in V. Then we get a holomorphic function $\tilde{f} = f \circ \tilde{\theta}$ on B^{2n}, bounded by A. We get bounds on all the derivatives of $\tilde{f}$ on an interior ball, and also a lower bound on the derivative of $\tilde{f}$ on its zero set. Then it is clear that the fundamental form $\tilde{\beta}$ of the zero set in the rescaled metric g_k is bounded and, scaling back, this goes over to $|\beta_W| \leq Ck^{1/2}$. Finally the estimate on the curvature tensor of W_k arises from the fact that

$$R_W = R_V|_W - \beta_W \beta_W^*.$$

Observe that the volume of W_k is proportional to k, and the integral of the form Tr $(R_{W_k}) \wedge \omega^{n-2}$, which represents $c_1(W_k) \cup h^{n-2}$, over W_k grows quadratically with k, since

$$\langle c_1(W_k) \cup h^{n-2}, W_k \rangle = \langle k c_1(TV) h^{n-1} - k^2 h^n, [V] \rangle,$$

so

$$\frac{1}{Vol(W_k)} \int_{W_k} |R_{W_k}| \geq C^{-1} k,$$

and the estimate of Corollary 33 is optimal, to the extent that the maximum value of the curvature of the W_k grows at the same rate as the average value. It is an interesting exercise to write down explicit equations defining a sequence W_k satisfying the estimates of (32) in the case where $V = \mathbf{CP}^2$ say. It would be interesting to find the best constants in (32) and (33)—what are the "smoothest" curves of high degree?

To adapt our results to prove Proposition 32 we need to replace the approximately holomorphic sections constructed before by genuine holomorphic sections. We will prove:

Proposition 34. *There are positive constants a, b, ϵ such that for all sufficiently large k and each point p in V there is a holomorphic section σ of L^k over V with $|\sigma(p)| = 1$,*

$$e^{-bd_k(p,q)^2} \leq |\sigma(q)| \leq e^{-ad_k(p,q)^2}$$

if $d_k(p,q) \leq \epsilon k^{1/6}$, and

$$|\sigma(q)| \leq C e^{-ak^{1/3}}$$

if $d_k(p,q) \geq \epsilon k^{1/6}$.

Given this Proposition, the proof of (32) follows from the argument used for our main result. We cover V with balls as before and obtain a finite collection of sections σ_i. We need to show first that $\sum_i |\sigma_i(q)|$ is bounded, independent of k. To see this, we split up the sum into the sum over sections σ_i where the corresponding centre p_i has distance less than $\epsilon k^{1/6}$ from q, and the part where the distance is greater than

$\epsilon k^{1/6}$. The first part is estimated just as in Section 3. The second part is bounded by the fact that there are only $O(k^n)$ balls in total so the contribution is less than $Ck^n e^{-ak^{1/3}}$, which is certainly bounded as $k \to \infty$. The other crucial point comes where we need that for any D, and sufficiently large k:

$$|\sigma_i(q)| \leq e^{-aD^2},$$

if $d_k(q, p_i) \geq D$, and this follows immediately from the statement of (34).

To construct these "concentrated" holomorphic sections we use standard Hodge Theory methods. Recall that there is a Weitzenbock formula on $\Omega^{0,1}(L^k)$ of the shape

$$\Delta_{\overline{\partial}} = \square'' + R + k,$$

where $\Delta_{\overline{\partial}}$ is the Laplacian $\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$, $\square''$ is a semi-positive operator, and R is an algebraic operator involving the Ricci tensor of V, independent of k; see the discussion in [1] for example. It follows that once k is sufficiently large $\Delta_{\overline{\partial}}$ is invertible, and the L^2 operator norm of the inverse $G = \Delta_{\overline{\partial}}^{-1}$ is $O(k^{-1})$; here we are working with the fixed metric g on V. Now if $\tilde{s}$ is any smooth section of L^k, and ξ is the section

$$\xi = -\overline{\partial}^* G \overline{\partial}\, \tilde{s},$$

then $\tilde{s} + \xi$ is holomorphic, and we have an L^2 estimate

$$\|\xi\|_{L^2}^2 = \langle \overline{\partial}^* G \overline{\partial} \tilde{s}, \overline{\partial}^* G \overline{\partial} \tilde{s} \rangle = \langle G \overline{\partial} \tilde{s}, \overline{\partial} \tilde{s} \rangle \leq C k^{-1} \|\overline{\partial}\, \tilde{s}\|_{L^2}^2.$$

For the rest of the proof we will work in the rescaled metric g_k, where the estimate transforms to:

$$\|\xi\|_{L^2(V, g_k)} \leq C \|\overline{\partial}\, \tilde{s}\|_{L^2(V, g_k)}. \tag{35}$$

This gives a way to construct holomorphic sections $\tilde{s} + \xi$ starting with approximately holomorphic sections $\tilde{s}$, where ξ is a small correction term. We manufacture suitable approximately holomorphic sections by a variant of the mathod of Section 2.

Lemma 36. *There are constants a, b such that for any point p of V there is a holomorphic section τ of L^k over the g_k-ball of radius $k^{1/3}$ such that*

$$\exp(-b d_k(p, q)^2) \leq |\tau(q)| \leq \exp(-a d_k(p, q)^2).$$

We begin by considering a holomorphic section u of L in a neighbourhood of p, with $|u(p)| = 1$. Recall that the curvature form ω is given by $\overline{\partial}\partial \log |u|^2$. It is clear first that we can choose u so that the derivative of $\log |u|^2$ vanishes at p. So, in local co-ordinates ζ_α centred at p,

$$\log |u|^2 = 1 + \sum g_{\alpha\beta}\zeta_\alpha\overline{\zeta}_\beta + \sum h_{\alpha\beta}\zeta_\alpha\zeta_\beta + \sum \overline{h}_{\alpha\beta}\overline{\zeta}_\alpha\overline{\zeta}_\beta + O(|\zeta|^3).$$

The condition that $\omega = \overline{\partial}\partial \log |u|^2$ means that $g_{\alpha\beta}$ is the metric tensor. Multiplying u by a suitable holomorphic, quadratic, function, we can suppose that the co-efficients $h_{\alpha\beta}$ all vanish, so that

$$\log |u|^2 = 1 - |\zeta|^2 + O(|\zeta|^3).$$

Now consider the holomorphic section u^k of $L^{\otimes k}$ over this ball, and change to the dilated co-ordinates $z_\alpha = \sqrt{k}\zeta_\alpha$. Then we have:

$$\left(1 - \frac{|z|^2}{k} - C\frac{|z|^3}{k\sqrt{k}}\right)^k \leq |u^k| \leq \left(1 - \frac{|z|^2}{k} + C\frac{|z|^3}{k\sqrt{k}}\right)^k.$$

Now we have elementary inequalities:

$$e^{-x-x^2/k} \leq (1 - x/k)^k \leq e^{-x}$$

when x/k is sufficiently small. Putting these together we get

$$e^{-|z|^2 - C|z|^3/\sqrt{k}} \leq |u|^2 \leq e^{-|z|^2 + C|z|^3/\sqrt{k}},$$

if $z/\sqrt{k}$ is small enough. Now the estimate stated in the Lemma follows immediately.

We now define a section $\tilde{s}$ by multiplying by a cut-off function β with derivative supported in the annulus of outer radius $k^{1/6}$ and inner radius $\frac{1}{2}k^{1/6}$, as in Section 2. We have

$$|\overline{\partial}\, \tilde{s}| \leq Ck^{-1/6}e^{-ak^{1/3}}.$$

So we get a global holomorphic section $\sigma = \tilde{s} + \xi$ of L^k with

$$\|\xi\|_{L^2(V, g_k)} \leq const.k^{(n-1)/6}e^{-ak^{1/3}}.$$

We can go from this L^2 estimate to a L^∞ estimate. It is easy to see that on any g_k-ball of radius 1 the L^2 norm of a holomorphic section of

$L^{\otimes k}$ controls the L^∞ norm. There are two cases to consider. If q is a point with $d_k(p,q) < \frac{1}{2}k^{1/6} - 1$, then the unit ball about q does not met the support of $\overline{\partial}\,\tilde{s}$, so ξ is holomorphic there. Thus we get

$$(37) \qquad |\xi(q)| \le Ck^{(n-1)/6}e^{-ak^{1/3}} \text{ if } d_k(p,q) \le \tfrac{1}{2}k^{1/6} - 1.$$

On the other hand, if $d_k(p,q) \ge \frac{1}{2}k^{1/6} - 1$, then in the g_k-unit ball about q we have $|\tilde{s}| \le \exp(-a(\frac{1}{2}k^{1/6} - 2)^2) \le Ce^{-ak^{1/3}}$, so working with the holomorphic section $\tilde{s} + \xi$ yields

$$(38) \qquad |(\tilde{s} + \xi)(q)| \le C\exp(-ak^{1/3}) \text{ if } d_k(p,q) \ge \tfrac{1}{2}k^{1/6} - 1.$$

Now (34), with a suitable choice of constants, follows from (36), (37), (38) by elementary inequalities.

The Lefschetz hyperplane theorem

We shall now prove the analogue of the Lefschetz Theorem on the topology of hypersurfaces, in our symplectic setting.

Proposition 39. *Let W_k be the zero-set of a section s of $L^k \to V$, satisfying the conditions of Theorem 5. When k is sufficiently large the inclusion $i : W_k \to V$ induces an isomorphism on homotopy groups π_p for $p \le n-2$ and a surjection on π_{n-1}.*

For example, in the case where $n = 2$, so W_k is a surface in a 4-manifold V, we see that W_k is connected when k is large.

The proof of Proposition 39 is a small modification of the usual "Morse Theory" proof in the Kähler case [2]. We consider the function

$$\psi = \log|s|^2$$

on $V \setminus W_k$. This tends to $-\infty$ at W_k, and the standard Morse theory arguments show that it suffices to prove that at any critical point $p \in V \setminus W_k$ the index of the Hessian H_p (the dimension of a maximal negative eigenspace) is at least n. Note that it is not necessary to suppose that ψ has non-degenerate critical points. Now consider a real quadratic form H on a complex vector space $\mathbf{C}^n$. Let Π be the quadratic form

$$\Pi(x) = H(x) + H(Ix),$$

and suppose that Π is negative definite. If the index of H were less than n, there would be a real subspace $P \subset \mathbf{C}^n$ with $dimP > n$ on which H is positive semi-definite. But then, by the dimension formula, $P \cap IP$ is non-trivial, and Π is non-negative on $P \cap IP$ giving a contradiction. So it suffices to show that the form Π_p obtained from the Hessian H_p of ψ is negative definite, for each critical point p.

Now recall that we have operators $\partial_V, \overline{\partial}_V$ on the differential forms on V, defined by the almost-complex structure. If we choose osculating co-ordinates centred on a critical point p as in Section 2, we can express these as $\partial + \overline{\mu}\overline{\partial}, \overline{\partial} + \mu\partial$, where $\partial, \overline{\partial}$ are the usual operators in these co-ordinates, and μ vanishes at p. One sees then that if the first derivative of ψ vanishes at p, then $\overline{\partial}_V \partial_V \psi = \overline{\partial}\partial\psi$ at the critical point p. It follows that the quadratic form Π_p can be identified with $\overline{\partial}_V \partial_V \psi$, just as in the case of a complex manifold. Now we can compute, in terms of the operators defined by the connection on L^k and then Hermitian inner product $\langle\ ,\ \rangle$ on the fibres,

$$\begin{aligned} \partial_V \psi &= \partial_V \log |s|^2 \\ &= \partial_V \left(\tfrac{1}{|s|^2} (\langle \partial_L s, s\rangle + \langle s, \overline{\partial}_L s\rangle) \right), \end{aligned}$$

and, at a critical point p,

$$\overline{\partial}_V \partial_V \psi = \frac{1}{|s|^2} \left(\langle \overline{\partial}_L \partial_L s, s\rangle - \langle \partial_L s, \partial_L s\rangle + \langle \overline{\partial}_L s, \overline{\partial}_L s\rangle + \langle s, \partial_L \overline{\partial}_L s\rangle \right).$$

Now $(\overline{\partial}_L \partial_L + \partial_L \overline{\partial}_L) s = i\omega s$ so

$$\begin{aligned} \overline{\partial}_V \partial_V \psi = ik\omega + \frac{1}{|s|^2} \big(& -\langle \partial_L \overline{\partial}_L s, s\rangle + \langle s, \partial_L \overline{\partial}_L s\rangle \\ & - \langle \partial_L s, \partial_L s\rangle + \langle \overline{\partial}_L s, \overline{\partial}_L s\rangle \big). \end{aligned}$$

Go, as usual, to rescaled co-ordinates on a ball around p. We know that $k\omega$ is positive definite and close to the standard form in these co-ordinates. We also know that $\overline{\partial}_L s$ and $\nabla \overline{\partial}_L s$, hence also $\partial_L \overline{\partial}_L s$ are small, in fact $O(k^{-1/2})$. So $\overline{\partial}_V \partial_V \psi$ is negative definite (for large k) so long as $|s|$ is not small, in fact so long as $|s| \geq C^{-1} k^{-1/2}$ for a suitable constant. To see that this holds, we use the controlled transversality condition once again. Since p is a critical point of ψ we have

$$|\partial_L s| = |\overline{\partial}_L s| \leq const k^{-1/2}.$$

Now over our ball we represent sections in terms of the standard trivialisation, $s = g\sigma_i$ where $|\sigma_i|$ is bounded below and $|\nabla \sigma_i|$ is bounded above. The controlled transversality condition we achieved in Section 3 says that the function g and its derivative $\partial_V g$ cannot both be small at the same point. Now

$$\partial_V s = (\partial_V g)\ \sigma_i + g(\partial_L \sigma_i),$$

so the fact that $\partial_L s$ is $O(k^{-1/2})$ implies that $|g| \geq \eta$ for some fixed η, once k is large. This completes the proof.

Convergence of currents

We will now consider the asymptotic behaviour of the submanifolds W_k we have constructed, as $k \to \infty$. We will show that, considered as currents of degree 2 (the dual of the space of smooth $(2n-2)$-forms on V), we have $W_k \sim k\omega/2\pi$. More precisely we show:

Proposition 40. *There is a constant C such that for any test form $\psi \in \Omega^{2n-2}(V)$,*

$$\left| \int_{W_k} \psi - \frac{k}{2\pi} \int_V \omega \wedge \psi \right| \leq C k^{1/2} \|d\psi\|_{L^\infty(V)}.$$

Here the L^∞-norm is computed with the fixed metric g on V.

Let s be the section of L^k cutting out W_k. The 1-form $A = s^{-1}\nabla s$ on the complement of W_k has an integrable singularity (since the function $|z|^{-1}$ is integrable in a neighbourhood of the origin in $\mathbf{C}$), so may be regarded as a current on V. It is a singular connection form for L^k. The relation between this connection form, the curvature $k\omega$ and the zero set W_k is encapsulated in the equation of currents:

$$dA = W_k - \frac{k}{2\pi}\omega.$$

That is, for any test form ψ,

$$\int_{W_k} \psi - \frac{k}{2\pi} \int_V \psi \wedge \omega = - \int_V A \wedge d\psi.$$

So our result follows if we show that

$$\int_V |A| d\mu \leq C\sqrt{k},$$

for some constant C independent of k. Now transform to the dilated metric g_k, and the standard cover of V by g_k-unit balls. The volume form of V scales by k^{-n} and the number of balls is $O(k^n)$, while the norm of ∇s scales by $k^{-1/2}$. So it suffices to show that the contribution

$$\int_B \left|\frac{\nabla s}{s}\right|,$$

(defined with the metric g_k) from each ball is uniformly *bounded.* We know that the covariant derivative ∇s is uniformly bounded, so it suffices to control the integral of $|s|^{-1}$. As usual, we can represent the section s in terms of our standard trivialisation of L^k over the ball, and reduce to considering functions. In sum, the result we want can be obtained easily from the following:

Proposition 41. *Given $\rho > 0$ let $\mathcal{K}_\rho$ be the set of complex-valued functions f on $2B^{2n} \subset \mathbf{C}^n$ with $\|\overline{\partial} f\|_{C^1} \leq \rho/2, \|f\|_{C^0} \leq 1$ and such that $|\partial f| \geq \rho$ at all points where $|f| \leq \rho$. Then for any ρ there is a constant C such that*

$$\int_{B^{2n}} \frac{1}{|f|} d\mu \leq C$$

for all f in $\mathcal{K}_\rho$.

For any f we first divide the integral of $|f|^{-1}$ into the contribution I_1 from the region in B^{2n} where $|f| \leq \rho$ and the contribution, I_2, from the region where $|f| \geq \rho$. Clearly I_2 is bounded by 2Vol $(B^{2n})\rho^{-1}$, so it suffices to control I_1. For $\sigma \in \mathbf{C}$ with $|\sigma| \leq \rho$ let $Z_\sigma = f^{-1}(\sigma) \cap B^{2n}$. Now we know that at a point where $|f| \leq \rho$ we have $|\overline{\partial} f|\rho/2 < \rho \leq |\partial f|$, so the derivative of f is surjective, by the discussion in the introduction. Thus the sets Z_σ are smooth codimension-2 submanifolds. At any point on $Z_\sigma(p)$ we let $J_f(p)$ be the modulus of the determinant of the derivative of f, restricted to the 2-dimensional subspace in $\mathbf{C}^n$ normal to Z_σ at that point (defined by the area forms on this normal plane and $\mathbf{C}$). Then we have a "co-area" formula::

$$\int_{|f|\leq\rho} \frac{1}{|f|} d\mu = \int_{|\sigma|\leq\rho} \frac{I(\sigma)}{|\sigma|} d\mu_\sigma,$$

where

$$I(\sigma) = \int_{Z_\sigma} J_f^{-1} d\nu,$$

and $d\nu$ is the usual induced measure on the submanifold $Z_\sigma \subset B^{2n}$. Now it is simple linear algebra to show that

$$J_f = \left((|\partial f|^2 + |\overline{\partial} f|^2)^2 - 4|\langle \overline{\partial f}, \overline{\partial} f\rangle|^2\right)^{1/2} \geq |\partial f|^2 - |\overline{\partial} f|^2 \geq 3\rho^2/4,$$

and this lower bound on J_f gives

$$I(\sigma) \leq const.Vol(Z_\sigma).$$

Thus it suffices to find a bound on the volume of the Z_σ for all $|\sigma| \leq \rho$ and $f \in \mathcal{K}_\rho$. To get this we argue by contradiction. If there is a sequence f_i in $\mathcal{K}_\rho$ and a sequence σ_i so that the volume of $Z_{\sigma_i}(f_i)$ increases to infinity we may suppose, choosing a subsequence if necessary, that the f_i converges in C^1 over the interior region B^{2n} (by the usual elliptic estimates for the $\overline{\partial}$-operator), to some C^1 limit f and that the σ_i converge to a limit σ. Then the crucial point is that the volumes of $f_i^{-1}(\sigma_i)$ converge to that of $f^{-1}(\sigma)$, giving a contradiction. This continuity of the volume of the zero set with respect to C^1 convergence (given the bounds $|\overline{\partial} f_i| \leq \rho/2, |\partial f| \geq \rho$) follows easily from the usual implicit function theorem, which shows that $Z_{\sigma_i}(f_i)$ is obtained from $Z_\sigma(f)$ by a map which is C^1-close to the identity.

References

[1] S. K. Donaldson, *Yang-Mills invariants of four-manifolds*, Geometry of low-dimensional manifolds: 1 (ed. Donaldson and Thomas), Cambridge Univ. Press, Cambridge, 1990.

[2] P. Griffiths & J. Harris, *Principles of algebraic geometry*, John Wiley, New York, 1978.

[3] M. Gromov, *Partial differential relations*, Springer, Berlin, 1986.

[4] L. Hormander, *Complex analysis in several variables*, North-Holland, Amsterdam, 1973.

[5] P. B. Kronheimer & T. S. Mrowka, *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. **30** (1994) 215-221.

[6] C. H. Taubes, *More constraints on symplectic manifolds from the Seiberg-Witten equations*, Math. Res. Letters **2** (1995) 9-14.

[7] ———, *The Seiberg-Witten and Gromov invariants* , Math. Res. Letters **2** (1995) 221-238.

[8] ———, *From the Seiberg-Witten equations to pseudoholomorphic curves*, Preprint.

[9] G. Tian, *On a set of polarised Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990) 99-130.

[10] E. Witten, *Monopoles and four-manifolds*, Math. Res. Letters **1** (1995) 769-96.

[11] Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Ann. **4** (1983) 495-515.

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