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A CONSTRUCTION OF SINGULAR SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION USING ASYMPTOTIC ANALYSIS

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Abstract

The aim of this paper is to prove the existence of weak solutions to the equation $\Delta u + u^p = 0$ which are positive in a domain $\Omega \subset \mathbb{R}^N$, vanish at the boundary, and have prescribed isolated singularities. The exponent p is required to lie in the interval $(N/(N-2), (N+2)/(N-2))$. We also prove the existence of solutions to the equation $\Delta u + u^p = 0$ which are positive in a domain $\Omega \subset \mathbb{R}^n$ and which are singular along arbitrary smooth k-dimensional submanifolds in the interior of these domains provided p lies in the interval $((n-k)/(n-k-2), (n-k+2)/(n-k-2))$. A particular case is when $p = (n+2)/(n-2)$, in which case solutions correspond to solutions of the singular Yamabe problem. The method used here is a mixture of different ingredients used by both authors in their separate constructions of solutions to the singular Yamabe problem, along with a new set of scaling techniques.

1. Introduction and statements of main results

In this paper we construct solutions with prescribed singularities for the semilinear elliptic equation $\Delta u + u^p = 0$ and other closely related equations, for a certain range of values of the exponent p, in a variety of situations. The solutions will have singularities prescribed along a disjoint union of submanifolds of varying dimension. We now describe our results, starting with the simplest case when the prescribed singular set is discrete.

Suppose Ω is any bounded open set in $\mathbb{R}^N$, with smooth boundary. Consider the equation

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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A weak solution of (1) is a function u which solves this equation on all of Ω in the sense of distributions. In particular, any such solution must belong to $L^p(\Omega)$. The singular set of a weak solution u, $\operatorname{sing}(u)$, is the complement in Ω of the set of points where u is continuous, and hence smooth.

Our first result concerns the case when the singular set is finite. We single it out since its proof is slightly simpler than the more general case.

Theorem 1. *Let Ω be as above, and suppose that $\Sigma = \{x_1, \dots, x_K\} \subset \Omega$ is any finite set of points. Suppose also that the exponent p lies in the range*

$$\frac{N}{N-2} \leq p < \frac{N+2}{N-2}. \tag{2}$$

Then there is a K parameter family of positive weak solutions u of (1) *with $\operatorname{sing}(u) = \Sigma$. In fact, the solution space of this equation is locally a K dimensional real analytic variety.*

It is known that if the exponent p is less than $N/(N-2)$, then any weak solution of (2) must be smooth on all of Ω. The existence of solutions of this equation with prescribed isolated singularities when p lies in the interval $N/(N-2) \leq p < p_0$, where p_0 is some value close to $N/(N-2)$ (and in particular, less than $(N+2)/(N-2)$), has already been solved by the second author in [10] and in [11]. When $p = (N+2)/(N-2)$, which is the so-called critical exponent, the problem becomes conformally invariant. There is then a loss of compactness and the problem consequently becomes much more difficult. Solutions now correspond to metrics of constant nonnegative scalar curvature which are complete in a neighbourhood of the singular points. It is geometrically more natural in this case to replace the domain by S^N (or in fact, any other compact manifold of nonnegative scalar curvature); the operator Δ then needs to be replaced by the conformal Laplacian $\Delta - \frac{N-2}{4(N-1)}R_0$, where R_0 is the scalar curvature of the background manifold. An additional source of difficulties in this case is that the position of the singularities is no longer necessarily arbitrary. The general existence result for this geometric problem (when the background manifold is the sphere) was obtained by R. Schoen [14]. The recent work of the first author, along with D. Pollack and K. Uhlenbeck [7] examines the moduli space of solutions for this problem.

In any event, Theorem 1 extends the result of [10] and [11] to the full range of subcritical exponents.

It is also possible to prove existence of solutions of (1) which are singular along submanifolds of higher dimension. Now let $\Sigma = \cup_{i=1}^{K}\Sigma_i$, where each $\Sigma_i \subset \Omega$ is a smooth submanifold without boundary of dimension $k_i \geq 0$. We shall suppose that $\Omega \subset \mathbb{R}^n$, since we reserve the symbol N (or more properly $N_i = n - k_i$) for the dimension of the normal space to each Σ_i.

Theorem 2. *Suppose that the exponent p satisfies*

$$\frac{n-k_i}{n-k_i-2} \leq p < \frac{n-k_i+2}{n-k_i-2},$$

or in other words

$$n - \frac{2p+2}{p-1} < k_i \leq n - \frac{2p}{p-1},$$

for $i = 1, \dots, K$. Then there is a positive weak solution u of (1) with $sing(u) = \Sigma$. Provided at least one of the $k_i > 0$, there is an infinite dimensional space of solutions for this problem.

Again, the existence of solutions for this equation with higher dimensional singular sets of dimension k has been solved by Y. Rebaï [13] when the exponent p lies in the interval $(n-k)/(n-k-2) < p < p_k$, where p_k is close to the lower endpoint of this interval, and in particular, less than $(n-k+2)/(n-k-2)$. We shall only prove this result when $p > (n-k_i)/(n-k_i-2)$ since it would complicate the notation to cover the borderline case where p attains the lower limit of this interval of values. However, the arguments below could be extended without undue difficulty to cover this value as well.

A special case of Theorem 2 is the singular Yamabe problem. This occurs when the exponent p attains the critical value $(n+2)/(n-2)$. In this case, the dimensions of the Σ_i are allowed to lie in the range $0 < k_i < (n-2)/2$. As discussed above, it is more natural now to replace Ω by the round sphere (S^n, g_0), or indeed by any compact manifold (M, g_0) with (constant) nonnegative scalar curvature, and to replace the Laplacian by the conformal Laplacian $L_0 = \Delta - \frac{n-2}{4(n-1)} R(g_0)$. The relevant equation now is

$$-L_0 u = u^{\frac{n+2}{n-2}} \tag{3}$$

on M. At least when the background manifold is the sphere, it follows from work of R. Schoen and S. T. Yau [15] that solutions to this singular Yamabe problem (for which the corresponding conformally related metrics $g = u^{4/(n-2)} g_0$ are complete and have bounded Ricci curvature) exist only if the dimension of the singular set is less than or equal to $(n-2)/2$. Thus, Theorem 2 implies

Theorem 3. *Let (M, g_0) be any compact manifold with constant nonnegative scalar curvature. Let $\Sigma \subset M$ be any finite disjoint union* *of smooth submanifolds Σ_i of dimensions k_i with $0 < k_i \leq \frac{n-2}{2}$. Then there is an infinite dimensional family of complete metrics on $M \setminus \Sigma$ with constant positive scalar curvature.*

Together with the result of [12], which treats the case $k_i = \frac{n-2}{2}$ when n is even, this theorem settles the question of existence of solutions to the singular Yamabe problem (with constant positive scalar curvature) whenever Σ is a finite disjoint union of smooth submanifolds with dimensions greater than zero, but less than or equal to $(n-2)/2$.

Finally, we may apply the arguments of [8] to slightly refine Theorem 2 :

Theorem 4. *Let Σ be any finite disjoint union of $C^{3,\alpha}$ submanifolds in Ω of dimensions k_i satisfying the restrictions above. Then* (1) *has an infinite dimensional family of solutions.*

The basic idea of Theorems 2 and 3 is that solutions of (1) with positive dimensional singular sets may be obtained as perturbations of solutions to the problem on the fibres of the normal bundle of Σ. The exponent $\frac{n+2}{n-2}$ is subcritical for the induced problem on these fibres, which are of lower dimension than the ambient space. In particular, the problem with critical exponent may be reduced to a subcritical one.

The paper is organized as follows. First we analyze the asymptotics of rotationally invariant solutions of the problem on $\mathbb{R}^N \setminus \{0\}$. These are then used to construct approximate solutions for each of the problems. At this point we then give a more detailed outline of the strategy of the proof. The main task is to analyze the linearization of the nonlinear operator around these approximate solutions, and to prove that these linearizations are surjective on appropriate function spaces. This occupies the bulk of the paper. After this is accomplished, the exact solutions are obtained by a fixed point argument. The deformation spaces for the solutions of these operators, and the arguments necessary to replace Ω by a manifold M are discussed in the last section.

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2. Singular radial solutions on $\mathbb{R}^N \setminus \{0\}$

In this section, we recall some well known facts which appear for example in [2].

Proposition 1. *For any exponent* $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$*, there exists a one parameter family of weak solutions* u_ϵ*,* $\epsilon > 0$*, for the equation*

$$-\Delta u = u^p \quad in \quad \mathbb{R}^N, \tag{4}$$

such that the u_ϵ *are radial, singular only at the origin, and satisfy the following properties :*

- $u_\epsilon(r) > 0$ for $0 < r < \infty$.
- u_ϵ *can be written as*

$$u_\epsilon(x) \equiv |x|^{-\frac{2}{p-1}} v_1(-\log(|x|/\epsilon)),$$

where the function v_1 *is bounded independently of* ϵ*.*

- $\lim_{t\to+\infty} v_1(t) = v_\infty > 0$*, where*

$$v_\infty^{p-1} = \frac{2}{p-1}\left(N - \frac{2p}{p-1}\right), \tag{5}$$

and in particular, is independent of ϵ*.*

- $\lim_{t\to-\infty} e^{-t(N-\frac{2p}{p-1})} v_1(t) < +\infty$*; in particular, for* $|x|$ *large*

$$u_\epsilon(x) = c(\epsilon^{N-\frac{2p}{p-1}} + O(|x|^{-1}))|x|^{2-N},$$

where the constant c is independent of ϵ*.*

- *Finally,* $||v_1||_{L^\infty}^{p-1} < \frac{p+1}{2} v_\infty^{p-1}$.

Proof. First define a new independent variable $t = -\log|x|$ and set

$$u(x) \equiv |x|^{-\frac{2}{p-1}} v(-\log(|x|)). \tag{6}$$

Then the new function $v(t)$ satisfies

$$\partial_t^2 v - \left(N - 2\frac{p+1}{p-1}\right)\partial_t v - \frac{2}{p-1}\left(N - \frac{2p}{p-1}\right) v + v^p = 0. \tag{7}$$

Now look at the phase-plane portrait for this equation in the (v, v_t) plane. The two equilibrium points are $(0,0)$ and $(v_\infty, 0)$, where v_∞ is defined in (5); the first of these is a saddle point and the second a stable equilibrium. There is a single orbit issuing from $(0,0)$ and tending to $(v_\infty, 0)$ as $t \to \infty$. Let $v_1(t)$ be one of the functions corresponding to this orbit; it is determined only up to the choice of its initial Cauchy data $(v_1(0), v_1'(0))$. Now let $v_\epsilon(t) = v_1(t + \log\epsilon)$. As ϵ varies over $(0,\infty)$ we obtain all solutions of (7) which converge to 0 as $t \to -\infty$ and to v_∞ as $t \to \infty$.

We now obtain an upper bound on the function $v_1(t)$. It is easy to see that the trajectory $(v_1(t), v_1'(t))$ is contained within the homoclinic orbit of the Hamiltonian system

$$\partial_t^2 w - \frac{2}{p-1}\left(N - \frac{2p}{p-1}\right) w + w^p = 0 \tag{8}$$

which tends to $(0,0)$ as t tends both to $+\infty$ and $-\infty$. Let $(w_1(t), w_1'(t))$ parametrize this orbit. Then we conclude that

$$\sup v_1 \leq \sup w_1.$$

The conservation of Hamiltonian energy for (8) now shows that

$$\frac{1}{2}(\partial_t w_1)^2 - \frac{1}{p-1}\left(N - \frac{2p}{p-1}\right) w_1^2 + \frac{1}{p+1} w_1^{p+1} = 0. \tag{9}$$

w_1 attains its supremum when $w_1' = 0$, so we obtain the upper bound

$$v_1^{p-1}(t) < \frac{p+1}{2} v_\infty^{p-1} \qquad \text{for all} \qquad t \in \mathbb{R}.$$

This ends the proof of the Proposition.

Remark 1. For any $\epsilon > 0$, whenever $u(x)$ is a solution of (4), $\epsilon^{-\frac{2}{p-1}} u(\epsilon^{-1} x)$ is another ' of this same equation. This dilation invariance corresponds exactly to the translation invariance of the equation1 (7) for v. We let $u_1(x)$ be the solution corresponding to v_1 and

$$u_\epsilon(x) \equiv \epsilon^{-\frac{2}{p-1}} u_1(\epsilon^{-1} x)$$

Later we will also need the following :

Remark 2. By this rescaling we can always assume that, for any given constant $\alpha > 0$, v_1 may be chosen so that

$$\sup_{t \leq 0} v_1(t) \leq \alpha. \tag{10}$$

The final remark relates the construction of approximate solutions here to those used in [8]:

Remark 3. The stable point $(v_\infty, 0)$ corresponds to the singular solution

$$u_0(x) \equiv v_\infty |x|^{-\frac{2}{p-1}}.$$

This solution is invariant by the scaling described above. The fact that u_1 is not dilation invariant is crucial in the construction of approximate solutions in this paper. The approximate solutions in [8] were constructed using u_0, but because of its dilation invariance, only local solutions near the singular set could be proved to exist there.

3. Function spaces

In this section we define the weighted Hölder spaces $\mathcal{C}_\nu^{k,\alpha}(\Omega \setminus \Sigma)$ appropriate for this problem. Roughly speaking, the functions in these spaces are products of powers of the distance to Σ with functions whose Hölder norms are invariant with respect to scaling by dilations from any point on Σ.

We shall use local Fermi coordinates around each component of Σ to define these spaces. When Σ_i is a point, these are simply (geodesic) polar coordinates (r, θ) around that point. When Σ_i is higher dimensional, let $\mathcal{T}_\sigma^{(i)}$ be the tubular neighbourhood of radius σ around Σ_i. It is well known that $\mathcal{T}_\sigma^{(i)}$ is a disk bundle over Σ_i, and is diffeomorphic to the disk bundle of radius σ in the normal bundle $N\Sigma_i$. Using the metric, this diffeomorphism is canonical. The Fermi coordinates in this tubular neighbourhood will be constructed as coordinates in the normal bundle, transferred to $\mathcal{T}_\sigma^{(i)}$ via this fixed diffeomorphism. Here r is the distance to Σ_i, which is well defined in $\mathcal{T}_\sigma^{(i)}$ and smooth away from Σ_i provided σ is small enough, y is a local coordinate system on Σ_i, and θ is the angular variable on the sphere in each normal space $N_y\Sigma_i$. Let $B_{N,\sigma}$ denote the ball of radius σ in $N_y\Sigma_i$. We shall let x denote the rectangular coordinate in these normal spaces, so that $r = |x|$ and $\theta = x/|x|$.

Let us also fix a function $\rho > 0$ in $\mathcal{C}^\infty(\Omega \setminus \Sigma)$ with ρ equal to the polar distance r in each $\mathcal{T}_\sigma^{(i)}$. Let w be a function in this tubular neighbourhood, and define

$$||w||_{0,\alpha,0}^{\mathcal{T}_\sigma^{(i)}} = \sup_{z \in \mathcal{T}_\sigma^{(i)}} |w| + \sup_{z,\tilde{z} \in \mathcal{T}_\sigma^{(i)}} \frac{(r+\tilde{r})^\alpha |w(z) - w(\tilde{z})|}{|r-\tilde{r}|^\alpha + |y-\tilde{y}|^\alpha + (r+\tilde{r})^\alpha |\theta - \tilde{\theta}|^\alpha}.$$

Here $z, \tilde{z}$ are any two points in $\mathcal{T}_\sigma^{(i)}$ and (r, y, θ), $(\tilde{r}, \tilde{y}, \tilde{\theta})$ are their Fermi coordinates.

Definition 1. The space $\mathcal{C}_0^{k,\alpha}(\Omega \setminus \Sigma)$ is defined to be the set of all $w \in \mathcal{C}^{k,\alpha}(\Omega \setminus \Sigma)$ for which the norm

$$||w||_{k,\alpha,0} \equiv ||w||_{k,\alpha,\Omega_{\sigma/2}} + \sum_{i=1}^{K} \sum_{j=0}^{k} ||\nabla^j w||_{0,\alpha}^{\mathcal{T}_\sigma^{(i)}}$$

is finite. Here $\Omega_{\sigma/2} = \Omega \setminus \cup_{j=1}^K \mathcal{T}_{\sigma/2}^{(i)}$. Then, for any $\gamma \in \mathbb{R}$,

$$\mathcal{C}_\gamma^{k,\alpha}(\Omega \setminus \Sigma) = \{w = \rho^\gamma \bar{w} : \bar{w} \in \mathcal{C}_0^{k,\alpha}(\Omega \setminus \Sigma)\}.$$

Thus functions in $\mathcal{C}_\gamma^{k,\alpha}(\Omega \setminus \Sigma)$ are allowed to blow up like ρ^γ near the Σ_i. This space is endowed with the natural norm $||w||_{k,\alpha,\gamma} \equiv ||\rho^{-\gamma} w||_{k,\alpha,0}$. Note that when Σ_i is positive dimensional, functions in

$\mathcal{C}^{k,\alpha}_{\gamma}$ may be differentiated in the 'tangential' direction only at the expense of giving up a power of ρ. Equivalently, their derivatives with respect to up to k-fold products of the vector fields $r\partial_r$, $r\partial_y$, ∂_θ blow up no faster than ρ^γ.

We collect a few essentially trivial remarks about these spaces :

Lemma 1.

1. *If* $\gamma \in (-N, 0)$ *then* $\mathcal{C}^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) \subset L^p(\Omega)$ *for* $p \in (1, -N/\gamma)$.
2. *For* $w \in \mathcal{C}^{k,\alpha}_{\gamma}$ *and* $v \in \mathcal{C}^{k,\alpha}_{\gamma'}$, *then* $wv \in \mathcal{C}^{k,\alpha}_{\gamma+\gamma'}$ *and also*
$$||wv||_{k,\alpha,\gamma+\gamma'} \leq 4||w||_{k,\alpha,\gamma}||v||_{k,\alpha,\gamma'}.$$
3. *Given* $w \in \mathcal{C}^{k,\alpha}_{\gamma}$ *with* $w \geq 0$ *and if* $p > k+1$, *then* $w^p \in \mathcal{C}^{k,\alpha}_{p\gamma}$ *and*
$$||w^p||_{k,\alpha,p\gamma} \leq c_p||w||^p_{k,\alpha,\gamma},$$
for some constant $c_p > 0$ *only depending on* p.
4. *If* $w \in \mathcal{C}^{k+1,0}_{\gamma}$ *and* $|\nabla w| \in \mathcal{C}^{k,0}_{\gamma-1}$, *then for any* $\alpha \in (0,1)$, $w \in \mathcal{C}^{k,\alpha}_{\gamma}$.

We shall also need a slightly better estimate for the norm of w^p.

Lemma 2. *For any* $\eta > 0$, $\gamma > -\frac{2}{(p-1)}$, *there exists some* $\theta > 0$, *depending only on* γ *and* η, *such that if* $||w||_{0,\alpha,\gamma} < \theta$ *then*
$$||w^p||_{0,\alpha,\gamma-2} \leq \eta||w||_{0,\alpha,\gamma}.$$

Proof. The proof relies on the simple fact that
$$||w^p||_{0,\alpha,p\gamma} \leq c\eta^{p-1}||w||_{0,\alpha,\gamma}.$$
Since $\gamma > -\frac{2}{p-1}$, we see that $p\gamma \geq \gamma - 2$, and the result follows immediately.

Definition 2.
$$\mathcal{C}^{k,\alpha}_{\gamma,\mathcal{D}}(\Omega \setminus \Sigma) = \{w \in \mathcal{C}^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) : w = 0 \text{ on } \partial\Omega\}.$$

Finally, we shall also need to use Hölder spaces on $\mathbb{R}^N \setminus \{0\}$, respectively $\mathbb{R}^n \setminus \mathbb{R}^k$, which have different decay properties at the origin, respectively at $\mathbb{R}^k$, than at ∞. Now we simply use the global polar coordinates (r,θ) in $\mathbb{R}^N \setminus \{0\}$, and cylindrical coordinates (r,θ,y) in $\mathbb{R}^n \setminus \mathbb{R}^k = \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^k$.

Definition 3. For any $\gamma, \gamma' \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_{\gamma,\gamma'}(\mathbb{R}^N \setminus \{0\})$ consists of all functions w for which the norm
$$||w||_{k,\alpha,\gamma,\gamma'} \equiv \sup_{B_2(0)} ||w||_{k,\alpha,\gamma} + \sup_{\mathbb{R}^N \setminus B_1(0)} ||w||_{k,\alpha,\gamma'}$$
is finite. The definition of $\mathcal{C}^{k,\alpha}_{\gamma,\gamma'}(\mathbb{R}^n \setminus \mathbb{R}^k)$ is similar; we need only replace $B_j(0)$ by the tube of radius j around $\mathbb{R}^k$, $j = 1, 2$.

Although these Hölder spaces are our primary tools, we shall also need to refer on occasion to a family of weighted L^2 spaces and their associated Sobolev spaces. These will be needed only for nonisolated singularities, so we restrict ourselves to that case.

Definition 4. The weighted space $r^\delta L^2$ on $\mathbb{R}^n \setminus \mathbb{R}^k$ is defined by

$$\begin{aligned} r^\delta L^2(\mathbb{R}^n \setminus \mathbb{R}^k; r^{N-1}\, dr\, d\theta\, dy) =& \{w \in L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^k) : \\ & \int |w|^2\, r^{N-1-2\delta}\, dr\, d\theta\, dy < \infty\}. \end{aligned}$$

The space $\rho^\delta L^2(\Omega \setminus \Sigma)$ (relative to standard Euclidean volume measure) is defined similarly, using the function ρ above.

There are associated weighted Sobolev spaces $r^\delta H^s_e(\mathbb{R}^n \setminus \mathbb{R}^k)$ when s is a positive integer, where the subscript e signifies that these spaces are defined with respect to differentiations by the vector fields $r\partial_r$, $r\partial_y$ and ∂_θ. (When s is an arbitrary real number, they may be defined by duality and interpolation.) Note that

$$\begin{aligned} & \mathcal{C}^{k,\alpha}_{\gamma,\sigma}(\mathbb{R}^n \setminus \mathbb{R}^k) \subset r^\delta L^2(\mathbb{R}^n \setminus \mathbb{R}^k) \quad \text{provided} \\ & \qquad\qquad \sigma + (N-2)/2 < \delta < \gamma + (N-2)/2 \\ (11) \qquad & \\ & \mathcal{C}^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) \subset \rho^\delta L^2(\Omega \setminus \Sigma) \quad \text{provided} \quad \delta < \gamma + (N-2)/2. \end{aligned}$$

4. Construction of approximate solutions

Now that the family of radial solutions $u_\epsilon(x)$ to (4) on $\mathbb{R}^N \setminus \{0\}$ has been introduced, we can construct the approximate solutions for these problems. We also derive estimates of how far these solutions differ from exact solutions in terms of the weighted Hölder norms of the last section.

4.1. Approximate solutions with isolated singularities

Approximate solutions for (1) which are singular precisely at the points of $\Sigma = \{x_1, \ldots, x_K\}$ are constructed by superimposing appropriately translated and dilated copies of the singular radial solutions. The K free parameters in the exact solution correspond to the K different dilation parameters we can prescribe independently at each of the singular points. First, choose $\chi(x) \in \mathcal{C}^\infty_0(\mathbb{R}^N)$ with $\chi = 1$ when $|x| \le 1$ and $\chi = 0$ when $|x| \ge 2$, set $\chi_R(x) \equiv \chi(x/R)$. Also choose $R > 0$ with $R < R_0 \equiv \inf_{i \ne j}(\mathrm{dist}(x_i, x_j))/2)$. Let $\bar\epsilon = (\epsilon_1, \cdots, \epsilon_K\}$ be a K-tuple of dilation parameters. Now define

$$\begin{aligned} \bar{u}_{\bar{\epsilon}}(x) &= \sum_{i=1}^{K} \chi_R(x - x_i) u_{\epsilon_i}(x - x_i) \\ &= \sum_{i=1}^{K} \chi\left(\frac{x - x_i}{R}\right) \epsilon_i^{-\frac{2}{p-1}} u_1 \left(\frac{x - x_i}{\epsilon_i}\right). \end{aligned} \tag{12}$$

Set $f_{\bar{\epsilon}} = \Delta \bar{u}_{\bar{\epsilon}} + \bar{u}_{\bar{\epsilon}}^p$. After some computation, we get

$$\begin{aligned} f_{\bar{\epsilon}} = &\sum_{i=1}^{K} u_{\epsilon_i}(x - x_i) \Delta \chi_R(x - x_i) + 2 \nabla u_{\epsilon_i}(x - x_i) \nabla \chi_R(x - x_i) \\ &+ \sum_{i=1}^{K} (\chi_R^p - \chi_R)(x - x_i) u_{\epsilon_i}^p (x - x_i). \end{aligned}$$

Lemma 3. *For any $\gamma \in \mathbb{R}$, there exists a constant c, depending on R and γ, such that*

$$||f_{\bar{\epsilon}}||_{0,\alpha,\gamma-2} \leq c \epsilon_0^{N - \frac{2p}{p-1}}, \quad \textit{provided each} \quad \epsilon_i \leq \epsilon_0 \leq 1.$$

Proof. For $|x - x_i| \in (R, 2R)$, we get

$$|\bar{u}_{\bar{\epsilon}}(x)| \leq c\, \epsilon_0^{N - \frac{2p}{p-1}}$$

and

$$|\nabla \bar{u}_{\bar{\epsilon}}(x)| \leq c\, \epsilon_0^{N - \frac{2p}{p-1}},$$

for some constant $c > 0$ depending on N and R. The result follows at once.

4.2. Approximate solutions in the general case

Now suppose $\Sigma = \cup_{i=1}^{K} \Sigma_i$ where each Σ_i is a smooth submanifold in Ω of dimension k_i. To simplify the notation here, we assume that all $k_i > 0$.

In terms of the Fermi coordinates introduced in §3, the Euclidean Laplacian on $\mathcal{T}_\sigma^{(i)}$ can be written locally in terms of the Laplacians for Σ_i and $N_y \Sigma_i$:

$$\Delta = \Delta_N + \Delta_{\Sigma_i} + e_1 \cdot \nabla^2 + e_2 \cdot \nabla, \tag{13}$$

where e_1 and e_2 satisfy

$$||e_1||_{0,\alpha,1} + ||e_2||_{0,\alpha,0} \leq c,$$

for some constant $c > 0$ which does not depend on α, nor on x, y. This is discussed in [8] and [3].

Now choose a smooth cut-off function χ on $\mathbb{R}^N$ as before, which only depends on $|x|$, and such that $\chi(x) = 0$ for $|x| > 2$ and $\chi(x) = 1$ for $|x| < 1$. Also, set $\chi_R(x) \equiv \chi(x/R)$. If $0 < \epsilon_i < 1$ and $R < \sigma/2$, then define, in some neighborhood of $y_0 \in \Sigma_i$, the function

$$\bar{u}_{\epsilon_i}(x, y) \equiv \epsilon_i^{-\frac{2}{p-1}} u_1(x/\epsilon_i)\chi_R(x) \equiv u_{\epsilon_i}(x)\chi_R(x).$$

Since this function only depends on the variable in the normal space, and is independent of the angular variable θ, it is clear that it may be defined globally on all of $\mathcal{T}_\sigma^{(i)}$. Now let $\bar{u}_{\bar{\epsilon}} = \sum_{i=1}^K \bar{u}_{\epsilon_i}$.

As before, let

$$f_{\bar{\epsilon}} = \Delta\bar{u}_{\bar{\epsilon}} + \bar{u}_{\bar{\epsilon}}^p.$$

Using (13), we compute that

$$f_{\bar{\epsilon}} = e_1(x,y)\cdot\nabla^2\bar{u}_{\bar{\epsilon}} + e_2(x,y)\cdot\nabla\bar{u}_{\bar{\epsilon}} + \bar{u}_{\bar{\epsilon}}\Delta_N\chi_R + 2\nabla\chi_R\nabla\bar{u}_{\bar{\epsilon}} + (\chi_R^p - \chi_R)\bar{u}_{\bar{\epsilon}}^p.$$

Lemma 4. *There exists some $c > 0$ depending on γ but independent of $\epsilon_0 < 1$ such that if each $\epsilon_i \leq \epsilon_0$, then*

$$||f_{\bar{\epsilon}}||_{0,\alpha,\gamma-2} \leq c\epsilon_0^q,$$

where $q = \frac{p-3}{p-1} - \gamma$.

Proof. The estimate follows at once from similar estimates already used in Lemma 3 and also from the estimates given in (13).

This exponent q is strictly positive by our assumptions on p, provided $\gamma > \frac{-2}{p-1}$.

5. Outline of the proof

In both the cases of solutions with isolated or more general singularities, we wish to solve the equation (1) by perturbing the approximate solutions $\bar{u}_{\bar{\epsilon}}$ to this problem we obtained in the preceding section. That is, we wish to represent the exact solution u as a sum $u = \bar{u}_{\bar{\epsilon}} + v$, where v is small compared to $\bar{u}_{\bar{\epsilon}}$. Thus we wish to solve

$$\Delta(\bar{u}_{\bar{\epsilon}} + v) + (\bar{u}_{\bar{\epsilon}} + v)^p = 0,$$

or, what is the same thing,

$$L_{\bar{\epsilon}}v + f_{\bar{\epsilon}} + Q(v) = 0. \tag{14}$$

Here

$$Q(v) = (\bar{u}_{\bar{\epsilon}} + v)^p - \bar{u}_{\bar{\epsilon}}^p - p\bar{u}_{\bar{\epsilon}}^{p-1}v \tag{15}$$

is the remainder term, which is quadratically small if v is smaller than $\bar{u}_{\bar{\epsilon}}$ in an appropriate Hölder norm, and

$$L_{\bar{\epsilon}} = \Delta + p\bar{u}_{\bar{\epsilon}}^{p-1} \tag{16}$$

is the linearization about this approximate solution.

We shall show that there is some $\epsilon_0 > 0$ and some constant $0 < c < 1$ such that for any $\epsilon \le \epsilon_0$, if $c\epsilon \le \epsilon_i \le \epsilon$ for all i, there is an exact solution of (1) which is a small perturbation of $\bar{u}_{\bar{\epsilon}}$. Writing $u = \bar{u}_{\bar{\epsilon}} + w$ as before, where $w \in \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma)$, we can ensure that u is singular at Σ provided $\nu > -2/(p-1)$. In addition, we shall be able to prove that the solution space is a K-dimensional manifold.

When any of the $k_i > 0$, then we shall show later that the solution space is infinite dimensional.

The analysis of this linearization is the fundamental issue of this paper. It is easy to see that

$$L_{\bar{\epsilon}} : \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \tag{17}$$

is bounded for any $\nu \in \mathbb{R}$ and $\bar{\epsilon}$. We must show that (17) is surjective for some $\nu > -2/(p-1)$, provided that each ϵ_i is sufficiently small. Once this is shown, then it follows that $L_{\bar{\epsilon}}$ has a bounded right inverse

$$G_{\bar{\epsilon}} : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma). \tag{18}$$

We must also show that the norm of this map is bounded independently of the ϵ_i for $\epsilon_i < \epsilon_0$. However, for the range of values of ν for which (17) is surjective, the operator $L_{\bar{\epsilon}}$ is not injective on $\mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma)$, so that $G_{\bar{\epsilon}}$ is not uniquely defined. As usual, a good choice for a right inverse is the one which maps into a fixed complement of the nullspace of $L_{\bar{\epsilon}}$, for example, the orthogonal complement with respect to some weighted L^2 structure.

To prove surjectivity of (17) we prove both that its range is closed and that the cokernel is trivial. When Σ is a discrete set, the closedness of the range is rather elementary, but when some component of Σ has positive dimension this fact is somewhat deeper. Fortunately, the construction of pseudodifferential right parametrices for the general class of elliptic 'edge operators,' of which $L_{\bar{\epsilon}}$ is a particular example, falls within the scope of the theory developed by the first author in [6]. Existence of such a right parametrix with compact remainder implies that the range of $L_{\bar{\epsilon}}$ is closed in all cases, and that its cokernel is at most finite dimensional for a certain range of values of ν. To show that this cokernel is trivial, and that the norm of $G_{\bar{\epsilon}}$ is uniformly bounded for sufficiently small ϵ_i, we employ scaling arguments. These arguments proceed by contradiction, showing that if first surjectivity, and secondly uniform surjectivity were to fail, then counterexamples for increasingly small ϵ_i could be rescaled to obtain some element of the nullspace of the global operator L_1 on $\mathbb{R}^N \setminus \{0\}$, or $\mathbb{L}_1$ on $\mathbb{R}^n \setminus \mathbb{R}^k$ which we show cannot exist. Once these results are established, the rest of the proof of the existence of solutions follows from a standard fixed point argument.

We remark here that we have proved these results concerning the uniform surjectivity of $L_{\bar{\epsilon}}$ in slightly greater generality than is used in the nonlinear analysis. In particular, we are able to show that $L_{\bar{\epsilon}}$ is uniformly surjective provided all $\epsilon_i \leq \epsilon_0$ for ϵ_0 sufficiently small. However, we are only able to prove the fixed point theorem if $a\epsilon < \epsilon_i < \epsilon$ for some $\epsilon \leq \epsilon_0$, and for some $a \in (0,1)$, i.e., when all the ϵ_i are mutually relatively bounded.

6. The globalized linearization

In this section we analyze the behaviour of the operator

$$L_1 = \Delta + pu_1^{p-1}$$

on $\mathbb{R}^N \setminus \{0\}$, and later the corresponding induced operator on $\mathbb{R}^n \setminus \mathbb{R}^k$. In polar coordinates, L_1 takes the form

$$L_1 = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta + \frac{V_p(r)}{r^2}, \tag{19}$$

where $V_p(r) = r^2 \cdot pu_1(r)^{p-1}$. From Proposition 1 above we have

$$\lim_{r\to 0} V_p(r) = pv_\infty^{p-1} = \tfrac{2p}{p-1}\left(N - \tfrac{2p}{p-1}\right) \equiv A_p, \tag{20}$$

$$V_p(r) \sim cr^{(2-N)(p-1)+2} \quad \text{as} \quad r \to \infty. \tag{21}$$

Notice that the exponent $(2-N)(p-1)+2$ is negative precisely when $p > N/(N-2)$.

6.1. Indicial roots

L_1 has a regular singularity at $r = 0$ and, because $V_p(r)$ tends to 0 as r tends to infinity, it also has one at $r = \infty$. Hence the asymptotic behaviour of solutions to $L_1 w = 0$ are determined by the indicial roots of this operator at these points. At $r = 0$, the indicial roots for L_1 are

$$\gamma_j^\pm = \frac{2-N}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_j - A_p}. \tag{22}$$

Here the λ_j are the eigenvalues of the Laplacian on S^{N-1}, counted with their multiplicity. More precisely

$$\lambda_0 = 0, \quad \lambda_j = N-1, \ j = 1, \dots N, \quad \text{etc.}$$

The indicial roots for L_1 at $r = \infty$ are the same as for the Laplacian itself, since V tends to 0 at ∞; these values are

$$\tilde{\gamma}_j^{\pm} = \frac{2-N}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_j}, \qquad j = 0, 1, 2, \dots. \tag{23}$$

Notice that these numbers are integral and assume all integer values except $-1, -2, \dots, 3-N$.

Because L_1 has a regular singularity at 0 and ∞, its mapping properties are well-known, cf. [1], [6].

Proposition 2. *The bounded linear map*

$$L_1 : \mathcal{C}^{2,\alpha}_{\gamma,\gamma'}(\mathbb{R}^N \setminus \{0\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\gamma-2,\gamma'-2}(\mathbb{R}^N \setminus \{0\})$$

is Fredholm provided $\gamma \notin \{\gamma_j^{\pm}\}$ *and* $\gamma' \notin \{\tilde{\gamma}_j^{\pm}\}$.

We will let $\gamma' = -1$ or 0 since we are only interested in solutions which decay at ∞, or at least are bounded. We could replace this -1 by any value of γ' in $(2-N, 0)$. However, when $N = 3$, we need to choose $\gamma' \in (-1, 0)$. We shall not comment on this further, and persist in letting $\gamma' = -1$, with this understanding.

6.2. Numerology

We record some facts about these indicial roots and some of the other constants that arise frequently for later reference.

Lemma 5. *Let p be any number in the range $N/(N-2) < p < (N+2)/(N-2)$.*

1. *The functions $-2/(p-1)$ and A_p are monotone in p and*

$$2 - N > \frac{-2}{p-1} > \frac{2-N}{2} \ \textit{and}$$
$$0 < A_p \equiv \frac{2p}{p-1}\left(N - \frac{2p}{p-1}\right) < \frac{N^2-4}{4}.$$

2. *There is a number p^* in $(\frac{N}{N-2}, \frac{N+2}{N-2})$ such that for $N/(N-2) < p \le p^*$, the indicial roots $\gamma_0^{\pm}$ are real, with*

$$-2/(p-1) < \gamma_0^- < (2-N)/2 < \gamma_0^+,$$

while if $p^* < p < (N+2)/(N-2)$, then $\gamma_0^{\pm}$ are both complex with real part $(2-N)/2$.

3. $\gamma_1^- < -\frac{2}{p-1}$, *in fact* $\gamma_1^- = -\frac{2}{p-1} - 1$. *Hence in particular,* $\gamma_1^\pm$ *are always real.*

6.3. Injectivity of L_1 on $\mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^N \setminus \{0\})$

Fix two weights ν and μ with

$$\frac{-2}{p-1} < \nu < \Re(\gamma_0^-) \leq \frac{2-N}{2} \leq \Re(\gamma_0^+) < \mu \tag{24}$$

and $\mu + \nu = 2 - N$. These weights are 'dual' in a sense to be explained later. The interval in which μ lies is chosen to guarantee injectivity of L_1 on $\mathcal{C}^{2,\alpha}_{\mu,0}$; ν is determined by this duality, but also chosen not to be too negative in accordance with the nonlinear problem.

Proposition 3. *The only solution* $w \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^N \setminus \{0\})$ *of the equation* $L_1 w = 0$ *is the trivial solution* $w = 0$.

Proof. Let $\{\phi_j(\theta), \lambda_j\}$ be the eigenfunctions and eigenvalues for Δ_θ, the Laplacian on S^{N-1}. Then if w is any solution of $L_1 w = 0$, it decomposes into the infinite sum

$$w(r,\theta) = \sum_{j=1}^{\infty} w_j(r)\phi_j(\theta),$$

where the w_j solve

$$L_{1,j} w_j \equiv \frac{\partial^2}{\partial r^2} w_j + \frac{N-1}{r}\frac{\partial}{\partial r} w_j + \frac{1}{r^2}(V_p(r) - \lambda_j) w_j = 0.$$

The value $j = 0$ is omitted in the sum above since every solution of $L_{1,0} w_0 = 0$ blows up faster than r^μ near $r = 0$.

Now, by the growth restrictions on w near 0 and infinity, each w_j must be bounded by $C_j r^{\gamma_j^+}$ as $r \to 0$, and must decay like r^{2-N-j} as $r \to \infty$. We shall show that this forces w_j to vanish.

First of all, notice that if $\lambda_j > \sup V_p$ then solutions of $L_{1,j} w_j = 0$ satisfy the maximum principle. For these values of j it is also true that $\gamma_j^+ > 0$, and so w_j tends to 0 both at 0 and infinity, hence must vanish identically. Thus there are only finitely many values of j for which w_j may possibly not vanish. To deal with most of these, we use an integral estimate which will show that $w_j = 0$ for $\lambda_j > N - 1$. Multiply the equation $L_{1,j} w_j = 0$ by $r^{N-1} w$ and integrate from κ to R. Integrate by parts and use that $\mu > (2-N)/2$ and that w_j decays at least as fast as r^{1-N} as $r \to \infty$ to see that the boundary terms converge to zero as $\kappa \to 0$ and $R \to \infty$, and that the integral converges. The result is that

$$\int_0^\infty -(\partial_r w_j)^2 r^{N-1} + (V_p(r) - \lambda_j) w_j^2 r^{N-3}\, dr = 0.$$

Now rearrange terms and estimate V by its supremum:

$$\begin{aligned}\int_0^\infty r^{N-1}|\partial_r w_j|^2\,dr &= \int_0^\infty (V_p(r)-\lambda_j)r^{N-3}|w_j|^2\,dr\\ &\le (\sup V_p-\lambda_j)\int_0^\infty r^{N-3}w_j^2\,dr.\end{aligned}$$

To proceed further, we note that there is a general inequality for any function which decays like r^μ at 0 and like r^{1-N} at infinity:

$$\int_0^\infty r^{N-3}|w|^2\,dr \le \frac{4}{(N-2)^2}\int_0^\infty r^{N-1}|\partial_r w|^2\,dr.$$

To prove this, observe that

$$\begin{aligned}\int_0^\infty r^{N-3}w^2\,dr &= \frac{1}{N-2}\int_0^\infty w^2\partial_r r^{N-2}\,dr\\ &= -\frac{2}{N-2}\int_0^\infty r^{N-2}w\partial_r w\,dr,\end{aligned}$$

where again the boundary terms vanish in the integration by parts because of the assumed rates of decay. Now use the Cauchy Schwarz inequality to prove the claim.

Use this inequality in the estimate for w_j above to get

$$\frac{(N-2)^2}{4}\int_0^\infty r^{N-3}w_j^2\,dr \le (\sup V_p-\lambda_j)\int_0^\infty r^{N-3}w_j^2\,dr.$$

If $\sup V_p-\lambda_j<(N-2)^2/4$, then this inequality shows that w_j must vanish identically. By Proposition 1) and Lemma 5

$$\sup V_p \le \frac{p+1}{2}\frac{2p}{p-1}\left(N-\frac{2p}{p-1}\right).$$

The function of p on the right is dominated by its value at $p=(N+2)/(N-2)$, so that

$$\sup V_p < \frac{N}{N-2}\cdot\frac{N^2-4}{4}=\frac{N(N+2)}{4}.$$

But now

$$\frac{N(N+2)}{4}-\lambda_j<\frac{(N-2)^2}{4}\Longrightarrow \lambda_j>\frac{3}{2}N-1,$$

and this occurs for all j except $j=0,1,\dots,N$.

Thus we have shown that $w_j=0$ for $j>N$, and because of the restrictions on the growth at $r=0$, we also know that $w_0=0$. It remains to show that $w_j=0$, $j=1,\dots N$. It turns out that we can write down an explicit solution of $L_{1,j}w = 0$ for this range of j. In fact, differentiate the equation $\Delta u_1 + u_1^p = 0$ with respect to $\partial/\partial x_j$ to get

$$L_1 \frac{\partial u_1}{\partial x_j} = 0.$$

Since u_1 depends only on r, $\partial u_1/\partial x_j = (\partial u_1/\partial r)\theta_j$, where $\theta_j = x_j/|x|$. Since $-\Delta_\theta \theta_j = \lambda_j \theta_j$, $j = 1, \dots N$, $u_1'(r)$ solves $L_{1,j}w = 0$. But $u_1'(r)$ decays like r^{1-N} as $r \to \infty$ and blows up like $r^{-2/(p-1)-1}$ as $r \to 0$. If w_j solves $L_{1,j}w_j = 0$ and decays like r^{1-N} at infinity and like $r^{\gamma_j^+}$ at 0, then some nontrivial linear combination of u_1' and w_j must decay faster than r^{1-N} at infinity; since the singularities of these functions at 0 cannot cancel, this linear combination is nonvanishing. This is a contradiction, since no solution of $\mathcal{L}_{1,j}$ can decay faster than r^{1-N} at infinity. Hence $w_j = 0$, and the proof is complete.

6.4. Injectivity of $\mathbb{L}_1$ on $\mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^k)$

The argument that the induced operator $\mathbb{L}_1 = L_1 + \Delta_y$ on $\mathbb{R}^n \setminus \mathbb{R}^k$ is injective on functions which blow up no faster than r^μ and decay like r^{-1} as $r \to \infty$ is somewhat more complicated, but rests on the results of the last section.

Proposition 4. *The only solution $w \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^k)$ which satisfies $\mathbb{L}_1 w = 0$ is $w = 0$.*

Proof. As in the previous subsection, we analyze this equation by reducing it to a family of ODE's and studying these separately. This is done by first taking the eigenfunction decomposition in θ, as before, but then also taking the Fourier transform in y. Letting η be the variable dual to y in the Fourier transform, we obtain the family of operators

$$\mathbb{L}_{j,|\eta|^2} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{V_p(r) - \lambda_j}{r^2} - |\eta|^2.$$

If $\mathbb{L}_1 w = 0$ and $w \in \mathcal{C}^{2,\alpha}_{\mu,0}$, then

$$w(r,\theta,y) = \sum_{j=0}^{\infty} \int e^{iy\cdot\eta}\, \hat{w}_j(r,\eta)\, dy,$$

where $\mathbb{L}_{j,|\eta|^2}\hat{w}_j = 0$ and $\hat{w}_j \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^+)$ and depends distributionally on η. Clearly, if $\mathbb{L}_{j,|\eta|^2}\hat{w}_j = 0$ with $\hat{w}_j \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^+)$ implies $\hat{w}_j = 0$ for every j, then $\mathbb{L}_1 w = 0$ with $w \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^k)$ implies $w = 0$ as well.

First observe that the integration by parts argument of the last subsection may be repeated essentially verbatim to show that all solutions of this equation are trivial unless $j = 0, 1, \dots, N$. Furthermore, since $\mu > \Re(\gamma_0^\pm)$, there are no solutions in $\mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^+)$ for $j = 0$, either. Thus we are left to consider only the case $j = 1, \dots N$, where $\lambda_j = N - 1$.

For simplicity, in this argument, let us drop the index j in our notation and replace $|\eta|^2$ by a parameter E. Then $\mathbb{L}_{j,|\eta|^2}$ becomes $\mathbb{L}_E$, and so forth. Now let us recall that any solution of $\mathbb{L}_E w = 0$ has a Frobenius series expansion (convergent in some interval, since V_p is real analytic) of the form

$$w \sim \sum_{j=0}^{\infty} a_j r^{\gamma^- + j} + \sum_{j=0}^{\infty} b_j r^{\gamma^+ + j}. \tag{25}$$

There are additional terms, with logarithmic factors, in the second series in the case when the two indicial roots differ by an integer. Since the argument is quite analogous in that case, we only consider the case when these extra terms do not appear. The two leading coefficients, a_0 and b_0 are free, and may be specified arbitrarily; all the a_j for $j > 0$ are determined in terms of a_0 by a recursion formula, and similarly for the b_j. In particular, if $a_0 = 0$, then all $a_j = 0$. Hence, since $\mu > \gamma^-$, we have that $w \in \mathcal{C}^{2,\alpha}_\mu$ near $r = 0$ if and only if $a_0 = 0$.

We shall show by a sort of continuity argument, treating both p and E as parameters, that if w is a nontrivial solution on $\mathbb{R}^+$ growing at most polynomially as $r \to \infty$ then a_0 never vanishes.

For any two numbers $0 \leq r_0 < r_1 < \infty$ there is a map C_{r_0,r_1} which sends the Cauchy data $(w(r_0), w'(r_0))$ of a solution w at r_0 to its Cauchy data $(w(r_1), w'(r_1))$ at r_1. When $r_0 = 0$, this Cauchy data is replaced by the pair of leading coefficients (a_0, b_0) in (23). If $0 < r_0$ then it is well-known that C_{r_0,r_1} is invertible and depends smoothly on the parameters (p, E). It is also true that if r_1 is in the interval of convergence of the series (23) then C_{0,r_1} is invertible and still depends smoothly on (p, E).

Amongst the two dimensional family of solutions of $\mathbb{L}_E w = 0$, there is one solution, unique up to multiplication, which decays at infinity. Actually, if $\mathbb{L}_E w = 0$ and if w is of polynomial growth as $r \to \infty$, then w decays like $r^q e^{-r\sqrt{E}}$ for q determined by the coefficients of $\mathbb{L}_E$. We fix the solution uniquely by requiring that $w(1) = 1$. This makes sense because, by the normalization in §2, the function $V_p(r)$ is less than some fixed constant, in particular less than $N-1$, for $r \geq 1$. Thus solutions of $\mathbb{L}_E w = 0$ satisfy the maximum principle on $[1, \infty)$. Hence any solution which vanishes at $r = 1$ cannot decay as $r \to \infty$.

For this decaying solution w look at the leading asymptotic coefficient $a_0 = a_0(p, E)$ in its series expansion at $r = 0$. We write out this dependence on the two parameters p and E explicitly since we need to show that $a_0(p, E) \neq 0$ for (p, E) in the strip

$$(N/(N-2), (N+2)/(N-2)) \times [0, \infty).$$

For if this holds, then $w \notin C^{2,\alpha}_{\mu,0}(\mathbb{R}^+)$ and our result will follow.

We first show that a_0 is smooth in (p, E). We do this by using the maps C_{r_0,r_1} discussed above. First consider the Cauchy data $(1, w'(1))$ of the decaying solution w. Since the coefficient of order zero of $\mathbb{L}_E$ is negative on $[1, \infty)$, w may be constructed by the 'shooting method' there, i.e., by solving the boundary problem $\mathbb{L}_E w_R = 0$ on $[1, R]$, with $w_R(1) = 1$ and $w_R(R) = 0$, and letting $R \to \infty$. By the maximum principle, the limit must exist, and since it is bounded it agrees with w. Since the Cauchy data $(1, w'_R(1))$ depends smoothly on (p, E) for each R, it is not difficult to see that the limiting values $(1, w'(1))$ do as well. Now use the composition of the (invertible) maps C_{0,r_0} for r_0 very small, and $C_{r_0,1}$ to conclude that (a_0, b_0) depends smoothly on (p, E).

Now we proceed with the rest of the argument. First observe that $a_0(p, 0) \neq 0$ for any p by the result of the previous subsection. In fact, writing out the explicit solution $\partial u_1/\partial r$ more explicitly, we can check that $a_0(p, 0) > 0$. Next, note that for p sufficiently close to its lower limit $N/(N-2)$, the function V stays uniformly small, and in particular less than $N - 1$ for all r. Hence we can again apply the integration by parts argument of the previous subsection, but now using that $\sup(V_p - N + 1) < 0$, which immediately shows that $a_0 \neq 0$ for all $p < N/(N-2) + \kappa$ for every E and for some $\kappa > 0$. By continuity from $E = 0$, $a_0 > 0$ for (p, E) in this smaller strip. Finally, we can apply a similar argument when E is sufficiently large, regardless of the value of p. In fact, all we need is that the supremum of $V_p(r) - N + 1 - r^2 E$ is less than $(N-2)^2/4$, and this is clearly true for any p, for E large enough. Thus, $a_0 > 0$ also for (p, E) in this range of values.

Finally, suppose that there is some value of (p, E), with $p \in (N/(N-2), (N+2)/(N-2))$, for which $a_0 = 0$. Using the regularity of a_0 in these parameters, there is some (p_0, E_0) for which $a_0(p_0, E_0) = 0$ and $(a_0)_E(p_0, E_0) = 0$ (i.e., both a_0 and its derivative with respect to E vanish there). Differentiate the equation $\mathbb{L}_E w = 0$ and set $(p, E) = (p_0, E_0)$. If $\tilde{w} = \partial w/\partial E$ then

$$\mathbb{L}_E \tilde{w} = 2w.$$

Since $a_0 = 0$, the right side of this equation blows up only like r^{γ^+} as $r \to 0$. Thus $\tilde{w}$ is the sum of two terms; the first blows up no faster than $r^{\gamma^+ + 2}$, while the second is a solution of the homogeneous equation and might blow up like r^{γ^-}. However, since the leading coefficient of $\tilde{w}$ is the derivative of the leading coefficient of w, i.e., it is $(a_0)_E$, which vanishes, we see that this solution of the homogeneous equation is absent and $\tilde{w}$ blows up no faster than $r^{\gamma^+ + 2}$. Now multiply the equation by $w \cdot r^{N-1}$ and integrate from 0 to infinity. Using the exponential decay as r tends to infinity, and that $w \sim r^{\gamma^+}$, $\tilde{w} \sim r^{\gamma^+ + 2}$ as r tends to 0, we see that this integral is well-defined, since $2\gamma^+ > 2 - N$. Furthermore, it is permissible to integrate by parts to get

$$0 = \int_0^\infty (\mathbb{L}_E w)\tilde{w}\, r^{N-1}\, dr = \int_0^\infty w(\mathbb{L}_E \tilde{w}) r^{N-1}\, dr = 2\int_0^\infty |w|^2 r^{N-1}\, dr.$$

Hence $w = 0$ for this value of (p, E), which is a contradiction.

6.5. Surjectivity of L_1 and $\mathbb{L}_1$ on $\mathcal{C}^{2,\alpha}_{\nu,-1}$

From the results of the last subsection we may deduce the following:

Proposition 5. *Let $-\frac{2}{p-1} < \nu = 2 - N - \mu < \Re(\gamma_0^-)$ as before. Then the maps*

$$L_1 : \mathcal{C}^{2,\alpha}_{\nu,-1}(\mathbb{R}^N \setminus \{0\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,-3}(\mathbb{R}^N \setminus \{0\})$$

and

$$\mathbb{L}_1 : \mathcal{C}^{2,\alpha}_{\nu,-1}(\mathbb{R}^n \setminus \mathbb{R}^k) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,-3}(\mathbb{R}^n \setminus \mathbb{R}^k)$$

are surjective. Furthermore, the first map is Fredholm and its nullspace is one dimensional, while the second map has an infinite dimensional nullspace.

Proof. We shall only sketch the proof of this result and provide references to papers where analogous facts are proved thoroughly. As discussed already in §5, the proof of this result is much simpler for the former of the two maps. In fact it is not difficult to write down a right inverse for L_1 directly in terms of solutions of the homogeneous problem $L_1 w = 0$. In order to do this, it is crucial to use the fact that any solution which is bounded by a multiple of r^μ near $r = 0$ does not decay as r tends to infinity, and any solution which decays at infinity must decay at least as fast as r^{2-N}. Note that this is equivalent to the injectivity of L_1 on $\mathcal{C}^{2,\alpha}_{\mu,-1}$. The right inverse for L_1 may be written as a sum of the right inverses for $L_{1,j}$ on each eigencomponent; each of these right inverses may be constructed explicitly from the solution which blows up like $r^{\gamma_j^+}$ as $r \to 0$ which is unbounded as $r \to \infty$ and the solution blowing up like $r^{\gamma_j^-}$ as $r \to 0$ and decaying as $r \to \infty$. It is then not difficult to show that this sum of right inverses for each eigencomponent is bounded on the weighted Hölder spaces, cf. [1] for this procedure for a closely related operator. This proves that L_1 on $\mathcal{C}^{2,\alpha}_{\nu,-1}$ is not only Fredholm but surjective.

Any solution of $L_1 w = 0$ in $\mathcal{C}^{2,\alpha}_{\nu,-1}$ must be radial, by considering its growth as $r \to 0$. There is a two dimensional space of solutions of $L_{1,0} w_0 = 0$. By considering a suitable nontrivial linear combination of any two basis elements of this space we obtain a solution which decays like r^{2-N} as $r \to \infty$; there cannot be two independent solutions with this decay rate for the same reasons as in the previous subsection. This solution grows like some combination of $r^{\gamma_0^\pm}$ as $r \to 0$, but in either case is in $\mathcal{C}^{2,\alpha}_{\nu,-1}$. Hence the nullspace is one dimensional.

To establish the corresponding facts for $\mathbb{L}_1$ we use Hilbert space techniques and the construction of a pseudodifferential right parametrix for $\mathbb{L}_1$ from [6]. $\mathbb{L}_1$ is surjective on $\mathcal{C}^{2,\alpha}_{\nu,-1}$ provided its range is both closed and dense. The existence and boundedness of this right parametrix gives the closedness of the range. Then duality, coupled with Proposition 4, yields that the cokernel of this map is trivial, so that the right parametrix may be replaced by a right inverse.

It is most natural to construct the pseudodifferential right parametrix for $\mathbb{L}_1$ relative to the spaces $r^\delta L^2$ because of the central role of the Fourier transform in this construction. Once the parametrix is obtained, it is then necessary to show its boundedness on the weighted Hölder spaces. These steps are carried out in detail in [6], to which we refer the reader, cf. also [8]. Here we simply show that $\mathbb{L}_1$ satisfies the hypotheses necessary for that machinery to apply.

The main result of [6] implies that $\mathbb{L}_1$ has closed range on $r^\delta L^2$ provided $\delta \notin \{\delta_j^\pm\}$, where $\delta_j^\pm = \gamma_j^\pm + (N-2)/2$. Choose a number $-\delta_\nu$ just slightly smaller than $\nu + (N-2)/2$, in particular so that $-2/(p-1) + (N-2)/2 < -\delta_\nu < \nu + (N-2)/2 < 0$. Then $-\delta_\nu \notin \{\delta_j^\pm\}$ and $\mathcal{C}^{2,\alpha}_\nu(\Omega\backslash\Sigma) \subset \rho^{-\delta_\nu}L^2(\Omega\backslash\Sigma)$. It is also proved in [6] that $\mathbb{L}_1$ is essentially surjective on $r^{-\delta_\nu}L^2$, i.e., it has closed range and at most a finite dimensional cokernel, and that it is essentially injective, i.e., has closed range and at most a finite dimensional kernel, on $r^{+\delta_\nu}L^2$. These two facts are consequences of one another by duality since $r^\delta L^2$ is the dual space of $r^{-\delta}L^2$ and $\mathbb{L}_1$ is self-adjoint on r^0L^2. The crucial hypothesis that must be satisfied in order for these last two conclusions to be true involves the 'model Bessel operator' $\hat{L}_1$ for $\mathbb{L}_1$. This operator is obtained by freezing coefficients of $\mathbb{L}_1$ at $r = 0$ in an appropriate sense, taking the Fourier transform in y, and finally rescaling by setting $s = r|\eta|$. The operator obtained in this way is

$$\hat{L}_1 = \frac{\partial^2}{\partial s^2} + \frac{N-1}{s}\frac{\partial}{\partial s} + \frac{A_p + \Delta_\theta}{s^2} - 1.$$

This crucial hypothesis is that $\hat{L}_1$ is injective as a map on $s^{\delta_\nu}L^2(\mathbb{R}^+ \times S^{N-1}; s^{N-1}\,ds\,d\theta)$, or equivalently, surjective as a map on $s^{-\delta_\nu}L^2(\mathbb{R}^+ \times S^{N-1}; s^{N-1}\,ds\,d\theta)$.

$\hat{L}_1$ may be analyzed directly to show that this hypothesis is satisfied. In fact, introducing the eigenfunction expansion with respect to Δ_θ, as usual, we obtain a family of ordinary differential operators, the solutions of which may be determined explicitly in terms of Bessel functions.

Analogously to the situation for L_1 we find that any solution which blows up no faster than r^ν as $r \to 0$ grows exponentially as $r \to \infty$. Using these solutions we can construct a right inverse for $\hat{L}_1$ explicitly.

Since $\hat{L}_1$ satisfies the hypothesis, we conclude that $\mathbb{L}_1$ itself has closed range, with at most a finite dimensional cokernel, as a map on $\mathcal{C}^{2,\alpha}_{\nu,-1}$. The cokernel of this map may be identified with the kernel of $\mathbb{L}_1$ as a map on $r^{\delta_\nu}L^2$ or $\mathcal{C}^{2,\alpha}_{\mu,-1}$. But from the previous subsection, we know that this cokernel is trivial; hence $\mathbb{L}_1$ is surjective, as desired.

Finally, we may explicitly exhibit an infinite dimensional nullspace of $\mathbb{L}_1$ in $\mathcal{C}^{2,\alpha}_{\nu,-1}$. For, if $\mathbb{L}_1 w = 0$ and w is bounded by r^ν as $r \to 0$, then only the eigencomponent w_0 may be nonvanishing, just as for L_1. Now, taking the Fourier transform in y, we see that $\mathbb{L}_{0,|\eta|^2}\hat{w}_0 = 0$. Hence $\hat{w}_0(r,\eta) = A(\eta)\hat{W}_0(r,\eta)$, where $\hat{W}_0$ is the unique solution of this equation which decays (exponentially if $\eta \neq 0$) as $r \to 0$, normalized so that $\hat{W}_0(1,\eta) = 1$. The coefficient $A(\eta)$ is allowed to be arbitrary, so long as the corresponding w_0 is bounded uniformly, for each r, as $|y| \to \infty$. Clearly there is an infinite dimensional freedom in choosing such coefficients; for example, we could let $A(\eta)$ be an arbitrary element of the Schwartz space. This completes the proof.

7. Injectivity of $L_{\bar{\epsilon}}$ on $\mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(\Omega \setminus \Sigma)$

We are now able to turn to the first of our main tasks, to show that $L_{\bar{\epsilon}}$ is surjective on $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$ when all the ϵ_i are sufficiently small. By an argument identical to the one indicated in the proof of Proposition 5, this surjectivity is equivalent to the injectivity of this operator acting on functions growing like r^μ at the singular points, where

$$(2-N)/2 < \mu = 2 - N - \nu < \gamma_1^+ < 0,$$

as in the previous section. This injectivity is what we shall prove.

7.1. Preparatory lemmas

There are three lemmas which are used in the rescaling proofs, both in the case when the singular set is discrete and when it is positive dimensional. We shall adopt the notation

$$\Omega_{\bar{\epsilon}} = \Omega \setminus \cup_{i=1}^K B_{\epsilon_i}(\Sigma_i).$$

Here, using slightly different notation than in previous sections, $B_{\epsilon_i}(\Sigma_i)$ is the tubular neighbourhood around Σ_i of radius ϵ_i, and, in case Σ is discrete, Σ_i simply equals the point x_i.

The first Lemma states that $L_{\bar{\epsilon}}$ satisfies the maximum principle on $\Omega_{\bar{\epsilon}}$.

Lemma 6. *There exists some $\epsilon_0 > 0$ such that if all $\epsilon_i < \epsilon_0$, then after a normalization of the initial radial solution u_1, the operator $L_{\bar\epsilon}$ satisfies the maximum principle on $\Omega_{\bar\epsilon}$. That is, if $w \leq 0$ on $\partial\Omega_{\bar\epsilon}$ and $L_{\bar\epsilon} w \geq 0$, then $w \leq 0$ on $\Omega_{\bar\epsilon}$.*

Proof. We give the proof of this result in the case where Σ is discrete. The general case, where Σ has positive dimension can be treated similarly. Set $w^+ = \max\{w, 0\}$ on $\Omega_{\bar\epsilon}$ and $w^+ = 0$ on each $B_{\epsilon_i}(\Sigma_i)$, so that $w^+ \in H_0^1(\Omega)$. Multiply the inequality $L_{\bar\epsilon} w \geq 0$ by w^+ and integrate by parts to get

$$\int_\Omega |\nabla w^+|^2 \leq \int_\Omega |w^+|^2 p\bar{u}_{\bar\epsilon}^{p-1} \leq p\alpha^{p-1} \int_\Omega (\sum_{i=1}^K |x - x_i|^{-2})|w^+|^2.$$

Here α is the supremum of $v_1 = r^{\frac{2}{p-1}} u_1$ for $r \geq 1$, which by Remark 2 can be taken as small as desired. Now, using the identity

$$(N-2)|x|^{-2} = \text{div}\,(\frac{x}{|x|^2}),$$

and integrating by parts, we obtain the inequality

$$\int_\Omega |x - x_i|^{-2}|w^+|^2 \leq \frac{4}{(N-2)^2} \int_\Omega |\nabla w^+|^2$$

If α is taken sufficiently small so that the inequality $4p\alpha^{p-1}K < (N-2)^2$ holds, we see that $w^+ = 0$, and hence $w \leq 0$, as desired.

The second Lemma uses this maximum principle to deduce estimates in the weighted Hölder spaces for solutions of $L_{\bar\epsilon} w_{\bar\epsilon} = f_{\bar\epsilon}$ in terms of their boundary values on $\partial\Omega_{\bar\epsilon}$.

Lemma 7. *There exists $\epsilon_0 > 0$ such that, if all $\epsilon_i < \epsilon_0$ then the following estimate holds. Let $f_{\bar\epsilon} \in C^{0,\alpha}_{\gamma-2}(\Omega_{\bar\epsilon})$, where $\gamma \in (2-N, 0)$ is fixed, and suppose that $w_{\bar\epsilon}$ is any solution to $L_{\bar\epsilon} w_{\bar\epsilon} = f_{\bar\epsilon}$ with $w_{\bar\epsilon} = 0$ on the 'outer boundary' of $\Omega_{\bar\epsilon}$, i.e., on $\partial\Omega_{\bar\epsilon} \cap \partial\Omega$. Then there exists a constant $c > 0$ independent of $\bar\epsilon$ such that*

$$||w_{\bar\epsilon}||_{2,\alpha,\gamma} \leq c\left(||f_{\bar\epsilon}||_{0,\alpha,\gamma-2} + \sum_{i=1}^K \epsilon_i^{-\gamma}||w_{\bar\epsilon}||_{0,0,\partial B_{\epsilon_i}(\Sigma_i)}\right),$$

where the first two norms are taken over $\Omega_{\bar\epsilon}$.

Proof. Define $\phi \in C^\infty(\Omega \setminus \Sigma)$ to be a positive smooth function for which, in some fixed neighbourhood $B_\sigma(\Sigma_i)$ for each i, $\phi(x) = \text{dist}\,(x, \Sigma_i)^\gamma$; here σ should also be chosen so that the approximate solution $\bar{u}_{\bar\epsilon}$ is supported in $\cup B_\sigma(\Sigma_i)$. For example, we can take $\phi = \rho^\gamma$, where ρ was the function introduced in §3. If some Σ_i is a point, $\Sigma_i = x_i$, then $\phi = |x - x_i|^\gamma$ in $B_\sigma(x_i)$. Then a simple computation shows that in $B_\sigma(x_i) \setminus B_{\epsilon_i}(x_i)$ we have

$$\begin{aligned} L_{\epsilon_i}|x - x_i|^\gamma &= \{\gamma(N - 2 + \gamma)|x - x_i|^{\gamma-2} + p\bar{u}_{\bar{\epsilon}}^{p-1}|x - x_i|^\gamma\} \\ &\leq -c|x - x_i|^{\gamma-2}, \end{aligned}$$

where the constant $c > 0$ can be chosen independent of $\bar{\epsilon}$, since $2 - N < \gamma < 0$ implies that $\gamma(N - 2 + \gamma) < 0$ and since $p\alpha^{p-1}$ can be chosen as small as desired. If Σ_i is positive dimensional, then a similar estimate holds. For this we simply need to use the expression (13) and the estimates for e_1 and e_2 there.

Let $A > 0$ denote the supremum of $w_{\bar{\epsilon}}$ on $\cup_{i=1}^K \partial B_\sigma(x_i)$. Hence ϕ, multiplied by a suitable constant times

$$\left(A + ||f_{\bar{\epsilon}}||_{0,\alpha,\gamma-2} + \sum_{i=1}^K \epsilon_i^{-\gamma}||w_{\bar{\epsilon}}||_{0,0,\partial B_{\epsilon_i}(\Sigma_i)}\right)$$

is a supersolution for the problem in $B_\sigma(x_i) \setminus B_{\epsilon_i}(x_i)$. Likewise, $-\phi$ multiplied by a similar constant is a subsolution.

We claim that, if $\epsilon_0 > 0$ is small enough, then A is bounded by a constant times

$$\left(||f_{\bar{\epsilon}}||_{0,\alpha,\gamma-2} + \sum_{i=1}^K \epsilon_i^{-\gamma}||w_{\bar{\epsilon}}||_{0,0,\partial B_{\epsilon_i}(\Sigma_i)}\right).$$

In order to prove this claim we will argue by contradiction and assume that we have a sequence of counterexamples to this assertion. Thus suppose that there is some sequence of K-tuples $\bar{\epsilon}^{(\ell)} = (\epsilon_1^{(\ell)}, \ldots, \epsilon_K^{(\ell)})$ such that for some subset of the indices $1, \ldots, K$, the corresponding $\epsilon_j^{(\ell)}$ tend to zero. For convenience, we take this subset to be $\{1, \ldots, J\}$, so that $\epsilon_j^{(\ell)} \to 0$ for $j \leq J$ and $\epsilon_j^{(\ell)} \geq c > 0$ for $j > J$. In addition, up to a subsequence we may assume that $\epsilon_j^{(\ell)}$ converges to some $\epsilon_j \geq c$ for all $j > J$. Suppose also that for each such $\bar{\epsilon}^{(\ell)}$ there is a function $w_{\bar{\epsilon}^{(\ell)}} \in \mathcal{C}_\gamma^{2,\alpha}(\Omega_{\bar{\epsilon}^{(\ell)}})$ with $L_{\bar{\epsilon}} w_{\bar{\epsilon}^{(\ell)}} = f_{\bar{\epsilon}^{(\ell)}}$ and such that $w_{\bar{\epsilon}^{(\ell)}} = 0$ on the 'outer boundary' of $\Omega_{\bar{\epsilon}}$ and $f_{\bar{\epsilon}^{(\ell)}} \in \mathcal{C}_{\gamma-2}^{0,\alpha}(\Omega_{\bar{\epsilon}})$ bounded independently of ℓ. First multiply the $w_{\bar{\epsilon}^{(\ell)}}$ by a suitable constant so that

$$\sup_{\cup_{i=1}^K B_\sigma(x_i)} |w_{\bar{\epsilon}^{(\ell)}}| = 1. \tag{26}$$

Using the super-solution constructed above, it is easy to see that $w_{\bar{\epsilon}^{(\ell)}}$ converges to a solution w of $\Delta w = 0$ in $\Omega \setminus \cup_{i=1}^J \Sigma_i \cup_{i=J+1}^K B_{\epsilon_i}(\Sigma_i)$. In addition, w also vanishes on $\partial\Omega \cup_{i=J+1}^K \partial B_{\epsilon_i}(\Sigma_i)$, and finally $w \in \mathcal{C}_\gamma^{2,\alpha}$. Since $\gamma > 2 - N$ it is well known that the singularities of w at $\cup_{i=1}^J \Sigma_i$

are removable. Therefore w solves $\Delta w = 0$ in $\Omega \setminus \cup_{i=J+1}^{K} B_{\epsilon_i}(\Sigma_i)$ with 0 boundary data, so $w = 0$ which contradicts (24). The claim follows.

We conclude that there is a constant $\delta > 0$ independent of $\bar{\epsilon}$ such that $L_{\bar{\epsilon}}(w_{\bar{\epsilon}} + \delta\phi) \leq f_{\bar{\epsilon}} - c\delta\phi \leq 0$ and $L_{\bar{\epsilon}}(w_{\bar{\epsilon}} - \delta\phi) \geq f_{\bar{\epsilon}} + c\delta\phi \geq 0$. By increasing δ to make $w_{\bar{\epsilon}} - \delta\phi < 0$ and $w_{\bar{\epsilon}} + \delta\phi > 0$ on $\partial\Omega_{\bar{\epsilon}}$, we deduce that $|w_{\bar{\epsilon}}| \leq \delta\phi$ in all of $\Omega_{\bar{\epsilon}}$, i.e., $w_\epsilon \in \mathcal{C}^{0,0}_{\nu}(\Omega_{\bar{\epsilon}})$, with norm independent of $\bar{\epsilon}$. Standard arguments using rescaled elliptic estimates show that $w_{\bar{\epsilon}} \in \mathcal{C}^{2,\alpha}_{\nu}(\Omega_{\bar{\epsilon}})$ with norm independent of $\bar{\epsilon}$.

The third Lemma shows that the weighted Hölder norm with exponent μ for a solution of the fixed homogeneous equation $L_1 w = 0$ on a fixed tubular neighbourhood $B_\sigma(\Sigma_i)$ is controlled by the norm of w on $\partial B_\sigma(\Sigma_i)$. This is quite obvious when Σ_i is a point, since the nullspace of L_1 is finite dimensional then. When Σ_i has positive dimension, this is a more substantial result, and it follows directly from the results of [6]. Nonetheless, we give an elementary proof, which covers both cases. For convenience here we drop the subscript i. As usual, (r, θ, y) are Fermi coordinates on this tubular neighbourhood.

Lemma 8. *Suppose w_ℓ is a sequence of solutions of $L_1 w_\ell = 0$ in $B_\sigma(\Sigma)$, with $w_\ell \in \mathcal{C}^{2,\alpha}_{\mu}(B_\sigma(\Sigma))$ and $|w_\ell| \leq A$ on $\partial B_\sigma(\Sigma)$ uniformly in ℓ. Then $||w_\ell||_{2,\alpha,\mu}$ cannot diverge as $\ell \to \infty$.*

Proof. First of all, observe that by the rescaled Schauder estimates, it suffices to show that the supremum of $r^{-\mu}|w_\ell|$ over B_σ does not diverge. Suppose, to the contrary, that it does; furthermore, suppose that this supremum takes the value C_ℓ, which tends to infinity, and is attained at some point $x_\ell \in B_\sigma$. Let $(r_\ell, \theta_\ell, y_\ell)$ be the Fermi coordinates of x_ℓ. Since w_ℓ can be bounded by some new constant A' on $B_\sigma(\Sigma) \setminus B_{\sigma/2}(\Sigma)$, the r_ℓ must converge to zero. For if they did not, then consider the rescaled function $\bar{w}_\ell = C_\ell^{-1} w_\ell$. This still solves $L_1 \bar{w}_\ell = 0$. Furthermore, the supremum of $r^{-\mu}\bar{w}_\ell$ on $B_\sigma(\Sigma)$ is equal to one, and is attained in some fixed annulus $B_\sigma(\Sigma) \setminus B_\beta(\Sigma)$. Thus, the $\bar{w}_\ell$ converge to a nonzero limit $\bar{w}_\infty$, which is a solution of $L_1 \bar{w}_\infty = 0$ in $B_\sigma(\Sigma)$. However, the supremum of $\bar{w}_\ell$ on $B_\sigma \setminus B_{\sigma/2}$ tends to zero, so that $\bar{w}_\infty = 0$ on the annulus $B_\sigma \setminus B_{\beta/2}$, which is a contradiction.

Now, since we have established that $r_\ell \to 0$, we may again rescale. If $\Sigma = x_i$ is a point (which is the origin in these coordinates), consider the function $\tilde{w}_\ell(r, \theta) = C_\ell^{-1} r_\ell^{\mu} w_\ell(r/r_\ell, \theta)$. If Σ is higher dimensional, then some subsequence of the points y_ℓ converge to $y_\infty \in \Sigma$ (since Σ is compact). Choose Fermi coordinates centred around this point, so that $y_\infty = 0$, and these coordinates are defined for $|y| \leq \tau$. In this case let $\tilde{w}_\ell = C_\ell^{-1} r_\ell^{\mu} w_\ell(r/r_\ell, \theta, (y - y_\ell)/r_\ell)$. In the former case, $\tilde{w}_\ell$ satisfies $L_{1/r_\ell}\tilde{w}_\ell = 0$ on $B_{\sigma/r_\ell}(0)$, and is bounded by r^μ there. The same is true in the latter case, except that the operator L_{1/r_ℓ} must be replaced by one for which the error terms e_1, e_2 are replaced by very small translates, and which are in any case still tending to zero when $\ell \to \infty$. This rescaling has been chosen so that $r^{-\mu}|\tilde{w}_\ell|$ attains its supremum on $\partial B_1(\Sigma)$. As before, pass to a limit, $\tilde{w}_\infty$. By the previous remark, $\tilde{w}_\infty \neq 0$.

When $\Sigma = 0$, then it is also true that $\tilde{w}_\infty \in \mathcal{C}^{2,\alpha}_{\mu,\mu}(\mathbb{R}^N \setminus \{0\})$, and is a solution of

$$\left(\Delta + \frac{1}{r^2} p v_\infty^{p-1}\right) \tilde{w}_\infty = 0$$

there. But the solutions of this equation are of the form

$$\sum_{j=0}^{\infty} \left(a_j r^{\gamma_j^+} + b_j r^{\gamma_j^-}\right) \phi_j(\theta).$$

It is clear that no function of this form can be bounded by r^μ both as $r \to 0$ and $r \to \infty$. Hence we arrive at a contradiction again, so the assertion of the lemma must be true when Σ is a point.

In case $\dim \Sigma = k > 0$, we may still take a limit, and get a function $\tilde{w}_\infty$ which solves the same equation, but now on $\mathbb{R}^n \setminus \mathbb{R}^k$. It also satisfies the estimate $\sup r^{-\mu}|\tilde{w}_\infty| < \infty$. However, introducing a decomposition into eigenfunctions for the spherical Laplacian Δ_θ yields the uncoupled system of equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r} + \Delta_{\mathbb{R}^k} + \left(p v_\infty^{p-1} - \lambda_j\right) \frac{1}{r^2}\right) w_j = 0,$$

where we have renamed the eigencomponents simply w_j. Each such eigencomponent satisfies $\sup r^{-\mu}|w_j| < \infty$ uniformly as $|y| \to \infty$. Taking the Fourier transform of this equation in y reduces it to an equation of Bessel type:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \left(A_p - \lambda_j - r^2|\eta|^2\right) \frac{1}{r^2}\right) \hat{w}_j = 0.$$

This equation may be solved explicitly, and as discussed earlier it is easy to see that for $\eta \neq 0$ the only solution of this equation which grows no faster than r^μ as $r \to 0$ must increase exponentially as $r \to \infty$. Thus $\hat{w}_j(r, \eta) = 0$ for $\eta \neq 0$, and so it is polynomial, and hence constant in y (since w_j is bounded in y). Therefore, the previous argument applies to show $w_j = 0$ for all j. Hence $\tilde{w}_\infty = 0$, which is again a contradiction. This proves the Lemma in all cases.

7.2. Injectivity

Theorem 5. *If ϵ_0 is sufficiently small, and if each $\epsilon_i < \epsilon_0$, then*

$$L_{\bar{\epsilon}} : \mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-2}(\Omega \setminus \Sigma)$$

is injective.

Proof. As in the previous lemma, the proof uses rescaling to argue by contradiction. The idea is quite simple. If the result of this theorem were false, there would exist a sequence of counterexamples, corresponding to some sequence of the $\bar{\epsilon}^{(\ell)}$, with at least some subset of the $\epsilon_j^{(\ell)}$ decreasing to zero. By rescaling these functions and passing to a limit, a counterexample to Propositions 3 and 4 would be obtained.

Thus suppose that there is some sequence of K-tuples $\bar{\epsilon}^{(\ell)} = (\epsilon_1^{(\ell)}, \dots, \epsilon_K^{(\ell)})$ such that (possibly passing to a subsequence) for some fixed subset of the indices $1, \dots, K$, the corresponding $\epsilon_j^{(\ell)}$ tend to zero. For convenience as before, we take this subset to be $\{1, \dots, J\}$, so that $\epsilon_j^{(\ell)} \to 0$ for $j \le J$ and $\epsilon_j^{(\ell)} \ge c > 0$ for $j > J$. Suppose also that for each such $\bar{\epsilon}^{(\ell)}$ there is a function $w_\ell \in \mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(\Omega \setminus \Sigma)$ with $L_{\bar{\epsilon}^{(\ell)}} w_\ell = 0$. First multiply the w_ℓ by a suitable constant so that

$$\sup_{\partial \Omega_{\bar{\epsilon}^{(\ell)}}} \rho(x)^{-\mu} |w_\ell| = 1. \tag{27}$$

From Lemma 7, we get that

$$\sup_{\Omega_{\bar{\epsilon}^{(\ell)}}} \rho(x)^{-\mu} |w_\ell| \le C \sup_{\cup \partial B_{\epsilon_i^{(\ell)}}(\Sigma_i)} \rho^{-\mu} |w_\ell| \le C.$$

For some subsequence (which we assume is the full sequence), the supremum of $\rho(x)^{-\mu}|w_\ell|$ on $\cup \partial B_{\epsilon_i^{(\ell)}}(\Sigma_i)$ is attained on some fixed $\partial B_{\epsilon_j^{(\ell)}}(\Sigma_j)$.

There are several possibilities, which we examine in turn. For simplicity, we initially consider the case where Σ is discrete. First suppose that for the sphere where supremum above is attained, the index j satisfies $j \le J$. Fix this index j, and for convenience, translate Ω so that $x_j = 0$. Then $\epsilon_j^{(\ell)} \to 0$. Now rescale, setting

$$\bar{w}_\ell(x) = (\epsilon_j^{(\ell)})^{-\mu}\, w_\ell(\epsilon_j^{(\ell)} x).$$

This function is defined on $B_{\sigma/\epsilon_j^{(\ell)}}(0)$ and solves $(\Delta + p\bar{u}_1^{p-1})\bar{w}_\ell = 0$ there. Here $\bar{u}_1$ is the radial solution $u_1(r)$ multiplied by a cutoff function $\chi(\epsilon_j^{(\ell)} x/R)$, which is equal to one on an increasingly large set as $\epsilon_j^{(\ell)}$ tends to zero. By construction, the supremum of $r^{-\mu}|\bar{w}_\ell|$ is bounded by C on $B_{\sigma/\epsilon_j^{(\ell)}}(0) \setminus B_1(0)$.

We can apply Lemma 8 to conclude that the $r^{-\mu}|\bar{w}_\ell|$ are bounded uniformly on $B_1(0)$. Now let $\ell \to \infty$. Since their norms are bounded, the $\bar{w}_\ell$ tend to a limit $\bar{w}_\infty \in \mathcal{C}^{2,\alpha}_\mu(B_1(0))$, which is defined with $r^{-\mu}|\bar{w}_\infty|$ bounded on all of $\mathbb{R}^N \setminus \{0\}$. In addition, it is a solution of $L_1 \bar{w}_\infty = 0$

there. But $\bar{w}_\infty$ is not identically zero, since its supremum on $\partial B_1(0)$ is one. This contradicts Proposition 3.

This last argument still leads to a contradiction if it is only true that for some $j \leq J$ (and some subsequence of the ℓ), the supremum of $\rho^{-\mu}|w_\ell|$ on $\partial B_{\epsilon_j^{(\ell)}}(x_j)$ is bounded below by some positive number $C' > 0$, independently of ℓ.

Having excluded these cases, we can now assume that

$$\sup_{\partial B_{\epsilon_j^{(\ell)}}(x_j)} \rho(x)^{-\mu}|w_\ell| \longrightarrow 0$$

for every $j \leq J$.

The norms $||w_\ell||_{2,\alpha,\mu}$ (over all of Ω) are bounded. This is clear in the region $\Omega_{\bar{\epsilon}^{(\ell)}}$ by (25), and by Lemma 8 it is also true for the balls with index $j > J$. For the balls with index $j \leq J$, note that the rescaled functions $\bar{w}_\ell = (\epsilon_j^{(\ell)})^{-\mu} w_\ell(\epsilon_j^{(\ell)} x + x_j)$ on $B_1(0)$ have the same weighted norm there as the unrescaled functions do over $B_{\epsilon_j^{(\ell)}}(x_j)$. Since the supremum of $\bar{w}_\ell$ over $\partial B_1(0)$ is bounded by $(\epsilon_j^{(\ell)})^{-\mu}$, which tends to zero as $\ell \to \infty$ (and in particular, is bounded), we can apply Lemma 8 again to conclude boundedness of the weighted norms of w_ℓ in these balls too.

Now (pass to an appropriate subsequence and) let ℓ tend to infinity. The w_ℓ converge to a nonzero limit w_∞, since they have supremum equal to one on $\partial B_{\epsilon_j^{(\ell)}}(x_j)$ for some $j > J$, i.e., on a sphere which has radius bounded away from zero. This limit is an element of $\mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(\Omega \setminus \Sigma)$, and solves $L_{\bar{\epsilon}'} w_\infty = 0$, where $\bar{\epsilon}'$ has $\epsilon'_j = 0$ for $j \leq J$. Hence the potential $p\bar{u}_{\bar{\epsilon}'}^{p-1}$ in $L_{\bar{\epsilon}'}$ is singular only at the points x_j for $j > J$. Since $w_\infty \in \mathcal{C}^{2,\alpha}_\mu$ locally near each x_j, and since $\mu > 2 - N$, it easy to check that w_∞ is a weak solution of $L_{\bar{\epsilon}'} w = 0$ in a full neighbourhood of the points x_j with $j \leq J$. Since the operator is smooth at these points, a standard removable singularities theorem shows that w_∞ is smooth except at the x_j, $j > J$. This means that w_∞ is a solution of this operator with singularities at some discrete set Σ' with strictly fewer elements than Σ. Now we may proceed by induction, the case $K = 1$ already having been treated by the proof above. This completes the proof when Σ is discrete.

The modifications necessary to handle the general case are rather minor. In fact, it is only necessary to modify the way in which the rescaling is done and invoking Proposition 4 at the appropriate place. Thus, starting the proof in the same way, if the functions $r^{-\mu}|w_\ell|$ attain their maximum at a point $z_\ell = (r_\ell, \theta_\ell, y_\ell)$ in Fermi coordinates around some Σ_i, then we use the rescalings

$$\bar{w}_\ell(r,\theta,y) = (\epsilon_j^{(\ell)})^\mu w_\ell(r/\epsilon_j^{(\ell)},\theta,(y-y_\ell)/\epsilon_j^{(\ell)}).$$

The rest of the proof is analogous to the previous case.

Remark 4. It is actually possible to prove that in the case of non-isolated singularities, the rescaled limiting function $\bar{w}_\infty$ is independent of y. This may be done by using some regularity results from [6] to show that the tangential oscillation of the initial sequence w_ℓ may be bounded independently of $\bar{\epsilon}$.

8. Uniform surjectivity of $L_{\bar{\epsilon}}$ on $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$

The second main step in the linear analysis is to show that there is some choice of right inverse $G_{\bar{\epsilon}}$ of $L_{\bar{\epsilon}}$ on $\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma)$ which has norm bounded independently of $\bar{\epsilon}$ provided each $\epsilon_j < \epsilon_0$. As we have indicated before, the subtlety here is that $G_{\bar{\epsilon}}$ is not unique, since $L_{\bar{\epsilon}}$ is not injective on $\mathcal{C}^{2,\alpha}_\nu$.

The usual choice for $G_{\bar{\epsilon}}$ is as the right inverse whose range equals the orthogonal complement of the nullspace of $L_{\bar{\epsilon}}$. Of course, since we are working in Hölder spaces, this orthogonal complement is meaningless. However, what amounts to the same thing is to require that the range of $G_{\bar{\epsilon}}$ lies in the range of a fixed adjoint $L^*_{\bar{\epsilon}}$ of $L_{\bar{\epsilon}}$. Again, this adjoint depends on some Hilbert space structure, but once we have chosen an appropriate one with respect to which the adjoint is taken, we can forget about it and simply use this adjoint.

By (11) and using the notation of §6.5, $\mathcal{C}^{2,\alpha}_\nu$ is contained in $\rho^{-\delta_\nu}L^2(\rho^{N-1}dr\,d\theta\,dy)$. In the following, the space L^2 will always be taken relative to the background Euclidean measure. We shall consider the spaces $\rho^\delta L^2$ and $\rho^{-\delta}L^2$ to be dual with respect to the natural pairing

$$\rho^\delta L^2 \times \rho^{-\delta}L^2 \ni (\phi,\psi) \longrightarrow \int \phi\psi.$$

Relative to this pairing, the adjoint of $L_{\bar{\epsilon}} : \rho^{-\delta_\nu}L^2 \to \rho^{-\delta_\nu-2}L^2$ is just $L_{\bar{\epsilon}} : \rho^{\delta_\nu+2}L^2 \to \rho^{\delta_\nu}L^2$. We have proved that the former of these maps is surjective and the latter one is injective. Using the fixed isomorphisms

$$\rho^{2\delta} : \rho^{-\delta}L^2 \longrightarrow \rho^\delta L^2,$$

we may identify this adjoint, $L^*_{\bar{\epsilon}}$, as

$$L^*_{\bar{\epsilon}} = \rho^{-2\delta_\nu}L_{\bar{\epsilon}}\rho^{2\delta_\nu} : \rho^{-\delta_\nu+2}L^2 \longrightarrow \rho^{-\delta_\nu}L^2.$$

It is not hard to see that Lemma 6 is valid for both $L_{\bar{\epsilon}}$ and $L^*_{\bar{\epsilon}}$.

Now form the fourth order operator

$$\mathcal{L}_{\bar{\epsilon}} = L_{\bar{\epsilon}} \circ L_{\bar{\epsilon}}^* = L_{\bar{\epsilon}}\rho^{-2\delta_\nu} L_{\bar{\epsilon}}\rho^{2\delta_\nu} : \rho^{-\delta_\nu+2}L^2 \longrightarrow \rho^{-\delta_\nu-2}L^2.$$

This map is an isomorphism. Hence there exists a bounded two-sided inverse

$$\mathcal{G}_{\bar{\epsilon}} : \rho^{-\delta_\nu-2}L^2 \longrightarrow \rho^{-\delta_\nu+2}L^2.$$

In particular, $\mathcal{L}_{\bar{\epsilon}}\mathcal{G}_{\bar{\epsilon}} = I$, which means that $G_{\bar{\epsilon}} = L_{\bar{\epsilon}}^*\mathcal{G}_{\bar{\epsilon}}$ is a right inverse to $L_{\bar{\epsilon}}$ which maps into the range of $L_{\bar{\epsilon}}^*$ as desired. Henceforth, $G_{\bar{\epsilon}}$ will always denote this particular right inverse.

The main goal of the work in [6] is to study inverses and generalized inverses for degenerate 'edge operators,' of which $L_{\bar{\epsilon}}$ and $\mathcal{L}_{\bar{\epsilon}}$ are particular examples. The methods are quite geometric, albeit microlocal in origin. One of the main results implies that $G_{\bar{\epsilon}}$ and $\mathcal{G}_{\bar{\epsilon}}$ have Schwartz kernels which are rather simple distributions on the space obtained by blowing up the polar coordinate compactification of $(\Omega \setminus \Sigma) \times (\Omega \setminus \Sigma)$ along a submanifold of the boundary. One advantage of having such a concrete description of these operators is that various mapping properties can be read off quite easily. In particular, from [6] it follows immediately that

$$\mathcal{G}_{\bar{\epsilon}} : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{4,\alpha}_{\nu+2,\mathcal{D}}(\Omega \setminus \Sigma)$$

and

$$G_{\bar{\epsilon}} : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$$

are bounded. In the former of these the subscript $\mathcal{D}$ denotes a more general set of elliptic boundary conditions for this fourth order operator, i.e., the domain is restricted to functions u which vanish along with $L_{\bar{\epsilon}}^* u$ at $\partial\Omega$.

We can now turn to the proof that the norm of $G_{\bar{\epsilon}}$ does not blow up as the components of $\bar{\epsilon}$ tend to zero. This uniform surjectivity is an immediate consequence of the following two results.

Proposition 6. *Let $\bar{\epsilon}$ be some K-tuple with all $\epsilon_j < \epsilon_0$. Then provided ϵ_0 is sufficiently small, there is no solution of the system of equations $L_{\bar{\epsilon}}u = 0$, $u = L_{\bar{\epsilon}}^* v$ with $u \in \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$ and $v \in \mathcal{C}^{4,\alpha}_{\nu+2,\mathcal{D}}(\Omega \setminus \Sigma)$.*

Proof. This is really just a corollary of Theorem 5. In fact, suppose that u and v satisfy this system. Then $L_{\bar{\epsilon}}L_{\bar{\epsilon}}^* v = 0$. Recalling that $L_{\bar{\epsilon}}^*$ is identified with $\rho^{-2\delta_\nu}L_{\bar{\epsilon}}\rho^{2\delta_\nu}$ here, multiply this equation by $\rho^{2\delta_\nu}v = w$ and integrate with respect to standard Euclidean measure. Then the integrations by parts in

$$\int w\, L_{\bar{\epsilon}}\rho^{-2\delta_\nu}L_{\bar{\epsilon}}w = \int |L_{\bar{\epsilon}}w|^2\rho^{-2\delta_\nu} = 0$$

are valid because $\partial_r^j w \in C^{4-j,\alpha}_{\nu+2\delta_\nu+2-j}$ for $j \leq 4$, and also because w satisfies Dirichlet conditions on $\partial\Omega$. (Here r is the Fermi polar distance coordinate near each Σ_i. Also note that we can choose ρ to be constant near $\partial\Omega$ so that $\rho^{2\delta_\nu} v$ and $L^*_{\bar{\epsilon}}(\rho^{2\delta_\nu} v)$ still vanish at the boundary.) Hence $L_{\bar{\epsilon}} w = 0$. But since $w \in C^{2,\alpha}_{\mu'}$ for some $\mu' > \Re(\gamma_0^\pm)$, we may conclude from Theorem 5 that $w = 0$ provided ϵ_0 is small enough.

Theorem 6. *Let $\bar{\epsilon}^{(\ell)}$ be any sequence of K-tuples, with each $\epsilon_i^{(\ell)} \leq \epsilon_0$. Let $f_{\bar{\epsilon}^{(\ell)}}$ be any sequence of functions in $C^{0,\alpha}_{\nu-2}(\Omega\backslash\Sigma)$ with norm uniformly bounded as ℓ tends to infinity. Let $u_{\bar{\epsilon}^{(\ell)}} \in C^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$ be the unique solution of $L_{\bar{\epsilon}} u_{\bar{\epsilon}^{(\ell)}} = f_{\bar{\epsilon}^{(\ell)}}$ which also satisfies $u_{\bar{\epsilon}^{(\ell)}} \in ran\ (L^*_{\bar{\epsilon}^{(\ell)}})$, i.e., $u_{\bar{\epsilon}^{(\ell)}} = L^*_{\bar{\epsilon}^{(\ell)}} v_{\bar{\epsilon}^{(\ell)}}$ for some $v_{\bar{\epsilon}^{(\ell)}} \in C^{4,\alpha}_{\nu+2,\mathcal{D}}$. Then the norm of $u_{\bar{\epsilon}^{(\ell)}}$ in $C^{2,\alpha}_\nu$ is bounded uniformly as $\ell \to \infty$.*

Proof. This argument is again by contradiction, and is very similar to the ones in the last section. Clearly we only need to consider the case where some subset of the $\epsilon_j^{(\ell)}$ tend to zero. As before, we assume that $\epsilon_j^{(\ell)} \to 0$ for $j = 1, \dots, J$ and $\epsilon_j^{(\ell)} \geq c > 0$ for $j = J+1, \dots, K$. To start, by hypothesis

$$\sup_\Omega \rho^{-\nu+2} |f_{\bar{\epsilon}^{(\ell)}}| \leq C,$$

for all ℓ. Applying Lemma 7, we get

$$\sup_{\Omega\backslash\cup B_{\epsilon_i^{(\ell)}}(x_i)} \rho^{-\nu} |u_{\bar{\epsilon}^{(\ell)}}| \leq C + C \sup_{\cup\partial B_{\epsilon_i^{(\ell)}}(x_i)} \rho^{-\nu} |u_{\bar{\epsilon}^{(\ell)}}|.$$

Now define

$$A_{\bar{\epsilon}} = \sup_{\cup B_{\epsilon_i^{(\ell)}}} \rho^{-\nu} |u_{\bar{\epsilon}^{(\ell)}}|.$$

We resize the functions $u_{\bar{\epsilon}^{(\ell)}}$, $v_{\bar{\epsilon}^{(\ell)}}$, $f_{\bar{\epsilon}^{(\ell)}}$ by setting

$$\tilde{u}_{\bar{\epsilon}^{(\ell)}} = u_{\bar{\epsilon}^{(\ell)}} / A_{\bar{\epsilon}}, \quad \tilde{v}_{\bar{\epsilon}^{(\ell)}} = v_{\bar{\epsilon}^{(\ell)}} / A_{\bar{\epsilon}}, \quad \tilde{f}_{\bar{\epsilon}^{(\ell)}} = f_{\bar{\epsilon}^{(\ell)}} / A_{\bar{\epsilon}},$$

so that $L_{\bar{\epsilon}^{(\ell)}} \tilde{u}_{\bar{\epsilon}^{(\ell)}} = \tilde{f}_{\bar{\epsilon}^{(\ell)}}$ and $L^*_{\bar{\epsilon}^{(\ell)}} \tilde{v}_{\bar{\epsilon}^{(\ell)}} = \tilde{u}_{\bar{\epsilon}^{(\ell)}}$. If $A_{\bar{\epsilon}}$ stays bounded as ℓ tends to infinity, then we are finished. If not, as we now assume, then $||\tilde{f}_{\bar{\epsilon}^{(\ell)}}||_{0,\alpha,\nu-2}$ tends to zero.

We wish to take a limit of the equations $L_{\bar{\epsilon}^{(\ell)}} \tilde{u}_{\bar{\epsilon}^{(\ell)}} = \tilde{f}_{\bar{\epsilon}^{(\ell)}}$ and $L^*_{\bar{\epsilon}^{(\ell)}} \tilde{v}_{\bar{\epsilon}^{(\ell)}} = \tilde{u}_{\bar{\epsilon}^{(\ell)}}$. The main point is to show that $\tilde{u}_{\bar{\epsilon}^{(\ell)}}$ and $\tilde{v}_{\bar{\epsilon}^{(\ell)}}$ tend to limits. For each ℓ, choose a point z_ℓ where $\rho^{-\nu} \tilde{u}_{\bar{\epsilon}^{(\ell)}}$ attains its maximum. By passing to a subsequence, we can assume that z_ℓ stays in some fixed $B_{\bar{\epsilon}_i^{(\ell)}}(\Sigma_i)$.

The simpler case is when $i > J$, so that ϵ_i does not tend to zero. Since by the above and Lemma 8 the supremum of $\rho^{-\nu} |\tilde{u}_{\bar{\epsilon}^{(\ell)}}|$ in $B_{\bar{\epsilon}_i^{(\ell)}}(\Sigma_i)$ cannot diverge and stays bounded away from zero. Then we can directly pass to a limit as ℓ tends to infinity and obtain a function $u \in C^{2,\alpha}_{\nu,\mathcal{D}}(\Omega\backslash\Sigma)$

which is nonvanishing and satisfies $L_{\bar{\kappa}} u = 0$. Here $\bar{\kappa}$ is the $(K-J)$-tuple comprised of the limiting values of the $\epsilon_j^{(\ell)}$ for $j > J$. Since u blows up no faster than ρ^ν at the Σ_j with $j \leq J$, and since the limiting term of order zero in the operator is smooth at these submanifolds, the same removable singularities theorem as we used before shows that u must be smooth at those points. Next, since the norm of $\tilde{v}_{\bar{\epsilon}^{(\ell)}}$ is uniformly bounded, we can also pass to a (weak) limit and obtain some function $v \in \mathcal{C}^{4,\alpha}_{\nu+2,\mathcal{D}}(\Omega \setminus \Sigma)$ with $L^*_{\bar{\kappa}} v = u$. Since u is nonvanishing, v must also be. In addition, by elliptic regularity, v is smooth at the Σ_j, $j \leq J$. However, this is a contradiction to Proposition 6. Thus we may assume that for the index i for which the maximum of $\rho^{-\nu}|\tilde{u}_{\bar{\epsilon}^{(\ell)}}|$ is attained in $B_{\bar{\epsilon}_i^{(\ell)}}(\Sigma_i)$, the radius $\epsilon_i^{(\ell)}$ tends to zero.

Let us first treat the case where Σ_i is a point. Translate the whole problem so that $\Sigma_i = \{0\}$. Next, rescale the two equations by the factor $\bar{\epsilon}_i^{(\ell)}$; in particular, $\tilde{u}_{\bar{\epsilon}^{(\ell)}}$ is replaced by

$$\bar{u}_{\bar{\epsilon}^{(\ell)}} = (\bar{\epsilon}_i^{(\ell)})^{-\nu} \tilde{u}_{\bar{\epsilon}^{(\ell)}}(\bar{\epsilon}_i^{(\ell)} x),$$

and similarly for $\tilde{v}_{\bar{\epsilon}^{(\ell)}}$ and $\tilde{f}_{\bar{\epsilon}^{(\ell)}}$. For convenience, replace $\bar{\epsilon}_i^{(\ell)}$ by ϵ. Then in the ball $B_{\sigma/\epsilon}(0)$ we have $\bar{f}_{\bar{\epsilon}^{(\ell)}}$ tending to zero in $\mathcal{C}^{0,\alpha}_{\nu-2}$, and $\bar{u}_{\bar{\epsilon}^{(\ell)}}$ having norm in $\mathcal{C}^{2,\alpha}_{\nu}$ bounded uniformly by one. Replace the rescaling of the function ρ now by the polar variable r in this expanding sequence of balls. In the new rescaled coordinates, let the supremum of $r^{-\nu}|\bar{u}_{\bar{\epsilon}^{(\ell)}}|$ be attained at some point, which we still call x_ℓ. Then $|x_\ell|$ is bounded. By the argument used in the previous section, we see that $|x_\ell|$ must also stay bounded away from zero, otherwise we could rescale by the factor $|x_\ell|^{-1}$ and arrive at a contradiction, as in that section. Thus, there is some uniform lower bound $0 < C \leq |x_\ell| \leq 1$, and hence we can pass to a limit of the $\bar{u}_{\bar{\epsilon}^{(\ell)}}$ in $\mathcal{C}^{2,\alpha}_{\nu}$. Let us call the limiting function $\bar{u}$. Then

$$L_1 \bar{u} = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$

We wish to show that $\bar{v}_{\bar{\epsilon}^{(\ell)}}$ also tends to a limit. To see this, first note, that by the maximum principle again,

$$\sup_{B_{\sigma/\epsilon}(0)\setminus B_1(0)} r^{-\nu}|\bar{v}_{\bar{\epsilon}^{(\ell)}}| \leq C + C \sup_{\partial B_1(0)} |\bar{v}_{\bar{\epsilon}^{(\ell)}}|.$$

Define

$$A'_\epsilon = \sup_{B_1(0)} r^{-\nu}|\bar{v}_{\bar{\epsilon}^{(\ell)}}|.$$

If A'_ϵ stays bounded as $\epsilon \to 0$, we can take a limit of the $\bar{v}_{\bar{\epsilon}^{(\ell)}}$ to get a function $v \in \mathcal{C}^{4,\alpha}_{\nu+2}(\mathbb{R}^N \setminus \{0\})$ such that $L^*_1 v = u$ there.

So, suppose not, i.e., suppose A'_ϵ tends to infinity. We resize all the functions once again, by letting $u'_{\bar{\epsilon}^{(\ell)}} = \bar{u}_{\bar{\epsilon}^{(\ell)}}/A'_\epsilon$, and so on. Then

$L_1 u'_{\bar\epsilon(\ell)} = f'_{\bar\epsilon(\ell)}$ and $L_1^* v'_{\bar\epsilon(\ell)} = u'_{\bar\epsilon(\ell)}$. By construction, the norm of $f'_{\bar\epsilon(\ell)}$ tends to zero in $\mathcal{C}^{0,\alpha}_{\nu-2}$ and the norm of $u'_{\bar\epsilon(\ell)}$ tends to zero in $\mathcal{C}^{2,\alpha}_{\nu}$, but the norm of $v'_{\bar\epsilon(\ell)}$ in $\mathcal{C}^{4,\alpha}_{\nu+2}$ stays bounded. If the supremum of $r^{-\nu-2}|v'_{\bar\epsilon(\ell)}|$ occurs at some point x_ℓ, then exactly the same arguments as above show that $|x_\ell|$ stays bounded away from 0 and ∞. Hence once again we could pass to a limit to get a solution $V \in \mathcal{C}^{4,\alpha}_{\nu+2}(\mathbb{R}^N \setminus \{0\})$ such that $L_1^* V = 0$. But we know that L_1^* has no elements in its nullspace which decay at infinity, which implies that $V = 0$, a contradiction.

Finally, then, we have arrived at the situation above, that there exist $u \in \mathcal{C}^{2,\alpha}_{\nu}(\mathbb{R}^N \setminus \{0\})$ and $v \in \mathcal{C}^{4,\alpha}_{\nu+2}(\mathbb{R}^N \setminus \{0\})$ such that $L_1 u = 0$, $L_1^* v = u$. Combining these two equations, we get $L_1 L_1^* v = 0$. But since v decays at infinity, we can multiply this equation by v and integrate. Integration by parts now shows that $L_1^* v = 0$, which we have already observed implies that $v = 0$, hence $u = 0$ as well. This is the final contradiction. The only alternative is that the $\mathcal{C}^{2,\alpha}_{\nu}$ norm of $u_{\bar\epsilon(\ell)}$ can not blow up.

The case when some Σ_i are of positive dimension is treated very similarly. In fact, only the rescaling needs to be done slightly differently, but in the same manner as at the end of the proof of Theorem 5.

9. The fixed point argument

We are now in a position to complete most of the proofs of Theorem 1 and Theorem 2; what will remain after this section is the assertions about the moduli of solutions. We need to find a solution $w \in \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega\setminus\Sigma)$ to the equation

$$L_{\bar\epsilon} v + Q(v) + f_{\bar\epsilon} = 0 \tag{28}$$

where $f_{\bar\epsilon}$ is the error term introduced in §4, either for the case when Σ is discrete, or in the more general case, and Q is the quadratic remainder (15). We do this by the standard contraction mapping argument. We will define a continuous operator $\mathcal{K}$ from the space $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega\setminus\Sigma)$ into itself and then prove that this map is a contraction on some small ball in this space.

If v is a solution of (26), then $\bar u_{\bar\epsilon} + v$ is a weak solution of

$$\begin{cases} -\Delta(\bar u_{\bar\epsilon} + v) = |\bar u_{\bar\epsilon} + v|^p & \text{in} \quad \Omega \\ \bar u_{\bar\epsilon} + v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{29}$$

Assuming a solution exists, let us show that $\bar u_{\bar\epsilon} + v$ is positive in $\Omega \setminus \Sigma$. On the one hand, for x near Σ_i, there exists some $R > 0$ such that if

$\rho(x) \le R\epsilon_i$ then

$$c_1 \rho(x)^{-\frac{2}{p-1}} \le \bar{u}_{\bar{\epsilon}}(x) \le c_2 \rho(x)^{-\frac{2}{p-1}} \tag{30}$$

On the other hand, since $v \in \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma)$, we have $v(x) \le c\rho(x)^{\nu}$. But since $\nu > -2/(p-1)$, it follows that $\bar{u}_{\bar{\epsilon}} + v > 0$ near each Σ_i; by the maximum principle we see that $\bar{u}_{\bar{\epsilon}} + v > 0$ in all of $\Omega \setminus \Sigma$, and hence $\bar{u}_{\bar{\epsilon}} + w$ is a positive solution of (1) which is singular at all points of Σ.

It remains to prove the existence of a solution to (26). We shall first treat the case where Σ is finite. First observe that $||f_{\bar{\epsilon}}||_{0,\alpha,\nu-2} \le C\epsilon_0^{N-\frac{2p}{p-1}}$ by Lemma 3. Let A denote a common upper bound for the norm of $G_{\bar{\epsilon}}$ for ϵ_0 sufficiently small. Then

$$||G_{\bar{\epsilon}} f_{\bar{\epsilon}}||_{2,\alpha,\nu} \le AC\epsilon_0^{N-\frac{2p}{p-1}}. \tag{31}$$

In view of this estimate, we shall work in the ball

$$\mathcal{B}(\epsilon_0, \beta) \equiv \{v \in \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma) : ||v||_{2,\alpha,\nu} \le \beta\epsilon_0^{N-\frac{2p}{p-1}}\}.$$

We have already shown that $G_{\bar{\epsilon}}(f_{\bar{\epsilon}}) \in \mathcal{B}(\epsilon_0, \beta)$, for β large enough. At this point we shall fix β large enough so that $G_{\bar{\epsilon}} f_{\bar{\epsilon}} \in \mathcal{B}(\epsilon_0, \beta/2)$. We shall now make a further restriction on the ϵ_i, namely that, for some fixed constant $a \in (0,1)$, we have

$$a\epsilon_0 \le \epsilon_i \le \epsilon_0 \tag{32}$$

for $i = 1, \dots, K$. This new restriction appears to be needed in the proof of the following Lemma.

Lemma 9. *There exists some constant $c > 0$ independent of $\epsilon_0 \ll 1$ such that*

$$||Q(v_2) - Q(v_1)||_{0,\alpha,\nu-2} \le \frac{1}{2A} ||v_2 - v_1||_{2,\alpha,\nu},$$

for all $v_1, v_2 \in \mathcal{B}(\epsilon_0, \beta)$. In particular, taking $v_2 = 0$ we see that $G_{\bar{\epsilon}} Q(v) \in \mathcal{B}(\epsilon_0, \beta/2)$ and hence that the operator $\mathcal{K}$ defined by $\mathcal{K}(v) = -G_{\bar{\epsilon}}(Q(v) + f_{\bar{\epsilon}})$ maps $\mathcal{B}(\epsilon_0, \beta)$ to itself, and is a contraction on this ball for ϵ_0 sufficiently small.

Proof. We first establish that there exists some $\tau > 0$, independent of $\epsilon_0 \ll 1$, such that for any $v \in \mathcal{B}(\epsilon_0, \beta)$ we have

$$x \in \cup_{i=1}^{K} B(x_i, \tau) \Longrightarrow |v(x)| \le \frac{1}{4}\bar{u}_{\bar{\epsilon}}(x).$$

Indeed, by Lemma 1, we know that there exist constants $c_1, c_2 > 0$ and a radius $R > 0$ such that

$$c_1 |x|^{-\frac{2}{p-1}} \le u_{\epsilon_i}(x) \le c_2 |x|^{-\frac{2}{p-1}} \quad \text{if} \quad |x| \le R\epsilon_i,$$

$$c_1\epsilon_i^{N-\frac{2p}{p-1}}|x|^{2-N} \le u_{\epsilon_i}(x) \le c_2\epsilon_i^{N-\frac{2p}{p-1}}|x|^{2-N} \quad \text{if} \quad R\epsilon_i \le |x| \le \tau.$$

The claim follows at once from these estimates and the fact that $v \in \mathcal{B}(\epsilon_0, \beta)$, so that

$$|v(x)| \le \beta\epsilon_0^{N-\frac{2p}{p-1}}\rho^\nu(x) \le c\beta\epsilon_i^{N-\frac{2p}{p-1}}\rho^\nu(x)$$

by (30). We also note that

$$\bar{u}_{\bar{\epsilon}}(x) \le c\rho(x)^{-\frac{2}{p-1}}$$

for all $x \in \Omega$.

Since $|v/\bar{u}_{\bar{\epsilon}}| \le 1/4$ in each $B(x_i, \tau)$, we may use a Taylor expansion to obtain

$$|Q(v_2) - Q(v_1)|(x) \le c|\bar{u}_{\bar{\epsilon}}|^{p-2}(x)(|v_1|(x) + |v_2|(x))|v_2 - v_1|(x).$$

So for $x \in B(x_i, \tau)$ we have

$$\begin{aligned}
&\rho(x)^{2-\nu}|Q(v_2) - Q(v_1)|(x) \\
&\quad \le c\rho(x)^{2-\nu-2\frac{p-2}{p-1}+2\nu}(||v_1||_{2,\alpha,\nu} + ||v_2||_{2,\alpha,\nu})||v_2 - v_1||_{2,\alpha,\nu} \\
&\quad = c\rho(x)^{\nu+\frac{2}{p-1}}(||v_1||_{2,\alpha,\nu} + ||v_2||_{2,\alpha,\nu})||v_2 - v_1||_{2,\alpha,\nu} \\
&\quad \le c\tau^{\nu+\frac{2}{p-1}}\beta\epsilon_0^{N-\frac{2}{p-1}}||v_2 - v_1||_{2,\alpha,\nu}.
\end{aligned}$$

The coefficient here may be taken as small as desired by choosing ϵ_0 sufficiently small. Outside the union of these balls we use the estimates

$$\bar{u}_{\bar{\epsilon}}(x) \le c\epsilon_0^{N-\frac{2p}{p-1}} \quad \text{and also} \quad |v(x)| \le c\epsilon_0^{N-\frac{2p}{p-1}},$$

where the constant $c > 0$ depends on τ but not on v. For $\rho(x) \ge \tau$ we can neglect all factors involving $\rho(x)$, hence for all $x \in \Omega_\tau$ we have

$$\begin{aligned}
\rho(x)^{2-\nu}|Q(v_2) - Q(v_1)|(x) &\le C(\bar{u}_{\bar{\epsilon}}^{p-1} + |v|^{p-1})|v_2 - v_1| \\
&\le C\epsilon_0^{(p-1)(N-\frac{2p}{p-1})}|v_2 - v_1| \\
&\le \frac{\beta}{BA}\epsilon_0^{(N-2)p-N}||v_2 - v_1||_{2,\alpha,\nu}
\end{aligned}$$

for any constant $B > 0$, provided ϵ_0 is chosen small enough.

We also need to estimate the Hölder norm of $Q(v_1) - Q(v_2)$. By the fourth part of Lemma 1, it suffices to estimate $\rho^{3-\nu}|\nabla(Q(v_1) - Q(v_2))|$. We shall only sketch this briefly. First, in each $B(x_i, \tau)$ we compute that

$$\nabla Q(v) = p\left((\bar{u}_{\bar{\epsilon}} + v)^{p-1} - \bar{u}_{\bar{\epsilon}}^{p-1} - (p-1)\bar{u}_{\bar{\epsilon}}^{p-1} v\right)(\nabla \bar{u}_{\bar{\epsilon}}) \\ + p\left((\bar{u}_{\bar{\epsilon}} + v)^{p-1} - \bar{u}_{\bar{\epsilon}}^{p-1})(\nabla v)\right).$$

For $x \in \Omega_\tau$ there is a similar expression, except that we must be more careful about the absolute value signs. However, in Ω_τ we estimate each term individually and conclude that

$$\rho^{3-\nu}|\nabla Q(v_1) - \nabla Q(v_2)| \leq C\epsilon_0^{(N-2)p-N-1}||v_2 - v_1||_{2,\alpha,\nu}$$

there. As before, this provides the proper estimate in this set. In $B(x_i, \tau)$ we use a Taylor expansion in the expression above to conclude the proper estimate. This completes the proof of the Lemma when Σ is discrete.

To carry out the proof in the more general case, only fairly minor changes need to be made in the argument above. The most important one, already observed in §4, is that the weight parameter ν must now be constrained to lie in the smaller interval

$$(33) \qquad \frac{-2}{p-1} < \nu < \min\{\frac{-2}{p-1} + 1 = \frac{p-3}{p-1}, \Re(\gamma_0^-)\}.$$

Furthermore, the estimate in Lemma 4 is slightly weaker than that in Lemma 3, hence we must replace the exponent $N - \frac{2p}{p-1}$ in the above argument by

$$q = \frac{p-3}{p-1} - \nu.$$

Thus we show that $\mathcal{K}$ is a contraction of the balls

$$\mathcal{B}(\epsilon_0, \beta) = \{v \in \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma) : ||v||_{2,\alpha,\nu} \leq \beta\epsilon_0^q\}.$$

The details of the argument need very little change since the nonlinear term $Q(v)$ is the same as before. This completes the proof in all cases.

Using this Lemma and the remarks preceding it, we may apply the standard fixed point argument for contraction mappings to see that there exists a unique solution v of the equation

$$v = -G_{\bar{\epsilon}}\left((|v + \bar{u}_{\bar{\epsilon}}|^p - \bar{u}_{\bar{\epsilon}}^p - p\bar{u}_{\bar{\epsilon}}^{p-1} w) + f_{\bar{\epsilon}}\right)$$

in $\mathcal{B}(\epsilon_0, \beta)$ for every ϵ_0 small enough, and for every K-tuple $\bar{\epsilon}$ satisfying (30). Fixing some ϵ_0 which is small enough for this argument to work, and replacing ϵ_0 by $\epsilon \leq \epsilon_0$ in this whole argument shows that we can find a solution which is a small perturbation of $\bar{u}_{\bar{\epsilon}}$ for every ϵ and $\bar{u}_{\bar{\epsilon}}$ for which $a\epsilon \leq \epsilon_i \leq \epsilon \leq \epsilon_0$.

10. The proofs completed

We have now completed the proof of the existence of solutions for Theorems 1 and 2. In this section we indicate the arguments leading to the determination of the deformation space of solutions for either problem. We also indicate the small changes needed to complete the proof of Theorem 3.

10.1. The deformation spaces

Included in the statements of Theorems 1 and 2 are the assertions that the space of all solutions to the equation (1) on a domain $\Omega \subset \mathbb{R}^N$ with isolated singularities at $\{x_1, \dots, x_K\} \subset \Omega$ is K dimensional, and in general that if any component Σ_i of the singular set is positive dimensional, then there is an infinite dimensional family of solutions with the same singular set. We recover both of these statements using the implicit function theorem. The latter case is analogous to the one studied in [8], and the former mirrors the situation in [7].

In both cases, the main point is that if u is the positive solution to (1) constructed by the procedure of this paper, then the linearization $L = \Delta + pu^{p-1}$ to the equation at u is nondegenerate in the sense that it is surjective as a map from $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$ to $\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma)$. This means, using the implicit function theorem, that all solutions to the equation which can be obtained as perturbations off of u by terms growing no faster than ρ^ν are parametrized locally by elements of the nullspace of this linearization in $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}$. We shall prove this surjectivity in the next proposition, or rather, we will prove the dual statement as in §6 that the linearization is injective on $\mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}$.

Proposition 7. *If $u = \bar{u}_{\bar{\epsilon}} + v$ is a solution of (1), as constructed in the last section, with all $\epsilon_i \in (a\epsilon, \epsilon)$, and $v \in \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma) \cap \mathcal{B}(\epsilon, \beta)$, then if ϵ is sufficiently small, the linearization $L_u = \Delta + pu^{p-1}$ is injective as a map on $\mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(\Omega \setminus \Sigma)$.*

Proof. Suppose not, i.e., suppose there exists some sequence $\bar{\epsilon}^{(\ell)}$ with $\epsilon_i^{(\ell)} \in (a\epsilon^{(\ell)}, \epsilon^{(\ell)})$ for some sequence $\epsilon^{(\ell)} \to 0$, and a solution $u_{(\ell)} = \bar{u}_{\bar{\epsilon}^{(\ell)}} + v$ such that the corresponding linearization $L_{u_{(\ell)}}$ is not injective. Thus there exists some $\phi^{(\ell)} \in \mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}$ such that $L_{u_{(\ell)}}\phi^{(\ell)} = 0$. We make exactly the same rescalings as in the proof of Theorem 5, to arrive at the same contradiction. (Note, however, that since the $\epsilon_i^{(\ell)}$ must all tend to zero simultaneously, the proof simplifies somewhat.) The only point that needs checking is that the operator obtained in the limit of these rescalings is just L_1 or $\mathbb{L}_1$, according as whether Σ is discrete or not. The rescaling sending $u_\epsilon(r)$ to $u_1(r)$ is $u_1(r) = \epsilon^{2/(p-1)}u_\epsilon(\epsilon r)$. Applying this same rescaling to $v(r)$ (suitably translated so that the origin corresponds to the appropriate point of Σ) yields the function

$v_\epsilon(r) = \epsilon^{2/(p-1)}v(\epsilon r)$. Since $|v(r)| \le \beta\epsilon^q\rho^\nu$ for some positive exponent q, we see that $|v_\epsilon(r)| \le \beta\epsilon^{q+\nu+\frac{2}{p-1}}\rho^\nu$ in some ball $B_{\sigma/\epsilon}$. Since $\nu + \frac{2}{p-1} > 0$ and $q > 0$, this term tends to zero with ϵ. Hence the limiting operator is just $\Delta + pu_1^{p-1}$, as desired. As shown in §7.2, this leads to a contradiction, so the proposition is proved.

The next step is to compute the size of the nullspace of L_u on $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$. We have already indicated that when some Σ_i is positive dimensional this nullspace is infinite dimensional, and hence there is an infinite dimensional space of solutions to the nonlinear problem. But when Σ is a finite set, then L is Fredholm and the nullspace is finite dimensional. We may calculate its dimension using a relative index theorem.

Proposition 8. *If Σ is discrete, the nullspace of L_u is K dimensional. If any Σ_i has positive dimension, then the nullspace is infinite dimensional.*

Proof. The latter statement, that the nullspace is infinite dimensional when some $\dim \Sigma_i > 0$ follows from the theory of [6]. In fact, this nullspace is parametrized by functions (of a certain negative distributional order!) along each of positive dimensional components of Σ. This infinite dimensionality can also be seen as a consequence of the fact that the globalized linearization $\mathbb{L}_1$ from §6.4 has infinite dimensional nullspace, as was pointed out in Proposition 5.

That the nullspace of L_u on $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}$ is K dimensional when Σ is discrete is proved in exactly the same manner as the analogous fact in [7]. We sketch this argument briefly here. Once again, L^2 techniques are used. Let $\operatorname{ind}(\delta)$ denote the index of L_u on $\rho^\delta L^2$. From the self-adjointness of L_u on $L^2(\Omega \setminus \Sigma)$ (with respect to Euclidean measure), we find that $\operatorname{ind}(-\delta) = -\operatorname{ind}(\delta)$, for any $\delta \notin \{\delta_j^\pm\}$. Although the index itself requires global information to calculate, it is well known that the 'relative index,' i.e., the difference $\operatorname{ind}(\delta') - \operatorname{ind}(\delta'')$ depends only on asymptotic data at the points of Σ. Using the relative index theorem proved by Melrose, we see that the relative index $\operatorname{ind}(-\delta) - \operatorname{ind}(\delta)$ equals $2K$ (the details are written out carefully in [7]). Combining these two facts, we find that $\operatorname{ind}(-\delta) = K$. However, since we have proved that L_u has no cokernel as a map from $\mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}$, we conclude that the nullspace of L_u is K dimensional.

Using these facts, and a standard implicit function theorem argument as in [7], we conclude the following result:

Theorem 7. *When Σ is a discrete set, then the solution space to the equation (1) is locally a K dimensional real analytic variety. The solutions constructed in this paper lie in the smooth set of this variety.*

The fact that this solution space is a real analytic variety (possibly with singularities) may be deduced using the Ljapunov-Schmidt reduction argument, as in [4]. But since we have produced smooth points in this variety, having found solutions u for which the corresponding linearization L_u is surjective, we conclude that the top stratum of this variety is K dimensional, hence that almost every solution is nondegenerate in this sense.

Finally, we may sharpen the deformation result when Σ is not necessarily discrete, applying the implicit function argument from [8] to conclude

Theorem 8. *Let $\Sigma \subset \Omega$ be any union of $C^{3,\alpha}$ submanifolds Σ_i of dimensions k_i satisfying the restrictions of Theorem 2. Then the equation (1) has an infinite dimensional family of solutions.*

10.2. The singular Yamabe problem on manifolds of positive scalar curvature

The modifications in the arguments of this paper required to solve the singular Yamabe problem on an arbitary compact manifold (M, g_0), where $R(g_0) \geq 0$, with singular set prescribed on an arbitrary finite disjoint union of smooth submanifolds Σ_i of dimensions greater than zero and less than (or equal to) $(n-2)/2$ are very minor. The equation that now must be solved is

$$\Delta_{g_0} v - \frac{n-2}{4(n-1)} R(g_0) v + \frac{n-2}{4(n-1)} v^{\frac{n+2}{n-2}} = 0,$$

$$v > 0 \text{ on } M \setminus \Sigma, \qquad \operatorname{sing}(v) = \Sigma.$$

The operator in the first two terms here, i.e., the linear part, is called the conformal Laplacian L_{g_0} associated to the metric g_0. The linearization of this operator around the approximate solution $\bar{u}_{\bar{\epsilon}}$ is

$$\Delta - \frac{n-2}{4(n-1)} (R(g_0) - p\bar{u}_{\bar{\epsilon}}^{p-1}).$$

Now, the rescalings of this linearization may be effected in exactly the same ways in local Fermi coordinate systems around the submanifolds Σ_i. The extra term, of order zero, in this operator disappears in these rescalings; the resulting model operator we need to study is exactly the operator $\mathbb{L}_1$ from §6. Thus we prove, as before, that this linearization is uniformly surjective, provided all components of $\bar{\epsilon}$ are small enough. The fixed point argument shows that (26) has a solution v. The only point of the whole argument that needs special comment is to indicate where we use the assumption that the conformal class of the metric g_0 is positive. This is when we show that $\bar{u}_{\bar{\epsilon}} + v$ remains positive on all of $M \setminus \Sigma$; at this step we use that $\Delta - \frac{n-2}{4(n-1)} R(g_0)$ satisfies the maximum principle – and this is only true if $R(g_0)$ is nonnegative. This completes the proof of Theorem 3.

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