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GLOBAL ASYMPTOTIC LIMIT OF SOLUTIONS OF THE CAHN–HILLIARD EQUATION

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Abstract

We study the asymptotic limit, as $\varepsilon \searrow 0$, of solutions of the Cahn–Hilliard equation

$$u_t^\varepsilon = \Delta(-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon))$$

under the assumption that the initial energy

$$\int_\Omega \left[\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,0)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,0))\right]$$

is bounded independent of ε. Here $f = F'$, and F is a smooth function taking its global minimum 0 only at $u = \pm 1$. We show that there is a subsequence of $\{u^\varepsilon\}_{0<\varepsilon\le 1}$ converging to a weak solution of an appropriately defined limit Cahn–Hilliard problem. We also show that, in the case of radial symmetry, all the interfaces of the limit have multiplicity one for almost all time $t > 0$, regardless of initial energy distributions.

1. Introduction

In this paper, we shall study the asymptotic limit, as $\varepsilon \searrow 0$, of the solutions of the Cahn–Hilliard equation

$$(1.1)\qquad \begin{cases} u_t^\varepsilon(x,t) = \Delta v^\varepsilon(x,t), & (x,t)\in\Omega\times(0,\infty),\\ v^\varepsilon = -\varepsilon\Delta u^\varepsilon + \varepsilon^{-1}f(u^\varepsilon), & (x,t)\in\Omega\times[0,\infty),\\ \frac{\partial}{\partial n}u^\varepsilon = \frac{\partial}{\partial n}v^\varepsilon = 0, & (x,t)\in\partial\Omega\times[0,\infty),\\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x\in\Omega. \end{cases}$$

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Here Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, and $f(u)$ is the derivative of a potential F satisfying

$$(1.2) \qquad \begin{cases} \text{(a) } F \in C^3(\mathbb{R}),\ F(\pm 1) = 0, \text{ and } F(u) > 0 \\ \qquad\qquad\qquad \text{for all } u \neq \pm 1; \\ \text{(b) } F' = f \text{ and for some } p > 2 \text{ and } c_0 > 0, \\ \qquad f'(u) \geq c_0 |u|^{p-2} \text{ if } |u| \geq 1 - c_0. \end{cases}$$

For the initial data u_0^ε, we assume

$$(1.3) \qquad \begin{cases} \displaystyle\sup_{0<\varepsilon\leq 1} \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_0^\varepsilon(x)|^2 + \frac{1}{\varepsilon}F(u_0^\varepsilon(x))\Big)dx \leq \mathcal{E}_0 < \infty, \\ \displaystyle\frac{1}{|\Omega|}\int_\Omega u_0^\varepsilon(x) = m_0 \in (-1,1) \qquad \forall\, \varepsilon \in (0,1]. \end{cases}$$

Note that (1.1) differs from the usual Cahn–Hilliard equation (see [20]) only in the scaling of time so that t here represents t/ε in the usual formulation. Equation (1.1) is widely accepted as a good model to describe the complicated phase separation (in the original time scale) and coarsening (in our current time scale) phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. Here v^ε is the chemical potential, and u^ε is a scaled concentration where $u^\varepsilon = \pm 1$ represents the two stable concentrations. The parameter ε is the "interaction length" which is very small. The Neumann boundary conditions reflect the conservation of mass and insulation from the outside. For more physical background, derivation, and discussion of the Cahn–Hilliard equation and related equations, see [7], [8], [9], [18], [19], [20], [34], [39], [52] and the references therein.

The Cahn–Hilliard equation (1.1) is a mass preserved and a gradient flow with the energy functional

$$(1.4) \qquad \mathcal{E}^\varepsilon(t) := \int_\Omega e^\varepsilon(u^\varepsilon)dx, \quad e^\varepsilon(u^\varepsilon) := \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon).$$

In fact, one can direct verify the following identities: for all $t > 0$,

$$(1.5) \qquad \frac{d}{dt}\int_\Omega u^\varepsilon(\cdot,t) = 0, \quad \frac{d}{dt}\mathcal{E}^\varepsilon(t) = -\int_\Omega |\nabla v^\varepsilon(\cdot,t)|^2.$$

The evolution of the concentration undergoes two stages called phase separation and phase coarsening, respectively. During the first stage, the alloy becomes a fine–grained mixture of two different phases, each of which corresponds to a stable concentration configuration. This stage usually takes a relative short time during which the nucleation, spinodal decomposition, and formation of the phases can be observed. In terms of equation (1.1), the solution u^ε quickly approximates the value 1 in one region $\tilde{\Omega}_t^+$ and the value -1 in another region $\tilde{\Omega}_t^-$ whereas the remaining region $\tilde{\Gamma}_t \equiv \Omega \setminus (\tilde{\Omega}_t^+ \cup \tilde{\Omega}_t^-)$ is a thin region, usually considered as a hypersurface called *interface*. At the end of the first stage, one can formally show that the energy $\mathcal{E}^\varepsilon(t)$ defined in (1.4) is proportional to the total area of the interface.

When the phase regions are formed, the evolution of the concentration enters into the second stage during which the phase regions are coarsened, the originally fine–grained structure becomes less fine, and the geometric shapes of the phase regions become simpler and simpler, eventually tending to regions of minimum surface area. In terms of the Cahn–Hilliard equation (1.1), this phenomenon corresponds to the behavior of the solution that the interface moves and eventually tends to a surface having minimum surface area (whereas its enclosed region has a fixed volume).

It was formally derived by Pego [53] that, as $\varepsilon \searrow 0$, the function v^ε tends to a limit v, which, together with a free boundary

$$\Gamma \equiv \cup_{0 \le t \le T}(\Gamma_t \times \{t\}),$$

solves the following free boundary problem:

$$\left\{\begin{array}{lll} \Delta v & = 0 & \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T], \\ \frac{\partial}{\partial n} v & = 0 & \text{on } \partial\Omega,\ t \in [0,T] \\ v & = \sigma\kappa & \text{on } \Gamma_t,\ t \in [0,T], \\ \mathcal{V} & = \frac{1}{2}[\frac{\partial}{\partial n} v]_{\Gamma_t} & \text{on } \Gamma_t,\ t \in [0,T]. \end{array}\right. \tag{1.6}$$

Here

$$\sigma = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}}\, ds, \tag{1.7}$$

κ and $\mathcal{V}$ are, respectively, the mean curvature and the normal velocity of the interface Γ_t, n is the unit outward normal to either $\partial\Omega$ or Γ_t, $[\frac{\partial}{\partial n} v]_{\Gamma_t} = \frac{\partial}{\partial n} v^+ - \frac{\partial}{\partial n} v^-$ and v^+ and v^- are respectively the restriction of v on Ω_t^+ and Ω_t^-, the exterior and interior of Γ_t in Ω. Also $u^\varepsilon \to \pm 1$ in $\Omega_t^\pm$ for all $t \in [0,T]$. Under the assumption that (1.6) has a smooth classical solution, rigorous justification of this Pego's result was recently carried out by Alikakos, Bates and the author in [1], using asymptotic expansions and spectral analysis. Using energy methods, Stoth [61] recently obtained a global (in time) convergence result for the case of three-dimensional radial symmetry and Dirichlet boundary conditions.

The main purpose of this paper is to formulate a weak solution to the free boundary problem (1.6) and to show that the solutions of (1.1) approach, as $\varepsilon \searrow 0$, to weak solutions of (1.6).

For completeness, we continue to discuss the dynamics of (1.1) after the second stage, though it is not the concern of this paper. Notice that equilibria of (1.6) are either a single sphere or spheres of same radii lying in Ω or intersecting Ω orthogonally. In case when the equilibrium is a sphere or spheres of same radii lying in Ω, there may not be a corresponding equilibrium of the Cahn–Hilliard equation (1.1). In fact, Alikakos and Fusco [2, and the reference therein] showed that if the "interface" of the solution of (1.1) is close to a single sphere lying in Ω, then the interface will move superslowly (with a speed of order $O(e^{-\frac{c}{\varepsilon}})$) toward the closest point on $\partial\Omega$. (Bronsard and Hilhorst[11] proved a similar result in the one–dimensional case). They referred such kind of solutions as "bubbles". After a super long time, the bubble will touch the boundary $\partial\Omega$. At this moment, the bubble will quickly collapse to a "half" bubble orthogonally attached to $\partial\Omega$. We believe that this collapse process will take $O(\varepsilon)$ time (in the time scale as (1.1)), though its detailed dynamics is totally unclear. Once the half bubble is formed, it will move along the boundary finding a final destination which minimizes its surface area (in Ω). Here we say a final destination since it corresponds to a local minimizer of the energy functional $\mathcal{E}^\varepsilon(\cdot)$. For related results, see Kohn and Sternberg [47]. Here we would like to point out that the motion of a half bubble is again described by the free boundary problem (1.6) with the extra constraint that the interface Γ_t intersects $\partial\Omega$ orthogonally, though rigorous verification is still under way.

Problem (1.6) is often referred as the Mullins–Sekerka problem. In studying solidification/liquidation of materials of zero specific heat, Mullins and Sekerka [50] first studied the linear stability of a special radially symmetric solution of (1.6) in $\mathbb{R}^3$ and showed that the spherical shape (of the interface) is stable when the radius of the interface is small, and otherwise unstable.

Problem (1.6) is also called (two phase) Hele–Shaw problem since if one replaces v by a constant in one of the region enclosed by Γ_t, it becomes the (one phase) Hele–Shaw problem (with surface tension) arising from the study of the pressure of immiscible fluid in the air [35].

Concerning the existence of smooth solutions of the free boundary problem (1.6), recently the author [23] established the local (in time) existence of a solution in the two-dimensional case and, when the initial curve is nearly circular, the global existence and long time behavior of the solution. Very recently, Hong, Yi and the author [26] established the local existence of a unique smooth solution to (1.6) in any space dimension. We would like to mention that it was Duchon and Robert [33] who first established the local existence of the one phase Hele–Shaw problem in the setting that the initial curves are given by a graph $y = f(x) \in H^{5/2}(\mathbb{R}^1)$. In case f is sufficiently flat, they also established the global existence and long time behavior. An extension of their result to the case where the initial curves are small analytic perturbations of a circle was recently carried out by Constantine and Pugh [30].

Another gradient flow for the same energy functional $\mathcal{E}^\varepsilon(\cdot)$ in (1.4) is the Allen–Cahn equation

$$u_t^\varepsilon = \Delta u^\varepsilon - \varepsilon^{-2} f(u^\varepsilon),$$

which originally was introduced by Allen and Cahn [4] to describe the motion of antiphase boundaries. It was formally derived that, as $\varepsilon \searrow 0$, the zero level set of u^ε approaches a surface which moves with a normal velocity $\mathcal{V}$ equal to the mean curvature κ of the surface; see, Allen and Cahn [4], Fife[39], Rubinstein, Sternberg, and Keller[54]. Rigorous justification of this limit has been successfully carried out in recent years. The one-dimensional case was extensively examined by Fife & Hsiao [41], Carr & Pego [21], [22], Fusco & Hale [41], Fusco [43], Bronsard & Kohn [12], etc. The radial symmetric case was shown by Bronsard & Kohn [13] whereas the general case was proven by de Mottoni & Schatzman [31], [32], Chen [25], Chen & Elliott [27], Nochetto, Paolini, & Verdi [51] and others, under the assumption that classical solutions of $\mathcal{V} = \kappa$ exist. Finally, it was Evans, Soner & Souganidis [36] who first established a global result: for all time $t \geq 0$, the limit of the zero level set of the solution of the Allen–Cahn equation is contained in the generalized solution of the motion by mean curvature flow established in [29], [37], [57]. More recently, Ilmanen [46] showed that this limit is actually one of the Brakke's motion by mean curvature solution [10], which is a subset of the unique generalized solution of the mean curvature flow established in [29], [37]. More recently, Soner established more delicate result [58] with more general initial data [59]. Related results for area preserved

Allen–Cahn equation can be found in [14, and the reference therein], [28].

Though both the Allen–Cahn equation and the Cahn–Hilliard equation are gradient flows of the same energy functional, their dynamics are pretty different since the former does not preserve the mass. Many celebrated PDE tools such as the maximum and comparison principles can be used for the former but not for the latter, thereby causing intrinsic difficulties in studying the Cahn–Hilliard equation. Nevertheless, some of the tools such as the energy method [12], [13], [46], [57] and the varifold approach [46] are shared by both equations (up to certain degree).

Another dynamics related to the energy functional $\mathcal{E}^\varepsilon(\cdot)$ in (1.4) is the phase field system:

$$\varepsilon[\alpha^\varepsilon u_t^\varepsilon - \Delta u^\varepsilon] + \varepsilon^{-1} f(u^\varepsilon) = \sigma^\varepsilon \ell^\varepsilon T^\varepsilon, \qquad c^\varepsilon T_t^\varepsilon - \Delta T^\varepsilon = -\ell^\varepsilon u_t^\varepsilon,$$

which models the solidification process. Here α^ε, σ^ε, ℓ^ε, and c^ε are non–negative parameters, T^ε is the temperature and u^ε is a phase order parameter with $u^\varepsilon \sim -1$ and $u^\varepsilon \sim 1$ corresponding to solid and liquid phases respectively. Notice that if $\alpha^\varepsilon = c^\varepsilon = 0$ and $\sigma^\varepsilon = \ell^\varepsilon = 1$, the phase field system becomes the Cahn–Hilliard equation, and if $\alpha^\varepsilon = 1$ and $\sigma^\varepsilon = 0$, it becomes the Allen–Cahn equation. Convergence results for various situations of the non–negative parameters $\alpha^\varepsilon, \sigma^\varepsilon$, ℓ^ε, and c^ε were formally derived by Caginalp and others [15, and the references therein], [16]. All Caginalp's formal asymptotic limits (which include the Allen–Cahn and the Cahn–Hilliard limits) were recently rigorously verified, under the assumption that the corresponding limit problems have unique local smooth solutions, by Caginalp and the author [17], by using a method similar to that used in [1] and a spectral estimate in [24]. More recently, Soner[60] studied the global (in time) behavior of the phase field system for the case $\alpha^\varepsilon = \sigma^\varepsilon = c^\varepsilon = 1$ and $\ell^\varepsilon = \sqrt{F(u^\varepsilon)}$. He showed that a subsequence of $\{(u^\varepsilon, T^\varepsilon)\}_{\varepsilon \in (0,1]}$ converges to an appropriately defined weak solution of the Mullins–Sekerka problem with kinetic undercooling. Here we shall use some of his varifold approaches.

This paper is organized as follows: In Section 2, we shall recall several definitions from geometric measure theory. Then we define a weak solution of the limit Cahn–Hilliard equation which can be regarded as a generalized solution to (1.6). Also, we state our main result. In Section 3 we establish certain ε–independent estimates for the solution of (1.1). These estimates allow us to draw convergent subsequences of $\{(u^\varepsilon, v^\varepsilon)\}$. With the help of a key result, Theorem 3.6, we show that the limit is a weak solution to the limit Cahn–Hilliard problem. Section 4 is devoted to the proof of Theorem 3.6 which concerns with the upper bound of the discrepancy measure $\xi^\varepsilon(u^\varepsilon)dxdt$ where

$$\xi^\varepsilon(u^\varepsilon) := \Big[\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon}F(u^\varepsilon)\Big]. \tag{1.8}$$

It is displayed in a general context so that it may be used for other similar problems such as the Allen–Cahn equation, phase–field systems, etc. Finally in Section 5, we study the radially symmetric case, which provides more complete result than that in [61] and explains certain important features of the Cahn–Hilliard dynamics.

In the rest of this paper, $\mathcal{E}^\varepsilon(\cdot)$, $e^\varepsilon(u^\varepsilon)$, σ, and $\xi^\varepsilon(u^\varepsilon)$ are defined as in (1.4), (1.7), and (1.8).

2. Preliminary

2.1. Basic notation.

In the sequel, $B(x,R)$ denotes a ball centered at x with radius R in $\mathbb{R}^N$, $B_R = B(0,R)$, and B'_R is a ball in $\mathbb{R}^{N-1}$ centered at the origin $0'$. Also, S^{N-1} is the unit sphere in $\mathbb{R}^N$, and $\vec{\nu}$ a generic element in S^{N-1}. If $n = (n^1, \cdots, n^N)$, we denote by $n \otimes n$ the matrix $(n^i n^j)_{N\times N}$. We use I to denote the identity matrix $(\delta_{ij})_{N\times N}$. For any $N\times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A : B := \mathrm{trace}(A^T B) = \sum_{i,j=1}^N a_{ij}b_{ij}$.

By $C_0^m(D)$ we denote the space of m-th differentiable functions with compact support in D where D can be open or closed. Note that if D is compact, then $C_0^m(D) = C^m(D)$. We use ψ to denote a generic test function in $C_0^m(D)$, and $\vec{Y}$ a generic vector valued test function in $C_0^m(D;\mathbb{R}^N)$. The action of a functional on a test function will be denoted by $\langle\cdot,\cdot\rangle$.

We assume that Ω is a smooth bounded open domain in $\mathbb{R}^N$ ($N \geq 2$). The inner product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and the usual $L^q(\Omega)$ norm by $\|\cdot\|_{q,\Omega}$. We denote by χ_E the characteristic function of a set E.

For reader's convenience, we recall several definitions from geometric measure theory [38], [55].

Radon measure. Let D be either an open or a closed domain. If L is a bounded linear functional on $C_0(D)$ satisfying $\langle L,\psi\rangle \geq 0$ whenever $\psi \geq 0$, the measure μ generated by

$$\mu(A) = \sup_{\psi\in C_0(A),|\psi|\leq 1} \langle L,\psi\rangle \quad \text{for all } A \text{ open in } D$$

is called a Radon measure on D. We use $\langle \mu, \psi \rangle$ ($\psi \in C_0(D)$) to denote the value $\int_D \psi d\mu$ ($= \langle L, \psi \rangle$), and use $\mathrm{spt}(\mu)$ to denote the support of μ.

If $\{\mu^j\}$ is a sequence of (Radon) measures on D, we say $\mu^j \to \mu$ as (Radon) measure on D if as $j \to \infty$, $\langle \mu^j, \psi \rangle \to \langle \mu, \psi \rangle$ for every $\psi \in C_0(D)$.

If μ is a Radon measure on $\bar{\Omega} \times [0, T]$ for every $T > 0$, we also call μ a Radon measure on $\bar{\Omega} \times [0, \infty)$.

BV functions. Let $u \in L^1(\Omega)$. If the distributional gradient Du defined by

$$\langle Du, \vec{Y} \rangle := (u, -\operatorname{div} \vec{Y}) \qquad \forall \vec{Y} \in C_0^1(\Omega; \mathbb{R}^N)$$

can be extended as a bounded linear functional over $C_0(\Omega; \mathbb{R}^N)$, then we say that u is a function of bounded variation, denoted by $u \in BV(\Omega)$. If $u \in BV(\Omega)$, we use $D_i u$ to denote the measure generated by the functional $(u, -\psi_{x_i})$ on $C_0(\Omega)$. We denote by $|Du|$ the Radon measure generated by

$$|Du|(A) = \sup_{\vec{Y} \in C_0(A; \mathbb{R}^N), |\vec{Y}| \le 1} \int_A u \operatorname{div} \vec{Y}, \qquad \forall A \text{ open } \subset \Omega.$$

One can show [38] that $D_i u$ is absolutely continuous with respect to $|Du|$, and there exists a $|Du|$–measurable unit vector valued function $\vec{\nu}$ such that $Du = \vec{\nu}|Du|$, $|Du|$–a.e. .

BV set. Let E be a set in Ω. If $\chi_E \in BV(\Omega)$, then we say E is a BV set, or a set of finite perimeter. We denote $D\chi_E = \vec{\nu}_E |D\chi_E|$. Clearly, in case ∂E is smooth, $\vec{\nu}_E$ is the unit inward normal of E on ∂E.

Varifold. Let $\mathrm{P} = S^{N-1}/\{\vec{\nu}, -\vec{\nu}\}$ be the set of unit normals of unoriented $N-1$ planes in $\mathbb{R}^N$. A varifold (or, more precisely, an $(N-1)$–varifold) V is a Radon measure on $\Omega \times \mathrm{P}$. If V is a varifold, the mass measure $\|V\|$ is a Radon measure on Ω defined by

$$\|V\|(A) = \iint_{A \times \mathrm{P}} dV(x, p).$$

First variation of a varifold. Let V be a varifold. Its first variation δV is a linear functional on $C_0^1(\Omega; \mathbb{R}^N)$ defined by

$$\langle \delta V, \vec{Y} \rangle := \iint_{\Omega \times \mathrm{P}} \nabla \vec{Y}(x) : \left(\mathrm{I} - p \otimes p \right) dV(x, p)$$

$\forall Y \in C_0^1(\Omega; \mathbb{R}^N)$.

Mean curvature vector. Let V be a varifold. If there is a $\|V\|$–measurable vector valued function $\vec{H}$ such that

$$-\langle \delta V, \vec{Y}\rangle = \langle \|V\|, \vec{H}\cdot\vec{Y}\rangle := \int_\Omega \Big(\vec{Y}(x)\cdot\vec{H}(x)\Big) d\|V\|(x)$$

$\forall \vec{Y} \in C_0^1(\Omega; \mathbb{R}^N)$, then we say that $\vec{H}$ is the mean curvature vector of V.

2.2. Definition of a weak solution.

Definition 2.1. A triple (E, v, V) is called a weak solution to the limit of the Cahn–Hilliard equation if the following holds:

1. $E = \cup_{t\geq 0}(E_t \times \{t\})$ is a subset of $\Omega \times [0,\infty)$ and $\chi_E \in C([0,\infty); L^1(\Omega)) \cap L^\infty([0,\infty); BV(\Omega))$;
2. $v \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega))$, i.e., $v \in L^2((0,T); H^1(\Omega))$ for every $T > 0$;

3. $V = V(x,p,t)$ is a Radon measure on $\Omega \times P \times (0,\infty)$ and for almost every $t \in (0,\infty)$, $V^t := V(\cdot,\cdot,t)$ is a varifold on Ω, and there exist a Radon measure μ^t on $\bar{\Omega}$, μ^t-measurable functions $c_1^t, \cdots, c_N^t$, and μ^t–measurable P valued functions $p_1^t, \cdots, p_N^t$ such that

$$\begin{aligned} &0 \leq c_i^t \leq 1 \ (i = 1, \cdots, N), \qquad \Sigma_{i=1}^N c_i^t \geq 1, \\ &\Sigma_{i=1}^N p_i^t \otimes p_i^t = \mathrm{I} \qquad \mu^t\text{–a.e.,} \end{aligned} \tag{2.1}$$

$$\frac{|D\chi_{E_t}|(x)dx}{d\mu^t(x)} \leq \frac{1}{2\sigma} \qquad \left(\sigma = \int_{-1}^1 \sqrt{\frac{F(s)}{2}} ds\right), \tag{2.2}$$

$$\begin{aligned} &\iint_{\Omega\times \mathrm{P}} \psi(x,p)dV^t(x,p) \\ &\quad = \sum_{i=1}^N \int_\Omega c_i^t(x)\psi(x,p_i^t(x))d\mu^t(x) \\ &\qquad\qquad \forall \psi \in C_0(\Omega \times \mathrm{P}); \end{aligned} \tag{2.3}$$

4. For every $T > 0$, almost every $t \in (0,\infty)$, and almost every

$\tau \in (0,t)$,

$$\int_0^T \int_\Omega \Big[-2\chi_{E_t}\psi_t + \nabla v \nabla \psi \Big] = \int_\Omega 2\chi_{E_0}\psi(\cdot,0) \tag{2.4}$$
$$\forall \psi \in C_0^1(\bar\Omega \times [0,T)),$$

$$-\langle D\chi_{E_t}, v\vec{Y}\rangle := \Big(\chi_{E_t}, \operatorname{div}(v\vec{Y})\Big) = \frac{1}{2}\langle \delta V^t, \vec{Y}\rangle \tag{2.5}$$
$$\forall \vec{Y} \in C_0^1(\Omega; \mathbb{R}^N),$$

$$\mu^t(\bar\Omega) + \int_\tau^t \int_\Omega |\nabla v|^2 \le \mu^\tau(\bar\Omega). \tag{2.6}$$

Before explaining our definition, first we introduce our main result.

2.3. Main result.

Theorem 2.1. *Assume that (1.2) and (1.3) hold. Let $(u^\varepsilon, v^\varepsilon)$, $\varepsilon \in (0,1]$, be the solution of (1.1). Then there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$, and the following holds:*

1. There exists $E \subset \Omega \times [0,\infty)$ such that $u^{\varepsilon_k} \longrightarrow -1 + 2\chi_E$ a.e. in $\Omega \times (0,\infty)$ and in $C^{1/9}([0,T]; L^2(\Omega))$ for any $T > 0$;

2. There exists $v \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega))$ such that

$$v^{\varepsilon_k} \longrightarrow v \quad \text{weakly in } L^2((0,T); H^1(\Omega)) \text{ for all } T > 0;$$

3. There exist a Radon measure μ and measures μ_{ij}, $i,j = 1, \cdots, N$, on $\bar\Omega \times [0,\infty)$ such that for every $T > 0$,

$$\begin{aligned} &e^{\varepsilon_k}(u^{\varepsilon_k})dxdt \longrightarrow d\mu(x,t) \text{ as a Radon measure on } \bar\Omega \times [0,T], \\ &\varepsilon_k u^{\varepsilon_k}_{x_i} u^{\varepsilon_k}_{x_j} dxdt \longrightarrow d\mu_{ij}(x,t) \text{ as measure on } \bar\Omega \times [0,T], \\ &\qquad\qquad i,j = 1, \cdots N. \end{aligned} \tag{2.7}$$

4. There exists a Radon measure V on $\Omega \times P \times (0,\infty)$ such that (E,v,V) is a weak solution of Definition 2.1., $d\mu^t(x)dt = d\mu(x,t)$ (μ^t as in (2.4) and μ as in (2.7)), and

$$\int_0^T \langle \delta V^t, \vec{Y}\rangle\, dt = \int_0^T \int_\Omega \nabla \vec{Y} : \Big[d\mu(x,t)\mathrm{I} - \Big(d\mu_{ij}(x,t)\Big)_{N\times N} \Big]$$
$$\forall \vec{Y} \in C_0^1(\Omega \times [0,T]; \mathbb{R}^N). \tag{2.8}$$

Observe from (2.8) that for V to be a varifold, one needs to show that $(\frac{d\mu_{ij}}{d\mu})_{N\times N} \le \mathrm{I}$. This will be our main task of Section 4.

In case of radially symmetry, we can identify the varifold V and the value of v on ∂E_t.

Theorem 2.2. *Assume that (1.2) and (1.3) hold, that $\Omega = B_1$, and that u_0^ε is radially symmetric. Then with the same notation as in Theorem 2.1.,*

$$d\mu = 2\sigma|D\chi_{E_t}|dxdt \qquad \text{as Radon measure on } \Omega \times [0,\infty),$$

$$\Big(d\mu_{ij}\Big)_{N\times N} = \vec{e}_r \otimes \vec{e}_r d\mu, \qquad \text{as Radon measure on } \bar{\Omega} \times [0,\infty),$$

$$dV(x,t,p) = 2\sigma|D\chi_{E_t}|dxdt\delta_{\vec{e}_r}dp$$

$$\text{as Radon measure on } \Omega \times [0,\infty) \times P,$$

$$v(x,t) = -\frac{\sigma(N-1)}{|x|}\vec{e}_r \cdot \vec{\nu}_{E_t} \quad \text{on } \operatorname{spt}(|D\chi_{E_t}|) \quad \text{for a.e. } t > 0$$

where $\vec{e}_r = \frac{x}{|x|}$ and $\delta_{\vec{e}_r}$ is the Dirac measure concentrated at $\{\vec{e}_r, -\vec{e}_r\} \in P$.

2.4. Remarks on the definition of weak solutions.

Assume that (E, v, V) is a weak solution of Definition 2.1.

1. Observe that (2.4) implies

$$(2.9) \qquad \begin{aligned} &\Big(2\chi_E\Big)_t = \Delta v && \text{in } \Omega \times (0,\infty) \text{ (in distribution sense)},\\ &\frac{\partial}{\partial n}v = 0 && \text{on } \partial\Omega \times [0,\infty) \text{ (in distribution sense)},\\ &\lim_{t\searrow 0} E_t = E_0. \end{aligned}$$

Hence, (2.4) is a weak formulation of all the equations in (1.6) except the third one.

2. Since $\chi_E \in C([0,\infty); L^1(\Omega))$, every E_t is uniquely defined. Also, by (2.9), we have $(\chi_E)_t \in L^2((0,\infty); H^{-1}(\Omega))$, which implies $\chi_E \in C^{1/3}([0,\infty); L^1(\Omega))$ by the assumption $\chi_E \in L^\infty([0,\infty); BV(\Omega))$ and the Sobolev imbedding (cf. [48]).

3. The definition of V in (2.4) can be written as

$$dV^t(x,p) = \Sigma_{i=1}^N c_i^t(x)\delta_{p_i^t(x)}d\mu^t(x)dp.$$

From (2.2), it follows that

$$d\|V^t\|(x) = \Sigma_{i=1}^N c_i^t(x)d\mu^t(x) \geq d\mu^t(x),$$

so that from (2.2), the function

$$m := \frac{d\|V^t\|(x)}{2\sigma|D\chi_{E_t}|(x)dx} \tag{2.10}$$

is $\|V^t\|$ measurable, $m \in [1,\infty)$ for $|D\chi_{E_t}|$–a.e, and $m \in [1,\infty) \cup \{\infty\}$ for $\|V^t\|$–a.e.

Now suppose that we have

$$\sup_{x\in\Omega, r>0} \frac{|D\chi_{E_t}|(B(x,r)\cap\Omega)}{r^{N-1}} < \infty.$$

Then by Theorem 7.1 of [60], we know that $\langle D\chi_{E_t}, v\vec{Y}\rangle$ is a bounded linear functional over $C_0(\Omega;\mathrm{IR}^N)$, namely, v is $|D\chi_{E_t}|$ measurable. Since $m = \infty$ for "$\|V^t\| \setminus |D\chi_{E_t}|$" a.e., $\frac{v}{m}$ is $\|V^t\|$ measurable, and therefore we can write (2.5) as, for a.e. $t > 0$,

$$-\langle \delta V^t, \vec{Y}\rangle = \langle \|V^t\|, \frac{v}{\sigma m}\vec{\nu}_{E_t}\cdot\vec{Y}\rangle \quad \forall \vec{Y} \in C_0^1(\Omega;\mathrm{IR}^N).$$

Hence, by the definition of the mean curvature vector,

$$\frac{v}{m}\,\vec{\nu}_{E_t} = \sigma\vec{H}_{V^t} \qquad \mu - a.e., \tag{2.11}$$

where $\vec{H}_{V^t}$ is the mean curvature vector of V^t. (Note that this implies $\vec{H}_{V^t} = 0$, "$\|V^t\| \setminus |D\chi_{E_t}|$" a.e.)

If we further assume that

$$\limsup_{r\searrow 0} \frac{|\mu^t|(B(x,r))}{r^{N-1}} > 0 \qquad \mu^t - \text{a.e.},$$

then, by the Allard theorem [3] or a less general theorem of Almgren [5], V^t is rectifiable (cf. [3], [5], [38] for definition). From the expression of V^t in (2.4) it thus follows that $c_1^t \equiv 1$, $c_2 = \cdots = c_N \equiv 0$, $\|V^t\| = \mu^t$, and p_1^t is the unit normal to the unoriented tangent plane of μ^t. In addition, $\vec{H}_{V^t} = \vec{H}_{|D\chi_{E_t}|}$ for $|D\chi_{E_t}|$ a.e. where $H_{|D\chi_{E_t}|}$ is the mean curvature vector of $|D\chi_{E_t}|$. (In case ∂E_t is smooth, it is the mean curvature vector of the hypersurface ∂E_t.) Hence, from (2.11) it follows

$$\frac{v}{m} = \sigma\kappa \qquad |D\chi_{E_t}|\text{–a.e.}, \tag{2.12}$$

where $\kappa = \vec{\nu}_{E_t}\cdot\vec{H}_{|D\chi_{E_t}|}$ is the "generalized" mean curvature of "spt($|D\chi_{E_t}|$)". Thus, if $m = 1$ for μ-a.e., i.e, if

$$\mu = 2\sigma|D\chi_{E_t}|, \quad \mu\text{-a.e. in } \Omega\times(0,\infty), \tag{2.13}$$

then (2.12) is a weak formulation of the third equation in (1.6).

In conclusion, if we have (2.13), then Definition 2.1 is an acceptable weak formulation of (1.6).

4. Generally, we cannot show (2.13) for the limit of the solutions of the Cahn–Hilliard equation, except for the case of radially symmetry as shown in Theorem 2.2.. In fact, equation $\mu = 2\sigma|D\chi_{E_t}|$ may not hold for every (x,t). For example, by adding phantom interfaces in the initial data (namely, oscillations on u_0^ε such that $2|D\chi_{E_0}|(\Omega) = |D\lim_{\varepsilon\searrow 0} u_0^\varepsilon|(\Omega) < \lim_{\varepsilon\searrow 0}|Du_0^\varepsilon|(\Omega)$), we can easily construct examples such that $\mu^0(\Omega) > 2\sigma|D\chi_{E_0}|(\Omega)$. Also, it may be possible that at some later time, $m > 1$ at certain lower dimensional set contained in $\text{spt}(\mu)$. Hence, we can regard allowing $m > 1$ as a special property which helps us to extend the classical solutions of (1.6) beyond the time where the topological changes occur.

5. On the other hand, m defined in (2.10) has to satisfy certain constraints, since otherwise, there would be too many weak solutions. For example, given smooth Γ_0 and any constant $m \geq 1$, if we let (v^m, Γ^m) be the unique (local) solution of (1.6) with σ replaced by $\sigma^m := \sigma m$, then one can easily check that if we define E_t^m to be the set enclosed by Γ_t^m, and define V^m by $(V^m)^t := 2\sigma^m \mathcal{H}^{N-1}\lfloor\Gamma_t^m\, \delta_{\vec{\nu}_{\Gamma_t^m}}$ where $\mathcal{H}^{N-1}\lfloor\Gamma_t^m$ is the $(N-1)$-dimensional Hausdorff measure restricted to Γ_t^m, then (E^m, v^m, V^m) will satisfy all the conditions to be a (local) weak solution, except the inequality (2.6).

Hence we impose (2.6) an an "entropy" condition to confine m. Here we provide the following example as our reasoning: Suppose that $m \geq 1$ is space independent, that

$$\cup_{t\in[0,T]}(\text{spt}(|D\chi_{E_t}|) \times \{t\})$$

and $\cup_{t\in[0,T]}(\text{spt}(\mu^t) \times \{t\})$ are smooth space–time hypersurfaces, and that $m = 1$ when $t = 0$ (i.e., $\mu^0(\bar{\Omega}) = 2\sigma|D\chi_{E_0}|(\Omega)$). Then one can easily calculate

$$\begin{aligned}
\frac{d}{dt}|D\chi_{E_t}|(\Omega) &= -\int_{\text{spt}(|D\chi_{E_t}|)} \kappa\mathcal{V} \\
&= -\frac{1}{2\sigma m}\int_{\text{spt}(|D\chi_{E_t}|)} v\left[\frac{\partial v}{\partial n}\right] \\
&= -\frac{1}{2\sigma m}\int_\Omega |\nabla v|^2.
\end{aligned}$$

Hence, comparing it with (2.6), we deduce that $2\sigma|D\chi_{E_t}|(\Omega) \geq \mu^t(\bar{\Omega})$ where equality is possible only if $m = 1$, μ-a.e.. Since we know that $\mu^t(\Omega) \geq 2\sigma|D\chi_{E_t}|(\Omega)$ for a.e. $t \geq 0$, we must have $m \equiv 1$ and $2\sigma|D\chi E_t|(\Omega) = \mu^t(\Omega) = \mu^t(\bar{\Omega})$ for a.e. $t \geq 0$. One notices that this argument works also for arbitrary function $m \geq 1$ if we know that $\kappa\mathcal{V} \geq 0$ on ∂E_t. One can check that the condition $\kappa\mathcal{V} \geq 0$ on ∂E_t always holds for radial symmetric smooth weak solutions. Therefore, we know that in the case of radial symmetry, a smooth weak solution of Definition 2.1 is a solution of (1.6).

However, we do not know in general if the condition (2.6) is sufficient to guarantee that a smooth weak solution of Definition 2.1 is actually a classical solution of (1.6). If (2.6) is not sufficient, then we need additional "entropy" conditions to confine m.

6. Clearly, a local (in time) classical solution of (1.6) is a local (in time) weak solution of Definition 2.1., if we define E^t to be the region enclosed by Γ_t and define V^t by $V^t = 2\sigma\mathcal{H}^{N-1}\lfloor\Gamma_t\delta_{\vec{\nu}_{E_t}}$.

3. Convergence

In this section we shall show that the family $\{(u^\varepsilon, v^\varepsilon)\}_{0<\varepsilon\leq 1}$ is weakly compact in some functional spaces so that we can draw convergent subsequences.

In the sequel, all positive constants independent of ε will be denoted by the same letter C.

3.1. Basic estimates.

The following estimates are a direct consequence of (1.5) and the properties of F in (1.2):

Lemma 3.1. *For every $\varepsilon \in (0,1]$ and every $t, \tau \geq 0$,*

$$\mathcal{E}^\varepsilon(t) + \int_\tau^t \int_\Omega |\nabla v^\varepsilon|^2 = \mathcal{E}^\varepsilon(\tau),$$

$$\int_0^\infty \int_\Omega |\nabla v^\varepsilon|^2 \leq \mathcal{E}_0,$$

$$\frac{1}{|\Omega|}\int_\Omega u^\varepsilon(\cdot, t) = m_0,$$

$$\int_\Omega |u^\varepsilon|^p \leq C(1+\mathcal{E}_0) \qquad \Big(p \text{ as in (1.2)(b)}\Big),$$

$$\int_\Omega (|u^\varepsilon| - 1)^2 \leq C\varepsilon\mathcal{E}_0.$$

3.2. Compactness of $\{u^\varepsilon\}_{0<\varepsilon\le 1}$.

To show the compactness of $\{u^\varepsilon\}$, it is convenient to introduce a function w^ε defined by

$$w^\varepsilon(x,t) = W(u^\varepsilon(x,t)), \qquad (x,t) \in \bar{\Omega} \times [0,\infty),$$

where

$$W(u) = \int_{-1}^{u} \sqrt{2\tilde{F}(s)}ds, \quad \tilde{F}(u) := \min\{F(u), 1+|u|^2\}, \quad u \in \mathbb{R}.$$

Observe that

$$\int_\Omega |\nabla w^\varepsilon(\cdot,t)| = \int_\Omega \sqrt{2\tilde{F}(u^\varepsilon)}|\nabla u^\varepsilon| \le \int_\Omega e^\varepsilon(u^\varepsilon) = \mathcal{E}^\varepsilon(t), \qquad \forall t \in [0,\infty).$$

Also by the properties of F, there are positive constants c_1 and c_2 such that

$$\begin{aligned} &c_1|u_1-u_2|^2 \le |W(u_1)-W(u_2)| \le c_2|u_1-u_2|(1+|u_1|+|u_2|), \\ (3.1) \qquad &\qquad\qquad\qquad \forall u_1,u_2 \in \mathbb{R}. \end{aligned}$$

Lemma 3.2. *There exists a positive constant C which is independent of ε such that*

$$\|w^\varepsilon\|_{L^\infty((0,\infty);W^{1,1}(\Omega))} + \|w^\varepsilon\|_{C^{1/8}([0,\infty);L^1(\Omega))} + \|u^\varepsilon\|_{C^{1/8}([0,\infty);L^2(\Omega))} \le C.$$

Proof. The idea to show the continuity of u^ε or w^ε in t is to use the equation $u_t^\varepsilon = \Delta v^\varepsilon$. For this purpose, let ρ be any fixed mollifier; namely,

$$\rho \in C^\infty(\mathbb{R}^N), \quad 0 \le \rho \le 1 \text{ in } \mathbb{R}^N, \quad \rho = 0 \text{ in } \mathbb{R}^N \setminus B_1, \quad \int_{\mathbb{R}^N} \rho = 1.$$

For any small $\eta > 0$, we define

$$u_\eta^\varepsilon(x,t) = \int_{B_1} \rho(y) u^\varepsilon(x-\eta y,t)dy, \quad x \in \Omega, t \ge 0.$$

Here we have assumed that u^ε has been extended to

$$\{x \notin \Omega \,|\, \text{dist}(x,\Omega) \le \eta_0\}$$

by

$$u^\varepsilon(s+\eta n(s),t) = u^\varepsilon(s-\eta n(s),t), \qquad s \in \partial\Omega, \eta \in [0,\eta_0], t \ge 0,$$

where η_0 is a small positive number, and $n(s)$ is the unit outward normal to $\partial\Omega$ at $s \in \partial\Omega$.

By the properties of mollifiers, for any $\eta \in (0, \eta_0)$ and every $t \geq 0$ we have

$$\|\nabla u^\varepsilon_\eta(\cdot,t)\|_{2,\Omega} \leq C\eta^{-1}\|u^\varepsilon(\cdot,t)\|_{2,\Omega} \leq C\eta^{-1}, \tag{3.2}$$

$$\int_\Omega |u^\varepsilon_\eta - u^\varepsilon|^2 dx \leq \int_\Omega \int_{B_1} \rho(y)|u^\varepsilon(x-\eta y,t) - u^\varepsilon(x,t)|^2\, dydx$$

$$\leq c_1 \int_\Omega \int_{B_1} \rho(y)|w^\varepsilon(x-\eta y,t) - w^\varepsilon(x,t)|\, dydx \tag{3.3}$$

$$\leq C\eta\|\nabla w^\varepsilon(\cdot,t)\|_{1,\Omega} \leq C\eta, \quad \Big(\text{by (3.1)}\Big). \tag{3.4}$$

For any $0 \leq \tau < t < \infty$, by using $u^\varepsilon(x,t) - u^\varepsilon(x,\tau) = \int_\tau^t u^\varepsilon_s(x,s)ds = \int_\tau^t \Delta v^\varepsilon(x,s)ds$, we can calculate

$$\begin{aligned}
&\int_\Omega (u^\varepsilon_\eta(x,t) - u^\varepsilon_\eta(x,\tau))(u^\varepsilon(x,t) - u^\varepsilon(x,\tau))dx \\
&= -\int_\tau^t \int_\Omega \nabla v^\varepsilon(x,s)(\nabla u^\varepsilon_\eta(x,t) - \nabla u^\varepsilon_\eta(x,\tau))dxds \\
&\leq 2\Big(\textstyle\int_\tau^t \int_\Omega |\nabla v^\varepsilon|^2\Big)^{1/2}(t-\tau)^{1/2}\sup_{s\in[0,\infty)}\|\nabla u^\varepsilon_\eta(\cdot,s)\|_{2,\Omega} \\
&\leq C\eta^{-1}(t-\tau)^{1/2}
\end{aligned}$$

by the estimates on ∇v^ε in Lemma 3.1and $\nabla u^\varepsilon_\eta$ in (3.2). This estimate, together with the estimate for $\|u^\varepsilon_\eta - u^\varepsilon\|_{2,\Omega}$ in (3.4) then yields

$$\begin{aligned}
\int_\Omega |u^\varepsilon(x,t) - u^\varepsilon(x,\tau)|^2 dx &\leq C(\eta + \eta^{-1}(t-\tau)^{1/2}) \\
&\leq C(t-\tau)^{1/4},
\end{aligned} \tag{3.5}$$

if we take $\eta = \min\{\eta_0, (t-\tau)^{1/4}\}$.

Finally, using the second inequality in (3.1) we obtain

$$\begin{aligned}
&\int_\Omega |w^\varepsilon(x,t) - w^\varepsilon(x,\tau)|dx \\
&\qquad \leq c_2\|u^\varepsilon(\cdot,t) \\
&\qquad\quad - u^\varepsilon(\cdot,\tau)\|_{2,\Omega}\Big(|\Omega|^{1/2} + \|u^\varepsilon(\cdot,t)\|_{2,\Omega} + \|u^\varepsilon(\cdot,\tau)\|_{2,\Omega}\Big) \\
&\qquad \leq C(t-\tau)^{1/8}.
\end{aligned}$$

This completes the proof of the lemma.

Lemma 3.3. *Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be any sequence satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$. Then there exist a subsequence $\{\varepsilon_{j_k}\}$ of $\{\varepsilon_j\}$, a non-increasing function $\mathcal{E}(t)$, and a set $E \subset \Omega \times [0,\infty)$ such that as $k \to \infty$,*

$$\begin{aligned}
\mathcal{E}^{\varepsilon_{j_k}}(t) &\to \mathcal{E}(t) && \textit{for all } t \geq 0,\\
w^{\varepsilon_{j_k}}(x,t) &\to 2\sigma\chi_E && \textit{a.e. in } \Omega \times (0,\infty)\\
& && \textit{and in } C^{1/9}([0,T]; L^1(\Omega)) \textit{ for all } T > 0,\\
u^{\varepsilon_{j_k}}(x,t) &\to -1 + 2\chi_E && \textit{a.e. in } \Omega \times (0,\infty)\\
& && \textit{and in } C^{1/9}([0,T]; L^2(\Omega)) \textit{ for all } T > 0.
\end{aligned}$$

In addition, there exists $C > 0$ such that $E_t := \{x \; ; \; (x,t) \in E\}$ satisfies the following:

1. *For any $0 \leq \tau < t < \infty$, $\int_\Omega |\chi_{E_\tau} - \chi_{E_t}| \leq C|t-\tau|^{1/4}$;*
2. *For any $t \in [0,\infty)$, $|E_t| = |E_0| = \frac{1+m_0}{2}|\Omega|$;*
3. *$\chi_E \in L^\infty([0,\infty); BV(\Omega))$ and for every $t \geq 0$,*

$$|D\chi_{E_t}|(\Omega) \leq \frac{1}{2\sigma}\mathcal{E}(t) \leq \frac{1}{2\sigma}\mathcal{E}_0.$$

Proof. The convergence of $\mathcal{E}^{\varepsilon_{j_k}}(t)$ follows from the monotonicity of the function $\mathcal{E}^\varepsilon(t)$.

Since bounded set in $W^{1,1}(\Omega)$ is precompact in $L^q(\Omega)$ for any $q \in [1, \frac{N}{N-1})$, from the estimate on w^ε in Lemma 3.2. we immediately conclude that there exist a subsequence $\{\varepsilon_{j_k}\}$ of $\{\varepsilon_j\}$ and a function $w(x,t)$ such that as $k \to \infty$,

$$\begin{aligned}
w^{\varepsilon_{j_k}}(x,t) \to w(x,t) \quad & \text{a.e. in } \Omega \times (0,\infty)\\
& \text{and in } C^{1/9}([0,T]; L^1(\Omega)) \text{ for all } T > 0.
\end{aligned}$$

Let $u(x,t)$ be the function defined by the relation $w(x,t) = W(u(x,t))$. Then in view of (3.1), we see that $u^{\varepsilon_{j_k}} \to u$ a.e. in $\Omega \times (0,\infty)$. Consequently, by the estimate for u^ε, we know that $u^{\varepsilon_{k_j}} \to u$ in $C^{1/9}([0,T]; L^2(\Omega))$. Furthermore, by the estimate for $(|u^\varepsilon| - 1)^2$ in Lemma 3.1 we have $|u| \equiv 1$, that is, there is a set $E \subset \Omega \times [0,\infty)$ such that $u = -1 + 2\chi_E$. This also implies that $w = 2\sigma\chi_E$.

Using the time estimate for u^ε in (3.6), we obtain, for all $0 \le \tau < t < \infty$,

$$\int_\Omega |\chi_{E_\tau} - \chi_{E_t}| = \int_\Omega |\chi_{E_\tau} - \chi_{E_t}|^2 = \lim_{k\to\infty} \frac{1}{4} \int_\Omega |u^{\varepsilon_{j_k}}(\cdot,t) - u^{\varepsilon_{j_k}}(\cdot,\tau)|^2$$
$$\le C|t-\tau|^{1/4}.$$

Also, since the average of $u^\varepsilon(\cdot,t)$ is m_0 for every ε and every t, we have $|E_t| = \frac{1+m_0}{2}|\Omega|$. Finally since $|Dw^\varepsilon(\cdot,t)|(\Omega) \le \mathcal{E}^\varepsilon(t)$, by the lower semicontinuity of the BV norm, $|D\chi_{E_t}|(\Omega) = \frac{1}{2\sigma}|Dw|(\Omega) \le \frac{1}{2\sigma}\mathcal{E}(t)$. This completes the proof of the lemma.

3.3. Weak compactness of $\{v^\varepsilon\}_{0<\varepsilon\le 1}$.

The following estimate depends only on the elliptic equation

$$v^\varepsilon = -\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon), \tag{3.6}$$

and the assumptions $\int_\Omega e^\varepsilon(u^\varepsilon) \le \mathcal{E}_0$ and $\int_\Omega u = m_0|\Omega|$ with $m_0 \in (-1,1)$.

Lemma 3.4. *There exist a large positive constant C and a small positive constant ε_0 such that for every $\varepsilon \in (0,\varepsilon_0]$*

$$\|v^\varepsilon(\cdot,t)\|_{H^1(\Omega)} \le C(\mathcal{E}^\varepsilon(t) + \|\nabla v^\varepsilon(\cdot,t)\|_{2,\Omega}) \qquad \forall t \in [0,\infty).$$

Proof. By the Sobolev imbedding, it suffices to estimate the average $\bar{v}^\varepsilon(t)$ of $v^\varepsilon(\cdot,t)$ over Ω. For simplicity, in the sequel, we shall suppress the t variable.

Let $\vec{Y}(x) \in C^1(\bar{\Omega};\mathbb{R}^N)$ be any function. Multiplying the equation (3.6) by $\vec{Y}\cdot\nabla u^\varepsilon$ and integrating over Ω we obtain

$$\begin{aligned}\int_\Omega \vec{Y}\cdot\nabla u^\varepsilon v^\varepsilon &= \int_\Omega \vec{Y}\cdot\nabla u^\varepsilon(-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1}f(u^\varepsilon)) \\ &= -\int_\Omega D\vec{Y} : \left(e^\varepsilon(u^\varepsilon)\,\mathrm{I} - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon\right) \\ &\quad + \int_{\partial\Omega} e^\varepsilon(u^\varepsilon)\vec{Y}\cdot\vec{n}_{\partial\Omega}.\end{aligned} \tag{3.7}$$

Integration by parts for the left–hand side yields

$$\begin{aligned}\int_\Omega \vec{Y}\cdot\nabla u^\varepsilon v^\varepsilon &= \int_{\partial\Omega} u^\varepsilon v^\varepsilon \vec{Y}\cdot\vec{n}_{\partial\Omega} - \int_\Omega \vec{Y}\cdot\nabla v^\varepsilon u^\varepsilon \\ &\quad - \int_\Omega (v^\varepsilon - \bar{v}^\varepsilon)u^\varepsilon \operatorname{div}\vec{Y} - \bar{v}^\varepsilon \int_\Omega u^\varepsilon \operatorname{div}\vec{Y}.\end{aligned}$$

Hence, for any smooth ψ with $\frac{\partial}{\partial n}\psi = 0$ on $\partial\Omega$, substituting $\vec{Y} = \nabla\psi$ gives the formula

(3.8)
$$\bar{v}^\varepsilon = \frac{1}{\int_\Omega \Delta\psi u^\varepsilon}\Big\{ \int_\Omega [D^2\psi : (e(u^\varepsilon)\mathrm{I} - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon) - u^\varepsilon \nabla\psi \cdot \nabla v^\varepsilon \\ - u^\varepsilon \Delta\psi(v^\varepsilon - \bar{v}^\varepsilon)]\Big\}.$$

Now we choose ψ. Let η be a small positive constant to be determined, and let u^ε_η be defined as in the previous subsection. Denote by $\bar{u}^\varepsilon_\eta$ the average of u^ε_η over Ω. We define ψ to be the unique solution to

$$\begin{aligned} -\Delta\psi =& u^\varepsilon_\eta - \bar{u}^\varepsilon_\eta \qquad \text{in } \Omega, \\ \tfrac{\partial}{\partial n}\psi =& 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega \psi = 0. \end{aligned}$$

Observe from the definition of u^ε_η that we have

$$\begin{aligned} \|u^\varepsilon_\eta\|_{\infty,\Omega} \le& 1 + \sup_{x\in\Omega} \int_{B(0,1)} \rho(y)\Big||u^\varepsilon(x-\eta y)| - 1\Big| dy \\ \le& 1 + C\eta^{-N/2}\|(|u^\varepsilon| - 1)\|_{2,\Omega} \le 1 + C\varepsilon^{1/2}\eta^{-N/2}. \end{aligned}$$

Similarly, we can show that

$$\|u^\varepsilon_\eta\|_{C^1(\Omega)} \le C\eta^{-1}(1 + \varepsilon^{1/2}\eta^{-N/2}),$$

so that by an elliptic estimate,

$$\|\psi\|_{C^2(\Omega)} \le C\|u^\varepsilon_\eta\|_{C^1(\Omega)} \le C\eta^{-1}(1 + \varepsilon^{1/2}\eta^{-N/2}).$$

Therefore, the numerator in (3.9) can be estimated from above by

$$\begin{aligned} &\left| \int_\Omega \left[D^2\psi : (e(u^\varepsilon)\mathrm{I} - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon) - u^\varepsilon\nabla\psi\cdot\nabla v^\varepsilon - u^\varepsilon\Delta\psi(v^\varepsilon - \bar{v^\varepsilon})\right] \right| \\ &\quad \le C\|\psi\|_{C^2(\Omega)}\left[\mathcal{E}^\varepsilon(t) + \|u^\varepsilon\|_{2,\Omega}\|\nabla v^\varepsilon\|_{2,\Omega} + \|u^\varepsilon\|_{2,\Omega}\|v^\varepsilon - \bar{v^\varepsilon}\|_{2,\Omega}\right] \\ &\quad \le C\eta^{-1}(1 + \varepsilon^{1/2}\eta^{-N/2})(\mathcal{E}^\varepsilon(t) + \|\nabla v^\varepsilon\|_{2,\Omega}). \end{aligned}$$

Using the definition of ψ, we can calculate the denominator in (3.9) by

$$\begin{aligned} \int_\Omega \Delta\psi u^\varepsilon =& \int_\Omega (u^\varepsilon_\eta - \bar{u}^\varepsilon_\eta)u^\varepsilon \\ =& \int_\Omega (u^\varepsilon_\eta - u^\varepsilon)u^\varepsilon + \int_\Omega (u^{\varepsilon 2} - 1) + |\Omega|(1 - \bar{u}^{\varepsilon 2}) \\ &+ |\Omega|\bar{u}^\varepsilon(\bar{u}^\varepsilon - \bar{u}^\varepsilon_\eta). \end{aligned}$$

Recall that $\bar{u}^\varepsilon = m_0 \in (-1,1)$, $\int_\Omega |u^{\varepsilon 2} - 1| \le C\sqrt{\varepsilon}$,

$$|\bar{u}^\varepsilon_\eta - \bar{u}^\varepsilon| \le C\|u^\varepsilon_\eta - u^\varepsilon\|_{2,\Omega} \le C\sqrt{\eta}.$$

We then have

$$\int_\Omega \Delta\psi u^\varepsilon \ge |\Omega|(1-m_0^2) - C(\sqrt{\varepsilon}+\sqrt{\eta}).$$

Therefore, from (3.9) we deduce that

$$|\bar{v^\varepsilon}| \le \frac{C\eta^{-1}(1+\varepsilon^{1/2}\eta^{-N/2})(\mathcal{E}^\varepsilon(t) + \|\nabla v^\varepsilon\|_{2,\Omega})}{|\Omega|(1-m_0^2) - C(\sqrt{\varepsilon}+\sqrt{\eta})}.$$

Taking η small but independent of ε, we then obtain the assertion of the lemma.

Corollary 3.5. *There exist positive constants C and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0]$ and all $T \ge 0$,*

$$\int_T^{T+1} \|v^\varepsilon\|^2_{H^1(\Omega)} \le C.$$

Consequently, for every subsequence $\{\varepsilon_j\}_{j=1}^\infty$ satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$, there exist a subsequence $\{\varepsilon_{j_k}\}$ and a function $v \in L^2_{\text{loc}}([0,\infty), H^1(\Omega))$ such that as $k \to \infty$,

$$v^{\varepsilon_{j_k}} \to v \qquad \textit{weakly in } L^2((0,T), H^1(\Omega)) \quad \forall T > 0.$$

Moreover,

$$\int_0^\infty \int_\Omega |\nabla v|^2 \le \mathcal{E}_0.$$

3.4. An upper bound for the discrepancy measure $\xi^\varepsilon(u^\varepsilon)dx$.

To obtain a weak solution, here we state a theorem concerning the upper bound of the discrepancy measure $\xi^\varepsilon(u^\varepsilon)dx$ defined in (1.8). Its proof will be given in the next section.

To state our theorem, for every $\varepsilon \in (0,1]$ we define

$$\mathcal{K}_\varepsilon := \Big\{(u,v) \in H^2(\Omega) \times L^2(\Omega) \,\Big|\, -\varepsilon\Delta u + \varepsilon^{-1} f(u) = v \text{ in } \Omega, \tfrac{\partial}{\partial n} u = 0 \text{ on } \partial\Omega\Big\}. \tag{3.9}$$

Also, we denote by w^+ the positive part of w, namely, $\max\{w,0\}$.

Theorem 3.6. *There exist a positive constant $\eta_0 \in (0,1]$ and continuous, non-increasing, and positive functions $M_1(\eta)$ and $M_2(\eta)$ defined on $(0,\eta_0]$ such that for every $\eta \in (0,\eta_0]$, every $\varepsilon \in (0, \frac{1}{M_1(\eta)}]$, and every $(u^\varepsilon, v^\varepsilon) \in \mathcal{K}_\varepsilon$, we have*

(3.10)
$$\int_\Omega \Big(\xi^\varepsilon(u^\varepsilon)\Big)^+ dx \le \eta \int_\Omega e^\varepsilon(u^\varepsilon) dx + \varepsilon M_2(\eta) \int_\Omega v^{\varepsilon 2}.$$

3.5. Convergence: Proof of Theorem 2.1.

With all the previous preparation, we can now prove Theorem 2.1.

Let $\{u_0^\varepsilon(\cdot)\}_{\varepsilon\in(0,1]}$ be a family of initial data satisfying (1.3). Let $(u^\varepsilon, v^\varepsilon)$ be the solution of (1.1) with initial data u_0^ε. By the previous estimates, we can draw a subsequence $\{\varepsilon_k\}_{k=1}^\infty$, such that as $k \to \infty$, $\varepsilon_k \searrow 0$ and the first three assertions of Theorem 2.1 hold.

Since $W(u^{\varepsilon_k}) \to 2\sigma\chi_E$ and $|DW(u^\varepsilon)| \le e^\varepsilon(u^\varepsilon)$ for every ε and every (x,t), by the lower semicontinuity of the BV norms, we have $|D\chi_{E_t}|dxdt \le d\mu$.

Sending k to ∞ in the differential equation $\Delta v^{\varepsilon_k} = (u^{\varepsilon_k})_t = (1+u^{\varepsilon_k})_t$ and using the convergence of u^{ε_k} and v^{ε_k}, we obtain the identity (2.4). Moreover, for any $\vec{Y} \in C([0,T]; C_0^1(\Omega; \mathbb{R}^N))$, integrating (3.7) (with $\varepsilon = \varepsilon_k$) from $t=0$ to $t=T$ and sending $k \to \infty$ yield

(3.11)
$$\int_0^T \int_\Omega 2\chi_E \operatorname{div}(v\vec{Y})\, dxdt = \int_0^T \int_\Omega D\vec{Y} : \Big(\mathrm{I}\; d\mu - (d\mu_{ij})_{N\times N}\Big).$$

To finish the proof of Theorem 2.1, it remains to construct V. To do this, we first study the measure $\mu(\cdot,t)$. Notice that for any $0 \le \tau < T < \infty$,

(3.12)
$$\int_\tau^T \int_{\bar\Omega} d\mu(x,s) = \lim_{k\to\infty} \int_\tau^T \int_\Omega e^{\varepsilon_k}(u^{\varepsilon_k}) dsdt = \int_\tau^t \mathcal{E}(s)\, ds.$$

One then can show that, for a.e. $t \in (0,\infty)$, there exists a Radon measure $\mu^t(x)$ on $\bar\Omega$ such that for any $g \in C(\bar\Omega)$, as function of t, $\int_{\bar\Omega} g(x) d\mu^t(x)$ is measurable in $t \in (0,\infty)$, and for any $0 \le \tau < t < \infty$

$$\int_\tau^t \int_{\bar\Omega} g(x) d\mu(x,s) = \int_\tau^t \int_{\bar\Omega} g(x) d\mu^s ds.$$

Therefore, in the sense of Radon measure,

$$d\mu(x,t) = d\mu^t(x)dt.$$

By (3.12), we have $\mu^t(\bar{\Omega}) = \mathcal{E}(t)$ for a.e. $t \in (0,\infty)$. Consequently, for a.e. $t \in (0,\infty)$ and a.e. $\tau \in (0,t)$,

$$\begin{aligned}\mu^t(\bar{\Omega}) = \mathcal{E}(t) = \lim_{k\to\infty} \mathcal{E}^{\varepsilon_k}(t) &= \lim_{k\to\infty} \left\{\mathcal{E}^{\varepsilon_k}(\tau) - \int_\tau^t \int_\Omega |\nabla v^{\varepsilon_k}|^2\right\} \\ &= \mathcal{E}(\tau) - \lim_{k\to\infty} \int_\tau^t \int_\Omega |\nabla v^{\varepsilon_k}|^2 \\ &\leq \mathcal{E}(\tau) - \int_\tau^t \int_\Omega |\nabla v|^2 = \mu^\tau(\bar{\Omega}) - \int_\tau^t \int_\Omega |\nabla v|^2.\end{aligned}$$

Next, we study the relation between μ_{ij} and μ. Observe that for any $\vec{Y}, \vec{Z} \in C(\bar{\Omega} \times [0,T]; \mathbb{R}^N)$,

$$\begin{aligned}\int_0^T \int_\Omega \vec{Y}^T \cdot \left(\varepsilon^k \nabla u^{\varepsilon_k} \otimes \nabla u^{\varepsilon_k}\right) \cdot \vec{Z} \leq & \int_0^T \int_\Omega |\vec{Y}|\, |\vec{Z}| e^{\varepsilon_k}(u^{\varepsilon_k}) \\ & + \int_0^T \int_\Omega |\vec{Y}|\, |\vec{Z}| \xi^{\varepsilon_k}(u^{\varepsilon_k}).\end{aligned}$$

Using Theorem 3.6, we know that as $k \to \infty$ the limit of the second term on the right–hand side is non–positive. Hence, by sending $k \to \infty$ we obtain

$$\int_0^T \int_{\bar{\Omega}} \vec{Y}^T \cdot \left(d\mu_{ij}\right)_{N\times N} \cdot \vec{Z} \leq \int_0^T \int_{\bar{\Omega}} |\vec{Y}||\vec{Z}| d\mu. \tag{3.13}$$

Therefore, in the sense of measure $|d\mu_{ij}(x,t)| \leq d\mu$. Consequently, there exist μ–measurable functions $\nu_{ij}(x,t)$ such that

$$d\mu_{ij}(x,t) = \nu_{ij}(x,t) d\mu(x,t) \qquad \mu - \text{a.e. } (x,t) \in \bar{\Omega} \times [0,\infty).$$

Clearly, by the definition of μ_{ij} and (3.13), we have

$$0 \leq \left(\nu_{ij}\right)_{N\times N} = \left(\nu_{ji}(x,t)\right)_{N\times N} \leq \mathrm{I}, \quad \mu - \text{a.e. } (x,t) \in \bar{\Omega} \times [0,\infty).$$

Thus, we can write

$$\left(\nu_{ij}\right)_{N\times N} = \Sigma_{i=1}^N \lambda_i \vec{\nu}_i \otimes \vec{\nu}_i \qquad \mu - \text{a.e.}$$

where λ_i, $i=1,\cdots,N$, are μ–measurable functions, and $\vec{\nu}_i$, $i=1,\cdots,N$, are μ measurable unit vectors, and they satisfy

$$
\begin{aligned}
(3.14) \qquad & 0 \le \lambda_i \le 1 \ (i=1,\cdots,N), \\
& \Sigma_{i=1}^N \lambda_i \le 1, \Sigma_{i=1}^N \vec{\nu}_i \otimes \vec{\nu}_i = \mathrm{I}, \qquad \mu - \text{a.e.}
\end{aligned}
$$

It then follows from equation (3.11) that for a.e. $t \in (0,\infty)$ and every $\vec{Y}(x) \in C_0^1(\Omega;\mathbb{R}^N)$,

$$
\begin{aligned}
2\int_\Omega \chi_{E_t} \operatorname{div}\left(v(\cdot,t)\vec{Y}\right) &= \int_\Omega D\vec{Y} : \left(I - \sum_{i=1}^N \lambda_i(x,t)\vec{\nu}_i(x,t) \otimes \vec{\nu}_i\right) d\mu^t(x) \\
&= \int_\Omega D\vec{Y} : \sum_{i=1}^N c_i^t(x)\Big[\mathrm{I} - \vec{\nu}_i(x,t) \otimes \vec{\nu}_i(x,t)\Big]\, d\mu^t(x),
\end{aligned}
$$

where

$$
c_i^t(x) = \lambda_i(x,t) + \frac{1}{N-1}\Big(1 - \Sigma_{j=1}^N \lambda_j(x,t)\Big).
$$

Clearly, for a.e. $t>0$, $0 \le c_i^t \le 1$ and $\sum_{i=1}^N c_i^t \ge 1$ for μ^t-a.e. Now define $p_i^t = \{\vec{\nu}_i(x,t), -\vec{\nu}_i(x,t)\} \in \mathrm{P}$ and V^t as in (2.4). Then V defined by $dV(x,t,p) = dV^t(x,p)dt$ satisfies the fourth assertion of Theorem 2.1. This completes the proof of Theorem 2.1.

Remark 3.1. If one can show that V^t is rectifiable, then for $\|V^t\|$ – –a.e., $V^{x,t}(\cdot) := V(x,t,\cdot)$ is the Dirac measure supported on the normal of the unoriented tangent plane of $\|V^t\|$. It thus follows that $c_1^t = 1, c_2^t = \cdots = c_N^t = 0$, $\|V^t\|$-a.e.. Consequently, $\lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_N = 0$, μ-a.e.. The definition of λ_i hence implies that $|\xi^\varepsilon(u^\varepsilon)|dxdt \to 0$ as Radon measure on $\Omega \times (0,\infty)$.

4. The elliptic equation $-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon) = v^\varepsilon$.

This whole section is devoted to proving Theorem 3.6. We first study the blow–up problem.

4.1. The equation $\Delta U = f(U)$.

Lemma 4.1. *Assume that $U \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies the equation*

$$
(4.1) \qquad \Delta U = f(U) \quad \text{in } \mathbb{R}^N.
$$

Then $U \in C^3(\mathbb{R}^N)$, $-1 \le U \le 1$ in $\mathbb{R}^N$, and

$$
(4.2) \qquad |\nabla U(x)|^2 \le 2F(U(x)) \qquad \forall\, x \in \mathbb{R}^N.
$$

Moreover, if the equality in (4.2) holds at some point in $\mathbb{R}^N$*, then the equality holds in all* $\mathbb{R}^N$*, and* U *either is trivial or is a planer wave; namely, either* U *is a constant function being* 1 *or* -1*, or there exist* $x_0 \in \mathbb{R}^N$ *and a unit vector* $\vec{e} \in S^{N-1}$ *such that*

$$U(x) = q((x - x_0) \cdot \vec{e}), \qquad x \in \mathbb{R}^N,$$

where $q(\cdot)$ *is the unique solution to the ODE*

$$\ddot{q} = f(q), \qquad q(0) = 0, \qquad q(\pm\infty) = \pm 1. \tag{4.3}$$

Lemma 4.2. *The assertion of Lemma 4.1. remains true if* $U \in H^1_{\rm loc}(\mathbb{R}^{N-1} \times [0,\infty))$ *satisfies*

$$\Delta U = f(U) \ \text{ in } \ \mathbb{R}^{N-1} \times (0,\infty),$$
$$\tfrac{\partial}{\partial n} U = 0 \ \text{ on } \ \mathbb{R}^{N-1} \times \{0\}.$$

Proof of Lemma 4.2. The assertion follows immediately from Lemma 4.1. if we extend U into $\mathbb{R}^N$ evenly.

Proof of Lemma 4.1. We shall prove the lemma in three steps.

Step 1. First we show that U is bounded. Let $\zeta(\cdot) \in C^\infty(\mathbb{R}^N)$ be a cut-off function; namely,

$$0 \le \zeta \le 1 \text{ in } \mathbb{R}^N, \qquad \zeta = 1 \text{ in } B_{1/2}, \qquad \zeta = 0 \text{ in } \mathbb{R}^N \setminus B_1.$$

For $k = \frac{2p}{p-2}$ and any fixed $x_0 \in \mathbb{R}^N$, multiplying (4.1) by $\zeta^k(x-x_0)U(x)$ yields

$$\begin{aligned} 0 &= \int \zeta^k U(-\Delta U + f(U)) \\ &= \int \Big[\zeta^k |\nabla U|^2 + \zeta^k U f(U) + k\zeta^{k-1} U \nabla U \cdot \nabla \zeta\Big] \\ &\ge \int \Big[\zeta^k |\nabla U|^2 + \zeta^k U f(U) - \frac{1}{2}\zeta^k |\nabla U|^2 - \delta \zeta^k |U|^p - C(\delta,p)|\nabla\zeta|^k\Big], \end{aligned}$$

where δ can be any small positive constant. Since $Uf(U) \ge c_1|U|^p - c_2$ for all $U \in \mathbb{R}$, by taking $\delta = c_1/2$ it follows from the last inequality that

$$\int \zeta^k (|\nabla U|^2 + |U|^p) \le C(c_1, c_2, p, \|\nabla\zeta\|_p),$$

so that, for any $x_0 \in \mathbb{R}^N$, $\|U\|_{H^1(B(x_0,1/2))} \le C$ where C depends only on c_1, c_2 and p. Consequently, by elliptic regularity theory (cf. [40]), we have

$$U \in C^3(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Since $f'(s) \geq c_0$ whenever $|s| > 1$, by comparing the function U with the auxiliary functions $\pm(1 + \delta_1 e^{\delta\sqrt{1+|x|^2}})$ for fixed small δ and $\delta_1 \in (0, \infty)$, one can show that $|U| \leq 1$.

Step 2. Next, we prove (4.2). In fact, L. Modica had proved in [49] that any $C^3(\mathbb{R}^N)$ bounded solution of (4.1) satisfies (4.2). Here for reader's convenience, we provide a self-contained proof.

We define, for every small positive constant δ,

$$W_\delta := \frac{1}{2}|\nabla U|^2 - F(U) - G_\delta(U),$$

where

$$G_\delta(u) := \delta\Big[1 + \int_{-1}^{u} \exp\Big(- \textstyle\int_{-1}^{s} \frac{|f(\hat{s})|+\delta}{2(F(\hat{s})+\delta)} d\hat{s}\Big)\, ds\Big].$$

One can directly calculate

$$\begin{aligned} \Delta W_\delta =& D^2U : D^2U + \nabla U \cdot \nabla(\Delta U) \\ & - (f + G_\delta{}')\Delta U - (f' + G_\delta{}'')|\nabla U|^2 \\ =& D^2U : D^2U - f(f + G_\delta{}') - 2G_\delta{}''(W_\delta + F + G_\delta) \end{aligned} \tag{4.4}$$

by substituting the relation $\Delta U = f(U)$ and $|\nabla U|^2 = 2(W_\delta + F + G_\delta)$. From the definition of W_δ, $\nabla W_\delta = D^2U \cdot \nabla U - (f + G_\delta{}')\nabla U$ which implies that

$$\begin{aligned} |\nabla U|^2 D^2U : D^2U \geq& |D^2U \cdot \nabla U|^2 = |\nabla W_\delta + (f + G_\delta{}')\nabla U|^2 \\ \geq& 2(f + G_\delta{}')\nabla U \cdot \nabla W_\delta + (f + G_\delta{}')^2|\nabla U|^2. \end{aligned}$$

It then follows from (4.4) that, when $|\nabla U| > 0$,

$$\begin{aligned} \Delta W_\delta -& \frac{2(f + G_\delta{}')\nabla U}{|\nabla U|^2} \cdot \nabla W_\delta + 2G_\delta{}'' W_\delta \\ \geq& (f + G_\delta{}')^2 - (f + G_\delta{}')f - 2G_\delta{}''(F + G_\delta) \\ =& (G_\delta{}')^2 + [G_\delta{}' f - 2G_\delta{}''(F + G_\delta{}')] \\ >& (G_\delta{}')^2, \end{aligned} \tag{4.5}$$

where in the last equality, we have used the fact that

$$G_\delta{}' f - 2G_\delta{}''(F + G_\delta) = G_\delta{}'\Big[f + \frac{|f| + \delta}{F + \delta}(F + G_\delta)\Big] > 0,$$

since $G_\delta \geq \delta$ and $G_\delta{}' > 0$ whenever $U \in [-1, 1]$.

We are now ready to show that $\sup_{\mathbb{R}^N} W_\delta \le 0$. In fact if $\eta := \sup_{\mathbb{R}^N} W_\delta > 0$, then there exists x_0 such that $w(x_0) > \frac{3}{4}\eta$. Consequently, for any positive integer k, there exists $y_k \in B(x_0, k)$ which attains the maximum of the function $w_k := W_\delta + \frac{1}{2}\eta\zeta(\frac{x-x_0}{k})$ in $\mathbb{R}^N$. Hence, at y_k, $w_k > \eta$, $\nabla w_k = 0$, $D^2 w_k \le 0$. This translates, via the definition of w_k, that at y_k:

$$\begin{aligned}
&|\nabla U|^2 \ge 2W_\delta = 2w_k - \eta\zeta > \eta,\\
&|\nabla W_\delta| = \frac{1}{2}\eta|\nabla\zeta| \le C\eta/k,\\
&\Delta W_\delta \le \frac{1}{2}\eta\Delta\zeta \le C\eta/k^2.
\end{aligned}$$

But this contradicts (4.5) since as $k \to \infty$, the left–hand side is $\le O(1/k)$ (noting that $G_\delta'' < 0$) whereas the right-hand side is

$$\ge (G_\delta')^2 \ge \delta^2 \exp(-2\int_{-1}^{1} \frac{|f|+\delta}{2(F+\delta)}) > 0.$$

Therefore, we must have $\sup_{\mathbb{R}^N} W_\delta \le 0$, which, by the definition of W_δ, implies that

$$\frac{1}{2}|\nabla U|^2 \le F + G_\delta \le F + 3\delta \qquad \text{in } \mathbb{R}^N.$$

Sending $\delta \to 0$ we thus obtain (4.2).

Step 3. Now we show the second assertion of the lemma. Assume that equality in (4.2) holds at some point in $\mathbb{R}^N$ and that $U \not\equiv \pm 1$. Then the function $W_0 := \frac{|\nabla U|^2}{2} - F(u)$ attains its maximum. However, the same calculation as before but with $\delta = 0$ shows that W_0 cannot attain a local maximum unless W_0 is a constant. Hence, we must have $W_0 \equiv 0$ in $\mathbb{R}^N$. Consequently, $|\nabla U|^2 = 2F(U)$ in $\mathbb{R}^2$.

Since we assumed that $U \not\equiv \pm 1$, by maximum principle, $U(x) \in (-1, 1)$ for all $x \in \mathbb{R}^N$. Noting that $q(\cdot)$ is monotonic, there exists a unique function z such that $U(x) = q(z(x))$. It then follows from the identity $|\nabla U|^2 = 2F(U)$ that $|\nabla z| = 1$ in $\mathbb{R}^N$; namely z is a distance function. Furthermore, substituting $U = q(z)$ into the equation $\Delta U = f(U)$ yields $\Delta z = 0$ in $\mathbb{R}^N$. Since z grows at most linearly, the properties of harmonic functions then imply that z is a linear function. This completes the proof.

Remark 4.1. A parabolic version of Lemma 4.1 was first obtained by Ilmanen [46] in studying the Allen–Cahn equation $U_t = \Delta U - f(U)$

in $\mathbb{R}^N \times (0,\infty)$. Assuming that $U(\cdot,0) \in (-1,1)$ and writing $U(x,t) = q(z(x,t))$, Ilmanen [46] showed that $|\nabla z| \le 1$ provided that $|\nabla z(\cdot,0)| \le 1$. Later, Soner [58] extended Ilmanen's result by dropping the crucial condition $|\nabla z(\cdot,0)| \le 1$. Since in both their papers they studied the function z, they need the technical assumption $(q''/q')' \le 0$, which is equivalent to the condition

$$f^2 \ge 2Ff' \qquad \forall u \in (-1,1). \tag{4.6}$$

Now suppose we are studying equation (4.1) in a bounded domain. From our proof we can see that $W_0 = |\nabla u|^2 - 2F(u)$ satisfies the maximum principle (but not minimum principle); namely, if W_0 obtains an interior maximum, then W_0 is a constant function. This conclusion is regardless of the condition (4.6) and does not need the assumption $|U| < 1$. On the other hand, with the assumption (4.6) and $|U| < 1$, one can show that the function $\bar{W}_0 := \frac{|\nabla U|^2}{2F(U)} - 1$ satisfies the maximum principle; namely, $\bar{W}_0$ cannot attain an interior non–negative maximum unless it is a constant function. Clearly when $F(U)$ is small inside the domain but large on the boundary, controlling $\bar{W}_0$ maybe more useful than controlling W_0.

4.2. The equation $-\Delta U + f(U) = V$.

Lemma 4.3. *Let $\hat{\Omega}$ be a domain given by*

$$\hat{\Omega} := \{(x', x_N) \in B_R; x_N > Y(x')\},$$

where $R \ge 2$ and Y' satisfies

$$Y(0') \le 0, \quad \nabla_{x'} Y(0') = 0', \quad \|D^2_{x'} Y\|_{C^0(B'_R)} \le R^{-3}. \tag{4.7}$$

Also, let $(U,V) \in H^2(\hat{\Omega}) \times L^2(\hat{\Omega})$ be any pair of functions satisfying

$$-\Delta U + f(U) = V \qquad \text{in } \hat{\Omega}, \tag{4.8}$$

$$\frac{\partial}{\partial n} U = 0 \quad \text{on } \{(x', x_N) \in B_R : x_N = Y(x')\}, \tag{4.9}$$

$$\|V\|_{2, B_R \cap \hat{\Omega}} \le R^{-1}. \tag{4.10}$$

Then for every $\eta > 0$, there exists a large positive constant $R(\eta)$ which is independent of U, V, and Y such that if $R \ge R(\eta)$, then

$$\int_{B_1 \cap \hat{\Omega}} \Big(|\nabla U|^2 - 2F(U)\Big)^+ \le \eta \int_{B_2 \cap \hat{\Omega}} \Big[|\nabla U|^2 + f^2(U) + F(U) + V^2\Big] + \int_{\{x \in B_1 \cap \hat{\Omega}\, ;\, |U| \ge 1 - \eta\}} |\nabla U|^2. \tag{4.11}$$

Proof. We need to distinguish the case where U is close to trivial functions 1 or -1 from the the case where U is far from trivial. For this purpose, we define

$$\hat{\Omega}_1 = \{x \in B_1 \cap \hat{\Omega}\,;\ |U| \le 1-\eta\}.$$

Denote by 2^* the number $\frac{2N}{N-2}$ if $N > 2$ and any number, say 7, if $N = 2$. Set $m = \frac{22^*}{2^*-2}$. We consider two separate cases: (i) $|\hat{\Omega}_1| \le \eta^m$ and (ii) $|\hat{\Omega}_1| \ge \eta^m$.

(i) First we consider the case where $|\hat{\Omega}_1| \le \eta^m$. Note that

$$\begin{aligned} \|\nabla U\|_{2,\hat{\Omega}_1} \le& |\hat{\Omega}_1|^{\frac{2-2^*}{22^*}} \|\nabla U\|_{2^*,\hat{\Omega}_1} \\ \le& C\eta \|\nabla U\|_{H^1(B_1\cap\hat{\Omega})} \end{aligned} \tag{4.12}$$

by Sobolev's imbedding and the assumption on the measure of $\hat{\Omega}_1$. On the other hand, a basic elliptic estimate (cf. [40]) shows that

$$\begin{aligned} \|\nabla U\|_{H^1(B_1\cap\hat{\Omega})} \le& C\Big[\|\Delta U\|_{2,B_2\cap\hat{\Omega}} + \|\nabla U\|_{2,B_2\cap\hat{\Omega}}\Big] \\ \le& C\Big[\|V\|_{2,B_2\cap\hat{\Omega}} + \|f\|_{2,B_2\cap\hat{\Omega}} + \|\nabla U\|_{2,B_2\cap\hat{\Omega}}\Big]. \end{aligned}$$

It then follows from (4.12) that

$$\|\nabla U\|_{2,\hat{\Omega}_1} \le C\eta\Big[\|V\|_{2,B_2\cap\hat{\Omega}} + \|f\|_{2,B_2\cap\hat{\Omega}} + \|\nabla U\|_{2,B_2\cap\hat{\Omega}}\Big].$$

Consequently,

$$\int_{B_1\cap\hat{\Omega}} |\nabla U|^2 \le C\eta^2 \int_{B_2\cap\hat{\Omega}} \Big[V^2+f^2+|\nabla U|^2\Big] + \int_{\{x\in B_1\cap\hat{\Omega}\,;\ |U|\ge 1-\eta\}} |\nabla U|^2,$$

and the assertion of the lemma follows. We remark that in this case, we need only $R(\eta) \ge 2$ and do not need the assumption that $\|V\|_{2,B_R\cap\hat{\Omega}}$ is small.

(ii) Next we consider the case $|\hat{\Omega}_1| \ge \eta^m$. We show the assertion of the lemma by a contradiction argument. If the assertion were not true, then there exists a sequence $\{(U^j, V^j, \Omega^j)\}_{j=2}^{\infty}$ such that for each $j \ge 2$, (U^j, V^j, Ω^j) satisfies (4.7)–(4.10) with $R = j$, but (4.11) is not true for (U^j, V^j). Of course, we have, for each $j \ge 2$, $|\hat{\Omega}^j| := |\{x \in B_1 \cap \Omega^j\,;\ |U^j| \le 1-\eta\}| \ge \eta^m$.

Using the same technique as in Step 1 in the proof of Lemma 4.1, we can show that for any fixed $r > 0$, if $j \ge r+2$, then $\|U^j\|_{H^2(B_r\cap\Omega^j)} + \|f\|_{2,B_r\cap\Omega^j}$ is bounded with a bound depending only on r.

Let Y^j be the function in (4.7) for Ω^j. We consider two separate cases: (a) $\liminf_{j\to\infty} Y^j(0) = -\infty$; (b) $d := \liminf_{j\to\infty} Y^j(0) > -\infty$.

In case (a) , we can select a subsequence $\{j^k\}$ from $\{j\}$ and a function $U \in H^2_{\text{loc}}(\mathbb{R}^N)$ such that as $k \to \infty$,

$$\begin{aligned}
Y^{j^k}(x') &\longrightarrow -\infty \text{uniformly in any compact subset of } \mathbb{R}^{N-1},\\
V^{j^k} &\longrightarrow 0 \ \text{ in } \ L^2(B_r), \qquad \forall r > 0,\\
U^{j^k} &\longrightarrow U \ \text{ in } \ H^2(B_r) \ \text{ and a.e. in } B_r, \qquad \forall r > 0,\\
f(U^{j^k}) &\longrightarrow f(U) \ \text{ in } \ L^q(B_r) \ \text{ and a.e. in } B_r, \qquad \forall r > 0, q \in [1,2),\\
F(U^{j^k}) &\longrightarrow F(U) \ \text{ in } \ L^1(B_r), \qquad \forall r > 0.
\end{aligned}$$

In addition, U satisfies

$$-\Delta U + f(U) = 0 \qquad \text{in } \mathbb{R}^N.$$

It then follows from Lemma 4.1 that

$$\begin{aligned}
\lim_{k\to\infty} \int_{B_1} \left(|\nabla U^{j^k}|^2 - 2F(U^{j^k})\right)^+ &= \int_{B_1} \left(|\nabla U|^2 - 2F(U)\right)^+\\
&= 0.
\end{aligned} \tag{4.13}$$

On the other hand, the assumption that $|\hat{\Omega}^{j^k}| \geq \eta^m$ will make the right-hand side of (4.11) uniformly (in j^k) positive. In fact, since $U^{j^k} \to U$ a.e. and $Y^{j^k}(0) \to -\infty$, we must also have $|\{x \in B_1 \ ; \ |U| \leq 1-\eta\}| \geq \eta^m$. Consequently,

$$\begin{aligned}
\lim_{k\to\infty} \eta \int_{B_1} \left(|\nabla U^{j^k}|^2 + F(U^{j^k})\right) &= \int_{B_1} \eta\left[|\nabla U|^2 + F(U)\right]\\
&\geq \eta^{m+1} \min_{s\in[-1+\eta, 1-\eta]} F(s).
\end{aligned} \tag{4.14}$$

But the last inequality and (4.13) imply that (4.11) must hold for U^{j^k} with large enough k. Hence, we obtain a contradiction, which shows that the assertion of the lemma holds.

The case (b) is similar. We can obtain a subsequence (U^{j^k}, Y^{j^k}) which converges to (U, d). The function U satisfies $-\Delta U + f(U) = 0$ in $\{x \ ; \ x_N > d\}$ and $\frac{\partial}{\partial x_N} U(x', d) = 0$ for all $x' \in \mathbb{R}^{N-1}$. Following the same argument as in case (a) and using Lemma 4.2 instead of Lemma 4.1 we derive the same conclusion. This completes the proof of the lemma.

4.3. Control of the bulk energy.

If we call the region where $|u^\varepsilon| > 1 - o(1)$ as the *bulk* region, and call the region where $|u^\varepsilon| < 1 - o(1)$ as the *interfacial* region, the following lemma shows that the bulk energy is small, comparing to the interfacial energy.

Lemma 4.4. *There exist positive constants C_0 and η_0 such that for every $\eta \in [0, \eta_0]$, every $\varepsilon \in (0, 1]$, and every $(u^\varepsilon, v^\varepsilon) \in \mathcal{K}_\varepsilon$,*

$$\begin{aligned}
&\int_{\{x\in\Omega\,;\,|u^\varepsilon|\geq 1-\eta\}} \Big[e^\varepsilon(u^\varepsilon) + \varepsilon^{-1} f^2(u^\varepsilon)\Big] \\
&\qquad \leq C_0\eta \int_{\{x\in\Omega\,;\,|u^\varepsilon|\leq 1-\eta\}} \varepsilon|\nabla u^\varepsilon|^2 + C_0\varepsilon\int_\Omega v^{\varepsilon 2}.
\end{aligned} \tag{4.15}$$

Proof. Let c_0 be as in (1.2)(b). For any $\eta \in [0, c_0/2]$, we define $g(u)$ such that $g(u) = f(u)$ if $|u| \geq 1 - \eta$, $g(u) = 0$ if $|u| \leq 1 - c_0$, and $g(u)$ is linear in the remaining part. Clearly, $0 \leq g^2 \leq fg$ for all u. From the identity

$$\begin{aligned}
\int_\Omega v^\varepsilon g(u^\varepsilon) &= \int_\Omega \Big[-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon)\Big] g(u^\varepsilon) \\
&= \int_\Omega \Big[\varepsilon g'(u^\varepsilon)|\nabla u^\varepsilon|^2 + \varepsilon^{-1} f(u^\varepsilon) g(u^\varepsilon)\Big],
\end{aligned}$$

we have, since $|\int_\Omega v^\varepsilon g(u^\varepsilon)| \leq \int_\Omega [\frac{\varepsilon}{2} v^{\varepsilon 2} + \frac{1}{2\varepsilon} g^2] \leq \int_\Omega [\frac{\varepsilon}{2} v^{\varepsilon 2} + \frac{1}{2\varepsilon} fg]$,

$$\begin{aligned}
&\int_{\Omega\cap\{|u|\geq 1-\eta\}} \Big[\varepsilon f'(u^\varepsilon)|\nabla u^\varepsilon|^2 + \frac{1}{2\varepsilon} f^2(u^\varepsilon)\Big] \\
&\qquad \leq \frac{\varepsilon}{2}\int_\Omega v^{\varepsilon 2} - \int_{\Omega\cap\{|u|<1-\eta\}} \varepsilon g'(u^\varepsilon)|\nabla u^\varepsilon|^2.
\end{aligned} \tag{4.16}$$

Since $|f(\pm(1-\eta))| = O(\eta)$, one has $g'(u) = O(\eta)$ when $|u| \leq 1 - \eta$. Also since $f' \geq c_0|u|^{p-2}$ when $|u| \geq 1 - c_0$, $F(u) \leq Cf^2(u)$ whenever $|u| \geq 1 - c_0$. The assertion of the lemma thus follows from (4.16).

4.4. Proof of Theorem 3.6.

Let $\eta > 0$ be any fixed small positive constant and $R = R(\eta)$ be as in Lemma 4.3. Assume that $\varepsilon \in (0, R^{-2}]$ is arbitrarily fixed.

Let $\{x_j\}_{j\in\mathcal{J}}$ be a maximal collection of points in $\bar\Omega$ such that

$$\inf_{i,j\in\mathcal{J}, i\neq j} |x_i - x_j| \geq \varepsilon.$$

Set $B^j = B(x_j, \varepsilon)\cap\Omega$. Clearly, $\cup_{j\in\mathcal{J}} B^j = \Omega$, and there exists a constant $C(N)$ depending only on the space dimension N such that

$$\sum_{j\in\mathcal{J}} \chi_{B(x_j,2\varepsilon)} \le C(N)$$

$$\text{and} \sum_{j\in\mathcal{J}} \chi_{B(x_j,R\varepsilon)} \le C(N)R^N \tag{4.17}$$

$$\text{for all } \; x \in \Omega.$$

For each $j \in \mathcal{J}$, we define

$$U^j(y) = u^\varepsilon(x_j + \varepsilon y), \quad V^j(y) = \varepsilon v^\varepsilon(x_j + \varepsilon y), \quad \Omega^j = \{y \mid x_j + \varepsilon y \in \Omega\}.$$

It is easy to check that

$$-\Delta_y U^j + f(U^j) = V^j \qquad \text{in } \; B_R \cap \Omega^j. \tag{4.18}$$

Notice that for each $j \in \mathcal{J}$, $\partial\Omega^j$ is isomorphic to the surface obtained by magnifying $\partial\Omega$ by a factor of ε, so that either $\partial\Omega^j \cap B_R$ is empty or $\partial\Omega^j \cap B_R$ can be represented, after a rotation, as a graph $y_N = Y^j(y_1, \cdots, y_{N-1})$ with $Y^j(0') \le 0$, $D_{y'}Y^j(0') = 0$, and

$$\|D_y^2 Y^j\|_{C^0(B'_R)} \le C(\|\partial\Omega\|_{C^2})\varepsilon^2.$$

Hence, by further assuming $\varepsilon^{1/2}C(\|\partial\Omega\|_{C^2}) \le 1$, one sees that (after a rotation) Y^j satisfies (4.7).

We decompose $\mathcal{J}$ into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ defined by

$$\mathcal{A} := \{j \in \mathcal{J} \; ; \; \|v^\varepsilon\|_{2,B(x_j,R\varepsilon)\cap\Omega} \le \varepsilon^{\frac{N}{2}-1}R^{-1}\},$$
$$\mathcal{B} := \mathcal{J} \setminus \mathcal{A} = \{j \in \mathcal{J} \; ; \; \|v^\varepsilon\|_{2,B(x_j,R\varepsilon)\cap\Omega} > \varepsilon^{\frac{N}{2}-1}R^{-1}\}.$$

First we consider the case where $j \in \mathcal{A}$. In this case, we have

$$\|V^j\|_{2,B_R\cap\Omega^j} = \varepsilon^{-N/2}\|\varepsilon v^\varepsilon\|_{2,B(x_j,R\varepsilon)\cap\Omega} \le R^{-1}.$$

It then follows from Lemma 4.3. that

$$\begin{aligned}
\int_{B_1\cap\Omega^j} &\Big(|\nabla U^j|^2 - 2F(U^j)\Big)^+ \\
&\le \eta \int_{B_2\cap\Omega^j} \Big(|\nabla U^j|^2 + F(U^j) + f^2(U^j) + (V^j)^2\Big) \\
&\quad + \int_{B_1\cap\{y\in\Omega^j; |U^j|\ge 1-\eta\}} |\nabla U^j|^2.
\end{aligned}$$

Transferring back to u^ε and v^ε we obtain

$$\int_{B^j}\left(\xi^\varepsilon(u^\varepsilon)\right)^+ \le \eta\int_{B(x_j,2\varepsilon)\cap\Omega}\left(e^\varepsilon(u^\varepsilon)+\varepsilon^{-1}f^2(u^\varepsilon)+\varepsilon v^{\varepsilon 2}\right)$$
$$+\int_{B(x_j,2\varepsilon)\cap\{x\in\Omega;|u^\varepsilon|\ge 1-\eta\}}\varepsilon|\nabla u^\varepsilon|^2.$$

Summing up $j\in\mathcal{A}$ and using (4.18) yields

$$\begin{aligned}\sum_{j\in\mathcal{A}}\int_{B^j}\left(\xi^\varepsilon(u^\varepsilon)\right)^+ \le& C(N)\eta\int_\Omega\left(e^\varepsilon(u^\varepsilon)+\varepsilon^{-1}f^2(u^\varepsilon)+\varepsilon v^{\varepsilon 2}\right)\\ &+C(N)\int_{\{x\in\Omega:|u^\varepsilon|\ge 1-\eta\}}\varepsilon|\nabla u^\varepsilon|^2\\ \le& C\eta\int_\Omega e^\varepsilon(u^\varepsilon)+C\varepsilon\int_\Omega v^{\varepsilon 2},\end{aligned}$$

where in the second inequality, we have used Lemma 4.4 and the fact that $f^2(u)\le CF(u)$ when $|u|\le 1$.

Next we consider the case where $j\in\mathcal{B}:=J\setminus\mathcal{A}$. By a local elliptic estimate, we have

$$\begin{aligned}\int_{B_1\cap\Omega^j}|\nabla U^j|^2\le& C\int_{B_2\cap\Omega^j}\left(f^2(U^j)+|V^j|^2+|U^j|^2\right)\\ \le& C+C\int_{B_2\cap\Omega^j}\left(|V^j|^2+f^2(U^j)\chi_{\{|U^j|\ge 1\}}\right).\end{aligned}$$

Transferring this estimate into $(u^\varepsilon,v^\varepsilon)$ and adding up $j\in\mathcal{B}$, we then obtain

$$\begin{aligned}\sum_{j\in\mathcal{B}}\int_{B^j}\varepsilon|\nabla u^\varepsilon|^2\le& C\varepsilon^{-1}\sum_{j\in\mathcal{B}}|B^j|+CC(N)\varepsilon\int_\Omega v^{\varepsilon 2}\\ &+CC(N)\varepsilon^{-1}\int_{\{x\in\Omega;|u^\varepsilon|\ge 1\}}f^2\\ \le& C\varepsilon^{-1}\sum_{j\in\mathcal{B}}|B^j|+C\varepsilon\int_\Omega v^{\varepsilon 2},\end{aligned}\tag{4.19}$$

where in the second inequality, we have used Lemma 4.4 with $\eta=0$ to control the integral involving f^2. Finally, since for every $j\in\mathcal{B}$, $\int_{B(x_j,R\varepsilon)\cap\Omega}v^{\varepsilon 2}\ge R^{-2}\varepsilon^{N-2}\ge\varepsilon^{-2}R^{-2}|B^j|/|B_1|$ where $|B_1|$ is the volume of the unit ball, follows that

$$\sum_{j\in\mathcal{B}}|B^j|\le\varepsilon^2|B_1|R^2\sum_{j\in\mathcal{B}}\int_{B(x_j,R\varepsilon)}v^{\varepsilon 2}\le\varepsilon^2|B_1|R^2C(N)R^N\int_\Omega v^{\varepsilon 2}$$

by (4.18). Substituting the last estimate into (4.19) we obtain

$$\sum_{j\in\mathcal{B}}\int_{B^j}\varepsilon|\nabla u^\varepsilon|^2 \le CC(N)[1+R^{N+2}]\varepsilon\int_\Omega v^{\varepsilon 2}.$$

Combining the estimates for the case $j\in\mathcal{A}$ and the case $j\in\mathcal{B}$ yields

$$\int_\Omega\Big(\xi^\varepsilon(u^\varepsilon)\Big)^+ \le C\eta\int_\Omega e^\varepsilon(u^\varepsilon)+\varepsilon\tilde{M}(\eta)\int_\Omega v^{\varepsilon 2}.$$

Renaming $C\eta$ as η we thus completes the proof of Theorem 3.6.

5. Case of radial symmetry.

In this section we shall restrict our attention to the case of radial symmetry. Hence, we assume that $\Omega = B_1$. We denote by S_r the sphere of radius r in $\mathbb{R}^N$ and by ω_N the area of unit sphere S_1. For convenience, we shall not distinguish functions of $x\in B_1$ from functions of $r\in[0,1)$. We do distinguish, however, the integrals of dx from that of dr, due to the consideration of singularities at the origin.

5.1. Equal partition of energy.

In the previous section, we have shown that the discrepancy measure $\xi^\varepsilon(u^\varepsilon)dxdt$ is non–positive in the limit. In this section we shall show that, in the case of radial symmetry, the limit is actually zero, as the following theorem proclaims.

Theorem 5.1. *Assume that $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon\in(0,1]}$ is a family of radially symmetric solutions of (1.1) with initial data satisfying (1.3). Then*

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega\Big|\xi^\varepsilon(u^\varepsilon)\Big|\,dxdt = 0.$$

Proof. Since $(u^\varepsilon, v^\varepsilon)$ is radially symmetric, $\xi^\varepsilon(u^\varepsilon)=\frac{\varepsilon}{2}u_r^{\varepsilon 2}-\frac{1}{\varepsilon}F(u^\varepsilon)$ and

(5.1)
$$-\varepsilon u^\varepsilon_{rr}-\frac{\varepsilon(N-1)}{r}u^\varepsilon_r+\frac{1}{\varepsilon}f(u^\varepsilon)=v^\varepsilon,\quad r\in(0,1),\ t\in(0,\infty).$$

Multiplying this equation by $r^{N-1}u^\varepsilon_r$ and integrating over $(0,r)$, we obtain

$$\int_0^r \hat{r}^{N-2}e^\varepsilon(u^\varepsilon)d\hat{r}+\frac{r^{N-1}}{N-1}\Big(\xi^\varepsilon(u^\varepsilon)+v^\varepsilon u^\varepsilon\Big)$$
$$-\int_0^r\hat{r}^{N-2}\Big(u^\varepsilon v^\varepsilon+\frac{\hat{r}}{N-1}v^\varepsilon_r u^\varepsilon\Big)d\hat{r}=0. \tag{5.2}$$

Integrating this identity from $r = 1/2$ to $r = 1$ then yields

$$
\begin{aligned}
\int_0^{1/2} \hat{r}^{N-2} e^\varepsilon(u^\varepsilon) d\hat{r} \le & C \int_{1/2}^1 r^{N-1} |\xi^\varepsilon(u^\varepsilon)| dr \\
& + C \int_0^1 (\hat{r}^{N-2} |v^\varepsilon u^\varepsilon| + \hat{r}^{N-1} |v_r^\varepsilon u^\varepsilon|) d\hat{r} \\
\le & C \int_\Omega e^\varepsilon(u^\varepsilon) \\
& + C(\|r^{-1/2} u^\varepsilon\|_{2,\Omega} \|r^{-1/2} v^\varepsilon\|_{2,\Omega} + \|u^\varepsilon\|_{2,\Omega} \|v_r^\varepsilon\|_{2,\Omega}).
\end{aligned} \tag{5.3}
$$

Since $\|r^{-1/2} v^\varepsilon\|_{2,\Omega} \le \|v^\varepsilon\|_{H^1(\Omega)}$ and

$$
\|r^{-1/2} u^\varepsilon\|_{2,\Omega} \le C + C \|r^{-1} F(u^\varepsilon)\|_{1,\Omega}^{1/2} \le C + C\sqrt{\varepsilon} \|r^{-1} e^\varepsilon(u^\varepsilon)\|_{1,\Omega}^{1/2},
$$

it then follows from (5.3) that

$$
\int_0^1 r^{N-2} e^\varepsilon(u^\varepsilon) dr \le C M^\varepsilon(t), \qquad M^\varepsilon(t) := 1 + \|v^\varepsilon\|_{H^1(\Omega)} + \varepsilon \|v^\varepsilon\|_{H^1(\Omega)}^2.
$$

As a consequence, this estimate implies that

$$
\int_{B_\delta} e^\varepsilon(u^\varepsilon) dx \le C \delta M^\varepsilon(t), \qquad \forall \delta \in (0,1), \tag{5.4}
$$

and that, via (5.2),

$$
\sup_{0<r<1} \left| r^{N-1} \Big(\xi^\varepsilon(u^\varepsilon) + v^\varepsilon u^\varepsilon \Big) \right| \le C M^\varepsilon(t). \tag{5.5}
$$

Hence, for any small δ and η,

$$
\begin{aligned}
\int_\Omega \left| \xi^\varepsilon(u^\varepsilon) \right| \le & \int_{B_\delta \cup \{|u^\varepsilon| \ge 1-\eta\}} e^\varepsilon(u^\varepsilon) dx \\
& + \int_{\Omega \cap \{r>\delta, |u^\varepsilon| \le 1-\eta\}} \Big[|v^\varepsilon|(1-\eta) + r^{1-N} C M^\varepsilon(t) \Big].
\end{aligned}
$$

The last integral can be controlled by

$$
C \delta^{(1-N)/2} M^\varepsilon(t) (\text{measure}\{|u^\varepsilon| \le 1-\eta\})^{1/2} \le C(\delta,\eta) \sqrt{\varepsilon} M^\varepsilon(t).
$$

Hence, using (5.4) to control the energy in B_δ and using Lemma 4.4 to control the energy in $\{|u^\varepsilon| \ge 1-\eta\}$ we then obtain

$$
\int_\Omega \left| \xi^\varepsilon(u^\varepsilon) \right| \le C_1 \Big\{ \delta + \eta + \varepsilon + C(\delta,\eta)\sqrt{\varepsilon} \Big\} M^\varepsilon(t),
$$

where C_1 is independent of $\varepsilon, \eta, \delta$ and t. The assertion of the theorem thus follows by integrating the last estimate in $(0,T)$ and sending first ε to 0 and then δ and η to 0.

Corollary 5.2. *Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be any sequence of positive numbers converging to 0. Also let $(u^{\varepsilon_j}, v^{\varepsilon_j})$ be the radially symmetric solution of (1.1) with initial data satisfying (1.3). Assume that as $j \to \infty$, $e^{\varepsilon_j}(u^{\varepsilon_j})dxdt$ converges, as Radon measure, to $d\mu(x,t)$. Then, for any $\psi(x,t) \in C_0([0,\infty); C(\bar\Omega))$,*

$$\begin{aligned}\int_0^\infty \int_{\bar\Omega} \psi d\mu &= \lim_{j\to\infty} \int_0^\infty \int_\Omega \varepsilon_j |\nabla u^{\varepsilon_j}|^2 \psi dxdt \\ &= \lim_{j\to\infty} \int_0^\infty \int_\Omega \frac{2}{\varepsilon_j} F(u^{\varepsilon_j}) \psi dxdt \\ &= \lim_{j\to\infty} \int_0^\infty \int_\Omega |\nabla W(u^{\varepsilon_j})| \psi dxdt,\end{aligned}$$

where $W(u) = \int_{-1}^{u} \sqrt{2F(s)}ds$.

5.2. No interfaces piling up.

The lower semicontinuity of the BV norm states that if $w^j \to w$ in $L^1(\Omega)$, then $|Dw|(\Omega) \le \liminf_{j\to\infty} |Dw^j|(\Omega)$. We only have inequality because certain oscillations of w^j may not be carried out to the limit function w.

In our case, there are possibilities leading to the discrepancy between the limit of the measure $\lim |DW(u^\varepsilon)|(\Omega)$ and the measure of the limit $|DW(\lim u^\varepsilon)|(\Omega) = 2\sigma |D\chi_{E_t}|(\Omega)$. One possibility is caused by the presence of phantom interfaces. That is, even if $u = \lim u^\varepsilon = -1$ a.e. near S_r so that $|DW(u)|(S_r) = 0$, for a sequence of ε, u^ε may go up and down (several times) near an $o(1)$ neighborhood of S_r. This instance up and down (known as phantom interfaces) produces energy which is carried to the limit of the measure, but not to the measure of the limit. Even if u does have a jump across S_r, so that $|DW(u)|(S_r) = 2\sigma|S_r|$, still u^ε can have arbitrary odd number of jumps (visually, interfaces piling up) so that $(\lim |DW(u^\varepsilon)|)(S_r) = (2m+1)|S_r|$, where m is a non-negative integer. Another possibility is that u^ε is uniformly away from ± 1 by a distance of order $O(\sqrt{\varepsilon})$ so that $\frac{1}{\varepsilon}F(u^\varepsilon)dxdt$ could carry non-trivial measure to the limit in the set where $|DW| = 0$. The second possibility, of course, has been ruled out by Lemma 4.4 for almost all time. Now in this subsection, under the assumption of radial symmetry, we shall rule out the first possibility; namely, there are no phantom or piling up of interfaces for almost all time.

Theorem 5.3. *Assume that $\{(u^{\varepsilon_j}, v^{\varepsilon_j})\}_{j=1}^{\infty}$ are radially symmetric solutions of (1.1) with initial data satisfying (1.3) and that as $j \to \infty$, $\varepsilon_j \searrow 0$, $e^{\varepsilon_j}(u^{\varepsilon_j})dxdt \to d\mu(x,t)$ as Radon measure on $\bar{\Omega} \times [0,T)$, and $u^{\varepsilon_j} \to -1 + 2\chi_E$ in $C^{1/9}([0,T]; L^1(\Omega))$ for any $T > 0$. Then for any $\psi \in C_0(\Omega \times [0,\infty))$,*

$$\int_0^{\infty} \int_{\Omega} \psi d\mu(x,t) = 2\sigma \int_0^{\infty} \int_{\Omega} \psi |D\chi_{E_t}| dxdt.$$

Clearly, Theorem 2.2. follows from Theorem 5.3 and the third remark in Subsection 2.4.

To prove Theorem 5.3, we need the following lemma which is a purely elliptic result; namely, it deals with functions of r that satisfy (5.1).

Lemma 5.4. *For every small positive constant δ, there exist a small positive constant $\varepsilon_0(\delta)$ and a large positive constant $C(\delta)$ such that for every $\varepsilon \in (0, \varepsilon_0(\delta)]$, if functions $u^\varepsilon(r)$ and $v^\varepsilon(r)$ satisfy (5.1), and*

$$\|v^\varepsilon\|_{H^1(B_1)} \le \delta^{-1}, \qquad \int_{B_1} e^\varepsilon(u^\varepsilon))dx \le \mathcal{E}_0 \tag{5.6}$$

then the following holds:

1. If $(a,b) \subset (\delta, 1]$ is an open interval where $|u^\varepsilon| < 1 - C(\delta)\sqrt{\varepsilon}$, then u^ε is strictly monotonic in (a,b) and $|b-a| \le C(\delta)\varepsilon|\ln \varepsilon|$.

2. Define $A^\varepsilon = \{r \in [2\delta, 1-2\delta]| \,;\, u^\varepsilon(r) = 0\}$. Then

$$\begin{aligned}
&\int_{2\delta}^{1-2\delta} r^{N-1} e^\varepsilon(u^\varepsilon)dr - C(\delta)\sqrt{\varepsilon} \\
&\quad \le 2\sigma \sum_{r \in A^\varepsilon} r^{N-1} \\
&\quad \le \int_{2\delta - C(\delta)\varepsilon|\ln\varepsilon|}^{1-2\delta+C(\delta)\varepsilon|\ln\varepsilon|} r^{N-1} e^\varepsilon(u^\varepsilon)dr + C(\delta)\sqrt{\varepsilon}.
\end{aligned}$$

3. For any $r \in A^\varepsilon$,

$$\left| v^\varepsilon(r) + sgn\Big(u_r^\varepsilon(r)\Big)\frac{\sigma(N-1)}{r} \right| \le C(\delta)\varepsilon^{1/8}.$$

4. If r_1 and r_2 are two different elements in A^ε, then

$$|r_1 - r_2| \ge \frac{1}{C(\delta)}.$$

Proof of Lemma 5.4. From the estimate (5.5), it follows

$$\|\xi^\varepsilon(u^\varepsilon(r))\|_{L^\infty((\delta,1))} \leq \hat{C}(\delta),$$

which implies, in (a,b), $|\varepsilon^2 u_r^\varepsilon| \geq 2F(u^\varepsilon) - \hat{C}(\delta)\varepsilon > 0$. Moreover, solving the equation $\varepsilon u_r^\varepsilon = \pm\sqrt{2F(u^\varepsilon) + O(\varepsilon)}$ we obtain

$$\begin{aligned} b - a = \Big| \int_{u^\varepsilon(a)}^{u^\varepsilon(b)} \frac{\varepsilon ds}{\sqrt{2F(s) + O(\varepsilon)}} \Big| \leq& \int_{-1+C\sqrt{\varepsilon}}^{1-C\sqrt{\varepsilon}} \frac{\varepsilon ds}{\sqrt{2F(s) - C\varepsilon}} \\ \leq& C(\delta)\varepsilon|\ln\varepsilon|. \end{aligned}$$

This proves the first assertion of the lemma.

Note that,

$$\begin{aligned} \int_a^b r^{N-1} e^\varepsilon(u^\varepsilon) = \int_a^b & r^{N-1}[|W_r(u^\varepsilon(r))| + O(\xi^\varepsilon)] \\ =& (a + O(\varepsilon|\ln\varepsilon|)^{N-1}[|W(u^\varepsilon(b)) - W(u^\varepsilon(a))| \\ &+ O(1)(b-a)]. \end{aligned}$$

Applying Lemma 4.4 with $\eta = C\sqrt{\varepsilon}$, after a routine calculation, we then obtain the second assertion of the Lemma.

Now we prove the third assertion. Let $r \in A^\varepsilon$ be arbitrary, and define

$$U(\rho) = u^\varepsilon(r + \varepsilon\rho), \quad \alpha = \varepsilon v^\varepsilon(r), \quad \beta = \varepsilon(N-1)/r.$$

Then U and $P = U_\rho$ satisfies the ODE system

(5.7)
$$\begin{cases} U' = P, \\ P' = f(U) - \alpha - \beta P + h^\varepsilon(\rho), \quad \rho \in (-\varepsilon^{-1/4}, \varepsilon^{-1/4}), \end{cases}$$

where

$$h^\varepsilon = \varepsilon\Big\{[v^\varepsilon(r) - v^\varepsilon(r+\varepsilon\rho)] + \frac{(N-1)\varepsilon\rho}{r(r+\varepsilon\rho)}P\Big\}.$$

Since $v_r \in L^2([\delta/2, 1])$ and $P = \varepsilon u_r^\varepsilon$ is bounded in $[-\varepsilon^{-1/4}, \varepsilon^{-1/4}]$ with a bound depending only on δ and $\mathcal{E}_0$ (by elliptic estimates and the energy bound for u^ε), we have

$$\|h^\varepsilon\|_{L^\infty([-\varepsilon^{-1/4}, \varepsilon^{-1/4}])} \leq C(\delta)\varepsilon^{11/8}.$$

Now consider the ODE system

$$(5.8) \qquad U' = P, \quad P' = f(U) - \alpha - \beta P$$

for small parameters α and β. One can use the phase plane technique to conclude the following:

1. Let $U^{\pm}(\alpha)$ be the unique root of $f(U) = \alpha$ near ± 1. Then $(U^+(\alpha), 0)$ and $U^-(\alpha), 0)$ are saddle stationary points of (5.8).

2. There exist positive constants α_0 and β_0 and a C^2 function $c(\beta)$ defined on $[-\beta_0, \beta_0]$ such that for every $\beta \in [0, \beta_0]$, the ODE system (5.8) has a heteroclinic orbit connecting $(U^+(\alpha), 0)$ and $(U^-(\alpha), 0)$ if and only if $\alpha = c(\beta)$ or $\alpha = c(-\beta)$. Namely, if we denote respectively by $\gamma_s^{\pm}(\alpha, \beta)$ and $\gamma_u^{\pm}(\alpha, \beta)$ the stable and unstable manifolds of $(U^{\pm}(\alpha), 0)$ (i.e, trajectory that enters or leaves $(U^{\pm}(\alpha), 0))$, then $\gamma_u^+ = \gamma_s^-$ if and only if $\alpha = c(\beta)$, and $\gamma_u^- = \gamma_s^+$ if and only if $\alpha = c(-\beta)$. In addition, $c(\beta) = -\sigma\beta + O(\beta^2)$ for small β.

3. If $\alpha \in [-\alpha_0, \alpha_0]$ and $\beta \in (0, \beta_0]$ satisfy the relation

$$|\alpha - c(\pm\beta)| \geq \beta^{9/8},$$

then for any trajectory of (5.8) with $U(0) = 0$ and $\pm P(0) \geq 0$, at lest one of the following is true:

(a) For some $t^* \in (0, C|\ln\beta|]$, $P(t)$ does not change sign in $(0, t^*)$ and $|U| \to \infty$ as $t \to t^*$;

(b) For some $t_* \in [-C|\ln\beta|, 0)$, $P(t)$ does not change sign in $(t_*, 0)$ and $|U| \to \infty$ as $t \searrow t_*$;

(c) $U \in (U^-(\alpha) + \frac{1}{C}\beta^{9/8}, U^+(\alpha) - \frac{1}{C}\beta^{9/8})$ either in $[0, \infty)$ or in $(-\infty, 0]$. In the former case the positive half trajectory rotates around certain points on $(U^-(\alpha), U^+(\alpha)) \times \{0\}$ infinitely many times, so that for some positive constant C depending only on α_0, β_0, and f,

$$\int_0^{jC\ln|\beta|} F(u)d\rho \geq j$$

for all positive integer j. A similar case happens in the latter case.

The proof is omitted. We refer interested readers to Smoller [56, Chapter §C §D], or Aronson & Weinberger [6], or Fife & Hsiao [41].

From the properties of the solution of (5.8), a perturbation argument then shows that if $|\alpha - c(\pm\beta)| \geq 2\beta^{9/8}$ (the signs $\pm$ go along with the signs of $P(0)$), then (5.7) could not have a solution in $(-\varepsilon^{-1/4}, \varepsilon^{1/4})$ such that $\int_{-\varepsilon^{1/4}}^{\varepsilon^{1/4}} F(U)d\rho$ is bounded by $\mathcal{E}_0$. Thus, we must have $\alpha = c(\pm\beta) + O(\beta^{9/8})$. This yields the third assertion of the lemma.

The last assertion follows from the third one since if r_1 and r_2 are neighboring elements of A^ε, then $u_r^\varepsilon(r_1)$ and $u_r^\varepsilon(r_2)$ have different signs so that

$$\begin{aligned}C\|v^\varepsilon\|_{H^1(\Omega)}\sqrt{|r_2-r_1|} \geq& |v^\varepsilon(r_1) - v^\varepsilon(r_2)|\\ =&\sigma(N-1)\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big) - O(\varepsilon^{1/8}).\end{aligned}$$

This completes the proof of the lemma.

Proof of Theorem 5.3. We prove the theorem by a contradiction argument. Assume the assertion is not true. Since

$$2\sigma|D\chi_{E_t}|dxdt \leq d\mu,$$

there exists $T > 0$ such that

$$\int_0^T \int_\Omega d\mu(x,t) > 2\sigma \int_0^T \int_\Omega |D\chi_{E_t}|dxdt.$$

Also since $\lim_{\delta\searrow 0}\int_0^T\int_{B_\delta} d\mu = 0$ (by the estimate (5.4)) and $\lim_{\delta\searrow 0}\int_0^T\int_{B_1\setminus\bar{B}(1-\delta)} d\mu = 0$ (by the properties of measures), there exists $\delta > 0$ such that

$$\int_0^T \int_{B_{1-3\delta}\setminus\bar{B}_{3\delta}} d\mu \geq 2\sigma \int_0^T \int_\Omega |D\chi_{E_t}(x)|dxdt + \delta(T + 2\mathcal{E}_0 + 1).$$

Consequently, by the definition of $d\mu$, there exists a large positive integer J such that for all $j \geq J$,

$$\int_0^T \int_{B_{1-2\delta}\setminus\bar{B}_{2\delta}} d\mu_t^{\varepsilon^j}(x)dt \geq 2\sigma \int_0^T \int_\Omega |D\chi_{E_t}|dxdt + \delta(T + 2\mathcal{E}_0),$$

where $d\mu_t^\varepsilon := e^\varepsilon(u^\varepsilon)dx$. Recalling that $\mu_t^\varepsilon(\Omega) = \mathcal{E}^\varepsilon(t) \leq \mathcal{E}_0$ for any ε and every t, we then have that

$$\text{measure}\{t\in[0,T]\ ;\ \mu_t^{\varepsilon_j}(B_{1-2\delta}\setminus\bar{B}_{2\delta}) \geq 2\sigma|D\chi_{E_t}|(\Omega)+\delta\} \geq 2\delta, \quad \forall j \geq J.$$

Also note that the set $\{t\in[0,T]\ ;\ \|v^\varepsilon\|_{H^1(\Omega)} \geq \delta^{-1}\}$ has measure $\leq \delta^2\int_0^T\|v^\varepsilon\|^2_{H^1(\Omega)} \leq \delta^2\mathcal{E}_0 \leq \delta$. Hence, for each $j \geq J$, there must exist $t_j \in [0,T]$ such that

$$\begin{aligned}(5.9)\qquad &\|\nabla v^{\varepsilon_j}(\cdot,t_j)\|_{2,\Omega} \leq \delta^{-1},\\ &\mu_{t_j}^{\varepsilon_j}(B_{1-2\delta}\setminus\bar{B}_{2\delta}) \geq 2\sigma|D\chi_{E_{t_j}}|(\Omega) + \delta.\end{aligned}$$

We now show that (5.9) is impossible for sufficiently large j.

For each $j \geq J$, we define

$$A^j := \{r \in [\delta, 1-\delta] \ ; \ r \in \operatorname{spt}(|D\chi_{E_{t_j}}(r)|)\},$$
$$A^{\varepsilon_j} := \{r \in [2\delta, 1-2\delta] \ ; \ u^{\varepsilon_j}(r, t_j) = 0\}.$$

Clearly, $|D\chi_{E_{t_j}}|(\Omega) \geq \sum_{r \in A^j} \omega_N r^{N-1}$. Also, by the first inequality in (5.10) and Lemma 5.4(2), there exists a large integer $J_1 \geq J$ such that

$$\mu_{t_j}^{\varepsilon_j}(B_{1-2\delta} \setminus \bar{B}_{2\delta}) \leq 2\sigma \sum_{r \in A^{\varepsilon_j}} \omega_N r^{N-1} + \frac{\delta}{2} \qquad \forall j \geq J_1.$$

Hence, by the second inequality in (5.10),

$$\sum_{r \in A^{\varepsilon_j}} \omega_N r^{N-1} \geq \sum_{r \in A^j} \omega_N r^{N-1} + \frac{\delta}{4\sigma}, \quad \forall j \geq J_1. \tag{5.10}$$

Since $u^{\varepsilon_j} \to -1 + 2\chi_E$ in $C^{1/9}([0,T]; L^1(\Omega))$,

$$h_j := \sqrt{\varepsilon_j} + \sup_{t \in [0,T]} \int_\delta^1 |u^{\varepsilon_j}(r,t) + 1 - 2\chi_{E_t}(r)| dr \to 0 \quad \text{as} \ j \to \infty.$$

We claim that (5.10) and the definition of h_j imply the existence of $J_2 \geq J_1$ such that

$$\min_{r_1, r_2 \in A^{\varepsilon_j}, r_1 \neq r_2} |r_1 - r_2| \leq 4h_j, \quad \forall j \geq J_2. \tag{5.11}$$

In fact if $A^{\varepsilon_j} \subset \cup_{r \in A^j}(r - 2h_j, r + 2h_j)$, since the total number of elements in A^j is bounded independent of j, then inequality (5.10) and the assumption $A^{\varepsilon_j} \subset \cup_{r \in A^j}(r - 2h_j, r + 2h_j)$ imply that A^{ε_j} has more elements than A^j for sufficiently large j (so that h_j is sufficiently small). Hence, for some $r \in A^j$, there are at least two elements of A^{ε_j} in $(r - 2h_j, r + 2h_j)$, which concludes (5.11). (This corresponds to piling up of interfaces.)

If the condition $A^{\varepsilon_j} \subset \cup_{r \in A^j}(r - 2h_j, r + 2h_j)$ does not hold, then there exists $r_1 \in A^j$ such that $r_1 \notin \cup_{r \in A^j}(r - 2h_j, r + 2h_j)$. Therefore, $\chi_{E_{t_j}} \equiv 1$ or $\equiv -1$ in the interval $(r_1 - 2h_j, r_1 + 2h_j)$, so that by Lemma 5.4 (1) and the definition of h_j, there must exist $r_2 \in A^{\varepsilon_j}$ such that $r_2 \in (r_1 - 2h_j, r_1) \cup (r_1, r_1 + 2h_j)$, and (5.11) follows. (This corresponds to phantom interfaces.)

However, (5.11) cannot hold for sufficiently large j since Lemma 5.4. (4) claims that (recalling the first inequality in (5.10))

$$\min_{r_1,r_2\in A^{\varepsilon_j},r_1\neq r_2} |r_1 - r_2| \geq \frac{1}{C(\delta)}$$

for all $j \geq J$ whereas $h_j \to 0$ as $j \to \infty$. This contradiction shows that the assertion of the theorem must be true. q.e.d.

Remark 5.1. Though $\mu^t(\bar{\Omega})$ is non–increasing in t, our Theorem 5.3 does not imply that $|D\chi_{E_t}|(\Omega)$ is non–increasing in t since $\mu^t(\bar{\Omega}) = \mu^t(\Omega) + \mu^t(\partial\Omega)$ where $\mu^t(\partial\Omega)$ may not be identically zero for a.e. $t > 0$. In fact, besides shrinking the radius of the interface to decrease the energy, moving an interface toward the boundary of Ω and then making it disappear also does the job. If the distance from the interface to the boundary is neither too large (so that it does not move away from the boundary quickly) nor to small (so that it does not disappear very fast), then it will stay there for a time interval. For example, let $u^\varepsilon(\cdot,0)$ be defined by $u^\varepsilon(r,0) = 1$ in $[0,1-d_\varepsilon]$, $= -1$ in $[1-d_\varepsilon+\varepsilon,1]$, and linear in $[1-d_\varepsilon, 1-d_\varepsilon+\varepsilon]$. If d_ε is not too small, due to the mass conservation, we believe that this interface will stay for a time interval $[0,T]$, and hence, although in the limit, $u^\varepsilon \to 1$ in $\Omega \times [0,\infty)$ so that $\mu^t(\Omega) = 0$, $\mu^t(\partial\Omega) = \lim \mathcal{E}^\varepsilon(t) > 0$ for a.e. $t \in [0,T]$. Of course, if d_ε is small, the interface moves toward the boundary $\partial\Omega$ and then disappears very fast so that in the limit, $\mu^t(\partial\Omega) = 0$ for all $t > 0$. Our analysis in this section may be extended to show that

$$\mu^t(\partial\Omega) = m(t)\omega_N \quad \text{for a.e. } t > 0,$$

where $m(t)$ is a non–negative integer valued non–increasing function.

Remark 5.2. Theorem 5.3 shows that, regardless of the distribution of the initial energy, for almost every t, there are no phantom interfaces (the spheres $(\Omega \cap \mathrm{spt}(\mu^t)) \setminus \mathrm{spt}(|D\chi_{E_t}))$ and all interfaces have multiplicity one (i.e., $2\sigma|D\chi_{E_t}| = \mu^t$ on $\mathrm{spt}(|D\chi_{E_t}|)$). This is a sharp contrast to the motion by the mean curvature equation as the limit of the Allen–Cahn equation, where there are phantom interfaces and interfaces of any odd finite integer multiplicity. For example, consider $u^\varepsilon(\cdot,0)$ defined by $u^\varepsilon(r,0) = 1$ in $[0,1] \setminus [1/2 - d^\varepsilon - \varepsilon, 1/2 + d^\varepsilon + \varepsilon]$, $= -1$ in $[1/2 - d^\varepsilon, 1/2 + d^\varepsilon]$ and linear in the rest of the interval $[0,1]$, where d^ε is not too small, say, $d^\varepsilon = \sqrt{\varepsilon}$. Clearly $u^\varepsilon(r,0) \to 1$ as $\varepsilon \to 0$. If one takes this $u^\varepsilon(r,0)$ as the initial data for the Cahn–Hilliard equation, we can conclude that $\mathcal{E}^\varepsilon(t) \to 0$ for every $t > 0$.

On the other hand, if one takes this as initial data for the Allen–Cahn equation, one can show that $u^\varepsilon \to 1$ in $C^\alpha([0,T];L^1(\Omega))$ as $\varepsilon \to 0$, but $e^\varepsilon(u^\varepsilon)dx \to 4\sigma\omega_N r^{N-1}\delta(r-r(t))dr$ where $r(t)$ is the solution of the motion by the mean curvature equation $r_t = -(N-1)/r$ with $r(0) = 1/2$. That is, the two (phantom) interfaces act as if they did not see each other.

5.3. Examples of the solutions of the limit problem.

In this subsection, we shall point out a few features of the solution of the limit problem (1.6) in radially symmetric case. For this purpose, we shall take for granted that the limits obtained from the solutions of the Cahn–Hilliard equation are classical solutions of (1.6). (We conjecture that Theorem 5.3 is sufficient to do this.)

Now assume, as before, that $\Omega = B_1$. Also assume that for every $t \geq 0$, $\mathrm{spt}(|D\chi_{E_t}|) = \cup_{j=0}^{J(t)}\{r_j(t)\}$ where $J(t) \geq 1$ is a finite integer and

$$1 > r_1(t) > \cdots > r_{J(t)} > 0 \qquad \forall t \geq 0.$$

We assume that $J(t)$ changes its value only at times when $r_1 \to 1$, or $r_{J(t)} \to 0$, or $r_j(t) - r_{j+1}(t) \to 0$ for some $j = 1, \cdots, J(t)-1$. (This is equivalent to assume that there is no nucleation of interfaces.)

We assume without loss of generality that $\sigma = 1$.

A. The ODE systems.

Since v is harmonic in $(r_j(t), r_{j+1}(t))$, using the interfacial condition in Theorem 2.2, we have (assuming WLOG that $\chi_{E_t} = 0$ near $r = 1$)

(5.12)

$$v(r,t) = \begin{cases} \frac{N-1}{r_1}, & r \in [r_1, 1], \\ (-1)^{j+1}\Big\{\frac{N-1}{r_j} - \big(\frac{N-1}{r_j} + \frac{N-1}{r_{j+1}}\big)\frac{r^{2-N}-r_j^{2-N}}{r_{j+1}^{2-N}-r_j^{2-N}}\Big\}, & \\ \qquad r \in [r_{j+1}, r_j],\ j = 1, \cdots, J-1, & \\ (-1)^{J+1}\frac{N-1}{r_J}, & r \in [0, r_J], \end{cases}$$

where $r_j = r_j(t)$ and $J = J(t)$. Here we understand that if $N = 2$, then r^{2-N} should be replaced by $\ln r$.

Hence the weak formulation of $(2\chi_E)_t = \Delta v$ yields

$$\dot{r}_j(t) := \frac{d}{dt}r_j(t) = (-1)^{j+1}[v_r(r_j(t)+0,t) - v_r(r_j(t)-0,t)].$$

It then follows that

(5.13)

$$\dot{r}_j(t) = -r_j^{1-N}(t)\Big\{g_{j-1/2}(t) + g_{j+1/2}(t)\Big\} \quad j = 1, \cdots, J(t)$$

where

(5.14)

$$g_{j-1/2} = \begin{cases} (N-2)(N-1)\frac{r_j^{-1}+r_{j-1}^{-1}}{r_j^{2-N}-r_{j-1}^{2-N}} & j = 2, \cdots, J(t), \\ 0 & j = 1, J(t)+1. \end{cases}$$

Here again, when $N = 2$, the quantity $\frac{N-2}{r_j^{2-N}-r_{j-1}^{2-N}}$ should be understood as $\frac{1}{\ln r_{j-1} - \ln r_j}$.

Lemma 5.5. *Let $J^0 \geq 2$ be an integer and $\{r_j^0\}_{j=0}^{J^0}$ be real numbers satisfying $1 > r_1^0 > \cdots > r_{J^0}^0 > 0$. Let $[0, t_1)$ be the maximal time interval where (5.13) has a smooth solution with $J(t) = J^0$ and $r_j(0) = r_j^0$, $j = 1, \cdots, J^0$. Then the following hold:*

1. *$\dot{r}_j(t) < 0$ for all $j = 1, \cdots, J(t)$ and all $t \in [0, t_1)$;*
2. *$t_1 < \infty$;*
3. *$\min_{1 \leq j \leq J^0-1} \inf_{t \in [0,t_1)} |r_j(t) - r_{j+1}| > 0$;*
4. *$\lim_{t \nearrow t_1} r_{J^0}(t) = 0$.*

Proof. The first and second assertions of the lemma follow directly from the ODE equations. In the sequel, we denote $r_j(t_1) = \lim_{t \nearrow t_1} r_j(t)$ for all $j = 1, \cdots, J^0$. Also, we denote $r_0(t) \equiv 1$.

We show the third assertion by a contradiction argument. Assume that the assertion is not true. Then there exists $i \in \{1, \cdots, J^0 - 1\}$ such that $r_{i-1}(t_1) > r_i(t_1) = r_{i+1}(t_1) \geq 0$. We show that this is impossible. In fact, since $r_{i-1} - r_i$ is bounded away from zero in $[0, t_1]$, subtracting the equations for r_i and r_{i+1} we have

$$\begin{aligned} \frac{d}{dt}\Big(r_i^N - r_{i+1}^N\Big) =& N(g_{i+3/2} - g_{i-1/2}) \\ =& N(N-2)(N-1)\left\{\frac{r_{i+1}^{-1} + r_{i+2}^{-1}}{r_{i+2}^{2-N} - r_{i+1}^{2-N}} - \frac{r_i^{-1} + r_{i-1}^{-1}}{r_i^{2-N} - r_{i-1}^{2-N}}\right\} \\ \geq& -C, \qquad \forall t \in [0, t_1) \end{aligned}$$

where C is a constant. (Here for $i = 1$ or $i = J^0 - 1$, the second equality needs obvious modification.) Integrating this inequality from t to $t_1 - \delta$ ($\delta > 0$) and then sending δ to 0, we obtain that

$$r_i^N(t) - r_{i+1}^N \leq C(t_1 - t) \qquad \forall t \in [0, t_1),$$

which implies

$$r_i^N(t) \leq -\int^t \frac{C}{t_1 - s} ds \to -\infty$$

as $t \to t_1$, which is impossible. This contradiction shows the third assertion of the lemma.

The last assertion of the lemma follows from the second and third assertions. q.e.d.

By Lemma 5.5, we can conclude the following:

1. No interfaces will collide.

2. The only possibility that an interface disappears is by approaching the origin.

3. If $J^0 = 1$, then it is an equilibrium.

4. If $J^0 = 2$, then there is a finite time t_1 such at t_1, the smaller interface disappears, and the dynamics reaches an equilibrium.

5. If $J^0 \geq 2$, then there exist $t_0 := 0 < t_1 < t_2 < t_{J^0-1} < t_{J^0} := \infty$ such that $J(t) = J^0 - i$ in $[t_i, t_{i+1})$ for all $i = 0, \cdots, J^0 - 1$; in particular, the dynamics reaches its equilibrium in the finite time t_{J^0-1}.

B. Motion of "phantom" interfaces.

Finally, we consider the special cases where there are interfaces that are very close initially. By abusing the language, we also call them "phantom" interfaces.

For simplicity and the purpose of illustration, consider the case where $J^0 = 3$, and

$$r_1^0 = 1/2, \qquad r_2^0 = 1/2 - \delta, \qquad r_3^0 = 1/4,$$
$$(\text{i.e. } E_0(r) = [0, 1/4] \cup [1/2 - \delta, 1/2]),$$

where δ is a very small parameter. Clearly, the first and second interfaces are "phantom" interfaces.

From the ODE system (5.13) and the proof of Lemma 5.5, we can easily show that the quantity $r_1(t) - r_2(t)$ and $r_3(t)$ vary in $O(1)$ magnitude, if r_2 is away from r_3; namely, before r_2 catches up r_3 in $O(\delta)$ time, the distance between r_1 and r_2 is $O(\delta)$. Hence, the pair of "phantom" interfaces $\{r_1, r_2\}$ move toward the origin with a speed of order $O(\delta^{-1})$, whereas the "real" interface r_3 does not show appreciable movement in $O(\delta)$ time.

After $O(\delta)$ time, the pair of "phantom" interfaces $\{r_1, r_2\}$ get very close to the "real" interface r_3, so that all the three interfaces cluster near $r = 1/4$. Now by the conservation of the mass, one can show that there is a new grouping: the interfaces r_2 and r_3 pair together as "phantom" interfaces moving toward the origin rapidly, whereas the interface r_1 is detached from the cluster, leaving behind as a "real" interface.

Finally, after another $O(\delta)$ time, the new pair of "phantom" interfaces $\{r_2, r_3\}$ disappear successively at the origin, and the system reaches its equilibrium.

Using a similar analysis, one can study the cases where there are arbitrarily number of interfaces which form a number of clusters initially. In this case, one has to use the smallest distance among neighboring interfaces as a criterion to distinguish "phantom" (cluster) or "real" interfaces.

For any cluster, if the number of interfaces are odd, then the interface with the largest radii will not move appreciably in a short time, whereas the remaining even number of interfaces move very fast toward the origin. In addition, if one groups these remaining even number of interfaces pair by pair, then the distances between these pairs may change significantly. (Hence, multiple time scales maybe needed.) If a cluster has even number interfaces, then all of them move towards the origin very fast, though the relative speeds of different pairs maybe very large. (Again, in this case multiple time scales are needed.)

When a cluster of interfaces approach an interface or a cluster of interfaces, if we consider all of them as a single cluster, then it moves by the way we just described in the preceding paragraph.

Finally, after a very short time, all the "phantom" interfaces are gone by disappearing at the origin, leaving all "real" interfaces, i.e, interfaces that are well separated.

Remark 5.3. We believe that the above described motion of "phantom" interfaces is actually the short time dynamics of the phantom interfaces of the radially symmetric solution u^ε of the Cahn–Hilliard equation (1.1) where δ can be arbitrarily small, say $O(\varepsilon)$. That is, phantom interfaces are not annihilated; they move toward the origin with a speed proportional to the inverse of their distance, so that they disappear after $o(1)$ (with respect to ε) time. Clearly this kind of motion of "phantom" interfaces is totally different from its one-dimensional counterpart, where "phantom" interfaces annihilate each other. Also, it is different from the Allen–Cahn dynamics where phantom interfaces can be either annihilated (if their distance is $o(\varepsilon|\ln\varepsilon|)$ or propagate as regular separate interfaces (if their distance, say, is $\geq \varepsilon|\ln\varepsilon|^2$).

Remark 5.4. In studying the motion by the mean curvature flow and its counterpart, the Allen–Cahn equation, a formula called monotonicity formula (cf. [46]) plays an essential role. In terms of the Allen-Cahn equation, this monotonicity formula ensures the "finite" propagation of the energy density $e^\varepsilon(u^\varepsilon)$; namely, the total energy in a ball B_r at any time in any time interval $[t_0, t_0 + \delta]$ is totally controlled by the energy in a ball B_{2r} at time t_0, where δ depends only on r but not on ε. Clearly, from the examples of radial symmetry of the Cahn–Hilliard equation, this kind of monotonicity formula may not be true since, for example, if initially ($t = 0$) there are only a pair of "phantom" interfaces (zero level set of u^ε) with $O(\varepsilon)$ distance located near $r = 3/4$, then in $O(\varepsilon)$ time, it will pass over all the balls of any size; therefore, the energy in $B_{1/4}$ at time $t \in (0, O(\varepsilon)]$ cannot be controlled by the energy in $B_{1/2}$ at $t = 0$. The lack of monotonicity formula is the main difficulty for us to establish a very close relation between the measure $2\sigma|D\chi_{E_t}|$ and the measure μ in Theorem 2.1.

References

[1] N. D. Alikakos, P. W. Bates & X. Chen, *The convergence of solutions of the Cahn–Hilliard equation to the solution of Hele–Shaw model,* Arch. Rational Mech. Anal. **128** (1994) 165–205.

[2] N. D. Alikakos & G. Fusco, *Slow dynamics for the Cahn–Hilliard equation in high space dimensions, Parts I: Spectral estimates, Partial Differential Equations, Part II: The motion of Bubbles,* to appear in Arch. Rational Mech. Anal.

[3] W. Allard, *On the first variation of a varifold,* Ann. of Math. **95** (1972) 417–491.

[4] S. Allen & J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening,* Acta Metallurgy **27** (1979) 1084–1095.

[5] F. J. Almgren, Jr. *The theory of varifolds,* Mimeographed Notes, Princeton, 1965.

[6] D. Aronson & Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve propagation;* Partial Differential Equations and Related Topics (J. A. Goldstein ed.), Lecture Notes in Math., Vol. 446, Springer, New York, 1975, 5–49.

[7] P. W. Bates & P. C. Fife, *The dynamics of nucleation for the Cahn–Hilliard equation,* SIAM J. Appl. Math. **53** (1993) 990–1008.

[8] J. F. Blowey & C. M. Elliott, *The Cahn–Hilliard gradient theory for the phase separation with non–smooth free energy, Part I: Mathematical analysis,* European J. Appl. Math. **2** (1991) 233–279.

[9] ———, *The Cahn–Hilliard gradient theory for the phase separation with non–smooth free energy, Part II: Numerical Analysis*, European J. Appl. Math. **3** (1992) 147–179.

[10] K. A. Brakke, *The Motion of A Surface by Its Mean Curvature,* Princeton Univ. Press, Princeton, 1978.

[11] L. Bronsard & D. Hilhorst, *On the slow dynamics for the Cahn-Hilliard equation in one space dimension,* Proc. Roy. Soc. London Ser. A **439** (1992) 669-682.

[12] L. Bronsard & R. V. Kohn, *On the slowness of the phase boundary motion in one space dimension,* Comm. Pure Appl. Math. **43** (1990) 983–997.

[13] ———, *Motion by mean curvature as the singular limit of Ginzburg–Landau dynamics,* J. Differential Equations **90** (1991) 211–237.

[14] L. Bronsard & B. Stoth, *Volume preserving mean curvature flow as a limit of nonlocal Ginzburg–Landau equation*, preprint.

[15] G. Caginalp, *Stefan and Hele–Shaw type models as asymptotic limits of the phase field equations,* Phys. Rev. A **39**, 5887–5896.

[16] ———, *An analysis of a phase field model of a free boundary,* Arch. Rational Mech. Anal. **92** (1986) 205–245.

[17] G. Caginalp & X. Chen, *Convergence of solutions of the phase–field equations to solutions of the sharp interface model,* preprint.

[18] J. W. Cahn, *On the spinodal decomposition,* Acta Metallurgy **9** (1961) 795–801.

[19] J. W. Cahn, C. M. Elliott & A. Novick–Cohen, *The Cahn–Hilliard equation with a concentration dependent mobility: Motion by minus Laplacian of the mean curvature,* preprint

[20] J. W. Cahn & J. E. Hilliard, *Free energy of a nonuniform system I. Interfacial free energy,* J. Chem. Phys. **28** (1958) 258–267.

[21] J. Carr & R. Pego, *Very slow phase separation in one-dimension,* Lecture Notes in Physics (M. Rascle ed.) Vol. 344, 1989, Springer, Berlin, 216–226.

[22] ———, *Invariant manifolds for metastable pattern in* $u_t = \varepsilon^2 u_{xx} - f(u)$, Proc. Roy. Soc. Edinburgh **116** (1990) 133–160.

[23] X. Chen, *The Hele–Shaw problem and area–preserving curve–shorting motions,* Arch. Rational Mech. Anal. **123** (1993) 117–151.

[24] ———, *Spectrums for the Allen–Cahn, Cahn–Hilliard, and phase–field equations for generic interface,* Comm. Partial Differential Equations **19** (1994) 1371–1395.

[25] ———, *Generation and Propagation of interface in reaction–diffusion equations,* J. Differential Equations. **96** (1992) 116–141.

[26] X. Chen, J. Hong & F. Yi, *Existence, uniqueness, and regularity of classical solutions to the Mullins–Sekerka free boundary problem*, preprint.

[27] X. Chen & C. M. Elliott, *Asymptotics for a parabolic double obstacle problem,* Proc. Royal Soc. London. Ser **A 444** (1994) 429–445.

[28] X. Chen, D. Hilhorst & E. Logak, *Asymptotic behavior of solutions of an Allen–Cahn equation with a non–local term*, preprint

[29] Y.G. Chen, Y. Giga & S. Goto, *Uniqueness and existence of viscosity solution of generalized mean curvature flow equations,* J. Differential Geom. **33** (1991) 749–786.

[30] P. Constantine & M. Pugh, *Global solutions for small data to the Hele–Shaw problem* , preprint.

[31] P. de Mottoni & M. Schatzman, *Evolution géométrique d'interfaces,* C. R. Acad. Paris Sci. Sér. I Math. **309** (1989) 453–458.

[32] ———, *Geometrical Evolution of developed interface,* to appear in Trans. Amer. Math. Soc..

[33] J. Duchon & R. Robert, *Évolution d'une interface par capillarité et diffusion de volume I. Existence locale en temps,* Ann. Inst. H. Poincaré, Anal. non linéaire **1** (1984) 361–378.

[34] C. M. Elliott & D.A. French, *Numerical studies of the Cahn–Hilliard equation for phase separation*, IMA J. Appl. Math. **38** (1987) 97–128.

[35] C. M. Elliott & J. Ockendon, *Weak and Variational Methods for Moving Boundary Problems,* Pitman, Boston, 1982.

[36] L. C. Evans, H. M. Soner & P. E. Souganidis, *Phase transitions and generalized motion by mean curvature,* Comm. Pure Appl. Math. **45** (1992) 1097–1123.

[37] L. C. Evans & J. Spruck, *Motion by mean curvature I,* J. Differential Geom. **33** (1991) 635–681.

[38] H. Federer, *Geometric Measure Theory*, Springer, New York, 1969.

[39] P. C. Fife, *Dynamical Aspects of the Cahn–Hilliard Equations,* Barret Lectures, University of Tennessee, Spring, 1991.

[40] D. Gilbarg & Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Ed., Springer, Berlin, 1983.

[41] P. C. Fife & L. Hsiao, *The generation and propagation of internal layers,* Nonlinear Anal. **70** (1988) 31–46.

[42] P. C. Fife & B. McLeod, *The approach of solutions of nonlinear diffusion equation to travelling front solutions*, Arch. Rational Mech. Anal. **65** (1977) 335–361.

[43] G. Fusco, *A geometric approach to the dynamics of* $u_t = \varepsilon^2 u_{xx} + f(u)$ *for small* ε, Lecture Notes in Phys. (Kirchgassner, ed) Vol. 359, 1990, Springer, Berlin, 53–73.

[44] G. Fusco & J. K. Hale, *Slow-motion manifolds, dormant instability, and singular perturbations,* J. Dynamics Differential Equations **1** (1989) 75–94.

[45] E. Guisti, *Minimal Surfaces and Functions of Bounded Variation,* Birkhäuser, Boston, 1984.

[46] T. Ilmanen, *Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993) 417–461.

[47] R. V. Kohn & P. Sternberg, *Local minimizers and singular perturbation,* Proc. Royal Soc. Edinburgh **111** (1989) 69–84.

[48] S. Luckhaus, *Solution of the two phase Stefan problem with Gibbs–Thomson relation for the melting temperature*, European J. Appl. Math. **1** (1990) 101–111.

[49] L. Modica, *A gradient bound and a Liouville theorem for nonlinear Poisson equations,* Comm. Pure Appl. Math. **38** (1985) 679–684.

[50] W. W. Mullins & J. Sekerka, *Morphological stability of a particle growing by diffusion or heat flow,* J. Appl. Math. **34** (1963) 322-329.

[51] R. H. Nochetto, M. Paolini & C. Verdi, *Optimal interface error estimates for the mean curvature flow,* preprint.

[52] A. Novick–Cohen, *On Cahn–Hilliard type equations*, Nonlinear Analysis **15** (1990) 797–814.

[53] R. L. Pego, *Front migration in the nonlinear Cahn–Hilliard equation,* Proc. Roy. Soc. London, Ser. A **422** (1989) 261–278.

[54] J. Rubinstein, P. Sternberg & J. B. Keller, *Fast reaction, slow diffusion and curve shorting,* SIAM J. Appl. Math. **49** (1989) 116–133.

[55] L. Simon, *Lecture Notes Geometric Measure Theory*, Proc. Centre Math. Anal., Austral. Nat. Univ., Vol. 3, 1983.

[56] J. Smoller, *Shock Waves and Reaction–Diffusion Equations,* Springer, Berlin, 1983.

[57] H. M. Soner, *Motion of a set by the mean curvature of its boundary,* J. Differential Equations, **1-1/2** (1993) 313–372.

[58] ———, *Ginzburg–Landau equation and motion by mean Curvature, I: Convergence,* Carnegie Mellon Univ. Res. Rep. No. 93-NA-026, 1995.

[59] ———, *Ginzburg–Landau equation and motion by mean curvature, II: development of initial interface,* J. Differential Equations, to appear.

[60] ———, *Convergence of the phase field equations to the Mullins–Sekerka problem with kinetic undercooling*, preprint.

[61] B. Stoth, *Convergence of the Cahn–Hilliard equation to the Mullins–Sekerka problem in spherical symmetry*, preprint.

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