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THE LENGTH OF A CUT LOCUS ON A SURFACE AND AMBROSE'S PROBLEM

JIN-ICHI ITOH

1. Introduction

There are many results about the cut locus $C(p)$ of a point p on a surface (M, g) going back to H.Poincaré's old paper [8]. S.Myers proved that if M is a real analytic sphere, $C(p)$ is a finite tree each of whose edges is an analytic curve with finite length [9]. It follows that the total length (1-dimensional Hausdorff measure) of $C(p)$ is finite. In the case of a C^∞ surface, $C(p)$ is somewhat complicated. In [3] H.Gluck and D.Singer constructed a C^∞ metric on S^2 so that there is a point p whose cut locus has infinitely many edges sharing a common end point and thus is not triangulable. Even in this case the total length of $C(p)$ is finite. Recently K.Shiohama and M.Tanaka showed that even on an Alexsandrov surface the cut locus of a point carries the structure of a local tree [10]. It is easy to construct an Alexandrov sphere so that the total length of a cut locus is infinite.

The purpose of this article is to study the relation between the length of a cut locus of a surface and the regularity of its metric. In the following, we will answer the question "When does $C(p)$ have infinite total length (1-dimensional Hausdorff measure) ?".

Theorem A. *Suppose (M, g) is a complete surface with a Riemannian metric of class C^2. Then any compact subset of the cut locus of $p \in M$ has finite 1-dimensional Hausdorff measure.*

Theorem B. *There is a $C^{1,1}$ metric on S^2 so that there is a point $p \in S^2$ whose cut locus $C(p)$ has infinite total length (1-dimensional Haudorff measure).*

In particular in the case of a compact surface, if the metric has C^2 regularity, the total lengths of the cut loci are all finite. If the metric loses C^2 regularity, then the cut loci may have infinite total length, and can further become what we know as a fractal set [7]. In the proof of Theorem A, we will show that the function, which assigns to each initial

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direction the distance to its cut point, is of bounded variation due to the fact that the function does not increase rapidly near its local minimum. In the proof of Theorem B, we will construct a sphere consisting of countably many flat triangles and constantly curved bi-angles.

W. Ambrose showed that the behavior of the curvature under parallel translation along the broken geodesics emanating from a point in a complete simply connected Riemannian manifold characterizes the manifold up to isometry, and posed the problem as to whether or not the behavior of the curvature along the unbroken geodesics emanating from a point was sufficient to so characterize the manifolds [1]. In the case of surfaces, we may formulate the problem as follows;

Ambrose's problem for surfaces. *Let $M, \bar{M}$ be complete Riemannian surfaces. Suppose that there are points $p \in M$ and $\bar{p} \in \bar{M}$, and a linear isometry $I : T_pM \to T_{\bar{p}}\bar{M}$ such that $G(\gamma(1)) = \bar{G}(\bar{\gamma}(1))$ for any geodesic $\gamma : [0,1] \to M$ emanating from p, where $\bar{\gamma} : [0,1] \to \bar{M}$ is the geodesic emanating from $\bar{p}$ with $\bar{\gamma}'(0) = I(\gamma'(0))$ and $G, \bar{G}$ denote the Gaussian curvature of $M, \bar{M}$. If M is simply connected, is there an isometric immersion $f : M \to \bar{M}$ with $f(p) = \bar{p}$ and $df_p = I$?*

In [5] J.Hebda answered the problem positively under the additional assumption that every compact subset of the cut locus of $p \in M$ has finite 1-dimensional Hausdorff measure. Thus Ambrose's problem for surfaces with C^2 metric is solved completely by Theorem A and Hebda's result.

Recently J.Hebda himself has proved Theorem A independently in [6]. But our method of proof is in essence different from his.

The author would like to express his thanks to M.Tanaka for his valuable comments.

2. Proof of Theorem A

Fix a point p and take a geodesic polar coordinate (r, θ) around p so that the Riemannian metric becomes

$$ds^2 = dr^2 + (f(r, \theta))^2 d\theta^2.$$

The function $f(r, \theta)$ of class C^1 satisfies $f(0, \theta) = 0$ and $f_r(0, \theta) = 0$. We denote the geodesic from p with initial direction θ by $\gamma_\theta(r)(= exp_p(r, \theta))$. On a geodesic γ_θ, the point of $f(r, \theta) = 0, \ r \neq 0$ becomes a conjugate point of p. The function $f(r, \theta)$ satisfies the differential equation

$$f_{rr}(r, \theta) + K(r, \theta) f(r, \theta) = 0,$$

where $K(r, \theta)$ is the Gaussian curvature at the point (r, θ). Furthermore

$f(r,\theta) = 0$ has local solutions $q = q(\theta)$ of class C^1 by the implicit function theorem. If for some θ, $q = q(\theta)$ has several positive values, let $q = q(\theta)$ denote the least such value; otherwise, let $q(\theta) = \infty$. For any positive number R, put $Q_R(\theta) := \min\{q(\theta), R\}$. Note that Q_R is a Lipschitz continuous function, and let $C_0(R)$ be the Lipschitz constant of Q_R. We denote the length of the minimal geodesic from p (to its cut point) along γ_θ by $\rho(\theta)$. For any positive number R, put $\rho_R(\theta) := \min\{\rho(\theta), R\}$.

To prove the Theorem we will show that ρ_R is a function of bounded variation. Denote the total variation of ρ (resp. ρ_R) by $V(\rho)$ (resp. $V(\rho_R)$), and define subsets E_0, E_1 of $U_pM(= S^1 = [0,2\pi]/0 \sim 2\pi)$ by

$$E_0 := \{\theta \in S^1 | \rho(\theta) = Q(\theta)\},$$

$$E_1 := \{\theta \in S^1 | \gamma_\theta|_{[0,2\rho(\theta)]} \text{ is a geodesic loop at } p\}.$$

Since E_0, E_1 are closed sets, $S^1 \setminus (E_0 \cup E_1)$ is a countable union of open intervals I_i which are mutually disjoint, i.e.,

$$S^1 \setminus (E_0 \cup E_1) = \bigcup_{i=1}^{\infty} I_i.$$

Lemma 1. *The set of locally minimal points of $\rho(\theta)$ is included in $E_0 \cup E_1$*

Proof of Lemma 1. Assume that $\rho(\theta)$ has a local minimum at θ_1 such that $\theta_1 \notin E_0 \cup E_1$. We denote a point $\gamma_{\theta_1}(\rho(\theta_1))$ by q. Then there is another minimizing geodesic γ_{θ_2} from p to q. We can take the unit tangent vector v at q which satisfies $\angle(v, -\dot\gamma_{\theta_1}|_q) = \angle(v, -\dot\gamma_{\theta_2}|_q) < \frac{\pi}{2}$ by the assumption. Let τ be the geodesic from q with initial direction v. Then the first variation formula yields a positive number δ such that $\tau|(0,\delta)$ is included in the metric ball whose center is p and radius is equal to $\rho(\theta_1)$. Thus, we can take a positive number ϵ such that for any $\theta \in (\theta_1, \theta_1 + \epsilon)$ or $(\theta_1 - \epsilon, \theta_1)$ the following holds: (1) $\rho(\theta) \geq \rho(\theta_1)$, (2) $a(\theta) < \rho(\theta_1)$ where $\gamma_\theta(a(\theta))$ is the first point on which γ_θ intersects with $\tau|(0,\delta)$, (3) $d(\gamma_\theta(\rho(\theta)), q)$ is less than the injectivity radius at q.

Let σ be the minimal geodesic from q to $\gamma_\theta(\rho(\theta))$. Then $\angle(-\dot\gamma_{\theta_2}|_q, \dot\sigma|_q) < \angle(-\dot\gamma_{\theta_2}|_q, v) < \frac{\pi}{2}$. Suppose that the metric ball whose center is p and radius is equal to $2\rho(\theta_1)$ has Gauss curvature bounded below by K_0. Compare the hinge $(\gamma_{\theta_2}, \sigma, \angle(-\dot\gamma_{\theta_2}|_q, \dot\sigma|_q))$ with the corresponding one on the constantly curved surface with Gauss curvature K_0. From the Toponogov's comparison theorem [2], it follows that $d(p, \gamma_\theta(\rho(\theta))) < \rho(\theta_1)$ contradicting the assumption.

Remark. For any interval $(\theta_a, \theta_b) \subset I_i$, take $\theta_c \in [\theta_a, \theta_b]$ such that $\rho(\theta_c)$ is a maximal value of ρ on $[\theta_a, \theta_b]$. Then, ρ is monotone nondecreasing on $[\theta_a, \theta_c]$ and ρ is monotone nondecreasing on $[\theta_c, \theta_b]$. Hence

$$V(\rho|_{[\theta_a,\theta_b]}) = 2\rho(\theta_c) - \rho(\theta_a) - \rho(\theta_b).$$

Now we will examine the variation of ρ_R near E_0 and E_1. From $\rho_R(\theta) \leq Q_R(\theta)$ and the Lipschitz continuity of Q_R we obtain the following Lemma 2 immediately.

Lemma 2. *For any $R > 0$ there is a positive constant $C_0(R)$ such that for any $\theta_0 \in E_0$ with $\rho(\theta_0) < R$ and for any θ*

$$\rho_R(\theta) - \rho_R(\theta_0) \leq C_0(R)|\theta - \theta_0|.$$

Lemma 3. *For any $R > 0$ there is a positive constant $C_1(R)$ such that for any $\theta_0 \in E_1$ with $\rho(\theta_0) < R$ and for any θ*

$$\rho_R(\theta) - \rho_R(\theta_0) \leq C_1(R)|\theta - \theta_0|.$$

Sublemma. *For any $R > 0$ there is a positive constant $A(R)$ such that for any $\theta_0 \in E_1$ with $\rho(\theta_0) < R$ and for any θ_1 with $\rho_R(\theta_1) \leq 2\rho_R(\theta_0)$,*

$$\rho_R(\theta_1) - \rho_R(\theta_0) \leq A(R)|\theta_1 - \theta_0|.$$

Proof of the Sublemma. When $\rho_R(\theta_1) \leq \rho_R(\theta_0)$, the Sublemma is trivial. Thus we can assume that $\rho_R(\theta_0) < \rho_R(\theta_1)$. Define a smooth curve σ from $\gamma_{\theta_0}(\rho_R(\theta_1))$ to $\gamma_{\theta_1}(\rho_R(\theta_1))$ by $\sigma(\theta) := exp_p(\rho_R(\theta_1)\gamma'_\theta(0))$. Then by the definition of distance,

$$\begin{aligned} &d(\gamma_{\theta_0}(\rho_R(\theta_1)), \gamma_{\theta_1}(\rho_R(\theta_1))) \\ &\leq |\int_{\theta_0}^{\theta_1} |\sigma'(\theta)| d\theta| \leq |\int_{\theta_0}^{\theta_1} |f(\rho(\theta_1), \theta)| d\theta| \leq A(R)|\theta_1 - \theta_0|, \end{aligned}$$

where $A(R) := \max\{|f(r,\theta)| \,|0 \leq r \leq R,\ 0 \leq \theta \leq 2\pi\}$. Hence from the triangle inequality it follows that

$$d(p, \gamma_{\theta_1}(\rho_R(\theta_1))) \leq d(p, \gamma_{\theta_0}(\rho_R(\theta_1))) + d(\gamma_{\theta_0}(\rho_R(\theta_1)), \gamma_{\theta_1}(\rho_R(\theta_1))).$$

On the other hand, we have

$$\begin{aligned} d(p, \gamma_{\theta_1}(\rho_R(\theta_1))) &= \rho_R(\theta_1), \\ d(p, \gamma_{\theta_0}(\rho_R(\theta_1))) &\leq \rho_R(\theta_0). \end{aligned}$$

Therefore

$$\rho_R(\theta_1) \leq \rho_R(\theta_0) + A(R)|\theta_1 - \theta_0|,$$

which completes the proof of the Sublemma.

Proof of Lemma 3. If $\rho_R(\theta) \leq 2\rho_R(\theta_0)$, Lemma 3 follows from the Sublemma. If $\rho_R(\theta) > 2\rho_R(\theta_0)$, we can take θ_a such that $\rho(\theta_a) = 2\rho(\theta_0)$ and $|\theta_a - \theta_0| < |\theta - \theta_0|$. We denote the injectivity radius at p by ι. Suppose that $|\theta_a - \theta_0| < \iota/A(R)$. Then from the Sublemma we get

$$\rho_R(\theta_a) \leq \rho_R(\theta_0) + A(R)|\theta_a - \theta_0| < \rho_R(\theta_0) + \iota \leq 2\rho_R(\theta_0),$$

which is a contradiction. Thus we can assume that $|\theta_a - \theta_0| \geq \iota/A(R)$, so that

$$\begin{aligned}\rho_R(\theta) - \rho_R(\theta_0) \leq R - \iota &\leq (R - \iota)\frac{A(R)}{\iota}|\theta_a - \theta_0| \\ &\leq (\frac{R}{\iota} - 1)A(R)|\theta - \theta_0|.\end{aligned}$$

Now we put $C(R) := \max\{A(R), (R/\iota - 1)A(R)\}$. Then Lemma 3 follows.

Combining the Remark, Lemma 2 and Lemma 3 yields the following Lemma 4 immediately.

Lemma 4. *For any intervals I_i, $V(\rho_R|_{I_i}) \leq C(R)m(I_i)$ where $C(R) := \max\{C_0(R), C_1(R)\}$, and $m(I)$ is the length of interval I. Furthermore*

$$\sum_{i=0}^{\infty} V(\rho_R|_{I_i}) \leq 2\pi C(R).$$

Proposition. *For any $R > 0$, ρ_R is a function of bounded variation.*

Proof. For any partition $\Delta : 0 = \theta_0 < \cdots < \theta_n = 2\pi$ of $[0, 2\pi]$, we will show that

$$V_\Delta = \sum_{i=1}^{n} |\rho_R(\theta_{i-1}) - \rho_R(\theta_i)| \leq 4\pi C(R).$$

We define the subsets Γ, Λ of $\{1, \dots, n\}$ by

$$\Gamma = \{i | (\theta_{i-1}, \theta_i) \cap (E_0 \cup E_1) = \emptyset\},$$

$$\Lambda = \{i | (\theta_{i-1}, \theta_i) \cap (E_0 \cup E_1) \neq \emptyset\}.$$

Of course the disjoint union of Γ and Λ coincides with $\{1, \dots, n\}$. Then

$$V_\Delta = \sum_{i \in \Gamma} |\rho_R(\theta_{i-1}) - \rho_R(\theta_i)| + \sum_{i \in \Lambda} |\rho_R(\theta_{i-1}) - \rho_R(\theta_i)|.$$

By Lemma 4 we get

$$\sum_{i \in \Gamma} |\rho_R(\theta_{i-1}) - \rho_R(\theta_i)| \leq 2\pi C(R).$$

For each $i \in \Lambda$, we take $\theta' \in [\theta_{i-1}, \theta_i]$ with $\theta' \in (E_0 \cup E_1)$. From Lemma 2 and Lemma 3, it follows that

$$\begin{aligned}|\rho_R(\theta_{i-1}) - \rho_R(\theta_i)| &\leq |\rho_R(\theta_{i-1}) - \rho_R(\theta')| + |\rho_R(\theta') - \rho_R(\theta_i)| \\ &\leq C(R)|\theta_{i-1} - \theta_i|,\end{aligned}$$

so that

$$\sum_{i \in \Lambda} |\rho_R(\theta_{i-1}) - \rho_R(\theta_i)| \leq 2\pi C(R),$$

which completes the proof of the Proposition.

In [4], P.Hartman proved that if ρ_R is of bounded variation for any $R > 0$, then ρ is absolutely continuous where ρ is finite valued. Hence the following Corollary 1.

Corollary 1. *ρ is absolutely continuous where ρ is finite valued.*

If M is compact, the following Corollary 2 is obvious.

Corollary 2. *Suppose (M, g) is a compact surface with a Riemannian metric of class C^2. Then ρ is a function of bounded variation and an absolutely continuous function. Furthermore the total length of the cut locus of $p \in M$ is finite.*

3. Proof of Theorem B

To begin with, on R^2, we will draw an infinite tree IT which will be preassigned as the cut locus. We take points $q_{(c_1,\cdots,c_k)}$ $(c_i = 0, 1)$ inductively as follows (see Figure 1):

(1) $q_{(\emptyset)} = (0,0)$, $q_{(0)} = (0,1)$, $q_{(1)} = (0,-1)$,

(2) $q_{(c_1,\cdots,c_k,c_{k+1})}$ is the point such that $d(q_{(c_1,\cdots,c_k)}, q_{(c_1,\cdots,c_k,c_{k+1})}) = (\frac{1}{2})^k$, and the angle from the line through $q_{(c_1,\cdots,c_{k-1})}$ and $q_{(c_1,\cdots,c_{k-1},c_k)}$ to the line through $q_{(c_1,\cdots,c_k)}$ and $q_{(c_1,\cdots,c_k,c_{k+1})}$ is equal to $-(\frac{1}{2})^k \frac{\pi}{2}$ if $c_{k+1} = 0$, and $(\frac{1}{2})^k \frac{\pi}{2}$ if $c_{k+1} = 1$.

Let IT be the union of all segments between $q_{(c_1,\cdots,c_k)}$ and $q_{(c_1,\cdots,c_k,c_{k+1})}$. Note that the length of IT is equal to

$$\sum_{k=0}^{\infty} 2^k (\frac{1}{2})^k = \infty.$$

Next, we will prepare the pieces and construct a sphere by attaching the pieces to each other. Let N be a positive number, at least greater than 2. We will determine the proper value of N at the end of this article. Take two points $p_{(\emptyset);0} = (N, 0)$ and $p_{(\emptyset);1} = (-N, 0)$ on R^2. For each $q_{(c_1,\cdots,c_k)}$ we define a point $p_{(c_1,\cdots,c_k)}$ inductively as follows:

(1) $p_{(c_1,\cdots,c_k)}$ is a point on the ray from $q_{(c_1,\cdots,c_{k-1})}$ to $q_{(c_1,\cdots,c_{k-1},c_k)}$,

$$(2) \qquad d(q_{(c_1)}, p_{(c_1)}) = d(q_{(0)}, p_{(\emptyset);0}), \quad (k=1),$$
$$d(q_{(c_1,\cdots,c_k)}, p_{(c_1,\cdots,c_k)}) = d(q_{(\underbrace{0,\cdots,0}_{k})}, p_{(\underbrace{0,\cdots,0}_{k-1})}), \quad (k \geq 2).$$

Take the point $p_{(c_1,\cdots,c_k);c_{k+1}}$ as the reflection of $p_{(c_1,\cdots,c_k)}$ with respect to the line $q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k,c_{k+1})}$. Each edge $q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k,c_{k+1})}$ of IT takes two triangles

$$\triangle p_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k,c_{k+1})}$$

and

$$\triangle p_{(c_1,\cdots,c_k);c_{k+1}}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k,c_{k+1})}.$$

We denote the union of all these triangles $\subset R^2$ by D (see Figure 1). Note that on any quadrilateral

$$\triangle p_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_{k+1})} \cup \triangle p_{(c_1,\cdots,c_k);c_{k+1}}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_{k+1})},$$

the cut locus of the two points $\{p_{(c_1,\cdots,c_k)}, p_{(c_1,\cdots,c_k);c_{k+1}}\}$ coincides with the segment $q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k,c_{k+1})}$.

Take bi-angles $B_{(c_1,\cdots,c_k);j}$ $(j=0,1)$ as follows.

(1) $B_{(c_1,\cdots,c_k);j}$ is the geodesic bi-angle on the constant curved sphere whose diameter is equal to

$$d(q_{(\underbrace{0,\cdots,0}_{k})}, p_{(\underbrace{0,\cdots,0}_{k})}).$$

(2) The vertices of $B_{(c_1,\cdots,c_k);j}$ are the north pole and south pole of the above constant curved sphere.

(3) Two angles of $B_{(c_1,\cdots,c_k);j}$ are equal to

$$\begin{cases} \angle(p_{(0);0}, q_{(0)}, p_{(\emptyset);0}) \quad \text{on } R^2 & \text{as} \quad k=1, \\ \angle(p_{(\underbrace{0,\cdots,0}_{k});0}, q_{(\underbrace{0,\cdots,0}_{k})}, p_{(\underbrace{0,\cdots,0}_{k-1});0}) \text{ on } \mathbf{R}^2 & \text{as} \quad k \geq 2. \end{cases}$$

Note that from the definition, the cut locus of one vertex of $B_{(c_1,\cdots,c_k);j}$ coincides with the other vertex. At the point $q_{(c_1)}$ there are two hinges $(p_{(\emptyset);0}, q_{(c_1)}, p_{(c_1);c_1})$ and $(p_{(\emptyset);1}, q_{(c_1)}, p_{(c_1);1-c_1})$ in ∂D. At each $q_{(c_1,\cdots,c_k)}$ there are exactly two hinges $(p_{(*)}, q_{(c_1,\cdots,c_k)}, p_{(*);*})$ and $(p_{(*);*}, q_{(c_1,\cdots,c_k)}, p_{(*);*})$ in ∂D. Attach $B_{(c_1,\cdots,c_k);j}$ $(j=0,1)$ to D, identifying the boundary of $B_{(c_1,\cdots,c_k);j}$ with the above hinges. By this attachment all $p_{(*)}$ and $p_{(*);j}$ become one point, and we will call this point p. Now we get a piecewise constantly curved sphere S^2 so that the cut locus of p coincides with the closure of IT .

Finally we will check that S^2 has a differential structure. From the construction, any point $\in S^2$ except p has a tangent plane. On the other

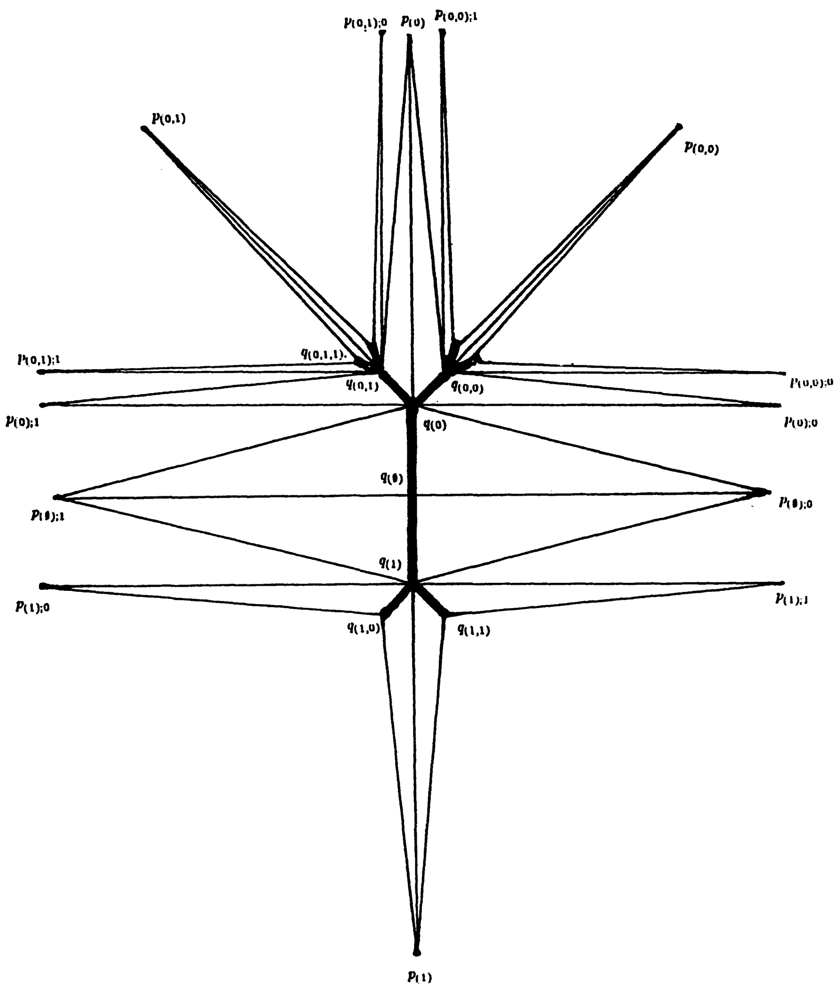

FIGURE 1

hand, the four angles $\angle(q_{(c_1)}p_{(\emptyset);j}q_{(\emptyset)})$ are the same. We denote this angle by ϕ_0. All the angles of the triangles $\triangle p_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_{k+1})}$, $\triangle p_{(c_1,\cdots,c_k);j}q_{(c_1,\cdots,c_k)}q_{(c_1,\cdots,c_{k+1})}$ at the vertices $p_{(c_1,\cdots,c_k)}$, $p_{(c_1,\cdots,c_k);j}$ are the same. We denote this angle by ϕ_k $(k \geq 1)$. It is trivial that

$$\phi_k = \sin^{-1}\left(\frac{d(q_{(\underbrace{0,\cdots,0}_{k})}, q_{(\underbrace{0,\cdots,0}_{k+1})})}{d(q_{(\underbrace{0,\cdots,0}_{k+1})}, p_{(\underbrace{0,\cdots,0}_{k})})}\sin\left(\frac{1}{2}\right)^k \frac{\pi}{2}\right).$$

Since

$$d(q_{(\underbrace{0,\cdots,0}_{k})}, q_{(\underbrace{0,\cdots,0}_{k+1})}) = (\frac{1}{2})^k,$$

$$N-2 < d(q_{(\underbrace{0,\cdots,0}_{k+1})}, p_{(\underbrace{0,\cdots,0}_{k})}) < 2N,$$

we have

$$\frac{1}{2N}(\frac{1}{2})^k \sin\left((\frac{1}{2})^k\frac{\pi}{2}\right) \leq \phi_k \leq \frac{1}{N-2}(\frac{1}{2})^k \sin\left((\frac{1}{2})^k\frac{\pi}{2}\right).$$

The angle of $B_{(c_1,\cdots,c_k);j}$ at the vertex is equal to ϕ_k. Hence, we get the following estimate of the total angle TA_p around $p \in S^2$:

$$4\sum_{k\geq 0} 2^k \frac{1}{2N}(\frac{1}{2})^k \sin\left((\frac{1}{2})^k\frac{\pi}{2}\right) \leq TA_p \leq 4\sum_{k\geq 0} 2^k \frac{1}{N-2}(\frac{1}{2})^k \sin\left((\frac{1}{2})^k\frac{\pi}{2}\right).$$

Therefore, we can take N so that the total angle around p coincides with 2π.

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KUMAMOTO UNIVERSITY, JAPAN