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ON THE MOD k INDEX THEOREM OF FREED AND MELROSE

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The purpose of this short note is to present an alternative approach to a formula of Freed-Melrose [6, Corollary 5.4], which expresses the topological index of vector bundles over Z/k-manifolds through geometric data.

Recall that Freed and Melrose proved their formula by first establishing a general index theorem for Z/k-manifolds and then making an application of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

Our approach is based on a result established jointly by Bismut and the author in [4] concerning the behaviour of the η-invariants under real embeddings. By such approach the use of the Atiyah-Patodi-Singer index theorem mentioned above is avoided. From our argument, it turns out immediately that for certain special dimensions, one can refine the Z/k index formula to a $2Z/2k$ formula. Furthermore, our method also suggests a promised new approach to the Atiyah-Patodi-Singer index theorem itself.

This paper is organized as follows. In Section 1, we recall the basic notation and facts about Z/k-manifolds. In Section 2, we give our approach to the Freed-Melrose formula in which we are interested. Section 3 containes a $2Z/2k$ refinement for dimension $8k+4$. In the final Section 4, we discuss the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2] from the point of view of our approach.

1. The topological index for Z/k-manifolds

Z/k-manifolds were introduced by Sullivan in his studies of geometric

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topology. We recall the basic definitions for completeness (cf. Freed [5] and Freed-Melrose [6]).

Definition 1.1. A compact Z/k-manifold is a compact manifold X with boundary ∂X, which has a decomposition $\partial X = \bigcup_{i=1}^{k}(\partial X)_i$ into k disjoint manifolds and k diffeomorphism $\pi_i : (\partial X)_i \to Y$ to a closed manifold Y.

Let $\pi : \partial X \to Y$ be the induced map. We use (X, Y, π) to denote this Z/k-manifold.

Convention 1.2. In what follows, we will call a (covariant) object α (e.g. vector bundles, metrics, connections, etc.) of X a Z/k object if there will be a corresponding object β of Y such that $\alpha\mid_{\partial X} = \pi^*\beta$.

For simplicity, we make the assumption that X is spin and of even dimension. Then Y is an odd dimensional manifold carrying an induced spin structure.

Let (X', Y', π') be another spin Z/k-manifold. We call an embedding $i_X : X \to X'$ a Z/k-embedding if there is an embedding $i_Y : Y \to Y'$ such that

$$\pi' i_X \mid_{\partial X} = i_Y \pi. \tag{1.1}$$

Let $K(X, Y, \pi)$ be the K-group of (X, Y, π) generated by Z/k complex vector bundles over (X, Y, π). Denote by $\widetilde{K}(X, Y, \pi)$ the corresponding reduced K-group. The classical construction of direct images in K-theory also works for $K(X, Y, \pi)$ (cf. [5], [6]). In particular, if $E \in K(X, Y, \pi)$ and $i : (X, Y, \pi) \to (X', Y', \pi')$ is an embedding between even dimensional compact spin Z/k-manifolds, then the direct image $i_! E$ lies in $\widetilde{K}(X', Y', \pi')$.

Example 1.2. Let n be a positive integer. Let $S^{n,k}$ be the Z/k-manifold obtained by removing k open balls D^n from the n-sphere. The identification map, which is obviously defined, will be denoted by $\pi_{n,k}$.

Lemma 1.3 (cf. Freed-Melrose [6]). *One has*

$$\widetilde{K}(S^{n,k}, S^{n-1}, \pi_{n,k}) = Z/k. \tag{1.2}$$

Now let E be a Z/k complex vector bundle over (X, Y, π), and let $i : (X, Y, \pi) \hookrightarrow (S^{n,k}, S^{n-1}, \pi_{n,k})$ be a Z/k embedding with n even. The existence of such an embedding is clear.

Definition 1.4. (cf. Freed [5] and Freed-Melrose [6]). The Z/k topological index of E is an element in Z/k given by

$$\text{ind}_{(k)}(E) = [i_! E] \in Z/k = \widetilde{K}(S^{n,k}, S^{n-1}, \pi_{n,k}). \tag{1.3}$$

Standard techniques in K-theory can be adapted here to show that $\text{ind}_{(k)}(E)$ does not depend on the embedding i. Furthermore, the following Riemann-Roch property still holds.

Proposition 1.5. *Let E be a Z/k complex vector bundle over (X, Y, π). Let $i_X : (X, Y, \pi) \hookrightarrow (X', Y', \pi')$ be a Z/k embedding between even dimensional compact spin Z/k manifolds. Then, one has*

$$\text{ind}_{(k)}(E) = \text{ind}_{(k)}(i_{x!} E). \tag{1.4}$$

2. The Freed-Melrose formula for the Z/k index

Let (X, Y, π) be as in Section 1 an even dimensional compact spin Z/k-manifold.

Let g^{TY} be a metric on TY. Let g^{TX} be a Z/k metric on TX such that g^{TX} is a product metric near ∂X and that

$$g^{TX} |_{T(\partial X)} = \pi^* g^{TY}. \tag{2.1}$$

Let ∇^{TX} (resp. ∇^{TY}) be the Levi-Civita connection of g^{TX} (resp. g^{TY}).

Let F be a Z/k complex vector bundle over Y. Let g^F be a metric on F and let ∇^F be a connection on F preserving g^F.

Let E be a Z/k vector bundle over (X, Y, π) such that $E |_{\partial X} = \pi^* F$. Let g^E be a metric on E such that it is a product metric near ∂X and that $g^E |_{\partial X} = \pi^* g^F$. Let ∇^E be a Z/k connection on E preserving g^E such that $\nabla^E |_{\partial X} = \pi^* \nabla^F$ and that ∇^E is a product connection near ∂X.

Let $D_{Y,F}$ be the Dirac operator coupled with F on Y.

If D is a self-adjoint Dirac operator, denote by $\overline{\eta}(D)$ the reduced η-invariant introduced by Atiyah-Patodi-Singer [2].

We will use the same notation as in Bismut-Zhang [4] to express the characteristic forms.

Theorem 2.1. (Freed-Melrose [6, Corollary 5.4]). *The following identity holds,*

$$\operatorname{ind}_{(k)}(E) = \int_X \hat{A}(TX, \nabla^{TX}) \operatorname{ch}(E, \nabla^E) - k\bar{\eta}(D_{Y,F}), \quad (\operatorname{mod} k). \tag{2.2}$$

Remark 2.2. The integrality of the right-hand side of (2.2) is non-trivial. It can be seen as a consequence of the index theorem of Atiyah-Patodi-Singer [2] for manifolds with boundary.

In what follows, we will give a proof of Theorem 2.1 directly, without refering to the Atiyah-Patodi-Singer index theorem.

Proof of Theorem 2.1. Let $i_X : (X, Y, \pi) \hookrightarrow (S^{n,k}, S^{n-1}, \pi_{n,k})$ be a Z/k embedding with n even. Note $i_Y : Y \hookrightarrow S^{n-1}$ the corresponding embedding of Y in S^{n-1}.

Let $\pi_X : N_X \to X$ (resp. $\pi_Y : N_Y \to Y$) be the normal bundle to X (resp. Y) in $S^{n,k}$ (resp. S^{n-1}). Then one has

$$\pi^* N_Y = N_X \,|_{\partial X} \, . \tag{2.3}$$

Let ξ_+, ξ_- be two Z/k complex vector bundles of a same dimension over $(S^{n,k}, S^{n-1}, \pi_{n,k})$ such that $\xi_+ - \xi_- \in \widetilde{K}(S^{n,k}, S^{n-1}, \pi_{n,k})$ is a representative of $i_! E$. Let μ_+, μ_- be two complex vector bundles over Y satisfying that $\xi_\pm \,|_{\partial S^{n,k}} = \pi_{n,k}^* \mu_\pm$. Then by (2.3), $\mu_+ - \mu_- \in \widetilde{K}(S^{n-1})$ is a representative of $i_! F$.

In view of Bismut-Zhang [4, Remark 1.1], we can and will assume that the following analogue of [4, 1.10], to which we also refer for relevant notation, holds, after constructing suitable Z/k metrics and connections on $(TS^{n,k}, TS^{n-1})$ and $\xi = \xi_+ \oplus \xi_-, \mu = \mu_+ \oplus \mu_-$ and a Z/k self-adjoint element $V_X \in \operatorname{End}(\xi)$ with corresponding element $V_Y \in \operatorname{End}(\mu)$,

$$\begin{aligned}&(\pi_X^* \operatorname{Ker} V_X, \pi_X^* h^{\operatorname{Ker} V_X}, j_Z V_X(x)) \\ &\quad \simeq (\pi_X^*(F^{N_{X*}} \otimes E), \pi_X^* g^{F^{N_{X*}\otimes E}}, \tau^{N_{X*}} \widetilde{c}(Z)), \text{ on } X \backslash \partial X,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(\pi_Y^* \operatorname{Ker} V_Y, \pi_Y^* h^{\operatorname{Ker} V_Y}, j_Z V_Y(y)) \\ &\quad \simeq (\pi_Y^*(F^{N_{Y*}} \otimes F), \pi_Y^* g^{F^{N_{Y*}\otimes F}}, \tau^{N_{Y*}} \widetilde{c}(Z)), \text{ on } Y,\end{aligned} \tag{2.4'}$$

Furthermore, V_X, V_Y are invertible on $S^{n,k}\backslash X, S^{n-1}\backslash Y$ respectively. Also, we can and will impose the product condition near $\partial S^{n,k}$ and ∂X for all objects, and the condition that the embedding $(X,Y) \hookrightarrow (S^{n,k}, S^{n-1})$ is totally geodesic.

Now let $\gamma^{S^{n,k}}, \gamma^{S^{n-1}}$ be the Chern-Simons currents on $S^{n,k}, S^{n-1}$ constructed in [3] and [4], corresponding to (2.4), (2.4′) respectively. Recall that they satisfy the following transgression formulas,

$$d\gamma^{S^{n,k}} = \mathrm{ch}(\xi, \nabla^{\xi}) - \hat{A}^{-1}(N_X, \nabla^{N_X})\,\mathrm{ch}(E, \nabla^E)\delta_X, \tag{2.5}$$

$$d\gamma^{S^{n-1}} = \mathrm{ch}(\mu, \nabla^{\mu}) - \hat{A}^{-1}(N_Y, \nabla^{N_Y})\,\mathrm{ch}(F, \nabla^F)\delta_Y, \tag{2.5′}$$

where ∇^{N_X} (resp. ∇^{N_Y}) is the orthogonal projection of $\nabla^{TS^{n,k}}|_X$ (resp. $\nabla^{TS^{n-1}}|_Y$) on N_X (resp. N_Y).

Furthermore one has

$$\gamma^{S^{n,k}}|_{\partial S^{n,k}} = \pi_{n,k}^{*}\gamma^{S^{n-1}}. \tag{2.6}$$

From (2.5), (2.6), one deduces that

$$\begin{aligned} &\int_{S^{n,k}} \hat{A}(TS^{n,k}, \nabla^{TS^{n,k}})\,\mathrm{ch}(\xi, \nabla^{\xi}) - \int_X \hat{A}(TX, \nabla^{TX})\,\mathrm{ch}(E, \nabla^E) \\ &= k\int_{S^{n-1}} \hat{A}(TS^{n-1}, \nabla^{TS^{n-1}})\gamma^{S^{n-1}}. \end{aligned} \tag{2.7}$$

On the other hand, use of [4, Theorem 2.2] yields

$$\bar{\eta}(D_{S^{n-1},\mu}) \equiv \bar{\eta}(D_{Y,F}) + \int_{S^{n-1}} \hat{A}(TS^{n-1}, \nabla^{TS^{n-1}})\gamma^{S^{n-1}}, \quad (\mathrm{mod}\, Z). \tag{2.8}$$

By (2.7), (2.8), we get

$$\begin{aligned} &\int_{S^{n,k}} \hat{A}(TS^{n,k}, \nabla^{TS^{n,k}})\,\mathrm{ch}(\xi, \nabla^{\xi}) - k\bar{\eta}(D_{S^{n-1},\mu}) \\ &\equiv \int_X \hat{A}(TX, \nabla^{TX})\,\mathrm{ch}(E, \nabla^E) - k\bar{\eta}(D_{Y,F}) \quad (\mathrm{mod}\, k). \end{aligned} \tag{2.9}$$

Now combining (2.9) with Proposition 1.5, we can reduce the proof of Theorem 2.1 to the case of the Z/k manifold $(S^{n,k}, S^{n-1}, \pi_{n,k})$. This, in tern, by using (2.9) again for the pair $(X, Y, \pi) = (S^{2,k}, S^1, \pi_{2,k})$, and by the Z/k Thom isomorphism theorem, which can be proved by adapting directly the usual proof of Thom isomorphism in ordinary K-theory, can be reduced to the case of $(S^{2,k}, S^1, \pi_{2,k})$. The proof of Theorem 2.1 can then be completed by the easy calculations already worked out in [5, 1.14]. q.e.d.

Remark 2.3. Although our proof is written out for spin manifolds, the same strategy applies to spin^c-manifolds as well. We leave this to the interested reader.

Remark 2.4. There is also a proof of (2.2) by N. Higson [7], using the K-theory of C^*-algebras and also the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

3. A mod 2 refinement for real vector bundles

The purpose of this section is to establish a mod 2 refinement of Theorem 2.1. In fact, by taking $k = 0$ in (2.2), one gets the well-known integrality of the characteristic number $< \hat{A}(TX)\,\mathrm{ch}(E), [X] >$.

Now if $\dim X \equiv 4 \pmod 8$, $k = 0$ and E is the complexification of a real vector bundle, one has the mod 2 refinement due to Atiyah and Hirzerbruch [1] that $< \hat{A}(TX)\,\mathrm{ch}(E), [X] >$ is an even integer. Our improvement of Theorem 2.1 paralleles this refinement of Atiyah-Hirzerbruch in the $k \neq 0$ case.

Thus from now on, we assume that (X, Y, π) is a compact spin Z/k-manifold of dimension $8m + 4$.

Let $\widetilde{KO}(X, Y, \pi)$ be the corresponding reduced KO group of Z/k real vector bundles over (X, Y, π).

Proposition 3.1. *Let n be a positive integer such that $n \equiv 4 \pmod 8$. Then one has $\widetilde{KO}(S^{n,k}, S^{n-1}, \pi_{n,k}) = 2Z/2k (= Z/k)$.*

Proof. This can be proved in the same way as Lemma 1.3 from the classical fact that $\widetilde{KO}(S^n) = 2Z$. q.e.d.

According to Proposition 3.1, and using the fact that an $8l$ dimensional spinor space is the complexification of real vector space, we can define the Z/k topological index of a Z/k real vector bundle E over (X, Y, π) as an element in $2Z/2k$:

$$\mathrm{ind}_{(k)}(E) \in 2Z/2k. \tag{3.1}$$

On the other hand, let E_C be the compexification of E. Then we can define metrics, connections and Dirac operators as in Section 2 for E_C.

Now we can state our improvement of Theorem 2.1 as follows.

Theorem 3.2. *The following identity holds,*

$$\operatorname{ind}_{(k)}(E) \equiv \int_X \hat{A}(TX, \nabla^{TX}) \operatorname{ch}(E_C, \nabla^{E_C}) - k\bar{\eta}(D_{Y,F_C}), \quad (\operatorname{mod} 2k), \tag{3.2}$$

where F_C is the compexification of F, which is the real vector bundle over Y corresponding to E.

Proof. The strategy of the proof of (3.2) is the same as our proof of Theorem 2.1. All that one needs to note is the following two points:

(i) Since an $8m+3$ dimensional spinor space carries a quaternionic structure, $\bar{\eta}(D_{Y,F_C})$ is in fact mod 2 continuous. So as argued similarly in [8], the formula here corresponding to (2.8) holds mod $2Z$;

(ii) Instead of reducing the proof of (2.2) to $(S^{2,k}, S^1, \pi_{2,k})$, here we use the fact that an $8l$ dimensional spinor space is the complexification of a real space, to reduce (3.2) to $(S^{4,k}, S^3, \pi_{4,k})$ for which (3.2) can also be verified easily.

We leave the details to the interested reader. q.e.d.

Remark 3.3. It seems that a similar modification of Freed-Melrose's and/or Higson's argument can also lead to such a mod 2 refinement.

4. Comments in relations with the Atiyah-Patodi-Singer index theorem

We assume $k = 1$ in this Section.

Recall that in this case, Theorem 2.1 is an immediate consequence of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

Now that we have given the direct proof of Theorem 2.1, we naturally hope that the idea of our approach would also be helpful in understanding the Atiyah-Patodi-Singer index theorem itself.

More precisely, in using the notation as in Section 1 and 2, let $D_{S^{n,1},\xi} + TV_X$ be the Dirac operator on $S^{n,1}$ coupled with the coefficient ξ satisfying the Atiyah-Patodi-Singer boundary condition [2]. Then one gets easily the following result.

Theorem 4.1. *Let T be a nonnegative real number. Then the following identity holds,*

$$\begin{aligned}\operatorname{ind}&(D_{S^{n,1},\xi}+TV_X)\\ &=\int_{S^{n,1}}\hat{A}(TS^{n,1},\nabla^{TS^{n,1}})\operatorname{ch}(\xi,\nabla^{\xi})-\bar{\eta}(D_{S^{n-1},\mu}+TV_Y).\end{aligned}\tag{4.1}$$

Proof. Formula (4.1) follows from the Atiyah-Patodi-Singer index theorem [2] for $D_{S^{n,1},\xi}+TV_X$ and the easy local index calculation that

$$\begin{aligned}&\lim_{t\to 0}\operatorname{Tr}_S[\exp(-t(D_{S^{n,1},\xi}+TV_X)^2)(x,x)]d\operatorname{vol}(x)\\ &\quad=\{\hat{A}(TS^{n,1},\nabla^{TS^{n,1}})\operatorname{ch}(\xi,\nabla^{\xi})\}^{max}(x,x),x\in S^{n,1}-S^{n-1}.\end{aligned}\tag{4.2}$$

q.e.d.

Now set $T=0$ in (4.1). One has,

$$\operatorname{ind}(D_{S^{n,1},\xi})=\int_{S^{n,1}}\hat{A}(TS^{n,1},\nabla^{TS^{n,1}})\operatorname{ch}(\xi,\nabla^{\xi})-\bar{\eta}(D_{S^{n-1},\mu}).\tag{4.3}$$

By (4.2), (4.3), we get

$$\begin{aligned}\operatorname{ind}(D_{S^{n,1},\xi}+TV_X)&+\bar{\eta}(D_{S^{n-1},\mu}+TV_Y)\\ &=\operatorname{ind}(D_{S^{n-1},\xi})+\bar{\eta}(D_{S^{n-1},\mu}).\end{aligned}\tag{4.4}$$

Clearly, the left-hand side of (4.4) does not depend on $T\in[0,+\infty)$.

Recall that the behaviour of $\bar{\eta}(D_{S^{n-1},\mu}+TV_Y)$ as T tends to ∞ has been studied in Bismut-Zhang [4]. This suggests that a new demonstration of the Atiyah-Patodi-Singer index theorem for Dirac operators could be achieved if we could

i) prove (4.4) directly;

ii) study the behaviour of $\operatorname{ind}(D_{S^{n,1},\mu}+TV_X)$ as T is sufficiently large.

We believe that such a strategy, which would yield a K-theoretic proof of the Atiyah-Patodi-Singer index theorem [2], is promising and would inevitably lead to better understandings of the role of Atiyah-Patodi-Singer boundary conditions [2] appearing at so many places in differential geometry and mathematical physics. (**Note added in proof:** see X. Dai & W. Zhang, C. R. Acad. Sci. Paris, (1) **319** (1994) 1293-1297.)

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