J.DIFFERENTIAL GEOMETRY
Vol. 42, No.3 November, 1995

MÖBIUS STRUCTURES ON SEIFERT MANIFOLDS. I

FENG LUO

1. Introduction

The purpose of this paper is to prove the following.

Main Theorem. *Suppose* $\mathbf{R}^2 \to W_{e,g} \to \Sigma_g$ *is the plane bundle over a closed orientable surface of genus g so that the Euler number of the fibration is e. If* $|e| \leq g-1$, *then there exists a complete hyperbolic metric on* $W_{e,g}$. *Furthermore, the conformal infinity of the hyperbolic structure is a Möbius structure on the associated circle bundle over surface* Σ_g.

In view of J. Milnor's theorem that there exists a flat $\mathrm{SL}(2, \mathbf{R})$ connection on $W_{e,g}$ if and only if $|e| \leq g-1$, one would expect that the result above is optimal. M. Kapovich [5] has made progress in this direction recently and has shown that if $|e|/(g-1)$ is too large, then there is no complete hyperbolic metric on $W_{e,g}$. The existence of such structures on $W_{e,g}$ for $e \neq 0$ was first proved by Gromov-Lawson-Thurston [3], Kapovich [4], and Kuiper [6]. The best result so far is obtained by Kuiper who showed that if $|e| \leq 2/3(g-1)$, then there exists a complete hyperbolic metric on $W_{e,g}$.

Our construction is based on a Fenchel-Nielsen type decomposition of $W_{e,g}$. The basic building block is $W = \mathbf{R}^2 \times P$ where P is a pair of pants, and the main objects are complete hyperbolic structures with totally geodesic boundary on W. Each boundary component of W has the induced complete hyperbolic structure which is characterized by the multiplier (a complex number of norm larger than 1) of the generator of the monodromy group. We call it the *multiplier* of the structure on the component of ∂ W. Similar to Fenchel-Nielsen's work on hyperbolic metrics on P, we have now the problem of constructing

Received February 2, 1994.

complete hyperbolic metrics on W with given three multipliers. We are not able to solve the problem in this paper. However, we constructed a complete hyperbolic structure on W so that the multipliers are three negative real numbers arbitrary near $(-ctg^2\pi/12, -ctg^2\pi/12, -ctg^2\pi/6)$. Gluing these structures along the boundary implies the main theorem above.

We will mainly work on the conformal infinity of the hyperbolic space H^4, namely the 3-dimensional Möbius geometry $(S^3, \mathrm{Mob}(S^3))$. The basic idea of the construction comes from Fenchel-Nielsen's lemma which states that for any three positive numbers a_1, a_2, a_3, there exists a unique hyperbolic structure on P so that the lengths of three boundary geodesics are the given numbers. Let us recall briefly the proof. Choose two geodesics l_1 and l_2 in the hyperbolic plane H^2 of distance $a_3/2$ apart. Let l_1' and l_2' be the curves of constant distances $a_2/2$ and $a_1/2$ to l_1 and l_2 respectively in the common region bounded by l_1 and l_2. Since l_1' and l_2' are circular arcs, there exists a geodesic l_3 tangent to both l_1' and l_2' (there are exactly two such geodesics which are symmetric about the common perpendicular to l_1 and l_2). Then the three geodesics l_1, l_2, and l_3 bound a common region in H^2 and their pairwise distances are $a_1/2, a_2/2$, and $a_3/2$. For each geodesic l in H^2, let H_l be the hyperbolic reflection about l. Then the group $< H_{l_1} \circ H_{l_2}, H_{l_2} \circ H_{l_3} >$ is a Schottky group uniformizing the hyperbolic structure on P with geodesic boundary of lengths a_1, a_2, and a_3.

Our generalization to S^3 is as follows. Given a circle C in S^3, let H_C be the sense preserving Möbius involution which leaves each point in C fixed. We call H_C the *half-turn* about C. Given three circles C_1, C_2, and C_3 in S^3, we form the *three-circle group* $H_{C_1,C_2,C_3} = < H_{C_1} \circ H_{C_2}, H_{C_2} \circ H_{C_3} >$. There is a very easy sufficient condition on C_1, C_2, and C_3 (Schottky condition), which implies that H_{C_1,C_2,C_3} is discrete and free: namely that C_1, C_2, and C_3 lie in three 2-spheres which bound a common region in S^3.

The complete hyperbolic structure on W is found among these three-circle groups.

There are four types of different configurations of pairwise disjoint three circles. See figure 1.1.

In this paper we will be interested in the first type where all pairs of circles are unlinked. The linked case is much more difficult due to the existence of elliptic elements in the three-circle group. We intend to study them in a subsequent paper. The most interesting problem is to decide when a three-circle group based on three pairwise linked circles is discrete. In this case, one obtains a discrete representation of a triangular group into SO(4,1).

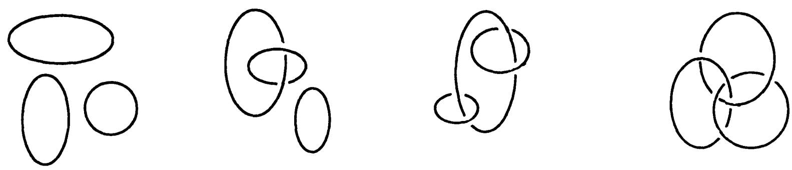

FIGURE 1.1

The organization of the paper is as follows. We recall some elementary properties of Möbius geometry in S^3 in section 2. In particular, it is shown that for any two circles in S^3, there is a third circle in S^3 orthogonal to each of them at two points. Given two unlinked circles C_1 and C_2, we also define a complex number (the principal multiplier of $H_{C_1}H_{C_2}$) associate to the pair which classifies the pair up to Möbius transformation. It can be shown that the set of all three circles $\{C_1, C_2, C_3\}$ modulo Möbius transformations so that their pairwise multipliers are fixed numbers forms a compact set (generically a compact surface). Section 3 is the main part of the paper. We study the configuration space $\mathcal{M}_3$ of three circles (C_1, C_2, C_3) modulo Möbius transformations so that each of these circles intersects a fourth circle C at two points, and $C_1 \cap$ C, $C_2 \cap$ C, and $C_3 \cap$ C bound three disjoint intervals in C. A parametrization of $\mathcal{M}_3$ is introduced, and the pairwise multipliers of (C_i, C_j) are expressed in term of the parametrization. We prove a local deformation result which is crucial to our construction. Call a triple of circles (C_1, C_2, C_3) *totally degenerate* if they are tangent to a fourth circle at three different points. We show that if (C_1, C_2, C_3) is totally degenerate and satisfies a mild condition, then there is a local deformation of it so that the resulting triple is in $\mathcal{M}_3$ and the deformation preserves the pairwise multipliers. The structure on W is found by a local deformation argument. In section 4, we discuss the gluing problem. Euler numbers of Möbius structures on $\Sigma \times S^1$ (Σ is an orientable compact surface) with trivial monodromy in S^1 fibers are defined. We show the additivity of the Euler number under gluing. This together with the special structure that we constructed on W implies the main theorem.

2. Elementary properties of Möbius geometry in dimension three

Recall that the base space of Möbius geometry is the unit 3-sphere S^3 in $\mathbf{R}^4$ or the 3-dimensional Euclidean space adding infinity $\bar{R}^3$. Möbius transformations are compositions of inversions about 2-spheres. They form the group Mob(S^3) ($\mathrm{SO}_+(4,1)$). These transformations preserve the set of circles and lines and the set of 2-spheres and planes. For simplicity, we will call lines (or planes) in $\mathbf{R}^3$ circles (2-spheres respectively). Given any circle C in $\bar{\mathbf{R}}^3$, the *half-turn* about C, denoted by H_C, is the orientation preserving Möbius involution leaving each point of C fixed. H_C may also be defined as the composition of two inversions about two 2-spheres intersecting orthogonally at C.

The goal of the section is to study the geometry of two circles C_1 and C_2 and its relation to the multiplers of the Möbius transformation $H_{C_1} \circ H_{C_2}$.

We will use the following terminologies. Two circles C_1 and C_2 are call *orthogonal* if they intersect at two points orthogonally; they are *unlinked* if they are disjoint and have zero linking number, and are *linked* if they are disjoint and have linking number one. For a set X $\subset S^3$ consisting of more than one point, sp(X) denotes the sphere of minimal dimension containing X. For instance, two circles C_1 and C_2 are *cosphere* if and only if sp$\{C_1, C_2\}$ is a 2-sphere. A pair of circles is called *standard* if it is Möbius equivalent to the pair (z-axis, unit circle in the xy-plan). Fix(h) denotes the fixed point set of h.

2.1. Let Mob$^+(S^3)$ be the group of sense preserving Möbius transformations. h $\in Mob^+(S^3)$ is called *hyperbolic* if $|Fix(h)| = 2$, *parabolic* if $|Fix(h)| = 1$, and *elliptic* if $|Fix(h)|$ is 0 or infinite. If h is hyperbolic, h is conjugate (in $Mob^+(S^3)$) to the transformation $x \mapsto rR_\theta x$, where $x \in \mathbf{R}^3$, $r \neq 1$ is a positive real number, and R_θ is the Euclidean degree θ rotation (counterclockwise in the xy-plane) about the z-axis. If h is parabolic, h is conjugate to $x \mapsto R_\theta x + (1,0,0)$ in $\mathbf{R}^3$. If h is elliptic, h is conjugate to an element in SO(4) $\subset$ SO(4,1) of the form $\begin{pmatrix} M_{\theta_1} & 0 \\ 0 & M_{\theta_2} \end{pmatrix}$ where $M_\theta = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$. To characterize these Möbius transformations up to conjugacy, we introduce the *multipliers* m(h) of h. For hyperbolic h, m(h) $= \{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$ where $\lambda = re^{i\theta}$. Each element in m(h) called a *multiplier* of the Möbius transformation h.

We call the element in m(h) with norm > 1 and argument in $[0, \pi]$ the *principal* multiplier of h. For parabolic h, m(h) $= \{e^{\pm i\theta}\}$; and for elliptic h, m(h) $= \{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_2}, e^{i\theta_1}), (e^{-i\theta_1}, e^{-i\theta_2}), (e^{-i\theta_2}, e^{-i\theta_1})\}$. Note that m(h) $=$ m(h^{-1}). Also a half-turn is the same as m(h) $= \{(1,-1), (-1, 1)\}$. Two elements h_1 and h_2 in $Mob^+(S^3)$ are conjugate if and only if m(h_1) $=$ m(h_2). If S is an oriented 2-sphere invariant under a hyperbolic element h and $h|_S \in$ Mob$^+$(S), then one may specify two multipliers of m(h) of the form $\{a, a^{-1}\}$ so that they are the multipliers of the two-dimensional Möbius transformation $h|_S$ with respect to the oriented 2-sphere. If furthermore a fixed point of h in S is specified, then we obtain exactly one multiplier which is the derivative of h at the fixed point.

From the classification, we deduce the following. If a hyperbolic element h has non-real multipliers, then h has a unique invariant circle (the z-axis) and a unique invariant 2-sphere (the xy-plan). Both of them contain Fix(h) and the restriction of h to each of them is orientation preserving. If h has negative real multipliers, then h has a unique invariant circle C so that $h|_C$ is sense preserving and a unique 2-sphere S so that $h|_S$ is sense preserving. For a parabolic element h with non-real multipliers, h has a unique invariant circle (the z-axis) containing Fix(h). For elliptic element h with non-real multipliers, h leaves a standard pair of circles invariant. Furthermore, the pair is unique if h has four distinct multipliers.

2.2. We prove in this section that for any two circles in S^3, there is a third circle which is orthogonal to both of them. We will also explain the geometric meaning of m($H_{C_1} \circ H_{C_2}$).

2.3. Lemma. *Given two circles C_1 and C_2 in S^3, there is a third circle C orthogonal to each C_i at two points, for i = 1,2.*

We call C a *common perpendicular* of C_1 and C_2.

Proof. If $C_1 \cap C_2 \neq \phi$, take a point of intersection to be the infinity for a Euclidean model of S^3. Then C_1 and C_2 are two lines in $\mathbf{R}^3$. Therefore, there is a line C in $\mathbf{R}^3$ orthogonal to both C_1 and C_2. If C_1 $\cap C_2 = \phi$, consider S^3 as the boundary of the unit 4-ball B^4 in $\mathbf{R}^4$, and let D_i be the 2-sphere in $\mathbf{R}^4$ intersecting S^3 orthogonally at C_i. Clearly (C_1, C_2) is linked if and only if $D_1 \cap D_2 \neq \phi$. If $D_1 \cap D_2 \neq \phi$, then $D_1 \cap D_2 \cap int(B^4) \neq \phi$. After a Möbius transformation of $\mathbf{R}^4$ leaving S^3 invariant, we may assume that $D_1 \cap D_2 = \{0, \infty\}$. Then C_1 and C_2 are great circles in S^3 with respect to the standard metric. Let C be a great circle which realizes the minimal spherical distance between C_1 and C_2. Then C is orthogonal to C_i at two points. Suppose finally that $D_1 \cap D_2 = \phi$. We consider int(B^4) as the hyperbolic 4-space. There is a hyperbolic geodesic l orthogonal to the two totally geodesic surfaces $D_1 \cap int(B^4)$ and $D_2 \cap int(B^4)$. Let the ends of l in ∂B^4 be {x, y}. Take a Euclidean model of S^3 so that x and y are the origin and the infinity respectively. Then C_1 and C_2 are two circles in $\mathbf{R}^3$, whose Euclidean centers are the origin. There is a straight line C passing through the origin intersecting both C_1 and C_2. Thus, C is orthogonal to each of C_i at two points. □

There are five different configurations of pair of circles in S^3 according to their relative positions. See Figure 2.1.

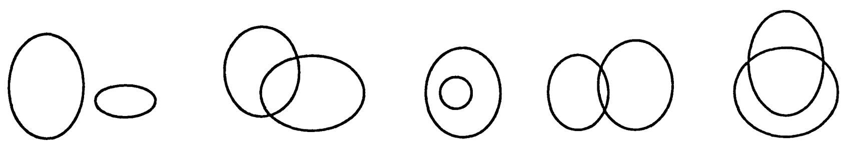

FIGURE 2.1

Below, we will discuss these five cases in detail. For simplicity, we also call m$(H_{C_1}H_{C_2})$ the multipliers of the pair of circles (C_1 , C_2) and denote it by m(C_1, C_2). It follows from the definition that m(C_1, C_2) = m(C_2, C_1) = m$(g(C_1), g(C_2))$ for $g \in Mob(S^3)$.

Case 1. (C_1, C_2) is a unlinked, not cosphere pair. By Lemma 2.3, after a Möbius transformation, we may assume that the common perpendicular C is the z-axis, and C_1 and C_2 are two circles centered at the origin and orthogonal to the z-axis. Then clearly, $H_{C_1}H_{C_2}$ is a hyperbolic transformation with fixed points 0, ∞ and leaves both C and the xy-plane invariant. To figure out m$(H_{C_1}H_{C_2})$, let θ be the dihedral angle between sp$(C_1$, C) and sp$(C_2$, C), and let r be the ratio of the radii of C_1 and C_2. Then a multiplier of $H_{C_1}H_{C_2}$ is $r^2e^{2i\theta}$. Thus m$(H_{C_1}H_{C_2})$ = { $r^{\pm 2}e^{\pm 2i\theta}$}. Since C_1 and C_2 are not cosphere, $\theta \in (0, \pi/2]$. Hence, the multipliers of $H_{C_1}H_{C_2}$ are not positive real. Furthermore, C and the xy-plane are the unique invariant circle and 2-sphere orthogonal to C_1 and C_2 respectively.

This also suggests the construction of C_1 and C_2 with given multipliers $r^{\pm 2}e^{\pm 2i\theta}$. Thus, each hyperbolic element of non-positive real multipliers of $Mob^+(S^3)$ is of the form $H_{C_1}H_{C_2}$ for some pair of unlinked, non-cosphere circles.

Case 2. (C_1, C_2) is a linked pair. By Lemma 2.3 we may assume that C is the z-axis, and C_1 and C_2 are two circles in R^3 centered at the z-axis. Let S_i be the 2-sphere obtained by rotating C_i about the z-axis. Then $C' = S_1 \cap S_2$ is another circle orthogonal to both C_1 and C_2. Furthermore, (C, C') is a standard pair. By the proof of Lemma 2.3, we may assume that C_1, C_2, C and C' are all great circles in S^3. Thus, $H_{C_1}H_{C_2} \in SO(4)$ leaves both C and C' invariant. To find the geometric meaning of m(C_1,C_2) = m($H_{C_1}H_{C_2}$), we should be a little careful about the orientations. Suppose S^3 is oriented, C_1, C_2 are so oriented that their linking number is 1, and C, C' are also oriented like so. Let θ_1 be the dihedral angle between sp(C_1, C') and sp(C_2, C') counted in the direction of C' from sp(C_1, C'), and θ_2 be the dihedral angle between sp(C_1, C) and sp(C_2, C) counted in the direction of C' from sp(C_1, C). Then m($H_{C_1}H_{C_2}$) = $\{(e^{2i\theta_1}, e^{2i\theta_2}), (e^{2i\theta_2}, e^{2i\theta_1}), (e^{-2i\theta_1}, e^{-2i\theta_2}), (e^{-2i\theta_2}, e^{-2i\theta_1})\}$. Given $\theta_1, \theta_2 \in [0, \pi]$, we may construct a linked pair (C_1, C_2) with multipliers $\{(e^{2i\theta_1}, e^{2i\theta_2}), (e^{i\theta_2}, e^{i\theta_1}), (e^{-2i\theta_1}, e^{-2i\theta_2}), (e^{-2i\theta_2}, e^{-2i\theta_1})\}$ as follows. Let C_1' and C_2' be two geodesic in H^2 intersecting at an angle θ_1 where H^2 is represenated as the half plane { (x,0,z) $|x > 0\}$. Take C_1 to be sp(C_1') and C_2 to be sp(C_2') rotated about the z-axis at an angle θ_2. Then the multipliers of this pair are $\{(e^{2i\theta_1}, e^{2i\theta_2}), (e^{2i\theta_2}, e^{2i\theta_1}), (e^{-2i\theta_1}, e^{-2i\theta_2}), (e^{-2i\theta_2}, e^{-2i\theta_1})\}$.

Case 3. (C_1, C_2) is a disjoint, cosphere pair. Then we may assume that C_1 and C_2 are two concentric circles in the xy-plane centered at the origin. $H_{C_1}H_{C_2}$ is a hyperbolic transformation with positive real multipliers $r^{\pm 2}$ where r is the ratio of the radii of C_1 and C_2.

Case 4. (C_1, C_2) is a pair of circles intersecting at two points. We may assume after a Möbius transformation conjugation that C_1 and C_2 are two lines in the xy-plane intersecting at the origin. Then $H_{C_1}H_{C_2}$ is a rotation about the z-axis at an angle 2θ where θ is the intersecting angle of C_1 and C_2. Thus $H_{C_1}H_{C_2}$ is an elliptic element with two multipliers.

Case 5. (C_1, C_2) consists of two circles intersecting at only one point. We may assume after a Möbius transformation that C_1 and C_2 are two lines in R^3 intersecting the z-axis orthogonally. Then $H_{C_1}H_{C_2}$ is the skew motion $x \mapsto R_{2\theta}x + (2a, 0, 0)$ where θ is the intersection angle of C_1 and C_2 at the infinity, and a is the distance between the intersection points of C_1 and C_2 with the z-axis.

From the above analysis, we have shown that each element h in $Mob^+(S^3)$ is a product of two half-turns, and the multipliers of h can be interpreted geometrically in terms of the relative position of the two circles. Furthermore, for two pairs of circles (C_1, C_2) and (D_1, D_2), there is a Möbus transformation taking (C_1, C_2) to (D_1, D_2) if and only if they have the same multipliers.

2.4. Pairs of spheres and circles and focal points.

Given two 2-spheres S_1 and S_2, the composition h of the inversions about these spheres is an element in $Mob^+(S^3)$. It is either a hyperbolic element with positive real multipliers, or a rotation about a circle, or a parabolic element with real multipliers. The first case corresponds to $|S_1 \cap S_2| = 0$, the second case to that $|S_1 \cap S_2|$ is infinite so that h is a rotation about the circle $S_1 \cap S_2$, and the last case to $|S_1 \cap S_2| = 1$. If $S_1 \cap S_2 = \phi$, we call Fix(h) the *focal points* of the pair (S_1, S_2). It is characterized by the following property. A circle C is orthogonal to both S_1 and S_2 if and only if C contains the focal points. In particular, the focal points of two disjoint spheres S_1 and S_2 are given by $C_1 \cap C_2$ where C_1 and C_2 are two distinct circles orthogonal to both S_1 and S_2.

We can also define the *focal points* of a pair (C, S) where C is a circle and S is 2-sphere disjoint from C. It is the pair of points {x,y} so that a circle C' is orthogonal to both C and S if and only if C' contains both x and y. One way to see the existence of focal points is to find a sphere A containing C and orthogonal to S. Then let S' be the 2-sphere orthogonal to A at C. One shows easily that $S' \cap S = \phi$. The focal points of (S, S') is then the focal points of (C,S). The other way is to define the focal points to be Fix($H_C \circ Inv_S$) where Inv_S is the inversion about S.

2.5. In this section we will derive a useful formula for calculating the multipliers of two unlinked circles in $\bar{R}^3$.

We will identify the complex plan $\mathbf{C}$ with the xy-plane in $\mathbf{R}^3$. For two distinct points a, b in $\mathbf{C} \cup \{\infty\}$, we use $[a, b]$ to denote the unique circle in $\bar{\mathbf{R}}^3$ intersecting $\mathbf{C} \cup \{\infty\}$ orthogonally at a and b.

2.6. Lemma. *(a). The restriction of the half-turn* $H_{[a,b]}$ *to* $\mathbf{C}$ *is given by* $z \mapsto \frac{(a+b)z-2ab}{2z-(a+b)}$.

(b). The multipliers of $H_{[a,b]} \circ H_{[c,d]}$ *restricted to* $\mathbf{C}$ *(with the natural orientation) are given by* $(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}})^2$ *where* μ *is the cross product*

$(a,b,c,d) = \frac{a-c}{a-d} : \frac{b-c}{b-d}$.

The proof is a direct computation.

2.7. We finish this section by listing two more properties of pairs of circles. The proofs are all simple.

(a). Given any pair of circles (C_1, C_2), there is a third circle C so that the half-turn about C interchanges C_1 and C_2.

(b). Given any pair of unlinked circles, there is a third circle tangent to both of them. Furthermore, if these two circles are not cosphere, then there exist exactly four distinct circles tangent to both of them.

3. Configuration space of three circles

3.1. In this section we will study the configuration space of triples of circles (of a specific type) in S^3 modulo Möbius transformations. We will introduce a coordinate for the configuration space and use these coordinates to calculate the multipliers of the pairs of circles in the triple.

We will be interested in the following type of configuration of three circles C_1, C_2, and C_3 in $\bar{\mathbf{R}}^3$.

3.2. Definition. A type I configuration of three circles is a collection of three circles C_1 C_2 and C_3 satisfying:

(1) there exists a circle C intersecting each C_i at two points;

(2) the three pairs of points $C_1 \cap$ C, $C_2 \cap$ C, and $C_3 \cap$ C bound three disjoint intervals in C.

We call C an *axis* of (C_1, C_2, C_3).

Note that (2) implies that C_i, C_j are unlinked. Furthermore, C_i, C_j are cosphere if and only if C_i, C_j and C are cosphere. The second condition (2) can be generalized to higher dimension. A collection of k codimension-1 -spheres S_1,...,S_k in S^n is said to be in *Schottky position* (*weak Schottky* respectively) if they bound k disjoint balls (k balls with disjoint interiors respectively) in S^n; a collection of k (n-2)-spheres C_1, ..., C_k in S^n is said to be in *Schottky position* (or *weak Schottky*) if they lie in k codimension-1 spheres which are in Schottky position (or *weak Schottky* respectively).

Given a finite collection of codimension-1-spheres $\{S_1, ..., S_k\}$ in weak Schottky position, the *natural orientation* on S_i is the induced orientation from the common region bounded by S_1,..., S_k.

3.3. Schottky Lemma. *(a) If $C_1,...,C_k$ are k codimension-2-spheres in S^n in Schottky position, and H_{C_i} is the half-turn about C_i, then the group Γ generated by the compositions of even number of $H_{C_1},..., H_{C_n}$ is a Schottky group. In particular, it is free and discrete. More generally, if there exists $(k-1)$ codimension-1-spheres $S_2,...,S_k$ so that $S_2,...,S_k$, $H_{C_1}(S_2),...,H_{C_1}(S_k)$ are in Schottky position, then the group Γ is Schottky.*

(b) If $C_1,...,C_k$ are k codimension-2-spheres in S^n in weak Schottky position, and H_{C_i} is the half-turn about C_i, then the group Γ generated by the compositions of even number of $H_{C_1},..., H_{C_n}$ is discrete and free.

The proof is as follows (known to many mathematicians, see for instance [1]). Γ is generated by $H_{C_1}H_{C_i}$ for i=2,..., k by definition. Since the collection $\{S_2, ... S_k, H_{C_1}(S_2) , ..., H_{C_1}(S_k)\}$ is in Schottky position, the generator $H_{C_1}H_{C_i}$ sends the exterior of the ball bounded by S_i to the ball bounded by $H_{C_1}(S_i)$. Thus the result follows from the Klein-Maskit combination theorem.

If we replace the Schottky position by weak Schottky position in the lemma, then the group Γ is still discrete and free by the same argument.

3.4. Lemma. *Suppose (C_1, C_2, C_3) is a type I configuration of three circles with an axis C. Then there exist unique three 2-spheres S_1, S_2 and S_3 in Schottky position so that S_i is orthogonal to C for all i, and S_i is orthogonal to C_j for $i \neq j$. Conversely, suppose three 2-spheres S_1, S_2 and S_3 are in Schottky position and three circles C_1, C_2 and C_3 satisfy that C_i is orthogonal to S_j for $i \neq j$. Then (C_1, C_2, C_3) is of type I.*

We call S_1, S_2 and S_3 the *dual spheres* of the triple (C_1, C_2, C_3).

Proof. By condition (2) in Definition 3.1, $C_i \cap$ C does not separate $C_j \cap$ C in C. Thus there exists a unique 2-sphere S_k $(k \neq i, j)$ orthogonal to C so that inversion about S_k leaves both $C_i \cap$ C and $C_j \cap$ C invariant. Hence S_k is orthogonal to C_i and C_j. Furthermore, (S_1, S_2, S_3) is in Schottky position because $(C_1 \cap$ C, $C_2 \cap$ C, $C_3 \cap$ C) is also so. Suppose conversely that S_1, S_2 and S_3 are in Schottky position. Then there exists uniquely a circle C orthogonal to S_i for i= 1,2,3. Since $C_j \perp S_i$ for i $\neq$ j, C $\cap$ C_i consists of two points which are the focal points of S_j and S_k. Furthermore, $(C_1 \cap$ C, $C_2 \cap$ C, $C_3 \cap$ C) is in Schottky position in C since (S_1, S_2, S_3) is also so. Thus, (C_1, C_2, C_3) is of type I. □

The following lemma shows the uniqueness of the axes.

3.5. Lemma. *Suppose that C_1, C_2 and C_3 form a type I configuration and that no pair (C_i, C_j) $(i \neq j)$ is cosphere. Then the axis of C_1, C_2, C_3 is unique. Conversely, if one pair (C_i, C_j) $(i \neq j)$ is cosphere, then the axis of (C_1, C_2, C_3) is not unique.*

Proof. Since (C_i, C_j) is not cosphere for each i ≠ j, there exists a unique 2-sphere S_k so that $S_k \perp C_i$ and $S_k \perp C_j$ $(k \neq i, j)$. Now, by the proof of the previous lemma, these S_k's can also be constructed using an axis C so that $S_k \perp$ C. Thus, C contains the six focal points of the pairs (S_i, S_j) $(i \neq j)$. This shows C is unique. Conversely, if (C_1, C_2) is cosphere, then C $\subset$ sp(C_1, C_2), and C_3 intersects sp(C_1, C_2) at two points. We can easily pick infinitely many $C \subset$ sp(C_1, C_2) passing through $C_3 \cap$ sp$(C_1\ C_3)$ and intersecting both C_1 and C_2 transversely.

3.6. We now introduce a parametrization of the space of all type I configurations modulo Möbius transformations. We fix an axis C and an orientation on C so that $C_1 \cap C$, $C_2 \cap C$ and $C_3 \cap C$ are in the order of the orientation. $\bar{\mathbf{R}}^3$ is oriented by the right-hand rule. The normal bundle of C has the induced orientation from $\bar{\mathbf{R}}^3$ and C. For simplicity, we assume that C is the positively oriented z-axis, and the normal bundle has the same orientation as the natural orientation on the xy-plane.

Let D_i be the disc in sp(C, C_i) bounded by C_i so that $D_i \cap D_j = \phi$ for all i ≠ j.

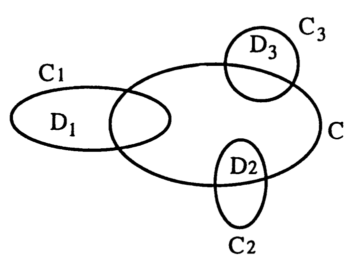

FIGURE 3.1

Each disc D_i is decomposed into two half discs D_i^+ and D_i^- by C where D_i^+ is the half disc so that the inner angle θ_i at its vertices is less than or equal to $\pi/2$. If $\theta_i = \pi/2$, we choose D_i^+ to be any of the two half discs.

We need the notion of hyperbolic distance for non-separating pairs of points in a circle. Given four distinct points a, b, c and d in a circle so that (a, b) does not separate (c, d), the *hyperbolic distance* between (a, b) and (c, d) is defined to be $lg\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}$ where λ is the cross ratio (a, b, c, d). Indeed, if l_1 and l_2 are two geodesics in a hyperbolic plane with infinity C so that ∂l_1 = (a, b) and ∂l_2 = (c, d), then the distance defined above is the hyperbolic distance between l_1 and l_2 in the hyperbolic plane.

Now for any type I circles (C_1, C_2, C_3) with axis C, its *real coordinate* $(\theta_1, \theta_2, \theta_3, \phi_{12}, \phi_{23}, \phi_{31}, d_{12}, d_{23}, d_{31})$ is defined as follows.

(1) θ_i is the angle between C_i and C so that $\theta_i \in (0, \pi/2]$ (i.e., θ_i is the inner angle of D_i^+);

(2) d_{ij} is the hyperbolic distance between $C_i \cap$ C, $C_j \cap$ C in C;

(3) ϕ_{ij} is the angle counted from D_i^+ to D_j^+ in the normal direction of C.

Thus, $\phi_{ij} \in [0,2\pi)$, and $\phi_{12}+\phi_{23}+\phi_{31}= 2\pi$ or 4π.

Note that (C_i, C_j) is cosphere if and only if ϕ_{ij} =0 or π. Also if $\theta_i=\pi/2$, then ϕ_{ij} is well defined up to the choice of D_i^+.

3.7. Lemma. *Given $(\theta_1, \theta_2, \theta_3, \phi_{12}, \phi_{23}, \phi_{31}, d_{12}, d_{23}, d_{31})$ satisfying $\phi_{12}+\phi_{23}+\phi_{31}= 2\pi$ or 4π, $\theta_i \in (0, \pi/2]$, $d_{ij}>0$ for all $i \neq j$, there exists a configuration of three circles C_1, C_2 and C_3 so that its real coordinate is $(\theta_i, \phi_{ij}, d_{ij})$. Furthermore, if $\phi_{ij} \neq 0, \pi$ for all i,j, then configuration is unique.*

Proof. Fix any circle C as the axis. Given three positive numbers d_{ij}, we construct three pairs of points (X_i, Y_i) in Schottky position in C so that d_{ij} is the hyperbolic distance between (X_i, Y_i) and (X_j, Y_j) for i $\neq$ j by Fenchel-Nielsen lemma. Now we construct a disc D_1 intersecting C at $\{X_1, Y_1\}$ at an angle θ_1. Then each of the rest of the discs D_i is determined since D_i intersects C at $\{X_i, Y_i\}$ at an angle θ_i and forms an dihedral angle ϕ_{1i} with D_1. By Lemma 3.5, C is unique if no two C_i, C_j are cosphere which is the same as $\phi_{ij} \neq 0, \pi$.

3.8. We now calculate the multipliers of C_i, C_j using the real coordinate. Recall that the multipliers of C_i and C_j are the same as the multipliers of the Möbius transformation $H_{C_i}H_{C_j}$ which consist of four complex numbers. To specify two of them, let S_k be the dual spheres constructed in Lemma 3.4. We will be interested in the the multipliers of $H_{C_i}H_{C_j}|S_k$ with respect to S_k in the natural orientation.

3.9. Proposition. *Suppose (C_1, C_2, C_3) forms type I configuration with real coordinate $(\theta_1, \theta_2, \theta_3, \phi_{12}, \phi_{23}, \phi_{31}, d_{12}, d_{23}, d_{31})$, and S_1, S_2*

and S_3 are the dual 2-spheres equipped with the natural orientations. Then for each (i, j) $\in \{(1,3),(3,2),(2,1)\}$, the multipliers of C_i and C_j with respect to the orientation of S_k are $(\frac{1+\sqrt{\lambda_{ij}}}{1-\sqrt{\lambda_{ij}}})^{\pm 2}$ where λ_{ij} are

$$\frac{(e^{d_{ij}-\sqrt{-1}\phi_{ij}}tg(\theta_i/2)+tg(\theta_j/2))(e^{d_{ij}-\sqrt{-1}\phi_{ij}}tg(\theta_j/2)+tg(\theta_i/2))}{(e^{d_{ij}-\sqrt{-1}\phi_{ij}}tg(\theta_i/2)tg(\theta_j/2)-1)(e^{d_{ij}-\sqrt{-1}\phi_{ij}}-tg(\theta_i/2)tg(\theta_j/2))}.$$

Proof. Let $p_k \in S_k \cap$ C be the point so that the inner norm of the common region Ω bounded by S_i's at p_k is the same as the orientation of C. Now conjugate C_i, C_j, C and S_k by an $h \in$ Mob$^+(S^3)$ sending $S_k\cap$ C to $\{0,\infty\}$, and p_k to 0 so that, the following hold:

(1) The oriented C is the positively oriented z-axis.

(2) S_k with the orientation is the xy-plane with the natural orientation (thus Ω is in { (x,y,z) $|z>0\}$).

(3) For (i, j) $\in$ {(1,2), (2,3), (3,1)}, $C_j \cap$ C = {$\pm$ (0, 0, 1)}, $C_i \cap$ C = $\{\pm(0,0,e^{d_{ij}})\}$. D_j^+ lies in the half-space { (x,0,z)$|x \le 0\}$, and D_i^+ in {(x,y,z)$|(x,y)=\lambda e^{-\sqrt{-1}\phi_{ij}}, \lambda \le 0\}$.

Recall the notation introduced in section 2.6: [a,b] is the unique circle in $\bar{\mathbf{R}}^3$ orthogonal to $\mathbf{C}\cup\{\infty\}$ at a, b. We obtain C_j=$[-tg(\theta_j/2), cot(\theta_j/2)]$ and

$$C_i = [tg(\theta_i/2)e^{d_{ij}-\sqrt{-1}\phi_{ij}}, -cot(\theta_i/2)e^{d_{ij}-\sqrt{-1}\phi_{ij}}].$$

By Lemma 2.6, the result thus follows.

3.10. We will study the degeneration of type I circles in this section. A triple of circles (C_1, C_2, C_3) is said to be *totally degenerate* if C_1, C_2, C_3 are tangent to a fourth circle C at three different points. Let D be a disc with boundary C so that its interior int(D) is considered as a model for the 2-dimensional hyperbolic space H^2. Take three horocycles C_1', C_2', C_3' in H^2 based on three different points. Then rotations of these C_i''s about C in S^3 give a totally degenerate triple. Conversely, all totally degenerate triples are obtained in this way. This leads to the study of the configuration space of three horocycles based on three different points in H^2 modulo hyperbolic isometries. There is a simple parametrization of the configuration space. Suppose h_1 and h_2 are two horocycles in H^2 based on distinct points. Then their *weighted distance* is defined as follows. Let l be the geodesic determined by the base points of h_1 and h_2, and let d be the hyperbolic distance between $l \cap h_1$ and $l \cap h_2$. Then the weighted distance between h_1 and h_2 is d if $h_1 \cap h_2 = \phi$ and is $-d$ otherwise.

3.11. Lemma. *Given any three real numbers a_1, a_2 and a_3, there exist three horocycles h_1, h_2 and h_3 based on three different points in H^2 so that the weighted distance between h_i, h_j is a_k, $i \neq j \neq k \neq i$. The triple of horocycles is unique up to isometry.*

Indeed, there exist three pairwise tangent horocycles in H^2. Thus, the result follows by a simple calculation of weighted distances.

We now parametrize the space of totally degenerate triples as follows. Suppose $(C_1,\ C_2,\ C_3)$ is a totally degenerate triple of circles tangent to C at three different points. We orient C so that $C_1 \cap C$, $C_2 \cap C$, and $C_3 \cap C$ are in the natural order of C. Take a disc D with $\partial D = C$ and consider int(D) as a model for the hyperbolic space H^2. Let (C_1', C_2', C_3') be the three horocycles in int(D) so that C_i is obtained by rotating C_i' positively at an angle ϕ_i about C. Then the *Möbius coordinate* of (C_1, C_2, C_3) is given by (z_1, z_2, z_3) where $z_k = e^{d_{ij}+\sqrt{-1}(\phi_i - \phi_j)}$. Here d_{ij} is the weighted distance between C_i and C_j, and (i,j,k) is a positive permutation of (1,2,3). By the definition, $z_1 z_2 z_3$ is positive real. Lemma 3.11 implies that for any three complex numbers z_1, z_2 and z_3 in $\mathbf{C} - \{0\}$ so that $z_1 z_2 z_3$ is positive real, there exists a triple of totally degenerate circles whose Möbius coordinate is (z_1, z_2, z_3).

The *dual spheres* of a totally degenerate triple of circles (C_1, C_2, C_3) are defined similarly. Namely, the 2-spheres S_1, S_2, S_3 satisfy that S_i is orthogonal to C_j, C_k and C for $i \neq j \neq k \neq i$. The dual spheres can be constructed as follows. Let (C_1', C_2', C_3') be the triple of horocycles constructed above. For each pair C_i', C_j' (i $\neq$ j), let l_k be the geodesic in int(D) with end points the tangent points of C_i and C_j with C. Then S_k is the unique 2-sphere orthogonal to sp(D) at sp(l_k). Furthermore, S_1, S_2 and S_3 bound a common region Ω in S^3.

3.12. Lemma. *The multipliers of C_i and C_j with respect to the naturally oriented S_k are $(\frac{1+\sqrt{\lambda_k}}{1-\sqrt{\lambda_k}})^{\pm 2}$ where $\lambda_k = z_k/(z_k - 1)$ for $i \neq j \neq k \neq i$. It is a negative real number if and only if $Re(z_k) = 1/2$.*

One can calculate the multipliers directly. We will however derive it from Proposition 3.8 and the proposition below.

3.13. Given a type I triple (C_1, C_2, C_3) with real coordinate $(\theta_i,\ \phi_{ij},\ d_{ij})$, the *Möbius coordinate* of it is defined to be $(z_1, z_2, z_3, a_1, a_2, a_3)$

where $z_k = tg(\theta_i/2)tg(\theta_j/2)e^{d_{ij}-\sqrt{-1}\phi_{ij}}$ and $a_k = tg^2(\theta_k/2)$, $i \neq j \neq k \neq i$. Thus $z_1z_2z_3$ is positive real and $|z_k|^2 > a_ia_j$, where (i,j,k) is a negative permutation of (1,2,3). The corresponding pairwise multipliers are given by $(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_i = (\frac{1+\sqrt{f_i}}{1-\sqrt{f_i}})^{\pm 2}$ and $f_i = \frac{(z_i+a_j)(z_i+a_k)}{(z_i-1)(z_i-a_ja_k)}$ by Proposition 3.9.

3.14. Proposition. *The Möbius coordinate* $(z_1, z_2, z_3, a_1, a_2, a_3)$ *converges to* $(w_1, w_2, w_3, 0, 0, 0) \in (C - \{0\})^3 \times C^3$ *if and only if the type I triple* (C_1, C_2, C_3) *(modulo Möbius transformations) converges to a totally degenerate triple of circles having coordinate* (w_1, w_2, w_3).

Proof. Let us first show the result for the special case where z_1, z_2, and z_3 are positive real numbers. The general case follows from the specific case. Thus, we are given a type I triple (C_1, C_2, C_3) so that they all lie in a 2-sphere. First let us assume that the Möbius coordinate converges, and show that the triple converges geometrically to a totally degenerate one. Let D be the disc so that ∂D is the axis of (C_1, C_2, C_3) and that D_i^+'s are in D. We consider int(D) as a model for the hyperbolic space H^2. Thus each C_i corresponds to a curve of constant curvature in H^2. Let l_k be the geodesic in H^2 orthogonal to $C_i \cap H^2$ and $C_j \cap H^2$. The existence of l_k follows from the assumption that $C_i \cap C$ for i = 1,2,3 bound three disjoint intervals in ∂D. We define the weighted distance between $C_i \cap H^2$ and $C_j \cap H^2$ as before, i.e., it is the hyperbolic distance between $C_i \cap l_k$ and $C_j \cap l_k$ if $C_i \cap C_j = \phi$, and is the negative of this hyperbolic distance otherwise. One calculates that the exponential of the weighted distance between $C_i \cap H^2$ and $C_j \cap H^2$ is given by $\frac{e^{d_{ij}} \sin\theta_i \sin\theta_j}{(\cos\theta_i+1)(\cos\theta_j+1)}$. Thus, if $(z_1, z_2, z_3, a_1, a_2, a_3)$ converges to $(w_1, w_2, w_3, 0, 0, 0)$, then the exponential of the weighted distances converge to w_i for i =1,2,3. Therefore, after a normalization, $(C_1 \cap H^2$, $C_2 \cap H^2$, $C_3 \cap H^2)$ converges to three horocycles based on three different points. The result follows. Now, if the triple converges, from the above calculation of weighted distances, we conclude that their coordinates converge as well.

The general case follows from the above special case since $(z_1, z_2, z_3, a_1, a_2, a_3)$ converges if and only if $(|z_1|, |z_2|, |z_3|, a_1, a_2, a_3)$ converges and $(arg(z_1), arg(z_2), arg(z_3))$ converges (mod$(\mathbf{Z})$).

3.15. Remark. By a *partially degenerate* triple of circles (C_1, C_2, C_3) we mean a degeneration of type I circles so that some C_i becomes tangent to the axis C. The above proposition still holds for partially degenerate triples.

3.16. We summarize the result as follows. The space of all triples of type I circles together with their degenerations is parametrized by $\{(z_1, z_2, z_3, a_1, a_2, a_3) | |z_i|^2 > a_j a_k, i \neq j \neq k \neq i,\ a_i \in [0,1], z_1 z_2 z_3$ is positive real$\}$ where the degeneration corresponds to $a_1 a_2 a_3 = 0$. The corresponding pairwise multipliers (with respect to the naturally oriented dual spheres) of the triple with coordinate $(z_1, z_2, z_3, a_1, a_2, a_3)$ are given by $(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_i = (\frac{1+\sqrt{f_i}}{1-\sqrt{f_i}})^{\pm 2}$ and $f_i = \frac{(z_i+a_j)(z_i+a_k)}{(z_i-1)(z_i-a_j a_k)}$.

This leads us to the study of the function f_i. The proof of the following lemma is a simple calculation.

3.17. Lemma.

(1)
$$\partial/\partial z \frac{(z+a)(z+b)}{(z-1)(z-ab)} = \frac{(1+a)(1+b)(ab-z^2)}{(z-1)^2(z-ab)^2}.$$

(2)
$$\partial/\partial a \frac{(z+a)(z+b)}{(z-1)(z-ab)} = \frac{z(z+b)(1+b)}{(z-1)(z-ab)^2}.$$

Our basic observation is the following.

3.18. Proposition. *Let* Π *be the map from*

$$\mathcal{M} = \{(z_1, z_2, z_3, a_1, a_2, a_3) | z_1 z_2 z_3 \text{ is positive real, } z_k \neq 1, \quad z_i \neq a_i a_j, \\ z_k^2 \neq a_i a_j, \quad z_i \neq -a_j \text{ for } i \neq j \neq k \neq i, \text{ and } a_i \in (-1,1)\}$$

to $\mathbf{C}^3$ *sending* $(z_1, z_2, z_3, a_1, a_2, a_3)$ *to* (f_1, f_2, f_3) *where* $f_i = \frac{(z_i+a_j)(z_i+a_k)}{(z_i-1)(z_i-a_j a_k)}$. *Then every point* $p = (p_1, p_2, p_3) \in \mathbf{C}^3$ *so that not all* p_i *are real is a regular value of* Π.

Proof. Suppose $\Pi(z_1, z_2, z_3, a_1, a_2, a_3) = $ p. The derivative of Π is the Jacobian matrix below restricted to the tangent space of the hypersurface $z_1 z_2 z_3 =$ positive real,

$$J = \begin{pmatrix} \partial f_1/\partial z_1 & 0 & 0 & 0 & \partial f_1/\partial a_2 & \partial f_1/\partial a_3 \\ 0 & \partial f_2/\partial z_2 & 0 & \partial f_2/\partial a_1 & 0 & \partial f_2/\partial a_3 \\ 0 & 0 & \partial f_3/\partial z_3 & \partial f_3/\partial a_1 & \partial f_3/\partial a_2 & 0 \end{pmatrix}.$$

Since not all p_i's are real, we may assume that at least two of z_i's are not real, say, z_2 and z_3 are not real numbers. Since $\partial f_i/\partial z_i \neq 0$

for all i by Lemma 3.17, to show that the rank of $D\Pi$ is six, it suffices to show that the real rank of $A_3 = \begin{pmatrix} Re(\partial f_3/\partial a_1) & Re(\partial f_3/\partial a_2) \\ Im(\partial f_3/\partial a_1) & Im(\partial f_3/\partial a_2) \end{pmatrix}$ or $A_2 = \begin{pmatrix} Re(\partial f_2/\partial a_1) & Re(\partial f_2/\partial a_3) \\ Im(\partial f_2/\partial a_1) & Im(\partial f_2/\partial a_3) \end{pmatrix}$ is two.

Suppose that the rank of A_3 is less than two. Then, by Lemma 3.17, there exists a real number μ so that $\mu(a_2 + z_3) = a_1 + z_3$. Since z_3 is not real, $\mu = 1$. Thus $a_1 = a_2$. Applying the same argument to A_2, we see that the rank of $D\Pi$ is six unless $a_1 = a_2 = a_3$.

To finish the proof, we will show that the rank of $D\Pi$ is still six at the point with $a_1 = a_2 = a_3$. To this end, we consider in the equation $z_1 z_2 z_3$ = positive real that z_1, z_2 and $|z_3|$ are free variables. Thus $z_3 = re^{i\theta}$ where r is a free variable. One calculates that $\partial f_3/\partial r = \frac{(1+a_1)(1+a_2)(a_1 a_2 - z_3^2)e^{i\theta}}{(z_3-1)^2(z_3-a_1a_2)^2}$. Thus, if we show that the real rank of the matrix $B = \begin{pmatrix} Re(\partial f_3/\partial r) & Re(\partial f_3/\partial a_2) \\ Im(\partial f_3/\partial r) & Im(\partial f_3/\partial a_2) \end{pmatrix}$ is two, then the result follows. Suppose otherwise, then there is a real number μ so that $\partial f_3/\partial r = \mu \partial f_3/\partial a_1$. By Lemma 3.17 and the formula above, we obtain that $z_3^2 - a_1^2 = \lambda(z_3 - 1)(z_3 + a_1)$ for some real number λ. Thus $z_3 - a_1 = \lambda(z_3 - 1)$. Since z_3 is not real, $\lambda = 1$. This implies that $a_1 = a_2 = a_3 = 1$ which is excluded in the definition of $\mathcal{M}$.

The following is crucial to our construction.

3.19. Theorem. *Suppose (C_1, C_2, C_3) is a triple of totally degenerate circles with coordinate (z_1, z_2, z_3) so that no two of $z_i/(z_i - 1)$'s are real and $z_1 z_2 z_3 \neq 0$. If there exist three positive real numbers a_1, a_2, a_3 so that $\sum_{i \neq j \neq k \neq i} (a_i + a_j)\frac{z_k - 1}{z_k}$ is real, then there exists a local deformation $(C_1(t), C_2(t), C_3(t))$, $t \in [0, 1)$ of (C_1, C_2, C_3) so that,*

(1) $C_i(0) = C_i$ for i= 1,2,3;

(2) $(C_1(t), C_2(t), C_3(t))$ is a type I configuration if $t > 0$;

(3) the pairwise multipliers of $C_i(t), C_j(t)$ are the same as the pairwise multiplers of C_i, C_j for $i \neq j$.

Proof. Consider the point $p = (z_1/(z_1 - 1), z_2/(z_2 - 1), z_3/(z_3 - 1))$ in $\mathbf{C}^3$. By the assumption, no two of the coordinates of p are real. Thus p is a regular value of Π from $\mathcal{M}$ to $\mathbf{C}^3$. Therefore $\Pi^{-1}(p)$ is a 2-dimensional submanifold of $\mathcal{M}$ containing $q = (z_1, z_2, z_3, 0, 0, 0)$.

Hence the result follows if we show that the tangent space $T_q\Pi^{-1}(p)$ contains a vector of the form $(v_1, v_2, v_3, a_1, a_2, a_3)$ where $a_i > 0$ for all

i. Indeed such a tangent vector produces a path $q(t) = (z_1(t), z_2(t), z_3(t), a_1(t), a_2(t), a_3(t))$ so that all $a_i(t) > 0$. Thus, they are the Möbius coordinates of some type I triple of circles for t small.

Suppose $(v_1, v_2, v_3, a_1, a_2, a_3) \in T_q\mathcal{M}$ which is in $T_q(\Pi^{-1}(p))$. Then we have $\sum_{i=1}^{3} v_i/z_i$ is real since $(z_1 + tv_1)(z_2 + tv_2)(z_3 + tv_3)$ is real infinitesimally. Furthermore, using Lemma 3.17, we find that

$$-\frac{v_i}{(z_i - 1)^2} + \frac{a_j + a_k}{z_i - 1} = 0 \quad \text{for} \quad i \neq j \neq k \neq i.$$

This implies that the only restriction on a_i is that

$$\sum_{i \neq j \neq k \neq i} (a_i + a_j)(z_k - 1)/z_k$$

is real.

4. The Euler number of Möbius structures

4.1. A Seifert manifold is an oriented compact 3-manifold with an S^1-action without a global fixed point. The S^1-fibers are oriented by the orientation of S^1. The quotient space of the S^1-action is an oriented orbifold and the 3-manifold is a Seifert fibration over the orbifold. By a *horizontal curve* in a Seifert manifold we mean a smooth curve intersecting each S^1- fiber transversely in at most one point. A *marking* in a Seifert manifold M is a collection of finitely many simple closed oriented horizontal curves one in each boundary component so that the orientations of the S^1-fiber and the marking curve determine the induced orientation on ∂M. We call a Seifert manifold together with a marking a *marked Seifert manifold.* The goal in this section is to define the Euler number (relative Euler number) of the fibration of marked Seifert manifolds and to define the Euler number of a Möbius structure with trivial monodromy in the S^1-fiber on a Seifert manifold.

4.2. Suppose M is a non-closed marked Seifert manifold with a marking consisting of curves $C_1, C_2, ..., C_n$. Then $\sum_{i=1}^{n}[C_i] = e[S^1]$ in $H_1(M, Q)$. We call e the *Euler number of the fibration* of the marked Seifert manifold. If M is closed, the Euler number of fibration is defined to be the usual one. We denote the Euler number by $e(M)$. Note that if M has no singular fibers, then $e(M)$ is an integer.

The gluing of two marked Seifert manifolds M_1 and M_2 along some components of their boundaries is defined as follows. Take orientation reversing diffeomorphisms from the specified boundary components of M_1 to the specified boundary components of M_2 so that S^1-fibers are mapped orientation preservingly to S^1-fibers, and marked curves are mapped orientation reversingly to the marked curves. The result of gluing by the diffeomorphisms is still a marked Seifert manifold denoted by $M_1 \#_\partial M_2$. We call $M_1 \#_\partial M_2$ a boundary connected sum of M_1 and M_2.

4.3. Lemma. *$e(M_1 \#_\partial M_2) = e(M_1) + e(M_2)$.*

Indeed, if $M_1 \#_\partial M_2$ has boundary components, then the formula is a direct consequence of the definition. To show the lemma for closed $M_1 \#_\partial M_2$, it suffices to show that if marking curves $C_1, ..., C_r$ in the gluing tori $\partial M_1 \bigcap \partial M_2$ are changed to a new system of horizontal simple closed curves $D_1, ..., D_r$, one in each of these tori, the resulting Euler number $e(M_1 \#_\partial M_2)$ remains the same. To see this, let C_i^+ and D_i^+ be the copies of C_i and D_i with correct orientation in ∂M_1, and C_i^- and D_i^- be the copies of C_i and D_i with the correct orientations in ∂M_2. Then if $[C_i^+] = [D_i^+] + k_i[S^1]$ in $H_1(\partial M_1, Z)$, we have $-[C_i^-] = -[D_i^-] + k_i[S^1]$ in $H_1(\partial M_2, Z)$ by applying the gluing map. Thus, $[C_i^+] + [C_i^-] = [D_i^+] + [D_i^-]$ in $H_1(M_1 \#_\partial M_2, Q)$.

4.4. We will first recall Dehn surgery on 3-manifolds and then calculate the Euler number of fibration resulting from Dehn surgery in this section. Suppose C is a marking curve in a boundary component S of a marked Seifert manifold M. Given two relative prime numbers p,q (q $\neq$ 0), a p/q-Dehn surgery on M along S is the gluing of the boundary of a solid torus $D^2 \times S^1$ to S so that the meridian $\partial D \times \{1\}$ is attached to a curve in S representing the homology class $p[S^1] + q[C]$ in $H_1(S, Z)$. It is well known that the result manifold is still a marked Seifert 3-manifold with the induced marking and orientation from M.

We define the *simple marked Seifert manifolds* as follows and we will call them simple manifold from now on for simplicity. Let P be an oriented pair of pants. Then a simple marked Seifert manifold N of type I is $P \times S^1$ with product orientation, product S^1-fibration and some marking curves $\{C_1, C_2, C_3\}$. If the Euler number of fibration is an integer n, we denote the type I manifold by N(n). A p_1/q_1-Dehn surgery on a component of ∂N(n) gives rise to a Seifert manifold of type II, denoted by N(n; p_1/q_1). A p_2/q_2-Dehn surgery on a component of ∂N(n; p_1/q_1) is a type III simple manifold, denoted by N(n; $p_1/q_1, p_2/q_2$). Lastly, a type IV simple manifold is one obtained by a p_3/q_3-Dehn surgery on a type III manifold, denoted by N(n; $p_1/q_1, p_2/q_2, p_3/q_3$). By the definition of the Euler number of fibration, we have $e(N(n; p_1/q_1, ..., p_i/q_i)) = n + p_1/q_1 + ... + p_i/q_i$ for i= 0,1,2,3. The orbit spaces of the simple manifolds are two-dimensional pair of pants of types I,II,III, IV listed below.

To construct all compact orientable hyperbolic 2-orbifolds by using simpler orientable hyperbolic orbifolds, we need three more exceptional simple orbifolds:

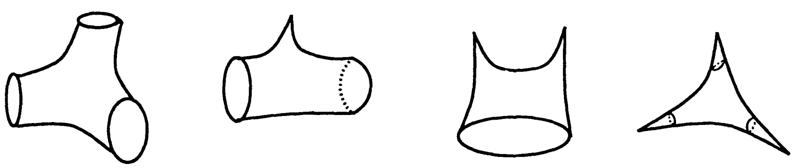

FIGURE 4.1

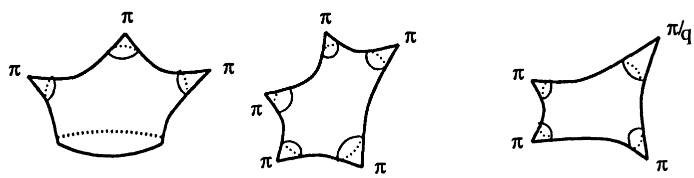

FIGURE 4.2

where the type V simple orbifold is a hyperbolic 2-orbifold on the closed disc with three cone points of angles π; the type VI simple orbifold is the one on the 2-sphere with five cone points of angles π; and the type VII simple orbifold is the one on the 2-sphere with three cone points of angle π and one cone point of angle $2\pi/q_4$ ($q_4 > 2$). We define the corresponding exceptional simple Seifert manifolds of types V,VI, VII similarly. They are denoted by $N_0(n; p_1/2, p_2/2, p_3/2)$, $N_0(n; p_1/2, ..., p_5/2)$, and $N_0(n; p_1/2, p_2/2, p_3/2, p_4/q_4)$. The Euler number of fibration is

$n + \sum(p_i/q_i)$.

Each compact hyperbolic 2-orbifold is a boundary union of simple hyperbolic 2-orbifolds. Thus, each marked Seifert manifold M over a hyperbolic orbifold is the boundary connected sum of simple ones, i.e., M $= M_1 \#_\partial ... \#_\partial M_k$ where each M_i is simple or exceptional simple. If M is a marked Seifert manifold over a hyperbolic orbifold of genus greater than zero, then M can be expressed as $M_1 \#_\partial ... \#_\partial M_k$ where each M_i is simple of types I, II, or III. By Lemma 4.3, this decomposition gives rise a way to calculate the Euler number of fibration of a Seifert manifold.

To finish the discussion of Euler number, we observe that the Euler number classifies the type I simple manifolds up to isomorphism. Here two marked Seifert manifolds M_1 and M_2 are said to be *isomorphic* if there is an orientation preserving diffeomorphism between them so that it preserves the oriented S^1-fibers and the markings.

4.5. Lemma. *Suppose N_1 and N_2 are two simple type I Seifert manifolds having the same Euler number of fibration. Then N_1 is isomorphic to N_2.*

The proof follows by examining the group G of orientation preserving diffeomorphisms of $P \times S^1$ preserving the S^1-fibers and their orientations. Let $\partial P = \{C_1, C_2, C_3\}$ where each C_i has the induced orientation. We also use C_i to denote the corresponding horizontal curve $C_i \times \{x_0\}$ in $\partial P \times S^1$. Given two integers p, q, let f: $P \to S^1$ be a smooth map representing the cohomology class $p[C_1]^* + q[C_2]^* \in H^1(P, Z) \tilde{=} [P, S^1]$ where $[C_i]^*$ is the dual class of $[C_j]$, i.e., $[C_i]^*([C_j]) = \delta_{ij}$, $1 \leq i, j \leq 2$. Then the diffeomorphism $\tilde{f}(x,t) = (x, f(x)t) : P \times S^1 \to P \times S^1$ is in G and sends $[C_1]$ to $[C_1] + p[S^1]$, $[C_2]$ to $[C_2] + q[S^1]$, and $[C_3]$ to $[C_3] + (-p - q)[S^1]$. This shows that G acts transitively on the space of all markings $\{C_1, C_2, C_3\}$ having the same Euler number. Thus the result follows.

4.6. Corollary. *Suppose $N(n; \frac{p_1}{q_1}, \ldots, \frac{p_i}{q_i})$ and $N(m; \frac{a_1}{b_1}, \ldots, \frac{a_i}{b_i})$ are two simple Seifert manifolds of the same type and have the same Euler number of fibration. If $p_j/q_j = a_j/b_j mod(Z)$ for all j, then these two marked Seifert manifolds are isomorphic.*

Note that $p_i/q_i - [p_i/q_i]$ is sometimes called the local Seifert invariant of the singular fiber corresponding to the core curve of the p_i/q_i-Dehn surgery.

4.7. We now define the Euler numbers of Möbius structures on Seifert manifolds.

A 3-manifold M is said to have a Möbius structure if M can be covered by open coordinate charts $\{U_\alpha, \Phi_\alpha\}$ so that Φ_α maps U_α to an open set in a closed ball in S^3, that the transition functions are restrictions of Möbius transformations, and that $\phi_\alpha(\partial M \cap U_\alpha)$ is in some 2-sphere. In particular, ∂M consists of 2-manifolds with the induced 2-dimensional Möbius structures. The global version of Möbius structure on M consists of two objects : the developing map (a local diffeomorphism) dev: $\tilde{M} \to S^3$ sending $\partial\tilde{M}$ to spherical submanifolds where $\tilde{M}$ is the universal cover of M, and a monodromy homomorphism $\rho : \pi_1(M) \to Mob(S^3)$ so that $\text{dev}(\gamma m) = \rho(\gamma)\ \text{dev}(m)$ for all m $\in \tilde{M}$ and $\gamma \in \pi_1(M)$.

Möbius structures on Seifert manifolds so that the monodromy is non-trivial in the S^1-fiber are all known. In particular, the Euler number of fibration of a closed Seifert manifold M supporting such a structure is zero. All Möbius structures that we are going to discuss below have trivial monodromy in S^1-fibers.

Suppose a Seifert manifold M has a Möbius structure with trivial monodromy in S^1-fibers. Then the induced Möbius structure on each boundary component of ∂M is Möbius isomorphic to the Möbius tours $T_\lambda = \mathbf{C}\text{-}\{0\}/(z \sim \lambda z)$ where λ is a non-zero complex number of norm larger than one. We call λ the *principal multiplier* of T_λ.

We now produce a marking curve on each component of ∂M as follow. Choose a sense preserving Möbius isomorphism between a component S of ∂M and T_λ so that the oriented S^1-fibers correspond to the quotient of circles $|z| = consts$ with the positive orientation (counterclockwise in $\mathbf{C}$). The marking in S is the inverse image of the quotient of the oriented horizontal curve $\{ \lambda^t | t \in [0,1]\}$ in T_λ.

Given a Möbius structure with trivial monodromy in S^1-fiber on a Seifert manifold M so that the principal multipliers of ∂M are $\lambda_1, ..., \lambda_m$ and the induced marking curves are $C_1, ..., C_m$, the *Euler number of the Möbius structure*, denoted by $e_S(M)$, is defined to be $e(M; C_1, ..., C_m) - \sum_{i=1}^m Arg(\lambda_i)/2\pi$. If M has no boundary component, then $e_S(M)$ is defined to be $e(M)$.

From the definition, one sees easily that if M has an $H^2 \times R^1$ geometric structure, then the Euler number of structure is always zero.

The basic property of e_S is the additivity under Möbius gluing. Suppose N and M are two compact Seifert 3-manifolds having Möbius structures with trivial monodromy in S^1-fibers. Let $h : \partial_0 N \to \partial_0 M$ be an orientation reversing local Möbius transformation from a collection of boundary components $\partial_0 N$ of N to a collection of boundary components $\partial_0 M$ of M so that h takes S^1-fiber to S^1-fiber and preserves the orientations in the fiber. Then $N\#_h M$ has a Möbius structure with trivial monodromy in S^1-fiber so that the restriction of the structure to N and M gives back to the original Möbius structures. Recall that a Möbius structure on M is *uniformizable* if there is an open domain D in S^3 and a 2-manifold S in ∂D which is contained in a unions of spheres and a discrete group Γ of Möbius automorphsims of $D \cup S$ so that $(D \cup S)/\Gamma$ is conformally equivalent to the Möbius structure on M. Now if both Seifert manifolds N and M have uniformizable Möbius structures with trivial monodromy groups, then the Möbius structure on $N\#_h M$ is also uniformizable whose monodromy group is the Maskit combination of the monodromy groups of N and M. Our major observation is the following.

4.8. Proposition. *Under the above assumption, $e_S(N\#_h M) = e_S(N) + e_S(M)$.*

Proof. We need

4.9. Lemma. *Suppose a diffeomorphism $\phi : T_\lambda \to T_\mu$ is an orientation reversing local Möbius transformation preserving the family of circles $S^1 = \{z||z| = const.\}$ up to orientation. Then $\lambda = \bar{\mu}$. Furthermore, if C_λ and C_μ are the marking curves in the tori, then $\phi_*([C_\lambda]) = [S^1] - [C_\mu]$ in the first homology group.*

Proof. Consider the lifting $\tilde{\phi}$: $\mathbf{C}$ -$\{0\} \to \mathbf{C}$-$\{0\}$ of the gluing map. By the assumption, $\tilde{\phi}$ (z) $= a/\bar{z}$ for some $a \in \mathbf{C}$-$\{0\}$. Thus $\lambda = \bar{\mu}$. The second statement follows from the definition.

We now use the lemma to show the proposition. To this end, we observe that if M,N are two marked Seifert manifolds, and $h : \partial_0 N \to \partial_0 M$ is an orientation reversing diffeomorphism preserving S^1-fibers up to orientation, then h takes the marking curves C_i in $\partial_0 N$ to a curve homologous to $S^1 - C_j'$ where C_j' is the marking curve in ∂M by the above lemma. Thus, $e(M\#_h N) = e(M) + e(N) - k$ where k is the number of components in $\partial_0 N$. To finish the proof, let $\lambda_1, ..., \lambda_k, \lambda_{k+1}, ..., \lambda_n$ be the principal multipliers of ∂N, and $\bar{\lambda}_1, .., \bar{\lambda}_k, \mu_{k+1}, ..., \mu_m$ be the principal multipliers of ∂M where the first k numbers are the multipliers of the k components of $\partial_0 N$ and $\partial_0 M$ respectively. Now, by the definition and the formula $1 = Arg(\lambda)/2\pi + Arg(\bar{\lambda})/2\pi$, we have

$$\begin{aligned} e_S(N\#_h M) =& e(N\#_h M) - \sum_{i=k+1}^{n} \frac{Arg(\lambda_i)}{2\pi} - \sum_{j=k+1}^{m} \frac{Arg(\mu_j)}{2\pi} \\ =& e(N) + e(M) - k - \sum_{i=k+1}^{n} \frac{Arg(\lambda_i)}{2\pi} - \sum_{j=k+1}^{m} \frac{Arg(\mu_j)}{2\pi} \\ =& e(N) - \sum_{i=1}^{n} \frac{Arg(\lambda_i)}{2\pi} + e(M) - \sum_{j=1}^{m} \frac{Arg(\mu_j)}{2\pi} \\ =& e_S(N) + e_S(M). \end{aligned}$$

4.10. Lemma. *Suppose M is a Seifert manifold with a Möbius structure having trivial monodromy in S^1-fibers, and $-M$ is the same manifold with reversed orientation but the same orientation on S^1-fiber. Then $e_S(-M) = -e_S(M)$.*

Indeed, the principal multipiers of the Möbius structure on the boundary components of $-M$ are the complex conjugates of the principal multipliers of ∂M, and the homology classes of the marking curves of $-M$ are equal to the homology class of $[S^1]$ subtracting the homology classes of the marking curves of M.

5. Proof of the Main theorem

We begin this section by proving that there exists a uniformizable Möbius structure on $N = P \times S^1$ with trivial monodromy in S^1-fibers so that the three principal multipliers of the boundary Möbius tori are negative real numbers arbitrary near $(-ctg^2\pi/12, -ctg^2\pi/12, -ctg^2\pi/6)$. Then by gluing several copies of N with the Möbius structure, we prove the main theorem.

5.1. Suppose a totally degenerate triple (C_1, C_2, C_3) has coordinate (z_1, z_2, z_3). Since the pairwise multipliers of C_i, C_j are given by $(\frac{1+\sqrt{c_k}}{1-\sqrt{c_k}})^2$ where $c_k = z_k/(z_k - 1)$, the multipliers are negative real if and only if $\mathrm{Re}(z_k) = 1/2$.

We now focus on a very special totally degenerate triple (C_1, C_2, C_3) with coordinate $(1/\sqrt{3}e^{\pi i/6},\ 1/\sqrt{3}e^{\pi i/6},\ e^{-\pi i/3})$. Their pairwise principal multipliers are $-ctg^2\pi/12$, $-ctg^2\pi/12$, and $-ctg^2\pi/6$. Suppose (C_1', C_2', C_3') is a triple of horocycles in H^2 corresponding to (C_1, C_2, C_3), i.e., their pairwise weighted distances are 0, $-lg\sqrt{3}$, and $-lg\sqrt{3}$. We take H^2 to be the half-space $\{(x,o,z)|x > 0\} \subset R^3$ and represent C_1', C_2', and C_3' in H^2 as follows:

C_1' is the Euclidean circle of radius $\sqrt{3}/2$ centered at $(\sqrt{3}/2, 0, \sqrt{3}/2)$;

C_2' is the Euclidean circle of radius $\sqrt{3}/2$ centered at $(\sqrt{3}/2, 0, -\sqrt{3}/2)$;

C_3' is the line $\{(x,0,z)|\ x = 1\} \cup \{\infty\}$.

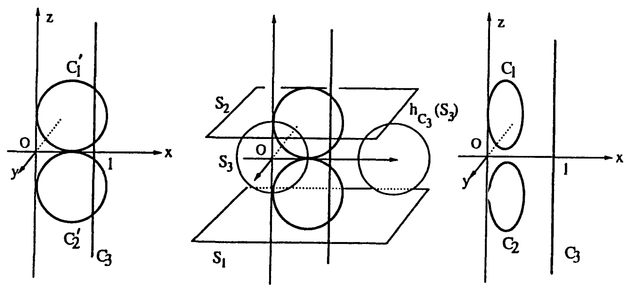

FIGURE 5.1

The z-axis is considered to be the axis C of C_1', C_2' and C_3'. Take C_3 to be C_3', C_1 to be C_1' rotated positively about the z-axis at an angle $\pi/6$, and C_2 to be C_2' rotated negatively about the z-axis at an angle $\pi/6$. The dual spheres S_1, S_2, S_3 (so that $S_i \perp C_j$ and $S_i \perp C_k$) are given by: $S_1 = \{(x,y,z)|z = -\sqrt{3}/2\}$, $S_2 = \{(x,y,z)|z = \sqrt{3}/2\}$, $S_3 = \{(x,y,z)|\sqrt{x^2+y^2+z^2} = \sqrt{3}/2\}$. These dual spheres bound three balls $D_1 = \{(x,y,z)|z \leq -\sqrt{3}/2\}$, $D_2 = \{(x,y,z)|z \geq \sqrt{3}/2\}$, $D_3 = \{(x,y,z)|\sqrt{x^2+y^2+z^2} \leq \sqrt{3}/2\}$ in S^3 where $\partial D_i = S_i$. Furthermore, these four balls D_1, D_2, D_3 and $H_{C_3}(D_3)$ have disjoint interiors. We also have $S_i \cap C_i = \phi$ for i = 1,2,3.

5.2. Lemma. *The three-circle group H_{C_1,C_2,C_3} is a Schottky group.*

Proof. The generator $H_{C_3}H_{C_1}$ ($H_{C_3}H_{C_2}$ respectively) leaves S_2 (S_1 respectively) invariant. C_1 is the circle passing through (0, 0, $\sqrt{3}/2$) and $(3/2, \sqrt{3}/2, \sqrt{3}/2)$ and orthogonal to S_2. Similarly, C_2 is the circle passing through (0,0, $-\sqrt{3}/2$) and $(3/2, -\sqrt{3}/2, -\sqrt{3}/2)$ and orthogonal to S_1.

Let B_ϵ (B'_ϵ respectively) be the 3-ball centered at $((1+\epsilon)/2, (1-\epsilon)\sqrt{3}/2, \sqrt{3}/2)$ ($((1+\epsilon)/2, -(1-\epsilon)\sqrt{3}/2, -\sqrt{3}/2)$ respectively) of radius $\sqrt{1-\epsilon+\epsilon^2}$. One checks easily the following:

(1) $C_1 \subset \partial B_\epsilon$, and $C_2 \subset \partial B'_\epsilon$.

(2) B_0 is tangent to $H_{C_3}H_{C_1}(B_0)$ and $H_{C_3}H_{C_2}(B'_0)$, and B'_0 is tan-

gent to $H_{C_3}H_{C_1}(B_0)$ and $H_{C_3}H_{C_2}(B_0')$, i.e., C_1, C_2, C_3 satisfy the weak Schottky condition with respect to C_3.

(3) For ϵ positive and small, $B_\epsilon \cap (B_\epsilon' \cup H_{C_3}H_{C_2}(S^3 - int(B_\epsilon'))) = \emptyset$, and B_ϵ intersects $H_{C_3}H_{C_1}(S^3 - int(B_\epsilon))$ transversely near (1,0, $\sqrt{3}/2$). Similarly, $B_\epsilon' \cap (B_\epsilon \cup H_{C_3}H_{C_1}(S^3 - int(B_\epsilon))) = \emptyset$, and B_ϵ' intersects $H_{C_3}H_{C_2}(S^3 - int(B_\epsilon'))$ transversely near (1,0, $-\sqrt{3}/2$).

Conditions (1) and (2) imply that the three-circle group H_{C_1,C_2,C_3} is discrete and free by Lemma 3.3.

To produce a Schottky condition for the three-circle group, we now modify these four spheres constructed in (2) above.

Recall that a *convex lens* in S^3 is a topological ball which is the intersection of two 3-balls, and a *concave lens* in S^3 is a topological ball which is the union of two 3-balls (or the same, the complement of the interior of a convex lens).

The goal now is to replace B_ϵ and B_ϵ' by lenses L_ϵ and L_ϵ' so that L_ϵ, L_ϵ', $H_{C_3}H_{C_1}(L_\epsilon)$ and $H_{C_3}H_{C_2}(L_\epsilon')$ satisfy the Schottky condition for the three-circle group.

Choose ϵ positive and very small and let A (A' respectively) be the ball centered at (1,0, $\sqrt{3}/2$) ((1, 0, $-\sqrt{3}/2$) respectively) of radius 2ϵ. Then $B_\epsilon \cap H_{C_3}H_{C_1}(S^3 - int(B_\epsilon))$ is a convex lens inside A. Similarly, $B_\epsilon' \cap H_{C_3}H_{C_2}(S^3 - int(B_\epsilon'))$ is a convex lens inside A'. Furthermore, $H_{C_3}H_{C_1}(A)$ is a 3-ball of radius o(ϵ) containing (2, $-\sqrt{3}$, $\sqrt{3}/2$). Both A and $H_{C_3}H_{C_1}(A)$ do not intersect D_1, D_3 and $H_{C_3}(D_3)$. Similarly, $H_{C_3}H_{C_2}(A')$ is a 3-ball of radius o(ϵ) containing (2, $\sqrt{3}$, $-\sqrt{3}/2$). Both A' and $H_{C_3}H_{C_1}(A')$ do not intersect D_2, D_3, and $H_{C_3}(D_3)$.

We now consider four lenses $L_\epsilon = B_\epsilon$ - int(A), $L_\epsilon' = B_\epsilon'$ - int(A'), $M_\epsilon = H_{C_3}H_{C_1}(S^3 - int(B_\epsilon)) \cup H_{C_3}H_{C_1}(A)$, and $M_\epsilon' = H_{C_3}H_{C_2}(S^3 - int(B_\epsilon')) \cup H_{C_3}H_{C_2}(A')$. For small ϵ these four lenses are disjoint, $H_{C_3}H_{C_1}$ sends L_ϵ to the complement of M_ϵ, and $H_{C_3}H_{C_2}$ sends L_ϵ' to the complement of M_ϵ' as in Figure 5.3.

This verifies the Schottky condition for the three-circle group H_{C_1,C_2,C_3}.

5.3. The condition in Theorem 3.19 that $\sum_{i\neq j\neq k\neq i}(a_i+a_j)(z_k-1)/z_k$ is real for the triple (C_1, C_2, C_3) with coordinate $(1/\sqrt{3}e^{\pi i/6}, 1/\sqrt{3}e^{\pi i/6}, e^{-\pi i/3})$ is $a_3 = 0$ which does not have positive solution in $a_i's$. However, a slight deformation of it, the triple with coordinate

$$(e^{(\pi-\delta)i/6}/(2cos(\pi-\delta)/6),\\ e^{(\pi-\delta)i/6}/(2cos(\pi-\delta)/6), e^{-(\pi-\delta)i/3}/(2cos(\pi-\delta)/3)),$$

where δ is a small positive number, satisfies the condition in Theorem 3.19. Since the Schottky condition is stable under perturbation, the new three-circle group H_{C_1,C_2,C_3} is still Schottky with respect to four lenses.

5.4. By the local deformation Theorem 3.19, we may deform the totally degenerate triple in 5.3 to produce a type I configuration of circles (C_1, C_2, C_3) so that their pairwise (principal) multipliers are negative real numbers arbitrary near $(-ctg^2\pi/12, -ctg^2\pi/12, -ctg^2\pi/6)$.

We claim that if ϵ is chosen sufficiently small, the new three-circle group H_{C_1,C_2,C_3} uniformizes a Möbius structure on $N = P \times S^1$ with a Schottky monodromy group.

First, the group H_{C_1,C_2,C_3} is a Schottky group due to the stability of Schottky condition.

To show that H_{C_1,C_2,C_3} uniformizes a Möbius structure, we will construct a fundamental region in $\bar{\mathbf{R}}^3$ so that H_{C_1,C_2,C_3} identifies some faces of the region with quotient space $P \times S^2$.

We begin by considering the undeformed group H_{C_1,C_2,C_3} constructed in Lemma 5.2. The four lenses L_ϵ, M_ϵ, L'_ϵ, and M'_ϵ interset the four balls D_1, D_2 D_3 and $H_{C_3}(D_3)$ bounded by the dual spheres S_1, S_2, S_3 and $H_{C_3}(S_3)$ in the following pattern:

(1) L_ϵ only intersets D_3 and D_2;
(2) M_ϵ only intersects D_2 and $H_{C_3}(D_3)$;
(3) L'_ϵ only intersects D_3 and D_1;
(4) M'_ϵ only intersects D_1 and $H_{C_3}(D_3)$.

Furthermore, each of the intersection above is a topological 3-ball, and the complement of the union of the interiors of these lenses L_ϵ, L'_ϵ, M_ϵ, M'_ϵ and the four balls D_1, D_2, D_3 and $H_{C_3}(D_4)$ is a topological solid tours.

After the deformation, the above four properties still holds for the type I configuration (C_1, C_2, C_3) since all of these are open conditions. Furthermore, the four balls D_1, D_2, D_3 and $H_{C_3}(D_3)$ are disjoint since the dual spheres for type I configuration satisfy the Schottky condition by Lemma 3.4. Let $\mathcal{S}$ be the solid torus which is the complement of the union of the interiors of these lenses L_ϵ, L'_ϵ, M_ϵ, M'_ϵ and the four balls D_1, D_2, D_3 and $H_{C_3}(D_4)$.

The boundary of $\mathcal{S}$ is a union of eight topological annuli which are the intersections of $\partial\mathcal{S}$ with the four lenses and the four balls D_1, D_2, D_3, $H_{C_3}(D_3)$. These annuli have disjoint interiors. The generator $H_{C_3}H_{C_1}$ of H_{C_1,C_2,C_3} identifies the annulus $\mathcal{S}\cap L_\epsilon$ with $\mathcal{S}\cap M_\epsilon$, and the other generator $H_{C_3}H_{C_2}$ identifies $\mathcal{S}\cap L'_\epsilon$ with $\mathcal{S}\cap M'_\epsilon$. The quotient space is topologically homeomorphic to $P\times S^1$. Furthermore, by Poincaré polyhedron theorem (see [10]), the quotient space has a Möbius structure. Each boundary component of $P \times S^1$ has the induced 2-dimensional Möbius structure corresponding to the quotient of $S_k - Fix(H_{C_i}H_{C_j})$ by the hyperbolic element $H_{C_i}H_{C_j}$.

Thus we have proved

5.5. Theorem. *There exists a uniformizable Möbius structure on $P \times S^1$ with trivial monodromy in S^1-fibers and a discrete free monodromy group so that the principal multipliers of the Möbius tori in the boundary are negative real numbers arbitrary near ($-ctg^2\pi/12$, $-ctg^2\pi/12, -ctg^2\pi/6$).*

5.6. Remark. The special totally degenerate triple used in Lemma 5.2 is found as follows. Consider the set of all triples of totally degenerate three circles with Möbius coordinate $(e^{i\theta}/(2cos\theta)$, $e^{i\phi}/(2cos\phi)$, $e^{-i(\theta+\phi)}/(2cos(\theta+\phi)))$ where $\theta, \phi \in (0, \pi/2)$ and $\theta+\phi < \pi/2$. The triple with coordinate $(1/\sqrt{3}e^{\pi i/6}$, $1/\sqrt{3}e^{\pi i/6}$, $e^{-\pi i/3})$ is the only one for which there exist two 2-spheres S_1 and S_2 containing C_1 and C_2 respectively so that S_1, S_2, $H_{C_3}S_1$, and $H_{C_3}S_2$ bound four 3-balls with disjoint interior (weak Schottky condition).

5.7. We now finish the proof of the main theorem.

Let N be $P \times S^1$ with the Möbius structure constructed above, and let A_1 A_2 and A_3 be the Möbius tori in ∂N. By the definition of Euler number of structure, $e_S(N) = 1/2 + n$ for some integer n. A concrete calculation shows that $|e_S(N)| = 1/2$. Thus, we may assume (by choosing an orientation on N) that $e_S(N) = 1/2$.

Given any integer e satisfying $|e| \leq g - 1$, there are two positive integers p and q so that $p + q = 2g - 2$ and $p - q = 2e$. Take p copies of N and q copies of $-N$. We decompose $W_{e,g}$ into a boundary union of 2g-2 copies of the simple type I manifolds (p of them are N's and q of them are $-N$'s) so that when two such simple manifolds are glued along two boundary components, these two components correspond to the same Möbius tori A_i. See for instance the figure below.

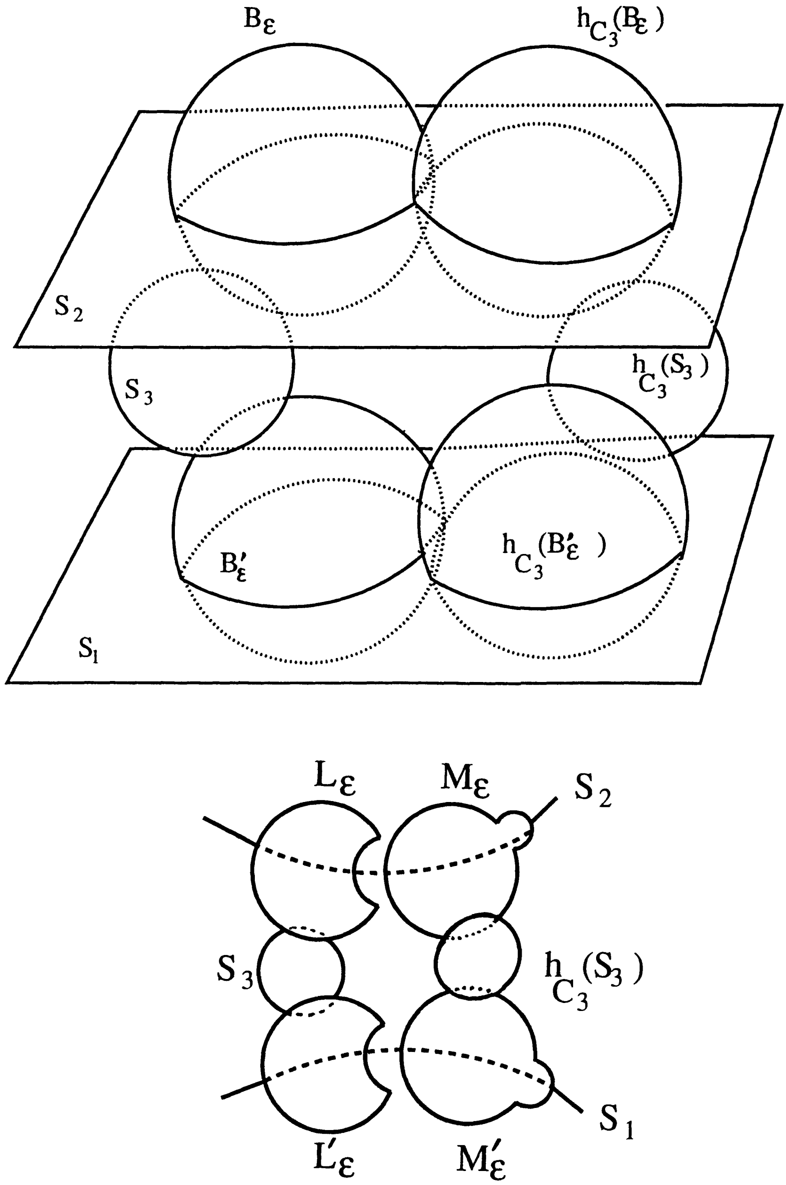
B_ε
$h_{C_3}(B_\varepsilon)$
S_2
S_3
$h_{C_3}(S_3)$
B'_ε
$h_{C_3}(B'_\varepsilon)$
S_1
L_ε
M_ε
S_2
S_3
$h_{C_3}(S_3)$
L'_ε
M'_ε
S_1

FIGURE 5.2

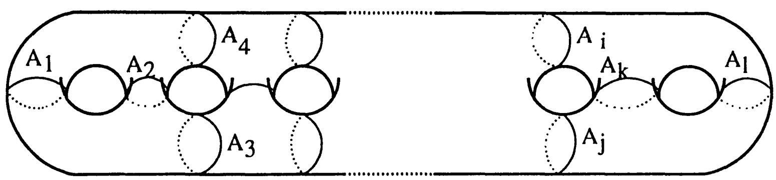

FIGURE 5.3

Now realize the gluing map between the boundary tori by an orientation reversing Möbius transformation (its existence is guaranteed since the multipliers are real) which preserving the S^1-fibers and their orientations. The result is then a Möbius structure on $W_{e,g}$ by Proposition 2.8. Furthermore, by Maskit combination theorem, the monodromy group is discrete and is isomorphic to the surface group $\pi_1(\Sigma_g)$.

Added in proof. We are informed by P. Waterman that he and Kuiper have found some (e.g.) with $|e| > g - 1$ so that $W_{e,g}$ supports complete hyperbolic metrics.

Acknowledgment

I would like to thank Mike Freedman, Jane Gilman, Troel Jϕrgensen and Peter Teichner for helpful discussions. I also thanks the referee for many useful suggestions.

References

[1] Gilman, J., *Kleinian groups of NSDC type*, Preprint.

[2] Goldman, W., *Conformally flat manifolds with nilpotent holonomy*, Trans. Amer. Math. Soc. **278** (1983) 573-583.

[3] Gromov, M., Lawson, B. & Thurston, W., *Hyperbolic 4-manifolds and conformally flat 3-manifolds*, Inst. Hautes Études Sci. Math. Publ. **68** (1988) 27-45.

[4] Kapovich, M., *Conformally flat structures on 3-manifolds*, J. Differential Geom. **38** (1993) 191-215.

[5] _____, *Intersection pairing and Thurston's norm on hyperbolic 4-manifolds.* I, Preprint.

[6] Kuiper, N., *Hyperbolic 4-manifolds and tesselations*, Inst. Hautes Études Sci. Math. Publ. **68** (1987) 47-76 .

[7] Luo, F., *Constructing conformally flat structures on some 3-manifolds*, Math. Ann. **294** (1992) 449-458.

[8] _____, *Möbius structres on Seifert 3-manifolds. II*, in preparation.

[9] _____, *Triangulations in Möbius geometry*, Trans. Amer. Math. Soc. **337** 181-193.

[10] Maskit, B., *Kleinian groups*, Springer, New York, 1987.

[11] Wilker, J. B., *Inversive geometry*, The geometric vein, ed. by C. Davis, et al., Springer, New York, 1981.

RUTGERS UNIVERSITY