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SYMPLECTIC PACKING CONSTRUCTIONS

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1. Introduction

Let V^{2n} be a symplectic manifold. A symplectic k-packing of V via equal balls consists of k symplectic embeddings of a $2n$-dimensional ball with disjoint images in the interior of V. If $\mathrm{Vol}\, V < \infty$, there is an upper bound to the radii of the balls which admit a symplectic k-packing since symplectic embeddings preserve volume. Some natural questions include: *For fixed k, what is the least upper bound for r such that there exists a symplectic packing via k embeddings of a ball of radius r? For which k is there a full packing, i.e., for which k can the volume of the image of the packing get arbitrarily close to the volume of V?*

Using his technique of pseudo-holomorphic curves, Gromov calculated that a packing of the 4-dimensional ball of radius 1, $B^4(1)$, via 2, 3, or 4 symplectic embeddings of a closed ball does not exist if $r \geq \sqrt{1/2}$ and that a packing via 5 or 6 embeddings cannot exist if $r \geq \sqrt{2/5}$, [2 (0.3.B)]. McDuff and Polterovich, in [6], combined the pseudo-holomorphic curve theory with the theory of symplectic blow ups and proved that a packing of $B^4(1)$ does not exist for 7 embeddings when $r \geq \sqrt{3/8}$ nor for 8 embeddings when $r \geq \sqrt{6/17}$. Moreover, they proved that these obstructions are sharp: there exist packings of $B^4(1)$ via $2, 3, 4, 5, 6, 7, 8$ symplectic embeddings of a closed ball of radius arbitrarily close to $\sqrt{1/2}$, $\sqrt{1/2}$, $\sqrt{1/2}$, $\sqrt{2/5}$, $\sqrt{2/5}$, $\sqrt{3/8}$, $\sqrt{6/17}$, respectively. For higher dimensional balls, Gromov calculated that a packing of $B^{2n}(1)$ via $k \leq 2^n$ embeddings cannot exist if $r \geq \sqrt{1/2}$. McDuff and Polterovich proved that for $k \leq 2^n$, there exists a packing

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via k embeddings of a closed ball of radius arbitrarily close to $\sqrt{1/2}$. This is a consequence of their discovery that $B^{2n}(1)$ can be fully packed by k^n embeddings, for all k. In other words, it is possible to find a packing of $B^{2n}(1)$ via k^n embeddings of a closed ball of radius arbitrarily close to $\sqrt{1/k}$, the radius given by the volume obstruction.

McDuff and Polterovich discovered the existence of these packings quite indirectly. In the following, elementary constructions are described for most of these maximal packings. In particular, constructions are given for

- full packings of B^{2n} via k^n closed balls, $\forall k \in \mathbb{Z}^+$ (Section 3);
- the "densest" packings of B^4 via $k \leq 6$ closed balls (Section 3, Section 5).

In dimension 4, the constructions for the above lead to packings via open balls.

Explicit constructions are useful for visualizing the full and maximal packings. In addition, it is often easy to study the packing problem for more general manifolds. For example, a construction is given for packing a product of two surfaces, Theorem 4.1. The following is a sampling of some new results. See Theorems 3.7, 3.9, 6.4, 6.3 for precise statements.

Let $E(r_1, r_2)$ be the standard ellipsoid of radii $\sqrt{r_1}, \sqrt{r_2}$:

$$E(r_1, r_2) := \left\{ \frac{1}{r_1}\left(x_1^2 + y_1^2\right) + \frac{1}{r_2}\left(x_2^2 + y_2^2\right) \leq 1 \right\} \subset \left(\mathbb{R}^4, \omega_0\right).$$

Theorem 1.1. *There exists a full packing of $B^4(1)$ via two embeddings of $E(1/2, 1)$.*

Theorem 1.2. $\operatorname{Int} E(2,1) \cap \{x_1 > 0\}$ *is symplectically equivalent to* $\operatorname{Int} B^4(1)$, *the open ball of radius* 1.

Theorem 1.3. *For all $\varepsilon > 0$, there exists a symplectic embedding*

$$\psi : E(1/(k+1), k) \to \operatorname{Int} B^4(1+\varepsilon), \qquad \forall k \in \mathbb{Z}^+.$$

Theorem 1.4. *There exist full packings of $E(2,1)$ via 2 embeddings of a ball and via 1 embedding of a ball and 1 embedding of an ellipsoid. Moreover, these full packings can be constructed so that symplectic embeddings of a ball from each packing are symplectically isotopic.*

2. Notation and Definitions

The following subsets of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$, will be encountered frequently:

$$
\begin{gathered}
D(r) := \{x^2 + y^2 \leq r\} \subset \mathbb{R}^2, \\
D(r_1, r_2, \ldots, r_n) := D(r_1) \times \cdots \times D(r_n), \qquad (2.1)\\
E(r_1, \ldots, r_n) := \left\{ \frac{1}{r_1}(x_1^2 + y_1^2) + \cdots + \frac{1}{r_n}(x_n^2 + y_n^2) \leq 1 \right\} \subset \mathbb{R}^{2n}, \\
B^{2n}(r) := E(r, \ldots, r).
\end{gathered}
$$

Notice that $B^{2n}(r)$ is the closed $2n$-dimensional ball of radius $\sqrt{r}$ and that

$$
\operatorname{area} D(r) = \pi r, \qquad \operatorname{Vol} E(r_1, \ldots r_n) = \frac{\pi^n r_1 r_2 \ldots r_n}{n!}.
$$

The cotangent bundle of T^n will be viewed as a quotient of $\mathbb{R}^{2n}$ with the induced symplectic form:

$$
T^*T^n = (\{x_1, x_2, \ldots, x_n\} / \sim) \times \{y_1, y_2, \ldots, y_n\},
$$

where, for fixed constants $c_1, \ldots, c_n$,

$$
(x_1, x_2, \ldots, x_n) \sim (x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n).
$$

In the following sections, packings via balls will be generalized to packings via ellipsoids and other symplectic manifolds. Thus it will be important to have the following notion of "shape approximation" which generalizes a sequence of balls of increasing radius.

Definition 2.2 (Shape Approximation). Given (U, ω), suppose that for $0 < r \leq 1$ there are diffeomorphisms φ_r of U into U satisfying

$$
\varphi_r^* \omega = r\omega, \qquad \varphi_1 = \mathrm{id}.
$$

Then define $U(r) = \varphi_r(U)$. Note that $U(r)$ is conformally equivalent to $U = U(1)$ and is defined up to symplectomorphism.

Examples 2.3. There exists such a 1-parameter family of diffeomorphism φ_r for any star convex subset of $\mathbb{R}^{2n}$

(1) For $U = B^{2n}(1) \subset \mathbb{R}^{2n}$, $U(r) = B^{2n}(r)$;

(2) For $U = E(\alpha_1, \ldots, \alpha_n) \subset \mathbb{R}^{2n}$, $U(r) = E(r\alpha_1, \ldots, r\alpha_n)$.

Definition 2.4 (Packings). Let (V, Ω), (U, ω) be symplectic manifolds of the same dimension. A *symplectic k-packing of V via U* is a set of embeddings $\{\psi_i\}_{i=1}^k$ satisfying

$$\psi_i : U \to \operatorname{Int} V, \qquad \psi_i^* \Omega = \omega, \quad i = 1, \dots, k,$$
$$\operatorname{Im} \psi_i \cap \subset \psi_j = \emptyset, \qquad i \neq j.$$

Definition 2.5 (Full Packings). Suppose that U is closed and $\operatorname{Vol} V < \infty$. V is said to have a *full k-packing via U* if $k \operatorname{Vol} U = \operatorname{Vol} V$ and for r arbitrarily close to 1, there exists a symplectic k-packing of V via $U(r)$. For U open, V is said to have a *full k-packing via U* if there exists $\{\psi_i\}_{i=1}^k$, a k-packing of V via U, so that $\operatorname{Int} V \subset \overline{\cup \operatorname{Im} \psi_i}$.

3. Full Packings of the Ball

McDuff and Polterovich discovered a beautiful pattern for the existence of full packings of the ball.

(3.1) Theorem (McDuff-Polterovich) [6 (1.5.C)]. *For every positive integer k, there is a full k^n-packing of B^{2n} via closed balls.*

McDuff and Polterovich discovered the existence of these full packings indirectly via an algebraic-geometric argument. Later, they discovered an explicit construction along the lines of the construction given in Section 5. The following, more elementary, alternate construction is based on the fact that $B^{2n}(r)$ is a subset of $D(r, \dots, r)$.

Construction 3.2. *Full Packings of $B^{2n}(1)$ via k^n Closed Balls.* Fix $k \in \mathbb{Z}^+$. To construct a full packing of $\operatorname{Int} B^{2n}(1)$ with k^n copies of $B^{2n}(r)$, for r arbitrarily close to $1/k$, first $D(1, \dots, 1)$ is packed with k^n copies of $D(r, \dots, r)$, $r < 1/k$. To construct this packing of the polydisc with polydiscs, begin by dividing $D(1)$ into k "pie shaped regions", $P_i(1/k)$, each of area π/k. More precisely, in polar coordinates,

$$P_i(1/k) := \left\{(\rho, \theta) \in D(1) : (i-1)\frac{2\pi}{k} \leq \theta < i\frac{2\pi}{k}\right\}, \quad i = 1, \dots k.$$

Given $r < \frac{1}{k}$, choose area preserving diffeomorphisms

$$\sigma_i : D(r) \to P_i(1/k), \qquad i = 1, \dots, k,$$

so that if $|z|^2 := x^2 + y^2 \leq \alpha$ then $|\sigma_i(z)|^2 \leq k\alpha + (\frac{1}{k} - r)$. There are k^n distinct ordered index sets $\{\{i_1, \dots, i_n\} : i_j \in \{1, \dots, k\}\}$. For each index set $I = \{i_1, \ldots, i_n\}$, define

$$\begin{aligned}\Psi_I^r = \Psi_{\{i_1,\ldots,i_n\}}^r &: D(r, \ldots, r) \\ &\to P_{i_1}(1/k) \times P_{i_2}(1/k) \times \cdots \times P_{i_n}(1/k) \subset D(1, \ldots, 1)\end{aligned}$$

by

$$\Psi_I^r(x_1, y_1, \ldots, x_n, y_n) = (\sigma_{i_1}(x_1, y_1), \ldots, \sigma_{i_n}(x_n, y_n)).$$

Since the images of σ_i and σ_j are disjoint when $i \neq j$, these embeddings are disjoint for distinct index sets. Thus it only remains to prove that the restriction of these embeddings to the $2n$-dimensional balls define embeddings into $\operatorname{Int} B^{2n}(1)$.

Lemma 3.2.1. $\Psi_I^r(B^{2n}(r)) \subset \operatorname{Int} B^{2n}(1)$.

Proof. Suppose $(x_1, y_1, \ldots, x_n, y_n) \in B^{2n}(r)$, $\alpha_i := x_i^2 + y_i^2$. By construction, it follows that

$$|\sigma_{i_j}(x_j, y_j)|^2 \leq k\alpha_j + \left(\frac{1}{k} - r\right),$$

and thus, since $\alpha_1 + \cdots + \alpha_n \leq r < 1/k$,

$$\begin{aligned}\textstyle\sum_{j=1}^n |\sigma_{i_j}(x_j, y_j)|^2 &\leq k(\alpha_1 + \cdots + \alpha_n) + \tfrac{n}{k} - nr \\ &\leq (k-n)r + \tfrac{n}{k} < (k-n)\tfrac{1}{k} + \tfrac{n}{k} = 1.\end{aligned}$$

This completes Construction (3.2).

Remark 3.3. As mentioned in the introduction, Gromov proved that there does not exist a symplectic packing of $B^{2n}(1)$ with $k \leq 2^n$ closed balls of radius $r \geq \sqrt{1/2}$. Notice that the full packing of $B^{2n}(1)$ with 2^n balls is done via closed balls of radius arbitrarily close to $\sqrt{1/2}$. Thus any $k \leq 2^n$ of these embeddings gives the maximal packing via k balls. See Yael Karshon's Appendix, [3], for an alternate construction of the maximal packing of B^{2n} via $k \leq n+1$ embeddings. ◇

In particular, the above construction gives full k^2-packings of $B^4(1)$ via closed balls, for all $k \in \mathbb{Z}^+$. In fact, the above construction can be extended to prove

Theorem 3.4. *There exists a full k^2-packing of $B^4(1)$ via open balls, for all $k \in \mathbb{Z}^+$.*

One way to construct a symplectic embedding of $\operatorname{Int} B^{2n}(r)$ is to choose a sequence $\{r_j\}$, $r_j < r_{j+1}$, $r = \sup r_j$, and construct ψ^j, a sequence of symplectic embeddings of $B^{2n}(r_j)$, so that

$$\psi^{j+1}|_{B^{2n}(r_j)} = \psi^j.$$

McDuff proved that all symplectic embeddings of a closed 4-dimensional ball are symplectically isotopic inside $\operatorname{Int} B^4(1)$, [5]. Thus the following criterion can be used to construct open packings.

(3.5) (McDuff, [5 (Corollary 1.2)]). *If there exists a sequence of symplectic embeddings*

$$\psi^j : B^4(r_j) \to V, \quad \operatorname{Im} \psi^j \subset \operatorname{Im} \psi^{j+1} \quad \forall j, \quad \sup r_j = r,$$

then $\operatorname{Int} B^4(r)$ *symplectically embeds into* V .

Thus Theorem 3.4 follows immediately from the following lemma.

Lemma 3.6. *Fix* $k \in \mathbb{Z}^+$ *and let* $P_1(1/k), \dots, P_k(1/k)$ *be as in (3.2). Given a sequence* $\{r_j\}$, $r_j < r_{j+1}$, $\sup r_j = 1/k$, *there exists a sequence of symplectic embeddings*

$$\Psi^j_{i_1,i_2} : B^4(r_j) \to \operatorname{Int} B^4(1) \cap (P_{i_1} \times P_{i_2})$$

such that $\operatorname{Im} \Psi^j_{i_1,i_2} \subset \operatorname{Im} \Psi^{j+1}_{i_1,i_2}$, *for all* j.

Proof. Let $P_i(1/k) \subset D(1)$, $i = 1, \dots, k$, be as in (3.2). Given $r_1 < r_2 < \cdots < 1/k$, there exists a sequence of area preserving diffeomorphisms

$$\sigma^j_i : D(r_j) \to P_i(1/k), \quad i = 1, \dots, k,$$

so that

(1) $|z|^2 \le \alpha \implies |\sigma^j_i(z)|^2 \le k\alpha + (1/k - r_j)$;

(2) $\operatorname{Im} \sigma^j_i \subset \operatorname{Im} \sigma^{j+1}_i \subset P_i(1/k)$;

(3) $\sigma^{j+1}_i(w) = \sigma^j_i(z) \implies |w|^2 \le |z|^2 + \frac{r_{j+1} - r_j}{2}$.

Construct

$$\Psi^j_I : D(r_j, r_j) \to D(1, 1)$$

as in (3.2). As before, condition (1) guarantees that Ψ^j_I maps $B^4(r_j)$ into $\operatorname{Int} B^4(1)$. Condition (2) guarantees that the images of the polydiscs are nested. (3) then guarantees that the images of the balls are nested. To see this, suppose $p \in \Psi^j_{i_1,i_2}(B^4(r_j))$. Then

$$p = \left(\sigma^j_{i_1}(z_1), \sigma^j_{i_2}(z_2)\right), \quad |z_1|^2 + |z_2|^2 \le r_j.$$

By (2) and (3), $p = (\sigma^{j+1}_{i_1}(w_1), \sigma^{j+1}_{i_2}(w_2))$ where

$$|w_1|^2 + |w_2|^2 \le |z_1|^2 + \frac{1}{2}(r_{j+1} - r_j) + |z_2|^2 + \frac{1}{2}(r_{j+1} - r_j) \le r_{j+1},$$

and thus $p \in \Psi_I^{j+1}(B(r_{j+1}))$. q.e.d.

Although there are full packings of B^{2n} with k^n balls, it is known that there are obstructions for fully packing B^{2n} with k equal balls, for some k. However, for a fixed k, it is always possible to choose ellipsoids which give a full k-packing of B^{2n}.

Theorem 3.7. *There exists a full K-packing of $B^{2n}(1)$ via $E(1/k_1, 1/k_2, \ldots, 1/k_n)$ where $k_i \subset \mathbb{Z}^+$, $K = k_1 k_2 \ldots k_n$.*

Proof. To pack $B^{2n}(1)$ with $E(1/k_1, \ldots, 1/k_n)$, divide $D(1) \subset (x_i, y_i)$-plane into k_i "pie shaped pieces" and proceed as in (3.2). q.e.d.

A nice aspect of the construction in (3.2) is that it is easy to visualize the images of the embeddings. In particular, this leads to information about "chopped" ellipsoids.

Theorem 3.8. *For all n, $B^{2n}(1) \cap \{x_1 > 0\}$ and $B^{2n}(1) \cap \{x_1 > 0\} \cap \{y_1 > 0\}$ can be fully packed by $E(1/2, 1, \ldots, 1)$ and $E(1/4, 1, \ldots, 1)$, respectively.*

Similarly both $E(2,1) \cap \{x_1 > 0\}$ and $B^4(2) \cap \{x_1 > 0\} \cap \{x_2 > 0\}$ can be fully packed via a ball. Applying the open packing construction of (3.6) leads to the following result. (See also Proposition 5.2.)

Theorem 3.9. *Both*

$$\operatorname{Int} E(2,1) \cap \{x_1 > 0\} \quad \textit{and} \quad \operatorname{Int} B^4(2) \cap \{x_1 > 0\} \cap \{x_2 > 0\}$$

are symplectically equivalent to $\operatorname{Int} B^4(1)$.

4. Full Packings of Products of Surfaces

Theorem 4.1. *Let Σ_1, Σ_2 be oriented surfaces of equal area with respect to the area form σ_j, $j = 1, 2$. Then $(\Sigma_1 \times \Sigma_2, \sigma_1 \oplus \sigma_2)$ can be fully packed via $2k^2$ closed or open balls, for all $k \in \mathbb{Z}^+$.*

McDuff and Polterovich's result that a 4-dimensional polydisc can be fully packed by $2k^2$ closed balls, [6], implies the above statement for closed balls. Theorem 4.1 will be proved by an explicit construction as opposed to the technique in [6]. In fact, McDuff and Polterovich proved that a $2n$-dimensional polydisc can be fully packed via $n!k^n$ balls, for all $k \in \mathbb{Z}^+$. A direct construction for this, and thus for the n-fold product of oriented surfaces of equal area, has been found by Boris Krouglikov, [4]. Krouglikov's construction is similar to the techniques found in Section 5. He has also found a beautiful construction that gives an alternate proof of McDuff and Polterovich's discovery that $B^{2m_1}(k_1)\times\cdots\times B^{2m_r}(k_r)$ can be fully packed via $\frac{(m_1+\cdots+m_r)!}{m_1!\ldots m_r!}k_1^{m_1}\ldots k_r^{m_r}$ embeddings of $B^{2m}(1)$, $m := \Sigma m_i$, $k_i \in \mathbb{Z}^+$.

Since each 4-dimensional ball can be fully packed via k^2 balls, it suffices to prove that $\Sigma_1 \times \Sigma_2$ can be fully packed via 2 balls. The idea of the construction is to find symplectic embeddings of 2 polydiscs into $\Sigma_1 \times \Sigma_2$ so that although the images of the polydiscs are not disjoint, the images of the embeddings restricted to the balls will be disjoint.

Proof. Without loss of generality, assume Σ_1, Σ_2 have area π with respect to σ_1, σ_2. Fix $0 < \epsilon < 1$ and for $i, j = 1, 2$, choose area preserving embeddings

$$\tau_j^i : D(1-\epsilon) \to \Sigma_j$$

so that

$$r_1 + r_2 = 1 - \epsilon \implies \tau_j^1\left(D\left(r_1\right)\right) \cap \tau_j^2\left(D\left(r_2\right)\right) = \emptyset.$$

Then consider the symplectic embeddings $\Psi_1, \Psi_2 : B^4(1-\epsilon) \to \Sigma_1 \times \Sigma_2$,

$$\Psi_1(x_1, y_1, x_2, y_2) = \left(\tau_1^1(x_1, y_1), \tau_2^2(x_2, y_2)\right),$$
$$\Psi_2(x_1, y_1, x_2, y_2) = \left(\tau_1^2(x_1, y_1), \tau_2^1(x_2, y_2)\right).$$

To show this is a full packing, it suffices to prove that these embeddings are disjoint. Suppose there exist (x_1, y_1, x_2, y_2), $(u_1, v_1, u_2, v_2) \in B^4(1-\epsilon)$ such that

$$\tau_1^1(x_1, y_1) = \tau_1^2(u_1, v_1), \quad \tau_2^2(x_2, y_2) = \tau_2^1(u_2, v_2).$$

If $x_1^2 + y_1^2 = r_1$, then by construction of τ_1^1, τ_1^2, $u_1^2 + v_1^2 > (1-\epsilon) - r_1$. Since (x_1, y_1, x_2, y_2), $(u_1, v_1, u_2, v_2) \in B^4(1-\epsilon)$, it then follows that

$$x_2^2 + y_2^2 \leq 1 - \epsilon - r_1, \quad u_2^2 + v_2^2 \leq r_1,$$

and thus, by construction, $\tau_2^2(x_2, y_2) \neq \tau_2^1(u_2, v_2)$. This completes the construction for the closed ball statement of Theorem 4.1. The open ball statement follows from arguments as in the proof of (3.6).

5. Maximal Packings of the 4-Ball with 5 or 6 Balls

Next, cases where there are not full packings are examined. In [2], Gromov proved that there does not exist a symplectic packing of

$B^4(1)$ with 5 or 6 closed balls of radius r when $r \geq \sqrt{\frac{2}{5}}$. By studying the existence of Kähler structures on $\mathbb{CP}^2$ with 5 and 6 points blown up, McDuff and Polterovich proved that there exists a packing with 5 and 6 closed balls of radius arbitrarily close to $\sqrt{\frac{2}{5}}$, [6]. Below are explicit constructions for these maximal packings via closed balls. A limit process produces a packing of $B^4(1)$ with 5 or 6 open balls of radius $\sqrt{\frac{2}{5}}$. This construction was inspired by McDuff and Polterovich's construction of full packings of $B^{2n}(1)$ with k^n balls, [7].

Definition 5.1. Let

$$\Box(\pi) := \{0 < x_1, x_2 < \pi\},$$
$$\Delta(r) := \{0 < y_1, y_2 : \ y_1 + y_2 < r\}.$$

In other words, $\Box(s\pi)$ is an open lagrangian square in the (x_1, x_2)-plane and $\Delta(r)$ is the open lagrangian triangle in the (y_1, y_2)-plane with vertices $(0,0)$, $(r,0)$, $(0,r)$. Notice that the volume of $\Box(\pi) \times \Delta(r)$ is the same as the volume of $B(r)$, the ball of radius $\sqrt{r}$. Similarly, let

$$\Box(s_1\pi, s_2\pi) := \{0 < x_1 < s_1\pi, \quad 0 < x_2 < s_2\pi\},$$
$$\Delta(\alpha, \beta) := \left\{0 < y_1, y_2 \ : \ y_2 < -\frac{\beta}{\alpha} y_1 + \beta\right\}.$$

Portions of the following lemma are due to McDuff and Polterovich.

Proposition 5.2. *$\Box(\pi) \times \Delta(r)$ is symplectically equivalent to* $\operatorname{Int} B^4(r)$.

Proof. There is a symplectic embedding $\psi : \Box(\pi) \times \Delta(r) \to \operatorname{Int} B^4(r)$ given by

$$\psi(x_1, y_1, x_2, y_2) = \left(\sqrt{y_1}\cos(2x_1), -\sqrt{y_1}\sin(2x_1), \sqrt{y_2}\cos(2x_2), -\sqrt{y_2}\sin(2x_2)\right).$$

Next, using an argument similar to (3.2), it is shown that $\psi(\Box(\pi) \times \Delta(r))$, and thus $\Box(\pi) \times \Delta(r)$, can be fully packed with closed balls of radius arbitrarily close to $\sqrt{r}$.

Let $SD(r) \subset D(r)$ be the "slit disc":

$$SD(r) = D(r) - \{x \geq 0, \ y = 0\}.$$

$SD(r,r)$ will denote the corresponding slit polydisc. Notice that $\psi(\Box(\pi) \times \Delta(r)) = \operatorname{Int} B^4(r) \cap SD(r,r)$. Given $\rho < r$, choose an area preserving

diffeomorphism
$$\sigma^\rho : D(\rho) \to SD(r)$$
so that if $x^2+y^2 \le \alpha$ then $|\sigma^\rho(x,y)|^2 \le \alpha+(r-\rho)$. Thus $\Psi^\rho : D(\rho,\rho) \to SD(r,r)$ defined by

$$\Psi^\rho(x_1,y_1,x_2,y_2) = (\sigma^\rho(x_1,y_1), \sigma^\rho(x_2,y_2))$$

is symplectic. Furthermore it is easy to check that $\Psi^\rho(B^4(\rho)) \subset \operatorname{Int} B^4(r) \cap SD(r,r)$.

Using an argument as in (3.6), for $\rho_1 < \rho_2 < \cdots < r$, there exists σ^{ρ_j} so that the associated Ψ^{ρ_j} satisfy $\operatorname{Im}\Psi^{\rho_j} \subset \operatorname{Im}\Psi^{\rho_{j+1}}$ and $\bigcup_j \operatorname{Im}\Psi^{\rho_j}(B^4(\rho_j)) = \operatorname{Int} B^4(r) \cap SD(r,r)$. Thus it follows from (3.5) that $\square(\pi)\times\triangle(r)$ is symplectically equivalent to $\operatorname{Int} B^4(r)$.

Corollary 5.3. *$\square(s_1\pi, s_2\pi)\times\triangle(r/s_1, r/s_2)$ is symplectically equivalent to $\square(\pi)\times\triangle(r)$ and thus to* $\operatorname{Int} B^4(r)$. *$\square(\pi)\times\triangle(\alpha,\beta)$ can be fully packed by $E(\alpha,\beta)$.*

Remark 5.4. Higher dimensional generalizations exist. For example, the product of a 3-dimensional lagrangian cube and a 3-dimensional lagrangian tetrahedron can be fully packed by a 6-dimensional ball. However, since it is not yet known if all embeddings of a 6-dimensional ball are symplectically isotopic inside $\operatorname{Int} B^6(1)$, the proof of (5.2) does not imply this higher dimensional lagrangian product is symplectomorphic to $\operatorname{Int} B^6(1)$. $\diamond$

To find the maximal packing with 5 or 6 balls, $\square(\pi)\times\triangle(\frac{2}{5})$ will be used as the domains. However instead of using $\square(\pi)\times\triangle(1)$ as the range, a region which is compactified in the (x_1,x_2)-coordinates will be used. More precisely, consider the compactification of the square to the lagrangian torus T^2,

$$T^2 := (\{0 \le x_1, x_2 \le \pi\}/\sim) = \{x_1,x_2\}/(\pi\mathbb{Z}\times\pi\mathbb{Z}).$$

Then
$$\square(\pi)\times\triangle(r) \subset T^2\times\triangle(r) \subset T^*T^2.$$

Proposition 5.5. *There exists a symplectic embedding of $T^2\times\triangle(r)$ into* $\operatorname{Int} B^4(r)$.

Proof. The map ψ of $\square(\pi)\times\triangle(r)$ into $\operatorname{Int} B(r)$ constructed in Proposition 5.2 extends to a symplectic embedding of $T^2\times\triangle(r)$ into $\operatorname{Int} B(r)$. Notice that $\operatorname{Im}\psi$ does not contain any points of the form

$(0,0,x_2,y_2)$ or $(x_1,y_1,0,0)$. Thus the problem of packing $B^4(1)$ with copies of $B^4(r)$ is reduced to the problem of packing $T^2 \times \Delta(1)$ with copies of $\square(\pi) \times \Delta(r)$.

Definition 5.6. $(\psi_{M,\tau})$. For

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \quad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathbb{R}^2$$

let $\psi_{M,\tau} : \square(\pi) \times \Delta(r) \to T^*T^2$ denote the symplectic embedding defined by

$$\psi_{M,\tau} := (dx_1 - cx_2,\ ay_1 + by_2 + \tau_1,\ -bx_1 + ax_2,\ cy_1 + dy_2 + \tau_2).$$

Construction 5.7. *Maximal Packings of $B^4(1)$ via 5 embeddings.* Combining Proposition 5.2 and Proposition 5.5, it follows that to pack $B^4(1)$ with 5 balls, it suffices to pack $\Delta(1)$ with 5 disjoint images of $\Delta\left(\frac{2}{5}\right)$ under the group generated by elements of $\mathrm{SL}(2,\mathbb{Z})$ and translations. For example, one maximal packing is given by $\psi_{M_1,\tau_1},\ldots,\psi_{M_5,\tau_5}$ for

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix};$$
$$M_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix}; \qquad M_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix};$$
$$M_5 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_5 = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix}.$$

See Figure 1. One can also construct a maximal packing using the following translations and elements of $\mathrm{SL}(2,\mathbb{Z})$:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 3/5 \\ 0 \end{pmatrix};$$
$$M_3 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 \\ 4/5 \end{pmatrix}; \quad M_4 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix};$$
$$M_5 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}.$$

See Figure 2.

The above packing of B^4 with 5 balls is constructed with elements of $\mathrm{SL}(2,\mathbb{Z})$, and thus the symplectic embeddings of $\square(\pi) \times \Delta(r)$ are restrictions of symplectic embeddings of $T^2 \times \Delta(r)$. In fact, it suffices to work with elements of $\mathrm{SL}(2,\mathbb{R})$, which define an embedding of $\square(\pi)$ into T^2.

Definition 5.8 ($ISL(\square(\pi))$). In the symplectic map $\psi_{M,\tau}$ defined in (5.6), the matrix M acts on the (y_1, y_2)-coordinates and M^*, the inverse of the transpose of M, acts on the (x_1, x_2)-coordinates. Let $\mathrm{pr} : \mathbb{R}^2 \to T^2$ be the quotient map and define $ISL(\square(\pi)) \subset \mathrm{SL}(2,\mathbb{R})$ by

$$ISL(\square(\pi)) := \{M \in \mathrm{SL}(2,\mathbb{R}) : \mathrm{pr} \circ M^* : \square(\pi) \to T^2 \text{ is injective}\}.$$

Lemma 5.9. *Given $M \in ISL(\square(\pi))$, $\tau \in \mathbb{R}^2$, the symplectic map $\psi_{M,\tau}$ from (5.6) defines a symplectic embedding of $\square(\pi) \times \Delta(r)$ into T^*T^2, for all r.*

Clearly $\mathrm{SL}(2,\mathbb{Z}) \subset ISL(\square(\pi))$. In addition, we have

Lemma 5.10. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ *is in* $ISL(\square(\pi))$ *if*

$$\{|d-b|, |a-c|\} = \{0, 1\}.$$

Proof. It is necessary to prove that given M, when (x_1, x_2), $(x_1', x_2') \in \square(\pi)$, $(x_1, x_2) \neq (x_1', x_2')$, there are no solutions to the system of equations

$$\begin{aligned} d(x_1 - x_1') - c(x_2 - x_2') &= \ell\pi, \\ -b(x_1 - x_1') + a(x_2 - x_2') &= j\pi, \end{aligned}$$

for $j, \ell \in \mathbb{Z}$.

Suppose $|a-c| = 1$, $|d-b| = 0$ and there is a pair of points (x_1, x_2), $(x_1', x_2') \in S\,\square\,(\pi)$ which solves the above system of equations. It then follows that

$$|(\ell+j)\pi| = |(d-b)(x_1-x_1') + (a-c)(x_2-x_2')| < |d-b|\pi + |a-c|\pi = \pi.$$

Thus it is clear that $\ell = -j$. It is easy to check that since M has a trivial kernel, the only solution to the above system when $j = 0$ is

$x_1 = x_1'$, $x_2 = x_2'$. So we can assume $|j| \geq 1$. Then using the fact that $ad - bc = 1$, the system

$$d(x_1 - x_1') - c(x_2 - x_2') = -j\pi$$
$$-b(x_1 - x_1') + a(x_2 - x_2') = j\pi$$

implies

$$|x_1 - x_1'| = |j(c-a)|\pi = |j|\pi \geq \pi$$

However since $(x_1, x_2), (x_1', x_2') \in \square(\pi)$, this is impossible.

A similar argument proves there are no solutions when $|a - c| = 0, |d - b| = 1$.

Example 5.11. Given any $\alpha \in \mathbb{R}$, the following matrices will be in $ISL(\square(\pi))$:

$$\begin{pmatrix} \alpha & 1 \\ -1+\alpha & 1 \end{pmatrix}; \quad \begin{pmatrix} -1+\alpha & -1 \\ \alpha & -1 \end{pmatrix}.$$

Construction 5.12. *Maximal packing of* $B^4(1)$ *via* 6 *embeddings* It is possible to construct a maximal packing of $B^4(1)$ by 6 balls using the following elements of $ISL(\square(\pi))$:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 3/5 \\ 0 \end{pmatrix};$$
$$M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix}; \quad M_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 3/5 \\ 0 \end{pmatrix};$$
$$M_5 = \begin{pmatrix} 1/2 & 1 \\ -1/2 & 1 \end{pmatrix}, \tau_5 = \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}; \quad M_6 = \begin{pmatrix} -1/2 & -1 \\ 1/2 & -1 \end{pmatrix}, \tau_6 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}.$$

See Figure 3. Any 5 of these embeddings give a maximal packing via 5 balls.

Remark 5.13. It is possible to construct a full k^2-packing of $B^4(1)$ via embeddings of the type ψ_{M_i,τ_i} where $M_i \subset \mathrm{SL}(2,\mathbb{Z})$ are either inclusion or rotation matrices. A full packing of $B^4(1)$ can also be constructed by packing $\square(\pi)$ with k^2 copies of $\square(\pi/k)$. See Figure 4 (a), (b) which illustrate full packings via 4 embeddings using these two methods. The same ideas can be applied to construct the full k^n-packings of B^{2n}.

6. Symplectic Isotopies and Folds

The construction from Section 5 will be extended and used to prove that there are multiple pairs of objects which can fully pack, say, an ellipsoid. In addition, it will be shown that the open ball can be asymptotically fully packed with "skinny ellipsoids".

The extension of the constructions in Section 5 are based on the following simple fact.

Lemma 6.1. $\mathrm{SL}(2,\mathbb{Z})$ *acts on the set* $ISL(\square(\pi))$ *by left multiplication: If* $M \in ISL(\square(\pi))$, $Z \in \mathrm{SL}(2,\mathbb{Z})$, *then* $Z \circ M \in ISL(\square(\pi))$.

Proof. Since $(Z \circ M)^* = Z^* \circ M^*$ it follows that $\mathrm{pr} \circ (Z \circ M)^* : \square(\pi) \to T^2$ is injective. Proposition 6.2 implies that if the image of $\triangle(r)$ under $\mathrm{SL}(2,\mathbb{Z})$ is changed by shearing one vertex along the line parallel to the opposite side, the resulting triangle can still "represent" a ball.

Proposition 6.2. *Let* $z_1, z_2 \in \mathbb{Z}$ *and suppose there exist* b, d *such that*

$$\begin{pmatrix} z_1 & b \\ z_2 & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

If $z_1 \neq 0$, *then for all* $x \in \mathbb{R}$,

$$\begin{pmatrix} z_1 & x \\ z_2 & \frac{z_2}{z_1}x + \frac{1}{z_1} \end{pmatrix}, \begin{pmatrix} x & z_1 \\ \frac{z_2}{z_1}x - \frac{1}{z_1} & z_2 \end{pmatrix} \in ISL(\square(\pi)).$$

If $z_1 = 0$, *then for all* $x \in \mathbb{R}$,

$$\begin{pmatrix} 0 & b \\ z_2 & x \end{pmatrix}, \begin{pmatrix} -b & 0 \\ x & z_2 \end{pmatrix} \in ISL(\square(\pi)).$$

Proof. By the above lemma, it suffices to find $Z \in \mathrm{SL}(2,\mathbb{Z})$ and $M \in ISL(\square(\pi))$ so that $Z \circ M$ equals each of the mentioned matrices. Note,

$$\begin{pmatrix} -b + z_1 & b \\ -d + z_2 & d \end{pmatrix} \begin{pmatrix} 1 & \frac{-b+x}{z_1} \\ 1 & \frac{x-b+z_1}{z_1} \end{pmatrix} = \begin{pmatrix} z_1 & x \\ z_2 & \frac{z_2}{z_1}x + \frac{1}{z_1} \end{pmatrix};$$

$$\begin{pmatrix} -b & b \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{x-z_2}{z_2} \\ 1 & \frac{x}{z_2} \end{pmatrix} = \begin{pmatrix} 0 & b \\ z_2 & x \end{pmatrix}.$$

It is easy to check that the other two types of matrices are also in $ISL(\square(\pi))$. q.e.d.

The above calculations are useful in visualizing many isotopies of a ball or ellipsoid. In addition, it becomes apparent how different pairs of objects can fully pack a given symplectic manifold.

Theorem 6.3. Int $E(2,1)$ *has full packings by*

(1) *two ellipsoids,* $E(2,1/2)$*;*

(2) *two balls,* $B^4(1)$*;*

(3) *one ball and one ellipsoid,* $B(1)$ *and* $E(2,1/2)$*.*

Moreover, the full packings for (2) and (3) can be constructed so that balls from each packing are symplectically isotopic.

Proof. From Corollary 5.3, $\square(\pi) \times \Delta(\alpha,\beta)$ can be fully packed with $E(\alpha,\beta)$. Thus, since $T^2 \times \Delta(2,1)$ symplectically embeds into Int $E(2,1)$, it suffices to find $M_i \in ISL(\square(\pi))$, $\tau_i \in \mathbb{R}^2$ so that

$$\psi_{M_i,\tau_i} : \square(\pi) \times \Delta(\alpha_i,\beta_i) \to T^2 \times \Delta(2,1), \quad i = 1,2$$

are packings, where α_i, β_i are chosen appropriately for (1) – (3). See Figure 5.

(1) corresponds to $\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = 1/2$. Consider

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

(2) corresponds to $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$. Consider

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad M_2 = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

(3) corresponds to $\alpha_1 = 2$, $\beta_1 = 1/2$, $\alpha_2 = 1$, $\beta_2 = 1$. Consider

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad M_2 = \begin{pmatrix} -2 & -2 \\ 1 & \frac{1}{2} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

To construct an isotopy between balls in the packings (2) and (3), first an isotopy in $ISL(\square(\pi))$ is constructed between

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 & -2 \\ 1 & \frac{1}{2} \end{pmatrix}.$$

Consider

$$M_t = \begin{pmatrix} -2 & -1-t \\ 1 & \frac{t}{2} \end{pmatrix}, \quad t \in [0,1].$$

Then for $\tau_t = \tau = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\psi_t := \psi_{M_t,\tau_t} : \square(\pi) \times \Delta(1) \subset T^2 \times \Delta(2,1)$. Let B be the image of a symplectic embedding of $B^4(1-\epsilon)$ into $\square(\pi) \times \Delta(1)$. $\psi_t(B)$ is the desired isotopy. q.e.d.

With the above construction, it is possible to see the "symplectic folding" phenomena described in [1 (1.2)].

Theorem 6.4. *For all $\varepsilon > 0$, there exists a symplectic embedding*

$$\psi : E\left((1-\varepsilon)k, (1-\varepsilon)\frac{1}{k+1}\right) \to \operatorname{Int} B^4(1), \quad \forall k \in \mathbb{Z}^+.$$

Proof. It is easy to check that it suffices to find a symplectic embedding

$$\psi_{M,\tau} : \square(k\pi, (1/k)\pi) \times \Delta(1, k/(k+1)) \to T^2 \times \Delta(1).$$

Consider

$$M = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \implies M^* = \begin{pmatrix} 1 & 0 \\ -1/k & 1 \end{pmatrix}.$$

Then M^* maps $\square(k\pi, (1/k)\pi)$ injectively into T^2. Moreover, since $M(\Delta(1, k/(k+1)))$ is the triangle spanned by $(0,0)$, $(1,0)$, and $(1/(k+1), k/(k+1))$, it follows that $\operatorname{Im} \psi_{M,\tau} \subset T^2 \times \Delta(1)$.

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References

[1] A. Floer, H. Hofer & K. Wysocki, *Applications of symplectic homology I*, Math. Z. **217** (1994) 577-606.

[2] M. Gromov, *Pseudo-holomorphic curves on almost complex manifolds*, Invent. Math. **82** (1985) 307-347.

[3] Y. Karshon, *Appendix to* [6], Invent. Math. **115** (1994) 425 - 429.

[4] B. Krouglikov, *Note on symplectic packing constructions*, preprint 1994.

[5] D. McDuff, *Blow ups and symplectic embeddings in dimension 4*, Topology **30** (1991) 409-421.

[6] D. McDuff & L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994) 405 - 425.

[7] _____, private communication.

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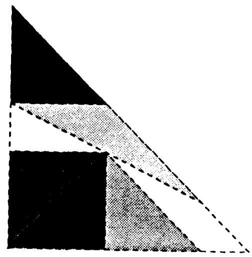


Figure 1

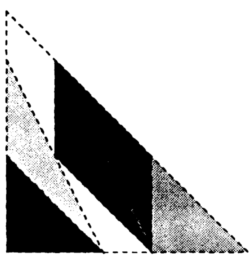


Figure 2

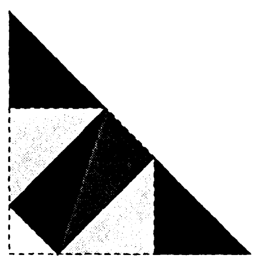
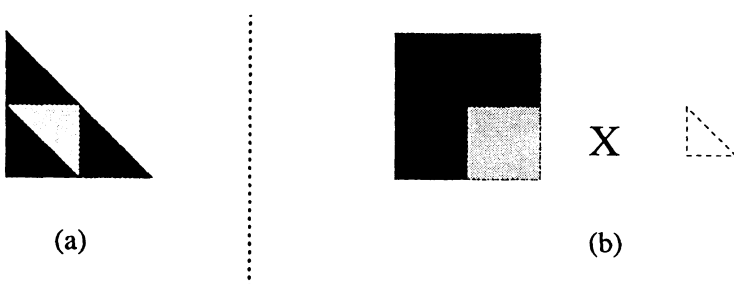


Figure 3



(a)

(b)

Figure 4

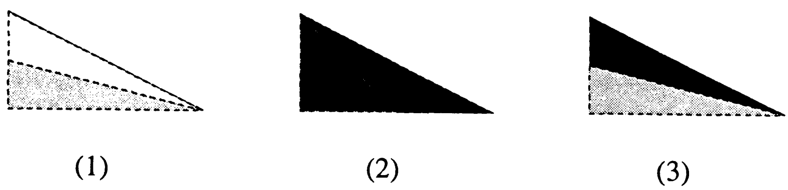


Figure 5