J.DIFFERENTIAL GEOMETRY
Vol. 42, No.2 September, 1995

LOCAL POSITIVITY OF AMPLE LINE BUNDLES

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Introduction

The purpose of this paper is to establish a lower bound on the Seshadri constants measuring the local positivity of an ample line bundle at a general point of a complex projective variety of arbitrary dimension.

Let X be an irreducible complex projective variety, and let L be a nef line bundle on X. Demailly [6] has introduced a very interesting invariant which in effect measures how positive L is locally near a given smooth point $x \in X$. This *Seshadri constant* $\epsilon(L, x) \in \mathbf{R}$ may be defined as follows. Consider the blowing up

$$f : Y = \mathrm{Bl}_x(X) \longrightarrow X$$

of X at x, and denote by $E = f^{-1}(x) \subset Y$ the exceptional divisor. Then f^*L is a nef line bundle on Y, and we put

$$\epsilon(L, x) = \sup\{\epsilon \geq 0 \mid f^*L - \epsilon \cdot E \text{ is nef}\}.$$

Here $f^*L - \epsilon E$ is considered as an $\mathbf{R}$-divisor on Y, and to say that it is nef means simply that $f^*L \cdot C' \geq \epsilon E \cdot C'$ for every irreducible curve $C' \subset Y$. For example, if L is very ample, then $\epsilon(L, x) \geq 1$ for every smooth point $x \in X$. Seshadri's criterion (cf. [10 (Chapter 1)]) states that L is ample if and only if there is a positive number $\epsilon > 0$ such that $\epsilon(L, x) \geq \epsilon$ for every $x \in X$. We refer to Section 1 below, as well as [6 (§6)], for alternative characterizations and further properties of Seshadri constants.

Received August 15, 1994, and, in revised form, November 8, 1994. The first author was partially supported by NSF Grant DMS 93-02512, the second author was supported by Deutsche Forschungsgemeinschaft, and the third author was partially supported by NSF Grant DMS 94-00815.

It was shown by an elementary argument in [7] that if S is a smooth projective surface, and L is an ample line bundle on S, then $\epsilon(L,x) \geq 1$ for all except perhaps countably many $x \in S$. This suggested the somewhat surprising possibility that there could be a similar lower bound on the local positivity of an ample line bundle at a general point of an irreducible projective variety of any dimension. Our main result shows that this is indeed the case:

Theorem 1. *Let L be a nef and big line bundle on an irreducible projective variety X of dimension n. Then*

$$\epsilon(L,x) \geq \frac{1}{n}$$

for all $x \in X$ outside a countable union of proper closed subvarieties of X. Moreover given any $\delta > 0$ the locus

$$\left\{ x \in X \;\middle|\; \epsilon(L,x) > \frac{1}{n+\delta} \right\}$$

contains a Zariski-open dense set.

More generally, we prove that if there exists a countable union $\mathcal{B} \subset X$ of proper closed subvarieties, plus a real number $\alpha > 0$ such that for $1 \leq r \leq n$:

$$\int_Y c_1(L)^r \geq (r \cdot \alpha)^r \quad \forall \; r\text{-dimensional } Y \subset X \;\text{ with } Y \not\subset \mathcal{B},$$

then $\epsilon(L,x) \geq \alpha$ for all sufficiently general $x \in X$. Examples constructed by Miranda show that given any $b > 0$, there exist X, L and x such that $0 < \epsilon(L,x) < b$. In other words, there cannot be a bound (independent of X and L) that holds at every point. On the other hand, it is unlikely that the particular constant appearing in Theorem 1 is optimal. In fact, it is natural to conjecture that in the setting of the Theorem one should have $\epsilon(L,x) \geq 1$ for a very general point $x \in X$.

Recent interest in Seshadri constants stems in part from the fact that they govern an elementary method for producing sections of adjoint bundles. Our bounds then imply the following, which complements the non-vanishing theorems of Kollár ([12 (§3)]):

Corollary 2. *Let L be a nef line bundle on a smooth projective variety X of dimension $n \geq 2$, and given an integer $s \geq 0$ suppose that*

$$\int_Y c_1(L)^r \geq (r(n+s))^r$$

for every r-dimensional subvariety $Y \subseteq X$ not contained in some fixed countable union $\mathcal{B} \subset X$ of proper subvarieties. Then the adjoint series $|\mathcal{O}_X(K_X + L)|$ generates s-jets at a general point $x \in X$, i.e., the evaluation map

$$H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is surjective. In particular,

$$h^0(X, \mathcal{O}_X(K_X + L)) \geq \binom{n+s}{n}.$$

It follows for example that if A is ample, then $\mathcal{O}_X(K_X + (ns + n^2)A)$ generates s-jets at almost all points $x \in X$. We remark that contrary to what one might expect from extrapolating the well-known conjectures of Fujita [9] on global generation and very ampleness, there cannot exist a linear function $f(s)$ (depending on n, but independent of X and A) such that $\mathcal{O}_X(K_X + f(s)A)$ generates s-jets for $s \gg 0$ at *every* point of X (Remark 1.7).

Similarly, we have

Corollary 3. *Suppose that L is a nef and big line bundle on a smooth projective variety X of dimension $n \geq 2$. Then for all $m \geq 2n^2$, the linear series $|K_X + mL|$ is very big*[1], *i.e., the corresponding rational map*

$$\phi_{|K_X+mL|} : X \dashrightarrow \mathbf{P}$$

maps X birationally onto its image.

For example, suppose that X is a smooth minimal variety of general type, i.e., K_X is nef and big. Then the pluricanonical rational maps

$$\phi_{|mK|} : X \dashrightarrow \mathbf{P}$$

are birational onto their images for $m > 2n^2$. This extends (with somewhat weaker numbers) the results of Ando [1] in the cases $n \leq 5$. More generally, if X is a general type minimal n-fold of global index r, then $|mrK_X|$ is again very big when $m > 2n^2$ (Corollary 4.6). As

[1]This terminology was suggested by Kollár to replace what used to be known as "birationally very ample"

above, one also has an analogue of Corollary 3 for the linear series $|K_X + L|$ involving intersection numbers of L with subvarieties of X.

The proof of Theorem 1 draws inspiration from two sources: first, the arguments used in [17], [4] and [14] to prove boundedness of Fano manifolds of Picard number one; and secondly, some of the geometric ideas occuring in [3], [18] and especialy [8]. Roughly speaking, if Theorem 1 fails, then given a general point $x \in X$ there exists a curve $C_x \subset X$ through x such that

$$\frac{\mathrm{mult}_x(C_x)}{(L \cdot C_x)} > n.$$

We start by fixing a divisor $E_x \in |kL|$ for $k \gg 0$ with suitably large multiplicity at x. If one could arrange that $C_x \not\subset E_x$, then one arrives right away at a contradiction by estimating $E_x \cdot C_x$ in terms of multiplicities at x. Unfortunately it does not seem to be immediate that one can do so. Instead, we use a gap construction to show that for an appropriate choice of $y = y(x)$, we can at least control the difference of the multiplicities of E_x at y and at a general point of C_y. The principal new ingredient is then an argument showing that we can *rechoose* the divisors E_x in such a way as to ensure that $C_y \not\subset E_x$ while keeping the multiplicity $\mathrm{mult}_y(E_x)$ of E_x at y fairly large, and then we are done. Stated somewhat informally, the main lemma here is the following:

> Suppose that $\{Z_t \subseteq V_t\}_{t \in T}$ is a family of subvarieties of a smooth variety X, parametrized by a smooth affine variety T, and assume that $\cup_{t\in T} V_t$ is dense in X. Suppose also there is given a family $\{E_t\}_{t\in T} \in |L|$ of divisors in a fixed linear series on X, with
>
> $$a = \mathrm{mult}_{Z_t}(E_t) \;\text{ and }\; b = \mathrm{mult}_{V_t}(E_t)$$
>
> for general $t \in T$. Then one can find another family of divisors $\{E'_t\}_{t\in T} \in |L|$ such that
>
> $$V_t \not\subseteq E'_t \;\text{ and }\; \mathrm{mult}_{Z_t}(E'_t) \geq a - b$$
>
> for general $t \in T$.

We refer to Proposition 2.3 for the precise statement and proof. Denoting by $\{s_t\} \in \Gamma(X, L)$ the family of sections defining E_t, the idea is to construct E'_t as the divisor of the section $Ds_t \in \Gamma(X, L)$, where D is a general differential operator of order b in t. For divisors on projective space (and other compactifications of group varieties), the process of differentiating in order to arrive at a proper intersection plays an important role e.g. in [3] and [8]. Our observation here is that the same idea works in a deformation-theoretic context, and we hope that the lemma may find other applications in the future.

Concerning the organization of the paper, we start in Section 1 with a quick review of general facts about Seshadri constants. In Section 2 we discuss multiplicities in a family of divisors, and show in particular that by differentiating in parameter directions one can lower the multiplicity of such a family along a covering family of subvarieties. The proof of the main result occupies Section 3, and finally we give some elementary applications in Section 4.

We have profitted from discussions with F. Campana, V. Maşek, A. Nadel and G. Xu, and have benefitted from several suggestions by J. Kollár. Nadel in particular stressed some years ago the relevance of techniques from diophantine approximation and transcendence theory to arguments of this type. We are especially endebted to M. Nakamaye for helping us to understand some of these arithmetically motivated ideas. In particular, the crucial Proposition 2.3 was inspired by a proof Nakamaye showed us of Dyson's lemma concerning singularities of curves in $\mathbf{P}^1 \times \mathbf{P}^1$.

0. Notation and conventions

(0.1). We work throughout over the complex numbers $\mathbf{C}$.

(0.2). We will say that a property holds at a *general* point of a variety X if it holds for a non-empty Zariski-open subset of X. It holds at a *very general* point if it is satisfied off the union of countably many proper closed subvarieties of X.

(0.3). If X is a projective variety of dimension n, and L is a line bundle on X, we denote by $L^n \in \mathbf{Z}$ the top self-intersection number of L. Given a subvariety $Z \subset X$ of dimension r, $L^r \cdot Z$ indicates the degree $\int_Z c_1(L)^r \in \mathbf{Z}$. Recall that a line bundle L is *numerically effective* or *nef* if $L \cdot C \geq 0$ for all effective curves $C \subseteq X$. Kleiman's criterion ([10 (Chapter 1)]) implies that a line bundle is nef if and only if it lies in the closure of the ample cone in the Néron - Severi vector space

$NS(X)_{\mathbf{R}}$. Recall also that a nef line bundle L is big if and only if $L^n > 0$. Similar definitions and remarks hold for (numerical equivalence classes of) $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors on X.

(0.4). For varieties X and T, $pr_1 : X \times T \longrightarrow X$, $pr_2 : X \times T \longrightarrow T$ denote the projections. If $Z \longrightarrow T$ is a mapping, Z_t denotes the fibre of Z over $t \in T$. Given a Zariski-closed subset (or subscheme) $Z \subset X \times T$, we consider the fibre Z_t of pr_2 as a subset (or subscheme) of X. Similarly, $Z_x \subset T$ is the fibre of Z over $x \in X$. If $V \subset X$ is a subvariety, $\mathcal{I}_V \subset \mathcal{O}_X$ denotes its ideal sheaf.

1. Seshadri constants

In this section, we recall briefly some of the basic facts about Seshadri constants. We start with a

Definition 1.1. ([6]) Let X be a projective variety, $x \in X$ a point and L a nef line bundle on X. Then the *Seshadri constant of L at x* is the real number

$$\epsilon(L, x) := \inf_{C \ni x} \left\{ \frac{L \cdot C}{\operatorname{mult}_x(C)} \right\},$$

where the infimum is taken over all reduced and irreducible curves $C \subset X$ passing through x. (We remark that it is enough here that L be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor.)

It is elementary (and standard) that (1.1) is equivalent to the alternative definition given in the Introduction:

Lemma 1.2. *Let X be a projective variety, L a nef line bundle on X, and $x \in X$ a smooth point. Let*

$$f : Y = Bl_x(X) \longrightarrow X$$

be the blowing up of X at x, and denote by $E = f^{-1}(x) \subset Y$ the exceptional divisor. Then

$$\epsilon(L, x) = \sup\{\epsilon \geq 0 \mid f^*L - \epsilon \cdot E \text{ is nef}\}.$$

Note that the supremum in Lemma 1.2 is actually a maximum.

There are interesting characterizations of Seshadri constants involving the generation of jets. Recall that given a line bundle B on a smooth variety X, and an integer $s \geq 0$, we say that the linear series $|B|$ *generates s-jets at* $x \in X$ if the evaluation map

$$H^0(X, \mathcal{O}_X(B)) \longrightarrow H^0(X, \mathcal{O}_X(B) \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is surjective, where $\mathcal{I}_x$ denotes the ideal sheaf of x. The following Proposition — which is a variant of [6 (Theorem 6.4)] — shows in effect that computing the Seshadri constant $\epsilon(L, x)$ is equivalent to finding a linear function $f(s)$ such that the adjoint series $|K_X + f(s)L|$ generates s-jets at x for all $s \gg 0$.

Proposition 1.3. *Let L be a nef and big line bundle on a smooth projective variety X of dimension n.*

(1.3.1). *If*

$$r > \frac{s}{\epsilon(L,x)} + \frac{n}{\epsilon(L,x)},$$

then $|K_X + rL|$ generates s-jets at $x \in X$. The same statement holds if $r = \dfrac{s+n}{\epsilon(L,x)}$ and $L^n > \epsilon(L,x)^n$.

(1.3.2). *Conversely, suppose there is a real number $\epsilon > 0$ plus a constant $c \in \boldsymbol{R}$ such that $|K_X + rL|$ generates s-jets at $x \in X$ for all $s \gg 0$ whenever*

$$r > \frac{s}{\epsilon} + c.$$

Then $\epsilon(L, x) \geq \epsilon$.

Proof. (1). [6 (Prop. 6.8)] This is a standard application of Kawamata-Viehweg vanishing for nef and big line bundles. In brief, let $f : Y = \mathrm{Bl}_x(X) \longrightarrow X$ be the blowing-up of X at x, with exceptional divisor $E \subset Y$. It suffices to show that

$$(*) \quad \begin{aligned} &H^1(X, \mathcal{O}_X(K_X + rL) \otimes \mathcal{I}_x^{s+1}) \\ &\qquad = H^1(Y, \mathcal{O}_Y(K_Y + rf^*L - (s+n)E)) = 0. \end{aligned}$$

Setting $\epsilon = \epsilon(L, x)$, one has the numerical equivalence

$$rf^*L - (s+n)E \equiv \frac{s+n}{\epsilon}(f^*L - \epsilon E) + (r - \frac{s+n}{\epsilon})f^*L.$$

Hence $rf^*L - (s+n)E$ is big and nef, and (*) follows from Vanishing.

(2). Let $C \ni x$ be a reduced and irreducible curve with $\mathrm{mult}_x(C) = m$. Fix $s \gg 0$ and let r be the least integer $> (s/\epsilon) + c$. The geometrical interpretation of the fact that $|K_X + rL|$ generates s-jets at x is that we can find a divisor $D_x \in |K_X + rL|$, with $\mathrm{mult}_x(D_x) = s$, having an arbitrarily prescribed tangent cone at x. In particular, we can choose D_x such that the tangent cones to D_x and C at x meet properly, and since C is irreducible it follows that D_x and C themselves meet properly. Then

$$C \cdot (K_X + rL) \geq \mathrm{mult}_x(C) \cdot \mathrm{mult}_x(D_x) = m \cdot s,$$

and hence

$$\frac{C \cdot L}{m} \geq \frac{s}{r} - \frac{C \cdot K_X}{rm}.$$

Since $r \leq (s/\epsilon) + c + 1$, the claim follows by letting $s \to \infty$.

Our main result (Theorem 3.1) will give a lower bound on the Seshadri constant of a nef and big line bundle at a very general point, i.e., a bound which holds off the union of countably many proper subvarieties. However the following Lemma shows that one then obtains a statement valid on a Zariski-open set:

Lemma 1.4. *Let L be a nef and big line bundle on an irreducible projective variety X. Suppose that there is a positive rational number $B > 0$ such that $\epsilon(L, y) > B$ for a very general point $y \in X$. Then the locus*

$$\Big\{ x \in X \;\Big|\; \epsilon(L, x) > B \Big\}$$

contains a Zariski-open dense set.

Proof. Assume for the time being that L is ample. Given a smooth point $x \in X$, let

$$f_x : Y_x = \mathrm{Bl}_x(X) \longrightarrow X$$

be the blowing-up of X at x, with exceptional divisor $E_x \subset Y_x$. Consider the $\mathbf{Q}$-divisor

$$M_x =_{\text{def}} f_x^* L - B \cdot E_x$$

on Y_x. It follows from the hypothesis and characterization (1.2) of Seshadri constants that $f_y^* L - \epsilon(L, y) \cdot E_y$ is nef, and hence M_y is ample. Since ampleness is an open condition in a flat family of line bundles, there exists a non-empty Zariski-open subset $U \subset X$ of smooth points of X such that M_x is ample whenever $x \in U$. But by (1.2) again, the ampleness of M_x implies that $\epsilon(L, x) > B$, which proves the assertion for ample L. Now following a suggestion of Kollár, we reduce the case of nef and big L to this one.

Recall that, for L big, we can write $L = A + R$ with an ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor R (cf. [16 (1.9)]). Since L is also nef, the $\mathbf{Q}$-divisors

$$A_m = L - \frac{R}{m} = A + \frac{m-1}{m}R$$

are again ample, and we have the estimate

$$L \cdot C \geq A_m \cdot C \geq \frac{m-1}{m} L \cdot C$$

for all irreducible curves $C \not\subseteq R$. Concerning the second inequality observe that $mA_m = mL - R = (m-1)L + (L-R)$ where $L - R$ is nef. Therefore

$$\epsilon(L, y) \geq \epsilon(A_m, y) \geq \frac{m-1}{m}\epsilon(L, y)$$

for $y \notin R$. Now pick a very general $y \notin R$ and write $\epsilon(L, y) = B + \gamma > B$. Chose an integer $m >> 0$ such that $(m-1)(B+\gamma) > mB$. Then we have

$$\epsilon(A_m, y) \geq \frac{m-1}{m}\epsilon(L, y) = \frac{m-1}{m}(B + \gamma) > B,$$

and by the above there exists a Zariski open subset $U \subset X$ such that $\epsilon(L, x) \geq \epsilon(A_m, x) > B$ for all $x \in U$.

Finally, for the convenience of the reader we recall from [7 (§3)], the examples of Miranda showing that the Seshadri constants of an ample line bundle can take on arbitrarily small positive values.

Proposition 1.5 (Miranda). *Given any positive number $b > 0$, there exist a projective variety X and an ample line bundle L on X such that*

$$0 < \epsilon(L, x) < b$$

for all x in a codimension two subset $V \subset X$.

Sketch of Proof. We first construct a line bundle N on a surface S having the required property. To this end, start with a reduced and irreducible plane curve $C \subset \mathbf{P}^2$ of degree $d \gg 0$ with a point $y \in C$ of multiplicity $m > \frac{1}{b}$. Fix a second integral curve $C' \subset \mathbf{P}^2$ of degree d meeting C transversely. Provided that d is sufficiently large, by taking C' generally enough we can assume that all the curves in the pencil spanned by C and C' are reduced and irreducible. Blow up the base-points of this pencil to obtain a surface S, admitting a map $f : S \longrightarrow \mathbf{P}^1$ with irreducible fibres, among them $C \subset S$. Observe that any of the exceptional divisors over $\mathbf{P}^2$ gives rise to a section $\Gamma \subset S$ of f which meets C transversely at one point. Fix an integer $a \geq 2$, and put $N = aC + \Gamma$. It follows from the Nakai criterion that N is ample. But $N \cdot C = 1$ whereas $\operatorname{mult}_y(C) = m > \frac{1}{b}$, so $\epsilon(N, y) < b$. As Viehweg pointed out, this gives rise automatically to higher dimensional examples. In fact, take for instance $X = S \times \mathbf{P}^{n-2}$ and $L = pr_1^*(N) \otimes pr_2^*(\mathcal{O}_{\mathbf{P}}(1))$. By considering the evident curve in $S \times \{z\}$, one sees that

$$\epsilon(L, (y, z)) \leq \epsilon(S, y) < b \text{ for all } z \in \mathbf{P}^{n-2}.$$

Thus it suffices to take $V = \{y\} \times \mathbf{P}^{n-2}$.

Remark 1.6. We do not know whether Seshadri constants can become arbitrarily small on a codimension-one subset of X. It is shown in [7] that this cannot happen when X is a surface.

Remark 1.7. A well-known conjecture of Fujita [9] asserts that if L is an ample line bundle on a smooth projective variety X of dimension n, then $\mathcal{O}_X(K_X+(n+1)L)$ is free and $\mathcal{O}_X(K_X+(n+2)L)$ is very ample. Extrapolating, one might be tempted to hope that for all $s \geq 0$:

$$(*) \qquad |K_X + (n + s + 1)L| \text{ separates } s\text{-jets at every point } x \in X.$$

However (1.3.2) and (1.5) show that (*) is not true in general. In fact, there cannot exist a linear function $f(s)$ (depending on n but independent of X and L) such that $|K_X + f(s)L|$ generates s-jets at all $x \in X$. However when X is a surface, it follows from [2] or [5] that there exists a *quadratic* function $f(s)$ such that $|K_X + f(s)L|$ separates s-jets at every $x \in X$. (Cf. [15 (§7)].)

Remark 1.8. Let X be a projective variety of dimension n, L a nef line bundle on X, and $x \in X$ a smooth point. Then

$$\epsilon(L, x) \leq \sqrt[n]{(L^n)} \text{ for every } x \in X.$$

In fact, if $f : Y = \mathrm{Bl}_x(X) \longrightarrow X$ is the blowing up of x, with exceptional divisor E, then $(f^*L - \epsilon(L, x) \cdot E)^n \geq 0$. As an interesting example, suppose that X is a simple abelian variety, and L is a principal polarization on X. Then Nakamaye has shown that $\epsilon(L,x) \geq 1$ for all x. Therefore one has the inequality:

$$1 \leq \epsilon(L,x) \leq \sqrt[n]{n!} \approx \frac{n}{e}. \tag{1.8.1}$$

Since X is homogeneous, $\epsilon(L,x)$ is independent of $x \in X$, so there is a real number $\epsilon(L)$ satisfying (1.8.1) canonically attached to a principally polarized abelian variety (X,L). However it is not obvious to us what the value of this invariant is, even for Jacobians or very general p.p.a.v.'s. Note that if $C \subset X$ is any curve, then $L \cdot C \geq n$. Hence (1.8.1) implies that the curves computing $\epsilon(L,x)$ in (1.1) cannot be smooth.

2. Multiplicity lemmas

This section is devoted to some preliminary results concerning multiplicity loci in a family of divisors. Proposition 2.3 — which allows one to reduce the multiplicity of a family of divisors along a covering family of subvarieties — is the crucial ingredient in the proof of our main Theorem. It is in this section that we make essential use of the fact that we are working in characteristic zero.

We start with some notation. If M is a smooth variety, and E is an effective divisor on M, then the function $x \mapsto \mathrm{mult}_x(E)$ is Zariski upper-semicontinuous on M. Given an irreducible subvariety $Z \subset M$, by $\mathrm{mult}_Z(E)$ we mean the value of $\mathrm{mult}_x(E)$ at a general point $x \in Z$. We refer to (0.4) for notation and conventions concerning projections from products, and fibres of morphisms.

The first lemma allows one to make fibrewise calculations of multiplicities. It is certainly a well-known fact, but we include a proof for the convenience of the reader.

Lemma 2.1. *Let X and T be smooth irreducible varieties, and suppose that $Z \subset X \times T$ is an irreducible subvariety which dominates T (under projection to the second factor). Let $E \subset X \times T$ be any effective divisor. Then for a general point $t \in T$, and any irreducible component $W_t \subseteq Z_t$ of the fibre Z_t, we have:*

$$\mathrm{mult}_{W_t}(E_t) = \mathrm{mult}_Z(E).$$

Proof. Consider more generally a mapping $f : M \longrightarrow T$ of smooth varieties, and suppose that $V \subset M$ is a smooth subvariety dominating T. Assume given an effective divisor $E \subset M$ with $\mathrm{mult}_V(E) = a$. We will show that for a general point $t \in T$, and any irreducible component $W_t \subseteq V_t$:

$$(*) \qquad \mathrm{mult}_{W_t}(E_t) = a.$$

The Lemma then follows by taking M to be an open subset of $X \times T$ on which Z is smooth, and setting $V = Z \cap M$.

To prove (*), note first that for $b \leq a$ the section $s \in \Gamma(M, \mathcal{O}_M(E))$ defining E lies in the subspace

$$\Gamma(M, \mathcal{O}_M(E) \otimes \mathcal{I}_V^b) \subseteq \Gamma(M, \mathcal{O}_M(E)).$$

Hence s determines a section

$$\delta_b(s) \in \Gamma(V, \mathcal{I}_V^b/\mathcal{I}_V^{b+1}(E)) = \Gamma(V, Sym^b(N^*_{V/M})(E))$$

of a twist of the b^{th} symmetric power of the conormal bundle to V in M. (One thinks of $\delta_b(s)$ as giving the b^{th} order terms in the Taylor expansion of s in the directions normal to V.) One checks e.g. by a calculation in local coordinates that $\delta_b(s) = 0$ for $b < a$ whereas $\delta_a(s) \neq 0$. Now fix a point $t \in T$ lying in the open subset of T over which the mappings $M \longrightarrow T$ and $V \longrightarrow T$ are smooth, and let $s_t = s|M_t \in \Gamma(M_t, \mathcal{O}_{M_t}(E_t))$ be the restriction of s to M_t, so that s_t is the section defining E_t. Then

$$\delta_b(s)|V_t = \delta_b(s_t) \in \Gamma(V_t, Sym^b(N^*_{V_t/M_t})(E_t)).$$

But since $V \longrightarrow T$ is dominating, a non-zero section of a locally free sheaf on V restricts to a non-zero section on each irreducible component $W_t \subseteq V_t$ of a general fibre. Hence $\delta_b(s_t) = 0$ for $b < a$ and $\delta_a(s_t) \neq 0 \in \Gamma(W_t, \mathcal{O}_{W_t}(E_t))$ for general $t \in T$. But as we have just seen, this implies that $\mathrm{mult}_{W_t}(E_t) = a$, as claimed.

Remark 2.2. Some readers may prefer to see the argument phrased in a more concrete manner. In the situation of (2.1) it is enough to show that for sufficiently general $t \in T$, and for any component $W_t \subset Z_t$, there exists at least one point $x \in W_t$ such that $\mathrm{mult}_x(E_t) =$

$\mathrm{mult}_{(x,t)}(E)$. Now since Z dominates T, given general points $t \in T$ and $x \in W_t \subset Z_t$ we can find a local analytic section of the projection $Z \longrightarrow T$, say $\sigma : U \longrightarrow Z$, defined in a (classical) neighborhood U of t in T, whose image passes through the point (x,t). Replacing T by U, and working analytically, we can assume given a holomorphic mapping $p : T \longrightarrow X$, and we are reduced to proving that

$$(*) \qquad \mathrm{mult}_{p(t)}(E_t) = \mathrm{mult}_{(p(t),t)}(E)$$

for general $t \in T$. But this follows easily from an explicit calculation in local holomorphic coordinates. [Choose coordinates x and t on X and T, and suppose p is given by $p = p(t)$. Defining $y = x - p(t)$, expand a local equation for E as a Taylor series in y and t.]

We now come to the main result of this section.

Proposition 2.3. *Let X and T be smooth irreducible varieties, with T affine, and suppose that*

$$Z \subseteq V \subseteq X \times T$$

are irreducible subvarieties such that V dominates X. Let L be a line bundle on X, and suppose given on $X \times T$ a divisor

$$E \in |pr_1^*(L)|.$$

Write

$$\ell = \mathrm{mult}_Z(E), \quad k = \mathrm{mult}_V(E).$$

Then there exists a divisor

$$E' \in |pr_1^*(L)|$$

on $X \times T$ having the property that

$$\mathrm{mult}_Z(E') \geq \ell - k, \quad \textit{and} \quad V \not\subseteq \mathit{Supp}(E').$$

Let $\sigma \in \Gamma(X \times T, pr_1^*(L))$ be the section defining E. In a word, the plan is to obtain E' as the divisor of a section

$$\sigma' = D\sigma \in \Gamma(X \times T, pr_1^*(L)),$$

where D is a general differential operator of order $\leq k$ on T. So we begin with some remarks about differentiating sections of the line bundle $pr_1^*(L)$ in parameter directions.

Let $\mathcal{D}_T^k$ be the (locally free) sheaf of differential operators of order $\leq k$ on T. Then sections of $\mathcal{D}_T^k$ act naturally on the space $\Gamma(X \times T, pr_1^*(L))$ of sections of $pr_1^*(L)$. Naively this comes about as follows. Choose local coordinates x and t on X and T, and let $g_{\alpha,\beta}(x)$ be the transition functions of L with respect to a suitable open covering of X. Then sections of $pr_1^*(L)$ are given by collections of functions $\sigma = \{s_\alpha(x,t)\}$ such that $s_\alpha(x,t) = g_{\alpha,\beta}(x)s_\beta(x,t)$. If D is a differential operator in the t-variables, then

$$Ds_\alpha(x,t) = g_{\alpha,\beta}(x)Ds_\beta(x,t).$$

Therefore the $\{Ds_\alpha(x,t)\}$ patch together to define a section $D\sigma \in \Gamma(X \times T, pr_1^*(L))$.

To say the same thing in a more invariant fashion, let $\mathcal{D}_{X\times T}^k(pr_1^*(L))$ denote the sheaf of differential operators of order $\leq k$ on $pr_1^*(L)$, i.e.,

$$\mathcal{D}_{X\times T}^k(pr_1^*(L)) = P_{X\times T}^k(pr_1^*(L))^* \otimes pr_1^*(L),$$

where $P_{X\times T}^k(pr_1^*(L))$ is the sheaf of principal parts associated to $pr_1^*(L)$. Observe that there is a canonical inclusion of vector bundles

$$(*) \qquad pr_2^*(\mathcal{D}_T^k) \hookrightarrow \mathcal{D}_{X\times T}^k(pr_1^*(L)).$$

In fact, it follows from the construction of bundles of principal parts plus the projection formula that one has an isomorphism:

$$P_{X\times T/X}^k(pr_1^*(L)) = pr_2^*(P_T^k(\mathcal{O}_T)) \otimes pr_1^*(L),$$

and then (*) is deduced from the surjection

$$P_{X\times T}^k(pr_1^*(L)) \longrightarrow P_{X\times T/X}^k(pr_1^*(L)).$$

On the other hand, a section $\sigma \in \Gamma(X \times T, pr_1^*(L))$ gives rise to a vector bundle map

$$\mathcal{D}_{X\times T}^k(pr_1^*(L)) \longrightarrow pr_1^*(L),$$

and hence by composition a homomorphism

$$j_\sigma : pr_2^*(\mathcal{D}_T^k) \longrightarrow pr_1^*(L).$$

Given $D \in \Gamma(T, \mathcal{D}_T^k)$,

$$D\sigma \in \Gamma(X \times T, pr_1^*(L))$$

is just the image of $pr_2^*(D) \in \Gamma(X \times T, pr_2^*(\mathcal{D}_T^k))$ under the map on sections determined by j_σ.

Proof of Proposition 2.3. Since T is affine, the vector bundle $\mathcal{D}_T^k$ is globally generated. Choose finitely many differential operators

$$D_\alpha \in \Gamma(T, \mathcal{D}_T^k)$$

which span $\mathcal{D}_T^k$ at every point of T. Let $\sigma \in \Gamma(X \times T, pr_1^*(L))$ be the section defining the given divisor E, and consider the algebraic subset

$$X \times T \supset B = \{(x,t) \mid D_\alpha\sigma(x,t) = 0 \ \ \forall\, \alpha\}$$

cut out by the common zeroes of all the sections $D_\alpha\sigma \in \Gamma(X \times T, pr_1^*(L))$.

We assert that

$$(*) \qquad V \not\subseteq B.$$

To verify this, we study the first projection

$$pr_1 : X \times T \longrightarrow X.$$

Fix any point $x \in X$, and consider the fibre $E_x \subset T$ of E over x. Assume that $E_x \neq T$ (which will certainly hold for general x), so that E_x is a divisor on T. Given $t \in T$, it follows from the fact that the D_α generate $\mathcal{D}_T^k$ at t that

$$(x,t) \in B \iff \mathrm{mult}_t(E_x) > k.$$

On the other hand, since V dominates X, Lemma 2.1 applies to pr_1 and we conclude that

$$\mathrm{mult}_t(E_x) = \mathrm{mult}_V(E) = k$$

for sufficiently general $(x,t) \in V$. This proves (*).

From (*) it follows that if $D \in \Gamma(T, \mathcal{D}_T^k)$ is a sufficiently general $\mathbf{C}$-linear combination of the D_α, then

$$\sigma' =_{\mathrm{def}} D\sigma \in \Gamma(X \times T, pr_1^*(L))$$

does not vanish on V. On the other hand, a differential operator of order $\leq k$ decreases multiplicities by at most k. Therefore if E' is the divisor of σ', then $\operatorname{mult}_Z(E') \geq \ell - k$, as required.

3. The main theorem

The purpose of this section is to prove:

Theorem 3.1. *Let L be a nef line bundle on an n-dimensional irreducible projective variety X. Suppose there exists a countable union $\mathcal{B} \subset X$ of proper subvarieties of X plus a positive real number $\alpha > 0$ such that*

$$(L)^r \cdot Y \geq (r \cdot \alpha)^r \tag{3.1.1}$$

for every irreducible subvariety $Y \subset X$ of dimension r $(1 \leq r \leq n)$ with $Y \not\subseteq \mathcal{B}$. Then

$$\epsilon(L, x) \geq \alpha$$

for all $x \in X$ outside the union of countably many proper subvarieties of X.

Observe that (3.1.1) implies that L is big. Recall also that a line bundle B is big if and only if there exist an ample divisor A and an effective divisor E such that $aB = A + E$ for some $a \gg 0$ (cf. [16 (1.9)]). Thus given a nef and big line bundle L on X, the restriction of L to $Y \not\subset E$ is again big, and hence the inequality (3.1.1) automatically holds with $\alpha = \frac{1}{n}$. Therefore (3.1) implies the first statement of Theorem 1 in the Introduction, and Lemma 1.4 yields the second assertion. Similarly, Corollary 2 follows from (3.1) and the second statement in (1.3.1). We will prove Corollary 3 in Section 4.

(3.2). Turning to the proof of Theorem 3.1, we start with some preliminary remarks and reductions. First, the statement is clear if $\dim X = 1$. Therefore we may – and do – assume inductively that the Theorem is known for all varieties of dimension $< n$.

Note next that there is no loss of generality in supposing that X is smooth. In fact, let

$$f : X' \longrightarrow X$$

be a resolution of singularities, and set $L' = f^*L$, so that L' is a nef line bundle on X'. Suppose that $Y \subset X$ is an r-dimensional subvariety of X, not contained in the fundamental locus of f. If $Y' \subset X'$ is the proper transform of Y, then $(L)^r \cdot Y = (L')^r \cdot Y'$. Simililarly, if $x \in X$ is a point over which f is an isomorphism, then one sees from (1.1) that

$$\epsilon(L, x) = \epsilon(L', x').$$

Thus it suffices to prove the theorem for X', so we will henceforth assume that X is smooth.

(3.3). Let $\beta > 0$ be a real number and let $x \in X$ be a point at which $\epsilon(L, x) < \beta$. Then there exists a reduced irreducible curve $C_x \subset X$ with

$$\beta \cdot \mathrm{mult}_x(C_x) > (L \cdot C_x).$$

Observe that the set of all pairs

$$\left\{ (C, x) \;\middle|\; C \subset X \text{ an integral curve, } \beta \cdot \mathrm{mult}_x(C) > (C \cdot L) \right\}$$

is parametrized by countably many irreducible quasi-projective varieties. This is a consequence of the existence of Hilbert schemes, plus the fact that in a flat family of curves, it is a constructible condition to be reduced and irreducible (cf. [11 (4.10)]). It follows to begin with that the set

$$U_\beta =_{\mathrm{def}} \left\{ x \in X \;\middle|\; \epsilon(L, x) < \beta \right\}$$

can be expressed as a countable union of locally closed subsets of X. Therefore to prove the Theorem, it is enough to show that U_α does not contain a Zariski-open subset of X. By the same token, it is even sufficient to show that for any small rational $\delta > 0$ the set $U_{\alpha-\delta}$ does not contain a Zariski-open subset. Indeed,

$$U_\alpha = \bigcup_{\delta \in \mathbf{Q}^+} U_{\alpha-\delta},$$

and the latter is a countable union.

We fix now $\delta \ll \alpha$ and set $\gamma = \alpha - \delta$. So the issue is to show that U_γ does not contain a Zariski-open subset.

(3.4). Assume to the contrary that U_γ does contain a Zariski-open subset, i.e., that there exists a Zariski-open subset $U \subseteq X$ such that

$$\epsilon(L, x) < \gamma < \alpha$$

for every $x \in U$. Then for every $x \in U$ there exists a reduced irreducible curve $C_x \subset X$ passing through x such that $\gamma \cdot \text{mult}_x(C_x) > (L \cdot C_x)$. We will say that C_x is a *Seshadri-exceptional* curve *based at* x.

From the discussion in (3.3) it follows that there is an irreducible family of Seshadri exceptional curves whose base-points sweep out an open subset of X. More precisely, there exists an irreducible quasi-projective variety T, a dominant morphism

$$g : T \longrightarrow X,$$

plus an irreducible subvariety

$$C \subset X \times T,$$

flat over T, such that for every $t \in T$ the fibre C_t is a Seshadri-exceptional curve based at $g(t) \in X$. In other words, $C_t \subset X$ is a reduced irreducible curve, passing through $g(t)$, with

$$\gamma \cdot \text{mult}_{g(t)} C_t > (L \cdot C_t).$$

Replacing T first by a suitable subvariety, and then by an open subset we can – and do – assume that T is smooth and affine, and that $g : T \longrightarrow X$ is quasi-finite. Write

$$\Gamma \subset X \times T$$

for the graph of g, and as in (0.4) given a subset $Z \subset X \times T$, denote by $Z_t \subset X$ the fibre of Z over $t \in T$, viewed as a subset of X.

(3.5). We next consider a construction analogous to one used by Kollár, Miyaoka and Mori in their proof [14] of the boundedness of Fano varieties of Picard number one.

Lemma 3.5.1. *Let $Z \subset X \times T$ be an irreducible closed subvariety dominating both X and T. Then one can construct an irreducible closed subvariety*

$$CZ \subset X \times T$$

having the following properties:

(3.5.2) *$Z \subseteq CZ$ and $\dim CZ \leq \dim Z + 1$.*

(3.5.3) *For generic $t \in T$, the fibre $(CZ)_t \subset X$ has the form*

$$(CZ)_t = \text{closure}\Big(\bigcup_{s \in S_t} C_s \Big),$$

where $S_t \subset g^{-1}(Z_t)$ is a closed subset of T, which dominates Z_t.

In other words, for general $t \in T$, $(CZ)_t$ is the closure of all the points on a family of Seshadri exceptional curves $\{C_s\}_{s \in S_t}$ based at a dense constructible subset of Z_t.

Proof. First, let

$$S' = (g \times id_T)^{-1}(Z) \subseteq T \times T.$$

The hypothesis that Z dominates X implies that $S' \neq \emptyset$. Fix an irreducible component S_1 of S' whose image under $g \times id_T$ dominates Z. Then $\dim S_1 = \dim Z$ since g is quasi-finite. Next, letting $\pi : C \longrightarrow T$ denote the projection of $C \subset X \times T$ onto the second factor, put

$$V_1 = (\pi \times id_T)^{-1}(S_1) \subseteq X \times T \times T.$$

Very concretely, V_1 may be described as the set

$$V_1 = \{(x, s, t) \mid x \in C_s,\ g(s) \in Z_t,\ (s,t) \in S_1\}.$$

The fibres of the projection $p : V_1 \longrightarrow S_1$ are irreducible curves, and hence V_1 is irreducible, with

$$\dim V_1 = \dim S_1 + 1.$$

Note also that p admits a section $\sigma : S_1 \longrightarrow V_1$ given by $\sigma(s,t) = (g(s), s, t)$.

Consider now the projection $pr_{13} : X \times T \times T \longrightarrow X \times T$ onto the first and third factors, and set

$$V = pr_{13}(V_1) \subseteq X \times T.$$

Then V is an irreducible constructible subset of $X \times T$, and V contains an open subset of Z [viz. an open subset of $(pr_{13} \circ \sigma)(S_1)$]. Given $t \in T$, let

$$S_t = g^{-1}(Z_t) \cap (S_1)_t,$$

where by $(S_1)_t$ we mean the fibre of $S_1 \subset T \times T$ over the second factor. Then by construction, for every $t \in T$: $V_t = \cup_{s \in S_t} C_s$. Finally, put

$$CZ = \text{closure}(V) \subseteq X \times T.$$

Thus property (3.5.2) is clear. As for (3.5.3), from the remark it follows that if $V \subseteq X \times T$ is an irreducible constructible subset dominating T, then for general $t \in T$,

$$\text{closure}(V_t) = (\text{closure}(V))_t.$$

This completes the proof of (3.5.1).

(3.6). The inductive hypothesis is now used to prove:

Lemma 3.6.1. *Let $Z \subset X \times T$ be a proper irreducible subvariety dominating both X and T, and consider the variety $CZ \subseteq X \times T$ constructed in (3.5). Then Z is a proper subvariety of CZ.*

Proof. Assume to the contrary that $CZ = Z$, and fix a very general point $t \in T$. Given a general point $x \in Z_t$, from (3.5.3) it follows that there exists a Seshadri exceptional curve C_s based at x such that C_s lies in $Z_t = (CZ)_t$. But this means that the restriction $L|Z_t$ of L to Z_t has small Seshadri constant at a general point, i.e.,

$$\epsilon(L|Z_t, x) < \gamma$$

for a dense open set of points $x \in Z_t$. But every component of Z_t has dimension $< n$. Therefore the induction hypothesis will give a contradiction once we show that $L|W_t$ satisfies (3.1.1) for any irreducible component $W_t \subset Z_t$. Since the morphism $Z \longrightarrow X$ is dominating, for sufficiently general $t \in T$ no component W_t of Z_t lies entirely in $\mathcal{B}$. Hence for very general $t \in T$, $\mathcal{B} \cap W_t$ is a countable union of proper subvarieties of W_t. On the other hand, if $Y \subseteq W_t$ is a subvariety of dimension r not lying in $\mathcal{B} \cap W_t$, then

$$(L|W_t)^r \cdot Y = L^r \cdot Y \geq (r \cdot \gamma)^r.$$

Hence (3.1.1) holds for $L|W_t$, as required.

(3.7). Much as in [14], (3.5.1) will be used to construct a chain of irreducible subvarieties $Z_i \subseteq X \times T$, as follows. Start with

$$Z_0 = \Gamma = \text{ graph}(g) \ , \qquad Z_1 = C \ \subseteq X \times T,$$

and then for $1 < i \leq n-1$ apply (3.5.1) inductively to form

$$Z_{i+1} = CZ_i \subset X \times T.$$

It follows from (3.6.1) that $Z_i \subsetneq Z_{i+1}$, and consequently Z_i has relative dimension i over T. In particular, $Z_n = X \times T$. Thus we have a chain

$$\Gamma = Z_0 \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = X \times T \tag{3.7.1}$$

of irreducible subvarieties of $X \times T$.

(3.8). We now come to the second construction, inspired by Nadel [17], Campana [4] and the gap arguments used in connection with zero estimates (cf. [3], [8], [18]). The idea is to choose a family of divisors $E_t \in |kL|$ ($k \gg 0$) having high multiplicity at $g(t) \in X$, and to study the multiplicities of E_t along the subvarieties $(Z_i)_t$ defined in (3.7).

We start with a pointwise description. Since $(L^n) \geq (\alpha n)^n > (\gamma n)^n$, a standard parameter count shows that if $k \gg 0$, then given any point $x \in X$ there exists a divisor

$$E_x \in |kL| \quad \text{with} \quad \operatorname{mult}_x(E_x) > k\gamma n.$$

In fact, by Riemann-Roch

$$h^0(\mathcal{O}_X(kL)) = k^n \frac{(L^n)}{n!} + o(k^n),$$

whereas it is $\frac{(k\gamma n)^n}{n!} + o(k^n)$ conditions to impose multiplicity $[k\gamma n + 1]$ at a given point. In particular, we may apply this with $x = g(t)$ to construct a divisor E_t having high multiplicity at the base of the Seshadri-exceptional curve C_t.

These remarks globalize in the following manner. Put $b = [k\gamma n + 1]$, and consider the projections $pr_1 : X \times T \longrightarrow X$, $pr_2 : X \times T \longrightarrow T$. Then for $k \gg 0$ the torsion-free $\mathcal{O}_T$-module

$$\mathcal{F} = pr_{2,*}(pr_1^*(kL) \otimes \mathcal{I}_\Gamma^b)$$

has positive rank, where $\mathcal{I}_\Gamma \subset \mathcal{O}_{X \times T}$ denotes the ideal sheaf of Γ. As T is affine, $\mathcal{F}$ is globally generated. We fix a non-sero section $\sigma \in \Gamma(T, \mathcal{F})$, and since

$$\Gamma(T, \mathcal{F}) = \Gamma(X \times T, pr_1^*(kL) \otimes \mathcal{I}_\Gamma^b),$$

σ gives rise to a divisor

$$E \in |\mathcal{O}_{X \times T}(pr_1^*(kL))| \quad \text{with} \quad \operatorname{mult}_\Gamma(E) > k\gamma n.$$

(3.9). Consider now the multiplicities

$$\mathrm{mult}_{Z_i}(E)$$

of E along the sets Z_i appearing in (3.7.1). We have

$$\mathrm{mult}_{Z_0}(E) = \mathrm{mult}_{\Gamma}(E) > k\gamma n \quad , \quad \mathrm{mult}_{Z_n}(E) = \mathrm{mult}_{(X\times T)}(E) = 0.$$

It follows that there is at least one index i $(0 \leq i \leq n-1)$ such that

$$\mathrm{mult}_{Z_i}(E) - \mathrm{mult}_{Z_{i+1}}(E) > k\gamma.$$

The heart of the argument is that we can now apply Propostion 2.3 to produce a new divisor, not containing Z_{i+1}, with relatively high multiplicity along Z_i.

Specifically, since Z_{i+1} dominates X, Proposition 2.3 implies the existence of a divisor

$$E' \in |\mathcal{O}_{X\times T}(\ pr_1^*(kL)\ )|$$

such that

$$\mathrm{mult}_{Z_i}(E') > k\gamma \quad , \quad Z_{i+1} \not\subseteq \mathrm{Supp}(E').$$

Fix a general point $t \in T$, and consider the divisor $E'_t \in |kL|$ on X. Then E'_t does not contain any component of $(Z_{i+1})_t$, whereas it follows from Lemma 2.1 that

$$\mathrm{mult}_{W_t}(E'_t) = \mathrm{mult}_{Z_i}(E') > k\gamma$$

for any irreducible component W_t of $(Z_i)_t$.

Consider finally a general point $x \in W_t$ for some irreducible component $W_t \subset (Z_i)_t$. Then $\mathrm{mult}_x(E'_t) > k\gamma$. On the other hand, from property (3.5.3) of the construction (3.5.1) of Z_{i+1} it follows that there is a Seshadri exceptional curve $C_s \subset (Z_{i+1})_t$ based at x such that

$$C_s \not\subseteq \mathrm{Supp}(E'_t).$$

Thus C_s meets E'_t properly, and we find:

$$\begin{aligned} k(L\cdot C_s) = E'_t \cdot C_s &\geq \mathrm{mult}_x(E'_t)\cdot \mathrm{mult}_x(C_s) \\ &> k\gamma \cdot \mathrm{mult}_x(C_s). \end{aligned}$$

This contradicts the fact that C_s is Seshadri exceptional, and completes the proof of Theorem 3.1.

4. Applications

In this section we give some simple applications of the main Theorem.

We begin with a criterion for birationality which, together with (3.1), implies Corollary 3 in the Introduction.

Lemma 4.1. *Let X be a smooth projective variety of dimension $n \geq 2$, and L a nef and big line bundle on X. Suppose that there exists a countable union $\mathcal{V} \subset X$ of proper subvarieties such that $\epsilon(L, x) \geq 2n$ for all $x \in X - \mathcal{V}$. Then the adjoint bundle $\mathcal{O}_X(K_X + L)$ is very big, i.e., the corresponding rational mapping*

$$\phi_{|K_X+L|} : X \dashrightarrow \mathbf{P}$$

maps X birationally onto its image.

Proof. We start with a general remark. Suppose that X is an irreducible projective variety, and B is a line bundle on X, with $H^0(X, B) \neq 0$, defining a rational mapping

$$\phi = \phi_{|B|} : X \dashrightarrow \mathbf{P}.$$

Then we claim that ϕ is birational onto its image if and only if there exists a countable union $\mathcal{V} \subset X$ of proper subvarieties such that ϕ is defined and one-to-one on $X - \mathcal{V}$. In fact, there exists in any event a Zariski-open subset $U \subset X$ (which in general may be empty) on which ϕ is defined and one-to-one. The stated condition implies that $U \neq \emptyset$, and so ϕ is generically one-to-one over its image, hence birational.

Returning to the situation of the Lemma, take $B = K_X + L$. We will prove momentarily that for any two distinct points $x, y \in X - \mathcal{V}$ one has the vanishing

$$(*) \qquad H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x \otimes \mathcal{I}_y) = 0.$$

But this means exactly that $\phi_{|K_X+L|}$ is defined and one-to-one on $X - \mathcal{V}$, and hence is birational onto its image, as claimed. As for (*), let $f : Y \longrightarrow X$ be the blowing up of X at x and y, and denote by

$E_x, E_y \subset Y$ the exceptional divisors. Since $\epsilon(L,x) \geq 2n$ and $\epsilon(L,y) \geq 2n$ by hypothesis, it is a consequence of (1.2) that the **Q**-divisors

$$\frac{1}{2}f^*L - nE_x, \ \frac{1}{2}f^*L - nE_y$$

are nef. Therefore $f^*L - nE_x - nE_y$ is nef. Moreover, since $n \geq 2$ we have $L^n \geq (2n)^n > 2n^n$ by (1.8), and so $f^*L - nE_x - nE_y$ is also big. Then just as in (1.3), (*) follows from vanishing on Y.

In the rest of this section we outline how these results can be generalized in the context of **Q**-divisors. To begin with, note:

Remark 4.2. Suppose that X is an irreducible projective variety, and L is a nef **Q**-Cartier **Q**-divisor on X satisfying the numerical hypotheses (3.1.1). Then $\epsilon(L,x) \geq \alpha$ for all smooth $x \in X$ outside the union of countably many proper subvarieties. In fact, choose a positive integer $m > 0$ such that mL is a Cartier divisor. Since $\epsilon(mL,x) = m \cdot \epsilon(L,x)$ for all $x \in X$, the assertion follows from (3.1).

We will henceforth deal with the following set-up:

Assumptions 4.3. X is a smooth irreducible projective variety of dimension $n \geq 2$, and L is a nef and big **Q**-divisor on X. We suppose that Δ is a fractional **Q**-divisor on X (i.e., $\llcorner \Delta \lrcorner = 0$) with normal crossing support. Finally we assume that N is an integral divisor on X satisfying the numerical equivalence $N \equiv L + \Delta$.

Arguing much as in the proof of (1.3.1), but using Kawamata-Viehweg vanishing for **Q**-divisors, one then finds first of all:

Proposition 4.4. *In the situation of (4.3) suppose that L satisfies the numerical hypothesis (3.1.1) of Theorem 3.1. If*

$$\alpha \geq n + s,$$

then $|K_X + N|$ generates s-jets at a very general point $x \notin Supp(\Delta)$.

The case $s = 0$ is proven (in more generality, and with slightly weaker numerical hypotheses) by Kollár in [12 (§3)]. As in [13 (§8)], this implies for example that if X is a smooth projective variety with generically large algebraic fundamental group, and L is any big line bundle on X, then $H^0(X, \mathcal{O}_X(K_X + L)) \neq 0$. It would be interesting to know whether one can use the cases $s > 0$ of Proposition 4.4 (or the birationality statement of Proposition 4.5 below) to obtain further information under suitable hypotheses on L. We note that from [13 (Lemma 8.2)], it follows that if X has generically large algebraic fundamental group, and L is an ample line bundle on X, then given any $\alpha > 0$ there exists an étale covering $m : X' \longrightarrow X$ such that m^*L satisfies the hypotheses of Theorem 3.1. However it is not immediately clear how to pass to useful information on X beyond the non-vanishing established by Kollár.

Arguing as in (4.1) one finds similarly:

Proposition 4.5. *In the set-up of (4.3), suppose that $\epsilon(L, x) \geq 2n$ for a very general point $x \in X$. Then $\mathcal{O}_X(K_X + N)$ is very big.*

In view of Remark 4.2, this applies in particular if L satisfies the numerical hypotheses of (3.1.1) with $\alpha \geq 2n$.

Finally, we give a simple application of (4.5) to pluricanonical maps of minimal varieties:

Corollary 4.6. *Let X be a minimal n-fold of general type having (global) index r, i.e., assume that X has only terminal singularities, that K_X is nef and big, and that rK_X is Cartier. Then the pluricanonical series $|mrK_X|$ is very big for $m \geq 2n^2 + 1$.*

Sketch of Proof. Let $f : Y \longrightarrow X$ be a log resolution of X. Since X has only terminal singularities, we can write

$$K_Y + \Delta \equiv f^*K_X + P,$$

where Δ is a fractional divisor (i.e., $\llcorner \Delta \lrcorner = \emptyset$) with normal crossing support, and P is integral, effective and f-exceptional. Hence

$$K_Y + \Delta + (mr - 1)f^*K_X \equiv f^*(mrK_X) + P.$$

By (3.1), $\epsilon(f^*\mathcal{O}_X(rK_X), y) \geq \frac{1}{n}$ for very general $y \in Y$. If $m > 2n^2$, then

$$\epsilon((mr-1)f^*K_X) \geq \frac{mr-1}{nr} \geq 2n + \frac{r-1}{nr},$$

and from (4.5) applied to $N \equiv \Delta + (mr-1)f^*K_X$ it follows that the linear series $|f^*(mrK_X) + P|$ is very big on Y. But since P is f-exceptional,

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(f^*(mrK_X) + P)) &= H^0(Y, \mathcal{O}_Y(f^*(mrK_X))) \\ &= H^0(X, \mathcal{O}_X(mrK_X)). \end{aligned}$$

Therefore $|mrK_X|$ is very big on X.

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