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TWISTOR SPACES, EINSTEIN METRICS AND ISOMONODROMIC DEFORMATIONS

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1. Introduction

The characteristic feature of twistor theory is its ability to convert questions in differential geometry and differential equations into equivalent ones in algebraic geometry. Moreover, the natural objects in algebraic geometry such as sheaf cohomology groups or vector bundles correspond in a remarkably fortuitous way to solutions of equations which have at some level a physical or geometrical significance. The fundamental example of this is the basic correspondence between an anti-self-dual conformal structure on a 4-manifold M and the holomorphic structure of a complex 3-manifold Z, its twistor space.

In this paper we shall study in depth a problem which goes one step further: to describe an anti-self-dual conformal structure not by algebraic geometry, but by topology, or indeed algebra – the representations of a fundamental group. The conformal structures which are amenable to this approach are those which admit $SU(2)$ as a symmetry group, with certain generic properties. Each such structure describes, and is determined by, a representation in $SL(2,\mathbf{C})$ of a free group on 3 generators. Building on the basic framework of this correspondence, we can introduce other geometrical structures, and in particular an Einstein metric in the conformal class. It turns out that the Einstein condition yields a considerable simplification of the representation, so much so that we can retrace our footsteps from the topology back to the differential geometry and write down new explicit solutions of anti-self-dual $SU(2)$-invariant Einstein metrics with 3-dimensional generic orbits. Among these are two families of complete metrics on the unit ball in $\mathbf{R}^4$: one two-parameter family consists of deformations of the

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hyperbolic metric, the other is a one-parameter family of deformations of the Bergmann metric.

To set the work in context, consider the traditional approach to a problem of this type. Given the Riemannian 4-manifold M with 3-dimensional orbits under an $SU(2)$ action, equations for the curvature can typically be written as a system of ordinary differential equations where the independent variable parametrizes the orbits. In fact, in a number of papers, Tod [34] showed that, in the important diagonal case, the equation which yields an anti-self-dual conformal structure is Painlevé's sixth equation. Tod's differential geometric arguments can be replaced by twistor theoretic ones, where Painlevé's equation makes its appearance in the context of isomonodromic deformations. Recall that this theory concerns itself with a system of first-order ordinary differential equations with simple poles on the projective line, or equivalently a meromorphic connection. An isomonodromic deformation is a deformation of the coefficients of the equation as the poles vary in order that the monodromy of solutions remains the same. It was R.Fuchs in 1907 who discovered Painlevé VI in this context for equations with 4 poles.

The relationship with twistor theory is through the action of $SU(2)$ on the twistor space. In the generic case, the three holomorphic vector fields generated by this action are linearly independent on an open set in Z, but become dependent on a divisor Y. On the Lie algebra level the action on the twistor space defines a homomorphism of vector bundles

$$\alpha : Z \times \mathfrak{g}^c \to TZ.$$

The inverse of α is a meromorphic $SL(2,\mathbf{C})$ connection with a pole on Y. Each twistor space contains a 4-parameter family of projective lines, generically meeting the divisor in 4 points. As the lines vary, the connection induces one with 4 poles on each line. The connection varies as the lines vary, but the monodromy is determined by the fixed holonomy of the connection on the ambient space $Z\backslash Y$. The monodromy, a representation in $SL(2,\mathbf{C})$ of the fundamental group of a quadruply punctured 2-sphere, is what determines the conformal structure, but it takes a considerable amount of effort to reconstruct that structure and to investigate its global behaviour, hence the length of the paper.

We begin in Section 2 with a brief survey of the essential features of twistor geometry, and follow it in Section 3 with the effect of an $SU(2)$ action. Different properties of the homomorphism α divide invariant anti-self-dual conformal structures into disjoint types. We show that the image of α is a subsheaf of rank 2 or 3 and produce a classification in the rank-2 case, Type I. Apart from the conformally flat metrics, the only possibility here is for a hyperkähler metric where $SU(2)$ fixes all complex structures, and this rapidly yields the Eguchi-Hanson metric or one of the Belinskii-Gibbons-Page-Pope family. In the rank-3 case, we have a non-trivial section $\Lambda^3\alpha$ of the anti-canonical bundle of Z which vanishes on a divisor Y either with multiplicity 1 or 2. The first case, Type III, yields Painlevé VI by the argument outlined above, and the second, Type II, Painlevé III as noted in [30]. Apart from the consideration of Einstein metrics, we deal exclusively in this paper with the generic Type III case.

Section 4 establishes the relationship between $SU(2)$-invariant twistor spaces and the isomonodromic deformation problem, and Section 5 derives the conformal structure from the residues A_1, A_2, A_3, A_4 of the connection at the four poles on $\mathbf{C}P^1$. A useful result in a practical direction is that the metric can be put in diagonal form if and only if the residues A_i are all conjugate. In this case, we can write down the conformal structure as

$$g = \frac{dx^2}{x(x-1)} + \frac{\sigma_1^2}{k + \operatorname{tr} A_1A_2} + \frac{(x-1)\sigma_2^2}{k + \operatorname{tr} A_2A_3} - \frac{x\sigma_3^2}{k + \operatorname{tr} A_3A_1},$$

where $\operatorname{tr} A_i^2 = k$. We deduce from the monodromy that this metric is conformal to an Einstein metric if and only if $k = 1/8$.

In Section 6 we discuss the monodromy group Γ and its relation to the real structure on Z in general, but also give a characterization in the case of an Einstein metric. The group Γ in the Einstein case is almost abelian: it lies in the normalizer of a connected abelian subgroup H of $SL(2,\mathbf{C})$, and so the kernel Γ_0 of the map to the Weyl group $\Gamma \to N(H)/H \cong \mathbf{Z}_2$ is abelian. When the scalar curvature vanishes, H is unipotent, otherwise semisimple. Beyond this, there is a subtle interplay between the properties of this monodromy, the real structure and the sign of the scalar curvature of the metric, the results of which are given in Theorem 5.

Given the relatively simple nature of the monodromy for an Einstein metric, we begin in Section 7 to work backwards to find explicit formulas for the metric. To do this, the essential remark is that because the monodromy has an abelian subgroup Γ_0 of index 2, passing to a double covering of $\mathbf{C}P^1$ branched over the four points gives a connection with abelian holonomy. Thus in Section 7 we work on this elliptic curve and calculate the pulled-back connection in terms of Weierstrass elliptic functions. Because the monodromy is abelian, it can be defined via the periods of differentials on the curve and the isomonodromic deformation problem solved. In so doing, we find a solution of the following particular case of Painlevé VI:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}\\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\frac{1}{8}-\frac{x}{8y^2}+\frac{x-1}{8(y-1)^2}+\frac{3x(x-1)}{8(y-x)^2}\right)\end{aligned}$$

depending on two constants c_1 and c_2. Explicitly, it can be written in terms of theta functions as

$$\begin{aligned}y(x) &= \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left(1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right)\\ &+\frac{\vartheta_1'''(\nu)\vartheta_1(\nu) - 2\vartheta_1''(\nu)\vartheta_1'(\nu) + 4\pi i c_1(\vartheta_1''(\nu)\vartheta(\nu) - \vartheta_1'^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu)+2\pi i c_1\vartheta_1(\nu))},\end{aligned}$$

where $\nu = c_1\tau + c_2$ and $x = \vartheta_3^4(0)/\vartheta_4^4(0)$. There are of course more convenient ways of expressing it in order to assess its properties. The monodromy can be written down explicitly in terms of the constants c_1 and c_2: the group is generated by ρ_1, ρ_2, ρ_3 where

$$\rho_j = \begin{pmatrix} 0 & a_j \\ -a_j^{-1} & 0 \end{pmatrix},$$

and $a_1 = e^{i\pi c_1}, a_2 = e^{i\pi(c_1+c_2)}, a_3 = e^{i\pi c_2}$. This is a solution which yields others by taking limiting values of the constants. In particular the case of zero scalar curvature gives

$$y(x) = \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left(1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right) - \frac{ik}{\pi(k\tau-1)\vartheta_4^4(0)}.$$

Section 8 is devoted to global properties of some of these Einstein metrics. Here, to get explicit information, we express the metric in terms of the solution $y(x)$ of the Painlevé equation. We use a formula of Tod for the conformal factor relating the metric g above to the Einstein metric, and make repeated use of identities and expansions for theta functions which are all culled from the classical text [33]. For generic real values of the constants c_1, ic_2 in the above solution we find an Einstein metric on the ball whose conformal structure extends across the boundary, and hence induces one on the 3-sphere. In this way, for every left-invariant conformal structure $\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2 + \lambda_3\sigma_3^2$ on the 3-sphere with distinct λ_i we find such a metric. On the other hand, if two coefficients coincide, then Pedersen [29] gave a solution, and if all are equal the hyperbolic metric is the required solution to the problem. Thus, in the language of LeBrun [25], every left-invariant conformal structure on S^3 has "positive frequency" (with the appropriate orientation). In the particular case $c_1 = 1/2$, we also obtain a family of complete metrics which induce not a conformal structure, but a CR-structure on the boundary. Together with the Bergmann metric on the ball in $\mathbf{C}^2$, these metrics show that every left-invariant CR-structure on S^3 can be induced from a complete Einstein metric on the ball.

Finally in Section 9, we identify the Einstein metrics in the case where the rank of α is 2, and thus, with the results of the preceding sections, give a complete list of anti-self-dual $SU(2)$-invariant Einstein metrics with 3-dimensional generic orbits. We then proceed to determine the complete metrics by looking at the various domains of definition of the function $y(x)$. The list consists of known metrics together with the new metrics on the ball constructed here.

It is useful to see other better known metrics in a more uniform context. In particular, the hyperkähler metric on the moduli space of two $SU(2)$-monopoles derived in [4] arises from a point of view more general than the study of a spectral curve or the minor works of Halphen [16].

Since the author embarked on this work, a number of other relationships between twistor theory, isomonodromic deformations and the Painlevé equations have appeared which shed further light on special features of these metrics. The paper of Mason and Woodhouse [27], relating the Painlevé equation to invariant solutions of the self-dual Yang-Mills equations highlights the existence of some distinguished anti-self-dual 2-forms, and [28] gives different approaches to deriving the metric.

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2.Twistor spaces and self-duality

The basic geometrical object we shall focus on in this paper is an oriented four-dimensional manifold M with an anti-self-dual conformal structure preserved by an action of the Lie group $SU(2)$. Since this group is compact, we can, if required, take it to preserve a metric in the conformal equivalence class.

The general class of anti-self-dual conformal structures is quite wide, but there are a number of special classes which deserve particular consideration:

- anti-self-dual Einstein metrics,
- scalar-flat Kähler metrics,
- hypercomplex structures.

Here an *anti-self-dual Einstein metric* consists of a metric in the anti-self-dual conformal equivalence class with Ricci tensor given by $R_{ij} = \Lambda g_{ij}$. A *scalar-flat Kähler metric* is a metric with zero scalar curvature which is Kählerian with respect to a complex structure on M. It is automatically anti-self-dual with respect to the canonical orientation. A *hypercomplex structure* is a quaternionic structure on the tangent bundle of M such that the action of i, j and k (denoted by I, J, K) each defines an integrable complex structure. There is a unique conformal structure determined by this action, and it is anti-self-dual.

There is a fourth important case which lies in all three of the above. This is the class of *hyperkähler metrics.* These are Riemannian metrics with holonomy group $SU(2)$.

Twistor theory translates the differential geometry of anti-self-dual conformal structures into a branch of holomorphic geometry. We briefly recall the situation. For details the reader may consult [6],[32],[8].

Given a Riemannian four-manifold M, its *twistor space* Z is the unit sphere bundle in the bundle of self-dual 2-forms Λ^2_+, or equivalently the projectivized spinor bundle $P(\Sigma_+)$. Each point z in the fibre over $m \in M$ defines a complex structure on the tangent space T_m, compatible with the metric. Using the Levi-Civita connection, a tangent space T_z of Z splits into horizontal and vertical subspaces, and the projection $p : Z \to M$ identifies the horizontal space with $T_{p(z)}$. This space has a complex structure defined by z, and the vertical subspace is the tangent space of the fibre $\cong S^2 \cong \mathbf{C}P^1$ which has its own natural complex structure. Thus Z is an almost complex manifold.

The almost complex structure on Z depends only on the conformal equivalence class of the metric, and is integrable if and only if the conformal structure is anti-self-dual, that is, if the self-dual part W_+ of the Weyl tensor vanishes. For an anti-self-dual four-manifold, then, the twistor space is a complex 3-manifold. The fibres of the projection p are, moreover, complex submanifolds. Each is a rational curve with normal bundle isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$, where $\mathcal{O}(k)$ is the unique holomorphic line bundle on $\mathbf{C}P^1$ of degree k. These curves belong to a four-parameter holomorphic family of rational curves in Z, and are called twistor lines. The antipodal map on each fibre gives Z a real structure, an antiholomorphic free involution τ. The fibres, parametrized by M, are real members of this family.

The reverse of this process holds, so that an arbitrary anti-self-dual four-manifold can be constructed from a complex three-manifold. Let Z be a complex three-manifold fibred by projective lines whose normal bundle is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and suppose Z admits a free antiholomorphic involution which transforms each fibre to itself. Then Z is the twistor space of some anti-self-dual manifold M.

It will be important in Section 5 to know how to find the conformal structure from this approach, and this is easiest done by considering the complexification $T_m \otimes \mathbf{C}$ of the tangent space at m. A conformal structure over the complex numbers is equivalent to defining the null cone (the set of tangent vectors of length zero) in the tangent space. If it is real with no real points other than 0, then it defines a positive definite conformal structure in T_m. In the situation of twistor theory, the complexified tangent space is the tangent space to the full holomorphic family of deformations of the curve $p^{-1}(m) \cong \mathbf{C}P^1$, and this is the space of holomorphic sections of the normal bundle N. A section of N is defined to be a null vector if it *vanishes at some point* of the fibre. Since by hypothesis $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, a section of N is a pair of linear functions $(a_1 z + a_2, a_3 z + a_4)$ in an affine parameter z on $\mathbf{C}P^1$. Its vanishing at some point is therefore given by the quadratic condition

$$a_1 a_4 - a_2 a_3 = 0,$$

and this is the conformal structure.

One of the remarkable features of this Penrose twistor approach is that natural geometrical properties of the four-manifold M are often reflected in equally natural holomorphic properties of the twistor space Z. We can illustrate this for the three classes of anti-self-dual structures isolated above:

Theorem 1. *Let Z be the twistor space of an anti-self-dual four-manifold M. Then*

- *an Einstein metric in the conformal class is defined by a holomorphic section θ of $T^*Z \otimes K^{-1/2}$ which is compatible with τ and restricts to a non-zero form on each fibre of Z,*
- *a scalar-flat Kähler metric in the conformal class is defined by a holomorphic section s of $K^{-1/2}$ compatible with τ and non-zero on each fibre of Z,*
- *a hypercomplex structure in the conformal class is defined by a holomorphic projection $\pi : Z \to \mathbf{C}P^1$ compatible with τ, and for which π is an isomorphism on each fibre of Z.*

Here $K^{-1/2}$ is a distinguished square root of the canonical bundle of Z.

Proof. The proofs are all in the literature. Note that there are always three aspects: a holomorphic object, a reality condition and a non-degeneracy condition.

For a proof in the case of Einstein metrics, see [35],[18],[8]. The scalar curvature of the Einstein metric is given by $\theta \wedge d\theta$ interpreted as a section of $\Lambda^3 T^* \otimes K^{-1} \cong \mathcal{O}$, i.e., a (necessarily constant) holomorphic function on Z. If the scalar curvature vanishes, then we have $\theta \wedge d\theta = 0$, which is the Frobenius integrability condition for the holomorphic distribution defined by the twisted 1-form θ. The non-degeneracy condition implies that a real twistor line is transversal to the resulting codimension-1 foliation, and so (at least in a neighbourhood of the line) Z fibres over the twistor line. This is the link with the hypercomplex (in fact hyperkähler) structure as viewed at the twistor space level.

The proof for scalar-flat Kähler metrics is due to Pontecorvo [31]. The zero-set of the section s of $K^{-1/2}$ is a divisor which meets each fibre of Z in two antipodal points. These points define the complex structure I and its conjugate structure $-I$ on M.

The case of hypercomplex manifolds can be found in [9]. Geometrically, the twistor space of a hypercomplex and a hyperkähler manifold look very similar, both fibring over $\mathbf{C}P^1$. The difference is the line bundle $\pi^*\mathcal{O}(1)$, which is arbitrary for a hypercomplex manifold, and is isomorphic to $K^{-1/2}$ in the hyperkähler case. In the latter case, the fibre $\pi^{-1}(z)$ for $z \in \mathbf{C}P^1$ is a divisor of a section of $K^{-1/2}$ and thus defines a scalar-flat Kähler metric.

3. The action of $SU(2)$

Our anti-self-dual manifold M will now be assumed to admit an action of the Lie group $SU(2)$, preserving the conformal structure, and with 3-dimensional orbits. Differentiably, the manifold is thus locally a product

$$M \cong (a, b) \times SU(2)/\Gamma$$

for some interval $(a, b) \subset \mathbf{R}$ and finite subgroup $\Gamma \subset SU(2)$. We may take the conformal structure to be defined by an invariant metric and thus on each orbit it is a left invariant metric, and hence given by an inner product $B(t)$ on the Lie algebra $\mathfrak{g}$ of $SU(2)$, for each $t \in (a, b)$. Taking a unit vector field normal to the orbits, we can write the metric in the form

$$g = f(t)dt^2 + B(t).$$

Using a standard orthonormal basis $\{\sigma_1, \sigma_2, \sigma_3\}$ of $\mathfrak{g}$, $B(t) = \sum_{l,m=1}^{3} b_{lm}(t)\sigma_l\sigma_m$. If the matrix b_{lm} can be chosen to be diagonal for all t, then we say that g has *diagonal form:*

$$g = f(t)dt^2 + a(t)^2\sigma_1^2 + b(t)^2\sigma_2^2 + c(t)^2\sigma_3^2. \tag{1}$$

A useful characterization of diagonal metrics is provided by the following:

Proposition 1. *Let M be a Riemannian four-manifold with a free isometric action of $SU(2)$. Its metric can be put in diagonal form if and only if there is a free isometric action of the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ which commutes with $SU(2)$ and preserves each orbit.*

Proof. First suppose g has diagonal form. Since the action is free, the orbits are isomorphic to $SU(2)$, so we have a product decomposition

$M \cong (a,b) \times SU(2)$. The basis $\{\sigma_1, \sigma_2, \sigma_3\}$ with respect to which the metric is diagonal identifies $SU(2)$ with the group of unit quaternions. Consider the right action of $G = \{\pm 1, \pm i, \pm j, \pm k\}$. Combined with the left action which leaves the metric invariant, we have the conjugation action, and i acts on the Lie algebra by conjugation as

$$(\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, -\sigma_2, -\sigma_3)$$

which clearly preserves the metric (1), as do the corresponding actions of j and k.

Conversely, given the action of G on $M \cong (a,b) \times SU(2)$, since it commutes with the left action and preserves the orbits, it is obtained by right multiplication. The conjugation action then fixes both the form $B(t)$ and the Killing form. As above, G acts through the adjoint representation as $\mathbf{Z}_2 \times \mathbf{Z}_2$. But if P, Q are commuting rotations with $P^2 = Q^2 = 1$, then the axes of P, Q and PQ are orthogonal. These are therefore orthogonal with respect to both $B(t)$ and the Killing form, and thus diagonalize $B(t)$. Since P, Q are independent of t, the metric is diagonalized.

We turn now to the specific case where M is anti-self-dual. Since $SU(2)$ preserves the conformal structure on M, its natural lifting to an action on the twistor space Z preserves the complex structure, the real structure τ and the twistor lines. In particular, each element of the Lie algebra $\mathfrak{g}$ defines a holomorphic vector field on Z, so if we denote by $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$, we have a homomorphism of holomorphic vector bundles

$$\alpha : \mathcal{O} \otimes \mathfrak{g}^c \to TZ. \tag{2}$$

Proposition 2. *Let M be an $SU(2)$-invariant anti-self-dual four manifold with 3-dimensional orbits. Then the following hold:*

1. *The image of α is a subsheaf of rank 2 or 3.*
2. *If the rank is 2, then M is locally hypercomplex.*

Proof. 1. The group $SU(2)$ has 3-dimensional orbits on M, and hence its lifting to the bundle $p : Z \to M$ must also have 3-dimensional orbits. Now for $z \in Z$ let

$$\mathfrak{h} = \ker \alpha_z \subset \mathfrak{g}^c.$$

Since

$$\dim(\mathfrak{h} \cap \overline{\mathfrak{h}}) \geq \dim \mathfrak{h} + \dim \overline{\mathfrak{h}} - \dim \mathfrak{g}^c,$$

if $\dim \mathfrak{h} \geq 2$, then $\mathfrak{h} \cap \overline{\mathfrak{h}}$ is non-zero, and contained in the real Lie algebra $\mathfrak{g}$. But since $SU(2)$ has zero-dimensional stabilizers in its action on Z, its Lie algebra maps *injectively* to the tangent space T_z. Hence $\dim \mathfrak{h} \leq 1$, and the image sheaf has rank 2 or 3.

2. If the rank is 2, then $1 \leq \dim \mathfrak{h} < 2$, from the proof of (1), so $\dim \mathfrak{h} = 1$ for all points $z \in Z$. The image is therefore a rank-2 subbundle $E \subset TZ$. Moreover, since $\mathfrak{g}$ is a Lie algebra, E satisfies the Frobenius integrability condition, and defines a foliation. Note that if the $SU(2)$-action can be integrated to an $SL(2,\mathbf{C})$-action, then the leaves of the foliation are just the orbits of $SL(2,\mathbf{C})$.

We shall prove that the foliation is always transverse to a real twistor line. First we use the non-holomorphic decomposition of the tangent bundle of Z into horizontal and vertical parts, using a metric which is preserved by the action. Define

$$\alpha^H : Z \times \mathfrak{g}^c \to (TZ)^H$$

to be the horizontal component of each holomorphic vector field from $\mathfrak{g}^c$. Now let $\tilde{X}$ be the lifting to Z of a vector field X on M generated from the $SU(2)$ action, that is, $\tilde{X}_z = \alpha_z(a)$ for $a \in \mathfrak{g}$. By definition,

$$dp(\tilde{X}) = dp(\tilde{X}^H) = X,$$

so since the orbits on M are 3-dimensional, X is non-vanishing at each point and thus so is $\tilde{X}^H$. In other words, α^H is injective on $\mathfrak{g} \subset \mathfrak{g}^c$.

We now repeat the argument in the proof of (1) using the kernel of α^H instead of the kernel of α, and deduce that the image of α^H is a complex (not necessarily holomorphic) subbundle of TZ of rank 2.

This however means that α maps surjectively onto the normal bundle of the fibre of $p : Z \to M$. Thus the leaves of the foliation meet a real twistor line transversely.

Using this, we can, in a neighbourhood of the line, identify the quotient space of the foliation with the line and obtain a holomorphic fibration of this neighbourhood over $\mathbf{C}P^1$. From Theorem 1 this implies that there is a local hypercomplex structure in the conformal class.

We can go further and completely classify the rank 2 case in Proposition 2:

Theorem 2. *Let M be an $SU(2)$-invariant anti-self-dual four manifold with 3-dimensional orbits and such that the rank of α is 2. Then the conformal structure is either*

1. *conformally flat, or*
2. *the Eguchi-Hanson metric, or*
3. *a member of the Belinskii-Gibbons-Page-Pope family.*

Proof. From Proposition 2, we know that there is a hypercomplex structure. Moreover, the orbits of the $SU(2)$ action on the twistor space are contained in the fibres of the holomorphic projection $\pi : Z \to \mathbf{C}P^1$. This means that $SU(2)$ preserves each complex structure I or J or K of the hypercomplex family. Using the conformal structure, choose an invariant vector field normal to the orbits. Then integrating the vector field, we have a product decomposition $M \cong (a,b) \times SU(2)/\Gamma$ with local coordinates (t, x_1, x_2, x_3) such that the normal vector field is $\partial/\partial t$. Now define the vector fields X_1, X_2, X_3 by

$$X_1 = I\frac{\partial}{\partial t}, \qquad X_2 = J\frac{\partial}{\partial t}, \qquad X_3 = K\frac{\partial}{\partial t}. \tag{3}$$

The complex structure I defines the two vector fields Z_1, Z_2 of type $(1,0)$:

$$Z_1 = \frac{\partial}{\partial t} - iX_1, \qquad Z_2 = X_2 - iX_3,$$

and its integrability is equivalent to

$$[Z_1, Z_2] = a_1 Z_1 + b_1 Z_2 \tag{4}$$

for functions a_1, b_1. But I, J, K are compatible with the conformal structure, so X_1, X_2, X_3 are tangent to the $SU(2)$ orbits. Also, since I, J, K and $\partial/\partial t$ are invariant, these are invariant vector fields on the orbit, hence elements of the Lie algebra $\mathfrak{g}$. Expanding (4) then gives

$$\frac{dX_2}{dt} - i\frac{dX_3}{dt} - i[X_1, X_2] - [X_1, X_3] = b_1(t)(X_2 - iX_3),$$

and together with the similar expressions giving the integrability of J

and K, we obtain

$$\frac{dX_1}{dt} + [X_2, X_3] = b(t)X_1,$$
$$\frac{dX_2}{dt} + [X_3, X_1] = b(t)X_2, \tag{5}$$
$$\frac{dX_3}{dt} + [X_1, X_2] = b(t)X_3.$$

Now put

$$\tilde{X}_i = f(t)X_i \qquad \textit{where} \qquad f(t) = \exp(-\int^t b(u)du)$$

and (cf [21])

$$s(t) = -\int^t f(v)dv,$$

then the equations (5) become *Nahm's equations*

$$\frac{d\tilde{X}_1}{ds} = [\tilde{X}_2, \tilde{X}_3],$$
$$\frac{d\tilde{X}_2}{ds} = [\tilde{X}_3, \tilde{X}_1], \tag{6}$$
$$\frac{d\tilde{X}_3}{ds} = [\tilde{X}_1, \tilde{X}_2].$$

By a rotation, these equations for the Lie algebra of $SU(2)$ can be solved by setting

$$\tilde{X}_1 = f_1 i, \qquad \tilde{X}_2 = f_2 j, \qquad \tilde{X}_3 = f_3 k,$$

giving equations equivalent to Euler's equations for a spinning top:

$$f_1' = 2f_2 f_3, \qquad f_2' = 2f_3 f_1, \qquad f_3' = 2f_1 f_2.$$

The conformal structure is determined from the condition that $\partial/\partial t, X_1, X_2, X_3$ have the same length. Using the substitutions above, it can then be written as

$$ds^2 + f_1^2\sigma_1^2 + f_2^2\sigma_2^2 + f_3^2\sigma_3^2.$$

Euler's equations have two integrals $f_1^2 - f_2^2 = a$ and $f_1^2 - f_3^2 = b$ so that

$$f_1' = 2\sqrt{(f_1^2 - a)(f_1^2 - b)}.$$

The general solution is an elliptic function, and gives the Belinskii-Gibbons-Page-Pope metrics of [7]. When $a = 0$ and $b \neq 0$ one obtains the Eguchi-Hanson metric ([12],[10]), and when $a = b = 0$, this is conformally flat. In all these cases there is a hyperkähler metric in the conformal equivalence class. This actually follows from Ashtekar's approach [2] since the invariant vector fields $\tilde{X}_i$ on $SU(2)/\Gamma$ are volume-preserving.

Having dispensed with the rank-2 case, we shall now proceed to the more general case where α is generically an isomorphism.

4. Isomonodromic deformations

We assume from now on that the image sheaf of

$$\alpha : \mathcal{O} \otimes \mathfrak{g}^c \to TZ$$

has rank 3. This means that on the complement of an analytic subset of Z, α is an isomorphism of holomorphic vector bundles. It fails to be an isomorphism where $\Lambda^3\alpha = 0$. But $\Lambda^3\alpha \in H^0(Z, Hom(\Lambda^3\mathfrak{g}^c, \Lambda^3 T)) \cong H^0(Z, K^{-1})$ is a section of the anticanonical bundle K^{-1}. Its divisor is always non-empty, since it has degree 4 on each twistor line. This is because the normal bundle of a twistor line is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and its tangent bundle $\mathcal{O}(2)$, so on any twistor line we have

$$K^{-1} \cong \Lambda^2 N \otimes \mathcal{O}(2) \cong \mathcal{O}(4).$$

Note that in the case that the $SU(2)$ action extends to a holomorphic $SL(2,\mathbf{C})$ action, then Y is simply the union of lower-dimensional orbits.

Since α is compatible with the real structure, $\Lambda^3\alpha$ is real, and is either identically zero on the line, vanishes with multiplicity two at a pair of antipodal points, or vanishes non-degenerately at four points, forming antipodal pairs. The first case cannot hold for all lines, so for a generic line we have a non-trivial intersection.

Proposition 3. *Let M be an $SU(2)$-invariant anti-self-dual manifold with 3-dimensional orbits, and suppose the image sheaf of α has* *rank 3. The zero set of $\Lambda^3\alpha$ meets a generic real twistor line in two points if and only if the conformal structure of M admits an $SU(2)$-invariant scalar-flat Kähler metric.*

Proof. Let Y be the zero-set of $s = \Lambda^3\alpha$, and suppose that s vanishes non-degenerately at $z \in Y \subset Z$. Consider all the twistor lines passing through z. The tangents of these span the tangent space T_zZ. (One way to see this is to blow up z. The lines through z then lift to lines with normal bundle $N \otimes \mathcal{O}(-1) \cong \mathcal{O} \oplus \mathcal{O}$, and the resulting two-parameter family meets the exceptional divisor in an open set (cf. [18])). Thus s vanishes non-degenerately on a generic twistor line. Those that meet Y tangentially form an analytic subset, but this cannot contain the real points M of the full holomorphic family. Hence, if s is non-degenerate, a generic real twistor line meets Y in four points.

Thus, if Y meets a generic line in two points, the divisor of s must have non-trivial multiplicity. If $\tilde{D}$ is the multiple divisor, then $\tilde{D} = 2D$ where D is a divisor of $K^{-1/2}$, since $K^{-1/4}$, (which exists locally), is a quaternionic bundle and has no real sections. But D is $SU(2)$-invariant, and thus from Theorem 1 it defines an $SU(2)$-invariant scalar-flat Kähler metric.

Conversely, such a metric is defined by an $SU(2)$-invariant section of $K^{-1/2}$. Since its divisor D contains the orbits of this group, the homomorphism α has rank 2 on D, and so D is a component of the divisor $\tilde{D}$ of $\Lambda^3\alpha$. Now consider the complementary divisor $D' = \tilde{D} - D$ of $K^{-1} \otimes K^{1/2} \cong K^{-1/2}$. This is also invariant, and if $D' \neq D$ we have a pencil of such invariant divisors. But this implies that α is of rank 2 on an open set, contradicting the hypothesis on α. Thus $D' = D$ and $\tilde{D} = 2D$, and a generic twistor line meets the zero-set Y in two points.

Example. To illustrate the different cases, consider the example of S^4 and its twistor space $\mathbf{C}P^3$. A conformal transformation of the sphere induces the action of an element of $PGL(2,\mathbf{H}) \subset PGL(4,\mathbf{C})$ on $\mathbf{C}P^3$, and thus an $SU(2)$ action is a quaternionic representation of dimension 2. Let V be the standard one-dimensional quaternionic representation space $SU(2) \cong Sp(1)$ and let 1 denote the trivial 1-dimensional representation. There are then three 2-dimensional quaternionic representations:

(1) $V \oplus 1$, (2) $V \oplus V$, (3) S^3V.

In case 1 the generic orbit of the corresponding $SL(2,\mathbf{C})$ action on $\mathbf{C}P^3$ is two-dimensional, so α has rank 2. In case 2, a generic orbit is

3-dimensional, and thus the rank of α is 3, and the divisor Y is given by $Y = \{(v,w) \in V \oplus V : \omega(v,w) = 0\}$ where ω is the skew form on V invariant by $SL(2,\mathbf{C})$. This is a quadric in $\mathbf{C}P^3$, and so Y meets a line (a twistor line in this case) in two points. For the final case, it is best to think of this as the action of $SL(2,\mathbf{C})$ on the space of homogeneous cubic polynomials $p(z_0, z_1)$. The generic orbit is three-dimensional, so the rank of α is 3, and the divisor Y is the discriminant locus of cubics $q(z_1/z_0) = z_0^{-3} p(z_0, z_1)$ with a repeated root. This is a singular quartic surface, which meets a generic twistor line in 4 points.

Note that if we consider the above action as $PSL(2,\mathbf{C})$ acting on cubics $q(z)$ by Möbius transformations, the stabilizer Γ of a generic cubic is the group of permutations of the three roots, i.e.,

$$\Gamma \cong S_3, \tag{7}$$

the symmetric group on three letters.

Remark. The case of $SU(2)$-invariant scalar flat Kähler metrics has been studied in [30] and [11]. It leads to Painlevé's third equation. Although this is close in context to the current paper, we shall not investigate it further, apart from considering the Einstein metrics in such a conformal class later on. On the complement of Y, α is an isomorphism. We set

$$A = \alpha^{-1} : TZ \to \mathcal{O} \otimes \mathfrak{g}^c; \tag{8}$$

thus A is a holomorphic 1-form with values in $\mathfrak{g}^c$, and can therefore be considered as a connection on the trivial bundle.

Proposition 4. *The connection A on $Z \backslash Y$ is flat.*

Proof. We need to show that the curvature $dA + A^2$ vanishes, and it is sufficient to evaluate this 2-form on vectors of the form $\alpha(a)$ for $a \in \mathfrak{g}^c$, since they span the tangent space at each point. Now

$$dA(\alpha(a), \alpha(b)) = \alpha(a) \cdot A(\alpha(b)) - \alpha(b) \cdot A(\alpha(a)) - A([\alpha(a), \alpha(b)]),$$

but $A(\alpha(a)) = a$ and $A(\alpha(b)) = b$ are constant, so the right-hand side is

$$-\alpha^{-1}[\alpha(a), \alpha(b)] = -[a, b]. \tag{9}$$

since α is a Lie algebra homomorphism from $\mathfrak{g}^c$ to the vector fields on Z. But

$$A^2(\alpha(a),\alpha(b)) = A(\alpha(a))A(\alpha(b)) - A(\alpha(b))A(\alpha(a)) = [a,b],$$

which, together with (9), gives the result.

Remark. If the $SU(2)$ action extends to a holomorphic $SL(2,\mathbf{C})$ action, $Z\backslash Y$ is an open orbit $SL(2,\mathbf{C})/\Gamma$ for some subgroup Γ. The connection A is then simply the Maurer-Cartan form $A = -dgg^{-1}$, and the holonomy group of the flat connection is Γ. We cannot necessarily make that global assumption here, however. If H is the holonomy of A, then the assumption requires at the very least that $SL(2,\mathbf{C})/H$ should be a manifold.

The connection A becomes singular on Y. In the generic case where s is non-degenerate, the connection has a *logarithmic singularity*. This means that in any gauge, its connection matrix has a simple pole whose residue vanishes as a 1-form restricted to Y. It thus defines a connection along Y.

Proposition 5. *If $\Lambda^3\alpha$ vanishes non-degenerately on the divisor Y, then the connection $A = \alpha^{-1}$ has a logarithmic singularity along Y.*

Proof. In local coordinates, α is represented by a holomorphic function $B(z)$ with values in the space of 3×3 matrices. The divisor Y is then the zero set of $\det B$. If $\det B$ has a non-degenerate zero at $z \in Y$, then its null-space is one-dimensional at z, so the kernel of α is one-dimensional.

Now for any square matrix B, let $B^\vee$ denote the transpose of the matrix of cofactors. Then it is well-known that

$$BB^\vee = (\det B)I.$$

Hence in local coordinates

$$A = \alpha^{-1} = \frac{B^\vee}{\det B},$$

and so A has a simple pole along Y. We need to show that the residue restricts to zero as a 1-form on Y. For this, consider the invariant description of $B^\vee$. We have on Z

$$\Lambda^2\alpha : \Lambda^2\mathfrak{g}^c \to \Lambda^2 T,$$

and using the identifications $\Lambda^2\mathfrak{g}^c \cong \mathfrak{g}^{c*}$ and $\Lambda^2 T \cong T^* \otimes \Lambda^3 T$, $B^\vee$ represents the dual map of $\Lambda^2\alpha$:

$$(\Lambda^2\alpha)^* : T \to \mathfrak{g}^c \otimes \Lambda^3 T.$$

Now the image of α_z is the tangent space $T_z Y$ by the definition of α. Thus the image of $\Lambda^2\alpha_z$ is $\Lambda^2 T_z Y$ which means that $(\Lambda^2\alpha)^*$ annihilates TY, which is the required result.

The twistor space Z therefore has a meromorphic connection with a simple pole along the smooth part of Y. We can restrict it to a generic twistor line to obtain a flat connection on $\mathbf{C}P^1\backslash\{z_1, z_2, z_3, z_4\}$, the four points z_i being the intersection with Y. The essential fact about this connection concerns its holonomy group:

Proposition 6. *Let M_0 be a connected family of twistor lines in Z, each of which meets Y transversally. Then the holonomy representation of A restricted to each line is the same.*

Proof. Let

$$\pi_1(Z\backslash Y) \to SL(2, \mathbf{C})$$

be the holonomy representation of A. Then restricted to a twistor line we obtain

$$\pi_1(\mathbf{C}P^1\backslash\{z_1, \ldots, z_4\}) \to \pi_1(Z\backslash Y) \to SL(2, \mathbf{C}).$$

In a connected family, the homotopy class of the inclusion of the punctured projective line in $Z\backslash Y$ is unchanged, so the first homomorphism is independent of the twistor line in the family. Since the second homomorphism is fixed, the holonomy is unchanged.

Such a family of connections is called an *isomonodromic deformation* (see eg. [26]). The more general problem is to consider a $GL(m,\mathbf{C})$ connection on the trivial bundle over $\mathbf{C}P^1$ with simple poles at $z_1, \ldots, z_n$:

$$A = \sum_{i=1}^{n} \frac{A_i dz}{z - z_i},$$

An isomonodromic deformation $A_i(z_1, \ldots, z_n)$ for $(z_1, \ldots, z_n) \in U \subset \mathbf{C}^n$ is a family with constant holonomy. It necessarily satisfies the Schlesinger equation [26]:

$$dA_i + \sum_{j \neq i} [A_i, A_j] \frac{dz_i - dz_j}{z_i - z_j} = 0. \tag{10}$$

In our case, we have four points $z_1, \ldots, z_4$, and the $A_i \in \mathfrak{g}^c$ are trace-free 2×2 matrices. By a projective transformation we can make these points $0, 1, x, \infty$. Then

$$A(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-x},$$

and Schlesinger's equation becomes:

$$\begin{aligned} \frac{dA_1}{dx} &= \frac{[A_3, A_1]}{x}, \\ \frac{dA_2}{dx} &= \frac{[A_3, A_2]}{x-1}, \\ \frac{dA_3}{dx} &= \frac{-[A_3, A_1]}{x} - \frac{[A_3, A_2]}{x-1}, \end{aligned} \tag{11}$$

where the last equation is equivalent to

$$A_1 + A_2 + A_3 = -A_4 = \mathit{const}.$$

It is the differential equation (11) which is the analytical key to finding the $SU(2)$-invariant anti-self-dual conformal structures. It is applicable so long as we can show that the cross-ratio x of the four points $z_1, \ldots, z_4$ varies as the twistor line varies in its 4-parameter family.

Proposition 7. *Let M_0 be a connected family of twistor lines P in Z, each of which meets Y transversally. Then the cross-ratio of the four points $\{z_1, z_2, z_3, z_4\} = P \cap Y$ is a nonconstant function.*

Proof. From the proof of Proposition 5, at a smooth point $z \in Y$, the image of α_z is $T_z Y$. Since P meets Y transversally, this means that $\alpha : Z \times \mathfrak{g}^c \to TZ$ always maps *onto* the normal bundle to P. We therefore have a surjective homomorphism of holomorphic vector bundles

$$\beta : \mathcal{O} \otimes \mathfrak{g}^c \to N.$$

Since β is surjective, its kernel is a line bundle of degree $-\deg N = -2$, so we have an exact sequence of sheaves:

$$\mathcal{O}(-2) \to \mathcal{O} \otimes \mathfrak{g}^c \to N.$$

Under α, the kernel maps isomorphically to the sheaf of sections of the tangent bundle TP which vanish at the four points $P \cap Y$. From the long exact cohomology sequence we have

$$0 \to \mathfrak{g}^c \to H^0(P, N) \xrightarrow{\delta} H^1(P, \mathcal{O}(-2)) \to 0,$$

and since $H^0(P, N)$ is 4-dimensional and $\mathfrak{g}^c$ is 3-dimensional, the map δ is surjective. But $\alpha\delta$ is the Kodaira-Spencer map for deformations of the four points on P, so since it is non-trivial, the cross-ratio is non-constant.

The above provides a useful local parametrization of the twistor lines, and hence of the manifold M. The $SU(2)$ action on Z maps one twistor line P_1 biholomorphically to another P_2. Since the divisor Y is invariant, then the action makes the points of intersection correspond, and by projective invariance, the cross-ratio x is constant on each orbit, and from Proposition 7 is a non-constant function on the space of twistor lines M. The function x is real because the four points $P \cap Y$ on a real twistor line occur in antipodal pairs.

The holonomy representation is fixed by Z itself. A twistor line defines a connection with the same holonomy, determined by the triple

$$A_1(x), \qquad A_2(x), \qquad A_3(x), \tag{12}$$

which, as x varies, is a solution to the Schlesinger equation (11). By the $SU(2)$-invariance of the connection A, if $g \in SU(2)$, then

$$(g)^*(A) = g^{-1}Ag,$$

thus the action of $SU(2)$ on the triples is described by

$$A_i(x) \mapsto g^{-1}A_i(x)g. \tag{13}$$

We can now identify a point of M with the triple. This will be convenient in the next section for determining the conformal structure.

5. Residues and conformal structures

The connection

$$A = \sum_{i=1}^{4} \frac{A_i dz}{z - z_i}$$

on a twistor line has residues $A_1, \dots, A_4$ at the four points of intersection with Y. The invariants

$$\operatorname{tr} A_1^2, \qquad \operatorname{tr} A_2^2, \qquad \operatorname{tr} A_3^2, \qquad \operatorname{tr} A_4^2$$

play an important role both in the theory of isomonodromic deformations and in the geometrical interpretation here. Note that from Schlesinger's equation (10),

$$d(\operatorname{tr} A_i^2) = -2\sum_{j\neq i} \operatorname{tr}(A_i[A_i, A_j])\frac{dz_i - dz_j}{z_i - z_j} = 0,$$

so that the invariants are independent of z_1, z_2, z_3, z_4.

The real structure τ imposes relations on the invariants. Recall that τ arises from the antipodal map $z \mapsto -1/\bar{z}$ on the twistor line $P \cong \mathbf{C}P^1$ and, since we have an action of $SU(2)$, the real structure $a \mapsto -a^*$ on the Lie algebra $\mathfrak{g}^c$ of $SL(2,\mathbf{C})$. Thus, if $P \cap Y = \{z_1, -1/\bar{z}_1, z_2, -1/\bar{z}_2\}$, the connection A restricted to P can be written

$$A = \frac{A_1 dz}{z - z_1} + \frac{A_2 dz}{z + 1/\bar{z}_1} + \frac{A_3 dz}{z - z_2} + \frac{A_4 dz}{z + 1/\bar{z}_2},$$

and for reality we must have

$$(14) \qquad A_2 = -A_1^*, \qquad A_4 = -A_3^*.$$

This means in particular, that

$$(15) \qquad \operatorname{tr} A_2^2 = \overline{\operatorname{tr} A_1^2}, \qquad \operatorname{tr} A_3^2 = \overline{\operatorname{tr} A_4^2}.$$

Since A is a meromorphic differential on $\mathbf{C}P^1$, the sum of the residues is zero, so we must also have

$$(16) \qquad A_1 - A_1^* + A_2 - A_2^* = 0.$$

The following two results show how the invariants can determine geometrical properties of the conformal structure.

Theorem 3. *Let M be an $SU(2)$-invariant anti-self-dual four-manifold with 3-dimensional orbits and $\Lambda^3\alpha$ non-degenerate. Then the conformal structure has diagonal form if and only if*

$$\operatorname{tr} A_1^2 = \operatorname{tr} A_2^2 = \operatorname{tr} A_3^2 = \operatorname{tr} A_4^2 = k \in \boldsymbol{R}.$$

(Note that once the invariants are assumed to be equal, the real structure (15) implies that they are real.)

Proof. Suppose first that the conformal structure has diagonal form. From Proposition 1, for each point $m \in M$, there is an isometric action of the group $D = \mathbf{Z}_2 \times \mathbf{Z}_2$ fixing m. It acts on the 3-dimensional space $(\Lambda^2_m)_+$ via the diagonal matrices in $SO(3)$ and so, apart from vectors which lie on the axes of these rotations, each orbit has length 4. Now from the description in Section 2, the unit sphere in this space is the twistor line P corresponding to m, and since D acts isometrically, it preserves the intersection $P \cap Y$. For a generic twistor line, this consists of four distinct points, which are not the axes, and so D acts transitively on the four points. Since D preserves the conformal structure, it preserves the connection A, and hence acts transitively on the residues, which must therefore have the same invariants $\operatorname{tr} A_i^2$.

Conversely, suppose the residues have the same invariant k. By Proposition 1, we need to find an action of the quaternion group G on the universal covering of M. For this, it is useful to think of a point m as a triple as in (12), or more conveniently with regard to the real structure, as a quadruple of matrices with zero sum:

$$A_1, \qquad -A_1^*, \qquad A_2, \qquad -A_2^*,$$

associated to antipodal pairs of points $z_1, -1/\bar{z}_1, z_2, -1/\bar{z}_2$ as above.

If we consider $\mathbf{C}P^1 \cong S^2 \subset \mathbf{R}^3$, the antipodal pairs define two lines L_1, L_2 through the origin. The perpendicular to the plane of L_1, L_2 and the two bisectors of the angles between them define three orthogonal axes in $\mathbf{R}^3$. Rotations of π about these axes define a subgroup $D \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ of $SO(3)$ which acts transitively on the four points. To obtain an action of G on M, we need to lift the action to $SU(2)/\pm 1$ acting by conjugation on the matrices associated to the points.

Now if $a = -a^*$ is in the Lie algebra $\mathfrak{g}$ of $SU(2)$, a rotation by π fixing a is given by

$$P(x) = 2\frac{\operatorname{tr}(xa)a}{\operatorname{tr}(a^2)} - x.$$

Consider the element of D which sends z_1 to $-1/\bar{z}_1$ and z_2 to $-1/\bar{z}_2$. We want to lift this to an action which interchanges the corresponding residues.

Let $a = (A_1 - A_1^*) \in \mathfrak{g}$, and consider the rotation P above. Then

$$\begin{aligned} P(A_1) &= 2\frac{\operatorname{tr}(A_1^2 - A_1A_1^*)}{\operatorname{tr}(A_1 - A_1^*)^2}(A_1 - A_1^*) - A_1 \\ &= 2\frac{(k - \operatorname{tr} A_1A_1^*)}{(2k - 2\operatorname{tr} A_1A_1^*)}(A_1 - A_1^*) - A_1 \\ &= -A_1^* \end{aligned}$$

and so P interchanges the residues A_1 and $-A_1^*$. On the other hand from (16),

$$A_1 - A_1^* = -(A_2 - A_2^*)$$

so that replacing A_1 by A_2 above gives the same rotation P. Thus P interchanges both pairs of residues.

If $A_1 - A_1^* = 0$ then $A_2 - A_2^* = 0$ again from (16), and a rotation of π about the axis through $[A_1, A_2] \in \mathfrak{g}$ interchanges the same pairs.

Now consider the problem of interchanging A_1 and A_2, i.e. a pair corresponding to non-antipodal points on the sphere. From (16),

$$A_1 + A_2 = A_1^* + A_2^*$$

so that $A_1 + A_2$ is self-adjoint. Putting $a = i(A_1 + A_2) \in \mathfrak{g}$, a similar argument to the above provides the required rotation.

The inverse image in $SU(2)$ of this lifting of the action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ to the space of residues gives us the action of G to which we can apply Proposition 1 and deduce that the conformal structure can be put in diagonal form.

Theorem 4. *Let M be an $SU(2)$-invariant anti-self-dual four-manifold with 3-dimensional orbits and $\Lambda^3\alpha$ non-degenerate. If the conformal structure admits an Einstein metric, then*

$$\operatorname{tr} A_1^2 = \operatorname{tr} A_2^2 = \operatorname{tr} A_3^2 = \operatorname{tr} A_4^2 = 1/8.$$

The following corollary is immediate from Theorems 3 and 4.

Corollary 1. *An $SU(2)$-invariant self-dual Einstein metric with 3-dimensional orbits and $\Lambda^3\alpha$ nondegenerate can be put in diagonal form.*

Remark. By differential geometric arguments, the diagonalizability in fact holds for *any* Einstein metric with 3-dimensional orbits (see e.g. [34])

Proof. Suppose the conformal structure admits an Einstein metric. Then by Theorem 1, the twistor space has an invariant twisted 1-form θ, a section of $T^*Z \otimes K^{-1/2}$. Composing $\alpha : \mathcal{O} \otimes \mathfrak{g}^c \to TZ$ with $\theta : TZ \to K^{-1/2}$ gives

$$\beta : \mathcal{O} \otimes \mathfrak{g}^c \to K^{-1/2},$$

which, using the Killing form, we may consider as a section of the bundle $K^{-1/2} \otimes \mathfrak{g}^c$.

Now let $s = \Lambda^3\alpha$, a section of K^{-1}. We define

$$\phi = \beta s^{-1/2},$$

so that we obtain a 2-valued function $\phi : Z \to \mathfrak{g}^c$, branched around Y. More importantly, it is $SU(2)$-invariant. Now consider the connection $A = \alpha^{-1}$ and its covariant derivative ∇. By definition,

$$\nabla_{\alpha(a)}\phi = \alpha(a) \cdot \phi + [a, \phi].$$

But $SU(2)$-invariance of ϕ means that

$$\alpha(a) \cdot \phi + [a, \phi] = 0$$

for all $a \in \mathfrak{g}$. Thus ϕ is actually covariant constant with respect to the connection A.

Restrict to a twistor line, and suppose s vanishes at $z = 0$. Since s is assumed to vanish non-degenerately, ϕ has the local form

$$\phi(z) = z^{-1/2}\psi(z),$$

where $\psi(z)$ is holomorphic. Since ϕ is covariant constant, we have

$$\frac{d\phi}{dz} + [A, \phi] = 0,$$

and hence

$$-\frac{\psi}{2z} + \frac{d\psi}{dz} + [A, \psi] = 0.$$

But if A_i is the residue of A at $z = 0$, and ψ_0 is the first non-vanishing coefficient in the power series expansion of ψ, then

$$[A_i, \psi_0] = \frac{1}{2}\psi_0.$$

This implies that the eigenvalues of A_i are $\pm 1/4$ and hence $\operatorname{tr} A_i^2 = 1/8$.

We shall next derive an analytical form for the conformal structure in terms of the matrices A_1, A_2, A_3. Recall from Section 2 that the null cone of the structure consists of the sections of the normal bundle to a twistor line which vanish at some point.

First we choose a holomorphic 1-parameter family of twistor lines such that the cross-ratio x varies in $U \subset \mathbf{C}$:

$$f : \mathbf{C}P^1 \times U \to Z,$$

and parametrized such that $f(0), f(1), f(x), f(\infty)$ are the points of intersection with Y. Pulling back the connection A on Z, we obtain a connection on $\mathbf{C}P^1 \times U$ with a pole along the divisors $z = 0, z = 1, z = x, z = \infty$. Because the singularity is logarithmic, its residue vanishes on each divisor, and so gives a well-defined connection there. By parallel translation along the divisors, there is a trivialization such that

$$f^*(A) = A_1 \frac{dz}{z} + A_2 \frac{dz}{z-1} + A_3 \frac{dz - dx}{z - x}. \tag{17}$$

(This is the origin of the Schlesinger equation (10), which is essentially a restatement of the flatness of this connection (see [26]).)

Now recall that $A = \alpha^{-1}$ where $\alpha : \mathcal{O} \otimes \mathfrak{g}^c \to TZ$ identifies the tangent bundle to the twistor space with the trivial bundle $Z \times \mathfrak{g}^c$ on $Z \backslash Y$. We shall use this identification to describe tangent vectors, and determine the conformal structure by only considering the generic sections of the normal bundle which vanish outside Y. The formula (17) gives the tangent vector to a twistor line

$$f_*(\partial/\partial z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-x} = A(z) \tag{18}$$

and the vector field along the twistor line tangent to the deformation parametrized by x:

$$f_*(\partial/\partial x) = -\frac{A_3}{z-x}. \tag{19}$$

The complexified tangent space to a twistor line is defined by the sections of the normal bundle N which are the images of linear combinations of the tangent vector fields $f_*(\partial/\partial x)$ and $a \in \mathfrak{g}^c$. A section of N vanishes if the vector field is tangent to the twistor line, so from (18),(19) the null cone is defined by the condition

$$a - \xi \frac{A_3}{z - x} = \lambda A(z) \tag{20}$$

for some $z \in \mathbf{C}P^1$ and $\lambda \in \mathbf{C}$.

The null cone is given by an expression of the form

$$c\xi^2 + \xi G(b, a) + G(a, a) = 0,$$

where G is a non-degenerate symmetric form on $\mathfrak{g}^c$. To evaluate G, note that if $\xi = 0$, then for all z, $a = A(z)$ satisfies (20), thus $G(A(z), A(z)) = 0$ for all $z \in \mathbf{C}$. Equating coefficients, this gives

$$\begin{gathered} G(A_1, A_1) = 0, \qquad G(A_2, A_2) = 0, \qquad G(A_3, A_3) = 0, \\ G(A_1, A_2) = \gamma\, G(A_2, A_3) = (x - 1)\gamma\, G(A_3, A_1) = -x\gamma \end{gathered} \tag{21}$$

for some γ. If $\gamma = 0$, since G is nondegenerate, we must have the A_i all null and proportional. But then from (20) this is the unique null vector tangent to the complexified orbit, which is a contradiction.

Now substitute a defined by (20) into the equation of the null cone, and we obtain

$$0 = c\xi^2 + \xi G(b, \lambda A + \xi \frac{A_3}{z - x}) + G(\lambda A + \xi \frac{A_3}{z - x}, \lambda A + \xi \frac{A_3}{z - x})$$

for all ξ, λ and z. From the coefficients of ξ^2 and $G(A_3, A_3) = 0$ we obtain $c = 0$ and $G(b, A_3) = 0$. From the other coefficients we find

$$G(b, A_1) = -2\gamma, \qquad G(b, A_2) = 2\gamma.$$

Now from (21) we evaluate b:

$$b = \frac{A_1}{x} + \frac{A_2}{x - 1} + \frac{(2x - 1)A_3}{x(x - 1)}. \tag{22}$$

The conformal structure is thus defined by the metric

$$g = \xi G(b, a) + G(a, a).$$

From this expression, we see that the tangent vector field $X = \partial/\partial x - b$ is normal to the orbits, and

$$g(X, X) = -2G(b, b) + G(b, b) = -G(b, b) = -2/x(x-1)$$

from (22), where we have set γ, which is a non-zero factor of g, equal to 1. If $m(x)$ is an integral curve of X, we can parametrize the space of real twistor lines by $(x, h) \mapsto hm(x)$ for $h \in SU(2)$, and then the metric has the form

$$-\frac{2dx^2}{x(x-1)} + \tilde{G},$$

where $\tilde{G}$ is conjugate to G. Working with a general metric of this form is not easy, so suppose now that the metric can be put in diagonal form. Then the diagonal entries of $\tilde{G}$ are its eigenvalues relative to the Killing form, and since $\tilde{G}$ and G are conjugate, these are the same. If B denotes the matrix of the quadratic form $\operatorname{tr} a^2$, then we need to determine the roots of

$$\det(G - \lambda B) = 0.$$

Now the matrix $G_{ij} = G(A_i, A_j)$ is given from (21) by

$$G = \begin{pmatrix} 0 & 1 & -x \\ 1 & 0 & x-1 \\ -x & x-1 & 0 \end{pmatrix},$$

which is non-singular, so the A_i are linearly independent and we can use them as a basis. In this basis, $B_{ij} = \operatorname{tr} A_i A_j$. But from Theorem 3, if the metric is diagonal, $\operatorname{tr} A_i^2 = k$ for all residues, thus

$$B = \begin{pmatrix} k & \operatorname{tr} A_1 A_2 & \operatorname{tr} A_3 A_1 \\ \operatorname{tr} A_1 A_2 & k & \operatorname{tr} A_2 A_3 \\ \operatorname{tr} A_3 A_1 & \operatorname{tr} A_2 A_3 & k \end{pmatrix},$$

where moreover, $\operatorname{tr} A_4^2 = \operatorname{tr}(A_1 + A_2 + A_3)^2 = k$, and so

$$\operatorname{tr} A_1 A_2 + \operatorname{tr} A_2 A_3 + \operatorname{tr} A_3 A_1 = -k.$$

Now if $\sigma_1 = x_1 + x_2 + x_3$, the following identity is easily proven:

$$\det\begin{pmatrix}-\sigma_1 & x_3 & x_2\\ x_3 & -\sigma_1 & x_1\\ x_2 & x_1 & -\sigma_1\end{pmatrix} = -2(x_1+x_2)(x_2+x_3)(x_3+x_1). \tag{23}$$

Using this, the roots of $\det(G-\lambda B)$ are:

$$\lambda = \frac{1}{k+\operatorname{tr}A_1A_2},\quad \lambda = \frac{x-1}{k+\operatorname{tr}A_2A_3},\quad \lambda = -\frac{x}{k+\operatorname{tr}A_3A_1},$$

and using the fact that the standard basis i, j, k of $\mathfrak{g}$ satisfies $\operatorname{tr} i^2 = \operatorname{tr} j^2 = \operatorname{tr} k^2 = -2$, we find the conformal structure in diagonal form;

$$g = \frac{dx^2}{x(x-1)} + \frac{\sigma_1^2}{k+\operatorname{tr}A_1A_2} + \frac{(x-1)\sigma_2^2}{k+\operatorname{tr}A_2A_3} - \frac{x\sigma_3^2}{k+\operatorname{tr}A_3A_1}. \tag{24}$$

Remark. If we set

$$\begin{aligned}\Omega_1^2 &= -(k+\operatorname{tr}A_1A_2), \quad \Omega_2^2 = (k+\operatorname{tr}A_2A_3),\\ \Omega_3^2 &= (k+\operatorname{tr}A_3A_1),\end{aligned} \tag{25}$$

then the conformal structure (24) is defined by

$$g_0 = \frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2},$$

and differentiating (25) with respect to x, we find, using the Schlesinger equation for isomonodromic deformations (11),

$$\begin{aligned}2\Omega_1\frac{d\Omega_1}{dx} &= \frac{\operatorname{tr}([A_1,A_2]A_3)}{x(x-1)},\\ 2\Omega_2\frac{d\Omega_2}{dx} &= -\frac{\operatorname{tr}([A_1,A_2]A_3)}{x},\\ 2\Omega_3\frac{d\Omega_3}{dx} &= \frac{\operatorname{tr}([A_1,A_2]A_3)}{x-1}.\end{aligned} \tag{26}$$

But in the Lie algebra of $SL(2,\mathbf{C})$, we have the identity

$$(\operatorname{tr}([A_1,A_2]A_3))^2 = -2\det\begin{pmatrix}\operatorname{tr}A_1^2 & \operatorname{tr}A_1A_2 & \operatorname{tr}A_3A_1\\ \operatorname{tr}A_1A_2 & \operatorname{tr}A_2^2 & \operatorname{tr}A_2A_3\\ \operatorname{tr}A_3A_1 & \operatorname{tr}A_2A_3 & \operatorname{tr}A_3^2\end{pmatrix},$$

so from the determinant expansion (23),

$$\begin{aligned}(\operatorname{tr}([A_1, A_2]A_3))^2 &= -2\det B\\ &= -4(k + \operatorname{tr} A_1A_2)(k + \operatorname{tr} A_2A_3)(k + \operatorname{tr} A_3A_1).\end{aligned}$$

Thus $\operatorname{tr}([A_1, A_2]A_3)$ is a distinguished square root of $4\Omega_1^2\Omega_2^2\Omega_3^2$, and then the equations (26) can be written as

$$\begin{aligned}\Omega_1' &= -\frac{\Omega_2\Omega_3}{x(1-x)},\\ \Omega_2' &= -\frac{\Omega_3\Omega_1}{x},\\ \Omega_3' &= -\frac{\Omega_1\Omega_2}{1-x}.\end{aligned} \tag{27}$$

These are the equations derived by Tod [34] directly from the curvature of an anti-self-dual diagonalizable metric. The expression $\Omega_1^2 - \Omega_2^2 - \Omega_3^2$ is a constant from (27). From the definition of the Ω_i, it is given by

$$\Omega_1^2 - \Omega_2^2 - \Omega_3^2 = -3k - \operatorname{tr} A_1A_2 - \operatorname{tr} A_2A_3 - \operatorname{tr} A_3A_1 = -2k. \tag{28}$$

In particular from Theorem 4, for an Einstein metric $k = 1/8$, and so $\Omega_1^2 - \Omega_2^2 - \Omega_3^2 = -1/4$.

6. The monodromy group

The holonomy of the flat connection on $Z\backslash Y$ is obtained by parallel translation around closed paths and defines, after fixing a base point b, a representation of the fundamental group

$$\rho : \pi_1(Z\backslash Y) \to SL(2, \mathbf{C}).$$

On a twistor line, the holonomy may also be considered as the effect of analytic continuation of solutions to the system of ordinary differential equations

$$\frac{df}{dz} + \sum_{i=1}^{4} \frac{A_i f}{z - z_i} = 0$$

around closed paths through b hence the use of the classical term *monodromy*, which we shall use interchangeably with *holonomy* henceforth.

Changing the basepoint to b' effects an overall conjugation (by the monodromy along a path from b to b') of the monodromy representation.

For the punctured twistor lines above, we obtain a representation of the group $\pi_1(S^2\backslash\{z_1,\dots,z_4\})$. This is a free group on 3 generators $\gamma_1,\gamma_2,\gamma_3$, where γ_i is a simple loop from b passing once around z_i. Moving b close to z_i, it is easy to see that $\rho(\gamma_i)$ is conjugate to

$$\exp(-2\pi i A_i).$$

The fact that the invariants $\operatorname{tr} A_i^2$ are independent of $z_1,\dots,z_4$ is now evident from the isomonodromic property of the monodromy around γ_i.

Just as for the residues, the real structure also has implications for the monodromy. The connection A on the trivial principal bundle $Z\times SL(2,\mathbf{C})$ is invariant under the real structure

$$(z,g)\mapsto(\tau z,g^{*-1}),$$

thus if γ is a closed loop through $z\in Z$ with holonomy $h_\gamma\in SL(2,\mathbf{C})$, then $\tau(\gamma)$ is a loop through $\tau(z)$ and

$$h_{\tau(z)}=h_z^{*-1}. \tag{29}$$

We need to consider the monodromy on a real twistor line, where the connection has poles at z_1,z_2,z_3,z_4. To this end we take two antipodal points N and S on $\mathbf{C}P^1\cong S^2$ and two orthogonal great circles through N. This gives four paths $\beta_1,\beta_2,\beta_3,\beta_4$ in cyclic order from N to S which divide the sphere into four segments each of which we may assume contains one of the singular points. Choosing N as a base point, the generators γ_i of the fundamental group $\pi_1(S^2\backslash\{z_1,\dots,z_4\})$ are

$$\gamma_1=\beta_1\beta_2^{-1},\quad \gamma_2=\beta_2\beta_3^{-1},\quad \gamma_3=\beta_3\beta_4^{-1},\quad \gamma_4=\beta_4\beta_1^{-1}.$$

Note the obvious relation

$$\gamma_1\gamma_2\gamma_3\gamma_4=1, \tag{30}$$

which of course is carried through to the monodromy.

Now applying the antipodal map τ, we have

$$\tau(\beta_1)=\beta_3^{-1},\quad \tau(\beta_2)=\beta_4^{-1},\quad \tau(\beta_3)=\beta_1^{-1},\quad \tau(\beta_4)=\beta_2^{-1},$$

and so

$$\begin{aligned}\tau(\gamma_1) &= \beta_3^{-1}\beta_4 = \beta_1^{-1}(\gamma_1\gamma_2\gamma_4)\beta_1,\\ \tau(\gamma_2) &= \beta_4^{-1}\beta_1 = \beta_1^{-1}(\gamma_1\gamma_2\gamma_3)\beta_1,\\ \tau(\gamma_3) &= \beta_1^{-1}\beta_2 = \beta_1^{-1}(\gamma_2\gamma_3\gamma_4)\beta_1,\\ \tau(\gamma_4) &= \beta_2^{-1}\beta_3 = \beta_1^{-1}(\gamma_1\gamma_3\gamma_4)\beta_1.\end{aligned} \tag{31}$$

Setting $\rho_i = \rho(\gamma_i)$ in the monodromy representation, and σ the monodromy along β_1, we obtain

$$\begin{aligned}\rho_1^{*-1} &= \sigma^{-1}(\rho_1\rho_2\rho_4)\sigma,\\ \rho_2^{*-1} &= \sigma^{-1}(\rho_1\rho_2\rho_3)\sigma,\\ \rho_3^{*-1} &= \sigma^{-1}(\rho_2\rho_3\rho_4)\sigma,\\ \rho_4^{*-1} &= \sigma^{-1}(\rho_1\rho_3\rho_4)\sigma.\end{aligned} \tag{32}$$

In a more analytical form which will be useful later, we take the great circle $\beta_1\beta_3^{-1}$, parametrized by $\theta \in [0, 2\pi)$. The antipodal map is then $\theta \mapsto \theta + \pi$, and the reality of the connection is expressed from (14) as

$$A(\theta + \pi) = -A^*(\theta). \tag{33}$$

Now let $M(\theta)$ be a fundamental matrix solution of

$$\frac{dM}{d\theta} + AM = 0. \tag{34}$$

Writing $\tilde{M}(\theta) = M(\theta + \pi)$, and using (33), we obtain

$$\frac{d\tilde{M}}{d\theta} - A^*\tilde{M} = 0.$$

But

$$\frac{d}{d\theta}(M^{*-1}) = -M^{*-1}\frac{dM^*}{d\theta}M^{*-1} = A^*M^{*-1}$$

from (34). Hence there is a constant matrix H such that

$$M^{*-1}(\theta) = M(\theta + \pi)H. \tag{35}$$

It follows that

$$M(\theta + 2\pi) = M^{*-1}(\theta + \pi)H^{-1} = M(\theta)H^*H^{-1},$$

and so the monodromy around the great circle is defined by the matrix H^*H^{-1}, and H itself defines the monodromy σ in (32).

The study of the monodromy will lead us to a complete description of a very important case of $SU(2)$-invariant anti-self-dual geometry: the Einstein metrics in this class. To begin, we shall prove the following theorem, which characterizes these metrics by their monodromy. In the succeeding chapters we shall use this information to derive analytical formulae for the metrics.

Theorem 5. *Let M be an $SU(2)$-invariant anti-self-dual Einstein manifold with scalar curvature R and such that its twistor space Z has $\Lambda^3\alpha$ non-degenerate. Then the monodromy of the connection A is conjugate to the representation defined by:*

- *If $R < 0$, $\rho_j = \begin{pmatrix} 0 & a_j \\ -a_j^{-1} & 0 \end{pmatrix}$ where*

$$a_1 = \lambda, \quad a_2 = \lambda e^{i\theta}, \quad a_3 = e^{i\theta}, \quad a_4 = 1.$$

- *If $R = 0$, $\rho_j = \begin{pmatrix} -i & a_j \\ 0 & i \end{pmatrix}$ where*

$$a_1 = i\lambda, \quad a_2 = i\mu, \quad a_3 = i\mu - i\lambda, \quad a_4 = 0.$$

- *If $R > 0$, $\rho_j = \begin{pmatrix} 0 & a_j \\ -a_j^{-1} & 0 \end{pmatrix}$ where*

$$a_1 = e^{i\theta}, \quad a_2 = \lambda e^{i\theta}, \quad a_3 = \lambda, \quad a_4 = 1,$$

and θ, λ and μ are real.

Proof. First note that from Theorem 4, each ρ_j is conjugate to

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and consequently satisfies $\rho_j^2 = -1$. There are some basic algebraic consequences of this, given by the following lemmas.

Lemma 1. *The monodromy group Γ is non-abelian.*

Proof. Suppose Γ is abelian. Since $\rho_j^2 = -1$, then there is a basis such that

$$\rho_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and since ρ_2, ρ_3, ρ_4 commute with ρ_1, they are diagonal in the same basis and hence

$$\rho_j = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Parallel translation from the base point b thus preserves the subspaces defined by the two basis vectors, and defines two dual complementary line sub-bundles L and L^* of the trivial bundle away from the singular points.

Now if we choose a basis such that A_1 is diagonal, then it is easy to see that near $z = 0$ the covariant constant functions $f(z)$ relative to the connection A can be written as

$$f(z) = \begin{pmatrix} z^{1/4} a_1(z) \\ z^{-1/4} a_2(z) \end{pmatrix},$$

where a_1, a_2 are holomorphic. Thus the solutions with monodromy i and $-i$ are multiples of the first and second basis vectors respectively, and the line bundles L and L^* extend as the eigenspaces of the residues across the singular points.

But a subbundle of the trivial bundle must have non-positive degree, so that $\deg L \leq 0$ and $\deg L^* \leq 0$ and hence $\deg L = 0$. Since the degree is zero on $\mathbf{C}P^1$, L (and hence L^*) is trivial, and so given by a basis of $\mathbf{C}^2$. This basis thus diagonalizes all residues A_i, which are therefore proportional. But from (21) this means $G = 0$ which is a contradiction to the non-singularity of the metric.

Lemma 2. *The subgroup of $\pi_1(\mathbf{C}P^1 \backslash \{z_1, z_2, z_3, z_4\})$ consisting of the words of even length in the generators $\gamma_1, \gamma_2, \gamma_3$ maps under the monodromy representation to an abelian subgroup of $SL(2, C)$.*

Proof. We have the relations

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_4^2 = -1 \tag{36}$$

and

$$\rho_1 \rho_2 \rho_3 \rho_4 = 1. \tag{37}$$

We need to show that $\rho_i \rho_j$ commutes with $\rho_k \rho_l$ for all indices $i, j, k, l \in \{1, 2, 3\}$. Now since

$$(\rho_i \rho_j)^{-1} = \rho_j \rho_i$$

from (36), the ordering of the indices is immaterial.

By symmetry, it is enough to show that $\rho_1\rho_2$ commutes with $\rho_2\rho_3$. But

$$\rho_1\rho_2\rho_2\rho_3 = -\rho_1\rho_3 = \rho_1\rho_4^2\rho_3 = \rho_1\rho_4\rho_1\rho_2 = \rho_2\rho_3\rho_1\rho_2$$

using (36) and (37).

Let Γ_0 be the subgroup of the monodromy group Γ generated by words of even length. This is the image of a subgroup of index 2 in $\pi_1(\mathbf{C}P^1\backslash\{z_1, z_2, z_3, z_4\})$, and hence is either the whole group Γ or itself of index 2. However, Γ is non-abelian from Lemma 1, and Γ_0 abelian, so it is of index 2. If Γ_0 consisted of scalar matrices alone, then the whole monodromy group would be abelian, thus the abelian group $\Gamma_0 \subset SL(2,\mathbf{C})$ contains a non-trivial element and is contained in a 1-parameter subgroup. Such a subgroup is conjugate either to a group H of matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{or} \quad \pm\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

Since Γ_0 is normal in Γ, the group Γ lies in the normalizer $N(H)$ of H in $SL(2,\mathbf{C})$, and in fact $\Gamma/\Gamma_0 \cong N(H)/H \cong \mathbf{Z}_2$, the Weyl group. Thus the generators ρ_j of Γ are of the form

$$\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \quad or \quad \pm\begin{pmatrix} i & b \\ 0 & -i \end{pmatrix}$$

for $a \in \mathbf{C}^*$ and $b \in \mathbf{C}$.

It remains to determine the real structure and its relation to the sign of the scalar curvature.

Now the subgroup of words of even length in $\pi_1(\mathbf{C}P^1\backslash\{z_1, z_2, z_3, z_4\})$ is of index 2, and hence defines a double covering X of the punctured sphere. Since the representation ρ is abelian on this group, it factors through its abelianization, the homology group $H_1(X,\mathbf{Z})$. It is straightforward to see that this is isomorphic to $\mathbf{Z}^5$ and generated by

$$x_1 = \gamma_1\gamma_2, \quad x_2 = \gamma_1\gamma_3,$$

$$y_1 = \gamma_1^2, \quad y_2 = \gamma_2^2, \quad y_3 = \gamma_3^2.$$

From (31) the real structure acts on this as

$$\tau(x_1) = x_1, \quad \tau(x_2) = x_1^2 x_2^{-1} y_1^{-1} y_2^{-1},$$
$$\tau(y_1) = y_3^{-1}, \quad \tau(y_2) = y_1 y_2 y_3, \quad \tau(y_3) = y_1^{-1}. \tag{38}$$

Suppose we consider first the case $R \neq 0$. Then choosing a common eigenspace of the matrix $\rho_i \rho_j$ defines a homomorphism

$$\chi : H_1(X, \mathbf{Z}) \to \mathbf{C}^*.$$

Clearly the compatibility of the connection with the real structure implies that χ is real in a suitable sense. There are two possibilities, dependent on whether τ preserves or interchanges the eigenspaces. In turn this is determined by the sign of the scalar curvature:

Lemma 3. *If $R > 0$, then $\chi(\tau(x)) = \bar{\chi}(x)$, and if $R < 0$ then $\chi(\tau(x)) = \bar{\chi}(x)^{-1}$ for all $x \in H_1(X, Z)$.*

Proof. In twistor language, the scalar curvature of the metric defined by the twisted 1-form θ is given by

$$\theta \wedge d\theta \in H^0(Z, \bigwedge^3 T^*(K^{-1})) \cong H^0(Z, \mathcal{O}) \cong \mathbf{C}$$

as shown, for instance in [8].

To perform the calculation, let $s = \Lambda^3 \alpha$ and on some open set in $Z \backslash Y$ define the 1-form $\varphi = \theta / s^{1/2}$. Since φ is $SU(2)$-invariant, it satisfies

$$d\varphi(\alpha(x), \alpha(y)) = -\varphi(\alpha[x, y])$$

for $x, y \in \mathfrak{g}$. Choosing a standard orthonormal basis $\{x_1, x_2, x_3\} = \{i, j, k\}$ for the Lie algebra of $SU(2)$, this gives

$$(\varphi \wedge d\varphi)(\alpha(x_1), \alpha(x_2), \alpha(x_3)) = -2 \sum_{i=1}^{3} \varphi(\alpha(x_i))^2 = \operatorname{tr} \phi^2$$

using the notation of Theorem 4. Hence if R is the scalar curvature,

$$\operatorname{tr} \phi^2 = R. \tag{39}$$

Recall that ϕ is a 2-valued invariant function $\phi : Z \to \mathfrak{g}^c$.

Now let us consider the effect on the monodromy $\rho_1 \rho_2 = \rho(x_1)$ around the great circle $\beta_1 \beta_3^{-1}$. This encircles two branchpoints, and we can find a single-valued real branch of $s^{-1/2}$, and therefore we may assume that ϕ is single-valued. Now, as in Theorem 4, but using the parametrization by the angle θ, we have

$$\frac{d\phi}{d\theta} + [A, \phi] = 0,$$

and from (34)

$$\frac{dM}{d\theta} + AM = 0.$$

It follows that $P = M^{-1}\phi M$ is a constant. On the other hand, the reality condition on the twisted 1-form implies that $\phi(\theta+\pi) = -\phi^*(\theta)$, thus

$$P = M(\theta+\pi)^{-1}\phi(\theta+\pi)M(\theta+\pi) = -HM(\theta)^*\phi(\theta)^*M(\theta)^{*-1}H^{-1}$$

using (35). Hence

$$P^* = -H^{*-1}PH^*. \tag{40}$$

As remarked above, the monodromy around this loop is defined by H^*H^{-1}. From (40), it commutes with P. Hence we may write

$$H^*H^{-1} = c_1 I + c_2 P, \tag{41}$$

and so from (40)

$$H = H^{**} = H^*(\bar{c}_1 I + \bar{c}_2 P^*) = (\bar{c}_1 I - \bar{c}_2 P)H^* = (\bar{c}_1 I - \bar{c}_2 P)(c_1 I + c_2 P)H.$$

Now $P \in \mathfrak{g}$ so $\operatorname{tr} P = 0$. Thus

$$P^2 = \frac{1}{2}(\operatorname{tr} P^2)I = \frac{1}{2}(\operatorname{tr}\phi^2)I = \frac{1}{2}RI$$

from the definition of P and (39).

Hence,

$$(c_1\bar{c}_1 - \frac{1}{2}Rc_2\bar{c}_2) = 1 \quad \textit{and} \quad c_1\bar{c}_2 = \bar{c}_1 c_2.$$

Moreover since $\det(c_1 I + c_2 P) = 1$, we also have

$$c_1^2 - \frac{1}{2}Rc_2^2 = 1.$$

We deduce immediately that c_1 and c_2 are real and that $c_1^2 - \frac{1}{2}Rc_2^2 = 1$. But the eigenvalues of the monodromy $c_1 I + c_2 P$ are $c_1 \pm c_2\sqrt{R/2}$, thus if the scalar curvature R is positive, the eigenvalues of this monodromy element are of the form λ, λ^{-1} with λ real, and if the curvature is negative, they are λ, λ^{-1} with $\lambda\bar{\lambda} = 1$.

But now $\tau(x_1) = x_1$ so $\chi(\tau(x_1)) = \chi(x_1) = \lambda = \bar{\chi}(x_1)$ if R is positive, and $\bar{\chi}(x_1)^{-1}$ if R is negative.

To complete the proof of Theorem 5, suppose first that $R > 0$. Since P is semisimple, the monodromy elements $\rho_i\rho_j$ commute with P and are thus diagonalizable, so each ρ_j, being non-trivial in the normalizer of the corresponding diagonal subgroup, is of the form

$$\rho_j = \begin{pmatrix} 0 & a_j \\ -a_j^{-1} & 0 \end{pmatrix}$$

and

$$\rho_i\rho_j = \begin{pmatrix} -a_i/a_j & 0 \\ 0 & -a_j/a_i \end{pmatrix}.$$

The real structure gives

$$\begin{aligned} \bar{\chi}(x_1) &= \chi(\tau(x_1)) = \chi(x_1), \\ \bar{\chi}(x_2) &= \chi(\tau(x_2)) = \chi(x_1)^2\chi(x_2)^{-1}, \end{aligned}$$

using $\chi(y_i) = \rho_1^2 = -1$.

Thus

$$\begin{aligned} \frac{\bar{a}_1}{\bar{a}_2} &= \frac{a_1}{a_2}, \\ \frac{\bar{a}_1}{\bar{a}_3} &= \frac{a_1^2 a_3}{a_2^2 a_1}, \end{aligned}$$

and so

$$\frac{a_1}{a_2} = \lambda^{-1} \quad \frac{a_1}{a_3} = \lambda^{-1}e^{i\theta}$$

for $\lambda \in \mathbf{R}$. By a change of basis we can set $a_4 = 1$, and the relation $\rho_1\rho_2\rho_3\rho_4 = 1$ implies that $a_3 = a_2 a_1^{-1}$, which finally gives the full monodromy

$$a_1 = e^{i\theta}, \quad a_2 = \lambda e^{i\theta}, \quad a_3 = \lambda, \quad a_4 = 1.$$

The case of $R < 0$ is entirely similar.

It remains to deal with the case $R = 0$. Each generating monodromy element ρ_j is of the form

$$\rho_j = \pm \begin{pmatrix} i & a_j \\ 0 & -i \end{pmatrix},$$

and so the monodromy preserves the subspace spanned by the first basis vector. This defines a line bundle L on $\mathbf{C}P^1$ with a meromorphic connection whose residue at the point z_j is $\pm 1/4$. The line bundle is a subbundle of the trivial bundle on which the connection A is defined, and so must have non-positive degree. If it is trivial, then there is a vector $v \in \mathbf{C}^2$ which is an eigenvector for each residue A_j. But from the reality conditions (15) and (16), this implies that the A_j are simultaneously diagonal, which (as in Lemma 1) is impossible, so L must have negative degree. The degree is given by the sum of the residues of any connection, and so in this case is a number of the form $\pm\frac{1}{4} \pm \frac{1}{4} \pm \frac{1}{4} \pm \frac{1}{4}$, and we only obtain a negative integer, -1, by taking the negative sign at each pole. Hence

$$\rho_j = \begin{pmatrix} -i & a_j \\ 0 & i \end{pmatrix} \tag{42}$$

and

$$\rho_1\rho_2 = \begin{pmatrix} -1 & i(a_1 - a_2) \\ 0 & -1 \end{pmatrix}.$$

Thus $c_1 = -1$. From (41), we obtain

$$H + H^* = c_2 PH.$$

Now choose a unitary basis of eigenvectors for $H + H^*$, then in this basis

$$H = \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix},$$

moreover, since P is nilpotent and non-zero, one eigenvalue of $H + H^*$ is zero, and one non-zero, so $d + \bar{d} = 0$, say, and $a + \bar{a} \neq 0$. The condition $\det H = 0$ then forces $d = 0$ and $b\bar{b} = 1$. From this we deduce that in this unitary basis,

$$c_2 P = \begin{pmatrix} 0 & -b(a + \bar{a}) \\ 0 & 0. \end{pmatrix}.$$

But the first basis vector is the common $-i$ eigenspace for all the ρ_j and so these have the form (42) with respect to a *unitary* basis.

Now using the reality conditions (32) and $H = \sigma$, we obtain

$$\begin{aligned}
-\bar{a}_1 &= 2ia\bar{b} + a_1 - a_2 + a_4,\\
-\bar{a}_2 &= 2ia\bar{b} + a_1 - a_2 + a_3,\\
-\bar{a}_3 &= 2ia\bar{b} + a_2 - a_3 + a_4,\\
-\bar{a}_4 &= 2ia\bar{b} + a_1 - a_3 + a_4,
\end{aligned}$$

and from $\rho_1\rho_2\rho_3\rho_4 = 1$ we obtain $a_1 - a_2 + a_3 - a_4 = 0$.

Subtracting in pairs we find that $a_1 - a_2, a_2 - a_3, a_3 - a_4$ are imaginary. By a change of basis we can set $a_4 = 0$ and then a_1, a_2, a_3 are imaginary.

7. Elliptic curves and Einstein metrics

In this section, we shall adopt a geometrical approach to the monodromy for an Einstein metric, and thereby obtain explicit formulae. Our starting point is to consider the elliptic curve C defined as the double covering of $\mathbf{C}P^1$ branched over the four singular points $\{z_1, z_2, z_3, z_4\}$ of the connection A. If the corresponding ramification points on C are denoted by $\{w_1, w_2, w_3, w_4\}$, then

$$\pi_1(C\backslash\{w_1, w_2, w_3, w_4\}) \subset \pi_1(\mathbf{C}P^1\backslash\{z_1, z_2, z_3, z_4\})$$

is a subgroup of index two. As is readily seen, it is in fact the subgroup of words of even length in the generators $\gamma_1, \gamma_2, \gamma_3$ of the free group $\pi_1(\mathbf{C}P^1\backslash\{z_1, z_2, z_3, z_4\})$, so the covering X of the previous section is simply $C\backslash\{w_1, w_2, w_3, w_4\}$. The key to obtaining explicit solutions is now Lemma 2, which says that the image of this subgroup in the monodromy group is *abelian*. Put another way, if we pull back the connection A to C, then we obtain a connection with abelian holonomy, and we can then reduce the question of finding that monodromy to the calculation of periods of differentials on the curve C.

Let $\tilde{A}$ denote the connection A pulled back to C. Since the residues of A in the Einstein case have eigenvalues $\pm 1/4$, $\tilde{A}$ has poles at the points w_i with residue conjugate to

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

because of the local branching $z - z_i = (w - w_i)^2 + \ldots$. It follows that a local covariant constant section has the form

(43)
$$s = (w - w_i)^{-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \ldots \quad \text{or} \quad s = (w - w_i)^{1/2} \begin{pmatrix} a \\ b \end{pmatrix} + \ldots$$

First consider the case where the scalar curvature R is non-zero. From Theorem 5, we see that the monodromy of the connection on C preserves two 1-dimensional subspaces of $\mathbf{C}^2$, and so we obtain two line subbundles L_1 and L_2 of the trivial bundle over $C \backslash \{w_1, w_2, w_3, w_4\}$. Moreover, $\tilde{A}$ induces a connection on each of them. Consider a local covariant constant section s of L_1 in a neighbourhood of $w = w_i$. From (43) we see that L_1 extends to C, and the induced connection extends to a connection with a pole of residue $\pm 1/2$.

Now the sum of the residues of a meromorphic connection on a line bundle L is equal to $-\deg L$, so in our case this number is $\pm\frac{1}{2}\pm\frac{1}{2}\pm\frac{1}{2}\pm\frac{1}{2}$. On the other hand, L_1 is embedded as a holomorphic subbundle of $C \times \mathbf{C}^2$, and so its dual L_1^* must have at least two linearly independent sections. This means, for an elliptic curve, that $\deg L_1 \leq -2$, and hence is equal to $-2 = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}$, so that the residues must all be $1/2$.

The same holds for L_2. Note that the involution on C (the covering transformation of $C \to \mathbf{C}P^1$) interchanges L_1 and L_2, since the ρ_i, and hence all odd words, are non-trivial in the Weyl group $N(H)/H$ in the proof of Theorem 5. The subbundles L_1 and L_2 of $C \times \mathbf{C}^2$ coincide over the ramification points.

In a more analytical form, we represent the covering $C \to \mathbf{C}P^1$ as the meromorphic function on C given by the Weierstrass $\wp$ function:

$$z = \wp(u),$$

where $u \in \mathbf{C}/\Lambda \cong C$, and Λ is the lattice generated by $\{2\omega_1, 2\omega_3\}$. The involution on C is $u \mapsto -u$, and its fixed points, the ramification points of the covering, are $u = 0, \omega_1, \omega_2, \omega_3$ (we shall use the notation in [33] for formulas involving elliptic functions). Since $\wp$ is of degree 2, any line bundle of degree 2 on C is the pull back of $\mathcal{O}(1)$ by the meromorphic function $\wp(u - c)$ for some c. Thus we may take L_1 and

L_2 to be generated respectively by the two vectors in $\mathbf{C}^2$:

$$v_1 = \begin{pmatrix} 1 \\ \wp(u-c) \end{pmatrix} \quad \textit{and} \quad v_2 = \begin{pmatrix} 1 \\ \wp(u+c) \end{pmatrix}. \tag{44}$$

The connection $\tilde{A}$ preserves L_1, so there is a meromorphic 1-form $\theta_1 du$ on C such that

$$\left(\frac{d}{du} + \wp'(u)A\right)\begin{pmatrix} 1 \\ \wp(u-c) \end{pmatrix} = \theta_1 \begin{pmatrix} 1 \\ \wp(u-c) \end{pmatrix}. \tag{45}$$

Now $\theta_1 du$ is a connection form relative to the local trivialization v_1 of L_1. The meromorphic section v_1 has a double pole where $\wp(u-c) = \infty$, i.e., at $u = c$. As we have seen, the connection has simple poles at the ramification points with residue $1/2$, so θ_1 is an elliptic function with simple poles of residue $1/2$ at $u = 0, \omega_1, \omega_2, \omega_3$ and a simple pole with residue -2 at $u = c$. Now

$$\frac{\wp''(u)}{2\wp'(u)}$$

has simple poles of residue $1/2$ at $u = \omega_1, \omega_2, \omega_3$ and a simple pole of residue $-3/2$ at $u = 0$. Moreover,

$$\frac{2\wp'(c/2)}{\wp(u-c/2) - \wp(c/2)}$$

has a simple pole of residue 2 at $u = c$ and residue -2 at $u = 0$. Thus there is a constant κ such that

$$\theta_1 = \frac{\wp''(u)}{2\wp'(u)} - \frac{2\wp'(c/2)}{\wp(u-c/2) - \wp(c/2)} + \kappa. \tag{46}$$

The constants κ and c need to be determined from the monodromy of A. For a line bundle, however, the monodromy of a connection is easily obtained by integrating the connection form over closed loops, or their homology classes:

$$\rho(\gamma) = -\exp\left(\int_\gamma \theta_1 du\right).$$

Now consider the map on homology induced by the inclusion

$$i : H_1(C \backslash \{w_1, w_2, w_3, w_4\},\ \mathbf{Z}) \to H_1(C, \mathbf{Z}).$$

The classes $y_i = \gamma_i^2$ bound in C, so $H_1(C,\mathbf{Z}) \cong \mathbf{Z}^2$ is generated by $i(x_1), i(x_2)$. Moreover, since from (38) $\tau(x_1) = x_1$ and $\tau(x_2) = 2x_1 - x_2 - y_1 - y_2$, $i(x_1)$ is a real generator and $i(x_1 - x_2)$ an imaginary one. Thus the periods of $\theta_1 du$ over these cycles are determined by the monodromy elements $\rho_1\rho_2$ and $\rho_3\rho_2$ of the connection on $\mathbf{C}P^1$. In the fundamental rectangle, the cycles are represented by lines parallel to the x and y axes respectively.

To calculate the periods, we first use the general formula for the Weierstrass zeta function

$$\zeta(u+a) - \zeta(u-a) - 2\zeta(a) = \frac{-\wp'(a)}{\wp(u) - \wp(a)} \tag{47}$$

and the definition of the sigma function

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)}$$

to obtain

$$\int \frac{\wp'(a)}{\wp(u) - \wp(a)} du = -\log \frac{\sigma(u+a)}{\sigma(u-a)} + 2\zeta(a)u$$

and hence

$$\int \theta_1 du = \frac{1}{2}\log \wp'(u) + 2\log\frac{\sigma(u)}{\sigma(u-c)} - 4\zeta(c/2)u + \kappa u.$$

Finally, using

$$\sigma(u + 2\omega_1) = -e^{2\eta_1(u+\omega_1)}\sigma(u),$$
$$\sigma(u + 2\omega_3) = -e^{2\eta_3(u+\omega_1)}\sigma(u),$$

where $\eta_i = \zeta(\omega_i)$, we obtain the monodromy $e^{\lambda_1}, e^{\lambda_3}$ of θ_1 where:

$$\lambda_1 = 4\eta_1 c - 8\omega_1\zeta(c/2) + 2\kappa\omega_1,$$
$$\lambda_3 = 4\eta_3 c - 8\omega_3\zeta(c/2) + 2\kappa\omega_3.$$

From the Legendre relation $\eta_1\omega_3 - \eta_3\omega_1 = i\pi/2$, we find

$$\lambda_1\omega_3 - \lambda_3\omega_1 = 2\pi i c, \tag{48}$$
$$\lambda_1\eta_3 - \lambda_3\eta_1 = 4\pi i\zeta(c/2) - i\pi\kappa. \tag{49}$$

Each twistor line has a connection with the same monodromy, so λ_1 and λ_3 are fixed by the metric. Note that the ambiguity in the choice of logarithm is reflected in the two choices of lifting of the cycles $\gamma_1\gamma_2$ and $\gamma_3\gamma_2$. The equations (48) and (49) then define c and κ, and hence the connection form θ_1, in terms of the basic invariants ω_i, η_i of the elliptic curve. It will be more convenient to replace the parameters c and κ by ω and η defined by:

$$\omega = 2c, \qquad \text{and} \quad \eta = 4\zeta(c/2) - \kappa$$

thus giving

$$\begin{aligned}\lambda_1 &= 2\omega\eta_1 - 2\eta\omega_1,\\ \lambda_3 &= 2\omega\eta_3 - 2\eta\omega_3.\end{aligned}$$

Note that, from Theorem 5, λ_1 is imaginary and λ_3 real for a metric with scalar curvature $R < 0$, and vice versa for $R > 0$.

To derive the full connection, we can use (47) and (49) to rewrite (46) as

$$\begin{aligned}\theta_1 &= (\zeta(2u) - 2\zeta(u)) + 2(\zeta(u) - \zeta(u-c) - 2\zeta(c/2)) + \kappa\\ &= \zeta(2u) - 2\zeta(u-c) - \eta,\end{aligned} \tag{50}$$

and, since θ_2 is obtained by replacing c by $-c$, and taking the opposite period, we have

$$\theta_2 = \zeta(2u) - 2\zeta(u+c) + \eta.$$

These expressions, together with (45) provide a formula for the connection matrix A:

$$\begin{aligned}&\wp'(u)(\wp(u+c) - \wp(u-c))A\\ &= \begin{pmatrix}\theta_1 & \theta_2\\ \theta_1\wp(u-c) - \wp'(u-c) & \theta_2\wp(u+c) - \wp'(u+c)\end{pmatrix}\begin{pmatrix}\wp(u+c) & -1\\ -\wp(u-c) & 1\end{pmatrix}.\end{aligned} \tag{51}$$

In practice, this is very cumbersome to use, so we shall make use of a different approach to the isomonodromic deformation problem, which involves just a single function. A reference for this is [20].

First note that if, in the general context of isomonodromic deformations, the connection on $\mathbf{C}P^1$ is put in the form

$$A(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-x},$$

then each entry of the matrix $A_{ij}(z)$ is of the form $q(z)/[z(z-1)(z-x)]$ for some quadratic polynomial q. Now if, as in our case, $A_\infty = -(A_1 + A_2 + A_3)$ is diagonalizable, then there is a basis such that

$$A_\infty = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

and A_{12} can be written

$$A_{12}(z) = \frac{k(z-y)}{z(z-1)(z-x)}$$

for some $y \in \mathbf{C}P^1 \backslash \{0, 1, x, \infty\}$. If the $A_i(x)$ satisfy Schlesinger's equation (10), then the function $y(x)$ satisfies the *Painlevé equation*

$$\tag{52} \begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 \\ &\quad - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} \\ &\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right), \end{aligned}$$

where

$$\tag{53} \begin{aligned} \alpha &= (2\lambda - 1)^2/2, \\ \beta &= 2 \det A_1^2, \\ \gamma &= -2 \det A_2^2, \\ \delta &= (1 + 4 \det A_3^2)/2. \end{aligned}$$

For our purposes it is useful to note the geometrical form of the definition of $y(x)$ given by (52): the solution $y(x)$ to the Painlevé equation corresponding to an isomonodromic deformation $A(z)$ is the point $y \in \mathbf{C}P^1 \backslash \{0, 1, x, \infty\}$ at which $A(y)$ and A_∞ have a common eigenvector, corresponding to the eigenvalue λ at ∞.

Now in our example, $z = \wp(u) = \infty$ if $u = 0$, and from (45) we see that the vector

$$v = \begin{pmatrix} 1 \\ \wp(c) \end{pmatrix}$$

is an eigenvector of A_∞. Moreover, since, as noted in the course of its calculation, θ_1 has a simple pole at $u = 0$ with residue $+1/2$, the corresponding eigenvalue of A on $\mathbf{C}P^1$ is $+1/4$. Thus the values of the coefficients in the Painlevé equation (52) are $\alpha = 1/8, \beta = -1/8, \gamma = 1/8$ and $\delta = 3/8$.

To obtain $y(x)$, we must find the value of $\wp(u)$ at which v is again an eigenvalue of A. Equivalently, we want

$$\det\left(A\begin{pmatrix}1\\ \wp(c)\end{pmatrix}\begin{pmatrix}1\\ \wp(c)\end{pmatrix}\right) = 0.$$

The formula (52), after some manipulation, then gives the condition:

$$\theta_1 - \theta_2 = \frac{\wp'(u-c)}{\wp(u-c)-\wp(c)} - \frac{\wp'(u+c)}{\wp(u+c)-\wp(c)}.$$

Using (47) and the expressions for θ_1 and θ_2, this yields:

$$\begin{aligned}&2\zeta(u+c) - 2\zeta(u-c) - 2\mu\\ &\quad = \zeta(u) + \zeta(u-2c) - 2\zeta(u-c) - (\zeta(u+2c) + \zeta(u) - 2\zeta(u+c)),\end{aligned}$$

from which

$$-2\eta = \zeta(u-2c) - \zeta(u+2c) = -2\zeta(2c) + \frac{\wp'(2c)}{\wp(u)-\wp(2c)}.$$

Solving for $\wp(u)$, setting $\omega = 2c$ and using the Möbius transformation which takes the branch points e_1, e_2, e_3 to $0, 1, x$, we obtain the solution to the Painlevé equation as

$$y(x) = \frac{\xi - e_1}{e_2 - e_1}, \tag{54}$$

where

$$\xi = \wp(\omega) + \frac{\wp'(\omega)}{2(\zeta(\omega)-\eta)}, \qquad x = \frac{e_3 - e_1}{e_2 - e_1},$$

and

$$\begin{aligned}\omega &= k_1\omega_3 - k_3\omega_1,\\ \eta &= k_1\eta_3 - k_3\eta_1,\end{aligned}$$

for constants k_1, k_3.

Remark. In the above form, it is not entirely clear that y is a function of x alone. If we express the Weierstrass elliptic functions in terms of theta functions $\vartheta_\alpha(\nu,\tau)$, then $y(x)$ is also defined as

$$\begin{aligned} y(x) = {} & \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left(1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right) \\ & + \frac{\vartheta_1'''(\nu)\vartheta_1(\nu) - 2\vartheta_1''(\nu)\vartheta_1'(\nu) + 2\pi i k_1(\vartheta_1''(\nu)\vartheta(\nu) - \vartheta_1'^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu)+\pi i k_1\vartheta_1(\nu))}, \end{aligned}$$

where $\nu = (k_1\tau - k_3)/2$ and $x = \vartheta_3^4(0)/\vartheta_4^4(0)$. The above is the function which we shall use in the next sections to generate new solutions of the Einstein equations. Before doing that, however, we should for completeness consider the case $R = 0$, although the corresponding equations were solved already in the context of elliptic integrals in [4] and [5]. The advantage of our current viewpoint is the wider context in which the formula appears.

Consider then the case of $R = 0$. Here, from Theorem 5, the monodromy preserves a subbundle L of $\mathbf{C}P^1 \times \mathbf{C}^2$. From the proof of the theorem, L has degree -1 and so $L \cong \mathcal{O}(-1)$. We can therefore take L to be spanned by the vector

$$\begin{pmatrix} 1 \\ z \end{pmatrix}.$$

Now the connection preserves L and so

$$\nabla \begin{pmatrix} 1 \\ z \end{pmatrix} = \theta \begin{pmatrix} 1 \\ z \end{pmatrix}.$$

Again from the proof of the theorem, the meromorphic 1-form θ has simple poles, and residues $1/4$ over the finite branch points e_1, e_2, e_3. Thus

$$\theta = \frac{dz}{4(z-e_1)} + \frac{dz}{4(z-e_2)} + \frac{dz}{4(z-e_3)}.$$

If we now set

$$f(z) = (z-e_1)(z-e_2)(z-e_3)$$

and

$$s_1 = f^{-1/4}\begin{pmatrix} 1 \\ z \end{pmatrix},$$

then

$$\nabla s_1 = 0.$$

Now put

$$s_2 = f^{1/4}\begin{pmatrix} z^{-1} \\ 0 \end{pmatrix},$$

then $\nabla s_2 = as_1 + bs_2$, but since $< s_1, s_2 >= -1$ with respect to the skew pairing on $\mathbf{C}^2$, preserved by the $SL(2,\mathbf{C})$ connection A, we have

$$0 = d < s_1, s_2 >=< \nabla s_1, s_2 > + < s_1, \nabla s_2 >= b,$$

and so $\nabla s_2 = as_1$.

Thus the connection matrix A is given by

$$\left(\frac{d}{dz} + A\right)\begin{pmatrix} f^{-1/4} & f^{1/4}z^{-1} \\ f^{-1/4}z & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^{-1/4}a \\ 0 & f^{-1/4}az \end{pmatrix}.$$

Hence,

$$A = \begin{pmatrix} -f'/4f + 1/z + f^{-1/2}az & f'/2fz - 1/z^2 - f^{-1/2}a \\ -f^{-1/2}az^2 & f'/4f - 1/z - f^{-1/2}az \end{pmatrix}. \tag{55}$$

Each entry in this matrix $A(z)$ must be of the form $q(z)/f$ for some quadratic polynomial $q(z)$. Taking account of the powers of z in the denominator, we obtain

$$a = f^{-1/2}\left(\frac{f'}{2z} - \frac{f}{z^2} - \frac{z}{2} + c\right)dz \tag{56}$$

$$= d(f'/z) + f^{-1/2}(c - z/2)dz \tag{57}$$

for some constant c, to be determined from the monodromy.

To make this relationship, note that $f^{1/2} = \sqrt{(z-e_1)(z-e_2)(z-e_3)}$ is single valued on the elliptic curve C; in fact setting $z = \wp(u)$, then $\wp'(u) = 2f^{1/2}$. Thus the section $s_1 \otimes s_1$ (considered as a nilpotent element in the Lie algebra $\mathfrak{g}$ of $SL(2,\mathbf{C})$) is single valued and covariant constant. Moreover,

$$\nabla(s_1 \otimes s_2) = a(s_1 \otimes s_1),$$

so the two-dimensional subspace of $\mathfrak{g}$ spanned by $s_1 \otimes s_1$ and $s_1 \otimes s_2$ is preserved by the connection. We already know the monodromy is abelian but from this point of view, around a cycle x it is given by a matrix conjugate to

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad \alpha = \int_x a.$$

Using (57) we have

$$a = d(f'/z) + (2c - \wp(u))du,$$

and so the periods of this form over the real and imaginary cycles of C are

$$\lambda_1 = 4c\omega_1 + 2\eta_1, \tag{58}$$

$$\lambda_3 = 4c\omega_3 + 2\eta_3. \tag{59}$$

Now replacing the nilpotent element $s_1 \otimes s_1$ by a multiple of itself gives another basis of the same type, so only the ratio λ_1/λ_3 is an invariant of the monodromy. We can now compare with the form of the monodromy in Theorem 5. Here the monodromy around the two cycles is given by

$$\begin{pmatrix} -1 & \mu - \lambda \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix},$$

and so $\lambda_1/\lambda_3 = (\mu - \lambda)/\lambda$ and is in particular a real number k_0. Using the Legendre relation from (59) we obtain

$$c = \frac{-(k_0\eta_3 - \eta_1)}{2(k_0\omega_3 - \omega_1)}. \tag{60}$$

We need to find the corresponding solution to the Painlevé equation, and for this consider (55) as $z \to \infty$. It is easy to see, using (57), that the residue of A at infinity has the eigenvector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

corresponding to the eigenvalue $1/4$, and so the solution to the Painlevé equation is the point $z = y$ at which this vector is again an eigenvector.

From (55) this is where $f'/2fz - 1/z^2 - f^{-1/2}a = 0$, or from (57) where $z = 2c$. By (60), we therefore obtain

$$y(x) = \frac{\xi - e_1}{e_2 - e_1}, \tag{61}$$

where

$$\xi = \frac{-(k\eta_3 - \eta_1)}{(k\omega_3 - \omega_1)}.$$

We may summarize our calculations in the following theorem:

Theorem 6. *Let M be an $SU(2)$-invariant anti-self-dual Einstein manifold with scalar curvature R and such that its twistor space Z has $\Lambda^3\alpha$ non-degenerate. Then the corresponding solution $y(x)$ to the Painlevé equation is given as follows:*

- *For $R < 0$,*

  $$y(x) = \frac{\xi - e_1}{e_2 - e_1},$$

  where

  $$\xi = \wp(\omega) + \frac{\wp'(\omega)}{2(\zeta(\omega) - \eta)} \quad \text{and} \quad x = \frac{e_3 - e_1}{e_2 - e_1},$$

  and

  $$\omega = k_1\omega_3 - ik_3\omega_1,$$
  $$\eta = k_1\eta_3 - ik_3\eta_1$$

  for constants $k_1, k_3 \in \boldsymbol{R}$.

- *For $R = 0$,*

  $$y(x) = \frac{\xi - e_1}{e_2 - e_1},$$

  where

  $$\xi = \frac{-(k_0\eta_3 - \eta_1)}{(k_0\omega_3 - \omega_1)},$$

  and $k_0 \in \boldsymbol{R}$.

- *For $R > 0$,*

  $$y(x) = \frac{\xi - e_1}{e_2 - e_1},$$

where

$$\xi = \wp(\omega) + \frac{\wp'(\omega)}{2(\zeta(\omega) - \eta)} \quad \text{and} \quad x = \frac{e_3 - e_1}{e_2 - e_1},$$

and

$$\omega = ik_1\omega_3 - k_3\omega_1,$$
$$\eta = ik_1\eta_3 - k_3\eta_1$$

for constants $k_1, k_3 \in \boldsymbol{R}$.

Remarks.

1. In [19], some algebraic Einstein metrics with $R > 0$ were produced directly from the twistor construction, by using a relationship with Poncelet polygons. The connection with the Painlevé equation and isomonodromic deformations was central in that description, but the essential fact giving algebraic solutions was that the monodromy group was *finite*, in fact a binary dihedral group. In our discussion here, we have seen that the monodromy group for an Einstein metric has the basic property that it has an abelian subgroup of index two. Of course this is true in particular for the covering of the rotations in the binary dihedral group. The values of the constants k_1, k_3 in Theorem 6 relating to monodromy given by the dihedral group of symmetries of a regular k-gon are $k_3 = 2/k$ and $k_1 = 0$.

2. A possible interpretation of the algebraic metrics in [19] was given there in terms of the moduli space of charge 2 monopoles on a hyperbolic space of curvature $-1/p^2$ where $p + 1 = 1/(2k_3)$. The algebraicity of these metrics when $2/k_3 \in \mathbf{Z}$ is consistent with the fact that the corresponding solutions of the Bogomolny equations are algebraic. However, solutions exist for all values of the curvature, and so one expects appropriate metrics for all values of k_3. This is what the above formula yields. In fact, the asymptotic analysis of [19] giving the global structure of the metric is independent of whether k is an integer or not.

3. The natural metric on the moduli space of Euclidean charge 2 monopoles is hyperkähler and was calculated in [4]. The solution to the Painlevé equation which corresponds to it was derived as a limiting procedure of the dihedral metrics as $k \to \infty$ in [19]. By comparison with Theorem 6, it corresponds to the value of the constant $k_0 = 0$. We leave it as an exercise to relate the general solution of the $SU(2)$-

invariant hyperkähler metrics as discussed in [4] and [5] to the general solution of the corresponding Painlevé equation above.

8. Einstein metrics on the 4-ball

We shall now use the explicit formulae of the previous section to produce a family of complete anti-self-dual Einstein metrics with negative scalar curvature on the interior of the unit ball. There are two basic models for such metrics. The first is the hyperbolic metric on the unit ball

$$g = \frac{1}{(1-r^2)^2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).$$

It is the symmetric space $SO(4,1)/SO(4)$ with its natural metric. Away from the origin, this special metric is defined on $(0,1) \times S^3$ and as such can be written in the form

$$g = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{(1-r^2)^2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \tag{62}$$

The conformal structure extends over the boundary and induces the standard conformal structure on the sphere $r = 1$.

The second model is the Bergmann metric on the unit ball in $\mathbf{C}^2$. This is the symmetric space $SU(2,1)/U(2)$ and is the metric dual to the Fubini-Study metric on $\mathbf{C}P^2$. In diagonal form it can be given by

$$g = \frac{1}{2}\left(\frac{dr^2}{(1-r^2)^2} + \frac{r^2\sigma_2^2}{(1-r^2)^2} + \frac{r^2}{(1-r^2)}(\sigma_1^2 + \sigma_3^2)\right). \tag{63}$$

In this case the conformal structure does not extend across the boundary. Asymptotically, by putting $s = 1 - r$ and letting $s \to 0$, the metric is a multiple of

$$g = \frac{ds^2}{s^2} + \frac{1}{2s}\sigma_1^2 + \frac{1}{4s^2}\sigma_2^2 + \frac{1}{2s}\sigma_3^2 + \dots.$$

The conformal structure on an $SU(2)$ orbit degenerates: the coefficient of σ_2^2 decays faster than the others, and what is left is a conformal structure on the two-dimensional subbundle of the tangent bundle of S^3 annihilated by σ_2. This is a left-invariant CR-structure, whose existence is a natural consequence of the fact that the Bergmann metric is a Kähler metric respecting a *complex structure* which extends to the boundary.

We begin with the first model. Its essential features for our purposes are that it is invariant under $SO(4)$ and blows up on the 3-sphere $r = 1$, although the conformal structure extends smoothly across the sphere. In the terminology of [22], the sphere is the *conformal infinity* of the Einstein metric. The metrics we shall define will be invariant under $SU(2) \subset SO(4)$, and induce on their conformal infinity a left-invariant conformal structure on the 3-sphere $S^3 \cong SU(2)$.

The existence of such a metric in a collar neighbourhood of S^3 follows from a rather general twistorial result of LeBrun [22], and was the motivation for Pedersen's work [29], which succeeded in finding explicitly the metrics which are invariant under $U(2) \subset SO(4)$. Pedersen's metrics extend to the whole of the ball and the more general question of whether an arbitrary left invariant metric extends is the theme of Tod's paper [34]. From the work of Graham and Lee [15], (see the discussion in [25]), it is known that a metric sufficiently close to the bi-invariant metric will extend to a possibly non-anti-self-dual Einstein metric.

Our point of view is to select from our explicit solutions for $SU(2)$-invariant anti-self-dual Einstein metrics the ones with negative scalar curvature, and to investigate the global behaviour of the metric as the parameter x in (24) lies within a suitable interval. So far, we have only considered the conformal structure, for that is all the general twistor space gives us. In the anti-self-dual Einstein case it is the twisted form θ on the twistor space which defines the metric (see [8]) within the conformal structure, but there is a more direct expression given by Tod ([34]). With the conformal structure given by

$$g_0 = \frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2} \tag{64}$$

as in (24), the conformal factor to give an Einstein metric $g = e^{2u}g_0$ with scalar curvature Λ is

$$-4\Lambda e^{2u} = \frac{(8x\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2+\Omega_2^2) - (1-4\Omega_3^2)(\Omega_2^2-(1-x)\Omega_1^2)))}{(x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2-(1-x)\Omega_1^2))^2} \tag{65}$$

As we saw in Section 5, the Ω_i are derived from the residues of the connection matrix A, which in turn, from [20] is determined from the solution to the Painlevé equation. Explicitly, in the Einstein case, we have

$$\frac{dy}{dx} = \frac{y(y-1)(y-x)}{x(x-1)}\left(2z - \frac{1}{2y} - \frac{1}{2(y-1)} + \frac{1}{2(y-x)}\right), \tag{66}$$

which defines the auxiliary variable z, and then

$$\begin{aligned} \Omega_1^2 &= \frac{(y-x)^2 y(y-1)}{x(1-x)}\left(z - \frac{1}{2(y-1)}\right)\left(z - \frac{1}{2y}\right), \\ \Omega_2^2 &= \frac{y^2(y-1)(y-x)}{x}\left(z - \frac{1}{2(y-x)}\right)\left(z - \frac{1}{2(y-1)}\right), \\ \Omega_3^2 &= \frac{(y-1)^2 y(y-x)}{(1-x)}\left(z - \frac{1}{2y}\right)\left(z - \frac{1}{2(y-x)}\right). \end{aligned} \tag{67}$$

We shall take a solution to the Painlevé equation giving negative scalar curvature. From Theorem 6 this is

$$y(x) = \frac{\xi - e_1}{e_2 - e_1},$$

where

$$\xi = \wp(\omega) + \frac{\wp'(\omega)}{2(\zeta(\omega) - \eta)} \quad \text{and} \quad x = \frac{e_3 - e_1}{e_2 - e_1}, \tag{68}$$

and

$$\begin{aligned} \omega &= k_1\omega_3 - ik_3\omega_1 \\ \eta &= k_1\eta_3 - ik_3\eta_1 \end{aligned}$$

for constants $k_1, k_3 \in \mathbf{R}$. Note that the real structure implies that x is real and ω, η imaginary.

First consider the conformal structure. This is non-degenerate for $x \neq 0, 1, \infty$ so long as the Ω_i are finite and non-vanishing. From the Painlevé property of $y(x)$, the only branch points or essential singularities are at $x = 0, 1, \infty$, so $y(x)$ is meromorphic outside these values of x. If $y \neq 0, 1, x, \infty$, then from (67), Ω_i is finite. By considering the (holomorphic) linear factors in z in these formulae, if the Ω_i

vanish at all, they vanish in pairs. But the Ω_i satisfy the first order differential equation (27). If $\Omega_1 = \Omega_2 = 0$ at some point, then $\Omega_1 \equiv 0, \Omega_2 \equiv 0, \Omega_3 \equiv$ *const.* is the unique solution with this initial value. This is a constant solution, which is clearly not the case here.

In Propositions 8-13, we establish some basic properties of the function $y(x)$ which provide the tools for determining the global behaviour of these metrics. These properties are concerned with the behaviour of $y(x)$ at the special values $0, 1, x, \infty$.

Proposition 8. *Let $y(x)$ be a solution to the Painlevé equation (52), and suppose that y has a singularity at $x = x_0$ where $x_0 \neq 0, 1, \infty$. Then*

- *$y(x)$ has a simple pole of the form*

$$y(x) = \pm \frac{2x_0(x_0 - 1)}{(x - x_0)} = \dots.$$

- *If the residue is $2x_0(x_0 - 1)$, then the conformal structure (24) extends from the punctured ball $B^4\backslash 0 \cong S^3 \times (x_0, x_0 + \epsilon)$ to the whole ball B^4.*
- *If the residue is $-2x_0(x_0 - 1)$, then the conformal structure extends from $S^3 \times (x_0, x_0 + \epsilon)$ to $S^3 \times (x_0 - \epsilon, x_0 + \epsilon)$.*

Proof. Since $y(x)$ is meromorphic at $x = x_0$, the first part of the proposition follows by substitution of the Laurent polynomial in the Painlevé equation, and equating coefficients.

If the residue is $2x_0(x_0 - 1)$, then by substituting in (66), we obtain

$$z = (x - x_0)^2 + \dots,$$

from which, in (67),

$$\Omega_1^2 = -x_0(x_0 - 1)/(x - x_0)^2 + \dots,$$
$$\Omega_2^2 = x_0(x_0 - 1)^2/(x - x_0)^2 + \dots,$$
$$\Omega_3^2 = -x_0^2(x_0 - 1)/(x - x_0)^2 + \dots,$$

and then the conformal structure is given by

$$g_0 = \frac{dx^2}{x_0(1 - x_0)} - \frac{(x - x_0)^2\sigma_1^2}{x_0(x_0 - 1)} - \frac{(x - x_0)^2\sigma_2^2}{x_0(x_0 - 1)} - \frac{(x - x_0)^2\sigma_3^2}{x_0(x_0 - 1)} + \dots,$$

which, comparing with the hyperbolic metric (62) at $r = 0$, clearly extends over the puncture. Note also that if $x_0 > 1$, the conformal structure is negative definite.

If the residue is $-2x_0(x_0-1)$, then

$$z = \frac{-(x-x_0)}{2x_0(x_0-1)} + \dots,$$

and therefore

$$z - \frac{1}{2y} = -\frac{(x-x_0)}{2x_0(x_0-1)} + \frac{(x-x_0)}{2x_0(x_0-1)} + \dots.$$

thus it follows by substituting in (67), that the Ω_i^2 are regular at $x = x_0$. In fact, they are non-vanishing too, for if

$$z - \frac{1}{2y} = c(x-x_0)^n + \dots$$

with $n > 2$, we obtain

$$\begin{aligned} \Omega_1^2 &= A(x-x_0)^{n-2} + \dots, \\ \Omega_2^2 &= 1/4 + \dots, \\ \Omega_3^2 &= A(x-x_0)^{n-2} + \dots, \end{aligned}$$

which contradicts the fact (27) that $\Omega_1' = -\Omega_2\Omega_3/x(1-x)$. Hence the conformal structure is non-singular.

We need to consider the two possibilities for our particular solution. Here $y(x)$ has a pole if

$$\xi = \wp(\omega) + \frac{\wp'(\omega)}{2(\zeta(\omega)-\eta)}$$

is infinite, and this holds if and only if $\omega = 2n\omega_3$, for some integer n, or $\zeta(\omega) = \eta$.

Proposition 9. *Let $y(x)$ be the solution to the Painlevé equation given by (68). If $\zeta(\omega) = \eta$, then the residue is $-2x_0(x_0-1)$, and if $\omega = 2n\omega_3$ at $x = x_0$, then the residue of y(x) is $2x_0(x_0-1)$.*

Proof. It is sufficient to consider ξ near $\omega = k_1\omega_3 - ik_3\omega_1 = 0$ so that $\tau = \omega_3/\omega_1 = ik_3/k_1$.

By using the standard expansion of Weierstrass elliptic functions, coming from

$$\zeta(u) = \frac{1}{u} - \frac{g_2u^3}{60} + \dots$$

and its derivatives, we obtain $\xi = -\eta(x_0)/\omega + \dots$. But

$$\eta(x_0) = k_1\eta_3 - ik_3\eta_1 = k_1(\eta_3 - \tau\eta_1) = -k_1\pi i/2\omega_1$$

using the Legendre relation, so

$$\xi = \frac{\pi i}{2\omega_1^2(\tau - ik_3/k_1)} + \dots.$$

We need to relate the parameter τ in the upper half-plane to the cross ratio x. For this we use

$$x(\tau) = \frac{e_3 - e_1}{e_2 - e_1} = \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}.$$

Differentiating with respect to τ and using the heat equation

$$\vartheta_1'' = 4\pi i\frac{\partial}{\partial\tau}\vartheta_1$$

satisfied by theta functions, we find

$$\frac{dx}{d\tau} = \frac{x}{\pi i}\left[\frac{\vartheta_3''(0)}{\vartheta_3(0)} - \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right],$$

and from the relation

$$2\eta_1\omega_1 = -2e_\alpha\omega_1^2 - \frac{1}{2}\frac{\vartheta_{\alpha+1}''(0)}{\vartheta_{\alpha+1}(0)}$$

we obtain

$$\frac{dx}{d\tau} = \frac{1}{i\pi}4x\omega_1^2(e_3 - e_2). \tag{69}$$

Thus

$$\xi = \frac{2x_0(e_3 - e_2)}{(x - x_0)} + \dots$$

giving

$$y(x) = \frac{2x_0(x_0 - 1)}{(x - x_0)} + \dots.$$

For the second case, using the theta function $\vartheta_1(\nu, \tau)$, we have the standard relation

$$\zeta(u) - \frac{\eta_1 u}{\omega_1} = \frac{1}{2\omega_1}\frac{\vartheta_1'(\nu)}{\vartheta_1(\nu)}, \tag{70}$$

where $\nu = u/2\omega_1$, $\tau = \omega_3/\omega_1$. Using the definitions $\omega = k_1\omega_3 - ik_3\omega_1, \eta = k_1\eta_3 - ik_3\eta_1$ and the Legendre relation, we can then write

$$\zeta(\omega) - \eta = \frac{1}{2\omega_1}\left(\frac{\vartheta_1'}{\vartheta_1}(k_1\tau/2 - ik_3/2, \tau) + ik_1\pi\right). \tag{71}$$

Thus around a zero $\tau = \tau_0$ of $\zeta(\omega) - \eta$, we have an expansion:

$$\zeta(\omega) - \eta = \frac{1}{2\omega_1}\left[\frac{k_1}{2}\left(\frac{\vartheta_1'}{\vartheta_1}\right)' + \frac{\partial}{\partial\tau}\left(\frac{\vartheta_1'}{\vartheta_1}\right)\right](\tau - \tau_0) + \ldots.$$

By the heat equation we can express this as

$$\zeta(\omega) - \eta = \frac{1}{2\omega_1}\left[\frac{k_1}{2}\left(\frac{\vartheta_1'}{\vartheta_1}\right)' + \frac{1}{4\pi i}\left(\frac{\vartheta_1''}{\vartheta_1}\right)'\right](\tau - \tau_0) + \ldots.$$

However, differentiating the expression (70) leads to

$$\wp'(u) = -\frac{1}{8\omega_1^3}\left(\frac{\vartheta_1'}{\vartheta_1}\right)'' = -\frac{1}{8\omega_1^3}\left[\left(\frac{\vartheta_1''}{\vartheta_1}\right)' - 2\left(\frac{\vartheta_1'}{\vartheta_1}\right)'\left(\frac{\vartheta_1'}{\vartheta_1}\right)\right],$$

which, in consequence of the fact that

$$\frac{\vartheta_1'}{\vartheta_1} = -ik_1\pi$$

at τ_0, gives

$$\zeta(\omega) - \eta = \frac{i\omega_1^2}{\pi}\wp'(\omega)(\tau - \tau_0) + \ldots. \tag{72}$$

Substituting in the formula (68) for ξ we obtain

$$\xi = \frac{\pi}{2i\omega_1^2(\tau - \tau_0)} + \ldots,$$

thus

$$\xi = \frac{-2x_0(e_3 - e_2)}{(x - x_0)} + \ldots,$$

and finally

$$\begin{aligned} y(x) &= \frac{-2x_0(e_3 - e_2)}{(e_2 - e_1)(x - x_0)} + \ldots \\ &= \frac{-2x_0(x_0 - 1)}{(x - x_0)} + \ldots. \end{aligned} \tag{73}$$

Proposition 10. *If $y(x), y(x)-1$ or $y(x)-x$ vanishes at $x=x_0$, then*

$$y(x)=\mp\frac{(x-x_0)}{2(x_0-1)}+\dots,\quad y(x)=1\pm\frac{(x-x_0)}{2x_0}+\dots,$$

$$\text{or}\quad y(x)=x_0\mp\frac{(x-x_0)}{2}+\dots,$$

where the conformal structure is singular if the lower sign holds.

Proof. The method of proof is by expansion as in Proposition 8. If $y(x)=-(x-x_0)/2(x_0-1)+\dots$, then $z=z_0+\dots$ is regular, in which case

$$\Omega_1^2=\frac{x_0}{2(1-x_0)}\left(z_0+\frac{1}{2}\right)+\dots,$$

$$\Omega_2^2=\frac{(x-x_0)^2}{4(x_0-1)^2}\left(z_0+\frac{1}{2x_0}\right)\left(z_0+\frac{1}{2}\right)+\dots,$$

$$\Omega_3^2=\frac{x_0}{2(1-x_0)}\left(z_0+\frac{1}{2x_0}\right)+\dots.$$

Here Ω_2^2 vanishes to higher order than either Ω_1^2 or Ω_3^2, so the conformal structure can not be extended over either a zero-dimensional or two-dimensional orbit. The other cases are similar, with Ω_3^2 and Ω_1^2 vanishing.

Now y vanishes if $\xi=e_1$, which from (68) is when

$$e_1=\wp(\omega)+\frac{\wp'(\omega)}{2(\zeta(\omega)-\eta)}.$$

Using the identity

$$\zeta_1(u)\equiv\zeta(u+\omega_1)-\eta_1=\zeta(u)+\frac{\wp'(u)}{2(\wp(u)-e_1)},\tag{74}$$

this is equivalent to $\omega=\omega_1 \pmod{2\omega_1,2\omega_3}$ or $\zeta_1(\omega)=\eta$. But $\omega=k_1\omega_3-ik_3\omega_1$ is imaginary and ω_1 real, so only the second case may occur.

Proposition 11. *If $\zeta_1(\omega)=\eta$ at $x=x_0$, then the solution to the Painlevé equation has the form $y(x)=-(x-x_0)/2(x_0-1)+\dots$.*

Proof. The calculation is similar to Proposition 9. By (68) and (74), ξ satisfies

$$\frac{\wp'(\omega)(e_1-\xi)}{2(\xi-\wp(\omega))(e_1-\wp(\omega))}=\zeta_1(\omega)-\eta,\tag{75}$$

but we have the relation

$$\zeta_1(u) - \frac{\eta_1}{\omega_1}u = \frac{1}{2\omega_1}\frac{\vartheta_2'(\nu)}{\vartheta_2(\nu)}.$$

Following the arguments of Proposition 9 with the theta function ϑ_2, we obtain

$$\zeta_1(\omega) - \eta = \frac{i\omega_1^2}{\pi}\wp'(\omega + \omega_1)(\tau - \tau_0) + \dots,$$

and so from (75),

$$e_1 - \xi = \frac{i\omega_1^2}{\pi}\frac{2(e_1 - \wp(\omega))^2)\wp'(\omega + \omega_1)}{\wp'(\omega)}(\tau - \tau_0) + \dots,$$

and differentiating the formula

$$\wp(u + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(u) - e_1}$$

gives

$$\xi - e_1 = \frac{2i\omega_1^2}{\pi}(e_1 - e_2)(e_1 - e_3)(\tau - \tau_0) + \dots.$$

Hence, converting from τ to x,

$$\begin{aligned} y(x) = \frac{\xi - e_1}{e_2 - e_1} &\doteq \frac{-2i\omega_1^2}{\pi}(e_1 - e_3)(\tau - \tau_0) + \dots \\ &= \frac{(e_1 - e_3)}{2x_0(e_3 - e_2)}(x - x_0) + \dots \\ &= -\frac{(x - x_0)}{2(x_0 - 1)} + \dots. \end{aligned}$$

This discussion of the behaviour of $y(x)$ at some of the special values is sufficient for the regularity arguments which follow. However, we have so far only discussed the regularity of the conformal structure, and not the Einstein metric within that conformal class. The following proposition tells us when the metric is regular.

Proposition 12. *Suppose the conformal structure is non-singular. Then the Einstein metric in the conformal class is well defined if* $\zeta(\omega) - \eta \neq 0$.

Proof. From the twistor point of view, the metric is well-defined for those twistor lines on which the twisted 1-form θ is non-vanishing.

Recall, as in the proof of Theorem 4, that on the twistor space Z, we have

$$\alpha : \mathfrak{g}^c \to TZ \quad \text{and} \quad \theta : TZ \to K^{-1/2},$$

and the composition is defined by

$$x \mapsto \operatorname{tr}(\beta x)$$

for β a holomorphic section of $\mathfrak{g}^c \otimes K^{-1/2}$. Since $A = \alpha^{-1}$, it follows that θ vanishes on a twistor line if and only if

$$\operatorname{tr}(A\beta) = 0.$$

Now in terms of the eigenvectors v_1, v_2 (44), we can write, up to a constant multiple,

$$\beta = v_1 \otimes v_2 + v_2 \otimes v_1$$

using the skew form on $\mathbf{C}^2$ to identify with its dual. Thus

$$\operatorname{tr}(A\beta) = 2 < Av_1, v_2 >,$$

and now applying the formula (45) we see that

$$\text{(76)} \qquad \operatorname{tr}(A\beta) = \frac{\wp'(u-c)}{\wp'(u)} + \frac{(\wp(u+c) - \wp(u-c))}{\wp'(u)}\theta_1.$$

So $\operatorname{tr}(A\beta)$ vanishes if

$$\text{(77)} \qquad \theta_1 + \frac{\wp'(u-c)}{\wp(u+c) - \wp(u-c)} = 0.$$

From (50) and (47), this can be written as

$$\begin{aligned} 0 &= \zeta(2u) - 2\zeta(u-c) - \eta - \zeta(2u) + \zeta(2c) + 2\zeta(u-c) \\ &= \zeta(2c) - \eta = \zeta(\omega) - \eta \end{aligned}$$

from the definition of ω.

Proposition 13. *Let x_0,x_1 be two consecutive values of x at which ω is a lattice point. Then there is a unique point $\bar{x} \in (x_0, x_1)$ at which $\zeta(\omega) = \eta$.*

Proof. At x_0, x_1, $\vartheta_1(\omega, \tau)$ has a simple zero, and it follows that

$$f(x) = \frac{\vartheta_1'}{i\vartheta_1}(\omega, \tau) + k_1\pi$$

tends to $+\infty$ as $x \to x_0+$ and $-\infty$ as $x \to x_1-$. Since ϑ_1 only vanishes at the lattice points, $f(x)$ is regular in the interval and thus has an odd number (with multiplicity) of zeros, and hence at least one.

From (72), the sign of the derivative of f is determined by $\wp'(\omega)$, and this vanishes only when ω is a half-period. Since x_0 and x_1 correspond to neighbouring lattice points, there is only one point $\tilde{x} \in (x_0, x_1)$ at which $\wp'(\omega) = 0$. This cannot be a zero of $\zeta(\omega) - \eta$ by using $\zeta((2n+1)\omega_3) = (2n+1)\eta_3$ and the Legendre relation, so the zeroes of $f(x)$ are simple, and there is an odd number.

On the other hand, in each interval $(x_0, \tilde{x})$, $(\tilde{x}, x_1)$ there can be at most one zero, since the sign of f' is constant in each, thus the number of zeros is odd, less than two, and hence one.

We now put together these results to obtain:

Theorem 7. *Let $y(x)$ be the solution to the Painlevé equation (68), where $k_3 > 0$ and $0 \leq k_1 < 2$. Let $x_n = x(ik_3/(k_1 + 2n - 2))$, for each positive integer n. Then the following hold:*

1. *If $k_1 \leq 1$, the poles of $y(x)$ with negative residue are $\bar{x}_1, \bar{x}_2, \ldots$ with $\bar{x}_n \in (x_n, x_{n+1})$.*
2. *If $k_1 > 1$, the poles of $y(x)$ with negative residue are $\bar{x}_0, \bar{x}_1, \ldots$ with $\bar{x}_n \in (x_n, x_{n+1})$ and $x_0 = 1$.*
3. *The function $y(x)$ defines a complete anti-self-dual Einstein metric for $x \in (\bar{x}_n, x_{n+1}]$. The metric is defined on the unit ball, with $x = x_{n+1}$ the origin and $x = \bar{x}_n$ the boundary.*

Proof. To begin with, note that

$$x = \frac{e_3 - e_1}{e_2 - e_1} = \left[\frac{(1+q)(1+q^3)(1+q^5)\ldots}{(1-q)(1-q^3)(1-q^5)\ldots}\right]^8,$$

where $q = e^{i\pi\tau}$, so that for a real elliptic curve with $q = e^{-\pi\sigma}$, $x > 1$ and $\sigma \to 0$ corresponds to $x \to \infty$, whereas $\sigma \to \infty$ gives $x \to 1$. Moreover, since $dx/d\sigma$ is negative from (69), x is decreasing as a function of σ.

Take a solution $y(x)$, assuming without loss of generality that $k_3 > 0$ and $0 \leq k_1 < 2$. Then $\omega = k_1\omega_3 - ik_3\omega_1$ is a lattice point if and only if

$$\sigma \equiv \frac{\omega_3}{i\omega_1} = \frac{k_3}{k_1 + 2n - 2}$$

for n a positive integer. Let x_n be the corresponding values of $x(i\sigma)$. Then $\{x_n\}$ is an increasing sequence. By Proposition 13, there is a unique zero $\bar{x}_n$ of $\zeta(\omega) - \eta$ in each interval (x_n, x_{n+1}).

We wish to enumerate all such zeros, not just those for $x > x_1$. Suppose $x < x_1$. Then $\sigma > k_3/k_1$. As in Proposition 13, we define

$$f(\sigma) = \frac{\vartheta_1}{i\vartheta_1}(k_1 i\sigma/2 - ik_3/2, i\sigma) + k_1\pi.$$

The zeros of f are the zeros of $\zeta(\omega) - \eta = f(\sigma)/(2\omega_1)$ and hence the poles of $y(x)$ with negative residue.

Near $\nu = 0$, $\vartheta_1'/\vartheta_1 \sim \pi \cot \pi\nu$, which shows that $f(\sigma) \to -\infty$ as $\sigma \to k_3/k_1+$. On the other hand, the expansion

$$\vartheta_1(\nu) = \sum_{n=0}^{\infty} (-1)^n 2q^{(n+1/2)^2} \sin(2n+1)\pi\nu$$

implies that $f(\sigma) \to \pi(k_1 - 1)$ as $\sigma \to \infty$.

Suppose $k_1 < 1$. Then f is negative at both ends of the interval $(k_3/k_1, \infty)$. If it vanishes in the interval, it must do so with derivatives of both signs. However, $\wp'(\omega)$ vanishes only when $\omega = i(k_1\sigma - k_3)/2$ is a half period, and there are no such values when $\sigma > k_3/k_1$. Thus, as in Proposition 13, if f vanishes, the sign of its derivative is fixed. We conclude that $f(\sigma)$ is negative for all $\sigma > k_3/k_1$.

Now suppose $k_1 > 1$. In this case f is positive as $\sigma \to \infty$, and hence has an odd number of zeros in the interval. Since ω is a half-period when $\sigma = k_3/(k_1 - 1)$, we can repeat the argument of Proposition 13 and deduce that there is a unique such zero.

Finally consider $k_1 = 1$. Since $\omega = \omega_3 - ik_3\omega_1$, the points x_n are given by

$$\sigma = \frac{k_3}{2n-1},$$

and so the largest such value is $\sigma = k_3$. Now for $\sigma > k_3$ we need to know if

$$f(\sigma) = \frac{\vartheta_1'}{i\vartheta_1}(\tau/2 - ik_3/2, \tau) + \pi \tag{78}$$

vanishes. But (see [33])

$$\vartheta_1(\nu + \tau/2) = iq^{-1/4}e^{-i\pi\nu}\vartheta_4(\nu),$$

where $q = e^{i\pi\tau}$, so differentiating with respect to ν shows (78) vanishes if and only if

$$\vartheta_4'(-ik_3/2) = 0.$$

But we have an expansion

$$\frac{1}{4\pi}\frac{\vartheta_4'}{\vartheta_4}(\nu) = \sum_1^\infty \frac{q^{2s-1}\sin 2\pi\nu}{1 - 2q^{2s-1}\cos 2\pi\nu + q^{4s-2}}.$$

The denominator can be written, for $\nu = -ik_3/2$, as

$$e^{-(4s-2)\pi\sigma}(e^{(2s-1)\pi\sigma} - e^{\pi k_3/2})(e^{(2s-1)\pi\sigma} - e^{-\pi k_3/2}),$$

and if $\sigma > k_3 > 0$, this is positive, and so in this range ϑ_4' is non-zero. Hence f is negative for all $\sigma > k_3$.

It follows that the poles of $y(x)$ with negative residue are $\bar{x}_1, \bar{x}_2 \ldots$ if $k_1 < 1$, and $\bar{x}_0, \bar{x}_1 \ldots$ if $k_1 > 1$ where $\bar{x}_n \in (x_n, x_{n+1})$ (setting $x_0 = 1$).

To deal with Part 3 of the Theorem, note that from Proposition 9, $y(x)$ has a pole with positive residue at x_n and negative residue at $\bar{x}_n$, so $y \to -\infty$ at both ends of the interval $(\bar{x}_n, x_{n+1})$. It follows that $y(x)$ must be negative in the whole interval, for if the graph crosses the x-axis it must do so at some point with a positive derivative. However, from Proposition 11, only a negative derivative can occur. This means that in particular, $y, y-1$ and $y-x$ are non-zero in the interval, and so the conformal structure is non-singular. Since it is negative definite at $x = x_n$, it is negative definite everywhere. From Proposition 12, the Einstein metric is well-defined for $x \in (\bar{x}_n, x_{n+1})$.

There remains the question of the behaviour of the Einstein metric at the puncture and at the boundary, where we know it is singular. On the twistor space, the twisted 1-form is already well-defined at the boundary, and by Hartog's theorem extends over the twistor line corresponding to the puncture. This means that the *inverse* of the conformal factor (65) is smooth, but may vanish. To determine if it vanishes, we could calculate more expansions, but we can also rely on more general facts. The most important feature is that the conformal structure is everywhere regular. Choosing an Einstein metric within the conformal class can be done by solving a (conformally invariant) linear differential equation: if the metric is $g = f^{-2}g_0$, then f satisfies the equation

$$H_0 f + Ric_0 f = 0,$$

where H_0 is the trace-free part of the hessian, and Ric_0 the trace-free Ricci tensor of some metric in the conformal class. This overdetermined equation severely restricts the behaviour of f at a zero. For an isolated zero, we must have, in normal coordinates, $f = cr^2 + \dots$. However, the scalar curvature of the Einstein metric is given by

$$f^3(6\Delta(f^{-1}) + Rf^{-1}).$$

Since this is a negative constant, f can not in fact vanish at an isolated point. Thus the metric is non-singular at the puncture. On the boundary hypersurface f must have a simple zero (see also [15]), which makes the metric complete.

The metric constructed in Theorem 7 depends on two parameters k_1, k_3 effectively describing the monodromy group. It is natural to ask how the conformal structure on the boundary of the ball depends on the parameters. For this we must calculate Ω_1^2, Ω_2^2 and Ω_3^2 at the point where $\zeta(\omega) = \eta$. From Proposition 12 this corresponds to the conformal factor becoming singular, i.e., to the vanishing of the denominator:

$$x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2 - (1-x)\Omega_1^2) = 0. \tag{79}$$

This provides one relation, and we also have the conserved quantity (28),

$$\Omega_1^2 - \Omega_2^2 - \Omega_3^2 = -1/4. \tag{80}$$

A third equation can be obtained by considering the determinant of the connection matrix at $x = \bar{x}_n$. This is a meromorphic quadratic differential $(\det A)dz^2$ which around $z = 0$ has an expansion:

$$\det A = -\frac{1}{16z^2} + \frac{1}{z}\left(\operatorname{tr} A_1A_2 + \frac{1}{x}\operatorname{tr} A_1A_3\right) + \dots$$

and so, using (25), evaluating the coefficient of $1/z$ gives a linear equation in Ω_3^2 and Ω_2^2. Using $z = (w-e_1)/(e_2-e_1)$, the poles are at $w = e_i$, and the connection form is

$$\sum_{i=1}^{3} \frac{A_i\, dw}{(w-e_i)}$$

Now at $x = \bar{x}_n$, θ_1 is given by (77), and it follows from (45) that

$$\wp'(u)Av_1 = -\frac{\wp'(u-c)}{\wp(u+c) - \wp(u-c)}v_2,$$

$$\wp'(u)Av_2 = \frac{\wp'(u+c)}{\wp(u+c) - \wp(u-c)}v_1,$$

and so

$$\wp'(u)^2 \det A = \frac{\wp'(u+c)\wp'(u-c)}{(\wp(u+c)-\wp(u-c))^2}$$
$$= \wp(2c) - \wp(2u) = \wp(\omega) - \wp(2u)$$

by a well-known identity. Expanding the right-hand side in $\wp(u) = w$ around $w = e_1$ gives

$$\det A = -\frac{1}{16(w-e_1)^2} + \frac{1}{(w-e_1)}\frac{(e_1 + 2\wp(\omega))}{8(e_1-e_2)(e_1-e_3)} + \ldots,$$

thus

$$\frac{\operatorname{tr} A_1A_2}{(e_1-e_2)} + \frac{\operatorname{tr} A_1A_3}{(e_1-e_3)} = -\frac{(e_1+2\wp(\omega))}{8(e_1-e_2)(e_1-e_3)},$$

which yields the equation

$$\Omega_3^2 - x\Omega_1^2 = \frac{(1+x)}{12} - \frac{\wp(\omega)}{4(e_1-e_2)} = \frac{a}{4}. \tag{81}$$

The expansions about the other poles lead to similar equations which are linearly equivalent, given the relation (80).

We need to consider the signs involved here. For this, note the fact that, if u is imaginary, $\wp(u)$ is real and $\wp(u) \to -\infty$ if $u \to 0$ or $2\omega_3$. The maximum in between occurs at the turning point $u = \omega_3$, with value e_3 and thus $\wp(\omega) < e_3$. From (81), this means that

$$a > \frac{(1+x)}{3} - \frac{e_3}{(e_1-e_2)} = x.$$

Continuing, by (81),(79) and (80) we obtain a quadratic equation for Ω_3^2 which yields the two solutions $\Omega_3^2 = 1/4$ and $\Omega_3^2 = a(a-x)/4(1-x)$. Since from (64), for $x > 1$, Ω_3^2 must be negative to get a definite metric, it is the second case which holds and gives

$$\Omega_1^2 = \frac{a(1-a)}{4x(x-1)}, \quad \Omega_2^2 = \frac{(1-a)(x-a)}{4x}, \quad \Omega_3^2 = \frac{a(x-a)}{4(x-1)}.$$

Using (64), and the fact that $a > x$, the positive definite conformal structure on the sphere is then defined by

$$(a-x)\sigma_1^2 + a\sigma_2^2 + (a-1)\sigma_3^2, \tag{82}$$

and since $a > x$, it follows that $a > a - 1 > a - x$.

Remark. Note that the function $y(x)$ depends only on k_1 modulo $2\mathbf{Z}$, as befits its dependence on the monodromy $e^{\pi i k_1}$. However, for each choice of logarithm $k_1 + 2n$, there is a metric on the ball, inducing a different conformal structure on the boundary sphere, for x can be read off from the conformal structure in (82). The analytic continuation of a single solution $y(x)$ to the Painlevé equation thus defines many Einstein metrics.

The obvious question to ask is whether every invariant conformal structure on $SU(2)$ can be obtained in this way. Let the conformal structure be given by

$$\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2 + \lambda_3\sigma_3^2.$$

Clearly, from (82), since $x \neq 0, 1$, the λ_i must be distinct. Assuming this, choose a basis such that $\lambda_2 > \lambda_3 > \lambda_1 > 0$ and define

$$x = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3}.$$

Then $x > 1$. Now consider the elliptic curve in Weierstrass form

$$\wp'(u)^2 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3),$$

with $e_1 = \lambda_2 - (\lambda_1 + \lambda_2 + \lambda_3)/3, e_2 = \lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3)/3, e_3 = \lambda_1 - (\lambda_1 + \lambda_2 + \lambda_3)/3$, so that

$$x = \frac{e_3 - e_1}{e_2 - e_1}.$$

Take ω to satisfy

$$\wp(\omega) = -(\lambda_1 + \lambda_2 + \lambda_3)/3.$$

Now ω must be imaginary: as above, for imaginary values of u, $\wp(u)$ lies in the interval $(-\infty, e_3)$, and since

$$e_3 = \lambda_1 - (\lambda_1 + \lambda_2 + \lambda_3)/3 > -(\lambda_1 + \lambda_2 + \lambda_3)/3,$$

the value $-(\lambda_1 + \lambda_2 + \lambda_3)/3$ is achieved at two imaginary points $\pm\omega$. This choice, by a simple manipulation, leads to the conformal structure (82) with a satisfying

$$\frac{(1+x)}{12} - \frac{\wp(\omega)}{4(e_1 - e_2)} = \frac{a}{4},$$

as in formula (81).

Finally define η by

$$\zeta(\omega) = \eta.$$

Since $\zeta(u)$ is real and odd, η is imaginary. Now define real numbers

$$k_1 = 2(\omega\eta_1 - \omega_1\eta)/i\pi,$$
$$k_3 = -2(\omega\eta_3 - \omega_3\eta)/\pi.$$

For the sphere to arise as the boundary of one of the metrics of Theorem 7, we need to show that $k_3 \neq 0$. Suppose $k_3 = 0$. Then since ω is defined up to sign and addition of $2n\omega_3$ and $y(x)$ depends only on the monodromy, we can take $k_1 \leq 1$. In fact, since the monodromy is non-abelian, $k_1 < 1$. The relation $\zeta(\omega) = \eta$ implies that

$$\frac{\vartheta_1'}{\vartheta_1}(k_1\tau/2, \tau) + ik_1\pi = 0.$$

By the identity

$$\vartheta_1(\nu/\tau, -1/\tau) = \frac{\sqrt{\tau}}{i\sqrt{i}}e^{\pi i\nu^2/\tau}\vartheta_1(\nu, \tau),$$

this is equivalent to

$$\vartheta_1'(k_1/2, -1/\tau) = 0.$$

But the expansion

$$\frac{1}{4\pi}\frac{\vartheta_1'(\nu)}{\vartheta_1(\nu)} = \frac{1}{4}\cot\pi\nu + \sum_{n=1}^{\infty}\frac{\sin 2\pi\nu}{q^{-2n} - 2\cos 2\pi\nu + q^{2n}} \tag{83}$$

shows that for a real elliptic curve with $\tau = i\sigma$, the right-hand side is positive if $k_1 < 1$. But then $\zeta(\omega) - \eta$ is never zero, which contradicts the definition of η.

From the above discussion, we can prove the following theorem:

Theorem 8. *Given any left-invariant conformal structure on the 3-sphere, there exists a unique complete anti-self-dual Einstein metric on the 4-ball which induces the given conformal structure on the sphere at infinity.*

Proof. Choose a basis such that the conformal structure is given by

$$\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2 + \lambda_3\sigma_3^2$$

with $\lambda_2 \geq \lambda_3 \geq \lambda_1 > 0$. If equality holds anywhere, then Pedersen's metric [29] or the hyperbolic metric provides the solution. Otherwise, use the solution of Theorem 7, with the values of k_1, k_3 given above.

As to uniqueness, LeBrun's twistor method [22] shows that the local metric in a collar neighbourhood is determined by the conformal structure: the twistor space is the space of null geodesics of the complexification of the conformal structure on the 3-manifold. Thus an $SU(2)$-invariant conformal structure generates an $SU(2)$-invariant Einstein metric, and hence one of the cases considered here. The only choice is then whether to take the solution for $x < \bar{x}_n$ in Theorem 7, or for $x > \bar{x}_n$. But the second case is an analytic continuation of the first, and if it can be completed to the ball, we would have an anti-self-dual conformal structure on a compact 4-manifold homeomorphic to S^4. Since the signature of the sphere is zero, but also represented by the $\mathcal{L}^2$ norm of the Weyl tensor, it must be conformally flat and hence the Einstein metric is the hyperbolic metric. This yields the standard conformal structure on S^3, and so any other metric is unique. Of course, since the upper and lower hemispheres are isometric the metric is also unique in this case too.

Remark.

1. The classification of Theorem 13 in the final section will show in fact that we do not need to prescribe the topology of the interior as the ball — there are no other complete anti-self-dual Einstein metrics which induce a left invariant conformal structure on the 3-sphere.

2. Since we have seen in the proof of Theorem 8 that the explicit metrics of Theorem 7 are the same as the twistor-theoretic ones whose existence was shown by LeBrun, it is useful to note how the isomonodromic deformation story fits in to his approach. We recall that the twistor space Z is the space of complex null geodesics on $SL(2,\mathbf{C})$ with respect to a left-invariant metric. Consider the action of $SL(2,\mathbf{C})$ on Z. The monodromy group Γ should in principle be the stabilizer of a generic point in Z, i.e., a null geodesic. A geodesic $\gamma(z)$ is stabilized by $g \in SL(2, C)$ if

$$g\gamma(z) = \gamma(z + k).$$

In the general context of geodesics of left-invariant metrics on Lie groups (see [1], for example) we set

$$\omega_c = \gamma^{-1}\frac{d\gamma}{dz}.$$

Then ω_c satisfies Euler's equation, which is of the form

$$\frac{d\omega_c}{dz} = B(\omega_c, \omega_c).$$

For $SU(2)$, the problem is the same as the motion of a rigid body, where ω_c is interpreted as the angular velocity in the body. It is clearly invariant by any left multiplication, in particular by g above and hence periodic. In fact, Euler's equation, as discussed in Section 3, can be solved with elliptic functions, and $\gamma(z)$ therefore satisfies a linear equation

$$\frac{d\gamma}{dz} = \gamma\omega_c$$

on an elliptic curve. In this context, g is a monodromy element for the solution. We leave it to the reader to link up this monodromy problem for a connection on an elliptic curve with our discussion in Section 7. The explicit formulae for the motion of a rigid body given in [33] as an application of elliptic functions may be of some help.

Let us consider now the function (68) with $k_1 = 1$ and $k_3 > 0$. This will provide a one-parameter family of metrics in some ways modelled on the Bergmann metric. In particular, it will induce a left-invariant CR-structure on the boundary.

Theorem 9. *Let $y(x)$ be the solution to the Painlevé equation (68) with $k_1 = 1$ and $k_3 > 0$. Let $x_1 = x(ik_3)$. Then $y(x)$ defines for $x \in (1, x_1]$ a complete anti-self-dual Einstein metric on the unit ball with $x = x_1$ the origin.*

Proof. From Theorem 7, $y(x)$ has no poles in $(1, x_1)$ and so is regular for all $x \in (1, x_1)$. Moreover, as in the proof of that theorem, $y(x) \to -\infty$ as $x \to x_1-$. As $x \to 1+$, $\sigma \to \infty$ and we can use the expansion

$$\vartheta_1(\nu) = \sum_{n=0}^{\infty} (-1)^n 2q^{(n+1/2)^2} \sin(2n+1)\pi\nu$$

and its derivatives to estimate $y(x)$. We find

$$y(x) = -1 - (2 + \cosh \pi k_3)s/4 + \dots, \tag{84}$$

where $x = 1 + s$. In particular, the limit is negative as $x \to 1+$. Since from Proposition 11, the graph of $y(x)$ can only cross the x-axis with negative derivative, it follows that $y(x)$ is always negative, and in particular never equal to $0, 1$ or x, hence the Einstein metric is well defined, and as in Theorem 7, extends over the puncture at $x = x_1$.

The behaviour as $x \to 1$ can be seen by using (64) and (65). We obtain

$$\begin{aligned}
\Omega_1^2 &= -\frac{1}{4}\cosh^2(\pi k_3/2) + \dots,\\
\Omega_2^2 &= \frac{1}{16}\sinh^2(\pi k_3/2)\cosh^2(\pi k_3/2) + \dots,\\
\Omega_3^2 &= -\frac{1}{4}\sinh^2(\pi k_3/2) + \dots,
\end{aligned}$$

and this yields a metric which is asymptotically a constant multiple of

$$\begin{aligned}
g = \frac{ds^2}{s^2} &+ \frac{4}{s\cosh^2(\pi k_3/2)}\sigma_1^2 + \frac{16}{s^2\sinh^2(\pi k_3/2)\cosh^2(\pi k_3/2)}\sigma_2^2\\
&+ \frac{4}{s\sinh^2(\pi k_3/2)}\sigma_3^2 + \dots,
\end{aligned}$$

which is clearly complete as $s \to 0$.

Note that the coefficient of σ_2^2 vanishes to order s^2 just as in the Bergmann metric. Moreover on the annihilator of σ_2, we have the conformal structure

$$h = \cosh^2(\pi k_3/2)\sigma_1^2 + \sinh^2(\pi k_3/2)\sigma_3^2.$$

From this, we evidently have the following:

Theorem 10. *Let h be a left-invariant CR-structure on S^3. Then there is a complete anti-self-dual Einstein metric on the ball which induces h on the boundary.*

Proof. Choosing a suitable basis, any invariant CR-structure can be represented in the form

$$\lambda_1\sigma_1^2 + \lambda_3\sigma_3^2$$

with $\lambda_1 \geq \lambda_3 > 0$. If $\lambda_1 = \lambda_3$, the Bergmann metric provides the required solution. Otherwise, take $k_3 > 0$ satisfying

$$\tanh(\pi k_3/2) = \sqrt{\lambda_3/\lambda_1}$$

and use the metric of Theorem 9.

9. A classification of Einstein metrics

In the previous sections we have seen how monodromy considerations enable us to write down explicitly the solutions to Einstein's equations which arise from Painlevé's sixth equation, whereas the general $SU(2)$-invariant anti-self-dual conformal structure is still as intractable as the general Painlevé transcendent. Recall that in Section 4 we mentioned the fact that the general anti-self-dual conformal structure with $\wedge^3\alpha$ vanishing with multiplicity 2 can be solved with Painlevé's third equation, and hence more Painlevé transcendents. It turns out that the Einstein condition again provides a simplification which we shall use in this section to complete the classification of anti-self-dual Einstein metrics with $SU(2)$ symmetry and 3-dimensional generic orbits.

We shall prove the following:

Theorem 11. *Let M be an $SU(2)$-invariant anti-self-dual Einstein manifold with 3-dimensional orbits, and suppose that on its twistor space $\wedge^3\alpha$ vanishes on a divisor D with multiplicity 2. Then the Einstein metric is defined by an $SU(2)$-invariant $U(1)$ monopole on a 3-dimensional space of constant curvature.*

Before beginning the proof, let us recall the particular construction of anti-self-dual conformal structures by means of $U(1)$ monopoles. This has its origin in the Gibbons-Hawking Ansatz for hyperkähler metrics [13], where the 3-space is $\mathbf{R}^3$ and was used in the case of S^3 by Pedersen [29] and H^3 by LeBrun [24]. A $U(1)$ monopole on a 3-manifold N^3 with Riemannian metric h is a connection A on a principal $U(1)$-bundle, and a function V such that V and the curvature F are related by the Bogomolny equations

$$F = - * dV.$$

This is essentially equivalent to V being harmonic.

When h is of constant curvature, the conformal structure

$$g_0 = Vh + V^{-1}(d\tau + A)^2$$

on the 4-dimensional principal $U(1)$-bundle over N is anti-self-dual. Moreover, when V and A are $SU(2)$-invariant, the metric

$$g = f^2(Vh + V^{-1}(d\tau + A)^2)$$

is Einstein for a suitable choice of function f (see [29]). In this symmetric situation, we have the following:

- If $N = \mathbf{R}^3$, then $V = \epsilon + m/r$ and $f = c$, which gives a metric with scalar curvature $R = 0$.
- If $N = S^3$, then $V = \epsilon + m \cot\rho$ and $f = c(\epsilon\cos\rho - m\sin\rho)^{-1}$, where ρ is the distance from the fixed point of the $SU(2)$ action on N. This gives an Einstein metric with scalar curvature $R = -3\epsilon/c^2$.
- If $N = H^3$, then $V = \epsilon + m \coth\rho$ and $f = c(\epsilon\cosh\rho + m\sinh\rho)^{-1}$, where ρ is the distance from the fixed point. This gives an Einstein metric with scalar curvature $R = 3\epsilon/c^2$

In all cases ϵ, m, and c are real constants.

Proof of theorem. We shall use the twistor approach. From this point of view, a $U(1)$ monopole over a space of constant curvature is give by a holomorphic principal $\mathbf{C}^*$ bundle over its minitwistor space $\mathbf{T}$ which is of degree 0 on each twistor line in $\mathbf{T}$ (see [17]). The minitwistor space of $\mathbf{R}^3$ is the tangent bundle of $\mathbf{C}P^1$ with twistor lines the sections; of S^3 the quadric $\mathbf{C}P^1 \times \mathbf{C}P^1 \subset \mathbf{C}P^3$ with twistor lines plane sections; and of H^3 an open set in the quadric. Thus we need to prove that, if Z is the twistor space of the Einstein manifold in the theorem, it can be expressed (at least locally around a twistor line) as a principal holomorphic $\mathbf{C}^*$ bundle over a minitwistor space $\mathbf{T}$, and such that the twistor lines of Z map into twistor lines of $\mathbf{T}$. In fact, the existence of a $\mathbf{C}^*$ action is a global assertion, and it is sufficient for our purposes for Z to fibre over $\mathbf{T}$ with the fibres orbits of a holomorphic vector field.

From the hypothesis of the theorem, we have $\Lambda^3\alpha$ vanishing with multiplicity 2 on D, thus $2D$ is a divisor of K^{-1} and (working locally or passing to a double covering) there is a holomorphic section v of $K^{-1/2}$ vanishing on D. Since D is $SU(2)$-invariant, so is v. But as in Theorem 4, the twisted 1-form θ on the twistor space of an Einstein manifold defines by composition with α a section β of $K^{-1/2} \otimes \mathfrak{g}^c$, and hence a 3-dimensional subspace $\mathfrak{g}^c \subset H^0(Z, K^{-1/2})$. The section v lies outside this subspace since it is invariant whereas the representation $\mathfrak{g}^c$ is irreducible. We thus have a 4-dimensional subspace U of the space of sections.

We claim that there are no base points for this system, that is there are no points where all sections vanish. Suppose for a contradiction that $z \in Z$ is such a point. Since s vanishes at z, this point lies in D. But the base locus is invariant by $SU(2)$, which has 3-dimensional orbits, hence a neighbourhood D_0 of z in D is contained in the locus. By reality, the same is true of the antipodal point $\tau(z)$. Thus all sections $u \in U$ vanish on $D_0 + \tau(D_0)$. Now let P be a real twistor line passing through z. Since $K^{-1/2}$ is of degree 2 on P, D meets it only in the points $z, \tau(z)$. Nearby twistor lines meet D in D_0 and $\tau(D_0)$, thus in a neighbourhood of P, all sections $u \in U$ vanish only on $D_0 + \tau(D_0)$. Thus each u can be written $u = fv$ for some holomorphic function f in a neighbourhood of P. But on any twistor space all holomorphic functions are constant, thus u is $SU(2)$-invariant, which we know not to be the case. Hence there are no base points.

Since there are no base points, we can use the system U to define a holomorphic map $p : Z \to \mathbf{C}P^3$ with the property that $p^*(\mathcal{O}(1)) \cong K^{-1/2}$. Since $K^{-1/2}$ has degree 2 on twistor lines, each twistor line maps to a conic in $\mathbf{C}P^3$. Now $s = \operatorname{tr}(\beta^2)$ is an $SU(2)$-invariant section of K^{-1}, and so its divisor is a component of the zero set of $\wedge^3\alpha$. Hence, at least in a neighbourhood of a twistor line, $s = \lambda v^2$ for some constant λ. But the equation

$$\operatorname{tr}(\beta^2) = \lambda v^2$$

defines a subvariety of degree 2 in $\mathbf{C}P^3$, a nonsingular quadric if $\lambda \neq 0$ and a quadric cone if $\lambda = 0$. Note that from (39), which is equally valid in the present context, $\lambda = 0$ iff the scalar curvature $R = 0$. Thus if $R \neq 0$, p maps Z to a quadric and the twistor lines map to conics in the quadric, i.e., to plane sections. If $R = 0$, p maps to the cone, but the complement of the vertex is isomorphic to $T\mathbf{C}P^1$, and the sections are plane sections missing the vertex. In either case, then, the image is the minitwistor space for a 3-manifold of constant curvature, and the twistor lines correspond.

It remains to find the vector field along the fibres. For this, note (see [8], for example), that any real holomorphic section of $K^{-1/2}$ corresponds to a section ω of the bundle Λ^2_+ on M satisfying a certain differential equation $\bar{D}_2\omega = 0$, and as a consequence of this and the Einstein condition, the vector field X dual to the 1-form $d^*\omega$ is a Killing vector field. Thus the section v generates an $SU(2)$-invariant vector field, which preserves the metric and hence induces a holomorphic vector field on the twistor space. Being invariant, it leaves fixed each section in U, and so acts along the fibres of the projection p.

The only thing left to check is that X is not zero. However, if this were so, then ω would satisfy $\bar{D}_2\omega = 0$ and $d\omega = 0$ and would then be covariant constant by the definition of $\bar{D}_2$. But then the holonomy would reduce to $U(2)$ (i.e., the metric would be Kähler) or be trivial. On the other hand, as we have remarked in Section 2, an anti-self-dual Kähler metric has zero scalar curvature, and since the metric is Einstein, this makes it hyperkähler. Since the Kähler form ω is invariant, the action of $SU(2)$ on the full 3-dimensional space of covariant constant forms is trivial. In twistor terms, however, this means that the rank of α is 2, the case excluded here and dealt with in Theorem 2. The theorem is thus proved.

We can now put together Theorems 2, 6, and 11 to obtain a list of all invariant anti-self-dual Einstein metrics:

Theorem 12. *Let M be an $SU(2)$-invariant anti-self-dual Einstein metric with 3-dimensional generic orbits. Then M is locally isometric to one of the following:*

1. *a 4-manifold of constant curvature,*
2. *the Eguchi-Hanson metric,*
3. *the Belinskii-Gibbons-Page-Pope metric,*
4. *an $SU(2)$-invariant $U(1)$-monopole over a 3-manifold of constant curvature,*
5. *a metric defined by the solution to Painlevé's sixth equation in Theorem 6.*

Remark. The metrics which appear in two places in the list are those which have two different $SU(2)$ actions, or equivalently which have an isometry group which is a compact Lie group of rank 2. Only $SO(4)$, $SO(5)$ and $SU(3)$ can act on a 4-manifold and these give the standard metrics on S^4 and $\mathbf{C}P^2$. For the solution of Painlevé VI which gives rise to these, see [19].

A more useful classification is to consider the global condition of *completeness.* In this case we obtain the following list.

Theorem 13. *Let M be a complete $SU(2)$-invariant anti-self-dual Einstein metric with 3-dimensional generic orbits.*

1. *If $R > 0$, then M is isometric to:*
   (a) *the constant curvature metric on S^4,*

(b) *the Fubini-Study metric on* $\mathbf{C}P^2$.

2. *If* $R = 0$, *then* M *is isometric to:*
   (a) *the flat metric on* $\boldsymbol{R}^4$,
   (b) *the Taub-NUT metric on* $\boldsymbol{R}^4$,
   (c) *the Eguchi-Hanson metric on the cotangent bundle of* S^2,
   (d) *the* $\mathcal{L}^2$ *metric on the moduli space of two* $SU(2)$*-monopoles on* R^3.
3. *If* $R < 0$, *then* M *is isometric to:*
   (a) *the metric of constant negative curvature on the unit ball in* $\boldsymbol{R}^4$,
   (b) *the Pedersen metric on the unit ball,*
   (c) *one of the metrics of Theorem 7 on the unit ball,*
   (d) *the Pedersen/LeBrun metric on a line bundle of degree* $-n$ *over* S^2,
   (e) *the Bergmann metric on the unit ball,*
   (f) *one of the metrics of Theorem 9 on the unit ball.*

Proof. For the purposes of the proof, we shall distinguish the three types of solutions as follows:

- Type I: $\Lambda^3\alpha \equiv 0$.
- Type II: $\Lambda^3\alpha \not\equiv 0$ and vanishes with multiplicity 2.
- Type III: $\Lambda^3\alpha \not\equiv 0$ and vanishes nondegenerately.

(1) First consider the case $R > 0$. It is a well-known consequence of Myers' theorem that a complete Einstein manifold with positive scalar curvature is compact. We can then use Theorem (13.30) of [8] to deduce that M is isometric to $\mathbf{C}P^2$ or S^4, without using directly the symmetry assumption.

(2) Next consider the case $R = 0$, and metrics of Type I. Here Theorem 2 tells us that the metric is either conformally flat, Eguchi-Hanson or Belinskii-Gibbons-Page-Pope. The latter is well-known to be incomplete, and the Eguchi-Hanson metric is complete. A conformally flat Einstein metric with zero scalar curvature is flat, and so M is covered by $\mathbf{R}^4$. Since $SU(2)$ is simply connected, its action on M lifts to $\mathbf{R}^4$, where it has a unique fixed point, the origin. But $SU(2)$ must commute with the covering transformations, which therefore have a fixed point which is a contradiction unless $M = \mathbf{R}^4$.

Now consider Type II metrics. Here the non-flat ones are given, up to a scalar multiple, by taking $V = 1 + m/r$. When $m > 0$, this gives the Taub-NUT metric (see e.g.[8]) which is complete, and when $m < 0$ a metric which is incomplete at $r = -m$.

The metrics of Type III, and their completeness properties, are discussed in [4], in fact in the same context as the Taub-NUT metric. The only complete one is the 2-monopole hyperkähler metric.

(3) Now consider the case $R < 0$. Beginning with Type I, the only possibility is a conformally flat metric and hence a manifold of constant negative curvature. Again, since the $SU(2)$ action on H^4 has a unique fixed point, M must be isometric to H^4.

As for the metrics of Type II, there are two cases in Theorem 11, involving trigonometric or hyperbolic functions. Since we require negative scalar curvature, the formula for R shows that we can take $c = 1, \epsilon = 1$ in the trigonometric case and $c = 1, \epsilon = -1$ in the hyperbolic case. In diagonal form the metrics (from [29]) can be written as

$$g = \frac{1}{(\cos\rho - m\sin\rho)^2}\Big[(1 + m\cot\rho)(d\rho^2 + 4\sin^2\rho(\sigma_1^2 + \sigma_2^2)) + \frac{4m^2}{(1 + m\cot\rho)}\sigma_3^2\Big],$$

or

$$g = \frac{1}{(-\cosh\rho + m\sinh\rho)^2}\Big[(-1 + m\coth\rho)(d\rho^2 + 4\sinh^2\rho(\sigma_1^2 + \sigma_2^2)) + \frac{4m^2}{(-1 + m\coth\rho)}\sigma_3^2\Big].$$

In both cases, near $\rho = 0$, the metric is given by

$$g = 4m(d(\rho^{1/2})^2 + (\rho^{1/2})^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)) + \ldots,$$

which clearly extends to the puncture $\rho = 0$, as remarked in [29].

In the first case, the conformal structure is regular, but the metric blows up at $\rho = \rho_0$, where $\cot\rho_0 = m$ and this induces the conformal structure

$$\sigma_1^2 + \sigma_2^2 + \frac{m^2}{1 + m^2}\sigma_3^2$$

on the sphere. If $m > 0$, the metric is regular for $0 < \rho < \rho_0$, and this is Pedersen's metric (also found analytically in [14]) on the ball, inducing a conformal structure $\sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2$ on the boundary when $\lambda < 1$. For $\rho > \rho_0$, the metric becomes singular and incomplete at $\rho = \rho_1$ where $\cot\rho = -1/m$, and for $\rho > \rho_1$ it changes signature. If $m < 0$, the behaviour is similar, replacing ρ by $\pi - \rho$.

In the second, hyperbolic, case when $m > 1$ the metric is singular at $\rho = \rho_0$, where $\coth\rho_0 = m$ and this induces the conformal structure

$$\sigma_1^2 + \sigma_2^2 + \frac{m^2}{m^2-1}\sigma_3^2$$

on the sphere. Since $\coth\rho > 1 > 1/m$, in this case the conformal structure is regular for all ρ. For $\rho < \rho_0$ this is the Pedersen metric on the ball inducing the conformal structure $\sigma_1^2+\sigma_2^2+\lambda\sigma_3^2$ on the boundary for $\lambda > 1$. We need to estimate the metric as $\rho \to \infty$. This is easily seen to be

$$g = \frac{4}{(m-1)}\left(e^{-2\rho}d\rho^2 + \frac{4m^2}{(m-1)^2}e^{-2\rho}\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right),$$

which is incomplete, and in fact has a conical singularity around a 2-sphere, for setting $r = e^{-\rho}$, the metric is a multiple of

$$dr^2 + \left(\frac{2m}{m-1}\right)^2 r^2\sigma_3^2 + \sigma_1^2 + \sigma_2^2.$$

It can be extended to a metric on a smooth manifold, the complex line bundle of degree n over S^2 if m satisfies $m = n/(n-2)$. In this case, the conformal structure was found by LeBrun in [23] in the context of scalar-flat Kähler metrics (see also [30]), but it is a continuation of Pedersen's metric. The result is a complete Einstein metric with conformal infinity the Lens space $S^3/\mathbf{Z}_n$. If $\rho < 0$, the metric is negative definite.

When $m = 1$ the metric is defined for all $\rho > 0$ and is the Bergmann metric (see [29]). If $m < 1$, the conformal factor is still non-vanishing, but the conformal structure is singular where $\coth\rho = 1/m$. Since the coefficients of both σ_1^2 and σ_2^2 vanish, the metric does not extend to any 2-dimensional orbit.

Finally consider the generic case Type III, which relies on the solution (68) to the Painlevé equation. From Proposition 8, we saw that the two types of poles of $y(x)$, with positive or negative residues, give rise respectively to punctures or boundaries of Einstein metrics. When the conformal structure becomes singular as in Proposition 10, with $\Omega_2^2 \to 0$, the conformal factor (65) approaches

$$\frac{\Omega_2(x_0 - 4(1 - x_0)\Omega_1^2)}{2\Omega_1\Omega_3(1 - x_0)^2}.$$

Here we have used $\Omega_1^2 - \Omega_2^2 - \Omega_3^2 = -1/4$, which also shows that all three do not vanish simultaneously. Since, as in Proposition 10, Ω_2^2 vanishes to higher order than either Ω_1^2 or Ω_3^2, the conformal factor tends to zero, and we do not obtain a complete metric at such a singularity.

Thus a complete Einstein metric must be defined on intervals in which the conformal structure is nonsingular, and whose end-points are either $1, \infty$ or poles of $y(x)$.

There are four possible domains of definition:

1. a finite interval (x_0, x_1),
2. an interval $(1, x_1)$,
3. an interval (x_1, ∞),
4. the whole interval $(1, \infty)$,

where x_0, x_1 are poles of $y(x)$, and there are no poles in the interval. We begin with the case $k_3 \neq 0$, assuming as usual $k_3 > 0$ and $0 \leq k_1 < 2$. Suppose first that $k_1 \neq 0, 1$.

From Theorem 7, $y(x)$ has poles at the points $x(ik_3/(k_1 + 2n - 2)) = x_n$ and as $n \to \infty$, $x_n \to \infty$, which means that any interval of the form (3) or (4) contains poles and so cannot define a complete metric.

For a finite interval, Proposition 13 describes the situation. On the interval $(\bar{x}_n, x_{n+1})$ we have a complete metric already. If such a metric existed on $(x_n, \bar{x}_n)$, the conformal structure would be an analytic continuation and we could produce an anti-self-dual conformal structure on a manifold homeomorphic to S^4. As argued in Theorem 8, this must be conformally flat, which is a contradiction.

Thus the metrics of Theorem 7 are the only complete ones occurring on a finite interval.

It remains to consider case (2), and for that we must estimate $y(x)$ as $x \to 1$, or equivalently $\sigma \to \infty$. In this limit, it is convenient to normalize the choice of k_3, k_1 by taking $0 < k_1 < 1$ and not fixing the sign of k_3. Since $y(x)$ depends only on the monodromy, this has no effect on the estimates.

Using the expansion

$$\vartheta_1(\nu) = \sum_{n=0}^{\infty} (-1)^n 2q^{(n+1/2)^2} \sin(2n+1)\pi\nu$$

with $\nu = i(k_1\sigma - k_3)/2$ we obtain an expansion in increasing powers of $e^{-\sigma}$

$$\frac{\vartheta_1'}{\vartheta_1}(\nu) = i\pi \left(-1 - 2e^{2\pi i\nu} + 2e^{-2\pi\sigma - 2\pi i\nu} + \dots\right), \tag{85}$$

and differentiating this to obtain expressions for $\wp(\omega)$ and $\wp'(\omega)$ we find

$$y(x) = 1 + \frac{k_1 e^{\pi k_3}(x-1)^{k_1}}{2^{4k_1-2}(k_1-1)} + \dots, \tag{86}$$

By this expansion for $y(x)$ near $x = 1$, the Einstein metric from (64) and (65) is of the form

$$g = s^{k_1-2}ds^2 + \frac{4s_1^k}{(k_1-1)^2}\sigma_2^2 + c(\sigma_1^2 + \sigma_3^2) + \dots, \tag{87}$$

where $x = 1 + s$ and c is a constant. Putting $t = s^{k_1/2}$ the metric can be seen to have a conical singularity of angle $\pi k_1/(1-k_1)$ around an $\mathbf{R}P^2$ orbit (cf [19]) or twice the angle around an S^2 orbit on a double covering. If the angle is $2\pi n$ for an integer n, then the metric is well defined on a quotient by a right action of $\mathbf{Z}_n$. But this requires the coefficients of σ_1^2 and σ_3^2 to be equal, which leads to the case of metrics of Type II. If the angle is $2\pi/n$, then the metric is defined on an n-fold covering branched over the orbit, but for topological reasons the generic orbit of $SU(2)$ is then S^3/G where G is the quaternion group, or a double covering.

Returning to the normalization with $k_3 > 0$, Proposition 13 shows that the first pole $x = x_0$ of $y(x)$ has positive residue if $0 < k_1 < 1$ and negative residue if $1 < k_1 < 2$. In the first case, $y(x) \to -\infty$ as $x \to x_0-$, so that x_0 is a puncture. But a smooth metric requires the orbit of $SU(2)$ to be a 3-sphere, and not a quotient as above. In the second case, $y(x) \to +\infty$ as $x \to x_0-$. But from (86), $y(x)$ approaches 1 from below as $x \to 1+$, thus the graph of $y(x)$ must meet the line $y = x$ with positive derivative, which gives a singular conformal structure from Proposition 10.

Hence the generic case $k_1 \neq 0, 1$, $k_3 > 0$ produces only the complete metrics of Theorem 7.

Consider next the special case $k_1 = 0$, where we can now take $k_3 > 0$. The arguments above hold except for the expansion as $x \to 1$. Here, ν is independent of σ and as $\sigma \to \infty$, we use the expansion

$$\frac{1}{4\pi}\frac{\vartheta_1'(\nu)}{\vartheta_1(\nu)} = \frac{1}{4}\cot \pi\nu + q^2 \sin 2\pi\nu \ldots, \tag{88}$$

which yields the following behaviour for $y(x)$ near $x = 1$:

$$y(x) = 1 + \frac{1}{2}(x-1) + \frac{\sinh^2(\pi k_3/2)}{2}(x-1)^2 + \ldots.$$

Since $y(x)$ has derivative $1/2$ at $x = 1$, if $y \to +\infty$ as $x \to x_1-$, the graph must meet the line $y = x$ at some point x_0 with $y(x) - x$ having positive derivative, giving a singular conformal structure from Proposition 10. On the other hand if $y \to -\infty$, then x_1 is the point $\sigma = k_3/2$ and as $\sigma \to k_3/2+$, the expansion (83) shows that $\vartheta_1'/i\vartheta_1$ is large and negative. But from (88) it is positive as $\sigma \to \infty$, so again $y(x)$ must have a pole in the interval.

Now consider the other special case $k_1 = 1$ and $k_3 > 0$. Again the arguments above deal with the case of an interval (x_0, x_1). Let $(1, x_1)$ be an interval on which a metric is defined and complete. Then x_1 must be the first pole of $y(x)$. From Proposition 13, $x_1 = x(ik_3)$. But then Theorem 9 describes a complete metric.

Finally consider the case $k_3 = 0$. We can now choose $0 \leq k_1 \leq 1$, and in fact since the monodromy is non-abelian, $k_1 \neq 0, 1$. In this case $\omega = k_1\omega_3$ and is never a period, so there are no punctures. The other pole of $y(x)$, for a boundary, occurs when

$$\frac{\vartheta_1'}{\vartheta_1}(k_1\tau/2, \tau) + ik_1\pi = 0.$$

Differentiating the identity

$$\vartheta_1(\nu/\tau, -1/\tau) = \frac{\sqrt{\tau}}{i\sqrt{i}} e^{\pi i\nu^2/\tau}\vartheta_1(\nu, \tau) \tag{89}$$

is equivalent to

$$\vartheta_1'(k_1/2, -1/\tau) = 0,$$

but the expansion (83) shows that this never occurs if $k_1 < 1$, thus $y(x)$ is regular on the whole interval $(1, \infty)$. As $x \to 1$, the asymptotic form (87) still holds, with $k_3 = 0$. The arguments advanced there hold too and imply that $k_1 = 2/3$ or $1/2$. In this case, with $k_3 = 0$, the monodromy is then from Theorem 5 finite, in fact the binary dihedral groups of the symmetries of a triangle or square. As discussed in ([19]), these cases are analytic continuations of the standard metrics on S^4 and $\mathbf{C}P^2$, and the Einstein metrics of negative scalar curvature in these families are the hyperbolic metric and Bergmann metric, dealt with elsewhere and not, in fact, admitting an $SU(2)$ action of Type III.

Since all cases of Type and monodromy have now been considered, the classification follows.

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