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# PINCHING BELOW $\frac{1}{4}$, INJECTIVITY RADIUS, AND CONJUGATE RADIUS

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*Dedicated to S. S. Chern and W. Klingenberg*

ABSTRACT. The injectivity radius of any simply connected, even dimensional Riemannian manifold $M^n$ with positive sectional curvature equals its conjugate radius. So far the corresponding result in odd dimensions has only been known under the additional hypothesis that $M^n$ is weakly $\frac{1}{4}$-pinched. Moreover, some famous examples due to M. Berger show that the statement is even false, unless $M^n$ is at least $\frac{1}{9}$-pinched. It has been a longstanding problem whether the pinching constant can be pushed below $\frac{1}{4}$ for odd dimensional manifolds or not. In this paper we prove that this is indeed possible. The pinching constant $\delta \in [\frac{1}{9}, \frac{1}{4})$ that is needed in our main theorem does not depend on the dimension. As an application we obtain a sphere theorem for simply connected, odd dimensional, $\delta_n$-pinched manifolds where the pinching constant $\delta_n$ is strictly less than $\frac{1}{4}$ and up to now still depends on the dimension.

## 1. Introduction

By the Theorem of Bonnet and Myers any complete Riemannian manifold $(M^n, g)$ with sectional curvature $K_M \geq \lambda^2 > 0$ is compact. Its injectivity radius $\operatorname{inj} M^n$ is bounded from above by its conjugate radius $\varrho_c$. If $M^n$ is even dimensional and simply connected, equality holds. This has been shown by Klingenberg [16] using Synge's Lemma in combination with a lifting argument:

$$\operatorname{inj} M^n = \varrho_c \geq \pi/\sqrt{\max K_M}. \tag{1}$$

We are going to concentrate on the *odd dimensional case* which is much more subtle.

By the Long Homotopy Lemma in Klingenberg's proof of the Sphere Theorem [17], the injectivity radius and the conjugate radius are still equal, provided that $M^n$ is simply connected and $\frac{1}{4}\max K_M < \min K_M$. This

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condition on $K_M$ is customarily referred to as *strict quarter pinching*. It has been proved by Cheeger and Gromoll [7] and also by Klingenberg and Sakai [19] that (1) continues to hold in the *weakly quarter pinched* case, i.e., for manifolds with $\frac{1}{4}\max K_M \le \min K_M$.

The main purpose of this paper is to relax this pinching condition even further.

**Theorem 1.1.** *There exists a constant $\varepsilon > 0$ such that the injectivity radius of any complete, simply connected Riemannian manifold $M^n$ with $\frac{1}{4(1+\varepsilon)^2}$-pinched sectional curvature $K_M$ is controlled by its conjugate radius $\varrho_c$ as in* (1)*:*

$$\operatorname{inj} M^n = \varrho_c \ge \pi/\sqrt{\max K_M}.$$

The estimates given in our proof show that the theorem actually holds for $\varepsilon = 10^{-6}$, which is certainly not optimal. It is known that the optimal pinching constant for the preceding theorem must be $\ge \frac{1}{9}$. The standard examples are the *Berger metrics* $g_\varepsilon$, $0 < \varepsilon \le 1$, on odd dimensional spheres. They are obtained by shrinking the length of all Hopf fibers to $2\pi\varepsilon$. The resulting manifolds $(\mathbb{S}^n, g_\varepsilon)$ are weakly $\varepsilon^2/(4-3\varepsilon^2)$-pinched. The conjugate radius of $g_\varepsilon$ is $\pi/\sqrt{4-3\varepsilon^2}$, and its injectivity radius equals its conjugate radius, if and only if $\frac{1}{3} \le \varepsilon^2 \le 1$.

The *Aloff-Wallach examples* [2], [9] show how delicate the odd dimensional case is. They are constructed from a particular left invariant metric $g_0$ on $SU(3)$ that is preserved under the right action of a fixed maximal torus $\mathbb{T}^2 \subset SU(3)$. The embedded, 1-parameter subgroups $i_{k,l}(\mathbb{S}^1) \subset \mathbb{T}^2$ define a family of pairwise nonisometric, homogeneous, 7-dimensional quotient manifolds $M^7_{k,l} := \left(SU(3)/i_{k,l}\mathbb{S}^1, g_0\right)$. This family contains an infinite sequence of manifolds $M^7_{k_\nu, l_\nu}$ with positive sectional curvature such that the corresponding pinching constants $\delta_\nu$ converge to $\frac{16}{29\cdot 37}$ and such that the injectivity radii $\operatorname{inj} M^7_{k_\nu, l_\nu}$ approach zero [15]. Aloff and Wallach have already pointed out that infinitely many of these 7-manifolds are topologically different. Recently Kreck and Stolz discovered that some of these examples are homeomorphic but not diffeomorphic [21].

Some partial results for almost $\frac{1}{4}$-pinched manifolds have been obtained by Klingenberg and Sakai in [20] using convergence methods. In contrast the proof of our estimate is based on direct comparison methods. Therefore our pinching constant is independent of the dimension.

Theorem 1.1 has strong implications when viewed in the context of the Berger Rigidity Theorem [3] and the Pinching Below-$\frac{1}{4}$ Theorem [4]. In fact, we can deduce the following sphere theorem.

**Theorem 1.2.** *For any odd positive integer $n$ there is a number $\varepsilon_n > 0$ such that any complete, simply connected Riemannian manifold $M^n$ with $\frac{1}{4(1+\varepsilon_n)^2}$-pinched sectional curvature $K_M$ is homeomorphic to the sphere $\mathbb{S}^n$.*

For the discussion of this result we scale the metric of $M^n$ such that the sectional curvatures satisfy $1 \leq K_M \leq 4(1+\varepsilon_n)^2$. Our estimate for the injectivity radius translates into $\frac{\pi}{2(1+\varepsilon_n)} \leq \operatorname{inj} M^n \leq \operatorname{diam} M^n$. The result is well known, provided that the diameter of the manifold under consideration is $\geq \frac{\pi}{2}$. If $\operatorname{diam} M^n > \frac{\pi}{2}$, the Diameter Sphere Theorem due to Grove and Shiohama [13] applies. If $\operatorname{diam} M^n = \frac{\pi}{2}$, we can refer to the Diameter Rigidity Theorem by Gromoll and Grove [11].

In fact, the proof in [11] works for $C^{1,\alpha}$-manifolds with $\operatorname{diam} M^n = \frac{\pi}{2}$ and curvature $\geq 1$ in distance comparison sense. Therefore this result can be combined with the $C^{1,\alpha}$-convergence theorem of Peters [25], and Theorem 1.2 follows in the standard way upon considering appropriate sequences of pinched manifolds. These arguments have been carried out in full detail by Durumeric [8].

Due to the use of the convergence theorem the pinching constant in Theorem 1.2 depends on the dimension of the manifold. It is an interesting question whether our estimate for the injectivity radius can be used to obtain a sphere theorem with a pinching constant below $\frac{1}{4}$ that is independent of the dimension.[1]

The paper is organized as follows: in §2 we introduce special, finite dimensional approximation spaces of the free loop space $\Omega M$. These spaces play a crucial role in the proof of Theorem 1.1, and to our knowledge they have not been considered in the literature so far. The purpose of §3 is to present the basic holonomy estimates. In §4 we recall the Long Homotopy Lemma and summarize the ingredients from Morse theory that our argument depends on. In §5 we use this material to construct two special homotopies that serve as the starting point for an indirect proof. The contradiction is achieved by analyzing the geometric properties of a new lifting construction. The details of this construction, which is independent of the material in §§2, 4, and 5, are presented in §6.

The next paragraphs describe the *basic idea* for the proof of Theorem 1.1. As usual we normalize the sectional curvature of $M^n$ such that $\frac{1}{4(1+\varepsilon)^2} \leq K_M \leq 1$.

[1] *Note added in proof.* In the meantime the authors have established such a result. It will be the subject of a forthcoming paper.

We assume that the theorem were wrong. This means that there exists a closed geodesic of length $2 \operatorname{inj} M^n < 2\varrho_c$. We combine the properties of this geodesic, the Long Homotopy Lemma, and some arguments from Morse theory, in order to study *shortly null homotopic*, closed curves. These are rectifiable, closed curves $c_0 \colon \mathbb{R}/\mathbb{Z} \to M^n$ that are freely null homotopic through a family of curves that are strictly shorter than $c_0$ itself. By Proposition 4.8 the set of all shortly null homotopic, closed curves $c_0$ has a nonempty boundary in the free loop space $\Omega M^*$. On this boundary there exist points that are represented by shortly null homotopic curves in $M^n$. Such a point is a closed geodesic $\hat{c}_0$ of length $\geq 2\varrho_c \geq 2\pi$. In fact, $\hat{c}_0$ is a saddle point of index $\leq 1$ for the energy functional $E \colon \Omega M \to \mathbb{R}$, and thus it has length $\leq 2\pi(1+\varepsilon)$ and first holonomy angle $\geq \frac{\pi}{1+\varepsilon}$.

Among these closed geodesics $\hat{c}_0$ we can find one that is of minimal length. In Proposition 5.3 this geodesic is then deformed in two consecutive steps into a *shortly null homotopic, closed, piecewise $C^2$-curve $\check{c}_{\tau_2}$ which has length $< 2\pi$, total absolute curvature $< \kappa_0 \leq \frac{1}{7}$, and first holonomy angle $> \frac{\pi}{3}$*, provided that $\varepsilon = \varepsilon(\kappa_0)$ is sufficiently small. A schematic picture of the whole deformation process is given in Figure 1 (p. 670). Up to slight modifications the first homotopy is defined by the gradient flow of the energy functional. If $\varepsilon$ is sufficiently small, it leads to a shortly null homotopic curve $\hat{c}_{\tau_0}$ which has length $< 2\pi$ and total absolute curvature $< \frac{1}{35}$. There is no way to ensure directly that the first holonomy angle remains large under this deformation. However, if it gets small, we can stop the deformation at an earlier curve $\hat{c}_{\tau_1}$ with first holonomy angle equal to $\frac{\pi}{2}$. This curve then serves as a starting point for a second special homotopy $\check{c}_t$ that decreases energy and length by some a priori given amount for all $\varepsilon$ that are sufficiently small. On the other hand it can be estimated that this deformation only spoils the total absolute curvature and the first holonomy angle by some amount that is proportional to $\varepsilon$. The construction of the second deformation relies on the fact that the first eigenvalue of the Hessian of the energy functional at $\check{c}_0 = \hat{c}_{\tau_1}$ is bounded from above by a strictly negative constant. It moves in the direction of a corresponding approximate eigenvector which is defined in terms of the holonomy. The estimates for the latter deformation are based on the fact that there is an upper bound for $\frac{d^2}{dt^2}E(\check{c}_t)$ which is strictly negative at $t = 0$ and for which we can establish a modulus of continuity without using bounds for the covariant derivative of the curvature tensor.

The contradiction in our indirect argument is obtained by means of a new *lifting construction*. It shows that any shortly null homotopic curve

$\check{c}_{\tau_2}$ of length $< 2\pi$ is the boundary of an immersed ruled surface $\Sigma$ with a conical singularity consisting of radial geodesics which are shorter than $\frac{\pi}{2}$. The details are given in Theorem 6.8. A small upper bound for the total absolute curvature of $\check{c}_{\tau_2}$ implies that the intrinsic and extrinsic geometric data are close to those of a totally geodesically immersed hemisphere of curvature $K_\Sigma = 1$. In particular, the first holonomy angle of the boundary curve $\check{c}_{\tau_2}$ must be small. In fact, Theorem 6.1 asserts that the first holonomy angle of $\check{c}_{\tau_2}$ must be $< \frac{\pi}{3}$, provided that its absolute curvature is $< \frac{1}{7}$.

Hence Theorem 1.1 holds with $\varepsilon = \varepsilon(\frac{1}{7})$. This number turns out to be slightly greater than $10^{-6}$.

## 2. The free loop space $\Omega M$ and its finite dimensional approximations

The purpose of this section is to introduce the appropriate spaces for the Morse theory and the deformation arguments in §5. In the first two subsections we collect some facts which are very close to the results in the book by Milnor [22, Chapter III]. Our presentation parallels that in [22] as closely as possible. The third subsection is devoted to the basic estimates for the gradient and the Hessian of the energy functional on our approximation spaces.

**2.1. Basic facts.** Let $(M^n, g)$ be any compact, $n$–dimensional Riemannian manifold. The *free loop space* $\Omega M^*$ of $M^n$ is the space of $C^0$–maps from $\mathbb{R}/\mathbb{Z}$ into $M^n$ equipped with the compact open topology which is the standard choice in homotopy theory. For analytical purposes it is better to consider the subspace

$$\Omega M := \left\{ c\colon \mathbb{R}/\mathbb{Z} \to M^n \;\middle|\; \begin{array}{l} c \text{ is absolutely continuous and} \\ c' := c_*(\frac{\partial}{\partial s}) \in L^2(\mathbb{R}/\mathbb{Z}, c^*TM) \end{array} \right\} \tag{2}$$

and work with the $H^1$–topology on $\Omega M$. Then the *energy functional* and the *length functional*

$$E(c) := \frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}} |c'(s)|^2\, ds = \frac{1}{2}\int_0^1 |c'(s)|^2\, ds,$$

$$L(c) := \int_{\mathbb{R}/\mathbb{Z}} |c'(s)|\, ds$$

are both continuous. The following result may be found in [22], [23]:

**Theorem 2.1** (Milnor). *The canonical embedding $\Omega M \to \Omega M^*$ is a homotopy equivalence. Moreover, the critical points of $E$ are the closed geodesics in $M^n$.*

In the language of global analysis the free loop space $\Omega M^*$ can be considered as the space of $C^0$–sections of the trivial fiber bundle $\mathbb{R}/\mathbb{Z} \times M^n \to \mathbb{R}/\mathbb{Z}$. A more general homotopy equivalence theorem that holds for inclusions $\mathfrak{M}(E) \to C^0(E)$, where $\mathfrak{M}$ is an appropriate section functor and $E$ is a smooth fiber bundle over a compact manifold, is due to Palais [24, Theorem 13.14].

Milnor's proof makes use of the following spaces of broken geodesics

$$\Omega_k M := \left\{ c \in \Omega M \;\middle|\; \begin{array}{l} c|_{[s_i, s_{i+1}]} \text{ is a geodesic with length} \\ L(c|_{[s_i, s_{i+1}]}) < \operatorname{inj} M^n \text{ for } 0 \le i < k \end{array} \right\}, \tag{3}$$

where the numbers $s_i = \frac{i}{k}$, $0 \le i \le k$, define the customary subdivision of $[0, 1]$. It is actually shown that the embedding of the direct limit $\varinjlim_k \Omega_k M$ of these $kn$–dimensional manifolds into $\Omega M$ is a *homotopy equivalence.*

**2.2. Spaces of long broken geodesics.** Since in our context we are assuming in addition that $K_M \le 1$, we shall find it more convenient to work with the sets

$$\Omega_k^\ell M := \left\{ c \in \Omega M \;\middle|\; \begin{array}{l} c|_{[s_i, s_{i+1}]} \text{ is a geodesic with length} \\ L(c|_{[s_i, s_{i+1}]}) < \ell \text{ for } 0 \le i < k \end{array} \right\}, \tag{4}$$

where $0 < \ell \le \pi$. Using the map $j_k \colon c \mapsto (c'(s_i))_{i=0}^{k-1}$, we identify each of these sets with its image $\hat{\Omega}_k^\ell M := j_k(\Omega_k^\ell M)$ in the $k$–fold Cartesian product $TM \times \cdots \times TM$. Note that

$$\hat{\Omega}_k^\ell M = \left\{ (v_i)_{i=0}^{k-1} \in \prod\nolimits_{i=0}^{k-1} TM \;\middle|\; p_{i+1} = \exp_{p_i}(\tfrac{1}{k} v_i),\ \tfrac{1}{k}|v_i| < \ell \right\}. \tag{5}$$

Here $p_i$ stands for the footpoint of the tangent vector $v_i \in TM$. By our hypotheses on $K_M$ the conjugate radius of $M^n$ is $\ge \pi$, and hence the set $\hat{\Omega}_k^\pi M$ turns out to be a *$kn$-dimensional submanifold* in $\prod_{i=0}^{k-1} TM$. In fact, the standard projection $\pi_k \colon TM \times \cdots \times TM \to M \times \cdots \times M$ restricts to a *local diffeomorphism* of $\hat{\Omega}_k^\pi M$ onto an open subset in the $k$–fold product of the manifold $M^n$. We shall think of $\hat{\Omega}_k^\pi M$ and $\Omega_k^\pi M$ as equipped with the metric $\hat{g}_k := \frac{1}{k} \pi_k^*(g \oplus \cdots \oplus g)$ and its pullback

$g_k := j_k^* \hat{g}_k$, respectively, rather than working with the restriction of the $H^1$-inner product. The induced topologies are the same though, since we are dealing with finite dimensional manifolds.

The geodesics in $(\Omega_k^\pi M, g_k)$ will be called $g_k$-geodesics in order to distinguish them from geodesics in $M^n$. Note that the metric spaces $(\Omega_k^\pi M, g_k)$ are not complete, yet the following compactness result holds:

**Lemma 2.2.** *Let $k \in \mathbb{N}$, and let $(M^n, g)$ be a Riemannian manifold with sectional curvature $K_M \leq 1$. Let $c_0 \in \Omega_k^\ell M$, $\ell \in (0, \pi]$, such that it represents a closed geodesic in $M^n$. Then for any $\zeta < \frac{1}{\sqrt{2k}}(\ell - \frac{1}{k}L(c_0))$ the ball $B(c_0, \zeta)$ is relatively compact in $(\Omega_k^\ell M, g_k)$.*

*Proof.* Each segment of $c_0$ has length $\ell_i(0) := \frac{1}{k}L(c_0)$. Consider a $g_k$-geodesic $c\colon [0, T] \to (\Omega_k^\ell M, g_k)$ that begins at the given $c_0$. For each $i \in \{0, \dots, k-1\}$ we let $L_i$ denote the length of the geodesic $c(s_i, .)\colon [0, T] \to (M^n, g)$ defined by $c$. Then the length of $c$ as a curve in $\Omega_k^\ell M \subset \Omega_k^\pi M$ is bounded by $\zeta$, if and only if

$$\frac{1}{k}\sum_{i=0}^{k-1} L_i^2 \leq \zeta^2.$$

By hypothesis $\zeta < \frac{1}{\sqrt{2k}}(\ell - \ell_i(0))$, and hence we compute that

$$\ell_i(t) := L(c_t|_{[s_i, s_{i+1}]}) \leq \ell_i(0) + L_i + L_{i+1} \leq \ell_i(0) + \sqrt{2k}\cdot\zeta < \ell$$

for $0 \leq t \leq T$ and $0 \leq i < k$. q.e.d.

The next objects of interest are the *sublevels* $\Omega M_{\leq\eta} := \Omega M \cap E^{-1}([0, \frac{1}{2}\eta^2])$ and $\Omega_k^\pi M_{\leq\eta} := \Omega_k^\pi M \cap E^{-1}([0, \frac{1}{2}\eta^2])$ of the energy functional. The corresponding open domains are denoted by $\Omega M_{<\eta} := \Omega M \cap E^{-1}([0, \frac{1}{2}\eta^2))$ and $\Omega_k^\pi M_{<\eta} := \Omega_k^\pi M \cap E^{-1}([0, \frac{1}{2}\eta^2))$.

**Proposition 2.3.** *Let $M^n$ be a compact Riemannian manifold with $K_M \leq 1$, and let $k \in \mathbb{N}$ and $\ell \in (0, \pi]$. Then for any $\eta < \ell\sqrt{k}$ the sublevel $\Omega_k^\ell M_{\leq\eta} \subset \Omega_k^\pi M$ is compact, and the canonical embedding*

$$\Omega_k^\ell M_{\leq\eta} \to \Omega M_{\leq\eta}$$

*is a homotopy equivalence, and so is the direct limit*

$$\varinjlim_k \Omega_k^\ell M \to \Omega M$$

*of these embeddings. Moreover, the critical points of* $E$ *in* $\Omega M_{\leq\eta}$ *and of* $E_k := E|_{\Omega_k^\pi M}$ *in the subset* $\Omega_k^\ell M_{\leq\eta} \subset \Omega_k^\pi M_{\leq\eta}$ *coincide. They are closed geodesics* $c$ *in* $M^n$, *and, in fact,* $\operatorname{hess} E|_c$ *and* $\operatorname{hess} E_k|_c$ *have the same Morse index and nullity.*

*Proof.* Just as in [22, §16] we estimate that for any $c \in \Omega M_{\leq\eta}$ the length of its segments with respect to the subdivision of $[0, 1]$ given by the points $s_i = \frac{i}{k}$ are bounded by

$$(6) \qquad L\left(c|_{[s_i, s_{i+1}]}\right)^2 \leq 2(s_{i+1} - s_i) \cdot E\left(c|_{[s_i, s_{i+1}]}\right) \leq \tfrac{2}{k} \cdot E(c) < \ell^2 .$$

Hence $\Omega_k^\ell M_{\leq\eta}$ is a compact subset in $\Omega_k^\ell M$. Moreover, under the exponential map each of the segments $c|_{[s_i, s_{i+1}]}$ can be lifted to a curve $\tilde{c}_i: [s_i, s_{i+1}] \to T_{c(s_i)} M^n$ which begins at $\tilde{c}_i(s_i) = 0$. Using these lifts we can define a *retraction map* $r_{k,1}: \Omega M_{\leq\eta} \to \Omega_k^\ell M_{\leq\eta}$ and a continuous family of maps $t \mapsto r_{k,t}$ connecting $r_{k,0} = \mathrm{id}_{\Omega M_{\leq\eta}}$ to $r_{k,1}$ as follows:

(7)
$$(r_{k,t} c)(s) := \begin{cases} \exp_{c(s_i)}\left(\frac{1}{t}\,(ks - i)\,\tilde{c}\left(\frac{i+t}{k}\right)\right) & \text{for} \quad i < ks \leq i + t, \\ c(s) & \text{for} \quad i + t \leq ks \leq i + 1. \end{cases}$$

The rest of the argument can be carried over verbatim from the proof of Theorem 16.2 in [22]. q.e.d.

The fact that such a *ruled surface construction* works up to the conjugate radius essentially goes back to Klingenberg's proof of the Long Homotopy Lemma. In particular, this construction is crucial for introducing the appropriate product of short loops in the proof of the Almost Flat Manifolds Theorem [14, 5].

**2.3. Gradient and Hessian.** Recall that any differentiable map $t \mapsto c_t \in \Omega_k^\pi M$ can be interpreted as a 2-parameter map $c: (s, t) \mapsto c_t(s) \in M^n$. The first two derivatives of the energy functional $E_k$ along this path in the loop space can be expressed in terms of the tangent fields $c'(s, t) := \frac{\partial}{\partial s} c(s, t)$, the broken Jacobi fields $\dot{c}(s, t) := \frac{\partial}{\partial t} c(s, t)$, and their derivatives:

$$(8) \qquad \tfrac{d}{dt} E_k(c_t) = \int_{\mathbb{R}/\mathbb{Z}} \langle c_t', \tfrac{\nabla}{\partial s} \dot{c}_t \rangle \, ds = \sum_{i=0}^{k-1} \langle c_t', \dot{c}_t \rangle \Big|_{s=s_i+0}^{s=s_{i+1}-0}$$

and

$$
\begin{aligned}
\frac{d^2}{dt^2} E_k(c_t) &= \int_{\mathbb{R}/\mathbb{Z}} \left( |\tfrac{\nabla}{\partial s} \dot{c}_t|^2 + \langle \tfrac{\nabla}{\partial t} \tfrac{\nabla}{\partial s} \dot{c}_t , c_t' \rangle \right) ds \\
&= \sum_{i=0}^{k-1} \langle c_t', \tfrac{\nabla}{\partial t} \dot{c}_t \rangle \Big|_{s=s_i+0}^{s=s_{i+1}-0} \\
&\quad + \int_{\mathbb{R}/\mathbb{Z}} \left( |\tfrac{\nabla}{\partial s} \dot{c}_t|^2 - \langle R(\dot{c}_t, c_t')c_t', \dot{c}_t \rangle \right) ds.
\end{aligned} \tag{9}
$$

The preceding equations are known as the *first* and the *second variation formulas.* Note that in (9) we have kept track of the boundary terms at the $k$ corner points of $c_t$, when performing the partial integration. As a result this formula is valid for any broken geodesic $c_t$ and not just for one that is a critical point of the energy functional and thus a geodesic in disguise.

We shall find it convenient to identify the broken Jacobi field $s \mapsto \dot{c}_t(s)$ that represents a tangent vector to $\Omega_k^\pi M$ at $c_t$ with its image $(\dot{c}_t(s_i))_{i=0}^{k-1}$ under $d\pi_k \circ dj_k$.

The subsequent calculations make substantial use of the metrics $g_k$ that we have put on the spaces $\Omega_k^\pi M$. Note that a homotopy $t \mapsto c_t$ is a $g_k$-geodesic in $\Omega_k^\pi M$, if and only if the $k$ curves $t \mapsto c_t(s_i)$, $0 \le i < k$, are geodesics in $M^n$. Hence (8) and (9) can be rewritten as

$$
\operatorname{grad} E_k|_{c_t} = -\left(kc_t'(s_i+0) - kc_t'(s_i-0)\right)_{i=0}^{k-1} \tag{8'}
$$

and

$$
\operatorname{hess} E_k|_{c_t}(X, Y) = \int_{\mathbb{R}/\mathbb{Z}} \left( \langle \tfrac{\nabla}{\partial s} X, \tfrac{\nabla}{\partial s} Y \rangle - \langle R(X, c_t')c_t', Y \rangle \right) ds, \tag{9'}
$$

where $X$ and $Y$ are arbitrary broken Jacobi fields with respect to the partition $s_0 < s_1 < \dots s_{k-1} < s_k = s_0 + 1$.

**Remark 2.4.** Because of formulas (8) and (8′) we only need the condition $\eta < \pi\sqrt{k}$ to make sure that *the flow of* $-\operatorname{grad} E_k$ *preserves each subset* $\Omega_k^\ell M_{<\eta} \subset \Omega_k^\pi M_{<\eta}$.

**Proposition 2.5** (Gradient and Exterior Angles). *Let* $c \in \Omega_k^\pi M$. *Suppose that none of its* $k$ *segments is a point curve. Then all its exterior* *angles*

$$\theta_i(c) \ := \ \arccos\left(\tfrac{\langle c'(s_i-0), c'(s_i+0)\rangle}{|c'(s_i-0)|\cdot|c'(s_i+0)|}\right) \ \in [0, \pi]$$

*are well defined, and the total absolute curvature* $\boldsymbol{\kappa}(c) := \sum_{i=0}^{k-1}\theta_i(c)$ *can be bounded in terms of the velocity and the gradient of the restricted energy functional:*

$$\boldsymbol{\kappa}(c)\cdot\min_s |c'(s)| \ \le \ \tfrac{1}{2}\pi\cdot\left\|\operatorname{grad} E_k|_c\right\|.$$

*Proof.* Since the exterior angles take values in the interval $[0, \pi]$, it is clear that $\frac{1}{\pi}\theta_i(c) \le \sin\frac{1}{2}\theta_i(c)$ for $0 \le i < k$. Thus, using just (8′) and the definition of the metric $g_k$ on the space of broken geodesics, it is straightforward to compute that

$$\begin{aligned}
4\min_s|c'(s)|^2\Big(\tfrac{1}{\pi}\sum_{i=0}^{k-1}\theta_i(c)\Big)^2 \ &\le \ 4k\min_s|c'(s)|^2\sum_{i=0}^{k-1}\sin^2\left(\tfrac{1}{2}\theta_i(c)\right)\\
&\le \ k\sum_{i=0}^{k-1}(1-\cos\theta_i(c))\Big(|c'(s_i+0)|^2+|c'(s_i-0)|^2\Big)\\
&\le \ k\sum_{i=0}^{k-1}|c'(s_i+0)-c'(s_i-0)|^2 \ = \ \left\|\operatorname{grad} E_k|_c\right\|^2.
\end{aligned}$$

**Proposition 2.6** (Bounds for the Hessian of $\boldsymbol{E_k}$). *Let* $c \in \Omega_k^{\pi/2}M$, *and let* $Y \in T_c\Omega_k^{\pi/2}M$ *be a broken Jacobi field along* $c$. *Suppose that* $0 \le K_M \le 1$. *Then*

$$-\tfrac{8}{\pi}E_k(c)g_k(Y, Y) \ \le \ \operatorname{hess} E_k|_c(Y, Y) \ \le \ 4k^2 g_k(Y, Y).$$

In order to prove this proposition, it is sufficient to apply the the following lemma to each segment of $c$ separately.

**Lemma 2.7.** *Let* $c\colon [a, b] \to (M^n, g)$ *be a geodesic in a Riemannian manifold with* $0 \le K_M \le 1$, *and let* $P_c\colon T_{c(a)}M \to T_{c(b)}M$ *denote the parallel transport along* $c$. *Suppose that* $L(c) \equiv (b-a)|c'| < \pi$. *Then for any Jacobi field* $Y$ *along* $c$ *the following inequality holds:*

$$\begin{aligned}
|c'|\cot(L(c))\big|Y(b)-P_c Y(a)\big|^2 &- 2|c'|\tan(\tfrac{1}{2}L(c))\langle Y(b), P_c Y(a)\rangle\\
&\le \int_a^b\Big(|\tfrac{\nabla}{ds}Y|^2 - \langle R(Y, c')c', Y\rangle\Big)\,ds\\
&\qquad\qquad \le \tfrac{1}{b-a}\big|Y(b)-P_c Y(a)\big|^2.
\end{aligned}$$

*Proof.* Since by hypothesis $Y$ is a Jacobi field along $c$, we know that

(10)
$$\begin{aligned} I(Y, Y) &:= \int_a^b \left(\left|\tfrac{\nabla}{ds} Y\right|^2 - \langle R(Y, c')c', Y\rangle\right) ds \\ &= \inf_{\substack{X \in C^1(c^*TM) \\ X(a)=Y(a),\ X(b)=Y(b)}} \int_a^b \left(\left|\tfrac{\nabla}{ds} X\right|^2 - \langle R(X, c')c', X\rangle\right) ds. \end{aligned}$$

Hence the assumed bounds for the sectional curvatures of $M^n$ imply that

(11)
$$\begin{aligned} &\inf_{\substack{X \in C^1(c^*TM) \\ X(a)=Y(a),\ X(b)=Y(b)}} \int_a^b \left(\left|\tfrac{\nabla}{ds} X\right|^2 - |c'|^2 \cdot |X|^2\right) ds \\ &\leq I(Y, Y) \leq \inf_{\substack{X \in C^1(c^*TM) \\ X(a)=Y(a),\ X(b)=Y(b)}} \int_a^b \left|\tfrac{\nabla}{ds} X\right|^2 ds. \end{aligned}$$

Using the standard arguments from the calculus of variations, it follows that the infima on the left- and right-hand sides of (11) are attained at the vector fields

$$\begin{aligned} X_{\text{lower}}(s) &:= \tfrac{\sin(b-s)|c'|}{\sin(b-a)|c'|} \cdot \hat{Y}_a(s) + \tfrac{\sin(s-a)|c'|}{\sin(b-a)|c'|} \cdot \hat{Y}_b(s), \\ X_{\text{upper}}(s) &:= \tfrac{b-s}{b-a} \cdot \hat{Y}_a(s) + \tfrac{s-a}{b-a} \cdot \hat{Y}_b(s). \end{aligned}$$

Here we have used the symbols $\hat{Y}_a$ and $\hat{Y}_b$ to denote the parallel vector fields along $c$ that are determined by the initial data $\hat{Y}_a(a) = Y(a)$ and $\hat{Y}_b(b) = Y(b)$, respectively.

Now it is a simple calculation to evaluate the integrals from (11) for $X_{\text{lower}}$ and $X_{\text{upper}}$, respectively, thereby concluding the proof of the lemma. q.e.d.

The lower bound for the Hessian of the restricted energy functional deserves particular interest for estimating the gradient of $E_k$ itself.

**Proposition 2.8** (Gradient Lines). *Let $(M^n, g)$ be a complete manifold such that $0 \leq K_M \leq 1$, and let $c\colon [a, b] \to \Omega_k^{\pi/2} M$, $t \mapsto c_t$, be a gradient line for the restricted energy functional $E_k$. Then*

$$\|\operatorname{grad} E_k|_{c_b}\|^2 \leq \|\operatorname{grad} E_k|_{c_a}\|^2 + \tfrac{16}{\pi}\left(E_k(c_a) - E_k(c_b)\right) \cdot \max_{a \leq t \leq b} E_k(c_t).$$

*Proof.* We may reparametrize the gradient line $t \mapsto c_t$ such that $\frac{d}{dt} E_k(c_t) = -1$. Then we are dealing with an arc in $\Omega_k^{\pi/2} M$, that is a solution of the differential equation

$$\tfrac{d}{dt}\, c_t \;=\; -\big\|\operatorname{grad} E_k|_{c_t}\big\|^{-2} \cdot \operatorname{grad} E_k|_{c_t}.$$

Because of the lower bound for the Hessian of $E_k$ provided by Proposition 2.6 we conclude that

$$\tfrac{d}{dt}\Big(\big\|\operatorname{grad} E_k|_{c_t}\big\|^2\Big) \;=\; -2 \cdot \operatorname{hess} E_k|_{c_t}\Big(\tfrac{\operatorname{grad} E_k|_{c_t}}{\|\operatorname{grad} E_k|_{c_t}\|}, \tfrac{\operatorname{grad} E_k|_{c_t}}{\|\operatorname{grad} E_k|_{c_t}\|}\Big) \;\le\; \tfrac{16}{\pi}\, E_k(c_t).$$

Since by our normalization $b - a = E_k(c_a) - E_k(c_b)$, the proposition follows upon integration.

## 3. Basic holonomy estimates

By definition the *holonomy* of a closed curve $c_0\colon \mathbb{R}/\mathbb{Z} \to M^n$ is the endomorphism $U_{c_0}\colon T_{c_0(0)}M \to T_{c_0(0)}M$ which carries the information by how much the parallel vector fields $W$ along $c_0$ fail to close up. In more technical terms one introduces $U_{c_0}$ as the unique linear map such that $U_{c_0} \cdot W_0 = W(1)$ where $s \mapsto W(s)$ is the vector field along the curve $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \xrightarrow{c_0} M^n$ that is obtained by solving the ordinary differential equation $\frac{\nabla}{ds} W(s) = 0$ with initial value $W(0) = W_0$. Clearly, $U_{c_0}$ is an orthogonal endomorphism, and, since in this paper we only care about orientable manifolds $M^n$, we even know that $U_{c_0} \in \mathrm{SO}(T_{c_0(0)}M) \cong \mathrm{SO}(n)$.

**3.1. The first holonomy angle.** There are a number of scalar invariants attached to the holonomy $U_{c_0}$ by means of the Jordan canonical form theorem. Using the notation $D(\psi)$ as a shorthand for the $2 \times 2$–matrix $\left(\begin{smallmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{smallmatrix}\right)$, this theorem states that $U_{c_0}$ is conjugate within the orthogonal group $\mathrm{O}(n)$ to some block diagonal matrix

$$\begin{pmatrix} D(\psi_1) & & \\ & \ddots & \\ & & D(\psi_{n/2}) \end{pmatrix} \quad\text{or}\quad \begin{pmatrix} 1 & & & \\ & D(\psi_1) & & \\ & & \ddots & \\ & & & D(\psi_{(n-1)/2}) \end{pmatrix},$$

where $0 \le \psi_1 \le \cdots \le \psi_{n/2} \le \pi$ or $0 \le \psi_1 \le \cdots \le \psi_{(n-1)/2} \le \pi$, respectively, depending on whether the dimension $n$ is even or odd. The numbers $\psi_1, \ldots, \psi_{[n/2]}$ obtained in this way are uniquely determined by the endomorphism $U_{c_0}$, and they will be called the *holonomy angles of the closed curve* $c_0$.[2]

We are particularly interested in the *first holonomy angle* $\psi_1(c_0)$. Note that this quantity can also be characterized in terms of the action of $U_{c_0}$ on the unit sphere $\mathbb{S}^{n-1} \subset T_{c_0(0)}M$. In this picture the eigenspace $\{Y \in T_{c_0(0)}M \mid U_{c_0} \cdot Y = Y\}$ corresponds to the fixed point set $\mathrm{Fix}(U_{c_0}) \subset \mathbb{S}^{n-1}$, and

(12)
$$\psi_1(c_0) = \begin{cases} \inf\{\sphericalangle(X, U_{c_0}X) \mid X \in \mathbb{S}^{n-1}\} & \text{for } n \equiv 0 \ (2), \\ \inf\left\{\sphericalangle(X, U_{c_0}X) \;\middle|\; \begin{array}{l} X \in \mathbb{S}^{n-1} \cap Y^{\perp} \text{ for} \\ \text{some } Y \in \mathrm{Fix}(U_{c_0}) \end{array}\right\} & \text{for } n \equiv 1 \ (2). \end{cases}$$

As it turns out, the properties of the holonomy $U_{c_0}$ of a curve $c_0$, that are essential for our argument, are measured precisely by the first holonomy angle $\psi_1(c_0)$.

It is straightforward to control the change in the first holonomy angle when the closed curve $c \in \Omega M$ varies.

**Proposition 3.1** (Continuity). *Let $(M^n, g)$ be a Riemannian manifold with $0 \le K_M \le 1$, and let $c\colon [a, b] \to \Omega M$, $t \mapsto c_t$, be a differentiable map of class $C^1$. Then*

$$|\psi_1(c_b) - \psi_1(c_a)| \le \tfrac{4}{3}\,\mathrm{Area}\Big(c(\mathbb{R}/\mathbb{Z} \times [a, b])\Big).$$

*Proof.* Consider the parallel transport $P_c\colon T_{c_a(0)}M \to T_{c_b(0)}M$ along the curve $t \mapsto c_t(0)$. Clearly,

$$(13) \qquad |\psi_1(c_b) - \psi_1(c_a)| \le d^{\infty}_{\mathrm{O}(n)}(U_{c_a}, P_c^{-1}U_{c_b}P_c),$$

where $d^{\infty}_{\mathrm{O}(n)}$ is the biinvariant metric on the orthogonal group given by

$$(14) \quad d^{\infty}_{\mathrm{O}(n)}(U_1, U_2) := \sup_{X \in \mathbb{S}^{n-1}} \sphericalangle(U_1X, U_2X), \quad \forall U_1, U_2 \in \mathrm{O}(n).$$

In order to bound the right-hand side of (13), it is sufficient to consider all unit vector fields $X$ along $c|_{[0,1]\times[a,b]}$, that satisfy $\frac{\nabla}{\partial s}X(s, t) = 0$ and

[2]Note that we have allowed for conjugation by orientation reversing elements. This gives some extra flexibility in the case of an even dimensional manifold, where otherwise the sign of the product $\sin\psi_1(c_0)\cdot\cdots\cdot\sin\psi_{n/2}(c_0)$ could be determined by means of the Pfaffian of $\log(U_{c_0})$, once the orientation of $M^n$ has been fixed.

$\frac{\nabla}{\partial t} X(0, t) = 0$. An easy computation shows that

$$\int_a^b \left|\tfrac{\nabla}{\partial t} X(1, t)\right| dt \;\le\; \sup_{s,t} \left\|R|_{c(s,t)}\right\| \cdot \operatorname{Area}\Big(c(\mathbb{R}/\mathbb{Z} \times [a, b])\Big).$$

Our bounds for $K_M$ imply that $\sup\|R\| \le \frac{4}{3}$, hence the proposition.

**3.2. Bounds for the first holonomy angle.** Our next goal is to bound the first holonomy angle $\psi_1$ for a single closed curve $c_0 \in \Omega M$. Such an estimate can be accomplished in terms of the *total absolute rotation*

$$\operatorname{rot}(W) \;:=\; \int_{\mathbb{R}/\mathbb{Z}} \left|\tfrac{\nabla}{ds} W\right| ds \tag{15}$$

of some appropriately chosen, closed unit vector fields $W$ along $c_0$. The relevant arguments are summarized in the following lemma and its corollary.

**Lemma 3.2** (Closed Vector Fields). *Let $c_0 \colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ be a closed, rectifiable curve, and let $W$ be a closed unit vector field of class $C^1$ along $c_0$. Then*

$$\sphericalangle\big(W(0), U_{c_0} W(0)\big) \;\le\; \operatorname{rot}(W).$$

*Proof.* Consider the solution $\hat{W} \colon [0, 1] \times [0, 1] \to c_0^* TM$ of the ordinary differential equation $\frac{\nabla}{\partial s} \hat{W}(s, t)$ with initial values $\hat{W}(s, s) = W(s)$, $0 \le s \le 1$. Evidently, $s \mapsto \hat{W}(s, 1)$ is a path in $\mathbb{S}^{n-1} \subset T_{c_0(0)} M$ which connects $\hat{W}(0, 1) \equiv U_{c_0} W(0)$ to $\hat{W}(1, 1) = W(1) \equiv W(0)$. A straightforward computation shows that

$$\tfrac{\nabla}{\partial t} \hat{W}(s, t)\big|_{t=t_0} \;=\; \tfrac{\nabla}{\partial t} \hat{W}(t_0, t)\big|_{t=t_0} \;=\; \tfrac{\nabla}{\partial t} \hat{W}(t, t)\big|_{t=t_0} \;=\; \tfrac{\nabla}{\partial t} W(t)\big|_{t=t_0} \quad ,$$

and therefore $\sphericalangle\big(W(0), U_{c_0} W(0)\big) \le L\big(s \mapsto \hat{W}(s, 1)\big) = \operatorname{rot}(W)$ as required.

**Corollary 3.3.** *Let $c_0 \colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ be a closed, rectifiable curve.*

(i) *Suppose that the dimension $n$ is even, and let $W$ be a closed unit vector field of class $C^1$ along $c_0$. Then*

$$\psi_1(c_0) \;\le \operatorname{rot}(W).$$

(ii) *Suppose that $n$ is odd, and let $W_\alpha$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, be a family of closed unit vector fields of class $C^1$ along $c_0$ such that $W_{\alpha+\pi} = -W_\alpha$ for each $\alpha$. Then*

$$\psi_1(c_0) \;\le\; \sup_{\alpha \in \mathbb{R}/2\pi\mathbb{Z}} \operatorname{rot}(W_\alpha).$$

*Proof.* Because of the variational characterisation of $\psi_1(c_0)$ given in (12) the corollary follows in the even dimensional case directly from the preceding lemma.

In the *odd dimensional case* the minimax characterisation of $\psi_1(c_0)$ from (12) ties in nicely with an elementary *degree argument*, which again reduces the claim to Lemma 3.2.

**Corollary 3.4.** *Let* $c_0: \mathbb{R}/\mathbb{Z} \to (M^n, g)$ *be a closed, rectifiable curve in an (odd dimensional) Riemannian manifold, and let* $W$ *be a closed unit vector field along* $c_0$. *Suppose that* $\operatorname{rot}(W) < \psi_1(c_0)$. *Then there exists a closed, parallel, unit vector field* $W^{\|}$ *along* $c_0$, *which is unique up to sign, and the angle between* $W$ *and* $W^{\|}$ *is bounded as follows:*

$$\sin \sphericalangle(W(s), W^{\|}(s)) \leq \frac{\sin\frac{1}{2}\operatorname{rot}(W)}{\sin\frac{1}{2}\psi_1(c_0)}, \quad \forall s \in \mathbb{R}/\mathbb{Z}. \tag{16}$$

*Proof.* Requiring that $\operatorname{rot}(W) < \psi_1(c_0)$, means that we are just considering odd dimensional manifolds $M^n$. Thus the existence and uniqueness of the parallel unit vector field $W^{\|}$ are evident from the Jordan canonical form of the holonomy $U_{c_0}$.

In order to obtain inequality (16), we note that $U_{c_0}$ extends to a parallel field of orthogonal endomorphisms $\hat{U}_{c_0(s)}: T_{c_0(s)}M \to T_{c_0(s)}M$ along $c_0$ and that each of these endomorphisms can be interpreted in exactly the same way as $U_{c_0} \equiv \hat{U}_{c_0(0)}$. We shall consider the isosceles triangle $W(s)$, $W^{\|}(s)$, $\hat{U}_{c_0(s)}W(s)$ in the unit sphere $\mathbb{S}^{n-1} \subset T_{c_0(s)}M$. Clearly, its angle at $W^{\|}(s)$ is $\geq \psi_1(c_0)$, the edge $W(s)$, $\hat{U}_{c_0(s)}W(s)$ has length $\leq \operatorname{rot}(W)$, and the edges $W(s)$, $W^{\|}(s)$ and $W^{\|}(s)$, $\hat{U}_{c_0(s)}W(s)$ have length equal to $\sphericalangle(W(s), W^{\|}(s))$. Hence inequality (16) follows upon applying the Law of Sines from spherical geometry. q.e.d.

In fact, the first holonomy angle $\psi_1$ of a closed curve $c \in \Omega M$ is even more significant when considered in the case that $c$ itself is a closed geodesic and that the sectional curvature of $M^n$ is pinched just below $\frac{1}{4}$. This fact has already been observed in [20].

**Proposition 3.5** (Morse Index). *Let* $c: \mathbb{R}/\mathbb{Z} \to (M^n, g)$ *be a closed geodesic in an odd dimensional, complete Riemannian manifold with* $\frac{1}{4(1+\varepsilon)^2} \leq K_M$, $\varepsilon > 0$. *Suppose that* $c$ *has Morse index* $\operatorname{ind}_E(c) < 2$. *Then*

$$\tfrac{1}{2(1+\varepsilon)} L(c) \leq \psi_1(c) \leq \pi. \tag{17}$$

Note that the Morse index does not change when we think of $c$ as a curve in some space $\Omega_k^\pi M$ of broken geodesics and consider the restricted energy functional $E_k$ instead. To put it differently, $\operatorname{ind}_E(c) = \operatorname{ind}_{E_k}(c)$.

*Proof.* The second inequality is clear by the definition of the first holonomy angle. For the first inequality, we observe that there exists a family $W_\alpha$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, of closed unit vector fields along $c$, that rotate with constant speed $\psi_1(c)$. In a more formal way these fields can be described by the equations

$$\tfrac{\nabla}{ds} W_\alpha(s) \;=\; \psi_1(c) \cdot W_{\alpha+\frac{\pi}{2}}(s)$$

and

$$W_\alpha(s) \;=\; \cos\alpha \cdot W_0(s) + \sin\alpha \cdot W_{\pi/2}(s).$$

Since $c'$ is parallel, we conclude that $W_\alpha(s) \perp c'(s)$ for all $s$. Evaluating the index form as given in $(9')$, we see that $\operatorname{hess} E|_c(W_\alpha, W_\alpha) \le \psi_1(c)^2 - \frac{1}{4(1+\varepsilon)^2} L(c)^2$. This inequality contradicts the hypothesis that $\operatorname{ind}_E(c) < 2$, unless (17) holds. q.e.d.

It should be understood that this proposition is most interesting, if the closed geodesic $c$ has length $\ge 2\pi$. In this special case the geometry of a neighborhood of $c$ appears to be fairly rigid. With just a little more effort we could for instance show that the *sectional curvatures* of the planes in $c^*TM$ that contain the tangent vector $c'(s)$ are on the average not much larger than $\frac{1}{4}$.

## 4. On the connected components of the sublevels of $\Omega M$

As pointed out in the introduction, the *classical approach to a lower bound for the injectivity radius of $M^n$* is to study the connected components of an appropriate sublevel of the energy functional. This is actually the sublevel $\Omega M_{<2\varrho_c}$ that is determined by the conjugate radius $\varrho_c$ of $M^n$. Let $\Omega M_{<2\varrho_c,0}$ be the *connected component* of $\Omega M_{<2\varrho_c}$ that contains the space $\Omega M_{\leq 0}$ of point curves.

**4.1. The Long Homotopy Lemma.** The basic results about null homotopic, short closed geodesics are due to Klingenberg. At first they had been established for geodesic loops with some fixed base point [17]. We need the version for the free loop space $\Omega M_{<2\varrho_c,0}$.

**Lemma 4.1** (c.f. [18],[7]). *Let $(M^n, g)$ be a compact Riemannian manifold, and let $\varrho_c$ denote its conjugate radius. Then the connected* *component* $\Omega M_{<2\varrho_c,0} \subset \Omega M_{<2\varrho_c}$ *does not contain any nontrivial, closed geodesic.*

The proof of either version of this lemma involves a lifting construction. In the next section we are going to modify this lifting construction substantially in order to normalize the free homotopy $t \mapsto c_t$ to the extent described in Theorem 6.8. For this reason we find it convenient to include the short proof of the preceding lemma.

*Proof.* The argument is by contradiction. Suppose that there exists a nontrivial closed geodesic $c_0 \in \Omega M_{<2\varrho_c,0}$. By definition such a geodesic $c_0$ is *null homotopic* in $\Omega M_{<2\varrho_c}$, i.e., there exists a path $c\colon [0, 1] \to \Omega M_{<2\varrho_c}$, $t \mapsto c_t$, that begins at the given $c_0$ and ends at some $c_1 \in \Omega M_{\leq 0}$. Without loss of generality we may assume that each $c_t\colon \mathbb{R}/\mathbb{Z} \to M^n$ is parametrized proportional to the arc length. In particular, for any $t \in [0, 1]$ both the arcs, $c_t|_{[-\frac{1}{2},0]}$ and $c_t|_{[0,\frac{1}{2}]}$, are strictly shorter than $\varrho_c$. Since the mapping

$$\pi \times \exp\colon B_{\varrho_c}TM \equiv \{(p, v) \in TM \mid |v| < \varrho_c\} \to M \times M \tag{18}$$

is clearly a local diffeomorphism, there exists a lift $\tilde{c}\colon [-\frac{1}{2}, \frac{1}{2}] \times [0, 1] \to B_{\varrho_c}TM$ of the mapping $(s, t) \mapsto (c_t(0), c_t(s))$ under $\pi \times \exp$ such that $\tilde{c}(0, t) = 0 \in T_{c(0,t)}M^n$.

In particular, the curves $t \mapsto \tilde{c}(-\frac{1}{2}, t)$ and $t \mapsto \tilde{c}(\frac{1}{2}, t)$ must coincide, since by construction $\tilde{c}(-\frac{1}{2}, 1) = \tilde{c}(\frac{1}{2}, 1) = 0 \in T_{c(1,0)}M^n$ and $(c_t(0), c_t(-\frac{1}{2})) = (c_t(0), c_t(\frac{1}{2}))$ for all $t \in [0, 1]$. This property contradicts the fact that by construction the points $\tilde{c}(-\frac{1}{2}, 0)$ and $\tilde{c}(\frac{1}{2}, 0)$ must also be disjoint. q.e.d.

The Long Homotopy Lemma has two immediate consequences that are useful in our investigation of the injectivity radius.

**Corollary 4.2.** *Let* $c_0 \in \Omega M_{<2\varrho_c,0}$, *and let* $c^0, c^1\colon [0, 1] \to \Omega M_{<2\varrho_c}$ *be two null homotopies of* $c_0$. *Then* $c^0$ *and* $c^1$ *can be connected through a continuous family of null homotopies* $c^\tau\colon [0, 1] \to \Omega M_{<2\varrho_c}$ *that is defined for all* $\tau \in [0, 1]$.

*Proof.* It is sufficient to show that the path obtained by composing the inverse of $c^0$ with $c^1$ can be retracted inside $\Omega M_{<2\varrho_c}$ into the space $\Omega M_{\leq 0}$ of point curves. By Morse theory the obstacle to the existence of such a retraction map is a closed geodesic $\gamma \in \Omega M_{<2\varrho_c,0}$ which cannot exist by Lemma 4.1.

**Corollary 4.3.** *Let* $(M^n, g)$ *be a compact Riemannian manifold, and suppose that its injectivity radius* $\operatorname{inj} M^n$ *is strictly less than its conjugate radius* $\varrho_c$. *Then the sublevel* $\Omega M_{<2\varrho_c}$ *consists of at least two connected components.*

*Proof.* Consider a point $p \in M^n$ where the injectivity radius function attains its minimal value. By hypothesis this value is strictly less than $\varrho_c$, and hence it is not hard to see (c.f. [6, Corollary 5.7]) that there exists a closed geodesic $c_0$ of length equal to $2 \operatorname{inj} M^n < 2\varrho_c$ through this point.

**Remark 4.4.** The Long Homotopy Lemma and the preceding corollary are the crucial steps in Klingenberg's bound for the injectivity radius in the strictly $\frac{1}{4}$-pinched case. The proof is indirect, and the contradiction is obtained using the standard saddle point arguments from Morse theory. In fact, a calculation similar to that of Proposition 3.5 shows that any closed geodesic of length $\geq 2\varrho_c \geq 2\pi/\sqrt{\max K}$ has index $\geq 2$, and thus $\Omega M_{<2\varrho_c}$ should be connected if $\pi_1(M^n) = 0$.

**4.2. Some ingredients from Morse theory.** Recall that we are working with $C^\infty$-metrics $g$, and hence the energy functional $E\colon \Omega M \to \mathbb{R}$ as well as its restrictions $E_k\colon \Omega_k^\pi M \to \mathbb{R}$ is smooth. However, they may have *degenerate critical points.* Since the hypotheses of Theorem 1.1 constitute an open condition, we could avoid these degeneracies referring to the Bumpy Metrics Theorem due to Abraham [1]. The degenerate Morse lemma from [12] provides a much more direct approach. Cheeger and Gromoll [7] have based their proof of the injectivity radius estimate in the weakly $\frac{1}{4}$-pinched case on this lemma.

Our proof of Theorem 1.1 follows from the arguments in [7] fairly closely, as far as the ingredients from Morse theory are concerned. In particular, we shall make use of the following two lemmas:

**Lemma 4.5** (c.f. [7, Lemma 2]). *Let* $f$ *be a smooth function on a finite dimensional, differentiable manifold* $X$, *and* $p$ *a possibly degenerate critical point of index* $\geq 2$ (*or a regular point*) *with* $f(p) = a$. *Then there exists a neighborhood* $N$ *of* $p$ *such that* $N \cap X_{<a}$ *is* (*pathwise*) *connected and dense in* $N \cap X_{\leq a}$.

**Lemma 4.6** (c.f. [7, Connectedness Lemma]). *Let* $f$ *be a smooth proper function on a finite dimensional manifold* $X$. *Suppose, for some regular value* $b$, *all critical points of* $f$ *in* $X_{<b} \setminus X_{\leq a}$ *have index* $\geq 2$ (*but are possibly degenerate*). *Let* $C_1, \dots, C_N$ *be the connected components of* $X_{\leq b}$. *Then* $C_1 \cap X_{\leq a}, \cdots, C_N \cap X_{\leq a}$ *are the connected components of* $X_{\leq a}$. *In particular, if* $X_{\leq b}$ *is connected, so is* $X_{\leq a}$.

We want to apply these lemmas to the finite dimensional approximation spaces of the free loop space $\Omega M$ that have been introduced in §2.

**Proposition 4.7.** *Let $\varepsilon > 0$, and let $(M^n, g)$ be a simply connected, odd dimensional Riemannian manifold with sectional curvature $K_M$ between $\frac{1}{4(1+\varepsilon)^2}$ and $1$. Then for each $\eta \geq 2\pi(1+\varepsilon)$ the sublevel $\Omega M_{\leq \eta}$ of the free loop space is connected, and so are all its finite dimensional approximation spaces $\Omega_k^\pi M_{\leq \eta}$ with $k > \eta^2/\pi^2$.*

*Proof.* By Proposition 2.3 all these spaces are homotopy equivalent. Consider an arbitrary broken geodesic $c_0 \in \Omega_k^\pi M_{\leq \eta}$, and let $c_1 \colon \mathbb{R}/\mathbb{Z} \to \{p_0\} \subset M^n$ be the point curve corresponding to the base point of $M^n$. We just need to show that $c_0$ and $c_1$ lie in the same connected component of some $\Omega_{\hat{k}}^\pi M_{\leq \eta}$ where $\hat{k}$ is a multiple of $k$.

Since the manifold $M^n$ is simply connected, there exists a piecewise differentiable map $c \colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to M^n$ such that $c_0 = c(., 0)$ and $c_1 = c(., 1)$. We pick $\hat{\eta} > 0$ and $\hat{k} \in k \cdot \mathbb{N}$ so large that $\max_{0 \leq t \leq 1} E(c_t) \leq \frac{1}{2}\hat{\eta}^2 < \frac{1}{2}\hat{k}\pi^2$. This ensures that each curve $c_t \equiv c(., t)$ is contained in $\Omega M_{\leq \hat{\eta}}$. Since $\Omega_{\hat{k}}^\pi M_{\leq \hat{\eta}}$ is contained in the fixed point set of the retraction map $t \mapsto r_{\hat{k}, t}$ used in the proof of Proposition 2.3, we conclude that $c_0$ and $c_1$ lie in the same connected component of $\Omega_{\hat{k}}^\pi M_{\leq \hat{\eta}} \subset \Omega M_{\leq \hat{\eta}}$. By Proposition 3.5 all closed geodesics in $M^n$ that have length $> \eta \geq 2\pi(1+\varepsilon)$ have Morse index $\geq 2$, and therefore we can apply the Connectedness Lemma to deduce that $c_0$ and $c_1$ are in fact contained in the same connected component of the space $\Omega_{\hat{k}}^\pi M_{\leq \eta}$. q.e.d.

We shall find it convenient to set things up for an indirect proof of Theorem 1.1. For this purpose we are interested in the properties of

$$\mathfrak{L}_1 := \left\{ \eta > 0 \,\middle|\, \begin{array}{c} \Omega M_{< \eta} \cap \Omega M_{\leq \eta, 0} \\ \text{is not connected} \end{array} \right\}. \tag{19}$$

By Proposition 2.3 this set can be characterized in terms of the finite dimensional approximation spaces of $\Omega M$ as follows:

$$\mathfrak{L}_1 \cap \left[0, \sqrt{k}\pi\right) = \left\{ \eta > 0 \,\middle|\, \begin{array}{c} \Omega_k^\pi M_{< \eta} \cap \Omega_k^\pi M_{\leq \eta, 0} \\ \text{is not connected} \end{array} \right\}, \quad \forall k \in \mathbb{N}. \tag{20}$$

Note that the $\Omega_k^\pi M_{\leq \eta, 0}$ that occur in this formula are compact subsets of the finite dimensional manifold $\Omega_k^\pi M$. Since $(\Omega_k^\pi M, g_k)$ is not complete, compactness would not necessarily hold, if $\eta$ were greater than $\sqrt{k}\pi$.

**Proposition 4.8.** *Let $\varepsilon > 0$, and let $(M^n, g)$ be a simply connected, odd dimensional Riemannian manifold with $\frac{1}{4(1+\varepsilon)^2} \leq K_M \leq 1$. Suppose* *that its injectivity radius* $\operatorname{inj} M^n$ *is strictly less than its conjugate radius* $\varrho_c$. *Then* $\mathfrak{L}_1$ *is a nonempty, finite set that is contained in the interval* $[2\varrho_c, 2\pi(1+\varepsilon)]$. *In particular,* $\varrho_c \leq \pi(1+\varepsilon)$.

Before we give the proof of this proposition, let us indicate why information about the set $\mathfrak{L}_1$ is useful.

**Lemma 4.9.** *Let* $\eta_1 \in \mathfrak{L}_1$, *and let* $(M^n, g)$ *be a Riemannian manifold with sectional curvature* $K_M \leq 1$. *Then for any* $k > 4(1+\varepsilon)^2$ *the closure* $\overline{\Omega_k^\pi M_{<\eta, 0}}$ *is contained in* $\Omega_k^\pi M_{\leq\eta, 0}$. *Moreover, its boundary with respect to the relative topology of* $\Omega_k^\pi M_{\leq\eta, 0}$ *is nonempty, and it consists of closed geodesics* $c_0$ *with length* $L(c_0) = \eta_1$ *and Morse index* $\operatorname{ind}_E(c_0) < 2$.

*Proof of Proposition* 4.8. Let $k$ be any integer that is greater than the maximum of $4(1+\varepsilon)^2$ and $4\varrho_c^2/\pi^2$, and set $\mathfrak{L}_{1,k} := \mathfrak{L}_1 \cap [0, \sqrt{k}\pi)$. It is clearly sufficient to show that all these sets $\mathfrak{L}_{1,k}$ are nonempty, discrete subsets of the interval $[2\varrho_c, 2\pi(1+\varepsilon)]$.

So we can use the characterization of $\mathfrak{L}_{1,k}$ given in (20), and hence the connectedness lemma quoted above can be applied. We conclude that for any $\eta \in \mathfrak{L}_{1,k}$ the number $\frac{1}{2}\eta^2$ is a critical value of $E_k$ such that $E_k^{-1}(\frac{1}{2}\eta^2)$ contains a closed geodesic of index $< 2$.

Since the critical values of proper $C^\infty$–functions are discrete, we see that $\mathfrak{L}_{1,k}$ is indeed a *discrete subset* of $[0, \sqrt{k}\pi)$. Using the information about the index obtained in Proposition 3.5, we conclude that $2\pi(1+\varepsilon)$ is an *upper bound* for $\mathfrak{L}_{1,k}$. The Long Homotopy Lemma on the other hand states that the domain $\Omega_k^\pi M_{<2\varrho_c, 0}$ does not contain any nontrivial, closed geodesic at all, and therefore $2\varrho_c$ is in fact a *lower bound* for $\mathfrak{L}_1$.

It remains to show that $\mathfrak{L}_{1,k}$ is *nonempty.* By Proposition 4.7 the sets $\Omega_k^\pi M_{\leq\eta}$, $\eta \in [2\pi(1+\varepsilon), \sqrt{k}\pi)$, are connected. Since the union of any family of nested connected sets is connected, we conclude that the domains $\Omega_k^\pi M_{<\eta}$ where $\eta$ varies in $(2\pi(1+\varepsilon), \sqrt{k}\pi)$, are also connected. However, by Corollary 4.3 the set $\Omega_k^\pi M_{<2\varrho_c}$ is not connected. Hence $\varrho_c \leq \pi(1+\varepsilon)$, and the number

$$\eta_1 := \sup\left\{\eta \in [0, \sqrt{k}\pi) \;\middle|\; \Omega_k^\pi M_{<\eta} \text{ is not connected}\right\}$$

is contained in the interval $[2\varrho_c, 2\pi(1+\varepsilon)]$. Evidently,

$$\Omega_k^\pi M_{\leq\eta_1} = \bigcap_{\eta \in (\eta_1, \sqrt{k}\pi)} \Omega_k^\pi M_{<\eta}$$

is connected, and thus we conclude that indeed $\eta_1 \in \mathfrak{L}_{1,k}$.

*Proof of Lemma* 4.9. It follows directly from the definition of $\mathfrak{L}_1$ that the open set $\Omega_k^\pi M_{<\eta_1} \cap \Omega_k^\pi M_{\le\eta_1,0} \setminus \Omega_k^\pi M_{<\eta_1,0}$ is nonempty. Clearly, this domain does not contain any adherence point of $\Omega_k^\pi M_{<\eta_1,0}$. Since $\Omega_k^\pi M_{\le\eta_1,0}$ is connected, we conclude that the closure of $\Omega_k^\pi M_{<\eta_1,0}$ has a nonempty boundary with respect to the topology of $\Omega_k^\pi M_{\le\eta_1,0}$. The properties stated for the elements contained in this boundary follow directly from Lemma 4.5.

## 5. Special homotopies

In this section we put things together to perform a major step in the proof of Theorem 1.1. We have already indicated that we want to pursue an indirect approach. The key point is that in Proposition 5.3 we shall exhibit a broken geodesic $c_0 \in \Omega_5^{\pi/2} M_{<2\pi,0}$ with very special properties. The proof of Theorem 1.1 can then be accomplished in the next section showing that a curve with these properties simply does not exist in $M^n$. In order to construct the broken geodesic $c_0$, we shall employ two quite special homotopies in the free loop space $\Omega M$ or rather in the finite dimensional approximation space $\Omega_5^{\pi/2} M$.

**Proposition 5.1** (First Special Homotopy). *Let $\varepsilon > 0$, and let $(M^n, g)$ be a simply connected, odd dimensional Riemannian manifold such that $\frac{1}{4(1+\varepsilon)^2} \le K_M \le 1$. Suppose that its injectivity radius* $\operatorname{inj} M^n$ *is strictly less than its conjugate radius $\varrho_c$. Then for any $\zeta > 0$, $k > 4(1+\varepsilon)^2$, and any $\ell \in \left(\frac{2\pi}{k}(1+\varepsilon), \frac{\pi}{2}\right]$ there exist a piecewise differentiable homotopy $c\colon [0, T] \to \Omega_k^\ell M$, $t \mapsto c_t$, and a partition $0 = t_0 < t_1 < \cdots < t_{3m} = T$ such that the following hold:*

(i) $E_k(c_{t_0}) > E_k(c_{t_1}) > \cdots > E_k(c_T) = 0$,

(ii) *each $c_{t_\nu}$, $\nu \equiv 0 \mod 3$, is a closed geodesic in $M^n$ or a point curve,*

(iii) $L(c_0) = \inf \mathfrak{L}_1 \le 2\pi(1+\varepsilon)$ *where $\mathfrak{L}_1$ is as in* (19),

(iv) $\operatorname{ind}_{E_k}(c_0) \le 1$,

(v) *the segments $c|_{[t_\nu, t_{\nu+1}]}$ where $\nu \equiv 0, 2 \mod 3$ are $g_k$-geodesics in the space $(\Omega_k^\ell M, g_k)$ that are shorter than $\zeta$,*

(vi) *the segments $c|_{[t_\nu, t_{\nu+1}]}$ where $\nu \equiv 1 \mod 3$ are integral curves of the vector field $-\operatorname{grad} E_k \in C^\infty(T\Omega_k^\ell M)$,*

(vii) $\psi_1(c_t) \ge \frac{\pi}{1+\varepsilon} - \frac{8k}{3}\zeta$ *for any $t \in [0, t_1]$,*

(viii) *for any $t \in [t_1, T]$ the curve $c_t$ lies in $\Omega_k^\ell M_{<\inf \mathfrak{L}_1, 0}$, and, moreover,*

$$\|\operatorname{grad} E_k|_{c_t}\| \le 4k^2\zeta + 4\sqrt{\pi}(1+\varepsilon)\cdot\sqrt{4\pi^2(1+\varepsilon)^2 - 2E_k(c_t)}.$$

We think of $c$ as an approximate gradient line. Indeed, if $\zeta$ is tiny, the segments of $c$ that are $g_k$-geodesics in the space $\Omega_k^\pi M$ have for instance almost no impact on the bound for $\|\operatorname{grad} E_k|_{c_t}\|$ given in (viii). We merely use these segments in order to avoid the more subtle discussion of the behavior of Morse trajectories in the neighborhood of a (possibly degenerate) critical point.

*Proof.* By hypothesis any closed geodesic $c_0 \in \Omega_k^\pi M_{\le 2\pi(1+\varepsilon)}$ is subdivided into so many segments that it actually lies in $\Omega_k^\ell M_{\le 2\pi(1+\varepsilon)}$. Moreover, we may assume that $\zeta^2 < \pi(1-\frac{\pi}{4})$.

The first step is to argue that by decreasing $\zeta$ if necessary, it is possible to assume in addition that

(a) the distance tube of radius $\zeta$ around the set of critical points of $E_k$ in $\Omega_k^\ell M_{\le 2\pi(1+\varepsilon)}$ is relatively compact in the domain $\Omega_k^\ell M \subset \Omega_k^\pi M$;
(b) for any pair of distinct critical values $\frac{1}{2}\eta^2$, $\frac{1}{2}\bar\eta^2 \le 2\pi(1+\varepsilon)$ the distance between the preimages $E_k^{-1}(\frac{1}{2}\eta^2)$ and $E_k^{-1}(\frac{1}{2}\bar\eta^2)$ is greater than $2\zeta$.

In fact, the first statement follows directly from Lemma 2.2, and the second one is a consequence of the compactness of $\Omega_k^\pi M_{\le 2\pi(1+\varepsilon)}$ as asserted in Proposition 2.3.

The second step is to construct a piecewise differentiable homotopy $c$ satisfying conditions (i)–(vi). By Proposition 4.8 the set $\mathfrak{L}_1$ introduced in (19) is nonempty and bounded from above by $2\pi(1+\varepsilon)$. We set $\eta_1 := \min \mathfrak{L}_1$. By Lemma 4.9 there exists a closed geodesic $c_0 \in \Omega_k^\ell M_{\le\eta_1} \subset \Omega_k^\pi M_{\le\eta_1}$ such that $L(c_0) = \eta_1$, $\operatorname{ind}_{E_k}(c_0) < 2$, and such that it can be approximated by elements of $\Omega_k^\ell M_{<\eta_1,0}$. We pick a broken geodesic $c_{t_1} \in \Omega_k^\ell M_{<\eta_1,0}$ such that $\operatorname{dist}(c_0, c_{t_1}) < \zeta$, and define $c|_{[0,t_1]}$ to be a minimal $g_k$-geodesic in the space $\Omega_k^\ell M$ from $c_0$ to $c_{t_1}$. The remaining segments of the homotopy $c$ will be constructed inductively.

Suppose that we already know $c|_{[0,t_\nu]}$ for some $\nu \equiv 1 \mod 3$, and suppose that $c_{t_\nu} \in \Omega_k^\ell M_{<\eta_1,0}$. At $c_{t_\nu}$ we start an integral curve of the vector field $-\operatorname{grad} E_k$. Since $E_k$ is monotonically decreasing along such an integral curve and yet bounded from below, it must eventually reach a value $t_{\nu+1} > t_\nu$ such that

- $\|\operatorname{grad} E_k|_{c_{t_{\nu+1}}}\|$ is at least as small as specified in (22), and
- there exists a critical point $c_{t_{\nu+2}}$ of $E_k$ with distance $\operatorname{dist}(c_{t_{\nu+1}}, c_{t_{\nu+2}}) < \zeta$ and energy $E_k(c_{t_{\nu+2}}) < E_k(c_{t_{\nu+1}})$.

By property (b) the curve $c_{t_{\nu+1}}$ lies in $\Omega_k^\ell M_{<\eta_1, 0}$, and by property (a) it is possible to connect $c_{t_{\nu+1}}$ to $c_{t_{\nu+2}}$ by a $g_k$-geodesic that is contained in $\Omega_k^\ell M_{<\eta_1, 0}$. In this way we extend the definition of $c$ from $[0, t_\nu]$ to the interval $[0, t_{\nu+2}]$.

If $c_{t_{\nu+2}}$ turns out to be a point curve, we stop and set $T := t_{\nu+2}$. Otherwise, the minimality of $\eta_1$ implies that $c_{t_{\nu+2}}$ can be approximated by broken geodesics which are strictly shorter. In particular, there exists a curve $c_{t_{\nu+3}} \in \Omega_k^\ell M_{<\eta_1, 0}$ such that $E_k(c_{t_{\nu+3}}) < E_k(c_{t_{\nu+2}})$ and $\operatorname{dist}(c_{t_{\nu+2}}, c_{t_{\nu+3}}) < \zeta$. Using the $g_k$-geodesic from $c_{t_{\nu+2}}$ to $c_{t_{\nu+3}}$, we extend the homotopy $c$ constructed so far to the interval $[0, t_{\nu+3}]$, and the induction step is complete.

The recursive construction of $c$ that we have set up by now terminates, since there are only finitely many critical values of $E_k$ below $\frac{1}{2}\eta_1^2$.

The third step is to verify properties (vii) and (viii) for this particular path $c$.

Using Proposition 3.5 we deduce from property (iv) that the first holonomy angle of the closed geodesic $c_0$ is $\geq \frac{\pi}{1+\varepsilon}$. We want to use the continuity properties of $\psi_1$ as established in Proposition 3.1. Since $K_M \leq 1$, a standard comparison argument shows that

$$\operatorname{Area}\Big(c(\mathbb{R}/\mathbb{Z} \times [0, t_1])\Big) \;\leq\; \tfrac{4\ell}{\pi} \cdot \sum_{i=0}^{k-1} L_i \;\leq\; \tfrac{4\ell}{\pi} \cdot \Big(k \sum_{i=0}^{k-1} L_i^2\Big)^{1/2} \;\leq\; 2k\zeta,$$

where for each $i \in \{0, \cdots, k-1\}$ the number $L_i$ denotes the length of the geodesic $c(s_i, .): [0, t_1] \to M^n$. To finish the proof of (vii), we merely need to combine the preceding estimates.

It follows directly from (i), (vi), and (b) that the curve $c_t$ lies in $\Omega_k^\ell M_{<\eta_1, 0}$ for any $t \in [t_1, T]$. Moreover, the upper bound for the Hessian of $E_k$ established in Proposition 2.6 implies that

$$E_k(c_t) \;\leq\; E_k(c_{t_\nu}) + \tfrac{1}{2}k^2\zeta^2 \tag{21}$$

for $t \in [t_\nu, t_{\nu+1}]$, $\nu \equiv 0, 2 \mod 3$. Since $\|\operatorname{grad} E_k|_{c_t}\|$ vanishes at $t = t_\nu$ or $t = t_{\nu+1}$, Proposition 2.6 enables us to compute for any $t$ as above that

$$
\begin{aligned}
\|\operatorname{grad} E_k|_{c_t}\| &\le \zeta\cdot\max\left\{\tfrac{8}{\pi}\left(E_k(c_{t_\nu})+\tfrac{1}{2}k^2\zeta^2\right),\ 4k^2\right\}\\
&\le \zeta\cdot\max\left\{16\pi(1+\varepsilon)^2+\tfrac{4}{\pi}k^2\zeta^2,\ 4k^2\right\} \le 4k^2\zeta.
\end{aligned}
\tag{22}
$$

The remaining segments of $c$ are integral curves of $-\operatorname{grad} E_k$. They are contained in $\Omega_k^\ell M_{<\eta_1}\subset\Omega_k^\ell M_{\le 2\pi(1+\varepsilon)}$, and the bound for $\|\operatorname{grad} E_k|_{c_t}\|$ provided by Proposition 2.8 settles the proof of (viii). q.e.d.

Recall that the *total absolute curvature* $\boldsymbol{\kappa}(c)$ of a broken geodesic $c$ is just the sum of its exterior angles $\theta_i(c)$, $0\le i<k$.

**Lemma 5.2.** *Let $(M^n, g)$ be a complete Riemannian manifold with sectional curvature $K_M\le 1$. Let $k\ge 3$, $0\le (k-1)\lambda\le\frac{1}{10}$, and set $\ell := \frac{2\pi}{k}(1+\lambda)$. Then the total absolute curvature $\boldsymbol{\kappa}(c)$ of any broken geodesic $c\in\Omega_k^\ell M$ that has energy $E(c)\ge 2\pi^2(1-\frac{1}{40k})$ is bounded as follows:*

$$
\boldsymbol{\kappa}(c) \le \tfrac{2}{7}\|\operatorname{grad} E_k|_c\|
$$

*Proof.* Note that $\min|c'(s)| = \min\{k\ell_0,\cdots,k\ell_{k-1}\}$ where $\ell_i$ denotes the length of the segment $c|_{[s_i,s_{i+1}]}$. By hypothesis these numbers are bounded by $\frac{2\pi}{k}(1+\lambda)$, and hence a direct computation shows that

$$
\begin{aligned}
4k\pi^2(1-\tfrac{1}{40k}) &\le 2k\cdot E_k(c) = \sum_{i=0}^{k-1}(k\ell_i)^2\\
&\le \min_{s\in\mathbb{R}/\mathbb{Z}}|c'(s)|^2+(k-1)\cdot\max_{s\in\mathbb{R}/\mathbb{Z}}|c'(s)|^2\\
&\le \min_{s\in\mathbb{R}/\mathbb{Z}}|c'(s)|^2+4\pi^2\cdot(k-1)(1+\lambda)^2.
\end{aligned}
$$

Finally, we conclude that

$$
\min_{s\in\mathbb{R}/\mathbb{Z}}|c'(s)| \ge 2\pi\cdot\sqrt{\tfrac{39}{40}-2(k-1)\lambda-(k-1)\lambda^2} \ge \tfrac{7}{4}\pi ,
$$

and the claimed bound for $\boldsymbol{\kappa}(c)$ follows from Proposition 2.5. q.e.d.

The next result is the first major step in the proof of Theorem 1.1. It will be necessary to restrict to broken geodesics with a small, a priori bounded number of corners. We are going to work in the space $\Omega_5^{\pi/2}M$.

**Proposition 5.3.** *Let $0<\varepsilon<10^{-6}$, and let $(M^n, g)$ be a simply connected, odd dimensional Riemannian manifold such that $\frac{1}{4(1+\varepsilon)^2}\le K_M\le 1$.*

*Suppose that its injectivity radius* $\operatorname{inj} M^n$ *is strictly less than its conjugate radius* $\varrho_c$. *Then there exists a broken geodesic* $c_0 \in \Omega_5^{\pi/2} M_{<2\pi,0}$ *such that its first holonomy angle and its total absolute curvature are bounded as follows*:

$$\text{(i)} \qquad \psi_1(c_0) > \frac{\pi}{3},$$

$$\text{(ii)} \qquad \boldsymbol{\kappa}(c_0) < \frac{1}{7}.$$

In Theorem 6.1 in the next section we shall see that such a curve $c_0$ does not exist if $0 \le K_M \le 1$. This is the contradiction that is necessary to settle the indirect proof of Theorem 1.1.

*Proof.* The inequalities which we use to estimate the total absolute curvature of the broken geodesic $c_0$ which we are going to construct require that the order of magnitude of $\varepsilon$ is as small as $10^{-6}$. A number of other inequalities depend on $\varepsilon$ as well; however, for these estimates it would be sufficient to have $\varepsilon < \frac{1}{64}$.

Let $\hat{c}$ denote the piecewise differentiable homotopy that is obtained by specializing Proposition 5.1 to the case that $k := 5$, $\ell := \frac{2\pi}{5}(1+2\varepsilon) < \frac{49}{99}\pi$, and $\zeta := \frac{1}{100}\sqrt{\varepsilon}$. Then 5.1(vii) implies that

$$\text{(23)} \qquad \psi_1(\hat{c}_t) \ge \frac{\pi}{1+\varepsilon} - \frac{40}{3}\sqrt{5}\cdot\zeta \ge \frac{2\pi}{3} \qquad , \ \forall t \in [0, t_1].$$

Consider $\tau_0 := \inf\{t \in [t_1, \hat{T}] \mid E_5(\hat{c}_t) \le 2\pi^2(1-\frac{\varepsilon}{10})\}$. By (21) the curve $\hat{c}_{t_1}$ has energy $E_5(\hat{c}_{t_1}) \ge 2\pi^2 - 12.5\zeta^2 \ge 1.99\pi$, and hence $E_5(\hat{c}_{\tau_0}) \ge 1.99\pi$, no matter whether $\tau_0 > t_1$ or $\tau_0 = t_1$. Thus Lemma 5.2 applies, and the gradient bound in Proposition 5.1(viii) yields:

$$\text{(24)} \qquad \begin{aligned} \|\operatorname{grad} E_5|_{\hat{c}_t}\| &\le 100\zeta + 8\pi\sqrt{\pi}(1+\varepsilon)\sqrt{2\varepsilon + \varepsilon^2 + \frac{25\zeta^2}{4\pi^2}} \\ &\le 50\cdot\sqrt{2\varepsilon} < \frac{1}{10} \end{aligned}$$

and

$$\kappa(\hat{c}_t) \le \frac{2}{7}\|\operatorname{grad} E_r|_{\hat{c}_t}\| < \frac{1}{35}$$

for all $t \in [t_1, \tau_0]$. If $\psi_1(\hat{c}_{\tau_0}) > \frac{\pi}{3}$, then we may set $c_0 := \hat{c}_{\tau_0}$. By the very definition of $\mathfrak{L}_1$ this curve lies in the domain $\Omega_5^\ell M_{<2\pi,0} \subset \Omega_5^\ell M_{<\inf \mathfrak{L}_1, 0}$, and we are done.

Otherwise, $t_1 < \tau_0$, and by an intermediate value argument there exists some $\tau_1 \in (t_1, \tau_0)$ with $\psi_1(\hat{c}_{\tau_1}) = \frac{\pi}{2}$. Note that $\boldsymbol{\kappa}(\hat{c}_{\tau_1})$ and $\|\operatorname{grad} E_5|_{\hat{c}_{\tau_1}}\|$

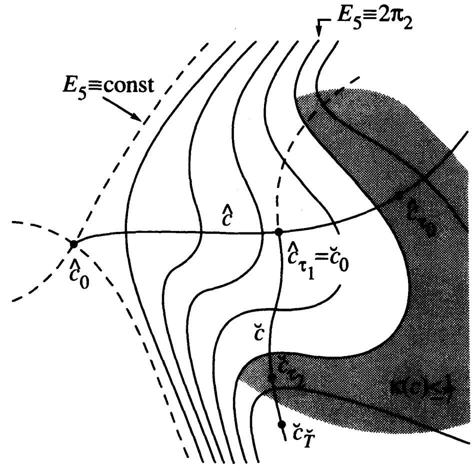


FIGURE 1. Special homotopies

are bounded as in (24). If $E(\hat{c}_{\tau_1}) < 2\pi^2$ we set $c_0 := \hat{c}_{\tau_1}$, and again the proof is finished referring to the definition of $\mathfrak{L}_1$.

It remains to handle the case that $E(\hat{c}_{\tau_1}) \geq 2\pi^2$. In this case we want to use the curve $\hat{c}_{\tau_1} \in \Omega_5^\ell M_{<\inf \mathfrak{L}_1, 0} \subset \Omega_5^{\pi/2} M_{<\inf \mathfrak{L}_1, 0}$ as initial point for another homotopy $\check{c}\colon [0, \check{T}] \to \Omega_5^{\pi/2} M_{<\inf \mathfrak{L}_1, 0}$, which is better adapted to the geometry of $\Omega_5^{\pi/2} M$. The basic idea is to work with a $g_k$-geodesic $\check{c}$ in the space $\Omega_5^{\pi/2} M$ such that $\operatorname{hess} E_5|_{\check{c}_0}$ is quite negative on the initial vector $\frac{\partial}{\partial t}\check{c}_t|_{t=0}$. The details of this construction are given in Proposition 5.4 below. The interplay between the two homotopies viewed as curves in the free loop space is illustrated in Figure 1.

Here we are dealing with the case that $k = 5$. The upper bound for $\varepsilon$ implies that $\check{T} := \frac{2}{375} < \frac{1}{24(1+\varepsilon)}$, and hence Proposition 5.4(ii), (v), and inequality (24) yield

$$\psi_1(\check{c}_t) > \tfrac{\pi}{3} \quad \text{and} \quad \|\operatorname{grad} E_5|_{\check{c}_t}\| \leq \tfrac{1}{10} + 75 \cdot t < \tfrac{1}{2}$$

for $0 \leq t < \check{T}$. Moreover, by Proposition 5.4(iii) we have $E_5(\check{c}_{\check{T}}) < 2\pi^2$. Since $E_5(\check{c}_0) \geq 2\pi^2$, there must also exist some $\tau_2 \in [0, \check{T})$ such that $1.99\pi^2 < E_5(\check{c}_{\tau_2}) < 2\pi^2$. Hence Lemma 5.2 is applicable, and the total absolute curvature $\boldsymbol{\kappa}(\check{c}_{\tau_2})$ is bounded by

$$\boldsymbol{\kappa}(\check{c}_{\tau_2}) \leq \tfrac{2}{7} \|\operatorname{grad} E_5|_{\check{c}_{\tau_2}}\| < \tfrac{1}{7}$$

as claimed. Since $\check{c}_0 \in \Omega_5^{\pi/2} M_{<\inf \mathfrak{L}_1, 0}$, we deduce from Proposition 5.4(iv) that $\check{c}_{\tau_2} \in \Omega_5^{\pi/2} M_{<\inf \mathfrak{L}_1, 0}$ as well, and we conclude the proof setting $c_0 := \check{c}_{\tau_2}$. q.e.d.

In order to pinpoint the places where the bound on the number $k$ of corners enters, we shall give the estimates for the second special homotopy for generic $k \in \mathbb{N}$ again. Wherever possible, we provide bounds that are stable when increasing $k$.

**Proposition 5.4** (Second Special Homotopy). *Let $\varepsilon > 0$, $T_\varepsilon := \frac{1}{24(1+\varepsilon)}$, $k \in \mathbb{N}$, and suppose that $\ell := \frac{2\pi}{k}(1 + 2\varepsilon) \leq \frac{49}{99}\pi$. Let $(M^n, g)$ be a compact Riemannian manifold with sectional curvature $\frac{1}{4(1+\varepsilon)^2} \leq K_M \leq 1$, and let $c_0 \in \Omega_k^\ell M_{\leq 2\pi(1+\varepsilon)}$ be a broken geodesic with first holonomy angle $\psi_1(c_0) = \frac{\pi}{2}$.*

*Then there exists a $g_k$-geodesic $c\colon [0, T_\varepsilon] \to \Omega_k^{\pi/2} M$, $t \mapsto c_t$, which begins at the given $c_0$, and which has the following properties:*

(i) $$\ell_i(t) := L(c_t|_{[s_i, s_{i+1}]}) \leq \left(\ell_i(0) + \tfrac{\pi}{2k} t\right)\left(1 + t^2\right) < \tfrac{\pi}{2},$$

(ii) $$\left|\psi_1(c_t) - \tfrac{\pi}{2}\right| \leq \tfrac{\pi}{2} \tfrac{32}{3}(1+\varepsilon) \cdot t\left(1+\tfrac{1}{3} t^2\right) + t^2\left(1+\tfrac{1}{2} t^2\right) < \tfrac{\pi}{6}.$$

*If, moreover, the total absolute curvature $\boldsymbol{\kappa}(c_0)$ is bounded by $\frac{1}{10}$, then the following estimates hold in addition:*

(iii) $$\begin{aligned} E_k(c_t) &\leq \pi^2(1 + \varepsilon)^2 \cdot \left(1 + \cos \tfrac{t}{(1+\varepsilon)\sqrt{2}}\right) \\ &\leq \pi^2\left(2(1+\varepsilon)^2 - \tfrac{1}{4} t^2 + \tfrac{1}{96} t^4\right), \end{aligned}$$

(iv) $$\tfrac{d}{dt} E_k(c_t) \leq 0, \text{ provided that } E_k(c_t) > \pi^2(1 + \varepsilon)^2,$$

(v) $$\left\|\operatorname{grad} E_k|_{c_t}\right\| \leq \left\|\operatorname{grad} E_k|_{c_0}\right\| + 15kt, \quad \text{if } \varepsilon < \tfrac{1}{20}.$$

**Remarks 5.5.** a) The estimates (ii) and (v) control how much the properties of $c_t$ deviate from those of $c_0$, and they are uniform in $\varepsilon$, at least as long as $\varepsilon$ is small. In the proof of Proposition 5.3 it has therefore been possible to make these deviations as small as necessary by restricting $t$ to a smaller interval.

b) Inequality (iii) implies that, no matter how small the parameter $t > 0$ is, one can guarantee that $E_k(c_t) < 2\pi^2$, if one picks $\varepsilon$ sufficiently small.

c) It seems inevitable that the bound for $\operatorname{grad} E_k|_{c_t}$ gets worse when $k$ increases.

The $g_k$-geodesic $c\colon [0, T_\varepsilon] \to \Omega_k^{\pi/2} M$ is constructed in a straightforward manner. The hypothesis that $\psi_1(c_0) = \frac{\pi}{2}$ means that there exist closed, rotating unit vector fields $W_\alpha$ along $c_0$ such that

$$\tfrac{\nabla}{ds} W_\alpha(s) = \psi_1(c) \cdot W_{\alpha+\frac{\pi}{2}}(s)$$

and

$$W_\alpha(s) = \cos\alpha \cdot W_0(s) + \sin\alpha \cdot W_{\pi/2}(s).$$

Shifting $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ if necessary, we may assume that $g_k(\operatorname{grad} E_k, W_0)|_{c_0} = 0$, which is in a sense the worst possible case. Now the idea is to require that

$$c_t(s_i) = \exp_{c_0(s_i)}(t \cdot W_0(s_i)), \quad \forall i \in \mathbb{Z}/k\mathbb{Z} \tag{25}$$

*Proof.* (i) Clearly, the first issue is to show that formula (25) defines indeed a map $c\colon [0, T_\varepsilon] \to \Omega_k^\pi M$. Since $\pi \times \exp\colon B_\pi TM \to M \times M$ is a local diffeomorphism of the ball bundle in $TM$ onto a neighborhood of the diagonal in $M^n$, we could apply the usual lifting arguments, provided we knew inequality (i) in advance.

Note that by hypothesis $\ell_i(0) \leq \ell < \frac{\pi}{2}$. So our plan is to employ a continuity argument. For this purpose it is sufficient to prove (i) for the largest interval $I_\varepsilon \subset [0, T_\varepsilon)$ on which we already know about the existence of the map $t \mapsto c_t \in \Omega_k^\pi M$.

For this purpose we shall consider in addition the unit vector field $\hat{W}$ along $c\colon \mathbb{R}/\mathbb{Z} \times I_\varepsilon \to M^n$ that is defined by

$$\tfrac{\nabla}{\partial t}\hat{W}(s, t) = 0, \qquad \hat{W}(s, 0) = W_0(s). \tag{26}$$

By the first variation formula we see that

$$\begin{aligned} \tfrac{d}{dt}\ell_i(t) &= \langle \hat{W}(s_{i+1}, t), \tfrac{c_t'(s_{i+1}-0)}{|c_t'(s_{i+1}-0)|} \rangle - \langle \hat{W}(s_i, t), \tfrac{c_t'(s_i+0)}{|c_t'(s_i+0)|} \rangle \\ &\leq \beta_i(t), \end{aligned} \tag{27}$$

where

$$\beta_i(t) := \int_{s_i}^{s_{i+1}} |\tfrac{\nabla}{\partial s} \hat{W}(s, t)| \, ds.$$

It is straightforward to compute the derivative of this function:

$$\begin{aligned} \tfrac{d}{dt}\beta_i(t) &\leq \int_{s_i}^{s_{i+1}} |\tfrac{\nabla}{\partial t} \tfrac{\nabla}{\partial s} \hat{W}(s, t)| \, ds \\ &\leq \int_{s_i}^{s_{i+1}} |R(\dot{c}_t, c_t')\hat{W}| \, ds \leq \tfrac{4}{3} \int_{s_i}^{s_{i+1}} |\dot{c}_t \wedge c_t'| \, ds. \end{aligned} \tag{28}$$

The construction of $c$ asserts that $|\dot{c}_t(s_i)| = 1$. Therefore the upper curvature bound for $M^n$ implies that the Jacobi field $\dot{c}_t$ along $c_t|_{[s_i, s_{i+1}]}$ is controlled by

$$(29) \quad \left|\dot{c}_t\left(\sigma + \tfrac{1}{2}(s_i+s_{i+1})\right)\right| \le \frac{\cos k\sigma\ell_i(t)}{\cos\frac{1}{2}\ell_i(t)} \quad \text{for } |\sigma| \le \tfrac{1}{2}(s_{i+1} - s_i) = \tfrac{1}{2k},$$

and hence

$$(30) \quad \int_{s_i}^{s_{i+1}} |\dot{c}_t \wedge c_t'| \, ds \le \int_{\frac{-1}{2k}}^{\frac{1}{2k}} \frac{\cos k\sigma\ell_i(t)}{\cos\frac{1}{2}\ell_i(t)} \cdot k\ell_i(t)\, d\sigma \le 2 \cdot \tan\tfrac{1}{2}\ell_i(t).$$

Combining this inequality with (28), we get that $\frac{d}{dt}\beta_i(t) \le \frac{8}{3}\tan\frac{1}{2}\ell_i(t)$. In order to integrate this differential inequality, we introduce the nondecreasing functions $\hat{\ell}_i(t) := \max\{\ell_i(\tau) \mid 0 \le \tau \le t\}$. Since we have initial values $\beta_i(0) = \frac{1}{k}\psi_1(c_0) = \frac{\pi}{2k}$, we conclude that

$$\tfrac{d}{dt}\hat{\ell}_i(t) \le \beta_i(t) \le \tfrac{\pi}{2k} + \tfrac{8}{3}t \cdot \tan\tfrac{1}{2}\hat{\ell}_i(t).$$

If we restrict our considerations to the interval $I_\varepsilon' := \{t \in I_\varepsilon \mid \hat{\ell}_i(t) < \frac{\pi}{2}\}$, we can simplify the right-hand side of the preceding differential inequality, and obtain

$$\tfrac{d}{dt}\hat{\ell}_i(t) \le \beta_i(t) \le \tfrac{\pi}{2k} + \tfrac{16}{3\pi}t \cdot \hat{\ell}_i(t).$$

This inequality can be integrated explicitly in consequence of $\hat{\ell}_i(0) = \ell_i(0)$:

$$\begin{aligned}\ell_i(t) &\le \exp\left(\tfrac{8}{3\pi}t^2\right)\left(\ell_i(0) + \tfrac{\pi}{2k}\int_0^t \exp\left(-\tfrac{8}{3\pi}\tau^2\right)d\tau\right)\\ &\le \left(\ell_i(0) + \tfrac{\pi}{2k}t\right)\left(1+t^2\right) < \tfrac{\pi}{2}.\end{aligned}$$

Here we have simplified the resulting formula using the fact that the integrand on the right-hand side of the first line is bounded by 1. Moreover, we have used the inequality $\exp(\frac{8}{3\pi}t^2) \le 1 + t^2$, which holds at least for all $t \in [0, \frac{1}{2})$. In this way we have proved (i) for all $t \in I_\varepsilon'$, and the standard continuity argument implies that $I_\varepsilon' = I_\varepsilon$. Referring to the continuity argument once more, we conclude that eventually $I_\varepsilon = [0, T_\varepsilon)$, and therefore $c$ is indeed defined on the closure of this interval.

(ii) This inequality is an easy consequence of what has already been established. Using (30) and (i), one computes that

$$\begin{aligned}\operatorname{Area}\Big(c([s_i\,,\,s_{i+1}]\times[0\,,\,t])\Big) &= \int_0^t\int_{s_i}^{s_{i+1}}|\dot{c}_\tau\wedge c'_\tau|\,ds\,d\tau\\ &\le \int_0^t \tfrac{4}{\pi}\ell_i(\tau)\,d\tau \le \tfrac{4}{\pi}\ell_i(0)\cdot t(1+\tfrac{1}{3}t^2)+\tfrac{1}{k}\cdot t^2(1+\tfrac{1}{2}t^2).\end{aligned}$$

Taking the sum over all $k$ segments of $c_t$, yields that

$$\operatorname{Area}\Big(c(\mathbb{R}/\mathbb{Z}\times[0\,,\,t])\Big) \le \tfrac{4}{\pi}L(c_0)\cdot t(1+\tfrac{1}{3}t^2)+t^2(1+\tfrac{1}{2}t^2). \tag{31}$$

Since $L(c_0)\le 2\pi(1+\varepsilon)$, inequality (ii) follows from Proposition 3.1.

(iii), (iv) Here the idea is to set up an appropriate system of differential inequalities, which relies on a negative pointwise upper bound for $\frac{d^2}{dt^2}E_k(c_t)$ along the initial segment of the deformation $t\mapsto c_t$. At $t=0$ this bound can be computed explicitly. In order to obtain such a bound at some $t>0$, we cannot refer to a modulus of continuity of $\operatorname{hess}E_k$, since bounds for the covariant derivative of the curvature tensor are not at our disposal. We avoid this problem by using a pointwise upper bound for $\frac{d^2}{dt^2}E_k(c_t)$ which is based on the $H^1$-seminorm of the vector field $\hat{W}(.\,,t)$ and on the total absolute curvature of the broken geodesic $c_t$ itself. The amount by which the $H^1$-seminorm of $\hat{W}(.\,,t)$ differs from the $H^1$-seminorm of $W_0=\hat{W}(.\,,0)$ can be controlled—just like the path dependence of parallel transport—in terms of a two-sided curvature bound and of an upper bound for the area of the surface $c\colon\mathbb{R}/\mathbb{Z}\times[0\,,\,t]\to M^n$.

Recall that $t\mapsto c_t\in\Omega_k^{\pi/2}M$ is a $g_k$-geodesic. Moreover, $\dot{c}_t$ is a broken Jacobi field along $c_t$, and it coincides with the unit vector field $\hat{W}$ constructed in (26) at all $k$ corners $c_t(s_i)$. Hence

(32)

$$\begin{aligned}\tfrac{d^2}{dt^2}E_k(c_t) &= \operatorname{hess}E_k(\dot{c}_t\,,\,\dot{c}_t)\\ &\le \int_{\mathbb{R}/\mathbb{Z}}|\tfrac{\nabla}{\partial s}\hat{W}|^2-\langle R(\hat{W}\,,\,c'_t)c'_t\,,\,\hat{W}\rangle\,ds\\ &\le \int_{\mathbb{R}/\mathbb{Z}}|\tfrac{\nabla}{\partial s}\hat{W}|^2-\tfrac{1}{4(1+\varepsilon)^2}\big(|\hat{W}|^2\cdot|c'_t|^2-\langle\hat{W}\,,\,c'_t\rangle^2\big)\,ds\\ &\le -\tfrac{1}{2(1+\varepsilon)^2}E_k(c_t)+h_{\hat{W}}(t)^2+\varphi_{\hat{W}}(t)^2,\end{aligned}$$

where

$$h_{\hat{W}}(t) := \left(\int_{\mathbb{R}/\mathbb{Z}} |\tfrac{\nabla}{\partial s}\hat{W}|^2\, ds,\right)^{1/2},$$

$$\varphi_{\hat{W}}(t) := \tfrac{1}{2(1+\varepsilon)}\left(\int_{\mathbb{R}/\mathbb{Z}} \langle \hat{W}(s,t), c_t'(s)\rangle^2\, ds\right)^{1/2}.$$

We claim that

$$\text{(33)}\qquad \tfrac{d}{dt} h_{\hat{W}}(t) \le \tfrac{8}{3}\cdot\sqrt{E_k(c_t)}, \qquad h_{\hat{W}}(0) = \tfrac{\pi}{2},$$

$$\text{(34)}\qquad \tfrac{d}{dt}\varphi_{\hat{W}}(t) \le \tfrac{1}{2(1+\varepsilon)} h_{\hat{W}}(t), \qquad \varphi_{\hat{W}}(0) \le \pi\sqrt{2}\cdot\sin\boldsymbol{\kappa}(c_0).$$

Let us postpone the proofs of these formulas. We shall explain first how (iii) and (iv) follow from (32)–(34). In fact, it is evident that inequality (iii) holds at $t = 0$. Moreover, its right-hand side is $< 2\pi^2(1+\varepsilon)^2$ for any $t \in (0, T_\varepsilon]$. Referring to the standard continuity argument, it is therefore justified to evaluate the differential inequalities (32)–(34) assuming that $E_k(c_t) < 2\pi^2(1+\varepsilon)^2$, even though we are still in the process of proving (iii). Thus we conclude that

$$\text{(33}'\text{)}\qquad h_{\hat{W}}(t) \le \tfrac{\pi}{2} + \tfrac{8}{3}t\cdot\sqrt{E_k(c_0)} \le \tfrac{2\pi}{3},$$

$$\text{(34}'\text{)}\qquad \varphi_{\hat{W}}(t) \le \pi\sqrt{2}\cdot\boldsymbol{\kappa}(c_0) + \tfrac{\pi}{4(1+\varepsilon)}t + \tfrac{4\pi}{3}t^2 \le \tfrac{\pi}{6},$$

and therefore (32) implies that

$$\text{(35)}\qquad \tfrac{d^2}{dt^2}\left(E_k(c_t) - \pi^2(1+\varepsilon)^2\right) \le -\tfrac{1}{2(1+\varepsilon)^2}\left(E_k(c_t) - \pi^2(1+\varepsilon)^2\right)$$

for $0 \le t < \frac{1}{24(1+\varepsilon)}$. By hypothesis $E_k(c_0) < 2\pi^2(1+\varepsilon)^2$ and $\frac{d}{dt}E_k(c_t)|_{t=0} = 0$. Moreover, $T_\varepsilon$ is much smaller than $\frac{\pi}{\sqrt{2}}(1+\varepsilon)$, and hence the Sturm Comparison Theorem yields that $\frac{d}{dt}E_k(c_t) \le 0$, as long as $E_k(c_t) > \pi^2(1+\varepsilon)^2$, and that

$$E_k(c_t) - \pi^2(1+\varepsilon)^2 \le \pi^2(1+\varepsilon)^2\cdot\cos\tfrac{t}{(1+\varepsilon)\sqrt{2}}$$

for all $t \in [0, T_\varepsilon]$. Hence inequalities (iii) and (iv).

It remains to verify (33) and (34). A straightforward calculation shows that

$$\begin{aligned}
\tfrac{d}{dt}\left(\tfrac{1}{2}h_{\hat W}(t)^2\right) &\le \int_{\mathbb{R}/\mathbb{Z}} \left|\tfrac{\nabla}{\partial s}\hat W(s,t)\right| \cdot \left|\tfrac{\nabla}{\partial t}\tfrac{\nabla}{\partial s}\hat W(s,t)\right| ds \\
&\le h_{\hat W}(t)\cdot\left(\int_{\mathbb{R}/\mathbb{Z}} |R(\dot c_t, c_t')\hat W(s,t)|^2\, ds\right)^{1/2} \\
&\le \tfrac{4}{3}h_{\hat W}(t)\cdot\left(\int_{\mathbb{R}/\mathbb{Z}} |\dot c_t \wedge c_t'|^2\, ds\right)^{1/2}.
\end{aligned}$$

The integrand on the last line can be evaluated by means of (29). Thus

$$\begin{aligned}
\tfrac{d}{dt}h_{\hat W}(t) &\le \tfrac{4}{3}\cdot\left(\sum_{i=0}^{k-1}\int_{\frac{-1}{2k}}^{\frac{1}{2k}} \tfrac{\cos^2 k\sigma\ell_i(t)}{\cos^2\frac{1}{2}\ell_i(t)}\cdot k^2\ell_i(t)^2\, ds\right)^{1/2} \\
&\le \tfrac{4}{3}\cdot\left(\sum_{i=0}^{k-1} 2k\cdot\ell_i(t)^2\right)^{1/2} \le \tfrac{8}{3}\cdot\sqrt{E_k(c_t)}
\end{aligned}$$

as required. In order to determine the initial value $h_{\hat W}(0)$, we note that $W_0 = \hat W(.,0)$ represents the first holonomy angle $\psi_1(c_0)$, and the argument for (33) is complete.

For the proof of (34) we note that $\frac{\nabla}{\partial t}\hat W$ vanishes identically. Hence

$$\begin{aligned}
\tfrac{d}{dt}\varphi_{\hat W}(t) &\le \tfrac{1}{2(1+\varepsilon)}\left(\int_{\mathbb{R}/\mathbb{Z}} \langle \hat W(s,t), \tfrac{\nabla}{\partial t}c_t'(s)\rangle^2\, ds\right)^{1/2} \\
&\le \tfrac{1}{2(1+\varepsilon)}\left(\int_{\mathbb{R}/\mathbb{Z}} \left|\tfrac{\nabla}{\partial s}\dot c_t(s)\right|^2 ds\right)^{1/2} \le \tfrac{1}{2(1+\varepsilon)}h_{\hat W}(t).
\end{aligned}$$

The estimate for the initial value $\varphi_{\hat W}(0)$ is again based on the fact that the unit vector field $W_0 = \hat W(.,0)$ represents $\psi_1(c_0)$. This means that $\sphericalangle(\hat W(s,0), c_0'(s)) \ge \frac{\pi}{2} - \sphericalangle(W^{\|}(s), c_0'(s))$. Since $\psi_1(c_0) = \frac{\pi}{2}$, Corollary 3.4 implies that

$$\cos^2\sphericalangle(\hat W(s,0), c_0'(s)) \le \sin^2\sphericalangle(W^{\|}(s), c_0'(s)) \le 2\cdot\sin^2\boldsymbol{\kappa}(c_0),$$

so that

$$\begin{aligned}
\varphi_{\hat W}(0)^2 &\le \tfrac{1}{2(1+\varepsilon)^2}\sin^2\boldsymbol{\kappa}(c_0)\cdot\int_{\mathbb{R}/\mathbb{Z}} |c_0'(s)|^2\, ds \\
&= \tfrac{1}{(1+\varepsilon)^2}\sin^2\boldsymbol{\kappa}(c_0)\cdot E_k(c_0) \le 2\pi^2\cdot\sin^2\boldsymbol{\kappa}(c_0).
\end{aligned}$$

(v) Clearly

$$\tfrac{d}{dt}\|\operatorname{grad} E_k|_{c_t}\| = \operatorname{hess} E_k|_{c_t}(\dot{c}_t, \tfrac{\operatorname{grad} E_k}{\|\operatorname{grad} E_k\|}) \leq \|\operatorname{hess} E_k|_{c_t} \cdot \dot{c}_t\|.$$

It remains to show that the right-hand side is bounded by $15k$. Proposition 2.6 asserts that $a \leq \operatorname{hess} E_k|_{c_t} \leq b$ where $a = -\frac{8}{\pi}E_k(c_t)$ and $b = 4k^2$. Thus it is straightforward to show that $\|\operatorname{hess} E_k|_{c_t} - \frac{1}{2}(a+b)\cdot \mathrm{Id}\| \leq \frac{1}{2}(b-a)$, and hence

$$\|\operatorname{hess} E_k|_{c_t} \cdot \dot{c}_t\|^2 \leq (a+b)\cdot \operatorname{hess} E_k|_{c_t}(\dot{c}_t, \dot{c}_t) - ab \cdot g_k(\dot{c}_t, \dot{c}_t).$$

By construction $g_k(\dot{c}_t, \dot{c}_t) = 1$. Since $k \geq 5$ and $\varepsilon < \frac{1}{20}$, the bound for $E_k|_{c_t}$ from (iii) yields that $a + b = 4k^2 - \frac{8}{\pi}E_k|_{c_t} \geq 100 - 16\pi(1+\varepsilon)^2 \geq 0$. Hence we can use the bound for $\frac{d^2}{dt^2}E_k|_{c_t} \equiv \operatorname{hess} E_k(\dot{c}_t, \dot{c}_t)$ from (35) to conclude that

$$\begin{aligned}\|\operatorname{hess} E_k|_{c_t} \cdot \dot{c}_t\|^2 &\leq (4k^2 - \tfrac{8}{\pi}E_k(c_t))\cdot(\tfrac{1}{2}\pi^2 - \tfrac{1}{2(1+\varepsilon)^2}E_k(c_t)) + \tfrac{32}{\pi}k^2 E_k(c_t)\\ &\leq 2k^2\pi^2 + (\tfrac{32}{\pi}k^2 - \tfrac{2k^2}{(1+\varepsilon)^2} - 4\pi)\cdot E_k(c_t) + \tfrac{4}{\pi(1+\varepsilon)^2}E_k(c_t)^2.\end{aligned}$$

Here the coefficients of $E_k(c_t)$ is positive, and hence we can employ the bound for $E_k|_{c_t}$ from (iii) to estimate the right-hand side. Upon collecting terms we obtain

$$\begin{aligned}\|\operatorname{hess} E_k|_{c_t} \cdot \dot{c}_t\|^2 &\leq 64k^2\pi(1+\varepsilon)^2 - 2k^2\pi^2 + 8\pi^3(1+\varepsilon)^2\\ &\leq 64k^2\pi(1+\varepsilon)^2 \leq (15k)^2,\end{aligned}$$

where the last two inequalities hold, since $k \geq 5$ and $\varepsilon \leq \frac{1}{20}$.

## 6. Shortly null homotopic curves with small total absolute curvature

In this section the central theme is to obtain some relationship between the holonomy and the total absolute curvature of a closed, piecewise regular $C^2$–curve $c_0\colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ in some Riemannian manifold with $0 < K_M \leq 1$, provided that the curve under consideration is strictly shorter than $2\pi$ and is null homotopic in this class of curves.

Recall that the total absolute curvature $\kappa(c_0)$ of a closed, piecewise regular $C^2$–curve $c_0\colon \mathbb{R}/\mathbb{Z} \to M^n$ is given by

$$\kappa(c_0) := \int_{\mathbb{R}/\mathbb{Z}} |k_{c_0}(s)| \cdot |c_0'(s)|\, ds + \sum_{\text{corners of } c_0} \theta_i(c_0), \tag{36}$$

where $k_{c_0}(s) := |c_0'(s)|^{-1} \cdot \frac{\nabla}{ds} \frac{c_0'(s)}{|c_0'(s)|}$ denotes the *curvature vector* of $c_0$, and the sum is taken over the exterior angles at all the corners at which $c_0$ fails to be of class $C^2$.

**Theorem 6.1.** *Let $c_0 \colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ be a piecewise regular $C^2$-curve in a compact Riemannian manifold with $0 < \frac{1}{4(1+\varepsilon)^2} \le K_M \le 1$. Suppose that $L(c_0) < 2\pi$ and that $\kappa(c_0) \le \frac{\pi}{6}$. Suppose also that there exists a free homotopy from $c_0$ to a point curve which consists entirely of curves $c_t$ of length strictly less than $2\pi$. Then the first holonomy angle of $c_0$ can be estimated as follows*:

$$\psi_1(c_0) \le \begin{cases} \kappa(c_0) & \textit{for } n \equiv 0 \ (2), \\ \frac{\sqrt{2}\cdot\kappa(c_0) + 2\cdot\sqrt{\kappa(c_0)}}{\sqrt{1 - \sin\kappa(c_0)}} & \textit{for } n \equiv 1 \ (2). \end{cases} \tag{37}$$

The preceding estimate asserts in particular that $\psi_1(c_0) < \frac{\pi}{3}$, provided that $\kappa(c_0) \le \frac{1}{7}$. These are the numerical values which contradict Proposition 5.3 and are therefore used to conclude the proof of Theorem 1.1.

It seems just fair to point out that the even dimensional case is easy. It is the odd dimensional case, which requires all the elaborate constructions in the subsequent subsections, and is only settled in Proposition 6.16 at the end of the entire section.

**6.1. Basic reduction steps.** The first observation is that it suffices to prove Theorem 6.1 for regular closed $C^2$-curves. The apparently more general version for just piecewise regular curves $c_0$ is then a consequence of the following well-known approximation result.

**Lemma 6.2.** *Let $\eta > 0$, and let $c_0 \colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ be a closed, piecewise regular $C^2$-curve. Then there exists an approximating sequence of closed, regular $C^2$-curves $c_\mu \colon \mathbb{R}/\mathbb{Z} \to (M^n, g)$ such that*

(i) $$\operatorname{dist}_{C^0}(c_\mu, c_0) \equiv \sup_{s \in \mathbb{R}/\mathbb{Z}} d_{M^n}(c_\mu(s), c_0(s)) \xrightarrow[\mu\to\infty]{} 0,$$

(ii) $$\lim_{\mu\to\infty} L(c_\mu) = L(c_0),$$

(iii) $$\lim_{\mu\to\infty} \kappa(c_\mu) = \kappa(c_0).$$

*Moreover, for any sufficiently large $\mu$, it is clear that the minimizing geo-*

*desics* $t \mapsto \hat{c}_\mu(s, t)$ *from* $\hat{c}_\mu(s, 0) := c_\mu(s)$ *to* $\hat{c}_\mu(s, 1) := c_0(s)$ *are uniquely determined, and they define a piecewise regular* $C^2$*-map* $\hat{c}_\mu \colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to (M^n, g)$ *such that*

(iv)
$$L(s \mapsto \hat{c}_\mu(s, t)) \le (1+\eta) \cdot \tfrac{1}{2}\big(L(c_\mu) + L(c_0)\big),$$

(v) $\quad \mathrm{Area}(\hat{c}_\mu) \le (1+\eta) \cdot \tfrac{1}{2}\big(L(c_\mu) + L(c_0)\big) \cdot \mathrm{dist}_{C^0}(c_\mu, c_0) \longrightarrow 0,$

(vi)
$$d^\infty_{\mathrm{O}(n)}(U_{c_0}, P_\mu U_{c_\mu} P_\mu^{-1}) \le \mathrm{Const}(c_0, M^n) \cdot \mathrm{Area}(\hat{c}_\mu) \xrightarrow[\mu \to \infty]{} 0.$$

Here $P_\mu \colon T_{c_\mu(0)}M \to T_{c_0(0)}M$ denotes the parallel transport along the geodesic segment $t \mapsto \hat{c}_\mu(0, t)$, and again $d^\infty_{\mathrm{O}(n)}$ stands for the angle distance on the orthogonal group as defined in (14).

The precise meaning of the condition that $\mu$ be large can be figured out in terms of $\mathrm{dist}_{C^0}(c_\mu, c_0)$ and the local geometry of $M^n$ in a neighborhood of $c_0$. In fact, the properties (iv)–(vi) follow from (i) by the standard comparison arguments. The standard approach to constructing a sequence $c_\mu$ which obeys conditions (i)–(iii) is to round off the corners of $c_0$ by an explicit local formula.

Note that for any $C^2$-curve $c_0$ the *total absolute rotation of its unit tangent field* $\mathfrak{t}_0(s) := \frac{c_0'(s)}{|c_0'(s)|}$ is just the total absolute curvature of the curve $c_0$ itself:

$$\mathrm{rot}(\mathfrak{t}_0) = \int_{\mathbb{R}/\mathbb{Z}} \left| \frac{\nabla}{ds} \frac{c_0'(s)}{|c_0'(s)|} \right| ds = \boldsymbol{\kappa}(c_0). \tag{38}$$

Hence in the even dimensional case Theorem 6.1 follows right away from Corollary 3.3(i). In the odd dimensional case we want to apply Corollary 3.3(ii). For this purpose we need to construct an appropriate family $(W_\alpha)_{\alpha \in \mathbb{R}/2\pi\mathbb{Z}}$ of closed unit vector fields with small total absolute rotation. Extending the unit tangent field $\mathfrak{t}_0$ of $c_0$ to such a family is precisely the nontrivial step in the proof of Theorem 6.1.

The idea is to span $c_0$ by some ruled surface with a conical singularity, which is approximately an immersed, totally geodesic hemisphere of curvature $\approx 1$. Then a natural candidate for $W_\alpha$ can be defined in terms of the unit tangent field $\mathfrak{t}_0$ and the gradient of the intrinsic distance to the center of this hemisphere.

The construction of this ruled surface depends crucially on the hypothesis that $c_0$ is null homotopic in $\Omega M_{<2\pi}$. It is based on an elaborate lifting argument which makes use of circumscribed balls to define the center of a subset $C$ contained in some local unwrappings $(\tilde{M}, \tilde{g}) := (B_\pi T_p M, \exp_p^* g)$ of $M^n$.

So our plan is to recall some basic facts about such circumscribed balls in §6.2, to continue with the details of the lifting construction in §6.3, and to study the geometry of the ruled surface in §6.4. Proposition 6.16 eventually provides the rigorous estimates for the family $(W_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}}$ of closed unit vector fields along $c_0$. It is only then that the proof of Theorem 6.1 is complete.

**6.2. Circumscribed balls of compact subsets $C \subset \tilde{M}$.** By construction the space $(\tilde{M}, \tilde{g})$ is simply connected and has curvature $K_{\tilde{M}} \leq 1$. However, it is not complete. In order to deal with this problem, we shall assume that the subset $C$ is contained in the ball $B(0, \frac{\pi}{2}) \subset \tilde{M}$. Then all the necessary constructions can be done inside some compact subset of $\tilde{M}$.

In fact, all balls $B(\tilde{p}, \varrho) \subset \tilde{M}$ with $0 < \varrho < \frac{\pi}{2}$ and $\tilde{p} \in B(0, \pi - \varrho)$ have compact closure and are strongly convex. This means that for any two points $\tilde{p}_1$, $\tilde{p}_2$ in such a ball $B(\tilde{p}, \varrho)$ there exists a unique geodesic $\tilde{\gamma}: [0, 1] \to B(\tilde{p}, \varrho)$ such that $\tilde{\gamma}(0) = \tilde{p}_1$ and $\tilde{\gamma}(1) = \tilde{p}_2$. It is known that $\gamma$ has minimal length in the class of all those curves which connect $\tilde{p}_1$ to $\tilde{p}_2$ and stay inside the ball $B(\tilde{p}, \varrho)$.

We define the *radius* $\operatorname{rad}(C)$ *of a closed subset* $C \subset B(0, \frac{\pi}{2}) \subset \tilde{M}$ as the infimum of some function $\operatorname{rad}_C: \tilde{M} \to [0, \infty)$ where

$$\operatorname{rad}_C(\tilde{q}) := \inf\{\varrho \mid C \subset B(\tilde{q}, \varrho)\}. \tag{39}$$

Evidently, $\operatorname{rad}_C(0) < \frac{\pi}{2}$, and by the triangle inequality $\operatorname{rad}_C(\tilde{q}) \geq \frac{\pi}{2}$ at all points $\tilde{q} \in \tilde{M} \setminus B(0, \frac{\pi}{2} + \operatorname{rad}_C(0))$. Thus

$$\operatorname{rad}(C) = \min\{\operatorname{rad}_C(\tilde{q}) \mid \tilde{q} \in \tilde{M}\},$$

and the set $\{\tilde{q} \in \tilde{M} \mid \operatorname{rad}_C(\tilde{q}) = \operatorname{rad}(C)\}$ of all points where this minimum is achieved is a compact subset in $\tilde{M}$. Evidently rad is *continuous* with respect to the Hausdorff distance:

$$\begin{aligned} |\operatorname{rad}(C_1) - \operatorname{rad}(C_2)| &\leq \sup_{\tilde{q}\in\tilde{M}} |\operatorname{rad}_{C_1}(\tilde{q}) - \operatorname{rad}_{C_2}(\tilde{q})| \\ &\leq \operatorname{dist}_H(C_1, C_2). \end{aligned} \tag{40}$$

The fundamental ingredient in discussing the set of the points $\tilde{q} \in \tilde{M}$ where $\mathrm{rad}_C$ is minimal any further is the following lemma.

**Lemma 6.3.** *Let $\tilde{p}_1, \tilde{p}_2 \in B(0, \frac{\pi}{2}) \subset \tilde{M}$, and let $\varrho_1, \varrho_2 \in (0, \frac{\pi}{2})$. Suppose that the distance $d := d_{\tilde{M}}(\tilde{p}_1, \tilde{p}_2)$ satisfies the inequality*

$$(41) \qquad 2 \cdot \sin^2(\tfrac{d}{4}) > \frac{\tan|\frac{1}{2}(\varrho_1 - \varrho_2)| \cdot \tan\frac{1}{2}(\varrho_1 + \varrho_2)}{1 + \tan|\frac{1}{2}(\varrho_1 - \varrho_2)| \cdot \tan\frac{1}{2}(\varrho_1 + \varrho_2)}.$$

*Then*

$$\mathrm{rad}\big(\overline{B(\tilde{p}_1, \varrho_1) \cap B(\tilde{p}_2, \varrho_2)}\big) < \min\{\varrho_1, \varrho_2\}.$$

*Proof.* By hypothesis $d < \pi$, and hence $\cos^2(\frac{d}{4}) > \frac{1}{2}$. Using inequality (41), we conclude that $\sin^2(\frac{d}{2}) \geq 2\sin^2(\frac{d}{4}) > \sin^2|\frac{1}{2}(\varrho_1 - \varrho_2)|$, and therefore all three triangle inequalities hold:

$$(42) \qquad |\varrho_1 - \varrho_2| < d \leq \varrho_1 + \varrho_2.$$

Thus we can define a number $\varrho \in (0, \frac{\pi}{2})$ by means of

$$(43) \qquad \cos\varrho \cdot \cos(\tfrac{d}{2}) = \cos|\tfrac{1}{2}(\varrho_1 - \varrho_2)| \cdot \cos\tfrac{1}{2}(\varrho_1 + \varrho_2).$$

Since

$$\begin{aligned}\cos(\min\{\varrho_1, \varrho_2\}) &= \cos\tfrac{1}{2}(\varrho_1 - \varrho_2) \cdot \cos\tfrac{1}{2}(\varrho_1 + \varrho_2) \\ &\quad + \sin|\tfrac{1}{2}(\varrho_1 - \varrho_2)| \cdot \sin\tfrac{1}{2}(\varrho_1 + \varrho_2) \\ &= \cos\varrho \cdot \cos(\tfrac{d}{2}) \\ &\quad \cdot (1 + \tan|\tfrac{1}{2}(\varrho_1 - \varrho_2)| \cdot \tan\tfrac{1}{2}(\varrho_1 + \varrho_2)),\end{aligned}$$

inequality (41) implies that the number $\varrho$ defined in (43) is in fact strictly less than $\min\{\varrho_1, \varrho_2\}$.

We consider the unique geodesic $\tilde{\gamma}\colon [0, 1] \to B(0, \frac{\pi}{2}) \subset \tilde{M}$ connecting $\tilde{p}_1 = \tilde{\gamma}(0)$ to $\tilde{p}_2 = \tilde{\gamma}(1)$, and introduce its midpoint $\tilde{m} := \tilde{\gamma}(\frac{1}{2})$. By the preceding discussion it is sufficient to show that

$$(44) \qquad d_{\tilde{M}}(\tilde{m}, \tilde{q}) \leq \varrho, \quad \forall \tilde{q} \in \overline{B(\tilde{p}_1, \varrho_1) \cap B(\tilde{p}_2, \varrho_2)}.$$

We shall first establish this claim in the special case that $K_{\tilde{M}} \equiv 1$. This means that $\tilde{M}$ is a punctured sphere. Given any point $\tilde{q} \in \overline{B(\tilde{p}_1, \varrho_1) \cap B(\tilde{p}_2, \varrho_2)}$, we apply the Law of Cosines to the hinges $\tilde{p}_1\tilde{m}\tilde{q}$ and $\tilde{q}\tilde{m}\tilde{p}_2$ as depicted in Figure 2 (next page). Then we obtain

$$\begin{aligned}(45) \qquad \cos\varrho_\mu \leq \cos d(\tilde{p}_\mu, \tilde{q}) &= \cos(\tfrac{d}{2})\cos d(\tilde{m}, \tilde{q}) \\ &\quad + \cos\tilde{\varphi}_\mu \cdot \sin(\tfrac{d}{2})\sin d(\tilde{m}, \tilde{q})\end{aligned}$$

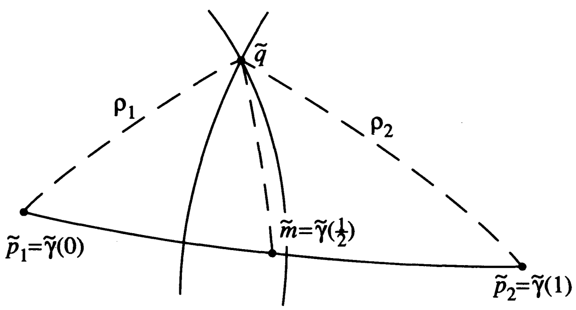


FIGURE 2. Configuration of the hinges

for $\mu = 1, 2$. Here $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ stand for the angle of the first and second hinge, respectively. Since $\tilde{\varphi}_1 + \tilde{\varphi}_2 = \pi$, the two inequalities add up to

$$\cos \varrho_1 + \cos \varrho_2 \leq 2 \cos(\tfrac{d}{2}) \cdot \cos d(\tilde{m}, \tilde{q}) \quad ,$$

which indeed proves inequality (44).

In order to handle the generic case, we observe that $K_{\tilde{M}} \leq 1$ and that the triangles $\tilde{p}_1 \tilde{m} \tilde{q}$ and $\tilde{q} \tilde{m} \tilde{p}_2$ can both be spanned by ruled surfaces. Hence the Alexandrov–Toponogov Triangle Comparison Theorem can be applied to both the hinges, $\tilde{p}_1 \tilde{m} \tilde{q}$ and $\tilde{q} \tilde{m} \tilde{p}_2$, and we can proceed with inequalities (46) as above. q.e.d.

**Proposition 6.4.** *The function* $\mathrm{rad}_C \colon \tilde{M} \to [0, \infty)$ *achieves its minimum* $\mathrm{rad}(C)$ *at precisely one point* $\tilde{m}(C) \in \tilde{M}$*, which we shall call the center of* $C$*. Moreover,* $\tilde{m}(C)$ *depends locally Hölder continuously on the closed set* $C$*. In fact, for any* $\varrho \in (0, \frac{\pi}{2})$ *and any two closed subsets* $C_1, C_2 \in B(0, \varrho) \subset \tilde{M}$ *the distance of their centers is bounded as follows:*

(46)
$$d_{\tilde{M}}(\tilde{m}(C_1), \tilde{m}(C_2)) < 4 \arctan \sqrt{\frac{\tan \varrho \cdot \tan \frac{1}{2} \mathrm{dist}_H(C_1, C_2)}{2 + \tan \varrho \cdot \tan \frac{1}{2} \mathrm{dist}_H(C_1, C_2)}} .$$

*Proof.* To demonstrate the uniqueness of the minimum, we set $\varrho_1 := \varrho_2 := \mathrm{rad}(C)$ and let $\tilde{p}_1, \tilde{p}_2$ be two points where the minimum is achieved. With these choices the right-hand side of (41) vanishes, and therefore Lemma 6.3 contradicts the definition of $\mathrm{rad}(C)$, unless $\tilde{p}_1 = \tilde{p}_2$.

Moreover, it is straightforward to compute (46) from (40) and (41). This proves local Hölder continuity of the map $C \mapsto \mathrm{rad}(C)$ as claimed. q.e.d.

In our applications the closed subset $C \subset B(0, \frac{\pi}{2})$ is always the image of some closed curve $\tilde{c}_t \colon \mathbb{R}/\mathbb{Z} \to B(0, \frac{\pi}{2})$. One of the standard proofs of the Fenchel inequality for $C^2$–curves in $\mathbb{R}^n$ makes use of the following property of closed curves in the standard sphere $\mathbb{S}^{n-1}$.

**Lemma 6.5.** *Let $c\colon \mathbb{R}/\mathbb{Z} \to \mathbb{S}^{n-1}$ be a closed curve of length $L(c) < 2\pi$. Then $c$ is contained in a hemisphere.*

Here we need a more general and quantitatively more precise statement.

**Lemma 6.6.** *Let $\tilde{c}_t\colon \mathbb{R}/\mathbb{Z} \to B(0, \frac{\pi}{2}) \subset \tilde{M}$ be a closed curve of length $L(\tilde{c}_t) < 2\pi$. Then the radius of the smallest circumscribed ball is bounded by*

$$\operatorname{rad}\big(\tilde{c}_t(\mathbb{R}/\mathbb{Z})\big) \;\le\; \tfrac{1}{4}\,L(\tilde{c}_t).$$

*Proof.* Suppose that the curve $\tilde{c}_t$ is parametrized proportional to the arc length, and let $\tilde{\gamma}\colon [0, 1] \to B(0, \frac{\pi}{2})$ be the unique geodesic from $\tilde{p}_1 := \tilde{c}_t(0)$ to $\tilde{p}_2 := \tilde{c}_t(\frac{1}{2})$. Note that $d := d_{\tilde{M}}(\tilde{p}_1, \tilde{p}_2) \le \frac{1}{2}L(\tilde{c}_t)$. Again we let $\tilde{m} := \tilde{\gamma}(\frac{1}{2})$ be the midpoint of $\tilde{\gamma}$. We consider an arbitrary point $\tilde{q} := \tilde{c}_t(s)$ and set $\varrho_1 := d_{\tilde{M}}(\tilde{p}_1, \tilde{q})$ and $\varrho_2 := d_{\tilde{M}}(\tilde{p}_2, \tilde{q})$. Clearly,

$$|\varrho_1 - \varrho_2| \;\le\; d \;\le\; \varrho_1 + \varrho_2 \;\le\; \tfrac{1}{2}L(\tilde{c}_t)\,,$$

and as in the proof of Lemma 6.3 we compute that

$$\begin{aligned}\cos d(\tilde{m}, \tilde{q}) &\ge \frac{\cos\frac{1}{2}(\varrho_1 - \varrho_2)}{\cos(\frac{d}{2})} \cdot \cos\tfrac{1}{2}(\varrho_1 + \varrho_2)\\ &\ge\; \cos\tfrac{1}{2}(\varrho_1 + \varrho_2) \;\ge\; \tfrac{1}{4}L(\tilde{c}_t).\end{aligned}$$

**6.3. Spanning shortly null homotopic curves by ruled surfaces.** Now for the null homotopies contained in $\Omega M_{<2\pi}$ we are in a position to describe the lifting construction which we have mentioned at the beginning of this section. The central result of this subsection is stated in Theorem 6.8 below.

**Lemma 6.7.** *Suppose that $(M^n, g)$ is complete and has curvature $K_M \le 1$, and let $c_0 \in \Omega M_{<2\pi, 0}$. Then for any null homotopy $c\colon [0, 1] \to \Omega M_{<2\pi, 0}$, $t \mapsto c_t$, which begins at the given curve $c_0$ there exists a uniquely determined pair $(m_c, \tilde{c})$ consisting of a path $m_c\colon [0, 1] \to M^n$ and a lift $\tilde{c}\colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to B_{\pi/2}TM$ of the mapping $(s, t) \mapsto \big(m_c(t), c_t(s)\big)$ under the local diffeomorphism $\pi \times \exp\colon B_\pi TM \to M \times M$ such that*

(i) $\{m_c(1)\} = c_1(\mathbb{R}/\mathbb{Z})$ ,
(ii) $\tilde{c}(s, 1) = 0 \in T_{m_c(t)}M$ *for all $s$, and*
(iii) $0 \in B_\pi T_{m_c(t)}M$ *is the center of the lifted curve* $\tilde{c}_t \equiv \tilde{c}(., t)$.

We postpone the proof of the lemma for a moment, and recall that by Corollary 4.2 any two null homotopies of $c_0$ inside $\Omega M_{<2\pi}$ may be connected through some path of null homotopies, which is entirely contained in $\Omega M_{<2\pi}$. Hence the pair $(m_0, \tilde{c}_0) := \big(m_c(0), \tilde{c}(.\,, 0)\big)$ does not depend on the particular choice of the short null homotopy $t \mapsto c_t$, and thus the lemma immediately implies

**Theorem 6.8.** *On any complete Riemannian manifold $(M^n, g)$ with $K_M \leq 1$ there exists a natural continuous map*

$$h^{\text{ruled}} \colon \Omega M_{<2\pi, 0} \to M^n \times \Omega TM$$
$$c_0 \mapsto (m_0, \tilde{c}_0)$$

*such that the following hold:*

(i) *the curve $\tilde{c}_0$ is a lift of $c_0$ under $\exp_{m_0} \colon B_{\pi/2} T_{m_0} M \to M^n$, and*
(ii) *$0 \in B_{\pi/2} T_{m_0} M$ is the center of $\tilde{c}_0$. In particular, $\tilde{c}_0$ does not lie in any open half space in $T_{m_0} M$.*

**Remarks 6.9.** (a) Property (i) means that there is a possibly singular ruled surface $\bar{c}_0 \colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to M^n$ given by $\bar{c}_0(s, \sigma) := \exp_{m_0}\big(\sigma \cdot \tilde{c}_0(s)\big)$ which spans the original curve $c_0$. Saying that the map $h^{\text{ruled}}$ is continuous is equivalent to saying that this *spanning ruled surface* depends continuously on $c_0$. In §6.4 we shall bound the total absolute curvature $\boldsymbol{\kappa}(c_0)$ in order to do the final computations on a ruled surface with a nondegenerate metric which has just a single conical singularity.

(b) In (ii) it does not matter whether we ask $0 \in B_{\pi/2} T_{m_0} M$ to be the center with respect to the Euclidean metric on $T_{m_0} M$ or with respect to the induced Riemannian metric $\exp_{m_0}^* g$.

Since $K_M \leq 1$, it is clear that $\varrho_c \geq \pi$. In the proof of Lemma 6.7 we shall make use of the precise structure of the fiber bundle $\pi \colon B_\pi TM \to M^n$, where we think of each fiber $B_\pi T_p M$ as a Riemannian manifold with metric $\exp_p^* g$. In the proof of the Long Homotopy Lemma we have made use of the fact that the map $\pi \times \exp \colon B_\pi TM \to M \times M$ is a local diffeomorphism. This means that there is some nice additional structure which can be summarized as follows

**Proposition 6.10.** *The sets $\big(\exp^{-1}(q)\big)_{q \in M^n}$ define a foliation of $B_\pi TM$ which induces a pseudogroup of isometries between the nearby fibers $(B_\pi T_{p_1} M, \exp_{p_1}^* g)$ and $(B_\pi T_{p_2} M, \exp_{p_2}^* g)$ of the standard projection $\pi \colon B_\pi TM \to M^n$.*

*In fact, for any* $(p, v) \in B_\pi TM$ *there exists an isometric immersion*

$$\iota_{(p,v)}\colon B_{\pi-|v|}T_pM \subset B_\pi T_pM \to B_\pi T_{\exp_p v}M$$

*such that*

(i) $\exp_{\exp_p v} \circ \iota_{(p,v)} = \exp_p|_{B_{\pi-|v|}T_pM}$,

(ii) $\iota_{(p,0)} = \mathrm{id}_{B_\pi T_pM}$ *for any* $p \in M^n$, *and*

(iii) $\iota_{(p,v)}$ *depends continuously on the point* $(p, v) \in B_\pi TM$.

*Moreover, the family* $(\iota_{(p,v)})_{(p,v)\in B_\pi TM}$ *of isometric immersions is uniquely determined by properties* (i)–(iii).

All these statements follow directly from the standard lifting arguments.

*Proof of Lemma* 6.7. All the essential ingredients have been provided in the previous subsection. The idea is to set up a continuity argument. We set

$$I := \left\{ t_0 \in [0, 1] \;\middle|\; \begin{array}{l} \text{the pair } (m_c, \tilde{c}) \text{ exists over } c|_{\mathbb{R}/\mathbb{Z}\times[t_0, 1]} \text{ and} \\ \text{satisfies conditions (i)–(iii) as far as it exists} \end{array} \right\}.$$

Clearly, $I \subset [0, 1]$ is an interval and $1 \in I$. Hence it is sufficient to find some $\delta > 0$ such that $B(t_0, \delta) \cap [0, 1]$ is contained in $I$ for any $t_0 \in I$.

Since $\ell := \sup\{L(c_t) \mid t \in [0, 1]\} < 2\pi$, it is possible to combine the bound for $\mathrm{rad}(\tilde{c}_{t_0}(\mathbb{R}/\mathbb{Z}))$ from Lemma 6.6 and the modulus of continuity of the path $t \mapsto c_t \in M^n$, $t \in [0, 1]$, in order to define some $\delta \equiv \delta(\ell)$ such that there exists a unique lift $t \mapsto (\tilde{c}_{t_0,t}\colon \mathbb{R}/\mathbb{Z} \to B_{\pi/2}T_{m_c(t_0)}M)$ of the path $t \mapsto c_t$ restricted to the interval $(t_0 - \delta, t_0 + \delta) \cap [0, 1]$. In particular, for these values of $t$ the centers $\tilde{m}(\tilde{c}_{t_0,t})$ are all well defined and depend continuously on $t$. By Proposition 6.10 we find that $m_c(t) := \exp_{m_c(t_0)}(\tilde{m}(\tilde{c}_{t_0,t}))$ and $\tilde{c}_t := \iota_{(m_c(t_0)),\tilde{m}(\tilde{c}_{t_0,t}))} \circ \tilde{c}_{t_0,t}$ are indeed the lifts asked for in the lemma, at least on the interval $(t_0 - \delta, t_0 + \delta) \cap [0, 1]$.

**6.4. On the geometry of the spanning ruled surfaces $\bar{c}_0$.** In this subsection we are going to investigate the ruled surface $\bar{c}_0\colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to M^n$ obtained in Theorem 6.8 above in more detail. We shall find it convenient to work with its lift $\hat{c}_0\colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to \tilde{M}$, which is given in terms of the linear structure of $T_{m_0}M$ by $\hat{c}_0(s, t) = t \cdot \tilde{c}_0(s)$, instead. By hypothesis $c_0$ is a closed regular $C^2$-curve in $\Omega M_{<2\pi,0}$. Clearly, the lifted curve $\tilde{c}_0\colon \mathbb{R}/\mathbb{Z} \to B(0, \frac{\pi}{2}) \subset \tilde{M}$ has the same regularity properties, and the total absolute curvatures of $c_0$ and $\tilde{c}_0$ coincide:

$$\boldsymbol{\kappa}(\tilde{c}_0) := \boldsymbol{\kappa}(c_0). \tag{47}$$

In Corollary 6.13 we shall see that the *ruled surface* $\Sigma$ described by the map $\hat{c}_0\colon \mathbb{R}/\mathbb{Z} \times [0, 1] \to \tilde{M} \equiv B_\pi T_{m_0} M$, $(s, t) \mapsto t \cdot \tilde{c}(s)$, is in fact *regular except for the conical singularity* at the origin, provided that $\boldsymbol{\kappa}(\tilde{c}_0) < \frac{\pi}{2}$.

The function $\tilde{\varrho}\colon \tilde{M} \to [0, \pi)$ defined by $\tilde{\varrho}(\tilde{q}) := d_{\tilde{M}}(0, \tilde{q})$ will play a crucial role in investigating the properties of the map $\hat{c}_0$. It is known to be a smooth function outside $\{0\} = \tilde{\varrho}^{-1}(0)$, where it induces the tensor fields

$$\tilde{v} := \operatorname{grad} \tilde{\varrho} = \operatorname{grad} d_{\tilde{M}}(.\,, 0) \quad \text{and} \quad A := \nabla \tilde{v} = \operatorname{Hess}_{\tilde{\varrho}}.$$

The integral curves of $\tilde{v}$ are the radial geodesics in $\tilde{M}$, and $A|_{\tilde{q}}$ extends the Weingarten map of the distance sphere around $0$ through $\tilde{q}$ to all of $T_{\tilde{q}}\tilde{M}$. It is a solution to the Riccati equation $\nabla_{\tilde{v}} A + A^2 + \tilde{R}(.\,, v)v = 0$ where $\tilde{R}$ denotes the Riemann curvature tensor of $\tilde{M}$. By the hypotheses of Theorem 6.1 we know that $0 < \frac{1}{4}\frac{1}{(1+\varepsilon)^2} < K_{\tilde{M}} \le 1$, and hence we have

**Lemma 6.11** (Bounds for $A$). *The Hessian of the distance function $\tilde{\varrho}$ introduced above is bounded by*

$$\text{(i)} \qquad \cot \tilde{\varrho} \cdot \langle .\,, P_{\tilde{v}} .\rangle \le \langle .\,, A.\rangle < \tilde{\varrho}^{-1} \cdot \langle .\,, P_{\tilde{v}} .\rangle,$$

*where $P_{\tilde{v}} := \mathrm{Id} - \langle .\,, \tilde{v}\rangle \tilde{v}$ denotes the orthogonal projector onto $\tilde{v}^\perp$. Moreover, for any pair of orthonormal vectors $\xi_1, \xi_2 \in \tilde{v}^\perp$ there is the estimate*

$$\text{(ii)} \qquad \begin{aligned} \langle \xi_1, A\xi_2\rangle^2 &\le \langle \xi_1, (A - \cot(\tilde{\varrho})P_{\tilde{v}})\xi_1\rangle \cdot \langle \xi_2, (A - \cot(\tilde{\varrho})P_{\tilde{v}})\xi_2\rangle \\ &\le (\tilde{\varrho}^{-1} - \cot \tilde{\varrho}) \cdot \langle \xi_2, (A - \cot(\tilde{\varrho})P_{\tilde{v}})\xi_2\rangle \\ &< \tfrac{2}{\pi} \cdot \langle \xi_2, A\xi_2\rangle. \end{aligned}$$

*Proof.* Inequality (i) is just a specialization of the standard Riccati Comparison Theorem using $0$ as a lower and $1$ as an upper curvature bound. In order to prove (ii), we merely observe that $\langle .\,, (A - \cot(\tilde{\varrho})P_{\tilde{v}}).\rangle$ is positive semidefinite, and hence it is possible to apply the Cauchy–Schwarz inequality. Note that the function $\varrho \mapsto (\varrho^{-1} - \cot \varrho)$, $\varrho \in [0, \pi]$, is monotonically increasing and concave, and its values at the boundary of the interval $[0, \pi]$ are $0$ and $\frac{2}{\pi}$. q.e.d.

It seems easiest to limit our investigation of the curve $\tilde{c}_0$ to the *domain*

$$(48) \qquad I := \left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{(i) } \tilde{c}_0(s) \neq 0 \text{ and} \\ \text{(ii) } \tilde{v}(s) := \tilde{v}|_{\tilde{c}_0(s)} \text{ and } \tilde{c}_0'(s) \\ \text{are linearly independent} \end{array} \right\}$$

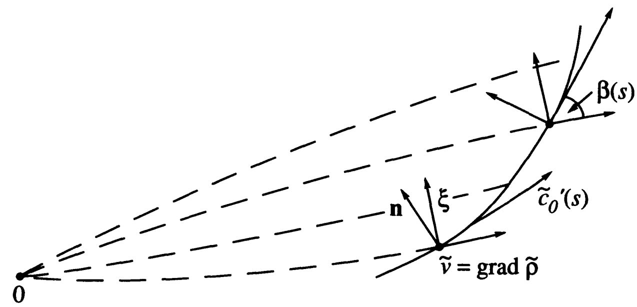


FIGURE 3. The ruled surface $\Sigma = \hat{c}_0(I \times [0, 1])$.

for a first approach. This restriction will be removed in Corollary 6.13. Note, however, that $I$ contains all the critical points of the function $s \mapsto \tilde{\varrho} \circ \tilde{c}_0(s)$, $s \in \mathbb{R}$, where this particular function is nonzero. Actually $s \mapsto \tilde{\varrho} \circ \tilde{c}_0(s)$ is periodic and does not vanish identically. Hence it always has a positive maximum. In particular, $I$ is always nonempty.

Clearly, the ruled surface $\Sigma$ is smooth along $\tilde{c}_0|_I$. We may thus introduce a smooth function $\beta\colon I \to (0, \pi)$ and a unit vector field $s \mapsto \xi(s)$ along $\tilde{c}_0|_I$, which is orthogonal to $\tilde{v}(s)$ for all $s \in I$, requiring that

$$(49) \qquad \tilde{c}_0'(s) = |\tilde{c}_0'(s)| \cdot (\cos\beta(s) \cdot \tilde{v}(s) + \sin\beta(s) \cdot \xi(s)).$$

Note that $\frac{d}{ds}\tilde{\varrho} \circ \tilde{c}_0(s) = |\tilde{c}_0'(s)| \cdot \beta(s)$. A straightforward computation shows that the curvature vector $k_{\tilde{c}_0}$ can be expressed as follows:

$$(50) \qquad \begin{aligned} |\tilde{c}_0'(s)| \cdot k_{\tilde{c}_0}(s) &\equiv \frac{\nabla}{ds} \frac{\tilde{c}_0'(s)}{|\tilde{c}_0'(s)|} \\ &= \cos\beta(s) \cdot \frac{\nabla}{ds}\tilde{v}(s) + \sin\beta(s) \cdot \frac{\nabla}{ds}\xi(s) \\ &\quad + \beta'(s) \cdot (-\sin\beta(s) \cdot \tilde{v}(s) + \cos\beta(s) \cdot \xi(s)). \end{aligned}$$

Since the vectors $\tilde{v}(s)$ and $\xi(s)$ span the tangent space $T_{\tilde{c}_0(s)}\Sigma$, it is clear that the unit normal field

$$(51) \qquad \mathfrak{n}(s) := -\sin\beta(s) \cdot \tilde{v}(s) + \cos\beta(s) \cdot \xi(s)$$

defines a *transverse orientation* on the segment $\tilde{c}_0(I) \subset \Sigma$. We may thus consider its *geodesic curvature* $k^g_{\tilde{c}_0}$, which is the signed quantity given by

$$
\begin{aligned}
|\tilde{c}_0'(s)|\cdot k^g_{\tilde{c}_0}(s) &= \langle \mathfrak{n}(s),\, \tfrac{\nabla}{ds}\tfrac{\tilde{c}_0'(s)}{|\tilde{c}_0'(s)|}\rangle \\
&= \beta'(s) - \sin^2\beta(s)\cdot\langle \tilde{v}(s),\, \tfrac{\nabla}{ds}\xi(s)\rangle + \cos^2\beta(s)\cdot\langle \xi(s),\, \tfrac{\nabla}{ds}\tilde{v}(s)\rangle \\
&= \beta'(s) + |\tilde{c}_0'(s)|\cdot\sin\beta(s)\cdot\langle \xi(s),\, A|_{\tilde{c}_0(s)}\xi(s)\rangle .
\end{aligned}
\tag{52}
$$

In fact, $k^g_{\tilde{c}_0}(s)\cdot\mathfrak{n}(s)$ is the *tangential component* of the curvature vector $k_{\tilde{c}_0}(s)$, and therefore $|k^g_{\tilde{c}_0}(s)| \le |k_{\tilde{c}_0}(s)|$. The size of the normal component $k^\perp_{\tilde{c}_0}(s) \in T_{\tilde{c}_0(s)}\Sigma^\perp$ of the curvature vector is also easy to compute; however, we do not need this quantity here.

**Lemma 6.12.** *Let $[b_0, b_1]$ be some interval contained in $I$. Then*

$$
\beta(b_1) - \beta(b_0) \le \int_{b_0}^{b_1} k^g_{\tilde{c}_0}(s)\cdot|\tilde{c}_0'(s)|\,ds \le \boldsymbol{\kappa}(\tilde{c}_0). \tag{53}
$$

*Proof.* By definition $\sin\beta(s) > 0$ for all $s \in I$. Moreover, the bilinear form $\langle\, .\, , A|_{\tilde{q}}\, .\rangle$ is positive definite for any $\tilde{q} \in B(0, \frac{\pi}{2}) \setminus \{0\}$, and thus the lemma follows directly from (52).

**Corollary 6.13.** *Suppose that $\boldsymbol{\kappa}(c_0) < \frac{\pi}{2}$. Then $I = \mathbb{R}$, and there is the following estimate for the angle:*

$$
|\beta(s) - \tfrac{\pi}{2}| \le \boldsymbol{\kappa}(\tilde{c}_0), \qquad \forall s \in \mathbb{R}. \tag{54}
$$

*In particular, the set $\Sigma = \hat{c}_0(\mathbb{R}/\mathbb{Z} \times [0, 1]) \subset \tilde{M}$ is a smooth surface with boundary $\partial\Sigma = \tilde{c}_0(\mathbb{R}/\mathbb{Z})$ and one conical singularity at $0$, and has curvature $\le 1$ in the distance comparison sense.*

*Proof.* The first step is to establish inequality (54) for all $s \in I$. We shall give an indirect proof.

Suppose that there were some $b_0 \in I$ with $\beta(b_0) < \frac{\pi}{2} - \boldsymbol{\kappa}(\tilde{c}_0)$. Using Lemma 6.12 we conclude that the function $s \mapsto \tilde{\varrho}\circ\tilde{c}_0(s)$ is strictly increasing on $[b_0, \infty)$, since by the continuity of

$$
\tfrac{d}{ds}\tilde{\varrho}\circ\tilde{c}_0(s) = |\tilde{c}_0'(s)|\cdot\cos\beta(s) > 0
$$

there is simply no way to reach a point $b_1 \in [b_0, \infty) \setminus I$ with $\frac{d}{ds}\tilde{\varrho}\circ\tilde{c}_0(s)|_{s=b_1} = -|\tilde{c}_0'(b_1)|$. Of course, this monotonicity property contradicts the periodicity of the curve $\tilde{c}_0$.

The case that there exists some $b_1 \in I$ with $\beta(b_1) > \frac{\pi}{2} + \boldsymbol{\kappa}(\tilde{c}_0)$ can be handled similarly.

The second step is to show that $I = \mathbb{R}$. We already know that $I$ is a nonempty, open subset. So let us suppose that $I$ has some boundary point $b_0$. The idea is that this possibility can be ruled out by inequality (54). Because of the continuity of the derivative $\frac{d}{ds}\tilde{\varrho} \circ \tilde{c}_0(s)$ at all points $\tilde{c}_0(s) \neq 0$, it remains to consider the case $\tilde{c}_0(b_0) = 0$, which is equally bad, since $b_0$ is an isolated zero of the function $\tilde{\varrho} \circ \tilde{c}_0$ and $\lim_{s \to b_0 + 0}\left(L(\tilde{c}_0|_{[b_0, s]})^{-1} \cdot \tilde{\varrho} \circ \tilde{c}_0(s)\right) = 1$.

By now we have shown that $\Sigma$ is a smooth ruled surface with boundary and one conical singularity at $0$. So the last step is to establish the upper curvature bound. Since $\Sigma$ is a ruled surface, the Gauss equations imply that $K_\Sigma \leq K_{\tilde{M}} \leq 1$ at all regular points. We claim that $\Sigma$ has curvature $\leq 0$ at the singularity as well. In fact, the intrinsic total angle $\sphericalangle_{\mathbf{0}}(\Sigma)$ at this point can be computed as the length of the curve $s \mapsto |\tilde{c}_0(s)|^{-1} \cdot \tilde{c}_0(s) \in T_{m_0}M$ with respect to the Euclidean metric in the tangent space. By Theorem 6.8 the curve in question does not lie in any open half space, and hence Lemma 6.5 implies that

$$\sphericalangle_{\mathbf{0}}(\Sigma) \geq 2\pi. \quad \text{q.e.d.} \tag{55}$$

**Remark 6.14.** By construction all the segments $t \mapsto \hat{c}_0(s, t)$ on $\Sigma$ are strictly shorter than $\frac{\pi}{2}$. Since $K_{\tilde{M}} \leq 1$, it follows directly from the Rauch Comparison Theorems that $\text{Area}(\Sigma) \leq L(\tilde{c}_0)$. Hence, the *local Gauss–Bonnet formula* yields

$$\begin{aligned} &\left(\sphericalangle_{\mathbf{0}}(\Sigma) - 2\pi\right) + \left(2\pi - L(\tilde{c}_0)\right) \\ &\qquad \leq 2\pi + \left(\sphericalangle_{\mathbf{0}}(\Sigma) - 2\pi\right) - \int_{\Sigma \setminus \{0\}} K_\Sigma \, dA \\ &\qquad = \int_{\mathbb{R}/\mathbb{Z}} k^g_{\tilde{c}_0}(s) \cdot |\tilde{c}_0'(s)| \, ds \leq \boldsymbol{\kappa}(\tilde{c}_0). \end{aligned} \tag{56}$$

By (55) the first term on the left-hand side of this inequality is nonnegative, and by the hypotheses of Theorem 6.1 the second term on the left-hand side is nonnegative as well. Actually, (56) is quite a strong statement about the shape of $\Sigma$, if the total absolute curvature $\boldsymbol{\kappa}(\tilde{c}_0)$ is small.

For instance, the fact that the tangent cone to $\Sigma$ at $0$ is not contained in any open half space ties in nicely with a small bound for the difference $\sphericalangle_{\mathbf{0}}(\Sigma) - 2\pi$. It is possible to deduce that the tangent cone of $\Sigma$ at $0$ lies in a small neighborhood of an approximate tangent plane.

Inequality (56) reveals its full power in the discussion of the limiting behavior when the total absolute curvature of some sequence of null homotopic curves $c_\mu \in \Omega M_{<2\pi,0}$ converges to $0$. In that case a suitable subsequence of the ruled surfaces $\bar{c}_\mu \colon \mathbb{R}/\mathbb{Z} \to M^n$ converges to a totally geodesically immersed hemisphere.

However, the behavior of the ruled surface $\Sigma$ near its boundary can be more easily controlled by direct estimates.

**Proposition 6.15.** *Suppose that the total absolute curvature $\boldsymbol{\kappa}(\tilde{c}_0)$ is strictly less than $\frac{\pi}{2}$. Then the unit vector field $s \mapsto \tilde{v}(s)$ along $\tilde{c}_0$ has total absolute rotation:*

$$\operatorname{rot}(\tilde{v}) \;\le\; \boldsymbol{\kappa}(\tilde{c}_0) + 2 \cdot \sqrt{\boldsymbol{\kappa}(\tilde{c}_0)}.$$

*Proof.* By Corollary 6.13 formulas (49)–(52) hold for all $s \in \mathbb{R}/\mathbb{Z}$. After decomposing $\frac{\nabla}{ds}\tilde{v}(s)$ into its tangential and normal components, we may use Lemma 6.11 to bound the second piece:

$$\begin{aligned}
\left|\tfrac{\nabla}{ds}\tilde{v}(s)\right| \;&\le\; \left|\langle \xi(s), \tfrac{\nabla}{ds}\tilde{v}(s)\rangle\right| + \sup_{X\in T^{\perp}\Sigma,\ |X|=1} \langle X, \tfrac{\nabla}{ds}\tilde{v}(s)\rangle \\
&= |\tilde{c}_0'(s)|\cdot \sin\beta(s)\cdot \langle \xi(s), A|_{\tilde{c}_0(s)}\xi(s)\rangle \\
&\qquad + \sup_{X\in T^{\perp}\Sigma,\ |X|=1} \langle X, \tfrac{\nabla}{ds}\tilde{v}(s)\rangle \\
&\le |\tilde{c}_0'(s)|\cdot \sin\beta(s)\cdot \Big(\langle \xi(s), A|_{\tilde{c}_0(s)}\xi(s)\rangle \\
&\qquad + \sqrt{\tfrac{2}{\pi}}\cdot \langle \xi(s), A|_{\tilde{c}_0(s)}\xi(s)\rangle^{1/2}\Big).
\end{aligned} \tag{57}$$

As in the proof of Lemma 6.12 we integrate this estimate, and combine (52) with the Cauchy–Schwarz inequality to conclude that

$$\operatorname{rot}(\tilde{v}) \;\equiv\; \int_{\mathbb{R}/\mathbb{Z}} \left|\tfrac{\nabla}{ds}\tilde{v}(s)\right| ds \;\le\; \boldsymbol{\kappa}_\Sigma(\tilde{c}_0) + \sqrt{\tfrac{2}{\pi}}\cdot L(\tilde{c}_0)^{1/2}\cdot \boldsymbol{\kappa}_\Sigma(\tilde{c}_0)^{1/2},$$

where

$$\boldsymbol{\kappa}_\Sigma(\tilde{c}_0) \;:=\; \int_{\mathbb{R}/\mathbb{Z}} k^{g}_{\tilde{c}_0}(s)\cdot |\tilde{c}_0'(s)|\, ds.$$

The latter quantity is bounded by the total absolute curvature of $\tilde{c}_0$, and by hypothesis the length of this curve is $< 2\pi$. q.e.d.

Next we shall consider a particular family of closed unit vector fields $W_\alpha$ along $\tilde{c}_0$ where $\alpha$ varies in $\mathbb{R}/(2\pi\mathbb{Z})$. Roughly speaking, we are going to interpolate the field $\tilde{v}$ and the unit tangent field of $\tilde{c}_0$. The precise definition is as follows:

$$(58) \qquad W_\alpha(s) := |w_\alpha^{\text{rough}}(s)|^{-1} \cdot w_\alpha^{\text{rough}}(s),$$

where

$$w_\alpha^{\text{rough}}(s) := \cos\alpha \cdot \tilde{v}(s) + \sin\alpha \cdot \frac{\tilde{c}_0'(s)}{|\tilde{c}_0'(s)|}.$$

Evidently, $W_{\alpha+\pi} = -W_\alpha$, and thus Corollary 3.3 can be applied, provided we find some uniform a priori bounds for $\operatorname{rot}(W_\alpha)$ for all $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$.

**Proposition 6.16.** *Suppose that the total absolute curvature $\boldsymbol{\kappa}(\tilde{c}_0)$ is strictly less than $\frac{\pi}{6}$. Then for each $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$ there is the following bound on the total absolute rotation of the unit vector field $W_\alpha$ along $\tilde{c}_0$:*

$$\operatorname{rot}(W_\alpha) \le \frac{\sqrt{2}\cdot\boldsymbol{\kappa}(\tilde{c}_0) + 2\cdot\sqrt{\boldsymbol{\kappa}(\tilde{c}_0)}}{\sqrt{1-\sin\boldsymbol{\kappa}(\tilde{c}_0)}}.$$

*Proof.* Using the bounds for the angle $\beta(s)$, that have been obtained in (54) in Corollary 6.13, it is easy to see that

$$|w_\alpha^{\text{rough}}(s)|^2 = 1 + \sin 2\alpha \cdot \cos\beta(s) \ge 1 - \sin\boldsymbol{\kappa}(\tilde{c}_0) \ge \tfrac{1}{2}.$$

Differentiating (58) yields that $|\frac{\nabla}{ds} W_\alpha(s)| \le |w_\alpha^{\text{rough}}(s)|^{-1} \cdot |\frac{\nabla}{ds} w_\alpha^{\text{rough}}(s)|$. Since by Proposition 6.15 integrals over the derivative of $w_\alpha^{\text{rough}}$ can be bounded in terms of the total absolute curvature $\boldsymbol{\kappa}(\tilde{c}_0)$, we conclude that

$$\begin{aligned}
\operatorname{rot}(W_\alpha) &\equiv \int_{\mathbb{R}/\mathbb{Z}} |\tfrac{\nabla}{ds} W_\alpha(s)|\, ds \\
&\le \frac{1}{\sqrt{1-\sin\boldsymbol{\kappa}(\tilde{c}_0)}} \cdot \int_{\mathbb{R}/\mathbb{Z}} |\cos\alpha| \cdot |\tfrac{\nabla}{ds}\tilde{v}(s)| + |\sin\alpha| \cdot |\tfrac{\nabla}{ds} \tfrac{\tilde{c}_0'(s)}{|\tilde{c}_0'(s)|}|\, ds \\
&\le \frac{|\cos\alpha| \cdot \left(\boldsymbol{\kappa}(\tilde{c}_0) + 2\sqrt{\boldsymbol{\kappa}(\tilde{c}_0)}\right) + |\sin\alpha| \cdot \boldsymbol{\kappa}(\tilde{c}_0)}{\sqrt{1-\sin\boldsymbol{\kappa}(\tilde{c}_0)}}.
\end{aligned}$$

It remains to determine the maximum of the right-hand side of this inequality with respect to the parameter $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. q.e.d.

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