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HYPERBOLIC MANIFOLDS WITH NEGATIVELY CURVED EXOTIC TRIANGULATIONS IN DIMENSION SIX

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0. Introduction

In this article we construct, given $\varepsilon > 0$, closed real hyperbolic manifolds of dimension 6 with exotic (smoothable) triangulations admitting Riemannian metrics with sectional curvatures in the interval $(-1, -\varepsilon, -1+\varepsilon)$.

A fundamental problem in geometry and topology is the following.

0.1. When are two homotopically equivalent manifolds diffeomorphic, PL homeomorphic, or homeomorphic?

When both manifolds in (0.1) are closed, hyperbolic, and of dimension greater than 2, Mostow's rigidity theorem states that they are isometric, in particular diffeomorphic. When both manifolds have strictly negative curvature, results of Eells and Sampson [4], Hartman [7], and Al'ber [1] show that if $f: M_1 \to M_2$ is a homotopy equivalence, then it is homotopic to a unique harmonic map. Lawson and Yau conjectured that this harmonic map is always a diffeomorphism (see problem 12 Yau [13], which asks for proof of (0.1), differentiably, for strictly negative curved manifolds). Farrell and Jones [5] gave counterexamples to this conjecture by proving the following. If M is a real hyperbolic manifold and Σ is an exotic sphere, then given $\varepsilon > 0$, M has a finite covering $\widetilde{M}$ such that the connected sum $\widetilde{M}\#\Sigma$ is not diffeomorphic to $\widetilde{M}$ and admits a Riemannian metric with all sectional curvatures in the interval $(-1-\varepsilon, -1+\varepsilon)$. Because there are exotic spheres only in dimensions 7 and up this does not give counterexamples to Lawson-Yau conjecture in dimension less than 7. The constructions here give counterexamples in dimension 6. Explicitly, we have the following theorem, that is a consequence of Theorem (3.1) and construction (3.2).

0.2. Theorem. *There are closed real hyperbolic manifolds M of dimension 6 such that the following holds. Given $\varepsilon > 0$, M has a finite cover*

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$\widetilde{M}$ *that supports an exotic (smoothable) PL structure that admits a Riemannian metric with sectional curvatures in the interval* $(-1-\varepsilon, -1+\varepsilon)$.

These manifolds are the ones that appear at the end of [11] for the real hyperbolic case.

Also, in [6] Farrell and Jones proved that (0.1) holds topologically when one manifold is nonpositively curved and has dimension greater than 4. And, again by [5], (0.1) does not hold, diffeomorphically, for dimensions greater than 6. Then it is natural to ask if (0.1) holds PL homeomorphically for nonpositively curved manifolds. (Note that [5] does not answer this because connected sum with spheres does not change PL structures). In §4 we show that, in general, this is not the case (for dimensions greater than 5). In fact, we obtain the following

0.3. Corollary. *For* $n \geq 6$, *there are closed nonpositively curved manifolds of dimension* n *that support exotic (smoothable) PL structures admitting Riemannian metrics with nonpositive sectional curvatures.*

These manifolds are simply the product of the manifolds in (0.2) with the m-torus.

Here is a short outline of the paper. First, in §1, we show how to change (concordance classes of) triangulations (modulo some closed subset) by cutting along a hypersurface and gluing back with a twist. Then, in §3, we take this hypersurface to be totally geodesic and search for one with a large tubular neighborhood width, so that we can use the same method as [5] (see §2) to provide this exotic triangulation with a Riemannian metric with sectional curvatures in $(-1-\varepsilon, -1+\varepsilon)$.

1. Triangulation lemmas

Recall that if M is a PL manifold and $C \subset M$, a closed subset (assume $m = \dim M \geq 6$ or $\dim M \geq 5$ and $\partial M \subset C$) then there is a one-to-one correspondence between $\check{H}^3(M, C; \mathbb{Z}_2)$ (this is Čech cohomology) and the set of concordance classes of PL structures on M that agree with the given one on a neighborhood of C. We can choose this correspondence to be such that it sends the given PL structure to 0. Next we sketch how this correspondence is given (see [8]).

Denote by τ_0 the given PL structure on M. Also denote by B_{TOP} and B_{PL} the stable classifying spaces for TOP and PL microbundle structures and $\mathrm{TOP}/PL \to B'_{PL} \to B_{\mathrm{TOP}}$ the fibration we obtain from the canonical map $B_{PL} \to B_{\mathrm{TOP}}$. Let τ be another PL structure on M that agrees with τ_0 on a neighborhood of C. Then there is an n such that $\tau \times \mathbb{R}^n$ is concordant to a PL structure θ, that makes $M \times \mathbb{R}^n$ a PL microbundle (trivial over a neighborhood of C) over M_{τ_0}. This gives a correspondence between concordance classes of PL structures on M that agree with τ_0 on a neighborhood of C and $\mathrm{TOP}/PL(\varepsilon(M)\,\mathrm{rel}\,C)$, the set of stable concordance classes $(\mathrm{rel}\,C)$ of PL microbundle structures of the trivial bundle $\varepsilon(M)$ over M_{τ_0} (see [8, p. 176]). But $\mathrm{TOP}/PL(\varepsilon(M)\,\mathrm{rel}\,C)$ is also in correspondence with $\mathrm{Lift}(f\,\mathrm{rel}\,C, F_o)$, the set of vertical homotopy classes of liftings of f to B'_{PL}, where $f\colon M \to B_{\mathrm{TOP}}$ classifies $\varepsilon(M)$ and F_0: {neighborhood of C} $\to B'_{PL}$ is a given lifting of $f|_{\text{neighborhood of } C}$ (it classifies $\tau_0|_{\text{neighborhood of } C}$). But $\varepsilon(M)$ is a trivial bundle so that we can choose f to be a constant map (and F_0 also constant because our PL microbundle structures are trivial over a neighborhood of C), hence $\mathrm{TOP}/PL(\varepsilon(M)\,\mathrm{rel}\,C)$ is in correspondence with $[M, C; \mathrm{TOP}/PL]$, the set of homotopy classes of maps from M to TOP/PL that send a neighborhood of C to a previously fixed point. But TOP/PL is an Eilenberg-MacLane space of type $(3, \mathbb{Z}_2)$, so that $[M, C; \mathrm{TOP}/PL]$ is in correspondence with $\check{H}^3(M, C; \mathbb{Z}_2)$. Note that this correspondence depends on which PL structure we are sending to zero in $\check{H}^3(M, C; \mathbb{Z}_2)$ and is also completely determined by this choice.

Given a concordance class of triangulations $[\tau]$ denote by $c_{[\tau]} = c_\tau \in \check{H}^3(M, C; \mathbb{Z}_2)$ the corresponding cohomology class, and also given a cohomology class c write $[\tau_c] = [\tau]_c$ for the corresponding concordance class of triangulations.

We have the following lemma.

1.1. Lemma. *Let $p\colon \widetilde{M} \to M$ be a covering, $C \subset M$ closed, and $m = \dim M \geq 6$ (or $\dim M \geq 5$ and $\partial M \subset C$). Suppose M as a PL structure τ_0 and denote by $\tilde{\tau}_0$ the pullback $p^*\tau_0$ of τ_0 and make these two triangulations correspond to zero in $\check{H}^3(\widetilde{M}, p^{-1}(C); \mathbb{Z}_2)$ and $\check{H}^3(M, C; \mathbb{Z}_2)$ respectively. Then $[\tau]_{p^*c} = [p^*\tau_c]$ for all $c \in \check{H}^3(M, C; \mathbb{Z}_2)$. Equivalently, $c_{p^*\tau} = p^*c_\tau$ for every PL structure τ on M.*

Note that if τ_1 and τ_2 are concordant PL structures on M, then $p^*\tau_1$ and $p^*\tau_2$ are also concordant.

Proof. Let τ be a PL structure on M $(\mathrm{rel}\,C)$. If θ is a PL structure that makes $M \times \mathbb{R}^n$ (for some n) a PL microbundle over M_{τ_0} concordant $(\mathrm{rel}\,C)$ to $\tau \times \mathbb{R}^n$, then $\tilde{p}^*\theta$ is a PL structure that makes $\widetilde{M} \times \mathbb{R}^n$ a PL microbundle over $\widetilde{M}_{\tilde{\tau}_0}$ concordant $(\mathrm{rel}\,p^{-1}C)$ to $p^*\tau \times \mathbb{R}^n$, where $\tilde{p} = (p, \mathrm{Id}_{\mathbb{R}^n})$. If $h\colon M \to \mathrm{TOP}/PL \subset B'_{PL}$ classifies θ, then hp classifies

$\tilde{p}^*\theta$. So, pulling back PL structures gives a map $[M, C; \mathrm{TOP}/PL] \to [\widetilde{M}, p^{-1}(C); \mathrm{TOP}/PL]$ given by $h \mapsto hp$. This completes the proof of the lemma.

Now, given a PL manifold M, we show how to change PL structures by cutting along a closed hypersurface N of M and gluing back with a twist.

Denote by M_χ the CAT ($= PL$ or DIFF) manifold obtained by cutting along N and identifying by χ the two copies of N we get, where N is a CAT closed hypersurface and $\chi: N \to N$ is a CAT isomorphism. In what follows we assume that the relative set is nice enough (for example, deformation retract of a subcomplex) so that we replace Čech cohomology by singular cohomology.

1.2. Lemma. *Let M be a PL orientable n-manifold, $n \geq 6$, N a closed PL hypersurface with a tubular neighbourhood $g: W \cong_{PL} N \times [-1, 1]$ of N in M, where $g(N) = N \times \{0\}$, and $J \subset N$ open with $\overline{J}$ compact. Then for every $c \in H^3(M, M\backslash J; \mathbb{Z}_2)$, there is a PL isomorphism $\chi: N \to N$, such that M_χ (that is, its PL structure) corresponds to c (by the correspondence that sends the given PL structure to 0) and χ is the identity outside a compact neighborhood of $\overline{J}$.*

Note that J is not open in M but $g^{-1}(J \times (-\delta, \delta))$ is, where $\delta < 1$, and $(M, M\backslash g^{-1}(J \times (-\delta, \delta)))$ is a deformation retract of $(M, M\backslash J)$.

Proof. Denote by τ_0 the given PL structure on M and make it correspond to $0 \in H^3(M, M\backslash J; \mathbb{Z}_2)$. Now, τ_c (a PL structure that corresponds to c) is a PL structure on W that agrees with τ_0 outside $g^{-1}(J \times (-\delta, \delta))$. In particular they agree on $g^{-1}((N\backslash J) \times [-1, 1])$, so that W_{τ_c} is a PL product there because W_{τ_0} is. By the s-cobordism theorem and the fact that the torsion of a homeomorphism is zero, we have that there is a PL homeomorphism $h: (W, \tau_c) \to N \times [-1, 1]$, such that

$$hg^{-1}|_{N\times\{-1\}\cup(\{N\backslash V\}\times[-1,1])} = \mathrm{Id}_{N\times\{-1\}\cup(\{N\backslash V\}\times[-1,1])},$$

where $\overline{J} \subset V \subset \overline{V} \subset N$, $\overline{V}$ is compact, and V is open. Let $\chi = (h^{-1}g)|_{g^{-1}(N\times\{1\})}$. Then we see that M_χ corresponds to τ_c (here to obtain M_χ we are cutting along $g^{-1}(N \times \{1\}) \subset W$), for we can define a PL homeomorphism $H: M_{\tau_c} \to M_\chi$ by

$$H(x) = \begin{cases} g^{-1}h(x), & x \in W, \\ x, & x \in M\backslash W. \end{cases}$$

Note that $\chi|_{N\backslash V} = \mathrm{Id}_{N\backslash V}$. This completes the proof of Lemma 1.2.

1.3. Remark. Note that if τ is smoothable, then, using now the differentiable s-cobordism theorem, we can choose χ to be smooth.

2. Geometric Lemma

Let M be a differentiable manifold and consider metrics A on $M \times I$, where $I = [1, 2]$ satisfying (recall that the tangent space at a point $(x, t) \in M \times I$ is isomorphic to $T_xM \oplus \mathbb{R}(\partial/\partial t)|_t)$.

2.1. (a) For any $v \in T_xM$, $A(v, \partial/\partial t) = 0$.

(b) $A(\partial/\partial t, \partial/\partial t) = 1$.

Equivalently, $A = S_t + dt^2$, where S_t is a metric on M depending on t.

2.2. Lemma. *Let M be compact and $A = S_t + dt^2$ a metric on $M \times I$ satisfying* (2.1). *Then given $\varepsilon > 0$ there is an L such that for $\alpha > L$ all the sectional curvatures of A_α lie in $(-1-\varepsilon, -1+\varepsilon)$, where A_α is the metric on $M \times I$ given by $A_\alpha(x, t) = \cosh^2(\alpha t)S_t + \alpha^2 dt^2$.*

The proof is the same as the proof of Lemma 3.5 of [5]. Just replace the function sinh by cosh and the $(m-1)$ sphere by any compact manifold.

3. Construction of the examples

First we proof the following theorem.

3.1. Theorem. *Consider the following data. For each $k = 1, 2, 3, \cdots$ we have closed hyperbolic manifolds $M_0(k)$, $M_1(k)$, $M_2(k)$, $M_3(k)$ such that the following hold.*

(a) $\dim M_0(k) = 6$, $\dim M_1(k) = 5$, $\dim M_2(k) = 3$, $\dim M_3(k) = 3$.

(b) *$M_2(k) \subset M_1(k) \subset M_0(k)$ and $M_3(k) \subset M_0(k)$. All the inclusions are totally geodesic.*

(c) *$M_2(k)$ and $M_3(k)$ intersect at one point transversally.*

(d) *For each k there is a finite covering map $p(k)\colon M_0(k) \to M_0(1)$ such that $p(k)(M_i(k)) = M_i(1)$, for $i = 0, 1, 2, 3$.*

(e) *$M_1(k)$ has a tubular neighborhood in $M_0(k)$ of width $r(k)$ and $r(k) \to \infty$ as $k \to \infty$.*

Then, given $\varepsilon > 0$, there is a K such that all $M_0(k)$, $k \geq K$, have exotic (smoothable) triangulations admitting Riemannian metrics with all sectional curvatures in the interval $(-1-\varepsilon, -1+\varepsilon)$.

Proof. Denote by $\sigma(k)$ the triangulation on $M_0(k)$ induced by the hyperbolic structure and make it correspond to zero in $H^3(M_0(k), \mathbb{Z}_2)$. Also, denote by $g(k)$ the restriction of the hyperbolic metric on $M_0(k)$ to the totally geodesic submanifold $M_1(k)$. Then the tubular neighborhood of width $r(k)$ of $M_1(k)$ in $M_0(k)$ is isometric to $M_1(k)\times[-r(k), r(k)]$ with metric, at a point (x, t), given by $(\cosh^2 t)g(k)+dt^2$ (note that hyperbolic n-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^{n-1}\times\mathbb{R}$ with metric $(\cosh^2 t)g+dt^2$, where g is the hyperbolic metric on $\mathbb{H}^{n-1}$).

Take now a tubular neighborhood $W(k)$ of $M_2(k)$ in $M_1(k)$. We can suppose that $p(k)|_{W(k)}: W(k)\to W(1)$ is a covering. Let the open set $U(k)$ be such that $\overline{U(k)}$ is a compact tubular neighborhood of $M_2(k)\times\{2\}$ in $W(k)\times(0, 3)$.

Consider the cohomology class $c(k)\in H^3(W(k)\times(0, 3), W(k)\times(0, 3)\backslash U(k); \mathbb{Z}_2)\cong H^3(W(k)\times(0, 3), W(k)\times(0, 3)\backslash M_2(k)\times\{2\}; \mathbb{Z}_2)$ dual to $M_2(k)\times\{2\}\subset W(k)\times(0, 3)$.

Denote by $\tau(k)$ the triangulation, modulo the complement of $U(k)$, on $W(k)\times(0, 3)$ corresponding to $c(k)$.

Let $f(k): W(k)\to W(k)$ be the PL isomorphism corresponding to $c(k)$ given by Lemma (1.2), so that the triangulation of $(W(k)\times(0, 3))_{f(k)}$, obtained by identifying $(x, 2)\in W(k)\times(0, 2]$ with $(f(k)(x), 2)\in W(k)\times[2, 3)$, corresponds to $c(k)$ (it is concordant to $\tau(k)$).

3.1.1. We have the following claims.

(1) $(p(k)|_{W(k)}\times\mathrm{Id}_{(0,3)})^*c(1)=c(k)$ and $\tau(k)=(p(k)|_{W(k)}\times\mathrm{Id}_{(0,3)})^*\tau(1)$.

(2) We can choose $f(k)$ such that it covers $f(1)$.

(3) We can suppose $\tau(k)$ to be smoothable and $f(k)$ a diffeomorphism.

(4) We can take $f(k)$ to be the identity outside a neighborhood $V(k)$ of $M_2(k)$, with $\overline{V(k)}\subset W(k)$ compact.

Proofs of the claims. (3) is true because in dimension 6 there is no obstruction for a PL structure to be smooth so that we can suppose $f(k)$ smooth (see Remark (1.3)). (4) follows from the fact that $\tau(k)$ and $\sigma(k)$ coincide outside $U(k)$ (see proof of Lemma (1.2)). (1) is true because $(p|_{W(k)})^{-1}(M_2(1))=M_2(k)$ (the pullback of the dual of a cycle is the dual of the inverse image (to see this just consider a sufficiently fine triangulation and its dual cell decomposition and pull back everything)). The second part of (1) follows from Lemma (1.1). For (2) note that the triangulation of $(W(k)\times(0, 3))_{f(k)}$ is $\tau(k)=(p(k)|_{W(k)}\times\mathrm{Id}_{(0,3)})^*\tau(1)$, and if $f(k)$ covers $f(1)$ then $(W(k)\times(0, 3))_{f(k)}$ covers $(W(1)\times(0, 3))_{f(1)}$ by a PL covering. Hence we can take as $f(k)$ a lifting of $f(1)$ (indeed, we could have defined $f(k)$ in this way). This completes the proof of the claims.

Consider the metric $A(1)$ on $W(1)\times[1, 2]$ defined by

$$A(1)=[\delta(t)f(1)^*(g(1))+(1-\delta(t))g(1)]+dt^2,$$

where δ is a smooth real function such that $0 \leq \delta(t) \leq 1$, $\delta(1) = 0$, $\delta(2) = 1$ and is constant near $1, 2$.

For $\varepsilon > 0$ let L be the constant given by Lemma (2.2), so that all sectional curvatures of $A(1)_\alpha$ lie in $(-1-\varepsilon, -1+\varepsilon)$, for $\alpha \geq L$. Note that, because $f(1)$ is the identity outside $V(1)$, we have that $A(1) = g(1)+dt^2$ outside $V(1) \times [1, 2]$ and then also $A(1)_\alpha = (\cosh^2(\alpha t))g(1) + \alpha^2 dt^2$ outside $V(1) \times [1, 2]$. Note that we cannot apply Lemma (2.2) directly because $W(1) \times [1, 2]$ is not compact, but we can apply the lemma to $M_1(1) \times [1, 2]$ because we can extend $A(1)$ to it. Define now a metric $B(1)$ on $W(k)\times(0, 3))_{f(1)}$ (that is $W(k)\times(0, 3)$ with triangulation $\tau(1)$) by

$$B(1) = \begin{cases} A(1), & t \in [1, 2], \\ g(1) + dt^2, & t \in (0, 1] \cup [2, 3). \end{cases}$$

Note that this metric is well defined since both definitions coincide on a neighborhood of $t = 1, 2$.

Thus $(W(1)\times(0, 3))_{f(1)}$ admits Riemannian metrics (the metric $B(1)_\alpha$ for $\alpha \geq L$) with all sectional curvatures in $(-1-\varepsilon, -1+\varepsilon)$. Remark that $B(1)_\alpha = (\cosh^2(\alpha t))g(1)+\alpha^2 dt^2$ outside a compact subset of $W(1)\times(0, 3)$ containing $M_2(1) \times \{2\}$.

Also, by defining $B(k) = p(k)^* B(1)$, we have that $(W(k) \times (0, 3))_{f(k)}$ (i.e., $W(k) \times (0, 3)$ with triangulation $\tau(k)$) admits Riemannian metrics (the metrics $B(k)_\alpha$ for $\alpha \geq L$) with all sectional curvatures in $(-1-\varepsilon, -1+\varepsilon)$. Note that we also have $B(k)_\alpha = (\cosh^2(\alpha t))g(k) + \alpha^2 dt^2$ outside a compact subset of $W(k) \times (0, 3)$ containing $M_2(k) \times \{2\}$. We try now to fit these models (i.e., $(W(k) \times (0, 3))_{f(k)}$ with the metrics $B(k)_\alpha$) on the $M_0(k)$, for large enough k.

Let K be such that $r(k) > 3L$ for $k \geq K$ (use hypothesis (e) here). We prove that $M_0(k)$ has exotic triangulations with Riemannian metrics with sectional curvatures in the interval $(-1-\varepsilon, -1+\varepsilon)$.

Because of (e) of the statement of the theorem, $M_1(k) \subset M_0(k)$ has a tubular neighborhood of width $r(k)$ isometric to $M_1 \times [-r(k), r(k)]$ with metric $(\cosh^2(t))g(k) + dt^2$. In what follows we make no distinction between the tubular neighborhood and $M_1(k) \times [-r(k), r(k)]$.

Consider

$$\begin{aligned} h(k)\colon W(k) \times (0, 3) \to W(k) \times (0, 3L) \subset W(k) \times (-r(k), r(k)) \\ \subset M_1(k) \times (-r(k), r(k)) \subset M_0(k) \end{aligned}$$

given by $(x, t) \mapsto (x, Lt)$.

Note that $h(k)$ is an isometry, where we are considering $W(k) \times (0, 3)$ with metric $\cosh^2(Lt)g(k) + L^2\,dt^2$, and $W(k) \times (0, 3L)$ with metric induced by the hyperbolic metric on $M_0(k)$.

Because the triangulation $(h(k)^{-1})^*\tau(k)$ coincides with $\sigma(k)$ outside a compact in $W(k) \times (0, 3L)$, we can extend it to all $M_0(k)$ by defining it to be $\sigma(k)$ outside $W(k) \times (0, 3L)$. Call this triangulation on $M_0(k)$, $\overline{\tau}(k)$. This (smoothable) trangulation corresponds to the cohomology class $\overline{c}(k) \in H^3(M_0(k), M_0(k)\backslash M_2(k) \times \{2L\}; \mathbb{Z}_2)$ dual to $M_2(k) \times \{2L\} \subset W(k) \times (0, 3L) \subset M_1(k) \times (-r(k), r(k)) \subset M_0(k)$ (the correspondence between PL structures and the third cohomology group is natural for restrictions to open sets; see [8, p. 195]. Define also a metric $\overline{B}(k)$, compatible with $\overline{\tau}(k)$, on $M_0(k)$ to be $(h(k)^{-1})^*B(k)_L$ on $W(k) \times (0, 3L)$ and the hyperbolic metric outside $W(k) \times (0, 3L)$. Note that all sectional curvatures of $\overline{B}(k)$ lie in $(-1-\varepsilon, -1+\varepsilon)$ (all sectional curvatures are -1 outside a compact subset of $W(k) \times (0, 3L)$).

So, given $\varepsilon > 0$, there is a K such that for $k \geq K$, $\overline{\tau}(k)$ is a triangulation on $M_0(k)$ that admits the Riemannian metric $\overline{B}(k)$ with all sectional curvatures in the interval $(-1-\varepsilon, -1+\varepsilon)$ and $\overline{\tau}(k)$ corresponds (by the correspondence that sends $\sigma(k)$ to zero) to $\overline{c}(k) \in H^3(M_0(k), M_0(k)\backslash M_2(k) \times \{2L\}; \mathbb{Z}_2)$ dual to $M_2(k) \times \{2L\}$.

But $\overline{c}(k)$ is not zero in $H^3(M_0(k); \mathbb{Z}_2)$. That is, if

$$i_3\colon H^3(M_0(k), M_0(k)\backslash M_2(k) \times \{2L\}; \mathbb{Z}_2) \to H^3(M_0(k); \mathbb{Z}_2)$$

is the inclusion, then $i_3(\overline{c}(k))$ is not zero because $M_2(k) \times \{2L\}$ is homologous to $M_2(k)$ and it intersects $M_3(k)$ tranversally at one point (by hypothesis (c)). This means that $\sigma(k)$ and $\overline{\tau}(k)$ are nonconcordant.

Finally we have to prove that $\overline{\tau}(k)$ is indeed not equivalent to $\sigma(k)$.

So suppose $f\colon (M_0(k), \overline{\tau}(k)) \to (M_0(k), \sigma(k))$ is a PL homeomorphism. We have two cases:

3.1.2. First case. Suppose f is homotopic to the identity. Let H_t, $0 \leq t \leq 1$, $H_0 = f$, $H_1 = \mathrm{Id}$ be a homotopy between f and the identity. Then the map $\overline{H}\colon M_0(k) \times [0, 1] \to M_0(k) \times [0, 1]$, defined by $\overline{H}(x, t) = (H_t(x), t)$ is homotopic to $\mathrm{Id}_{M_0(k)\times[0,1]}$, and because it is already a homeomorphism on $\partial(M_0(k) \times [0, 1])$, we may apply (1.6.1) of [6] to get a homotopy (which is constant on $\partial(M_0(k) \times [0, 1])$) of $\overline{H}$ to a homeomorphism $\widetilde{H}\colon M_0(k) \times [0, 1] \to M_0(k) \times [0, 1]$.

Since $\widetilde{H}_1 = \mathrm{Id}$, $\widetilde{H}_0 = f$, and $\sigma(k)$ and $\overline{\tau}(k)$ are nonconcordant, by pulling back the triangulation $\sigma(k) \times I$ of $M_0(k) \times [0, 1]$ using $\widetilde{H}$, we obtain a concordance between $\overline{\tau}(k)$ and $\sigma(k)$, a contradiction.

3.1.3. Second case. By the Mostow rigidity theorem, every homeomorphism from a compact hyperbolic manifold to itself is homotopic to a diffeomorphism, so that we have $f \sim g$, where $g: (M_0(k), \sigma(k)) \to (M_0(k), \sigma(k))$ is a diffeomorphism. Then the second case follows by applying the first case to $g^{-1}f \sim \mathrm{Id}_{M_0(k)}$. This completes the proof of Theorem (3.1).

Remark. The reason that Theorem (3.1) does not work for dimension 5 is that the triangulation Lemma (1.2) holds only for dimensions 6 and above. This is because the s-cobordism theorem is not true for dimension 5, so that we do not know if triangulations on $M^4 \times [0, 1]$, modulo the boundary, are products, where M^4 is a 4-manifold. Also, in Theorem (3.1) we need dimension less than 7 to ensure that the triangulations we obtain are smoothable.

3.2. We construct now, for every $n \geq 4$, manifolds $M_i(k)$, $i = 0, 1, 2, 3$ and $k = 1, 2, 3, \cdots$ with $\dim M_0(k) = n$, $\dim M_1(k) = n-1$, $\dim M_2(k) = n-3$, $\dim M_3(k) = 3$ satisfying (b), (c), (d), and (e) of the theorem. When $n = 6$ they will also satisfy (a).

Fix a positive prime number m and write $E = \mathbb{Q}(\sqrt{m})$. Denote by $\mathscr{O}_E$ the set of integers of E. Fix $l \in \mathscr{O}_E$ and define, for $k = 1, 2, \cdots$, the quadratic form $Q(k)$ on $\mathbb{R}^{n+1}$ by

$$Q(k)(x_1, \cdots, x_{n+1}) = l^{2(k-1)}x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 - \sqrt{m}x_{n+1}^2.$$

Define now groups

$$G_0 = \{g \in GL(n+1, \mathbb{R}): gH = H\} \quad \text{where } H = \{x \in \mathbb{R}^{n+1}: x_{n+1} > 0\},$$
$$G_1 = \{g \in G_0: ge_1 = e_1\},$$
$$G_2 = \{g \in G_0: ge_i = e_i, i = 1, 2, 3\},$$
$$G_3 = \{g \in G_0: ge_i = e_i, i = 4, 5, \cdots, n\},$$

and

$$H_0(k) = \{g \in G_0: Q(k)(gx) = Q(k)(x)\ \forall x \in \mathbb{R}^{n+1}\},$$
$$H_i(k) = H_0(k) \cap G_i, \qquad i = 1, 2, 3,$$
$$\Gamma_i(k) = H_i(k)_{\mathscr{O}_E}, \qquad i = 0, 1, 2, 3,$$

where the subindex $\mathscr{O}_E$ means that the entries of the matrices are in $\mathscr{O}_E$, and e_i is the vector in $\mathbb{R}^{n+1}$ whose jth coordinate is δ_i^j. Note that for all k, $H_i(k) = H_i(1)$ and $\Gamma_i(k) = \Gamma_i(1)$ for $i = 1, 2$, and write just H_1, H_2 and Γ_1, Γ_2 respectively.

Define also

$$X_0 = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}: Q(1)(x) = -\sqrt{m}, x_{n+1} > 0\},$$
$$X_1 = X_0 \cap \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}: x_1 = 0\},$$
$$X_2 = \{(x_1, \cdots, x_{n+1}) \in X_0: x_1 = x_2 = x_3 = 0\},$$
$$X_3 = \{(x_1, \cdots, x_{n+1}) \in X_0: x_4 = x_5 = \cdots = x_n = 0\},$$

and we consider X_0 with the metric, at a point $x \in X_0$, that is the restriction of $Q(1)$ to the hyperplane tangent to X_0 at x. This Riemannian metric is of constant curvature $-1/\sqrt{m}$. We remark that $X_2 \subset X_1 \subset X_0$, $X_3 \subset X_0$ where all the inclusions are totally geodesic, and also that $X_2 \cap X_3 = e_{n+1}$.

Consider the $n+1$ by $n+1$ diagonal matrices

$$D(k) = \operatorname{diag}\{l^{k-1}, 1, 1, \cdots 1\},$$

and note that $D(k)H_i(k)D(k)^{-1} = H_i(1)$, $i = 0, 1, 2, 3$.

Since $H_i(1)$ acts on X_i and $D(k)\Gamma_i(k)D(k)^{-1} \subset H_i(1)$ for $i = 0, 1, 2, 3$, we define

$$Y_i(k) = X_i/D(k)\Gamma_i(k)D(k)^{-1}, \qquad i = 0, 1, 2, 3.$$

Note that $Y_i(k) = Y_i(1)$ for all k and $i = 1, 2$ so write just Y_1 and Y_2.

3.2.1. Now for an ideal $\mathcal{J}$ of $\mathcal{O}_E$ consider the congruence subgroups

$$\Gamma_i(k)_{\mathcal{J}} = \{g \in \Gamma_i(k): g = \mathrm{Id} \bmod \mathcal{J}\}$$

for $i = 0, 1, 2, 3$. Also write

$$Y_i(k)_{\mathcal{J}} = X_i/D(k)\Gamma_i(k)_{\mathcal{J}}D(k)^{-1}, \qquad i = 0, 1, 2, 3.$$

3.2.2. We have the following facts.

1. For any nontrivial ideal $\mathcal{J}$ of $\mathcal{O}_E$, $\Gamma_i(k)_{\mathcal{J}}$ is a subgroup of finite index of $\Gamma_i(k)$ because $\mathcal{O}_E/\mathcal{J}$ is finite.

2. $\Gamma_i(k)$ is discrete (see the proof of step 1 of Lemma (3.2.3) or [10, p. 239]).

3. $Y_i(k)$ is compact (see [12] or [10, p. 238]).

4. For all but finite ideals $\mathcal{J}$, $GL(n+1, \mathcal{O}_E)_{\mathcal{J}}$ is torsion free (see [3; p. 113]), so that all $\Gamma_i(k)_{\mathcal{J}}$ are also torsion free. Thus all $Y_i(k)_{\mathcal{J}}$ are compact manifolds. Furthermore, for all but finite ideals $\mathcal{J}$, we have that if

$$\pi(k): X_0 \to X_0/D(k)\Gamma_0(k)_{\mathcal{K}}D(k)^{-1} = Y_0(k)_{\mathcal{J}}$$

is the projection, then

$$(*) \qquad \pi(k)X_i = X_i/D(k)\Gamma_i(k)_{\mathcal{J}}D(k)^{-1} = Y_i(k)_{\mathcal{J}}, \qquad i = 0, 1, 2, 3,$$

so that the $Y_i(k))_{\mathcal{J}}$ are (totally geodesic) submanifolds of $Y_0(k)_{\mathcal{J}}$ (see Proposition (2.2) of [11]).

Remark. To be able to apply (2.2) of [11] we need some remarks. Let σ_i, $i = 1, 2, 3$, be the following involutions: $\sigma_1(x_1, x_2, \cdots, x_{n+1}) = (-x_1, x_2, \cdots, x_{n+1})$, $\sigma_2(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) = (-x_1, -x_2, -x_3, x_4, \cdots, x_{n+1})$, and $\sigma_3(x_1, x_2, x_3, x_4, x_5, \cdots, x_n, x_{n+1}) = (x_1, x_2, x_3, -x_4, -x_5, \cdots, -x_n, x_{n+1})$. Note that X_i is the fixed point set of σ_i. We also have that $\sigma_i\Gamma_0(k)_{\mathcal{J}}\sigma_i = \Gamma_0(k)_{\mathcal{J}}$, $i = 1, 2, 3$, and the following two facts hold:

1. $\Gamma_0(k)_{\mathcal{J}}$ acts freely, because it is discrete and torsion free.
2. $\Gamma_i(k)_{\mathcal{J}} = \{g \in \Gamma_0(k)_{\mathcal{J}}: gX_i = X_i\} = \{g \in \Gamma_0(k)_{\mathcal{J}}: \sigma_i g\sigma_i = g\}$, $i = 1, 2, 3$. To see the first equality note that a group of orthogonal matrices with coefficients in $\mathcal{O}_E$ is finite. Thus we can apply (2.2) of [11] to obtain $(*)$.

3.2.3. Lemma. *The widths $r(k)$ of tubular neighborhoods of $(Y_1)_{\mathcal{J}}$ in $Y_0(k)_{\mathcal{J}}$ can be chosen such that $r(k) \to \infty$.*

Proof. We have three steps:

Step 1. We prove that

$$\begin{aligned}(X_0)_{\mathcal{O}_E} &= X_0 \cap \mathcal{O}_E^{n+1}\\ &= \{(x_1, \cdots, x_{n+1}): x_1^2 + \cdots + x_n^2 - \sqrt{m}x_{n+1}^2 = -\sqrt{m}, x_i \in \mathcal{O}_E\}\end{aligned}$$

is closed and discrete.

The proof of this is similar to the proof of the fact that $\Gamma_0(k)$ is discrete (see [2, p. 190]). So, to prove step 1 note first that $\mathcal{O}_E$ is not discrete in $\mathbb{R}$, but the map $\phi: \mathcal{O}_E \to \mathbb{R} \times \mathbb{R}$ defined by $x \mapsto (x, \overline{x})$, where $\overline{x}$ is the conjugate (i.e., $\overline{a + \sqrt{m}b} = a - \sqrt{m}b$), is a bijection of $\mathcal{O}_E$ in $\mathbb{R}^2$ whose image is closed and discrete.

Thus $\overline{\phi}: (X_0)_{\mathcal{O}_E} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a bijection and also has closed and discrete image. Since $x_1^2 + \cdots + x_n^2 - \sqrt{m}x_{n+1}^2 = -\sqrt{m}$ implies $\overline{x}_1^2 + \cdots + \overline{x}_n^2 + \sqrt{m}\overline{x}_{n+1}^2 = \sqrt{m}$, $\mathrm{proj}_2(\overline{\phi}((X_0)_{\mathcal{O}_E}))$ is compact, so that $(X_0)_{\mathcal{O}_E} = \mathrm{proj}_1(\overline{\phi}((X_0)_{\mathcal{O}_E}))$ is closed and discrete.

Step 2. We prove that for all $s \in \mathbb{R}^+$ there is a K such that $\|\gamma e_{n+1}\| > s$ for $k > K$ and $\gamma \in D(k)\Gamma_0(k)D(k)^{-1}\backslash\Gamma_1$, where the bars denote the Euclidean norm in $\mathbb{R}^{n+1}$.

So, take γ as before. Then $\gamma = D(k)\beta D(k)^{-1}$ for some $\beta \in \Gamma_0(k)$. Since γ is not in Γ_1 and also $(*)$ implies that $\Gamma_1 = \{g \in \Gamma_0(k): gX_1 = X_1\} = \{g \in \Gamma_0(k): gX_1 \cap X_1 \neq \varnothing\})$, by noting that $(l^{k-1}a_1)^2 + \cdots + a_n^2 - \sqrt{m}a_{n+1}^2 = -\sqrt{m}$ so that $\gamma e_{n+1} = (l^{k-1}a_1, a_2, \cdots, a_{n+1}) \in (X_0)_{\mathcal{O}_E}$, we obtain

$$\gamma e_{n+1} = D(k)\beta D(k)^{-1} e_{n+1} = D(k)\beta e_{n+1} = (l^{k-1} a_1, a_2, a_3, \cdots, a_{n+1})$$

with $a_i \in \mathscr{O}_E$ and $a_1 \neq 0$. Now $(X_0)_{O_E}$ is closed and discrete by step 1. Consequently $(X_0)_{\mathscr{O}_E} \cap B(0, s)$ is finite, where $B(0, s)$ is the ball in $\mathbb{R}^{n+1}$ with center at the origin and radius s. Thus the set $\mathrm{proj}_1((X_0)_{\mathscr{O}_E} \cap B(0, s))$ is finite, where $\mathrm{proj}_1(x_1, \cdots, x_{n+1}) = x_1$. By taking K large enough we have that, for $k > K$, l^{k-1} does not divide any of the nonzero elements of $\mathrm{proj}_1((X_0)_{\mathscr{O}_E} \cap B(0, s))$ so that $\gamma e_{n+1} = (l^{k-1} a_1, a_2, \cdots, a_{n+1})$ does not belong to $B(0, s)$ which means that $\|\gamma e_{n+1}\| > s$ for $k > K$.

Step 3. We complete the proof. Because of step 2, we have that $d(e_{n+1}, \{\gamma e_{n+1} : \gamma \in D(k)\Gamma_0(k)D(k)^{-1} \backslash \Gamma_1\}) \to \infty$ as $k \to \infty$, where d is the Euclidean distance. Thus it is easy to see that the same happens with the Riemannian metric of X_0 (both metrics induce the same topology), so that the lengths of closed geodesics, not in Y_1, at the point $o = \pi(k)(e_{n+1})$ go to infinity as k goes to infinity. Using the triangular inequality we hence complete the proof of the lemma.

We have found manifolds satisfying condition (e) of the theorem, we now pass to finite coverings to find manifolds satisfying (b), (c), and (d). We need the following result from [11, p. 122].

3.2.4. There are infinitely many ideals $\mathscr{I}$ of $\mathscr{O}_E$ such that the following two conditions hold:

1. $X_i/\Gamma_i(1)_{\mathscr{I}}$ is orientable, $i = 0, 1, 2, 3$.
2. If $\gamma \in \Gamma_0(1)_{\mathscr{I}}$ and $\gamma x \in X_2$, for some $x \in X_3$, then $\gamma = g_2 g_3$, where $g_i \in \Gamma_i(1)_{\mathscr{I}}$, $i = 2, 3$.

Thus we can suppose that the ideal $\mathscr{I}$ which we choose in Lemma (3.2.3) satisfy (3.2.4).

Remark. Statement 2 of (3.2.4) holds if and only if $Y_2(1)_{\mathscr{I}} \cap Y_3(1)_{\mathscr{I}}$ is one point.

Consider $\Gamma_i(k)_{(l^k)\cap\mathscr{I}}$, $i = 0, 1, 2, 3$, where (l^k) is the principal ideal generated by l^k. Since $(l^k) \cap \mathscr{I} \subset \mathscr{I}$, $\Gamma_i(k)_{(l^k)\cap\mathscr{I}} \subset \Gamma_i(k)_{\mathscr{I}}$. Denote by

$$\Sigma_i(k) = D(k)\Gamma_i(k)_{(l^k)\cap\mathscr{I}} D(k)^{-1}, \qquad i = 0, 1, 2, 3,$$

and note that $\Sigma_i(k)$ is a subgroup of $\Gamma_i(1)_{\mathscr{I}}$ for $i = 0, 1, 2, 3$. Moreover, from $\Gamma_i(1)_{(l^{2k})\cap\mathscr{I}} \subset \Sigma_i(k) \subset \Sigma_i(1) \subset \Gamma_i(1)_{\mathscr{I}}$ it follows that $\Sigma_i(k)$ has finite index in $\Gamma_i(1)_{\mathscr{I}}$ and $\Sigma_i(1)$. Write

$$M_i(k) = X_i/\Sigma_i(k), \qquad i = 0, 1, 2, 3.$$

We prove that these manifolds satisfy (b), (c), (d), and (e) of the theorem.

Note that all $M_i(k)$ are compact orientable manifolds because they are finite covers of the $X_i/\Gamma_i(1) = Y_i(1)_{\mathcal{J}}$ (for the orientability we use condition 1 of (3.2.4)). This also implies (b) and the fact that the dimensions are right.

Next we prove (d). Remark that $\Sigma_i(k) = \Sigma_0(k) \cap G_i = \Sigma_0(k) \cap \Gamma_i(k)_{\mathcal{J}}$, $i = 1, 2, 3$. This fact together with $(*)$ yields that if $\overline{\pi}(k): X_0 \to M_0(k)$ is the projection then $\overline{\pi}(k)(X_i) = M_i(k)$, $i = 1, 2, 3$. If $p(k)$ denotes the projection $p(k)$: $M_0(k) \to M_0(1)$, then (d) follows from $\overline{\pi}(1) = p(k)\overline{\pi}(k)$.

Since $(M_0(k), M_1(k))$ covers $(Y_0(k)_{\mathcal{J}}, (Y_1)_{\mathcal{J}})$, Lemma (3.2.3) implies that (e) holds.

Finally we prove (c). Let $\gamma \in \Sigma_0(k)$ be such that $\gamma x \in X_2$ for some $x \in X_3$. Then by (3.2.4), we have $\gamma = g_2 g_3$, $g_i \in \Gamma_i(1)_{\mathcal{J}}$, $i = 2, 3$.

Because $\gamma \in \Sigma_0(k) = D(k)\Gamma_0(k)_{(l^k)\cap\mathcal{J}}D(k)^{-1}$ there is a $\beta \in \Gamma_0(k)_{(l^k)\cap\mathcal{J}}$ such that $\gamma = D(k)\beta D(k)^{-1}$. Thus

$$\beta = [D(k)^{-1} g_2 D(k)][D(k)^{-1} g_3 D(k)].$$

But

$$D(k)^{-1} g_2 D(k) = g_2$$

so that

$$\beta = g_2[D(k)^{-1} g_3 D(k)],$$

which implies that $D(k)^{-1} g_3 D(k)$ has entries in $\mathcal{O}_E$, because β and g_2 have. Since $\beta = \mathrm{Id} \bmod (l^k)$, by noting that $\beta e_i = g_2 e_i$, $i = 4, \cdots, n$, and $\beta e_i = e_i \mod (l^k)$ and also that g_2 has determinant one (due to condition 1 of (3.2.4)), we have that $g_2 = \mathrm{Id} \bmod (l^k)$, and also $D(k)^{-1} g_3 D(k) = \mathrm{Id} \bmod (l^k)$. This means that $g_2 \in \Gamma_2(k)_{(l^k)}$ and $D(k)^{-1} g_3 D(k) \in \Gamma_3(k)_{(l^k)}$. Consequently

$$g_i \in D(k)\Gamma_i(k)_{(l^k)\cap\mathcal{J}}D(k)^{-1} = \Sigma_i(k) \subset \Sigma_0(k), \qquad i = 2, 3.$$

If $\overline{\pi}(k)$ denote the projection $X_0 \to M_0(k)$, then

$$\overline{\pi}(k)(x) = \overline{\pi}(k)(\gamma x) = \overline{\pi}(k)(g_2 g_3 x) = \overline{\pi}(k)(g_3 x).$$

But $g_3 x \in X_3$ for $x \in X_3$, and $g_3 x \in X_2$ since $g_2(g_3 x) = \gamma x \in X_2$, so that $g_3 x = X_2 \cap X_3 = e_{n+1}$, which implies that $\overline{\pi}(k)(x) = \overline{\pi}(k)(e_{n+1}) = o$.

4. Nonpositive curvature case in higher dimensions

Denote by T^m the m-torus $S^1 \times \cdots \times S^1$ with the canonical differentiable structure and the induced PL structure τ_{T^m}. We prove here that if we take one of the examples of §3 and multiply it by T^m, we still have exotic nonpositively curved triangulations. To see this we note that if (M, τ_0) and (M, τ_1) are two nonpositively curved triangulations on M, then $(M \times T^m, \tau_0 \times \tau_{T^m})$ and $(M \times T^m, \tau_1 \times \tau_{T^m})$ are also nonpositively curved. Moreover, if (M, τ_0) and (M, τ_1) are nonconcordant, then $(M \times T^m, \tau_0 \times \tau_{T^m})$ and $(M \times T^m, \tau_1 \times \tau_{T^m})$ are also so, for the Kunneth formula tells us that $\mathbb{Z}_2$-cohomology classes do not vanish when we take products. So, it remains to prove that these triangulations are not equivalent. To see this it is enough to prove the following (see (3.1.2) and (3.1.3)).

4.1. Proposition. *Let $f: M \times T^m \to M \times T^m$ be a homeomorphism, where M is a compact orientable hyperbolic manifold. Then $f \sim g$, where g is a diffeomorphism.*

Proof. Because $\pi_1(M)$ has trivial center (see [9]), we have that if $\phi: \pi_1(M \times T^m) \to \pi_1(M \times T^m)$ is an isomorphism, then there are isomorphisms $\phi_1: \pi_1(M) \to \pi_1(M)$, $\phi_2: \pi_1(T^m) \to \pi_1(T^m)$ and a homomorphism $\psi: \pi_1(M) \to \pi_1(T^m) \cong \mathbb{Z}^m$ such that

$$\phi = \phi_1 \oplus \phi_2 + 0 \oplus \psi.$$

4.1.1. Lemma. *Let M be a compact oriented differentiable manifold, and $\lambda: \pi_1(M) \to \mathbb{Z}^m$ a homomorphism. Then there is a diffeomorphism $h: M \times T^m \to M \times T^m$ such that for $h_*: \pi_1(M \times T^m) \to \pi_1(M \times T^m) \cong \pi_1(M) \oplus \mathbb{Z}^m$, we have*

$$h_* = \mathrm{Id}_{\pi_1(M \times T^m)} + 0 \oplus \lambda.$$

Proof. Since $H_1(M, \mathbb{Z})$ is the abelianization of $\pi_1(M)$, we have that λ factors through it:

$$\pi_1(M) \xrightarrow{\text{abelianization}} H_1(M, \mathbb{Z}) \xrightarrow{\bar{\lambda}} \mathbb{Z}^m,$$

i.e., the composite of these two maps is λ. Let ρ_i, $i = 1, \cdots, s$, be a basis for the free abelian group $H^1(M, \mathbb{Z}) \cong \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$. Then there are elements $a_i = (n_{i1}, \cdots, n_{im}) \in \mathbb{Z}^m$ such that $\bar{\lambda} = \sum a_i \rho_i$.

M is compact and oriented, so by Poincaré duality, there are $N_i \in H_{n-1}(M, \mathbb{Z})$ dual to ρ_i. We can represent N_i by an embedded $(n-1)$-dimensional closed submanifold (we denote it also by N_i). These N_i

have tubular neighborhoods $U_i \cong [0, 1] \times N_i$, and we make no distinction between U_i and their images. Define $g_i\colon U_i \times T^m \to U_i \times T^m$ by

$$g_i(t, x, \theta_1, \cdots, \theta_m) = (t, x, \theta_1 + 2\pi n_{1i}\delta(t), \cdots, \theta_m + 2\pi n_{mi}\delta(t)),$$

where δ is smooth such that $\delta' \geq 0$, $\delta(0) = 0$, $\delta(1) = 1$, and it is constant near $0, 1$. Define also $h_i\colon M \times T^m \to M \times T^m$ by

$$h_i(x) = \begin{cases} g_i(x), & x \in U_i \times T^m, \\ x, & x \in (M \times T^m)\backslash(U_i \times T^m). \end{cases}$$

These are well-defined diffeomorphisms, because the two definitions agree on a neighborhood of ∂U_i. Finally put $h = h_1 \cdots h_s$, which completes the proof of the lemma since $h_* = \mathrm{Id}_{\pi_1(M \times T^m)} + 0 \oplus \lambda$.

We complete now the proof of Proposition (4.1). Let $f\colon M \times T^m \to M \times T^m$ be a homeomorphism. Let ϕ_1, ϕ_2, and ψ be such that $f_* = \phi_1 \oplus \phi_2 + 0 \oplus \psi$. Let h be as in Lemma (4.1.2) where we take $\lambda = \psi\phi_1^{-1}$. Then $(h^{-1}f)_* = \phi_1 \oplus \phi_2$. By Mostow's rigidity theorem there are diffeomorphisms r_1 and r_2 inducing ϕ_1 and φ_2 respectively. Thus $(h^{-1}f)_* = (r_1 \times r_2)_*$ and by (1.6) of [6] $h^{-1}f \sim r_1 \times r_2$ or $f \sim h(r_1 \times r_2)$, which is a diffeomorphism.

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