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THE SPECTRUM OF DEGENERATING HYPERBOLIC 3-MANIFOLDS

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1. Introduction

According to the cusp closing theorem of Thurston [14, §5.8], a complete, three-dimensional, noncompact manifold M, of constant negative curvature -1 and finite volume, is a limit of a sequence of compact hyperbolic manifolds $M_i \to M$. The Laplacian on the limit manifold M has continuous spectrum filling the interval $[1, \infty)$ with multiplicity equal to the number of cusps. In this paper we investigate the rate of clustering of the eigenvalues of the Laplacian on M_i as i tends to infinity. The analogous question for surfaces has been studied by Wolpert [19], Hejhal [9], and Ji [10], and a sharp estimate of the accumulation rate was obtained by Ji and Zworski [11]. In addition, Colbois and Courtois [4], [5] proved that the eigenvalues below the bottom of continuous spectrum are limits of eigenvalues of compact approximating manifolds for both Riemann surfaces and hyperbolic three-manifolds. Problems of this kind do not arise in dimensions greater than or equal to four (cf. [7]), since the number of complete hyperbolic manifolds of volume less than or equal to a given constant is finite in this case.

Suppose M has only one cusp. Then, for large i, M_i will contain a metric tubular neighborhood of a short, simple, closed geodesic γ_i of length $l_i \to 0$ and of radius $R_i \to \infty$. Let Δ_i be the Laplacian on M_i, $\operatorname{Spec}(\Delta_i)$ its spectrum and $N_i(x) = \#\{\lambda \in \operatorname{Spec}(\Delta_i) | 1 \le \lambda \le 1 + x^2\}$. Our result is that

$$(1.1) \qquad N_i(x) = \frac{x}{\pi} R_i + O_x(1),$$

or equivalently (cf. (2.4))

$$(1.2) \qquad N_i(x) = \frac{x}{2\pi} \log\left(\frac{1}{l_i}\right) + O_x(1).$$

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If M has $q \geq 1$ cusps, then M_i will have q shrinking geodesics γ_i^1, γ_i^2, $\cdots$, γ_i^q surrounded by disjoint tubes of radii R_i^1, R_i^2, $\cdots$, R_i^q respectively and the equalities above hold with $R_i = \sum_{j=1}^{j=q} R_i^j$ and with $\log(1/l_i)$ replaced by $\sum_{j=1}^{j=q} \log(1/l_i^j)$.

The paper is organized as follows. In §2 we review certain aspects of the convergence $M_i \to M$ and the geometry of "thick and thin" decompositoin of hyperbolic three-manifolds. A reduction of the proof to the study of the Strum-Liouville problem

$$-u'' - \sinh^{-2}(r)u = \nu u,$$
$$u'(1) = u'(R) = 0,$$

as R tends to infinity is carried out in §3. After this reduction the proof of (1.1) is completed by invoking a theorem about ordinary differential equations proved in §4. The result of §4 is of independent interest. In particular, it can be used to rederive the theorem of Ji and Zworski [11], the analog of (1.2) for Riemann surfaces without appealing to scattering theory.

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2. Geometric preliminaries

From now on the term *hyperbolic manifold* will be used to refer to a complete oriented Riemannian manifold of three dimensions, finite volume and constant sectional curvature -1. A very readable survey of the geometry of such manifolds is contained in [7]. For a very thorough discussion of this topic see [14, Chaps. 4, 5, 6].

We will use the following notation. For a Riemannian manifold M and an interval I, $M_I = \{p \in M | \iota(p) \in I\}$, where $\iota(p)$ denotes the injectivity radius at $p \in M$. It is a consequence of Kazhdan-Margulis theorem [12], [14] that there exists a positive number μ such that for every hyperbolic manifold M, the set $M_{(\mu,\infty)}$ is nonempty and connected. $M_{(0,\mu]}$ consists of finitely many connected components. If a component C is not compact, it is isometric to the product $\mathbb{R}^+ \times F$ equipped with the metric

$$ds^2 = d\rho^2 + e^{-2\rho} ds_0^2, \tag{2.1}$$

where ds_0^2 is a flat metric on the two-dimensional torus F.

Compact components of $M_{(0,\mu]}$ are called *tubes*. They are metric tubular neighborhoods of simple closed geodesics in M of length smaller than or equal to 2μ. Let γ be such a geodesic and let $\tilde{\gamma}$ be one of its lifts to the universal covering $\widetilde{M} \cong \mathbb{H}^3$. The corresponding tube $T = T_\gamma$ in M is obtained as the quotient of a tubular neighborhood $\widetilde{T}$ of $\tilde{\gamma}$ by the cyclic group generated by $A = A_\gamma$, the deck transformation corresponding to γ. We use the Fermi coordinates (r, t, θ) in $\mathbb{H}^3$ based on $\tilde{\gamma}$. r denotes the distance from $\tilde{\gamma}$, t is the arclength along $\tilde{\gamma}$ and θ is the angular coordinate in the circle of unit vectors perpendicular to $\tilde{\gamma}$ at a point. To make a consistent choice of θ we choose and fix a parallel field of unit vectors perpendicular to $\tilde{\gamma}$. In terms of these coordinates the metric of $\mathbb{H}^3$ is expressed as

$$ds^2 = dr^2 + \cosh^2 r\,dt^2 + \sinh^2 r\,d\theta^2, \tag{2.2}$$

and the deck transformation A is given by $A(r, t, \theta) = (r, t+l(\gamma), \theta+\alpha)$ for some angle α. $\widetilde{T} = \{\tilde{p} \in \widetilde{M} | d(\tilde{p}, \tilde{\gamma}) \leq R\}$, and $T = \widetilde{T}/\langle A_\gamma \rangle$ is determined up to isometry by R, α, and $l = l(\gamma)$.

We need to establish relations between various quantities introduced above. Note first that

$$\mathrm{vol}(T_\gamma) = \int_0^l \int_0^R \int_0^{2\pi} \sinh\cosh r\,d\theta\,dr\,dt = \pi l \sinh^2 R.$$

Moreover, every tube T_γ contains an embedded ball of radius equal to $\frac{1}{2}\mu$ since the injectivity radius at a boundary point of T_γ is *equal* to μ. Thus there exist universal constants $c_1, c_2 > 0$ such that

$$c_1 \leq l \sinh^2 R \leq c_2 \mathrm{vol}(M). \tag{2.3}$$

In particular, if the volume is bounded and l approaches zero, then R tends to infinity, $R \sim \frac{1}{2}\log(1/l)$.

Recall that we are interested in a noncompact hyperbolic manifold M and a sequence of compact hyperbolic manifolds $(M_i)_{i=1}^\infty$ converging to M [7, §3]. It is known [14] that $\mathrm{vol}(M_i) \leq \mathrm{vol}(M)$.

If M has q cusps, we can find a positive number $\varepsilon \leq \mu$ and a sequence $\varepsilon_i \searrow 0$ such that each M_i contains exactly q geodesics $\gamma_i^1, \gamma_i^2, \cdots, \gamma_i^q$ of length less than or equal to ε_i and that for every i the injectivity radius at every point of $M_i \setminus \bigcup_{j=1}^q T_{\gamma_i^j}$ is greater than or equal to ε. Let $l(\gamma_i^j) = l_i^j$ and let R_i^j be the radius of $T_{\gamma_i^j}$. It follows from (2.3) that

$$R_i^j = \frac{1}{2}\log(1/l_i^j) + O(\log \mathrm{vol}(M)). \tag{2.4}$$

Thus M_i contains exactly q tubes of radii tending to infinity when $i \to \infty$. These tubes become cusps in the limit. The boundary of a tube with

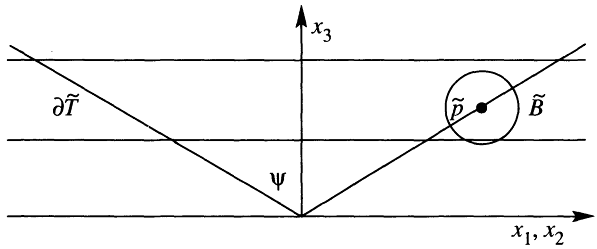

FIGURE 1

the induced metric is a flat torus. The tori $F_i^j = \partial T_{\gamma_i^j}$ are nondegenerate in the following sense.

Lemma 2.5. *There exists a universal constatn $\kappa > 0$ such that F_i^j with its induced metric has injectivity radius at every point greater than or equal to κ. Moreover $\mathrm{vol}(F_i^j) \leq c\,\mathrm{vol}(M)$ for a constant c independent of i. Therefore the tori F_i^j form a relatively compact family.*

Proof. Let $\gamma = \gamma_i^j$, $l = l_i^j$, $R = R_i^j$, $T = T_\gamma$. Then

$$\begin{aligned}\mathrm{vol}(F_i^j) &= \int_0^l \int_0^{2\pi} \sinh R \cosh R \, d\theta \, dt \\ &= 2\pi l \sinh R \cosh R \leq c\,\mathrm{vol}(T_\gamma) \leq c\,\mathrm{vol}(M),\end{aligned}$$

since for R bounded away from zero $\cosh R \leq c \sinh R$. This proves the second assertion. To prove the first one we use the upper half-space model of $\mathbb{H}^3$ taking x_3-axis for $\tilde{\gamma}$. The tube $\tilde{T}$ becomes a cone whose axis of rotation is the x_3-axis. Take $p \in \partial T = F$ and the embedded ball $B(p, \mu) \subset M_i$. Applying an appropriate power of $A_{\tilde{\gamma}}$ we can choose a lift $\tilde{p} = (x_1, x_2, x_3)$ of p with $1 \leq x_3 \leq 2$. Let $\tilde{B}$ be the lift of $B(p, \mu)$ centered at $\tilde{p}$. In our situation R tends to infinity, i.e., the angle ψ between the axis and the generators of the cone approaches $\pi/2$ (cf. Figure 1).

Recall that the induced metric on F is flat and observe that the set $B(p, \mu) \cap F$ is diffeomorphic to a disk. Thus, the injectivity radius of F at p is greater than or equal to the distance (with respect to the induced metric on F) from p to $\partial B(p, \mu) \cap F$. It therefore suffices to bound this distance from below independently of i. To do this we replace $B(p, \mu) \cap F$ with $\tilde{B} \cap \partial\tilde{T}$ and observe that, since the angle ψ is bounded away from zero, this set is quasi-isometric to its projection to the plane $x_3 = 1$ under $(x_1, x_2, x_3) \mapsto (x_1, x_2, 1)$. It is now clear that $d(\tilde{p}, \partial\tilde{B} \cap \partial\tilde{T})$ is bounded below independently of i, which proves the first assertion of the lemma.

Finally, the compactness of the family $\{F_i^j\}$ is a consequence of Mahler's criterion [13, Corollary 10.9]. q.e.d.

We conclude this section by introducing global coordinates on the tube T_γ and computing the Laplacian in terms of these coordinates. Let (r, t, θ) be the Fermi coordinates introduced above. Consider the mapping $f\colon (r, t, \phi) \mapsto (r, t, \theta)$, where $\theta = \phi + \alpha t/l$, $l = l(\gamma)$, and α is the twist angle of A_γ. Recall that $A(r, t, \theta) = (r, t + l(\gamma), \theta + \alpha)$. Thus, in terms of (r, t, ϕ), A_γ is given by $(r, t, \phi) \mapsto (r, t + l, \phi)$. It follows that $0 \le r \le R$, $t \in \mathbb{R}/l\mathbb{Z}$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ are well-defined functions on $T_\gamma = \widetilde{T}/\langle A_\gamma \rangle$. The pullback of the metric (2.2) is given by

$$ds^2 = dr^2 + \cosh^2 r\, dt^2 + \sinh^2 r\, d(\phi + \alpha t/l)^2. \tag{2.6}$$

We remark that near the boundary $F = \partial T_\gamma$ this metric is a very good approximation of the metric (2.1) of a cusp. Indeed, if $\rho = R - r$, then $4\sinh^2 r \sim 4\cosh^2 r \sim e^{2R} e^{-2\rho}$ since R tends to infinity so that

$$ds^2 \sim d\rho^2 + e^{-2\rho}\, ds_0^2, \tag{2.7}$$

where ds_0^2 is the metric of F.

For future reference note that

$$dV = \sinh r \cosh r\, dr\, d\phi\, dt \tag{2.8}$$

is the formula for the volume element on the tube. A straightforward calculation yields the expression for the Laplacian $\Delta = \Delta_i$.

$$\begin{aligned}\Delta u = &-\frac{1}{\sinh r \cosh r}\frac{\partial}{\partial r}\left(\sinh r \cosh r \frac{\partial u}{\partial r}\right)\\ &-\cosh^{-2} r \frac{\partial^2 u}{\partial t^2} - \left(\sinh^{-2} r - \frac{\alpha^2}{l^2}\cosh^{-2} r\right)\frac{\partial^2 u}{\partial \phi^2} + 2\frac{\alpha}{l}\cosh^{-2} r \frac{\partial^2 u}{\partial t \partial \phi}.\end{aligned}$$

We denote the first term by Lu and the sum of remaining terms by $\Delta_r u$. Thus

$$\Delta u = Lu + \Delta_r u \tag{2.9}$$

and

$$Lu = -\frac{1}{\sinh r \cosh r}\frac{\partial}{\partial r}\left(\sinh r \cosh r \frac{\partial u}{\partial r}\right) = -u'' - 2\coth(2r) u', \tag{2.10}$$

where the prime denotes the derivative with respect to r. The operator Δ_r is the Laplacian on the distance torus $F_r = \{p \in T \mid d(p, \gamma) = r\}$ with respect to the induced metric.

3. Reduction to a Sturm-Liouville problem

To simplify the notation we will assume that M has exactly one cusp. The modifications required in case of several cusps will be obvious. Recall that $N_i(x) = \#\{\lambda \in \mathrm{Spec}(\Delta_i) | 1 \leq \lambda \leq 1 + x^2\}$. We begin by proving the lower bound for $N_i(x)$.

Lemma 3.1. *For every* $x > 0$

$$N_i(x) \geq xR_i/\pi + O_x(1)$$

during the degeneration $M_i \to M$.

Proof. Let $\gamma = \gamma_i$ be the shrinking geodesic in M_i; and let $\mathscr{H} \subset C^\infty(M_i)$ be the space of functions vanishing on the complement of the tube T_γ and on $\{p \in M_i | d(p, \gamma) \leq 1\}$, and depending only on the distance r from γ in the tube. Consider the Sturm-Liouville problem

$$Lu = \nu u, \qquad u(1) = u(R) = 0 \tag{3.2}$$

on the interval $[1, R]$. Its eigenfunctions can be identified with elements of $\mathscr{H}$. Let $(\nu_k)_{k=1}^\infty$ be the sequence of eigenvalues of (3.2) and let $(\lambda_k)_{k=1}^\infty$ be the spectrum of the Laplacian on M_i, both ordered in an increasing way and with eigenvalues repeated according to their multiplicities. It is a consequence of the variational characterization of eigenvalues [3] that, for every k, $\nu_k \geq \lambda_k$. Upper eigenvalue bounds translate into lower bounds for the counting function $N_i(x)$. We remark here that the number of eigenvalues of M_i in the interval $[0, 1]$ is bounded by $c\,\mathrm{vol}(M_i) \leq c\,\mathrm{vol}(M)$ for a universal constant $c > 0$ [1], [2]. Therefore, if we allow all eigenvalues below the level $1+x^2$ in the definition of $N_i(x)$, the asymptotic behavior will change by a bounded amount which can be absorbed in $O_x(1)$ term. In addition, $\nu_1 > 1$ [6, Lemma 1]. Therefore

$$N_i(x) \geq D_i(x) - O(1),$$

where $D_i(x) = \#\{k | 1 \leq \nu_k \leq 1 + x^2\}$. To estimate $D_i(x)$ we transform (3.2) by the substitution

$$u = \sinh^{-1/2}(2r) f. \tag{3.3}$$

The eigenvalue problem (3.2) becomes

$$-f'' - \sinh^{-2}(2r) f = (\nu - 1) f, \qquad f(1) = f(R) = 0. \tag{3.4}$$

One easily sees that

$$\nu_k - 1 \leq \pi^2 k^2/(R-1)^2$$

by comparing $\nu_k - 1$ with eigenvalues of

$$-f'' = \kappa f, \qquad f(1) = f(R) = 0.$$

This proves that $N_i(x) \geq D_i(x) - O(1) \geq x(R_i - 1)/\pi - O(1) = xR/\pi - O_x(1)$. q.e.d.

The upper bound $N_i(x) \leq xR/\pi + O_x(1)$ will be reduced to a comparison with (3.4) as well. However, the reduction is more subtle. We first show that removing the complement of T_γ has no effect on clustering of eigenvalues. More precisely, we first show that the counting function for Neumann eigenvalues of the tube differs from $N_i(x)$ by a bounded amount when $i \to \infty$.

Let $T_1 = \{p \in M_i | d(p, \gamma) \leq R - 1\}$ and $M' = M_i \backslash T_1$. For an arbitrary Riemannian manifold Z with (a possibly empty) boundary we denote by $K(Z, x)$ the number of eigenvalues below level $1 + x^2$ of the Laplacian on Z for Neumann boundary conditions. If $Z = Z_1 \cup Z_2$ is the union of two manifolds with disjoint interiors, then

$$K(Z, x) \leq K(Z_1, x) + K(Z_2, x) \tag{3.5}$$

by the Neumann comparison of eigenvalues [3]. In particular, we have

$$N_i(x) \leq K(T_1, x) + K(M', x).$$

The following lemma reduces our problems of bounding $N_i(x)$ to analysis on the tube.

Lemma 3.6. *The counting function $K(M', x)$ satisfies $K(M', x) \leq c(x)$ with $c(x)$ independent of i.*

Proof. Recall that the injectivity radius satisfies $\iota(p) \geq \varepsilon$ for every point $p \in M_i \backslash T_\gamma$. Take a maximal set $\mathscr{P}$ of points whose pairwise distances are greater than or equal to ε. The collection of ε-balls centered at points of $\mathscr{P}$ covers $M_i \backslash T_\gamma$, and balls of radius $\varepsilon/2$ with centers at $p \in \mathscr{P}$ are disjoint. We take the covering $\mathscr{U}$ of M' consisting of all these balls and $S = T_\gamma \backslash T_1$ (see Figure 2, next page). The number of points in $\mathscr{P}$ can be estimated easily from above in terms of $\mathrm{vol}(M_i) \leq \mathrm{vol}(M)$ and ε. Similarly, there exists a positive constant m depending only on ε such that, for every $p \in M'$, the number of sets $U \in \mathscr{U}$ containing p is at most m. It follows from Lemma 2.5 and (2.7) that, for a given level z the number of eigenvalues of S below z can be bounded from above *independently* of i. The same is true trivially for every ε-ball in $\mathscr{U}$. Let $(\kappa_j)_{j=1}^\infty$ be the sequence of all Neumann eigenvalues of all sets of $\mathscr{U}$, and let $(\nu_j)_{j=1}^\infty$ be the eigenvalue sequence of M' for Neumann

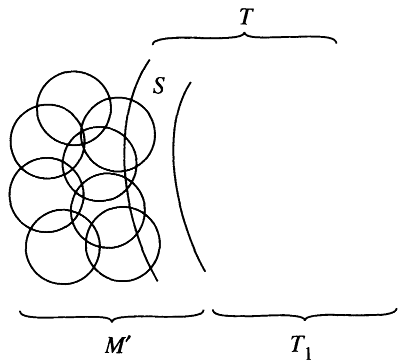

FIGURE 2

boundary conditions. We will show that

$$\nu_j \geq \kappa_j/m. \tag{3.7}$$

This implies the lemma, since, as remarked above, there are finitely many sets in $\mathscr{U}$, each has finitely many eigenvalues below the level $m(1+x^2)$, and all the bounds are uniform in i. It remains to prove (3.7). Consider the Hilbert spaces $H^1(M')$ and $\mathscr{H} = \bigoplus H^1(U)$, where $H^1(Z)$ denotes the Sobolev space of L^2 functions on Z with first derivatives in L^2, and the orthogonal direct sum extends over all sets of $\mathscr{U}$. We have the restriction map $F\colon f \mapsto (f|U)_{U\in\mathscr{U}}$. The eigenvalues ν_j have a min-max characterization in terms of Rayleigh-Ritz quotient

$$\mathscr{R}(f) = \frac{\int_{M'} |\mathrm{grad}(f)|^2}{\int_{M'} f^2}.$$

Similarly, κ_j are critical values of

$$\mathscr{R}_1(g) = \frac{\sum_{U\in\mathscr{U}} \int_U |\mathrm{grad}(g_U)|^2}{\sum_{U\in\mathscr{U}} \int_U |g_U|^2},$$

where $g = (g_U)$ is a typical element of $\mathscr{H}$. One verifies easily that

$$(1/m)\mathscr{R}_1(F(f)) \leq \mathscr{R}(f),$$

which proves (3.7) in view of min-max principle. q.e.d.

We now use a variant of separation of variables to estimate $K(T_1, x)$. Every function f on T_1 can be decomposed as $f = \overline{f} + \overline{\overline{f}}$ so that $\overline{f}$ depends only on r and, for every r, $\overline{\overline{f}}$ is orthogonal to constants on $F_r = \{p \in T_1 | d(p, \gamma) = r\}$. The value of $\overline{f}$ at a point is equal to the average of f over the torus F_r passing through that point, and $\overline{\overline{f}} = f - \overline{f}$. This decomposition is orthogonal with respect to the L^2 inner product, and it is preserved by the Laplacian. To prove the second assertion, it suffices to show that $\overline{\Delta f} = \Delta \overline{f}$. Clearly both $\overline{\Delta_r f}$ and $\Delta_r \overline{f}$ are equal to zero. Thus it is enough (cf. (2.9)) to prove that $\overline{Lf} = L\overline{f}$. This follows by differentiation under the integral sign since, by (2.8),

$$\overline{f} = \frac{1}{2\pi l} \int_{F_r} f \, d\phi \, dt.$$

We proceed to show that only eigenvalues belonging to eigenfunctions depending solely on r contribute to clustering. The following notation will be useful. Let $T_a = \{p \in T_\gamma | d(p, \gamma) \leq R - a\}$ for $a \in [0, R]$. Thus $\partial T_a = F_{R-a}$. Consider the space $H_0^1(T_a) \subset H^1(T_a)$ of functions satisfying $\overline{f} \equiv 0$. For such functions

$$\begin{aligned} \int_{T_a} |\mathrm{grad}(f)|^2 \, dV_\rho \, d\rho &\geq \int_0^{R-a} d\rho \int_{F_\rho} |\mathrm{grad}_\rho f|_\rho^2 \, dV_\rho \\ &\geq \int_0^{R-a} \nu(\rho) \, d\rho \int_{F_\rho} f^2 \, dV_\rho, \end{aligned} \tag{3.8}$$

where $\nu(\rho)$ denotes the smallest positive eigenvalue of Δ_ρ, and the subscript ρ is used to indicate that the corresponding object is computed with respect to the induced metric on F_ρ. We are going to need information about $\nu(\rho)$.

Lemma 3.9. *$\nu(r)$ is a decreasing function of r. Moreover for $r \in (0, R]$ and sufficiently large i, the eigenvalues $\nu(r)$ satisfy $\nu(r) \geq ce^{2(R-r)}$ with a positive constant c independent of i and r.*

Proof. Using (2.6) we compute

$$|\mathrm{grad}_r(g)|_r^2 = \sinh^{-2} r \left(\frac{\partial g}{\partial \phi}\right)^2 + \cosh^{-2} r \left(\frac{\partial g}{\partial t} - \frac{\alpha}{l}\frac{\partial g}{\partial \phi}\right)^2 \tag{3.10}$$

and the Rayleigh-Ritz quotient of g

$$\mathscr{R}_r(g) = \frac{\int_{F_r} |\mathrm{grad}_r(g)|_r^2 \, d\phi \, dt}{\int_{F_r} g^2 \, d\phi \, dt}. \tag{3.11}$$

The formulas above imply immediately that $\nu(r)$ is a decreasing function of r. The inequality $\nu(r) \geq ce^{2(R-r)}$ for $r = R$ follows from Lemma 2.5. To prove the general case we consider F_r as a fixed manifold with coordinates (t, ϕ) equipped with a family of Riemannian metrics depending on the parameter r. If R is sufficiently large, then $\sinh R/\sinh r \geq \frac{1}{2}e^{R-r}$ and $\cosh R/\cosh r \geq \frac{1}{2}e^{R-r}$. It follows from (3.10) and (3.11) that

$$|\operatorname{grad}_r(g)|_r^2 \geq \tfrac{1}{4}e^{2(R-r)}|\operatorname{grad}_R(g)|_R^2.$$

Therefore

$$\mathscr{R}_r(g) \geq \tfrac{1}{4}e^{2(R-r)}\mathscr{R}_R(g) \geq \tfrac{1}{4}\nu(R)e^{2(R-r)},$$

which completes the proof. q.e.d.

We now proceed as follows. For a fixed $x > 0$ we use the lemma above to choose $a > 0$ such that $\nu(R-a) \geq ce^{2a} > 1 + x^2$. This choice is determined by x alone, and from now on a is fixed. As above,

$$K(T_1, x) \leq K(T_a, x) + K(T_1 \backslash T_a, x).$$

Using Lemma 2.5 and (2.7) we see that the manifolds $T_1 \backslash T_a$, for different i, are mutually quasi-isometric with constants controlling the quasi-isometries independent of i. Thus the second summand above is bounded independently of i. If ϕ_λ is a Neumann eigenfunction of Δ on T_a belonging to an eigenvalue λ, both $\overline{\phi_\lambda}$ and $\overline{\overline{\phi_\lambda}}$ satisfy the equation $\Delta f + \lambda f = 0$. It now follows from Lemma 3.9 and (3.8) that, if $\overline{\overline{\phi_\lambda}} \neq 0$, then $\lambda > 1 + x^2$. In particular, the eigenfunctions whose eigenvalues contribute to $K(T_a, x)$ depend only on r.

The discussion above shows that $N_i(x) \leq K'(x) + O_x(1)$ where $K'(x)$ is the number of eigenvalues below the level $1 + x^2$ for the following Sturm-Liouville problem on $(0, R-a]$.

$$-\frac{1}{\sinh r \cosh r}(f' \sinh r \cosh r') = \mu_n u, \tag{3.12}$$

$$f'(R-a) = 0, \tag{3.13}$$

$$f(r) = 0(1) \text{ as } r \to 0.$$

The eigenvalues of this problem admit a variational characterization. Therefore all standard comparison theorems apply. We break the interval $(0, R-a]$ into the union of $(0, 1]$ and $[1, R-a]$ and use the requirement that the function be square-integrable as the boundary condition at 0 and Neumann condition $f' = 0$ at remaining end points. Neumann comparison of eigenvalues implies then that the counting function $K^N_{1,R-a}(x)$ for the eigenvalue problem on $[1, R-a]$ differs from $K'(x)$ by a bounded amount with bounds depending on x but not on i. We now wish to use the substitution (3.3). There is a slight complication in that the Neumann conditions are not preserved by this substitution. However Dirichlet and

Neumann eigenvalues, say $(\lambda_k)_{k=1}^{\infty}$ and $(\mu_k)_{k=1}^{\infty}$ respectively, for a Sturm-Liouville equation satisfy [16]

$$\mu_k \leq \lambda_k \leq \mu_{k+2}. \tag{3.14}$$

Therefore switching between these boundary conditions changes the counting function by a bounded amount (at most two in absolute value). We pass to Dirichlet boundary conditions on $[1, R-a]$, apply our substitution, than go back to Neumann conditions for the transformed equation. The new Sturm-Liouville problem

$$\begin{gathered} -u'' - \sinh^{-2} ru = \mu_n - 1, \\ u'(1) = u'(R-a) = 0 \end{gathered}$$

has the counting function $Q(x) = \#\{\mu_n | \mu_n \leq 1 + x^2\}$ satisfying $N_i(x) \leq Q(x) + O_x(1)$. According to Theorem 4.1, we have

$$Q(x) = (x/\pi)(R - a - 1) + O_x(1) = (x/\pi)R + O_x(1).$$

This concludes the proof of our main result (1.1) and (1.2).

4. A Sturm-Liouville problem on expanding interval

This section contains a proof of the following theorem.

Theorem 4.1. *Let $q(u)$ be a continuous, real-valued, integrable function on $[0, \infty)$. Consider the eigenvalue problem*

$$-f'' + q(u)f = \lambda f, \tag{4.2}$$

$$f'(0) = f'(R) = 0 \tag{4.3}$$

on $[0, R]$. Let $K_{0,R}^N(x)$ be the number of eigenvalues of this problem satisfying $0 \leq \lambda \leq x^2$. Then $K_{0,R}^N(x) = xR/\pi + O_x(1)$ as $R \to \infty$.

The theorem asserts that an integrable potential q is negligible; the behavior of the counting function is the same as that of the counting function for the problem with $q \equiv 0$. Classically, one studies $K_{0,R}(x)$ for fixed R and $x \to \infty$. We will retrace a classical argument, which goes back to Cauchy (cf. [15, Chapter 1]), to prove the theorem. A final comment before we begin the proof is that the Neumann conditions can be replaced by any selfadjoint boundary conditions without affecting the conclusion. We will use only Neumann and Dirichlet conditions in the proof.

Proof. Let $K_{a,b}^N$, $K_{a,b}^D$ be the counting functions of the eigenvalues of the equation (4.2) on the interval $[a, b]$ for Neumann and Dirichlet boundary conditions respectively. By Dirichlet-Neumann bracketing and (3.14)

$$K^N_{0,a}(x) + K^N_{a,R}(x) - 4 \le K^D_{0,a}(x) + K^D_{a,R}(x) \\ \le K^N_{0,R}(x) \le K^N_{0,a}(x) + K^N_{a,R}(x).$$

It follows that

$$K^N_{0,R}(x) - K^N_{a,R}(x) = O_{a,x}(1) \tag{4.4}$$

for $R \to \infty$. We will choose a later in such a way that it depends only on x. We proceed to estimate $K^N_{a,R}(x)$ for $R \to \infty$.

Let $\phi(u) = \phi(u, \lambda)$, $\chi(u) = \chi(u, \lambda)$ be solutions of (4.2) satisfying $\phi(a) = 1$, $\phi'(a) = 0$, and $\chi(R) = 1$, $\chi'(R) = 0$ respectively. ϕ and χ are holomorphic entire functions of the complex variable λ. The Wronskian $W(\phi, \chi) = \phi\chi' - \chi\phi'$ is independent of u and is an entire holomorphic function of λ denoted by $\omega(\lambda)$. Zeros of $\omega(\lambda)$ are precisely the eigenvalues of the Neumann problem on $[a, R]$. One verifies that, if $\lambda = s^2$, then

$$\phi(u) = \cos s(u-a) + \frac{1}{2}\int_a^u \sin s(u-y) q(y)\phi(y)\,dy$$

and

$$\chi(u) = \cos s(R-u) + \frac{1}{2}\int_u^R \sin s(y-u) q(y)\chi(y)\,dy.$$

Differentiation under the integral yields

$$\phi'(u) = -s\sin s(u-a) + \int_a^u \cos s(u-y) q(y)\phi(y)\,dy,$$

$$\chi'(u) = s\sin s(R-u) - \int_u^R \cos s(y-u) q(y)\chi(y)\,dy.$$

We set $s = \sigma + it$. The proof of [15, Lemma 1.7] gives the estimates

$$|\phi| \le e^{|t|(u-a)} \frac{1}{1 - |s|^{-1}\int_a^\infty |q(y)|\,dy}, \tag{4.5}$$

$$|\chi| \le e^{|t|(R-u)} \frac{1}{1 - |s|^{-1}\int_a^\infty |q(y)|\,dy}, \tag{4.6}$$

both valid provided $|s|^{-1}\int_a^\infty |q(y)|\,dy < 1$.

Let

$$B = \frac{|s|^{-1}\int_a^\infty |q(y)|\,dy}{1 - |s|^{-1}\int_a^\infty |q(y)|\,dy}.$$

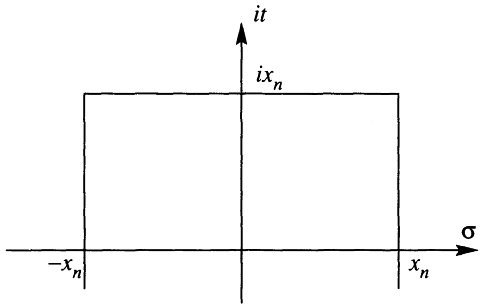

FIGURE 3

We substitute the inequalities (4.5) and (4.6) into the integral equations satisfied by ϕ, χ, and their derivatives to obtain

$$\begin{aligned}
\phi(u) &= \cos s(u-a) + O(Be^{|t|(u-a)}),\\
\chi(u) &= \cos s(R-u) + O(Be^{|t|(R-u)}),\\
\phi'(u) &= -s\sin s(u-a) + O(|s|Be^{|t|(u-a)}),\\
\chi'(u) &= s\sin s(R-u) + O(|s|Be^{|t|(R-u)}),
\end{aligned}$$

which hold if $|s|^{-1}\int_a^\infty |q(y)|\,dy < 1$. Thus we have

$$\omega(\lambda) = \phi\chi' - \phi'\chi = s\sin s(R-a) + O(|s|e^{|t|(R-a)}B). \tag{4.7}$$

We will work with $|s|$ bounded above and below and choose a sufficiently large. The exact choice of a will be specified below.

Let $x_n = (n+\frac{1}{2})\pi/(R-a)$ and let Γ_n be the closed contour in the λ-plane corresponding under $\lambda = s^2$ to the upper half of the boundary of the square in the s-plane determined by the lines $t = x_n$, $t = -x_n$, $\sigma = x_n$, and $\sigma = -x_n$ for a positive integer n (Figure 3). Along this contour

$$|\sin s(R-a)| > ce^{|t|(R-a)} \tag{4.8}$$

for a constant c independent of n. For given x and R choose n so that $x_n \le x < x_{n+1}$. Thus, if $R \to \infty$ with fixed x and a, then $n \to \infty$ and $|x - x_n| = O(1/(R-a))$.

We now specify the choice of a. Since $q(y)$ is integrable, we can choose a sufficiently large so that $|s|^{-1}\int_a^\infty |q(y)|\,dy < 1$ and, because of (4.8), so that the O term in (4.7) is smaller than $|s\sin s(R-a)|$ on Γ_n.

Write $F(\lambda) = s\sin s(R-a) = \sqrt{\lambda}\sin\sqrt{\lambda}(R-a)$ and $G(\lambda) = \omega(\lambda) - F(\lambda)$, the O term in (4.7). Then, by our choice of a, B is small so that

$|G(\lambda)| < |F(\lambda)|$ along Γ_n. By Rouche's theorem $\omega(\lambda)$ and $F(\lambda)$ have equal number of zeros inside the contour Γ_n. The zeros of $F(\lambda)$ are at

$$\pi^2 k^2/(R-a)^2, \qquad k \leq n.$$

Thus

$$K^N_{a,R}(x_n) = n+1 = x_n(R-a)/\pi + 1/2.$$

$K^N_{a,R}(x)$ is a nondecreasing function of x and $x_{n+1} - x_n = \pi/(R-a)$. Therefore

$$K^N_{a,R}(x) = x(R-a)/\pi + O(1) = xR/\pi + O_x(1).$$

This finishes the proof in view of (4.4).

Remarks. (a) Suppose $q(u)$ is an integrable function on the real line. Consider the Sturm-Liouville problem for equation (4.2) on $[-R, R]$ with $R \to \infty$. A slight modification of the proof above shows that the counting functions for Dirichlet and Neumann boundary conditions satisfy

$$K^D_{-R,R}(x) = 2xR/\pi + O_x(1),$$
$$K^N_{-R,R}(x) = 2xR/\pi + O_x(1)$$

respectively. This gives an alternative proof of Theorem 1 of [11].

(b) R. Sacksteder pointed out that it is possible to give a proof of Theorem 4.1 based on ideas of Wintner (cf. [8, §X.8], [17], [18]). Such a method would avoid use of complex analysis.

References

[1] P. Buser, *On Cheeger's inequality* $\lambda_1 \geq h^2/4$, Geometry of the Laplace Operator (R. Osserman & A. Weinstein, eds.), Proc. Sympos. Pure Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1980, 29–77.

[2] P. Buser, B. Colbois, & J. Dodziuk, *Tubes and eigenvalues for negatively curved manifolds*, J. Geom. Anal. **3** (1993) 1–26 .

[3] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, San Diego, 1984.

[4] B. Colbois & G. Courtois, *Les valuers propres inférieures à* 1/4 *des surfaces de Riemann de petite rayon d'injective*, Comment. Math. Helv. **64** (1989) 349–362 .

[5] ——, *Sur les petites valeurs propres des variete hyperboliques de dimension* 3, Prépublication de l'Institut Fourier, Grenoble 130 (1989).

[6] J. Dodziuk & B. Randol, *Lower bounds for* λ_1 *on a finite-volume hyperbolic manifold*, J. Differential Geometry **24** (1986) 133–139 .

[7] M. Gromov, *Hyperbolic manifolds according Thurston and Jørgensen*, Sem. Bourbaki 546 (1979).

[8] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[9] D. Hejhal, *Regular b-groups, degenerating Riemann surfaces and spectral theory*, Mem. Amer. Math. Soc. **437** (1990).

[10] L. Ji, *Spectral degeneration of hyperbolic Riemann surfaces*, preprint.

[11] L. Ji & M. Zworski, *The remainder estimate in spectral accumulation for degenerating hyperbolic surfaces*, preprint.

[12] D. Kazhdan & G. Margulis, *A proof of Selberg's hypothesis*, Math. Sb. **75** 117 (1968) 163–168 .

[13] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, New York, 1972.

[14] W. Thurston, *The geometry and topology of* 3*-manifolds*, Department of Mathematics, Princeton Univ., Princeton, NJ, 1980.

[15] E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations,* 1, Cambridge Univ. Press, London, 1946.

[16] H. Weinberger, *Variational methods for eigenvalue approximation*, SIAM, Philadelphia, 1974.

[17] A. Wintner, *Asymptotic integration constants*, Amer. J. Math. **68** (1946) 553–559 .

[18] ——, *Asymptotic integration of the adiabatic oscillator*, Amer. J. Math. **69** (1947) 251–272 .

[19] S. Wolpert, *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces*, Comm. Math. Phys. **112** (1987) 283–315.

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