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LINEAR HOLONOMY OF MARGULIS SPACE-TIMES

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To Aimee, with love

1. Introduction: Free discrete groups

If $\Gamma \subset \mathrm{Aff}(\mathfrak{R}^3)$ acts properly discontinuously on $\mathfrak{R}^3$, then Γ is either solvable or free up to finite index [3], [6]. If Γ is free and acts properly discontinuously on $\mathfrak{R}^3$, then Γ is conjugate to a subgroup of $\mathbf{H} = \mathbf{O}(2, 1) \ltimes \mathbf{V}$, where $\mathbf{V}$ is the group of parallel translations in $\mathbf{E} = \mathfrak{R}^{2,1}$ [3]. Let $\mathbf{G} = \mathrm{SO}(2, 1)$ and let $\mathbf{G}^o$ denote its identity component.

Complete affinely flat manifolds correspond to $\Gamma \subset \mathrm{Aff}(\mathfrak{R}^3)$ which act properly discontinuously and freely on $\mathbf{E}$. Define *Margulis space-times* as complete affinely flat 3-dimensional manifolds with free fundamental group; their existence was demonstrated by Margulis [4], [5].

Let $L\colon \mathrm{Aff}(\mathfrak{R}^3) \to \mathrm{GL}(n, \mathfrak{R})$ be the usual projection. If Γ acts properly discontinuously on $\mathbf{E}$, then $L(\Gamma)$ is conjugate to a free discrete group of $\mathbf{G}$; it was shown in [2].

Theorem 1. *For every Schottky group $G \subset \mathbf{G}^o$ there exists a free $\Gamma \subset \mathbf{H}$ which acts properly discontinuously on $\mathbf{E}$ and $L(\Gamma) = G$.*

$G \subset \mathbf{G}^o$ is a Schottky group if and only if all nonidentity elements are hyperbolic. The set of all Schottky groups in $\mathbf{G}^o$ is a proper subset of the set of all free discrete subgroups of $\mathbf{G}^o$. In particular, there are free discrete subgroups of $\mathbf{G}^o$, which contain parabolic elements.

We shall prove

Theorem 2. $G = L(\Gamma)$ *for some free finitely generated* $\Gamma \subset \mathrm{Aff}(\mathfrak{R}^3)$ *which acts properly discontinuously on* $\mathbf{E}$ *if and only if* G *is conjugate to a free finitely generated discrete subgroups of* $\mathbf{G}$.

For the affine manifold $\mathbf{M}$, the group of deck transformations Π acts on the universal cover $\widetilde{\mathbf{M}}$ by affine automorphisms. The developing map $D\colon \widetilde{\mathbf{M}} \to \mathbf{E}$ is a homeomorphism for complete $\mathbf{M}$. For every $\tau \in \Pi$ there

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is a unique affine automorphism $\phi(\tau)$ such that $D \circ \tau = \phi(\tau) \circ D$, and $\phi \colon \Pi \to \mathrm{Aff}(\mathfrak{R}^3)$ is called the *affine holonomy representation.*

$L \circ \phi \colon \Pi \to \mathrm{GL}(3, \mathfrak{R})$ is called the *linear holonomy representation.* Margulis conjectured that $L \circ \phi(\Pi)$ for any complete affine flat manifold with free fundamental group contains no parabolic elements. Theorem 2 shows that this conjecture is false. More generally, Theorem 2 is a classification of linear holonomy representations of complete affinely flat manifolds with free fundamental group.

2. Generalized Schottky groups

Classically, Schottky groups lie in $\mathrm{PSL}_2(\mathbf{C})$. We can consider their restriction to $\mathrm{PSL}_2(\mathfrak{R})$. We also allow for orientation reversing matrices and define a *Schottky group* $\mathbf{H} \subset \mathrm{PGL}_2(\mathfrak{R})$ of n generators as a free group which acts properly discontinuously on the hyperbolic plane $\mathbf{H}^2$ such that there exists a fundamental domain for its action which is bounded by $2n$ complete ultraparallel geodesics.

We define a *generalized Schottky group* of n generators in $\mathrm{PGL}_2(\mathfrak{R})$ to be a free group which acts properly discontinuously on $\mathbf{H}^2$ such that there exists a fundamental domain for this action, which is bounded by $2n$ complete nonintersecting geodesics.

The following theorem is well known.

Theorem 3. *$\Lambda \subset \mathrm{PGL}_2(\mathfrak{R})$ is a free finitely generated discrete subgroup if and only if it is a generalized Schottky group.*

Proof. By definition, a generalized Schottky group is a free finitely generated discrete group of $\mathrm{PGL}_2(\mathfrak{R})$.

Conversely, suppose that Λ acts properly discontinuously on $\mathbf{H}^2$. Topologically, $\mathbf{H}^2/\Lambda$ is an n-punctured surface of genus g, where $k = 2g - n - 1$, and k is the rank of Λ. $\mathbf{H}^2/\Lambda$ may be an unoriented surface.

We perform surgery on this surface along k nonintersecting puncture-to-puncture curves so that the resulting surface is simply connected. $n-1$ of these surgeries will be from one puncture to another. There will be $2g$ surgeries from a puncture to itself along curves which are not null homotopic.

We may now construct a fundamental domain $\mathbf{X}$ for the action of Λ on $\mathbf{H}^2$. $\mathbf{X}$ is bounded by the $2k$ nonintersecting curves corresponding to the k surgeries.

These curves may be straightened to complete nonintersecting geodesics. The region $\widetilde{\mathbf{X}}$ in $\mathbf{H}^2$ bounded by these $2k$ intersecting geodesics is also a fundamental domain for the action of Λ on $\mathbf{H}^2$. Thus, Λ is a generalized Schottky group. q.e.d.

Let $\pi: \mathbf{G} \to \mathrm{PGL}_2(\mathfrak{R})$ be the usual isomorphism. Generalized Schottky groups and Schottky groups in $\mathbf{G}$ are defined to be the preimages of generalized Schottky groups and Schottky groups in $\mathrm{PGL}_2(\mathfrak{R})$.

Elements of $\mathbf{G}$ act on $\mathbf{E}$ leaving the inner product

$$B(\mathrm{x}, \mathrm{y}) = \mathrm{x}_1\mathrm{y}_1 + \mathrm{x}_2\mathrm{y}_2 - \mathrm{x}_3\mathrm{y}_3$$

invariant. The associated cross product is

$$\mathrm{x} \boxtimes \mathrm{y} = \begin{bmatrix} \mathrm{x}_2\mathrm{y}_3 - \mathrm{x}_3 - \mathrm{y}_2 \\ \mathrm{x}_3\mathrm{y}_1 - \mathrm{x}_1\mathrm{y}_3 \\ \mathrm{x}_2\mathrm{y}_1 - \mathrm{x}_1\mathrm{y}_1 \end{bmatrix}.$$

Let $\mathbf{C} = \{\mathrm{x} \in \mathbf{E} | B(\mathrm{x}, \mathrm{x}) = 0\}$ be the cone invariant under the action of all $g \in \mathbf{G}$, and let

$$\mathbf{N} = \{\mathrm{x} \in \mathbf{C} | \mathrm{x}_3 > 0\}$$

be its upper nappe which is invariant under $\mathbf{G}^o$. A *conical neighborhood* $\mathbf{A}$ of $\mathbf{C}$ is a subset of $\mathbf{C}$ such that $\mathbf{A} \cap \mathbf{N}$ is c connected, and if $\mathrm{x} \in \mathbf{A}$ then $k\mathrm{x} \in \mathbf{A}$ for all $k \in \mathfrak{R}$. For $\mathrm{u}, \mathrm{v} \in \mathbf{A} \cap \mathbf{N} \cap \mathbf{S}^2$, the ordered pair (u, v) is the *bounding vector pair for* $\mathbf{A}$ if $B(\mathrm{u} \boxtimes \mathrm{v}, \mathrm{x}) \geq 0$ for all $\mathrm{x} \in \mathbf{A} \cap \mathbf{N}$.

Note that if (u, v) is the bounding vector pair for conical neighborhood $\mathbf{A}$, then (v, u) is the bounding vector pair for the conical neighborhood $\mathbf{C} - \mathbf{A}$.

For every generalized Schottky group in $\mathbf{G}^o$ there exist a set of independent generators $\{g_1, g_2, \cdots, g_n\}$ and corresponding $2n$ conical neighborhoods j $\{\mathbf{A}_1^{\pm}, \mathbf{A}_2^{\pm}, \cdots, \mathbf{A}_n^{\pm}\}$ whose interiors are disjoint and

$$g_i(\mathbf{A}_i^-) = \mathrm{cl}(\mathbf{C} - \mathbf{A}_i^+).$$

Here cl denotes closure in the usual topology.

Let $\rho(\mathrm{x}, \mathrm{y})$ denote the Euclidean distance between $\mathrm{x}, \mathrm{y} \in \mathbf{E}$. $\mathbf{S}^2$ is the Euclidean unit sphere centered at the origin, and $\mathbf{B}(\mathrm{x}, \delta)$ is the Euclidean δ-ball centered at x.

Generalized Schottky groups in $\mathbf{G}$ can contain *parabolic elements*, i.e., unipotent elements, and do contain *hyperbolic elements.* Hyperbolic elements in $\mathbf{G}$ are defined to have 3 distinct real eigenvalues $|\lambda_g| < 1 < |\lambda_g^{-1}|$, whose corresponding eigenvectors are x_g^-, x_g^o, and x_g^+. We choose the expanding eigenvector x_g^+ and the contracting eigenvector x_g^- so that both are in $\mathbf{N} \cap \mathbf{S}^2$. The invariant vector x_g^o is chosen so that $B(\mathrm{x}_g^o, \mathrm{x}_g^o) = 1$ and $\{sx_g^o, \mathrm{x}_g^-, \mathrm{x}_g^+\}$ is a right-handed basis for $\mathbf{E}$.

For any $y \in \mathbf{E}$ such that $B(y, y) > 0$, let x_y^- and x_y^+ be the elements of $\mathbf{N} \cap \mathbf{S}^2$ such that $B(y, x_y^\pm) = 0$ and $\{y, x_y^-, x_y^+\}$ is a right-handed basis of $\mathbf{E}$.

g is *ε-hyperbolic* if it is hyperbolic and $\rho(x_g^-, x_g^+) > \varepsilon$.

Lemma 1. *If $G \subset \mathbf{G}$ is a generalized Schottky group with generators $g_1, g_2, \cdots, g_n$, then for $g \in \Gamma$*

$$x_g^+ \in \mathbf{A}_{i_1}^{\operatorname{sign}(j_1)} \quad \text{and} \quad x_g^- \in \mathbf{A}_{i_m}^{-\operatorname{sign}(j_m)},$$

where $g = \Pi_{k=1}^m (g_{i_k}^{j_k})$ such that $i_k \in \{1, 2, \cdots, n\}$, $j_k \in \mathbf{Z} - \{0\}$, and $i_k \neq i_{k+1}$.

Proof. Consider the action of $g' = \pi(g)$ on the boundary of the hyperbolic plane $\partial \mathbf{H}^2$. Write the projectivization of all $x \in \mathbf{C}$ and $\mathbf{A} \subset \mathbf{C}$ as x' and $\mathbf{A}'$, respectively. Note that if $(g^{-1})' = f'$ then $(x_g^-)' = (x_f^+)'$, and if $d' = (g_i^n)'(g)'(g_i^{-n})'$ then $(x_d^+)' = (g_i^n)'(x_g^+)'$. Thus, we only need to consider x_g^+ where g is such that $i_1 \neq i_n$. By an appropriate change of generators we have $j_1 > 0$ and $i_1 = 1$.

We can show by induction that $g'((\mathbf{A}_1^+)') \subset (\mathbf{A}_1^+)'$. Hence Brouwer's fixed point theorem shows that g' has a fixed point in $(\mathbf{A}_1^+)'$, and $x_g^+ \in \mathbf{A}_1^+$. q.e.d.

In particular,

$$(1) \qquad g_1^{-j_i}(g(\mathbf{A}_1^+)) \subset \mathbf{A}_{i_2}^{\operatorname{sign}(j_2)} \quad \text{and} \quad g_2^{-j_2} g_1^{-j_i}(g(\mathbf{A}_1^+)) \subset \mathbf{A}_{i_3}^{\operatorname{sign}(j_3)}.$$

3. Separating wedges

For a conical neighborhood $\mathbf{A} \subset \mathbf{C}$ with bounding vector pair (u, v), let

$$\Theta(\mathbf{A}) = 2\arcsin(\rho(u, v)/2),$$

and $t(\mathbf{A}) = (u - v)$.

The *horizontal plane through* x is defined to be $\mathbf{H}_x = \{y | y_3 = x_3\}$. Denoting the origin by o, we say that y is a *horizontal vector* if $y \in \mathbf{H}_o$, that is $y_3 = 0$. $y, w \in \mathbf{H}_x$ for some x if and only if $y - w$ is a horizontal vector.

Note that $t(\mathbf{A})$ is a horizontal vector. $\Theta(\mathbf{A})$ can be interpreted as the angle between the projection of its bounding vectors u and v onto $\mathbf{H}_o$.

Define the *wedge*

$$\mathbf{W}(\mathbf{A}) = \left\{ w \in \mathbf{E} \,\middle|\, \begin{matrix} B(u \boxtimes v, \frac{1}{w_3} w) \geq 0 & \text{if } B(w, w) \leq 0, \text{ or} \\ B(u \boxtimes v, x_w^+) \geq 0 & \text{if } B(w, w) > 0, \end{matrix} \right\}.$$

See Figure 1.

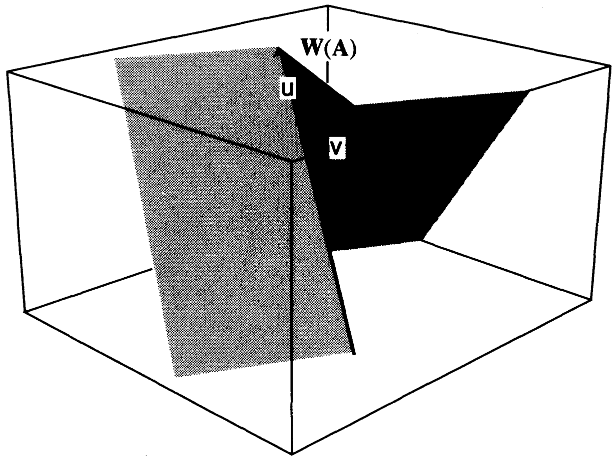

FIGURE 1. THE BOUNDARY OF A WEDGE.

The set of $y \in \mathbf{W}(\mathbf{A})$ such that $B(y, y) \leq 0$ is a closed set bounded by $\mathbf{A}$ itself and $\langle u, v \rangle$. For $y \in \mathbf{E}$ such that $B(y, y) > 0$, denote the half-planes tangent to $\mathbf{C}$ and containing y as

$$\mathbf{P}(y) = \{w \in \mathbf{E} | B(w, w) > 0 \text{ and } x_w^+ = x_y^+\}.$$

$\mathbf{P}(y) \subset \mathbf{W}(\mathbf{A})$ if and only if $x_v^+ \in \mathbf{A}$.

Define the *translated wedge.*

$$\mathbf{T}(\mathbf{A}) = [\mathbf{W}(\mathbf{A}) + t(\mathbf{A})].$$

(See [1] for more discussion.) By examining Figures 2–4 (see next page), we note that $\mathbf{T}(\mathbf{A}) \subset \mathbf{W}(\mathbf{A})$. In fact, $t(\mathbf{A})$ is the unique horizontal vector v such that for all $k \geq 0$, $(\mathbf{W}(\mathbf{A}) + kv) \subset \mathbf{W}(\mathbf{A})$.

Again citing Figures 2–4, we claim that if the interiors of conical neighborhoods $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ are mutually disjoint, then $\mathbf{T}(\mathbf{A}_i) \cap \mathbf{T}(\mathbf{A}_j) = \emptyset$, for $i \neq j$.

Lemma 2. *If G is a generalized Schottky group in $\mathbf{G}$ of rank n, then for all $1 \leq i \leq n$ there exist h_i and $\mathbf{X}_i$ such that $\mathbf{X}_i$ is a fundamental domain for the action of $\langle h_i \rangle$ on $\mathbf{E}$, $L(h_i) = g_i$, $G = \langle g_1, g_2, \cdots g_n \rangle$, and $(\mathbf{E} - \mathbf{X}_i)$ is a submanifold of $\mathbf{X}_j$ for $i \neq j$.*

Proof. Because G is a generalized Schottky group, there are generators g_i and associated conical neighborhoods $\mathbf{A}_i^{\pm} \subset \mathbf{C}$ whose bounding vector pairs we denote $(u_i^{\pm}, v_i^{\pm})$. $\mathbf{A}_i^{\pm}$ are chosen so that their interiors are disjoint and

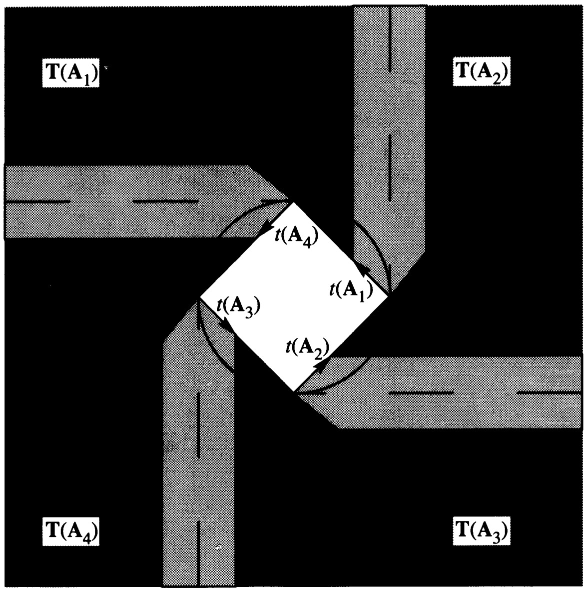

FIGURE 2. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES $(x_3 = c > 0)$.

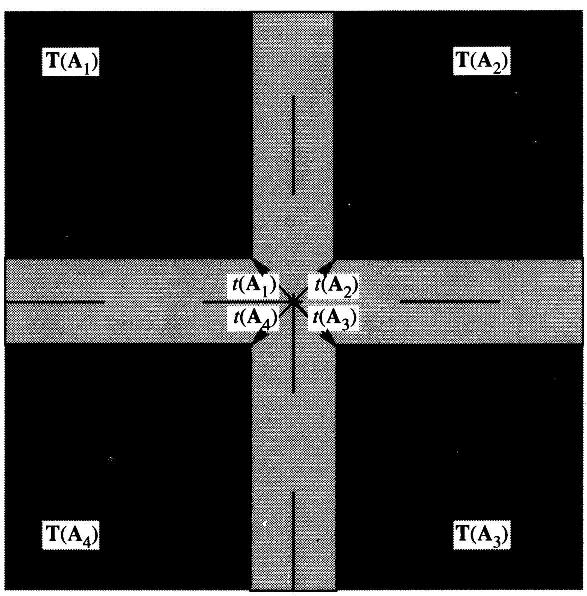

FIGURE 3. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES $(x_3 = 0)$.

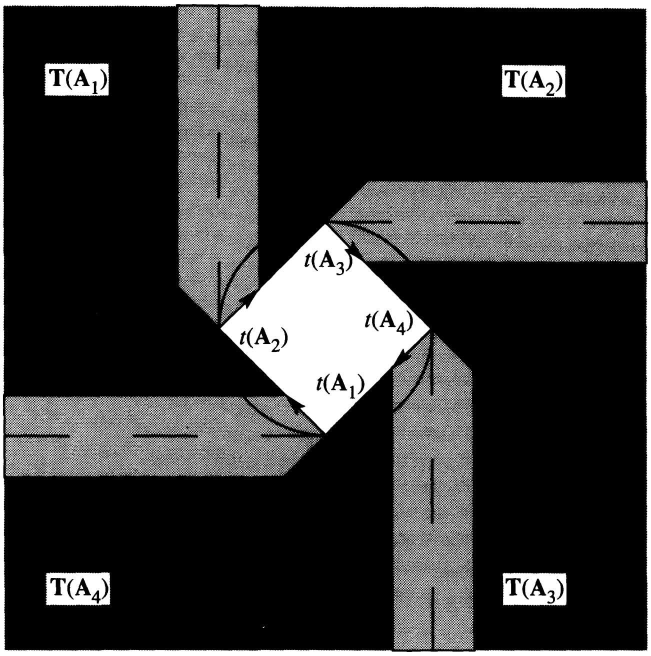

FIGURE 4. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES $(x_3 = c < 0)$.

$$g_i(\mathbf{A}_i^-) = \mathrm{cl}(\mathbf{C} - \mathbf{A}_i^+).$$

We claim that

$$g_i(\mathbf{W}(\mathbf{A}_i^-)) = \mathrm{cl}(\mathbf{E} - \mathbf{W}(\mathbf{A}_i^+)),$$

or more directly,

$$g_i(\mathbf{W}(\mathbf{A}_i^-)) = \mathbf{W}(g_i(\mathbf{A}_i^-)). \tag{2}$$

Remember that the set of $y \in \mathbf{W}(\mathbf{A}_i^-)$ such that $B(y, y) \le 0$ is a closed set bounded by $\mathbf{A}_i^-$ itself and $\langle u_i^-, v_i^- \rangle$.

$$g_i(\langle u_i^-, v_i^- \rangle) = \langle g_i(u_i^-), g_i(v_i^-) \rangle = \langle u_i^+, v_i^+ \rangle,$$

since g_i takes elements of bounding vector pairs to scalar multiples of elements of bounding vector pairs. Certainly for y such that $B(y, y) \le 0$, $y \in \mathbf{W}(\mathbf{A}_i^-)$ if and only if $g(y) \in \mathbf{W}(g_i(\mathbf{A}_i^-))$.

For $y \in \mathbf{W}(\mathbf{A}_i^-)$ such that $B(y, y) > 0$, it suffices to show that

$$g_i(\mathbf{P}(y)) = \mathbf{P}(g_i(y)). \tag{3}$$

In fact, it is enough to show that (3) is true for at least one y.

If g_i is hyperbolic, then $\mathbf{P}(\mathrm{x}^o_{g_i})$ can be written as

$$\{\mathrm{w} \in \mathbf{E} | \mathrm{w} = m(\mathrm{x}^+_{g_i}) + n(\mathrm{x}^o_{g_i}) \text{ for } m \in \mathfrak{R} \text{ and } n \in \mathfrak{R}^+\}.$$

The eigenvalue associated with $\mathrm{x}^+_{g_i}$ may be negative, but the eigenvalue associated with $\mathrm{x}^o_{g_i}$ is always 1, because $g \in \mathbf{G}$. $\mathbf{P}(\mathrm{x}^o_{g_i})$ is invariant under the action of g_i.

For parabolic g_i there exists a $j \in \mathbf{G}^o$ such that jg_i is hyperbolic. For y such that $B(\mathrm{y}, \mathrm{y}) > 0$ we know that

$$jg_i(\mathbf{P}(\mathrm{y})) = \mathbf{P}(jg_i(\mathrm{y})),$$

and

$$j^{-1}(jg_i(\mathbf{P}(\mathrm{y}))) = \mathbf{P}(j^{-1}(jg_i(\mathrm{y}))),$$

so that (3) is true for parabolic elements in $\mathbf{G}$.

Now choose

$$h_i(\mathrm{x}) = g_i(\mathrm{x}) + [-g_i(t(\mathbf{A}_i^-)) + t(\mathbf{A}_i^+)].$$

Then $h_i(t(\mathbf{A}_i^-)) = t(\mathbf{A}_i^+)$, and

$$\mathbf{X}_i = \mathrm{cl}[\mathbf{E} - \mathbf{T}(\mathbf{A}_i^-) - \mathbf{T}_i(\mathbf{A}_i^+)]$$

is a fundamental domain for the action of $\langle h_i \rangle$ on $\mathbf{E}$ since all of the translated wedges are distinct by the previous discussion. Further, $(\mathbf{E}-\mathbf{X}_i)$ is a 3-dimensional submanifold of $\mathbf{X}_j$. q.e.d.

Let $\mathbf{X} = (\bigcap_{i \in I} \mathbf{X}_i)$. Before showing that $\mathbf{X}$ is the fundamental domain for the action of Γ on $\mathbf{E}$, we will prove the following technical lemma. For $g \in \mathbf{G}^o$ and $\mathrm{p} \in \mathbf{E}$, We define $\mathbf{S}_g(\mathrm{p})$ to be the plane containing p and parallel to

$$\mathbf{S}_g = \langle \mathrm{x}^o_g, \mathrm{x}^+_g \rangle.$$

Lemma 3. *For ε-hyperbolic* $g \in \mathbf{G}$,

$$\mathbf{B}\left(g(\mathrm{p}), \frac{\varepsilon\delta}{2}\right) \cap \mathbf{S}_g(g(\mathrm{p})) \subset g(\mathbf{B}(\mathrm{p}, \delta) \cap \mathbf{S}_g(\mathrm{p})). \tag{4}$$

Proof. It is sufficient to consider $\mathrm{p} = 0$, and $\mathbf{S}_g(\mathrm{p}) = \mathbf{S}_g$. Let $\mathbf{Q}$ denote the rectangle in $\mathbf{B}(\mathrm{p}, \delta) \cap \mathbf{S}_g$ whose four vertices are the four of $[\langle \mathrm{x}^o_g \rangle \cup \langle \mathrm{x}^+_g \rangle] \cap \mathbf{B}(o, \delta)$. Note that

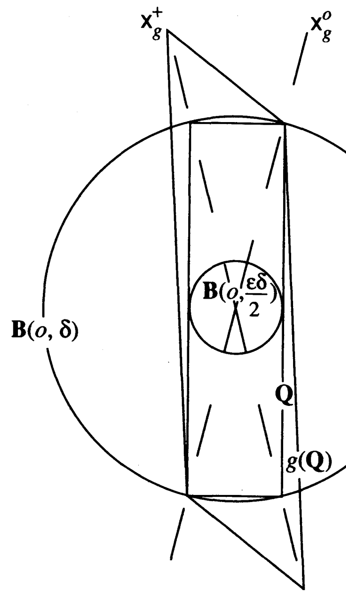

FIGURE 5. S_g.

$$\mathbf{B}(o, \varepsilon\delta/2) \cap \mathbf{S}_g \subset \mathbf{Q} \cap \mathbf{S}_g.$$

See Figure 5.

g is a linear map which fixes the vertices of $\mathbf{Q}$ on $\langle x_g^o \rangle$ and sends the vertices of $\mathbf{Q}$ on $\langle x_g^+ \rangle$ to points on $\langle x_g^+ \rangle$ further from the origin. Thus, $\mathbf{Q} \subset g(\mathbf{Q})$ and (4) follows. q.e.d.

The estimate in Lemma 3 is a lower bound of the *compression* by g parallel to $\mathbf{S}_g$. Note that (4) is independent of λ_g.

Theorem 4. *If $G \subset \mathbf{G}$ is a free discrete group, then there exists $\Gamma \subset \mathbf{G} \times \mathbf{V}$ which acts properly discontinuously on $\mathbf{E}$ such that $L(\Gamma) = G$.*

Proof. By Theorem 3, G is a generalized Schottky group.

We can choose g_i, h_i, and $\mathbf{X}_i$ as in Lemma 2. For $I = \{1, 2, \cdots, n\}$, it suffices to show that for the 3-dimensional manifold $\mathbf{X} = (\bigcap_{i \in I} \mathbf{X}_i)$ with boundary is a fundamental domain for the action of Γ on $\mathbf{E}$.

From the construction of $\mathbf{X}$ it is apparent that no two distinct points in the interior of $\mathbf{X}$ are Γ-equivalent. It remains to show that every element of $\mathbf{E}$ is Γ-equivalent to some point in $\mathbf{X}$.

Assume that there is a p not Γ-equivalent to any $y \in \mathbf{X}$. Certainly p is contained in one of the translated wedges $\mathbf{T}(\mathbf{A}_i^{\pm})$ since their union is the complement of $\mathbf{X}$. p is also Γ-equivalent to elements in all of the translated wedges $\mathbf{T}(\mathbf{A}_i^{\pm})$. Thus, we may assume that $p \in \mathbf{T}(\mathbf{A}_1^+)$ and that

$\Theta(\mathbf{A}_1^+) \leq \pi/2$, since the sum of the $\Theta(A_i^{\pm})$'s is not more than 2π.

Let $\mathbf{X}_0 = \mathbf{X}$ and

$$\mathbf{X}_{n+1} = \left[\mathbf{X}_n \cup \left(\bigcup_{i=1}^{n}(h_i(\mathbf{X}_n) \cup h_i^{-1}(\mathbf{X}_n))\right)\right].$$

This is a sequence of domains for which $\rho(\mathsf{p}, \mathbf{X}_{n+1}) \leq \rho(\mathsf{p}, \mathbf{X}_n)$. We can define $\gamma_n \in \Gamma$ such that $\gamma_n(\mathbf{X}) \subset \mathbf{X}_n$ and

$$\rho(\mathsf{p}, \mathbf{X}_n) = \rho(\mathsf{p}, \gamma_n(\mathbf{X})).$$

γ_n has word length n as a reduced word in the free group Γ. For $n \geq 1$, $\gamma_n(\mathbf{X}) \subset \mathbf{T}(\mathbf{A}_1^+)$ so the leading term of γ_n must be h_1.

Let $(\mathsf{u}_i^+, \mathsf{v}_i^+)$ be the bounding vector pair for $\mathbf{A}_i^+$, and let

$$\mathsf{w} = \mathsf{u}_i^+ - \mathsf{v}_i^+.$$

If $\mathsf{v} \in \mathbf{H}_o$ is parallel to a ray lying in $\mathbf{T}(\mathbf{A}_1^+)$, then the angle between w and v is less than or equal to $\pi/4$.

Let $\mathbf{L}_n \subset \mathbf{H}_\mathsf{p}$ be the line closest to p, which is Euclidean perpendicular to w and bounds a half-plane in $\mathbf{H}_\mathsf{p}$ containing the component of the complement of $\mathbf{X}_n \cap \mathbf{H}_\mathsf{p}$ which contains p. Note that

$$d_n = \rho(\mathsf{p}, \mathbf{L}_n) \geq \rho(\mathsf{p}, \mathbf{X}_n)$$

and $d_{n+1} \leq d_n$. To arrive at a contradiction it suffices to show that $(d_n - d_{n+1})$ is bounded away from 0. See Figure 6.

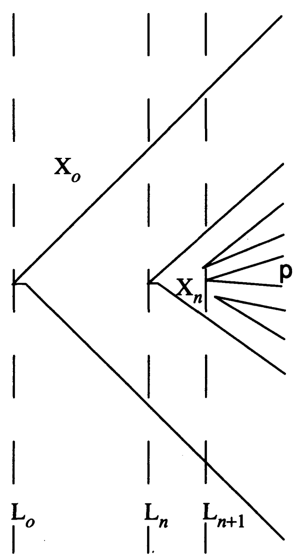

FIGURE 6. $\mathbf{H}_\mathsf{p}$.

There exists a $\delta > 0$ such that $\mathbf{B}(y, \delta) \subset \mathbf{X}_1$ for all $y \in \mathbf{X}$. Thus, $(d_1 - d_0) > \delta$. Choose

$$\varepsilon = \min\{\sqrt{2}\sin(\tfrac{1}{2}\Theta(\mathbf{A}_i^{\pm}))\}.$$

For $n \geq 1$, first suppose that γ_n is ε-hyperbolic. For every $x \in \mathbf{E}$, $\mathbf{S}_{L(\gamma_n)}(\gamma_n(x)) \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{T}(\mathbf{A}_1^+) \cap \mathbf{H}_p$ by Lemma 1. Every ball $\mathbf{B}(y, \delta)$ cannot be compressed by more than a factor of $\varepsilon/2$ in a direction parallel to $\mathbf{S}_{L(\gamma_n)}$ by the action of γ_n. Since the angle between $\mathbf{S}_{L(\gamma_n)}(\gamma_n(x)) \cap \mathbf{H}_p$ and the normal to $\mathbf{L}_n$ in $\mathbf{H}_p$ is at most $\pi/4$,

$$d_{n+1} \leq (d_n - \varepsilon\delta/2\sqrt{2}).$$

Now suppose γ_n is not ε-hyperbolic. There exists an $f_n \in \Gamma$ with word length ≤ 2 such that $f_n\gamma_n$ is ε-hyperbolic and has word length $n+1$ if f_n has word length 1 or $n+2$ if f_n has word length 2. It is enough to consider f_n having length 2.

f_n can be written as $h_{a_n}h_{b_n}$, where h_{a_n} and h_{b_n} are generators of Γ or their inverses. $\mathbf{S}_{L(f_n\gamma_n)}(f_n\gamma_n(x)) \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{T}(\mathbf{A}_a^+) \cap \mathbf{H}_p$. δ-balls are not compressed by more than a factor of $\varepsilon/2$ in the direction parallel to $\mathbf{S}_{L(f_n\gamma_n)}$ by the action of $f_n\gamma_n$.

We can define the *compression factor* for $g \in \mathbf{G}$,

$$C_g = \min_{v \in \mathbf{S}^2}\{\|g(v)\|/\|v\|\},$$

which is positive for all $g \in \mathbf{G}$. Let C_Γ be the minimum of the compression factors of the g_i's. Then $C_{f_n} \leq C_\Gamma^2$.

Thus, δ-balls are compressed by at most a factor of $C_\Gamma^2\varepsilon/2$ in the direction parallel to $[L(f_n^{-1}(S_{L(f_n\gamma_n)})]$ by the action of γ_n. From (1), $\mathbf{S}_{L(f_n\gamma_n)(\gamma_n(x))} \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{G}(\mathbf{A}_1^+) \cap \mathbf{H}_p$. In this case,

$$d_{n+1} \leq (d_n - C_\Gamma^2\varepsilon\delta/2\sqrt{2}).$$

There must be an $m \leq 2\sqrt{2}\,d_0/(C_\Gamma^2\varepsilon\delta)$ such that $p \notin \mathbf{X}_n$ but $p \in \mathbf{X}_{n+1}$.

4. The end

Theorem 4 proves Theorem 2 is one direction, and in this section we will prove the other direction of Theorem 2.

If $G = L(\Gamma)$ for some free $\Gamma \subset \mathrm{Aff}(\mathfrak{R}^3)$, then G is conjugate to a subgroup of $\mathbf{O}(2, 1)$ by [3]. Further, G is a discrete subgroup of $\mathbf{O}(2, 1)$ by [6].

Consider $G \subset \mathrm{PGL}_2(\mathfrak{R})$, and assume that $G \cap (\mathbf{O}(2, 1) - \mathbf{G}) \neq \emptyset$. There must be elements in $G \cap (\mathbf{O}(2, 1) - \mathbf{G})$ which have three distinct real eigenvalues and do not have 1 as an eigenvalue, but rather -1 is an eigenvalue. By an observation of Hirsch (see [3]), affine elements whose linear parts do not have 1 as an eigenvalue have fixed points. This contradicts the assumption that G acts properly on $\mathbf{E}$. Thus, G must be conjugate to a finitely generated free discrete subgroup in $\mathbf{G}$.

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