DIVISORS ON SOME GENERIC HYPERSURFACES

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In this paper we consider generic hypersurfaces of degree at least 5 in $\mathbb{P}^3$ and especially $\mathbb{P}^4$, and reduced, irreducible, but otherwise arbitrarily singular, divisors upon them. Our purpose is to prove that such a divisor cannot admit a desingularization having numerically effective anticanonical class.

Over the past decade or so, there has been considerable interest in various questions of what might be called "generic geometry", such as the following: given a variety X which is "generic" in some sense, suppose $f: Z \rightarrow X$ is a generically finite map from a smooth variety onto some subvariety $\bar{Z} \subset X$; then what can be said about the intrinsic geometry of Z?

Perhaps the first, and still the most famous, instance of this problem concerns the case where X is a generic quintic hypersurface in $\mathbb{P}^4$. There a conjecture of Clemens [1] is (equivalent to) the statement that Z as above must have nonnegative Kodaira dimension, i.e., cannot be birationally ruled (the usual statement of Clemens' conjecture is that X should contain only finitely many rational curves of given degree, obviously equivalent to the former statement). Coming from another direction, namely Faltings' work on the Mordell conjecture, etc., S. Lang has made a series of very general conjectures which, e.g., imply in the case of a quintic 3-fold X that Z as above cannot be an elliptic fibration, if $\bar{Z} = X$.

Along similar lines, Harris has conjectured that for X a generic surface of degree $d \geq 5$ in $\mathbb{P}^3$, Z as above cannot be a rational or elliptic curve. Harris' conjecture was recently proven by G. Xu [3], who also obtains more general bounds on the genus of Z in terms of the degree of $\bar{Z}$.

Now especially from a qualitative viewpoint, one common theme to the conjectures of Clemens and Harris stands out: that is some sort of "positivity" assertion on the canonical bundle K_Z. In dimension > 1 there are of course many ways to interpret such positivity, the one involved in Clemens'

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conjecture being in terms of Kodaira dimension. Along more "numerical" lines, one such interpretation, a very weak one, is the following: namely that the anticanonical bundle $-K_Z$ cannot be numerically effective, i.e., that there should exist at least one curve $C \subset Z$ with $K_Z . C > 0$. It is the latter that we will focus on here. We will prove the following.

Theorem. *A generic hypersurface of degree ≥ 5 in $\mathbb{P}^3$ or $\mathbb{P}^4$ does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle.*

Of course the $\mathbb{P}^3$ case here is just the above conjecture of Harris, first proven by Xu, for which we will give a different proof. In the $\mathbb{P}^4$ case, we note that the class of surfaces Z with $-K_Z$ numerically effective is well-understood and obviously contains the del Pezzo surfaces (with $-K_Z$ ample), those with $-K_Z$ numerically trivial, i.e., the abelian, $K3$, Enriques and hyperelliptic surfaces, as well as some birationally ruled surfaces. Hence our Theorem has some bearing on Clemens' conjecture, although the latter still seems out of reach. We also note that for degree ≥ 6 our theorem follows from Xu's result that Z has geometric genus ≥ 2. Thus the interesting case as one might expect is that of the quintic 3-fold.

Our methods revolve, in general, around considerations of normal bundles, as in Clemens [1]. Perhaps the main novel point in our approach is the surprising observation that at least in some relatively well-controlled, perhaps artificially so, situations such considerations work *better* in the case of subvarieties of dimension ≥ 2 than for curves, and it is this circumstance that, e.g., allows us to treat the case of quintic 3-fold, which has trivial canonical bundle, whereas the analogous assertion for the quartic surface is well known to be false. On a technical level, the difference is due to the fact that H^1 of suitably negative line bundles vanishes in dimension ≥ 2 but of course not for curves. On a more intuitive level, one could perhaps say that an exceptional surface is too heavy a piece of baggage for a fleet-moving generic hypersurface to carry along conveniently.

Finally, a word concerning the higher-dimensional case. The main "input" to our proof is the Ramanujam vanishing theorem on a surface; we have also used the Mori-Miyaoka uniruledness criterion, but only in a relatively elementary case of Fano-like manifolds. As these inputs admit generalizations to higher dimensions, it seems reasonable to expect, thought this is by no means immediate, that our theorem generalizes too, yielding the nonexistence of a divisor with nef anticanonical bundle on a generic hypersurface of degree $\geq r + 1$ in $\mathbb{P}^r$, $r \geq 5$. We hope to return to this matter elsewhere.

We now begin the proof of theorem, first in the case of $\mathbb{P}^3$. Thus let $X^2 \subset \mathbb{P}^3$ be a generic hypersurface of degree $d \geq 5$, and suppose $\bar{C} \subset X^2$ is an irreducible curve of (geometric) genus $g = 0$ or 1. The idea then is to view X^2 as $X^3 \cap H$, where $X^3 \subset \mathbb{P}^4$ is a generic but fixed hypersurface of degree d, and H is a hyperplane which we view as variable. As H moves, X^2 and $\bar{C}$ must move along and there are the following two possibilities:

(a) $\bar{C}$ fills up X^3; or

(b) $\bar{C}$ fills up an irreducible surface $\bar{S} \subset X^3$.

Assume to begin with that we are in case (b). Note that the generic hyperplane section $\bar{S} \cap H$ must contain $\bar{C}$. On the other hand by Bertini's theorem, $\bar{S} \cap H$ is irreducible. Hence we have $\bar{S} \cap H = \bar{C}$, i.e., $\bar{S}$ is a surface with rational or elliptic generic hyperplane section. Such surfaces are well known classically. Alternatively, one can use Reider's theorem [2] to conclude in any case that $\bar{S}$ must be a union of lines or conics. But an easy dimension count shows that a generic hypersurface X^3 of degree ≥ 5 contains at most finitely many lines and conics, which is a contradiction.

Now suppose case (a) above occurs. As X^3 has ample or trivial canonical bundle, obviously $\bar{C}$ cannot have genus $g = 0$. So suppose $g = 1$. Let C be the normalization of $\bar{C}$, and $f: C \to X^3$ the natural map. Let N_f be the normal sheaf of f, defined by the exact sequence

$$0 \to T_C \to f^* T_X \to N_f \to 0,$$

and put $M = N_f/\text{torsion}$. As $\bar{C} \subset H$ moves with H it follows that N_f and hence M are generically generated by global sections and moreover $h^0(N_f) \geq$. Also

$$c_1(M) = -\text{length}((\text{torsion}) - (d-5)\deg(\bar{C}) \leq 0,$$

and by the following observation we will easily get a contradiction (the added generality is for future use).

Lemma 1. *Let M be a torsion-free sheaf or a smooth variety C of dimension $n \leq 2$. Assume M is generically generated by global sections while for some ample divisor H on C, $H^{n-1}.c_1(M) \leq 0$. Then M is trivial.*

Proof. Let $M^{\vee\vee}$ be the double dual of M, which is locally free and contains M as subsheaf with quotient of finite length. Choosing $r = \text{rank}(M)$ general sections of M, we get an exact sequence

$$0 \longrightarrow r\mathscr{O}_C \stackrel{\varphi}{\longrightarrow} M^{\vee\vee} \longrightarrow \tau \longrightarrow 0$$

with τ supported by purely in codimension 1. Then

$$c_1(\tau).H^{n-1} = c_1(M^{\vee\vee}).H^{n-1} = c_1 H^{n-1} \leq 0.$$

As $c_1(\tau)$ is an effective divisor, it follows that $c_1(\tau) = 0$, so φ is an isomorphism in codimension 1 and hence an isomorphism. Thus $M^{\vee\vee} \simeq r\mathscr{O}$. Since $h^0(M) \geq r$ and $M \subseteq M^{\vee\vee}$, it follows easily that $M = M^{\vee\vee} \simeq r\mathscr{O}$. q.e.d.

Now in our case by degree considerations it follows that $M = N_f$ is trivial, contradicting $h^0(N_f) \geq 4$. This completes the proof of the Theorem in the $\mathbb{P}^3$ case.

Turning now to the $\mathbb{P}^4$ case, let $X^3 \subset \mathbb{P}^4$ be a generic hypersurface of degree $d \geq 5$, and $f: S \to \bar{S} \subset X^3$ a desingularization of a surface in X^3, with $-K_S$ nef. Without loss of generality, we may assume f to be minimal, i.e., no (-1)-curve is in a fiber of f. As before we consider X^3 as a generic hyperplane section $X^4 \cap H$ of $X^4 \subset \mathbb{P}^5$, which leads to the two cases (a), (b) as above. Assume first we are in case (a), where $\bar{S}$ moves with H, filling up X^4. Let $\tilde{f}: S \to X^4$ be the obvious map. Then we have exact sequences

(1) $0 \to T_S \to f^* T_{X^3} \to N_f \to 0,$

(2) $0 \to N_f \to N_{\tilde{f}} \to L \to 0, \qquad L := f^* \mathscr{O}(1).$

Note that the sequence (2) splits in a neighborhood of any fiber of f. Let τ be the torsion subsheaf of N_f (or, what is the same, of $N_{\tilde{f}}$), and note by (1) that τ must be supported purely in codimension 1. As $\bar{S}$ fills up X^4, $N_{\tilde{f}}$ is generically generated by global sections. Hence so is $N_{\tilde{f}}/\tau$, and in particular $c_1(N_{\tilde{f}}) - c_1(\tau)$ is nef. Note that

$$c_1(N_{\tilde{f}}) = K_S - (d-6)L.$$

Now if $d \geq 6$, then our assumption that $-K_S$ is nef yields immediately that $\tau = 0$, $d = 6$ and K_S is numerically trivial. Using Lemma 1, we conclude that $N_{\tilde{f}}$ is trivial too, contradicting $h^0(N_{\tilde{f}}) \geq 4$. Hence we may assume $d = 5$. Put $M = N_f^{\vee\vee}$ which is a line bundle on S. We have an exact sequence

$$0 \to \tau \to N_f \xrightarrow{\alpha} M \to \varphi \to 0$$

with supp φ finite.

Let N' be the direct image of the extension (2) by the map α. Thus we have a diagram:

$$0 \longrightarrow N_f \longrightarrow N_{\tilde{f}} \longrightarrow L \longrightarrow 0$$

(3)

$$\alpha \downarrow \qquad \beta \downarrow \qquad \|$$

$$0 \longrightarrow M \longrightarrow N' \longrightarrow L \longrightarrow 0$$

$$\downarrow \qquad \downarrow$$

(4)

$$\varphi \quad = \quad \varphi$$

$$\downarrow \qquad \downarrow$$

$$0 \qquad 0$$

Note that as $N_{\tilde{f}}$ is generically generated by global sections, so too is N'. We now make the following key claim.

Claim. The extension (4) does not split.

Proof of Claim. If not, let us identify $N' = M \oplus L$ and note that M must then be effective. From the linear equivalence $M \sim K_S - c_1(\tau)$ it then follows that $\tau = 0$ and $K_S = \mathscr{O}_S$; in particular, S is an abelian or $K3$ surface. Also, we can then identify $M = \mathscr{O}_S$, $N_f = I_{Z,S}$ for some finite subscheme $Z \subset S$. Next, since by generic generation of $N_{\tilde{f}}$, the image of the composite map

$$\gamma \colon N_{\tilde{f}} \to N' = \mathscr{O}_S \oplus L \to 0$$

must have a section, γ is surjective. Thus we have another exact sequence for $N_{\tilde{f}}$:

(5) $$0 \to L.I_{Z,S} \to N_{\tilde{f}} \to \mathscr{O}_S \to 0.$$

Now for a general point $P \in S$, sections of $\mathscr{O}_H(1)(-f(P))$ correspond to motions of H through $f(P)$, and their pullback to S must correspond to sections of $N_{\tilde{f}}(-P)$. Hence by (5) we have $h^0(L.I_{Z \cup P,S}) > 2$ and in fact $Z \cup P$ is contained in > 2 general hyperplane sections of S, which have just $\deg \bar{S}$ many points in common. Thus $\deg Z < \deg \bar{S}$. However, from (1), using $c_2(T_{X^3}) = 10H^2$, we compute readily that

$$\deg Z = c_2(N_f) 10 \deg(\bar{S}) - c_2(T_S),$$

and as $c_2(T_S) \in \{0, 24\}$ that is a contradiction, proving the Claim.

Now the Claim implies in particular that

$$0 \neq \mathrm{Ext}^1(L, M) = H^1(ML^{-1}) = H^1(K_S + (-K_S + L + c_1(\tau)))^*.$$

Put

$$B = -K_S + L + c_1(\tau).$$

As $-K_S$ is nef and L is nef and big, clearly B is big. In view of Ramanu-jam vanishing, B cannot be nef, i.e., there exists a reduced irreducible curve $A \subset S$ with $A.B < 0$. Clearly, any such A must occur in $c_1(\tau)$ considered as cycle, with multiplicity $a \geq 1$. As τ is the torsion subsheaf of N_f, i.e., the ramification sheaf of f, we have

(6) $A.L \geq \deg(f(A)) . \deg(f|_A)(a + 1) .$

Now as $A.B < 0$, of course $A^2 < 0$, hence the adjunction formula yields

$$-2 \leq 2p_a(A) - 2 = A.K_S + A^2 \leq A^2 < 0 .$$

If $A^2 = -1$, then $A.K_S = -1$ too, so $A.B < 0$ implies

$$A.L < a - 1 .$$

Hence by (6), $f(A)$ is a point, which contradicts our assumption that S is a minimal desingularization of $\bar{S}$. Thus we must have

$$A^2 = -2, \qquad A.K_S = 0, \qquad A.L < 2A .$$

Using (6) again, we conclude that $f(A)$ must be a line or a point. Write

$$c_1(\tau) = \tau_1 + \tau_2 + \tau_3 ,$$

where τ_1 (resp. τ_2; τ_3) is the sum of all the aA occurring in τ with $A.B < 0$ and $f(A)$ a point (resp. $A.B < 0$ and $f(A)$ a line; $A.B \geq 0$). If A occurs in τ_1, we saw above that the extension defining N' must split in a neighborhood of A. If A occurs in τ_2, then A is a (-2)-curve on S and $ML^{-1}.A > 0$. As is well-known, there is a neighborhood U of A such that $H^1(U, ML^{-1}) = 0$, so again the extension defining N' splits in a neighborhood of A. Thus we conclude that the extension defining N' yields a nontrivial element of

$$H^1(K_S + B - \tau_1 - \tau_2) = H^1(K_S + (-K_S + L + \tau_3)) ,$$

which is a contradiction, as $-K_S + L + \tau_3$ is evidently nef (and big). Hence the proof in case (a) is complete.

We now turn to case (b), where $\bar{S}$ extends to a divisor

$$\bar{D}^3 \subset X^4 \subset \mathbb{P}^5, \qquad \bar{S} = \bar{D}^3 \cap H .$$

For any $n \geq 5$, we may view X^4 as a generic linear space section $X^n \cap G$, with $X^n \subset \mathbb{P}^{n+1}$ a generic hypersurface of degree d and G a generic $\mathbb{P}^5$, and again we may ask whether $\bar{D}^3$ extends to a divisor $\bar{D}^{n-1} \subset X^n$. If it does not, then as before $\bar{D}^3$ moves in a family filling up X^n and therefore, of course, so does $\bar{S}$. Thus as before we obtain an extension

$$0 \longrightarrow N_f \longrightarrow N_{S \to X^n} \longrightarrow (n-3)L \longrightarrow 0,$$

in which the middle term is generically generated by global sections, and we may argue as above to get a contradiction. Hence we may assume $\bar{D}^3$ does extend to $\bar{D}^{n-1} \subset X^n$ for any n.

Now let $g_n : D^n \to X^{n+1}$ be a desingularization of $\bar{D}^n$. As

$$h^0(mK_S) = h^0(m(K_{D^3} + H)|_S) \le 1, \quad \forall m \ge 1,$$

it follows that $h^0(m(K_{D^3} + H)) \le 1$; in particular D^3 has Koraira dimension $-\infty$, hence is uniruled by Mori-Miyaoka. Let $h^3 : \mathbb{P}^1 \to D^3$ be a rational curve through a general point of D^3, and put

$$h_n^0 = i_n \circ h_3, \qquad i_n : D^3 \hookrightarrow D^n.$$

Note the exact sequence

$$0 \longrightarrow N_{h_3} \longrightarrow N_{h_n^0} \longrightarrow (n-3)h_3^* \mathscr{O}(1) \longrightarrow 0.$$

As N_{h_3} is semipositive and hence nonspecial, it follows in particular that for large n a general deformation h_n of h_n^0 will be linearly, hence projectively normal (possibly degenerate). Noting the exact sequence

$$0 \longrightarrow N_{h_n} \longrightarrow N_{g_r \circ h_n} \longrightarrow h_n^* N_g \longrightarrow 0,$$

by an easy calculation we have

$$c_1(h_n^* N_{g_n} j(-1)) = c_1(h_3^* N_{g_3}(-1)) = \deg h_3^*(K_{D^3}) \le -2.$$

Hence it follows that

$$H^1(h_n^* N_{g_n}(-1)) \ne 0, \qquad \mathscr{O}(1) := h_n^* \mathscr{O}(1),$$

a fortiori

$$H^1(N_{g_n \circ h_n}(-1)) \ne 0.$$

In view of the construction of h_n, the following then yields a contradiction.

Lemma 2. *Let $r : \mathbb{P}^1 \to X^n \subset \mathbb{P}^{n+1}$ be a projectively normal rational curve on a smooth hypersurface. Then there exists an extension X^m of X^n to $\mathbb{P}^{m+1}$ such that the evident map $r_m : \mathbb{P}^1 \to X^m$ has $h^1(N_{r_m}(-1)) = 0$.*

Proof. Consider a potential extension X^m, and let F be its homogeneous equation and $j : X^m \to \mathbb{P}^{m+1}$ the inclusion. Then we have an exact sequence

$$(*) \qquad 0 \longrightarrow N_{r_m}(-1) \longrightarrow N_{j \circ r_m}(-1) \longrightarrow r_m^* \mathscr{O}(d-1) \longrightarrow 0,$$

where $d = \deg X^n$, and the natural map

$$\delta : H^0(T_{\mathbb{P}^{m+1}}(-1)) \longrightarrow H^0(N_{j \circ r_m}(-1)) \longrightarrow H^0(r_m^* \mathscr{O}(d-1))$$

is just given by

$$\frac{\partial}{\partial X_i} \longmapsto \frac{\partial F}{\partial X_i}.$$

By projective normality, clearly for large m and general F, δ will be surjective. As $N_{j \circ r_m}(-1)$ is semipositive, its H^1 vanishes, hence the cohomology sequence of $(*)$ yields $H^1(N_{r_m}(-1)) = 0$.   q.e.d.

Since the above analysis of case (b) did not use the nefness of $-K_S$, S has nonpositive Kodaira dimension, hence

Corollary. *If* $\bar{S} \subset X_5^3 \subset \mathbb{P}^4$ *is a surface of nonpositive Kodaira dimension on a generic quintic, then for some generic quintic* $X_5^n \subset \mathbb{P}^{n+1}$, $n \geq 4$, *there exists a divisor* $\bar{D}^{n-2}$ *on the generic hyperplane section* $X_5^{n-1} = X_5^n \cap \mathbb{P}^n$ *such that* $\bar{D}^{n-2}$ *moves and fills up* X_5^n, *and* $\bar{S} \subset X_5^3$ *is the generic* $\mathbb{P}^4$-*section of* $\bar{D}^{n-2} \subset X_5^{n-1}$.

Acknowledgment

References

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