J. DIFFERENTIAL GEOMETRY
38 (1993) 559-583

DISCRETE PARABOLIC GROUPS

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0. Introduction

Let X be a complete simply connected Riemannian manifold of pinched negative curvature (i.e., all the sectional curvatures lie between two negative constants). The main aim of this paper is to show that any discrete group of isometries of X fixing some ideal points is finitely generated (Theorem 4.1). The only interesting case is that of a discrete parabolic group (preserving setwise some horosphere in X). In this case, by applying the Margulis Lemma (2.3), it follows that a discrete parabolic group contains a nilpotent subgroup of finite index. We may identify this subgroup as the group generated by those elements having "small rotational part". In fact, the notion of the rotational part of a parabolic isometry will be one of the main ingredients of the proof of the main theorem (see §3).

Conversely, it is well known that any (virtually) nilpotent discrete group of isometries must be "elementary". In particular, some finite-index subgroup must fix an ideal point. Thus, all discrete nilpotent groups are finitely generated. This rules out the possibility of groups such as the rational numbers occurring as discrete groups. (Note that there is no purely topological obstruction to this—see the end of §4.)

I suspect that one should be able to strengthen the conclusion of the main theorem, to show that the quotient space of a discrete parabolic group is always topologically finite, i.e., homeomorphic (or diffeomorphic) to the interior of a compact orbifold. However, I do not have a proof of this.

The case where X has constant negative curvature is a consequence of the Bieberbach Theorem (§1). Proofs of the Bieberbach theorem usually proceed by an induction on dimension, and so an argument along these lines would make essential use of the existence of totally geodesic subspaces. Thus, for the variable curvature case, we will need another approach.

Received July 12, 1991 and, in revised form, September 14, 1992. Supported by a Royal Society European Exchange Fellowship at Institut des Hautes Études Scientifique.

It turns out that both the upper and lower curvature bounds are essential for the main theorem. In fact, it is possible for a noncyclic free group to act as a discrete parabolic group in cases where we have an upper bound on curvature (away from 0), but no lower bound (away from $-\infty$), or vice versa. Examples are due to G. Mess, and recent work of Abresch and Schroeder, as we describe in §6.

The main result of this paper represents a step towards defining a notation of "geometrical finiteness" for manifolds of pinched negative curvature, as I have described in another paper [3]. The concept of geometrical finiteness first arose in the context of hyperbolic 3-manifolds, and readily generalizes to hyperbolic orbifolds of any dimension (see [4] for an exposition of this). More recently, some interest in the variable curvature case has arisen, particularly from the study of the symmetric spaces. A consequence of our main theorem here is that a geometrically finite group is finitely generated.

A brief summary of this paper is as follows. §1 reviews the case of constant curvature (i.e., hyperbolic space). §2 is a survey of simply connected manifolds of pinched negative curvature. In §3, we define the notion of the "rotational part" of a parabolic isometry. In §4, we reduce the main theorem to the case of abelian parabolic groups. We complete the proof in §5. Finally in §6, we describe counterexamples when there are no strict curvature bounds.

Notation. Let Y be a Riemannian manifold. We shall write $\operatorname{Isom} Y$ for the group of isometries of Y. Given $x \in Y$, we write $T_x Y$ for the tangent space at x. Thus, $TY = \bigcup_{x \in Y} T_x Y$ is the total tangent space. If $\gamma \in \operatorname{Isom} Y$, then γ_* is the induced map on TY.

Suppose that Y is complete and simply connected, and has nonpositive curvature. Then, any two points $x, y \in Y$ are joined by a unique geodesic $[x, y]$. More generally, if $x_0, x_1, \ldots, x_m \in Y$, we write $[x_0, x_1, \ldots, x_m]$ for the piecewise geodesic path $[x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{m-1}, x_m]$. We write $P(y, x)\colon T_x Y \to T_y Y$ for parallel transport along the geodesic $[x, y]$. We set

$$P(x_m, x_{m-1}, \cdots, x_0) = P(x_m, x_{m-1}) \circ \cdots \circ P(x_1, x_0)\colon T_{x_0} Y \to T_{x_m} Y.$$

In other words, $P(x_m, x_{m-1}, \cdots, x_0)$ is parallel transport along the path $[x_0, x_1, \cdots, x_m]$.

We shall use $\mathbf{E}^n$ to denote a Euclidean n-space, thought of as a Riemannian manifold (without any preferred coordinate system). On the other hand, $\mathbf{R}^n$ will be thought of as an inner-product space over the real numbers, $\mathbf{R}$.

1. The case of constant curvature

In order to give the heuristics of the proof of the main theorem, it will be useful to refer to the case of constant curvature. In this section, we give a brief survey of this case. Only the notation introduced in the next paragraph will be directly relevant to the rest of this paper.

Let $\langle\cdot\,,\cdot\rangle$ be the standard inner-product on $\mathbf{R}^n$, i.e., $\langle\xi\,,\zeta\rangle = \sum_{i=1}^n \xi_i\zeta_i$ where $\xi = (\xi_1\,, \cdots\,, \xi_n)$ and $\zeta = (\zeta_1\,, \cdots\,, \zeta_n)$. Let S^{n-1} be the unit sphere $\{\xi \in \mathbf{R}^n | \langle\xi\,,\xi\rangle = 1\}$. Given $\xi\,,\ \zeta \in S^{n-1}$, we write $\angle(\xi\,,\zeta) = \cos^{-1}\langle\xi\,,\zeta\rangle \in [0\,,\pi]$ for the angle between ξ and ζ. This gives the standard Riemannian metric on S^{n-1}, so that $\operatorname{Isom} S^{n-1}$ is the orthogonal group $O(n)$. Given $A \in O(n)$, write

$$|A| = \max\{\angle(\xi\,, A\xi) | \xi \in S^{n-1}\}.$$

Thus, for any $A\,,\ B \in O(n)$, we have

$$|A| = |A^{-1}|\,, \quad |AB| \le |A| + |B|\,, \quad |BAB^{-1}| = |A|.$$

Given any $\theta > 0$, write

$$U_\theta = \{A \in O(n) | \, |A| \le \theta\}.$$

Thus, as $\theta \to 0$, the sets U_θ form a base of closed neighborhoods of the identity in $O(n)$.

Let $\mathbf{E}^n$ be an n-dimensional Euclidean space. By a *trivialization* of the tangent bundle, we mean a map $\phi\colon \mathbf{E}^n \times \mathbf{R}^n \to T\mathbf{E}^n$ such that, for each $x \in \mathbf{E}^n$, the map $\phi_x = \phi(x\,,\cdot)\colon \mathbf{R}^n \to T_x\mathbf{E}^n$ is a linear isometry, which sends the standard inner-product on $\mathbf{R}^n$ to the Riemannian inner-product on $T_x\mathbf{E}^n$. A trivialization is *standard* if, for all $x\,,\ y \in \mathbf{E}^n$, the map $\phi_y\phi_x^{-1}\colon T_x\mathbf{E}^n \to T_y\mathbf{E}^n$ is just parallel transport, $P(y\,,x)$, along the geodesic $[x\,,y]$. Clearly, there is precisely an $O(n)$ worth of such standard trivializations. So, given a fixed trivialization, ϕ, we may define the *rotational part*, $\Theta_\phi(\gamma)$, of any isometry $\gamma \in \operatorname{Isom}\mathbf{E}^n$ according to

$$\gamma_* \circ \phi = \phi \circ (\gamma\,, \Theta_\phi(\gamma)).$$

This gives a homomorphism

$$\Theta_\phi\colon \operatorname{Isom}\mathbf{E}^n \to O(n).$$

Given any $\theta > 0$, and a subgroup Γ of $\operatorname{Isom}\mathbf{E}^n$, we let Γ_θ be the subgroup of Γ generated by those elements whose rotational parts lie in

U_θ. (We may write $\Gamma_\theta = \langle \gamma \in \Gamma \,|\, |\Theta_\phi(\gamma)| \leq \theta \rangle$.) Since U_θ is conjugacy invariant, Γ_θ is defined independently of ϕ. Moreover, Γ_θ is normal in Γ, and its index is finite and depends only on Θ and n (cf. Lemma 4.8). The following result is standard (see [13], [6] or [4]).

Bieberbach Theorem (1.1). *Suppose that the subgroup $\Gamma \subseteq \operatorname{Isom} \mathbf{E}^n$ is discrete (or equivalently acts properly discontinuously on $\mathbf{E}^n$). Then, there is a totally geodesic Γ- invariant subspace, μ, of $\mathbf{E}^n$, with the quotient μ/Γ compact. Moreover, there is some $\delta(n) > 0$, depending only on n, such that $\Gamma_{\delta(n)}$ is free abelian, and acts by translation on μ. Thus, $\mu/\Gamma_{\delta(n)}$ is a compact Euclidean torus.*

In particular, we see that Γ is finitely generated.

Now, let $\mathbf{H}^n$ be an n-dimensional hyperbolic space, and let $\mathbf{H}^n_I$ be the ideal sphere at infinity (see §2). We may represent $\mathbf{H}^n$ as the upper half-space $\mathbf{R}^n_+ = \{\xi \in \mathbf{R}^n | \xi_n > 0\}$, with Riemannian metric $|d\xi|/|\xi_n|$ where $\xi = (\xi_1, \cdots, \xi_n)$. Thus, $\mathbf{H}^n_I$ is the one-point compactification, $\partial \mathbf{R}^n_+ \cup \{\infty\}$, of $\partial \mathbf{R}^n_+ = \{\xi \in \mathbf{R}^n | \xi_n = 0\}$. If we put the standard Euclidean metric in $\mathbf{R}^n$, then any parabolic subgroup of $\operatorname{Isom} \mathbf{H}^n$, with ideal fixed point ∞, acts by Euclidean isometries on $\mathbf{R}^n_+ \cup \partial \mathbf{R}^n_+$. From the Bieberbach Theorem, we see that any discrete parabolic group acting on $\mathbf{H}^n$ is finitely generated.

2. Review of negative curvature

In this section, we review some relevant results in negative curvature. More details can be bound in [2] or [7].

Let (X, d) be a complete simply connected Riemannian manifold of pinched negative curvature. We shall normalise the Riemannian metric so that all sectional curvatures lie in the interval $[-\kappa^2, -1]$, where $\kappa \geq 1$. The exponential map based at any point $x \in X$ is a diffeomorphism from the tangent space $T_x Y$ onto X. Thus if X has dimension n, then it is diffeomorphic to an open n-ball. In fact, X can be naturally compactified by adding an *ideal sphere* X_I to X. Thus, X_I can be thought of as the set of equivalent classes of (semi-infinite) geodesic rays in X, where the two rays are regarded as equivalent if they remain a bounded distance apart in X (and hence, in fact, converge exponentially). An element of X_I will be called an *ideal point.* The set $X_C = X \cup X_I$ has a natural topological structure as a compact n-ball. By a *subspace* of X_C, we mean the closure, in X_C, of a totally geodesic subspace of X.

Now, X is a "visibility manifold". That is to say, any two distinct points x, $y \in X_C$ may be joined by a unique geodesic, which we shall write $[x, y]$. We shall call $[x, y]$ a *geodesic segment* if both x and y lie in X, a *geodesic ray* tending to y if $x \in X$ and $y \in X_I$, or a *bi-infinite geodesic* if x, $y \in X_I$. We shall always assume that geodesics are parametrized by the arc length. If $x_0, x_1, \cdots, x_m \in X_C$, we write $[x_0, x_1, \cdots, x_m]$ for the piecewise geodesic path $[x_0, x_1]\cup\cdots\cup[x_{m-1}x_m]$.

For each x in X, the Riemannian metric gives an inner-product $\langle\cdot, \cdot\rangle_x$ on the tangent space T_xX. We call $v \in T_xX$ a *unit vector* if $\langle v, v\rangle_x = 1$, and write T_x^1X for the unit tangent space $\{v \in T_xX | \langle v, v\rangle_x = 1\}$. Given v, $w \in T_x^1X$, we write $\angle(v, w) = \cos^{-1}\langle v, w\rangle_x \in [0, \pi]$ for the angle between v and w. Clearly, $\angle(v, w)+\angle(-v, w) = \pi$. If $S\colon T_xX \to T_yX$ is a linear isometry, then we write $|S| = \max\{\angle(v, Sv) | v \in T_x^1X\}$.

If x, $y \in X$, then parallel transport along the geodesic $[x, y]$ gives us an isometry

$$P(y, x)\colon T_xX \to T_yX.$$

More generally, if $x_0, x_1, \cdots, x_m \in X$, we write

$$P(x_m, m_{m-1}, \cdots, x_0) = P(x_m, x_{m-1}) \circ \cdots \circ P(x_1, x_0);$$

in other words, $P(x_m, x_{m-1}, \cdots, x_0)$ is parallel transport along $[x_0, x_1, \cdots, x_m]$.

Given $x \in X$ and $y \in X_C \setminus \{x\}$, write $\overrightarrow{xy}$ for the unit tangent vector at x in the direction of y. If $y \in X$, then clearly $P(y, x)\overrightarrow{xy} = -\overrightarrow{yx}$. If y, $z \in X_C \setminus \{x\}$, we write $y\hat{x}z = \angle(\overrightarrow{xy}, \overrightarrow{xz})$, i.e., the angle between $[x, y]$ and $[x, z]$.

Lemma 2.1. *For all* $\kappa \geq 1$, *there is some* $K = K(\kappa)$ *such that if* x, y *and* z *are any three points of* X, *then*

$$|P(x, z, y, x)| \leq K \min(d(x, y), d(y, z), d(z, x)).$$

Proof. Without loss of generality, $d(y, z) \leq \min(d(x, y), d(x, z))$. Note that the triangle $[x, y, z, x]$ spans a (ruled) surface, $S = \bigcup\{[x, w] | w \in [y, z]\}$, of area at most $d(y, z)$. The total angular displacement of a unit vector transported around the boundary of a surface S is at most the area of S times the norm of the Riemann curvature tensor. This norm is, in turn, bounded in terms of the pinching constants (cf. [5, Lemmas 6.2.1 and 6.7]). q.e.d.

We shall also need the following weak form of Toponogov's comparison theorem [2].

Proposition 2.2. *Let (X, d) be as above, and suppose that $x_1, x_2, x_3 \in X$. Let x_1', x_2', x_3' be points in the Euclidean plane, $(\mathbf{E}^2, d_{\text{euc}})$, such that $d(x_i, x_{i+1}) = d_{\text{euc}}(x_i', x_{i+1}')$, for each i (taking subscripts* mod 3). *Then, for each i, we have*

$$x_i \hat{x}_{i+1} x_{i+2} \leq x_i' \hat{x}_{i+1}' x_{i+2}',$$

where of course $x_i' \hat{x}_{i+1}' x_{i+2}'$ means the Euclidean angle between the segments $[x_{i+1}', x_i']$ and $[x_{i+1}', x_{i+2}']$.

In fact, we could replace the Euclidean plane by the hyperbolic plane in the above theorem.

Suppose $y \in X_I$. The set of all bi-infinite geodesics with an endpoint at y gives a 1-dimensional foliation $\mathscr{F}_y$, of X. This foliation is orthogonally integrable—there is a codimension 1 foliation $\mathscr{S}_y$ of X such that each leaf $\mathscr{S}_y$ meets each leaf of $\mathscr{F}_y$ orthogonally in a single point. Each leaf of $\mathscr{S}_y$ is a properly embedded C^2-submanifold of X, and is C^2-diffeomorphic to $\mathbf{R}^{n-1}$. Such a leaf is called a *horosphere* about y.

Suppose α, $\beta: [0, \infty] \to X_C$ are geodesic rays tending to y, such that $\alpha(0)$ and $\beta(0)$ lie in the same horosphere. Then, for any given $t \in [0, \infty)$, the points $\alpha(t)$ and $\beta(t)$ lie in the same horosphere. From standard comparison theorems, one may deduce that $d(\alpha(t), \beta(t))$ tends monotonically to 0 as $t \to \infty$. Moreover, there is some constant $C > 0$, such that $d(\alpha(t), \beta(t)) \leq Ce^{-t}$ for all t. This expresses the "exponential convergence" of geodesic rays referred to earlier.

Now, each isometry, $g \in \operatorname{Isom} X$, extends to homeomorphism of X_C, also denoted by g. We write $\operatorname{fix} g$ for the set of fixed points of g in X_C. Any such g is one of the following mutually exclusive types:

(0) g is the identity.

(1) g is *elliptic*, i.e., g is not the identity, and $X \cap \operatorname{fix} g \neq \varnothing$. In this case, $\operatorname{fix} g$ is a nonempty subspace of X_C.

(2) g is *parabolic*, i.e., $\operatorname{fix} g$ is a single point of X_I. In this case, g preserves setwise each horosphere about y.

(3) g is *loxodromic*, i.e., $\operatorname{fix} g$ consists of two distinct points x, $y \in X_I$. It thus preserves setwise the *loxodromic axis* $[x, y]$.

We may regard $\operatorname{Isom} X$ as a closed subgroup of the group of all diffeomorphisms of X in the C^1-topology. Given the subspace topology, $\operatorname{Isom} X$ is a locally compact topological group. A subgroup, Γ, of $\operatorname{Isom} X$ is discrete if and only if it acts properly discontinuously on X. In such a case, the torsion elements of Γ are precisely the elliptic ones. We write $\operatorname{fix} \Gamma = \bigcap_{g \in \Gamma} \operatorname{fix} g$ for the set of fixed points of Γ in X_C. If Γ is finite, then $\operatorname{fix} \Gamma$ is a nonempty subspace of X_C.

Given $p \in X_I$, we write

$$\operatorname{Isom}_p X = \{g \in \operatorname{Isom} X | gp = p\}.$$

If $\Gamma \subseteq \operatorname{Isom}_p X$ is discrete, then it is one of the following three types:

(1) Γ is finite.

(2) Γ contains a loxodromic element g, and preserves setwise the axis of g.

(3) Γ is infinite, and preserves setwise each horosphere about p.

In case (2), Γ is, group-theoretically, a semidirect product of an infinite cyclic group, and a finite subgroup of $O(n-1)$. In particular, Γ is finitely generated.

Definition. A discrete subgroup Γ of $\operatorname{Isom}_p X$ is said to be *parabolic* if it is infinite, and preserves some (and hence every) horosphere about p.

We shall show that every discrete parabolic group is finitely generated. It then follows from the discussion above that any discrete subgroup of $\operatorname{Isom}_p X$ is finitely generated.

We demanded, in the definition, that a parabolic group be infinite so as to accord with the usual notion. In fact, we shall make no use of this hypothesis anywhere in this paper. It is a consequence of the arguments presented in this paper (cf. the discussion following the proof of Lemma 4.9) that an (infinite) parabolic group necessarily contains a parabolic element.

The following result will be central to the proof. It is a slight rephrasing of [2, Theorem 9.5].

Margulis Lemma (2.3). *There is some universal $\omega > 0$, and for all $n \in \mathbf{N}$ and $\kappa > 0$, there exists $\varepsilon = \varepsilon(n, \kappa)$ such that the following holds.*

Suppose that (X, d) is a complete simply connected Riemannian n-manifold, all of whose sectional curvatures lie in $[-\kappa^2, 0]$. Suppose that Γ is a discrete subgroup of $\operatorname{Isom} X$, and that x is any point of X. Let $\Gamma(x)$ be the subgroup of Γ generated by all those $\gamma \in \Gamma$ which satisfy $d(x, \gamma x) \leq \varepsilon$ and have $\angle(v, P(x, \gamma x) \circ \gamma_* v) \leq \omega$ for all unit tangent vectors $v \in T_x^1 X$. Then, $\Gamma(x)$ is nilpotent.

3. Trivializations of the tangent bundle

Given an ideal point $p \in X_I$, we shall describe a preferred family of trivializations of the tangent bundle TX, indexed by the orthogonal group $O(n)$. This will allow us to define the "rotational part" of an isometry fixing p.

In §1 we defined the norm $|A|$ of an element $A \in O(n)$ as $\min\{\angle(\xi, A\xi) | \xi \in S^{n-1}\}$. Then, the map $[(A, B) \mapsto |A^{-1}B|]$ gives a bi-invariant metric on $O(n)$. Thus, to show that a sequence A_i converges, it is enough that $|A_j^{-1}A_i|$ can be made arbitrarily small for sufficiently large i and j. A similar discussion applies to automorphisms of the tangent space T_xX, at any $x \in X$, preserving the inner product $\langle \cdot, \cdot \rangle_x$.

Given $x_0, x_1, \cdots, x_m \in X$, we have defined $P(x_m, x_{m-1}, \cdots, x_0): T_{x_0}X \to T_{x_m}X$ as parallel transport along the piecewise geodesic path $[x_0, x_1, \cdots, x_m]$. Note that for any sequence of points $y_0, y_1, \cdots, y_r$ in X,

$$|P(y_0, \cdots, y_r, x_m, \cdots, x_0, y_r, \cdots, y_0)| = |P(y_r, x_m, \cdots, x_0, y_r)|.$$

Also, given $x, y, z \in X$, we have

$$|P(x, z, y, x)| = |P(z, y, x, z)| = |P(y, x, z, y)|.$$

Lemma 2.1 tells us that $|P(x, z, y, x)| \leq Kd(x, z)$. Note also that if x, y, z all lie along some geodesic in X, then $P(z, y, x) = P(z, x)$. In particular, of course, $P(x, y, x) = P(x, x)$ is the identity on T_xX.

Our next aim is to define $P(y, p, x): T_xX \to T_yX$ for $x, y \in X$, and $p \in X_I$.

Lemma 3.1. *Suppose $x, y \in X$ and $p \in X_I$. Let $(w_i)_{i\in\mathbf{N}}$ be any sequence of points in X tending to p. Then, the sequence of maps $P(y, w_i, x): T_xX \to T_yX$ converges.*

Proof. From the remarks above, it is enough to show that $|P(y, w_j, x)^{-1}P(y, w_i, x)| = |P(x, w_j, y, w_i, x)|$ can be made arbitrarily small for all sufficiently large i and j.

Suppose $\delta > 0$. Since the geodesic rays $[x, p]$ and $[y, p]$ converge, as we move towards p, we can find points $a \in [x, p]$ and $b \in [y, p]$, and $d(a, b) \leq \delta$. As $i \to \infty$, the geodesics $[x, w_i]$ and $[y, w_i]$ converge on $[x, p]$ and $[y, p]$ respectively. Thus, there is some $N \in \mathbf{N}$ such that for any $i \geq N$, we have $d(a, [x, w_i]) \leq \delta$ and $d(b, [y, w_i]) \leq \delta$. In other words, we can find $a_i \in [x, w_i]$ and $b_i \in [y, w_i]$ such that $d(a, a_i) \leq \delta$ and $d(b, b_i) \leq \delta$ (see Figure 1).

Suppose that $i, j \geq N$. Then, the piecewise geodesic path $[x, w_i, y, w_j, x]$ is spanned by the six triangles $[a_i, w_i, b_i, a_i]$, $[b_i, y, b_j, b_i]$, $[b_j, w_j, a_j, b_j]$, $[a_j, x, a_i, a_j]$, $[a_i, a_j, b_i, a_i]$ and $[b_j, b_i, a_j, b_j]$. Moreover, each of these six triangles has at least one side of length at

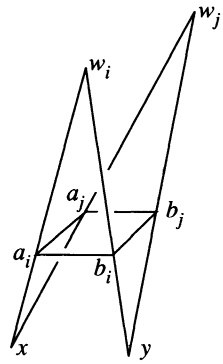

FIGURE 1

most 3δ. Applying Lemma 2.1, we conclude that

$$|P(x, w_j, y, w_i, x)| \leq 6K(3\delta) = 18K\delta,$$

which is arbitrarily small. q.e.d.

Since the sequence (w_i) was chosen arbitrarily, the limit of the maps $P(y, w_i, x)$ is well defined; we shall write it as $P(y, p, x)$. The map $P(y, w, x)$ depends continuously on w, as w varies in X_C. This may be seen by allowing the w_i to be the ideal points in the above proof, and using Lemma 3.2(1) below.

Lemma 3.2. *Suppose that $x, y, z \in X$ and $p \in X_I$. Then the following hold:*

(1) $|P(x, y) \circ P(y, p, x)| \leq Kd(x, y)$,

(2) $P(z, p, y) \circ P(y, p, x) = P(z, p, x)$,

(3) $P(y, p, x)\overrightarrow{xp} = \overrightarrow{yp}$,

(4) *if x, y and p all lie on some geodesic in X, then* $P(y, p, x) = P(y, x)$.

Proof. Let $w_i \to p$, with $w_i \in X$.

(1) By Lemma 2.1, we have $|P(x, y, w_i, x)| \leq Kd(x, y)$ for each i.

(2) For each i, $P(z, w_i, y, w_i, x) = P(z, w_i, x)$.

(3) If $w_i \notin \{x, y\}$, then $\angle(P(y, w_i, x)\overrightarrow{xw_i}, \overrightarrow{yw_i}) = x\hat{w}_i y$. As $i \to \infty$, we have $\overrightarrow{xw_i} \to \overrightarrow{xp}$, $\overrightarrow{yw_i} \to \overrightarrow{yp}$ and $x\hat{w}_i y \to 0$ (Proposition 2.2).

(4) We can choose each $w_i \in [x, p]$. Then $P(y, w_i, x) = P(y, x)$ for all i.

Examples. Let $X = \mathbf{H}^n$ be an n-dimensional hyperbolic space. Suppose $p \in X_i$. Any horosphere S about p is an $(n-1)$-dimensional Euclidean plane in the induced Riemannian metric. Suppose that $x \in S$, and that $v \in T_x^1 X$ is a unit vector, tangent to S, so that $\angle(v, \overrightarrow{xp}) = \pi/2$. Then, as y varies over S, the vectors $P(y, p, x)v$ define a Euclidean-parallel vector field on S.

More generally, suppose that X is one of the negatively curved symmetric spaces. Then, any horosphere S has the structure of a nilpotent Lie group with a left-invariant metric [8]. Let x, v be as above. Then, $\{P(y, p, x)v | y \in S\}$ is a left-invariant vector field on S.

We can now define our standard trivialization of the tangent bundle of X. By a *trivialization* of the tangent bundle, we mean a bundle isomorphism

$$\phi: X \times \mathbf{R}^n \to TX,$$

which respects the inner products on each of the fibers. For $x \in X$, we define

$$\phi_x: \mathbf{R}^n \to T_x X$$

by $\phi_x(\xi) = \phi(x, \xi)$. Then ϕ_x is a linear isometry.

Given x, $y \in X$, we define $D_\phi(y, x) \in O(n)$ by

$$D_\phi(y, x) = \phi_y^{-1} \circ P(y, x) \circ \phi_x.$$

Now, fix $p \in X_I$.

Definition. A trivialization $\phi: X \to \mathbf{R}^n \to TX$ is *standard* (with respect to p) if

$$\phi_y = P(y, p, x) \circ \phi_x$$

for all x, $y \in X$.

Now if ϕ is standard, from Lemma 3.2(1) it follows that $|D_\phi(y, x)| \leq Kd(x, y)$ for all x, $y \in X$. Also, if $y \in [x, p]$, then $D_\phi(y, x)$ is the identity in $O(n)$ (Lemma 3.2(4)). In fact, it is not hard to see that ϕ is standard precisely if these two conditions hold.

From Lemma 3.2(2), it is clear that for any $p \in X_I$, standard trivializations must exist. Moreover, if ϕ is standard, and T is any element of $O(n)$, then $\phi \circ (1_X, T)$ is also standard. Conversely, suppose that ϕ and ψ are both standard (with respect to p). Then, for all x, $y \in X$, we have $P(y, p, x) = \phi_y \phi_x^{-1} = \psi_y \psi_x^{-1}$. Thus, $\phi_x^{-1} \psi_x = \phi_y^{-1} \psi_y$, and so $\psi_x = \phi_x \circ T$ for some fixed $T \in O(n)$. In other words $\psi = \phi \circ (1_X, T)$. The rotation T is clearly unique for given ϕ and ψ.

We have defined $\mathrm{Isom}_p X$ earlier to be the subgroup of isometries of X fixing the point p. We are now in a position to define the *rotational part* of an element of $\mathrm{Isom}_p X$. Each $\gamma \in \mathrm{Isom}_p X$ induces a map $\gamma_*: TX \to TX$. If ϕ is a standard trivialization, then clearly $\gamma_* \circ \phi \circ (\gamma^{-1}, 1_{\mathbf{R}^n})$ is also.

Thus, there is a unique $T = O(n)$ such that $\gamma_* \circ \phi \circ (\gamma^{-1}, 1_{\mathbf{R}^n}) = \phi \circ (1_X, T)$. We write $\Theta_\phi(\gamma) = T$ so that we have

$$\gamma_* \circ \phi = \phi \circ (\gamma, \Theta_\phi(\gamma)).$$

Definition. Given $\gamma \in \operatorname{Isom}_p X$, we call $\Theta_\phi(\gamma)$ the *rotational part* of γ (with respect to ϕ).

From the formula defining $\Theta_\phi(\gamma)$ it is clear that this gives a homomorphism

$$\Theta_\phi\colon \operatorname{Isom}_p X \to O(n).$$

If we were to replace ϕ by a different standard trivialization, we would get another homomorphism, conjugate in $O(n)$ to Θ_ϕ. Thus, the norm $|\Theta_\phi(\gamma)|$ is defined independently of the choice of ϕ. It therefore makes sense to speak of an element of $\operatorname{Isom}_p X$ having "small rotational part".

The following statement is just a matter of unraveling the various definitions.

Lemma 3.3. *Suppose $p \in X_I$, and ϕ as a standard trivialization with respect to p. Suppose that $\gamma \in \operatorname{Isom}_p X$, and $x \in X$ and $\xi \in \mathbf{R}^n$. Let $v = \phi_x \xi \in T_x X$. Then*

$$\angle(v, P(x, \gamma x) \circ \gamma_* v) = \angle(D_\phi(\gamma x, x)\xi, \Theta_\phi(\gamma)\xi).$$

Proof. We have

$$P(x, \gamma x) \circ \gamma_* \circ \phi_x = \phi_x \circ D_\phi(x, \gamma x) \circ \Theta_\phi(\gamma).$$

Thus,

$$\begin{aligned}\angle(\phi_x\xi, P(x, \gamma x) \circ \gamma_*(\phi_x \xi)) &= \angle(\xi, D_\phi(x, \gamma x) \circ \Theta_\phi(\gamma)\xi) \\ &= \angle(D_\phi(\gamma x, x)\xi, \Theta_\phi(\gamma)\xi).\end{aligned}$$

Remark. Using Lemma 3.2(3), we could restrict attention to standard trivializations ϕ having the property that $\phi_x(0, 0, \cdots, 0, 1) = \overrightarrow{xp}$ for some (and hence all) $x \in X$. This reduces the rotational part of an element of $\operatorname{Isom}_p X$ to $O(n-1)$. However, we shall have no need to insist on this.

4. Reduction to the abelian case

Let $p \in X_I$. We have defined $\operatorname{Isom}_p X$ as the subgroup of isometries of X fixing p. The main result of this paper is the following.

Theorem 4.1. *Any discrete subgroup of* $\operatorname{Isom}_p X$ *is finitely generated.*

We showed in §2 that the only interesting case is that where this subgroup, Γ, is a discrete parabolic group. It is the aim of this section to reduce further to the case where Γ is abelian, in fact, isomorphic to a subgroup of $\mathbf{Q}^n$. The proof of Theorem 4.1 will be completed in §5.

We first recall some basic facts about nilpotent groups.

A group N is said to be *nilpotent* if, for some m, all m-fold commutators in N are trivial. The smallest such m is called the *class* of N. We define normal subgroups, Z_i, of N inductively as follows.

$$Z_0 = \{1\},$$
$$Z_{i+1} = \{y \in N | xyx^{-1}y^{-1} \in Z_i \text{ for all } x \in N\}.$$

This gives the *upper central series*

$$\{1\} = Z_0 \triangleleft Z_1 \triangleleft Z_2 \triangleleft \cdots \triangleleft Z_m = N,$$

where $m = \operatorname{class} N$. It is easy to see that any subgroup or quotient of a nilpotent group is nilpotent.

We quote the following results.

Proposition 4.2 [12]. *Suppose that N is nilpotent. If x, $y \in N$ both have finite order, then xy also has finite order.*

Thus, $T(N) = \{x \in N | x^r = 1 \text{ for some } r\}$ is a normal subgroup of N.

Proposition 4.3. *Suppose that T is a finitely generated nilpotent torsion group (every element has finite order). Then T is finite.*

Proof. We can assume, by induction on the class of T, that the commutator subgroup $[T, T]$ is finite. The quotient $T/[T, T]$ is a finitely generated abelian torsion group, and hence also finite.

Proposition 4.4 [12]. *Let N be a countable nilpotent group with upper central series $(Z_i)_{i=1}^m$. If the centre Z_1 is free abelian, then Z_i/Z_{i-1} is free abelian for all $i \in \{2, 3, \cdots, m\}$.*

In such a case, we define

$$\operatorname{rank} N = \sum_{i=1}^{m} \operatorname{rank}(Z_i/Z_{i-1}).$$

Clearly, class $N \leq \operatorname{rank} N$. Moreover, $\operatorname{rank} N$ is finite if and only if N is finitely generated.

The following is a theorem of Mal′cev.

Proposition 4.5 [10] **or** [5]. *Let N be a finitely generated torsion-free nilpotent group. Then N may be embedded as a discrete cocompact subgroup of a torsion-free nilpotent Lie group, G, of dimension $r = \operatorname{rank} N$.*

Proposition 4.6 [9]. *Let G be a torsion-free nilpotent Lie group. Then, the exponential map at the identity is a diffeomorphism of the Lie algebra onto G.*

(In fact the construction in [5] gives an explicit diffeomorphism of G with $\mathbf{R}^r$.) In particular, G is contractible. Propositions 4.5 and 4.6 together tell us that $\operatorname{rank} N$ is equal to the cohomological dimension of N. We deduce

Proposition 4.7. *Suppose that the nilpotent group N has free-abelian centre and acts properly discontinuously on some contractible n-manifold, then* $\operatorname{rank} N \leq n$.

Proof. We need only check that $\operatorname{rank} N$ is finite, or equivalently that N is finitely generated. However this is clear, since otherwise N would contain subgroups of arbitrarily large finite rank. q.e.d.

We now return to our manifold X. We want to define a subset of $\operatorname{Isom}_p X$ consisting of those elements of small rotational part.

Given any $\theta > 0$, we write

$$\mathscr{I}_p(\theta) = \{\gamma \in \operatorname{Isom}_p X \mid |\Theta_\phi(\gamma)| \leq \theta\},$$

where Θ_ϕ is the rotational part homomorphism defined in §3. As remarked in §3, $\mathscr{I}_p(\theta)$ is defined independently of our choice of trivializations ϕ.

Given any subset, Q, of a group G, we shall use the notation $\langle Q \rangle$ to denote the subgroup of G generated by Q.

Lemma 4.8. *For all $n \in \mathbf{N}$ and $\theta > 0$, there is $\nu = \nu(n, \theta) \in \mathbf{N}$ such that if G is any subgroup of* $\operatorname{Isom}_p X$, *then $\langle G \cap \mathscr{I}_p(\theta) \rangle$ has index at most ν in G.*

Proof. This is just group theory, so the argument is standard.

We have a homomorphism $\Theta: G \to H$, where H is a compact group, in this case $O(n)$. Now, $G' = \langle G \cap \mathscr{I}_p(\theta) \rangle = \langle g \in G | \Theta(g) \in U \rangle$, where $U = U_\theta = \{A \in O(n) | \, |A| \leq \theta\}$ is a neighborhood of the identity in H.

Let V be another neighborhood of the identity in H with $V^{-1}V \subseteq U$ (for example, take $V = U_{\theta/2}$). There is an upper bound ν on the number of right translates of V that can be packed disjointly into H. The number ν depends on U and H, and thus on θ and n. It is independent of G.

Now let $r \in \mathbf{N}$ be maximal such that we can find $g_1, g_2, \cdots, g_r \in G$ with the translates $V(\Theta(g_i))$ disjoint for $i \in \{1, 2, \cdots, r\}$. Clearly $r \leq \nu$. Now let g be any element of G. By maximality, $V(\Theta(g))$ must intersect $V(\Theta(g_i))$ for some $i \in \{1, 2, \cdots, k\}$. It follows that $\Theta(gg_i^{-1}) \in V^{-1}V \subseteq U$. Thus $gg_i^{-1} \in G'$ and so $g \in G'g_i$. We have shown that $\{Gg_i | 1 \leq i \leq r\}$ is a complete set of cosets for G' in G. Thus $[G : G'] \leq r \leq \nu$. q.e.d.

Now, let $\theta_0 = \omega/2$, where ω is the constant in the Margulis Lemma (2.3). Thus θ_0 depends only on n and κ. In view of Lemma 4.8, in order to prove the main theorem (4.1), we can restrict attention to discrete parabolic groups, Γ, which are generated by the subset $\Gamma \cap \mathcal{I}_p(\theta_0)$. We aim to show that such a group is finitely generated and nilpotent. We begin with the following.

Lemma 4.9. *If Γ is a discrete parabolic subgroup of $\mathcal{I}_p(\theta_0)$, with $\Gamma = \langle \Gamma \cap \mathcal{I}_p(\theta_0) \rangle$, then Γ is locally nilpotent, i.e., every finitely generated subgroup of Γ is nilpotent.*

Proof. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_r\}$ be a finite subset of $\Gamma \cap \mathcal{I}_p(\theta_0)$. Choose any $y \in X$. The geodesics $\gamma_i[y, p]$, $i \in \{1, 2, \cdots, r\}$, and $[y, p]$ itself, all converge exponentially. Thus, we can find some $x \in [y, p]$ such that, for all i, we have $d(x, \gamma_i x) \leq \min(\varepsilon, \theta_0/K)$, where $\varepsilon = \varepsilon(n)$ comes from the Margulis Lemma (2.3) and K comes from Lemma 3.2(1).

Now, choose any $\xi \in \mathbf{R}^n$, and set $v = \phi_x \xi$. By Lemma 3.3, for each i,

$$\angle(v, P(x, \gamma_i x) \circ \gamma_{i*} v) = \angle(D_\phi(\gamma_i x, x)\xi, \Theta_\phi(\gamma_i)\xi).$$

But $|\Theta_\phi(\gamma_i)| \leq \theta_0$, and by hypothesis and Lemma 3.2(1) we have

$$|D_\phi(\gamma_i x, x)| \leq K d(x, \gamma_i x) \leq K(\theta_0/K) = \theta_0.$$

Thus

$$\angle(D_\phi(\gamma_i x, x)\xi, \Theta_\phi(\gamma_i)\xi) \leq 2\theta_0 = \omega.$$

It follows that $\langle \gamma_1, \gamma_2, \cdots, \gamma_r \rangle$ is a subgroup of the nilpotent group $\Gamma(x)$ as defined in the Margulis Lemma. Thus, $\langle \gamma_1, \gamma_2, \cdots, \gamma_r \rangle$ is nilpotent.

Now, any finitely generated subgroup of Γ lies inside some subgroup of the form $\langle \gamma_1, \gamma_2, \cdots, \gamma_r \rangle$ for some r, and is thus itself nilpotent. q.e.d.

The next step will be to reduce to the case where Γ is torsion-free.

Let T be the set of elements of finite order in Γ. Let x and y be any elements of T. Then $\langle x, y \rangle$ is nilpotent. By Proposition 4.2, xy^{-1} has finite order. We deduce that T is a subgroup of Γ. Clearly, T is normal.

In summary, we know that T is a discrete torsion group in which every finitely generated subgroup is nilpotent. This implies that T is finite, as follows. Clearly T is countable, so we may write $T = \bigcup_{r=0}^{\infty} T_r$, where each T_r is a finitely generated subgroup of T, and $T_r \subseteq T_s$ for $r \leq s$. By Proposition 4.3, each T_r is, in fact, finite. Thus, its set of fixed points, $\operatorname{fix} T_r$, meets X in some nonempty totally geodesic subspace. Clearly, if $r \leq s$, then $\operatorname{fix} T_s \subseteq \operatorname{fix} T_r$. Thus, the dimension of $\operatorname{fix} T_r$ must stabilize, and so $\operatorname{fix} T = \bigcap_{r=0}^{\infty} \operatorname{fix} T_r$ must meet X. It follows that T is finite.

Now, $\sigma = X \cap \operatorname{fix} T$ is a totally geodesic subspace of X. Since T is a well-defined subgroup of Γ, we see that σ is Γ-invariant. Moreover, it is easily checked that T is precisely the pointwise stabilizer of σ. Thus, Γ/T acts freely and properly discontinuously on σ. Clearly, the notion of a horosphere is preserved by taking subspaces, and so we see that Γ/T acts intrinsically as a discrete parabolic group on σ. Thus, replacing Γ by Γ/T, and X by σ, we are reduced to the case where Γ acts freely on X, i.e., Γ is torsion free.

As before, let n be the dimension of X. Now, any set of n elements $\gamma_1, \gamma_2, \cdots, \gamma_n$ of Γ generate a torsion-free nilpotent group. By Proposition 4.7, this group has rank at most n. It follows that any n-fold commutator in the γ_i is trivial. Since these elements were chosen arbitrarily, we see that Γ is nilpotent.

We want to show that Γ is finitely generated. By Propositions 4.4 and 4.7, it is enough to show that Γ has free-abelian centre. We are thus reduced to considering torsion-free abelian groups. In fact, we may reduce Γ to a subgroup of the additive group $\mathbf{Q}^n$ as follows.

Let G be a torsion-free abelian group. Then, the tensor product $G \otimes \mathbf{Q}$ over the integers is also torsion-free. In fact, $G \otimes \mathbf{Q}$ has naturally the structure of a vector space over $\mathbf{Q}$. We may identify G with its image $G \otimes \mathbf{Q}$ under the injection $[g \mapsto g \otimes 1]$. Under this identification, $G \otimes \mathbf{Q}$ is spanned as a vector space by the elements of G. Thus, we may find a basis of $G \otimes \mathbf{Q}$ consisting entirely of elements of G. The subgroup H of G generated by all the elements in one such basis will be a free abelian group of rank equal to $\dim_{\mathbf{Q}}(G \otimes \mathbf{Q})$. Moreover, G/H will be a torsion group.

Suppose now that G acts freely and properly discontinuously on a contractable n-manifold. Then, from Proposition 4.7 we deduce that $\dim_{\mathbf{Q}}(G \otimes \mathbf{Q}) \leq n$. Thus, we may regard G as a subgroup of $\mathbf{Q}^n$.

In §5, we shall show that a discrete abelian parabolic group, G, is necessarily free abelian. However, if $n = \dim X \leq 3$, a simple topological argument will suffice.

If $\dim X = 2$, then clearly the quotient of any horosphere must be a topological circle, and so G is infinite cyclic. Suppose that $\dim X = 3$, and let S be some horosphere about p. Thus, S is homeomorphic to $\mathbf{R}^2$. We see that $F = S/\Gamma$ is a surface with $\pi_1 F = G$. However, surfaces with abelian fundamental groups are easily classified, and they are each topologically finite, i.e., homeomorphic to the interior of a compact manifold with boundary. (The essential point is that any surface admits a compact exhaustion by subsurfaces, F_i, whose fundamental groups inject. Thus each $\pi_1 F_i$ is abelian. Now any compact surface with abelian fundamental group is one of just a few possibilities, and so it is readily seen that the homeomorphism types of the F_i must stabilize.)

However, we cannot hope for a purely topological argument in higher dimensions. For example, we describe below a free, properly discontinuous action of the diadic rationals, $\mathbf{Z}[\frac{1}{2}] = \{h/2^k \mid h, k \in \mathbf{Z}\}$, on $\mathbf{R}^3$. This type of example is well known.

Let $N \equiv S^1 \times D^2$ be a solid torus, and let $\imath: N \to S^3$ be a standard embedding of N in the 3-sphere. Thus the closure of $S^3 \setminus N$ is also a solid torus. Let $f: N \to N$ be an embedding of N in itself, so that fN is the tubular neighborhood of a 2-strand closed braid, with one half-twist. Thus $f_* \pi_1 N$ has index 2 in $\pi_1 N$, and $\imath f N$ is unknotted in S^3. Let $M = S^3 \setminus \bigcap_{r=0}^{\infty} \imath f^r N$. It is easily seen that $\pi_1 M \equiv \mathbf{Z}[\frac{1}{2}]$. Moreover one can check that the universal cover $\widetilde{M}$ of M has the engulfing property, i.e., every compact set lies inside some 3-ball. Thus using the 3- dimensional annulus theorem, and the fact that an orientation-preserving diffeomorphism of the 2-sphere is isotopic to the identity, we see that $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$. Now $\mathbf{Z}[\frac{1}{2}]$ acts on $\widetilde{M} \equiv \mathbf{R}^3$ by covering translations. It is an interesting exercise to give an explicit description of this action on $\mathbf{R}^3$.

5. Abelian parabolic groups

The aim of this section is to complete the proof of the main theorem (4.1) by showing that every discrete abelian parabolic group is finitely generated. We begin with a discussion of abelian subgroups of $O(n)$.

We write $\langle \cdot, \cdot \rangle$ for the standard inner product on $\mathbf{R}^n$, so that $\langle \xi, \zeta \rangle = \cos \angle(\xi, \zeta)$ for all unit vectors $\xi, \zeta \in S^{n-1} \subseteq \mathbf{R}^n$. Complexifying, we can extend this to the standard Hermitian form on $\mathbf{C}^n$, i.e., $\langle z, w \rangle = z_1 \overline{w}_1 + z_2 \overline{w}_2 + \cdots + z_n \overline{w}_n$.

Suppose G is an abelian subgroup of $O(n)$. Then G acts on $\mathbf{C}^n$ preserving the Hermitian form. Now, $\mathbf{C}^n$ can be split as a direct sum, $\mathbf{C}^n = \bigoplus_{i=0}^k W_i$, of common eigenspaces of the elements of G. If we take the number, k, of common eigenspaces to be minimal, then this splitting is canonical (up to permutation of factors). In this case the eigenspaces are mutually orthogonal, and are paired under complex conjugation. An eigenspace is paired with itself if and only if all its eigenvalues (as g ranges over G) are equal to ± 1. We shall assume that W_0 is the eigenspace $\{z \in \mathbf{C}^n \mid gz = z \text{ for all } g \in G\}$, even though this may be trivial (all the other eigenspaces are assumed nontrivial). We can also assume that the set $\{W_0, W_1, \cdots, W_q\}$ contains precisely one eigenspace from each pair. For $i \in \{0, 1, \cdots, q\}$, let V_i be the subspace of $\mathbf{R}^n$ given by $V_i = \{z + \overline{z} | z \in W_i\}$. We may check the following.

Lemma 5.1. *The spaces V_i for $0 \leq i \leq q$ give a G-invariant splitting of $\mathbf{R}^n$ $(gV_i = V_i$ for all $g \in G)$. We have $V_0 = \{\xi \in \mathbf{R}^n | g\xi = \xi$ for all $g \in G\}$. Moreover, given $i \in \{0, 1, \cdots, q\}$, and $\xi, \zeta \in V_i \cap S^{n-1}$, we have $\angle(\xi, g\xi) = \angle(\zeta, g\zeta)$ for all $g \in G$.*

We extract the geometric information which we need from this result in the following lemma.

Lemma 5.2. *For any given abelian subgroup, G of $O(n)$, there exists some finite subset $\{g_1, g_2, \cdots, g_q\}$ of elements of G with the following property. Given any $\varepsilon > 0$, there is some $\eta > 0$ such that if $\xi \in S^{n-1}$ satisfies $\angle(\xi, g_i\xi) \leq \eta$ for each $i \in \{1, 2, \cdots, q\}$, then $\angle(\xi, g\xi) \leq \varepsilon$ for all $g \in G$.*

Proof. Let $\mathbf{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_q$ be the splitting given by Lemma 5.1. Then, given any $i \in \{1, 2, \cdots, q\}$, there is some $g_i \in G$ with $g_i\xi \neq \xi$ for some (and hence all) $\xi \in V_i \cap S^{n-1}$. Set $k_i = 1 - \langle \xi, g_i\xi \rangle$. Thus, $k_i > 0$, and is defined independently of the choice of $\xi \in V_i \cap S^{n-1}$. Let $k = \min\{k_i | 1 \leq i \leq q\}$.

Suppose we are given an arbitrary $\xi \in S^{n-1}$. We want to show that if $\langle \xi, g_i\xi \rangle$ is close to 1 for all $i \in \{1, 2, \cdots, q\}$, then $\langle \xi, g\xi \rangle$ is close to 1 for all $g \in G$. So suppose $\langle \xi, g_i\xi \rangle \geq 1 - \mu$ for all $i \in \{1, 2, \cdots, q\}$.

Write $\xi = \sum_{i=0}^q a_i\xi_i$, where $a_i \in \mathbf{R}$ and $\xi_i \in V_i$ for all i. Since $\xi \in S^{n-1}$, we have

$$\sum_{i=0}^{q} a_i^2 = 1.$$

Now, if $i \geq 1$, then

$$\begin{aligned} 1 - \mu \leq \langle \xi, g_i\xi \rangle &= \sum_{j=0}^{q} a_j^2 \langle \xi_j, g_i\xi_j \rangle \\ &\leq \sum_{j \neq i} a_j^2 + a_i^2(1 - k_i) = 1 - k_i a_i^2. \end{aligned}$$

Thus $a_i^2 \leq \mu/k_i \leq \mu/k$ for all $i \geq 1$.

Now, given any $g \in G$,

$$\begin{aligned}\langle \xi , g\xi \rangle &= \sum_{i=0}^{q} a_i^2 \langle \xi_i , g\xi_i \rangle \\ &\geq a_0^2 - \sum_{i=1}^{q} a_i^2 = 1 - 2\sum_{i=1}^{q} a_i^2 \\ &\geq 1 - 2q\mu/k .\end{aligned}$$

Given any $\varepsilon > 0$, define $\mu > 0$ by $\cos\varepsilon = 1 - 2q\mu/k$. Then $\eta > 0$ is given by $\cos\eta = 1 - \mu$. Hence the lemma follows. q.e.d.

We shall use Lemma 5.2 in the following form.

Lemma 5.3. *Let Γ be a subgroup of $\mathbf{Q}^m$, which spans $\mathbf{Q}^m$ as a vector space. Suppose that $\Psi\colon T \to O(n)$ is any homomorphism. Then, we can find $e_1, e_2, \cdots, e_m \in \Gamma$ which form a vector space basis for $\mathbf{Q}^m$ and have the following property. Given any $\varepsilon > 0$, there is some $\delta > 0$ such that for any $\xi \in S^{n-1}$ with $\angle(\xi, \Psi(e_i)\xi) \leq \delta$ for all $i \in \{1, 2, \cdots, m\}$, we have $\angle(\xi, \Psi(g)\xi) \leq \varepsilon$ for all $g \in \Gamma$.*

Proof. Let $G = \Psi(\Gamma)$ be the image of Γ in $O(n)$. We can find $\{g_1, g_2, \cdots, g_q\} \subseteq \Gamma$ so that $\{\Psi(g_1), \Psi(g_2), \cdots, \Psi(g_q)\} \subseteq O(n)$ has the property expressed in Lemma 5.2. Now, $\langle g_1, g_2, \cdots, g_q\rangle$ is free abelian. Let $\{e_1, e_2, \cdots, e_d\}$ be a free set of generators for this subgroup of Γ. Let N be the maximal word length of any of the g_i expressed in terms of $\{e_1, e_2, \cdots, e_d\}$. Now, given any $\varepsilon > 0$, let $\eta > 0$ be as in Lemma 5.2, and let $\delta = \eta/N$. Suppose we are given $\xi \in S^{n-1}$ such that, for each $i \in \{1, 2, \cdots, d\}$,

$$\angle(\xi, \Psi(e_i)\xi) \leq \delta ;$$

then for each $j \in \{1, 2, \cdots, q\}$, we have

$$\angle(\xi, \Psi(g_i)\xi) \leq N\delta \leq \eta .$$

Thus, by Lemma 5.2,

$$\angle(\xi, \Psi(g)\xi) \leq \varepsilon$$

for each $g \in \Gamma$.

Now, $e_1, e_2, \cdots, e_d$ are linearly independent over $\mathbf{Q}$, so we may extend arbitrarily to a basis $\{e_1, e_2, \cdots, e_m\} \subseteq \Gamma$ of $\mathbf{Q}^m$. q.e.d.

We stated at the beginning that the aim of this section is to show that discrete torsion-free abelian parabolic groups are finitely generated. In order to give the idea of the proof, we begin by considering a discrete abelian group, Γ, acting freely on $\mathbf{E}^n$. We want to show that Γ is free abelian. Of course, this is an immediate consequence of the Bieberbach Theorem (1.1). However, a proof along these lines would make essential use of the existence of totally geodesic subspaces of $\mathbf{E}^n$. Below, we give a proof that avoids this, and can be generalized to an argument for variable curvature.

As described in §4, we can regard Γ as a subgroup of $\mathbf{Q}^m$, with $\mathbf{Q}^m$ generated as a vector space by the elements of Γ. We shall, however, continue to use multiplicative notation for the group operations in Γ.

Let ϕ be a standard trivialization of the tangent bundle of $\mathbf{E}^n$, and let $\Theta = \Theta_\phi \colon \operatorname{Isom} \mathbf{E}^n \to O(n)$ be the corresponding rotational part homomorphism (see §1).

Let $\{e_1, e_2, \cdots, e_m\}$ be as in Lemma 5.3 (with $\Psi = \Theta$). Let H be the subgroup of Γ generated by $\{e_1, e_2, \cdots, e_m\}$. Thus, Γ/H is a torsion group. Our aim is to show that Γ/H is finite.

Take $\varepsilon = \pi/8$, and let $\delta > 0$ be as in Lemma 5.3. Choose any point $x \in \mathbf{E}^n$, and let

$$\lambda = \max\{d(x, e_i x) | 1 \leq i \leq m\}.$$

Let $l_0 = \frac{2}{3}\lambda \operatorname{cosec}(\delta/8)$, so that in any triangle, abc, with $d(a, b) \geq \frac{3}{4}l_0$, $d(a, c) \geq \frac{3}{4}l_0$ and $d(b, c) \leq \lambda$, we have $b\hat{a}c \leq \delta/4$. Let $l = \max(4m\lambda, l_0)$.

We claim that, for any $g \in \Gamma$, we have

$$gHx \cap N_l(x) \neq \varnothing.$$

In other words, each coset, gH, of H contains some element, h, with $d(x, hx) \leq l$. Since Γ acts properly discontinuously, it will follow that there are only finitely many such cosets.

So, suppose, for contradiction, that there is some $g_0 \in \Gamma$ with $g_0Hx \cap N_l(x) = \varnothing$. Let G be the subgroup of Γ generated by $H \cup \{g_0\}$. Thus G/H is finite cyclic.

Let $r = \max\{d(x, gHx) | g \in G\}$. Then $r > l$. Choose $k \in G$ such that $d(x, kHx) = r$. We must have $d(x, k^2Hx) \leq r$, and so there is some $h_0 \in H$ with $d(x, k^2h_0x) \leq r$.

Let $H' = \{h^2 | h \in H\}$. Then H' is a subgroup of index 2^m in H. Since $d(x, e_ix) \leq \lambda$ for each $i \in \{1, 2, \cdots, m\}$, every point in the orbit Hx lies within a distance $m\lambda$ of the orbit $H'x$. In particular, we can find $h \in H$ so that $d(h_0x, h^2x) \leq m\lambda$, and $d(k^2h_0x, k^2h^2x) \leq m\lambda$. Thus, $d(x, \gamma^2x) \leq r + m\lambda \leq r + \frac{r}{4} = \frac{5}{4}r$, where $\gamma = kh \in G$. Since $\gamma \in kH$, $d(x, \gamma x) \geq r$.

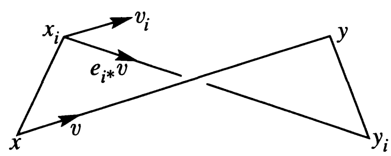

FIGURE 2

Now fix, for the moment, $i \in \{1, 2, \cdots, m\}$, and consider the four points x, $y = \gamma x$, $x_i = e_i x$ and $y_i = \gamma e_i x = e_i \gamma x$. Then we have

$$d(x, x_i) = d(y, y_i) \leq \lambda$$

and

$$d(x, y) = d(x_i, y_i) \geq r \geq l.$$

Thus, $d(x, y_i) \geq r - \lambda \geq r - \frac{r}{4} = \frac{3}{4}r$. We chose l large in relation to λ, so that we must have $y\hat{x}y_i \leq \delta/4$ and $x\hat{y}_i x_i \leq \delta/4$ (see Figure 2).

Let $v = \overrightarrow{xy}$, so that $e_{i*}v = \overrightarrow{x_i y_i}$. Let v_i be the parallel transport of v to the point x_i. Then, from the definition of rotational part, we have

$$\angle(\xi, \Theta(e_i)\xi) = \angle(v_i, e_{i*}v),$$

where $\xi = \phi_x^{-1}v = \phi_{x_i}^{-1}v_i \in S^{n-1}$. But $\angle(v_i, e_{i*}v) \leq y\hat{x}y_i + x\hat{y}_i x_i \leq \delta/4 + \delta/4 = \delta/2 < \delta$.

We have shown that

$$\angle(\xi, \Theta(e_i)\xi) \leq \delta$$

for each $i \in \{1, 2, \ldots, m\}$. Applying Lemma 5.3, we get that $\angle(\xi, \Theta(g)\xi) \leq \varepsilon = \pi/8$ for all $g \in \Gamma$. In particular, $\angle(\xi, \Theta(\gamma)\xi) \leq \pi/8$.

Now, let $z = \gamma^2 x$, so that $\gamma_* v = \overrightarrow{yz}$. The element γ was chosen so that $d(x, z) \leq \frac{5}{4}r$. Let $v' = \phi_y(\xi) \in T_y^1\mathbf{E}^n$, i.e., v' is the parallel transport of $v = \overrightarrow{xy}$ to y. From the definition of rotational part, we see that

$$\angle(-\overrightarrow{yx}, \overrightarrow{yz}) = \angle(v', \gamma_* v) = \angle(\xi, \Theta(\gamma)\xi) \leq \pi/8.$$

Thus $x\hat{y}z = \pi - \angle(-\overrightarrow{yx}, \overrightarrow{yz}) \geq \pi - \frac{\pi}{8} > \frac{3}{4}\pi$.

In summary, we have deduced that $d(x, y) = d(y, z) \geq r$, $d(x, z) \leq \frac{5}{4}r$ and $x\hat{y}z \geq \frac{3}{4}\pi$. Simple trigonometry shows that this is impossible.

We have contradicted the existence of $g_0 \in \Gamma$ with $d(x, g_0 Hx) \geq l$. This concludes the proof that Γ is free abelian.

The idea now is to give a similar argument in the case where Γ is a discrete abelian parabolic group acting freely on X. Let us consider the ingredients of the proof which we have just given. We have used an identification of the tangent spaces at different points, as well as the notion of the rotational part of an isometry. There are already provided for in X (§3). Our use of trigonometry allowed for some margin of error, and thus could be made to work in some perturbation of the metric. The idea therefore will be to carry out the argument in a small neighborhood of a suitably chosen point of X, where the metric can be assumed to be almost Euclidean.

Proposition 5.4. *Let Γ be a subgroup of $\mathbf{Q}^m$ (with multiplicative notation). If Γ acts as a discrete parabolic group on X, then Γ is free abelian.*

Proof. Let $\phi\colon X \times \mathbf{R}^n \to TX$ be a standard trivialization of the tangent bundle of X with respect to the fixed point p. Let $\Theta = \Theta_\phi\colon \Gamma \to O(n)$ be the rotational part homomorphism.

We can assume that $\mathbf{Q}^m$ is generated as a vector space by Γ. Let $\{e_1, e_2, \ldots, e_m\}$ be as in Lemma 5.3 (with $\Psi = \Theta$). Let $H = \langle e_1, e_2, \ldots, e_m\rangle \subseteq \Gamma$. Thus Γ/H is torsion, and we want to show that it is finite.

Let K be the constant of Lemmas 2.1 and 3.2(1), so that $|D_\phi(y, x)| \le Kd(x, y)$ for all $x, y \in X$. Let $l = \frac{1}{8}\pi/K$. Take $\varepsilon = \pi/8$, and let $\delta > 0$ be as in Lemma 5.3. Let $\eta_0 = \frac{3}{2}l\sin(\delta/8)$, so that in any Euclidean triangle abc with $d(a, b) \ge \frac{3}{4}l$, $d(a, c) \ge \frac{3}{4}l$ and $d(b, c) \le \eta_0$, we have $b\hat{a}c \le \delta/4$. Let $\eta = \min(\eta_0, \frac{1}{4}l/m, \frac{1}{4}\delta/K)$.

From the exponential convergence of geodesic rays, it is clear that we can find a point $a \in X$ with $d(a, e_i a) \le \eta$ for each $i \in \{1, 2, \cdots, m\}$. We claim that $gHa \cap N_l(a) \ne \varnothing$ for all $g \in \Gamma$. By discreteness, this will prove that Γ/H is finite.

Let $\beta\colon [0, \infty] \to X_C$ be the geodesic $[a, p]$, parametrized by arc-length. Thus, $\beta(0) = a$, and $\beta(\infty) = p$. For any $g \in \Gamma$, $d(\beta(t), g\beta(t))$ tends monotonically to 0 as t tends to ∞. Given any subset Δ of Γ, and $t \in [0, \infty)$, define

$$\begin{aligned} M(\Delta, t) &= d(\beta(t), \Delta\beta(t)) \\ &= \min\{d(\beta(t), g\beta(t)) | g \in \Delta\}. \end{aligned}$$

This minimum is attained since Γ acts properly discontinuously. Clearly, $M(\Delta, t)$ tends monotonically to 0 as $t \to \infty$. Moreover $M(\Delta, t)$ is continuous in t. To see this, fix $t_0 \ge 0$, and note that if $|t - t_0| \le 1$, then any $g \in \Delta$ which minimizes $d(\beta(t), g\beta(t))$ must satisfy $d(\beta(t_0), g\beta(t_0)) \le M(\Delta, t_0) + 2$. There are only finitely many such g, and for each, we have that $d(\beta(t), g\beta(t))$ is continuous in t.

We want to show that gHa meets $N_l(a)$ for each $g \in \Gamma$. So suppose, for contradiction, that there is some $g_0 \in \Gamma$ with $g_0Ha \cap N_l(a) = \varnothing$, i.e,

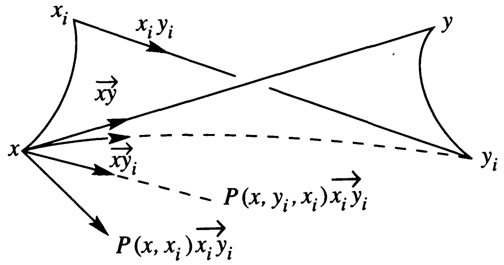

FIGURE 3

$M(g_0H, 0) > l$. Let G be the subgroup of Γ generated by $H \cup \{g_0\}$, so that G/H is finite cyclic.

Now, $L(t) = \max\{M(gH, t) | g \in G\}$ is continuous and tends monotonically to 0 as $t \to \infty$. Thus, there is some $t_0 \in [0, \infty)$ with $L(t_0) = l$. Let $b = \beta(t_0)$. Choose $k \in G$ such that $d(b, kHb) = l$. Now, $d(b, gHb) \leq l$ for all $g \in G$. In particular $d(b, k^2Hb) \leq l$, and so there is some $h_0 \in H$ with $d(b, k^2h_0b) \leq l$.

We chose $a \in X$ so that $d(a, e_ia) \leq \eta$ for each $i \in \{1, 2, \cdots, m\}$. Since $b \in [a, p]$, we have also $d(b, e_ib) \leq \eta$. Let $H' = \{h^2 | h \in H\}$, so that H' is a subgroup of index 2^m in H. Clearly, each point of the orbit Hb lies within a distance $m\eta$ of the orbit $H'b$. In particular, there is some $h \in H$ with $d(h_0b, h^2b) \leq m\eta$, and so $d(k^2h_0b, k^2h^2b) \leq m\eta$. Setting $\gamma = kh$, we see that $d(b, \gamma^2b) \leq l + m\eta \leq l + \frac{l}{4} = \frac{5}{4}l$. Now, $\gamma \in kH$ and $d(b, kHb) = l$. Thus $d(b, \gamma b) \geq l$.

Since $d(\beta(t), \gamma\beta(t))$ and $d(\beta(t), \gamma^2\beta(t))$ both tend monotonically to 0, we may find $x \in [b, p]$ with $d(x, \gamma x) = l$ and $d(x, \gamma^2 x) \leq \frac{5}{4}l$.

Let us fix, for the moment, $i \in \{1, 2, \cdots, m\}$ and consider the four points x, $y = \gamma x$, $x_i = e_ix$ and $y_i = \gamma e_ix = e_i\gamma x$ (see Figure 3). We have

$$d(x, x_i) = d(y, y_i) \leq \eta$$

and

$$d(x, y) = d(x_i, y_i) = l.$$

Thus $d(x, y_i) \geq l - \eta \geq l - \frac{l}{4m} \geq \frac{3}{4}l$. We chose $\eta \leq \eta_0 = \frac{3}{2}\sin(\delta/8)$, and so, applying the comparison theorem (Proposition 2.2) to the triangle yxy_i, we find that $y\hat{x}y_i \leq \delta/4$. Similarly, $x\hat{y}_ix_i \leq \delta/4$.

Consider now the four unit vectors $\overrightarrow{xy}$, $\overrightarrow{xy}_i$, $P(x, y_i, x_i)\overrightarrow{x_iy_i}$ and $P(x, x_i)\overrightarrow{x_iy_i}$, based at x. We have $\angle(\overrightarrow{xy}, \overrightarrow{xy_i}) = y\hat{x}y_i \leq \delta/4$ and

$\angle(\overrightarrow{xy_i}, P(x, y_i, x_i)\overrightarrow{x_iy_i}) = x\hat{y}_ix_i \leq \delta/4$. From Lemma 2.1 it follows that $|P(x_i, x, y_i, x_i)| \leq Kd(x, x_i) \leq K\eta \leq K(\delta/4K) = \delta/4$. Thus

$$\angle(P(x, y_i, x_i)\overrightarrow{x_iy_i}, P(x, x_i)\overrightarrow{x_iy_i}) = \angle(\overrightarrow{x_iy_i}, P(x_i, x, y_i, x_i)\overrightarrow{x_iy_i}) \leq \delta/4,$$

and therefore $\angle(\overrightarrow{xy}, P(x, x_i)\overrightarrow{x_iy_i}) \leq 3\delta/4$.

Let $v = \overrightarrow{xy}$, so that $\overrightarrow{x_iy_i} = e_{i*}v$. Let $\xi = \phi_x^{-1}v \in S^{n-1}$. Lemma 3.3 tells us that

$$\angle(D_\phi(x_i, x)\xi, \Theta(e_i)\xi) = \angle(v, P(x, x_i) \circ e_{e*}v) \leq 3\delta/4.$$

From Lemma 3.2(1), we have that

$$|D_\phi(x_i, x)| \leq Kd(x, x_i) \leq K\eta \leq K(\delta/4K) = \delta/4.$$

Thus, $\angle(\xi, D_\phi(x_i, x)\xi) \leq \delta/4$, and so

$$\angle(\xi, \Theta(e_i)\xi) \leq \delta/4 + 3\delta/4 = \delta.$$

Since this is true for each $i \in \{1, 2, \cdots, m\}$, applying Lemma 5.3, we get that $\angle(\xi, \Theta(g)\xi) \leq \varepsilon = \frac{\pi}{8}$ for all $g \in \Gamma$. In particular, $\angle(\xi, \Theta(\gamma)\xi) \leq \frac{\pi}{8}$.

From Lemma 3.2(1) it follows that

$$|D_\phi(y, x)| \leq Kd(x, y) = Kl = K(\pi/8K) = \frac{\pi}{8},$$

so that $\angle(\xi, D_\phi(y, x)\xi) \leq \frac{\pi}{8}$. Thus $\angle(D_\phi(y, x)\xi, \Theta(\gamma)\xi) \leq \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$, together with Lemma 3.3 means that

$$\angle(v, P(x, y) \circ \gamma_* v) \leq \frac{\pi}{4}.$$

We defined $v = \overrightarrow{xy}$, and so $\gamma_* v = \overrightarrow{yz}$ where $z = \gamma^2 x$. Thus

$$\angle(-\overrightarrow{yx}, \overrightarrow{yz}) = \angle(P(y, x)\overrightarrow{xy}, \overrightarrow{yz}) = \angle(v, P(x, y) \circ \gamma_* v) \leq \frac{\pi}{4},$$

and so

$$x\hat{y}z = \pi - \angle(-\overrightarrow{yx}, \overrightarrow{yz}) \geq \pi - \frac{\pi}{4} = \frac{3}{4}\pi.$$

However, we have $d(x, y) = d(y, z) = l$ and $d(x, z) \leq \frac{5}{4}l$. Applying the comparison theorem (Proposition 2.2), we find that

$$x\hat{y}z \leq 2\sin^{-1}(5/8) < \frac{3}{4}\pi.$$

Since we have contradicted the existence of $g_0 \in \Gamma$ with $d(a, g_0Ha) > l$, gHa meets $N_l(a)$ for all $g \in \Gamma$, as claimed.

6. Counterexamples

In this section, we describe examples of complete simply connected negatively curved riemannian manifolds which admit nonfinitely generated discrete parabolic group actions. In fact, one can arrange that the curvature be bounded either above (away from 0) or below (away from $-\infty$). The former case is due to G. Mess, and the latter case is related to recent work of Abresch and Schroeder [1]. I am indebted to V. Schroeder for bringing their paper to my attention.

Let $\mathbf{H}^2$ be the hyperbolic plane, with infinitesimal distance dx. We put a Riemannian metric on $X = \mathbf{H}^2 \times \mathbf{R}$ so that the infinitesimal distance ds at the point (x, t) is given by $ds^2 = f(t)^2 dx^2 + dt^2$, where $f: \mathbf{R} \to (0, \infty)$ is smooth. One may verify that all the sectional curvatures at the point (x, t) lie between

$$-\frac{1}{f}\frac{d^2 f}{dt^2} \quad \text{and} \quad -\frac{1}{f^2}\left(1 + \left(\frac{df}{dt}\right)^2\right).$$

Thus, X is negatively curved, provided f is convex. If f is bounded as $t \to \infty$, then the set of geodesic rays of the form $[t \mapsto (x, t)]: [0, \infty) \to X$ defines an ideal point p of X. Now, X is foliated by horospheres about p of the form $\mathbf{H}^2 \times \{t\}$ for $t \in \mathbf{R}$. Any isometry, γ, of $\mathbf{H}^2$ induces an isometry $[(x, t) \mapsto (\gamma x, t)]$ of X, preserving setwise each such horosphere. Thus, any infinite discrete group action on $\mathbf{H}^2$ gives us a discrete parabolic subgroup of $\operatorname{Isom} X$. There are plenty of noncyclic free subgroups, and hence also nonfinitely generated examples of such.

Note that if we set $f(t) = e^{-t}$, then all sectional curvatures lie in the interval $(-\infty, -1]$. If we set $f(t) = 1 + e^{-t}$, then they lie in the interval $(-1, 0)$.

We also remark that, by taking a surface group acting on $\mathbf{H}^2$, we can arrange that the quotient of a horosphere be compact, and so the quotient of a horoball is a "parabolic cusp", whose fundamental group is not virtually nilpotent. In the case of an upper curvature bound, such a cusp automatically has finite volume. In the case of a lower curvature bound the volume is infinite. However, one can construct examples in dimension 4 which have finite volume. The idea is to take a product with a circle, whose diameter tends to 0 as we move out the cusp. This construction is closely related to that described in [1], where examples are given of complete finite-volume negatively curved Riemannian 4-manifolds, with a lower bound on curvature, and whose ends are foliated by compact graph manifolds. The fundamental groups of such ends are not parabolic, but contain noncyclic free subgroups which are.

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