J. DIFFERENTIAL GEOMETRY
38 (1993) 485-500

PERESTROIKAS[1] OF OPTICAL WAVE FRONTS AND GRAPHLIKE LEGENDRIAN UNFOLDINGS

SHYŪICHI IZUMIYA

0. Introduction

It is well known that parallels of given smooth hypersurfaces in $\mathbb{R}^n$ have Legendrian singularities [2], [3]. If we are concerned about the way in which these parallels change as we alter the distance, then we are concerned about perestroikas of wave front sets. This problem has been considered by several people [1], [2], [4], [5]. Roughly speaking, it has been shown that generic perestroikas of the singularities of parallels of smooth hypersurfaces are generic perestroikas of Legendrian singularities. In [15] Zakalyukin proved that generic perestroikas of Legendrian singularities are stable perestroikas of Legendrian singularities in the case $n \leq 5$ and classified these perestroikas. In this paper we shall consider the realization problem of the perestroikas of Legendrian singularities. In relation to this problem, Arnol′d [2, p. 40] mentioned that "one may find in the literature the statement that the local perestroikas of the wave fronts generated by the general Legendre mapping over space-time and of the equidistants (i.e. parallel) of the smooth hypersurface are the same. It seems this has never been correctly proved." However, he has corrected his mistakes in his other book [2, p. 60]: "Indeed, the non-trivial perestroikas A_1 and A_2 change the number of connected components of the Legendrian manifold. Hence they cannot occur as perestroikas of equidistant hypersurfaces." This fact was already known to Bill Bruce in 1983 [4]. We have the following natural question.

Question. What sort of a class of one-parameter families of Legendrian immersion germs is the correct class to describe perestroikas of parallels of hypersurfaces?

Here we shall give a candidate of this class which we call *graphlike Legendrian unfoldings*. Roughly speaking, a graphlike Legendrian unfolding is a Legendrian submanifold germ with a submersive generating function.

Received June 22, 1992.

[1] The author uses this word in honor of Professor V. I. Arnol′d.

The general properties of graphlike Legendrian unfoldings will be studied in §2.

On the other hand, it should be natural to consider this problem in the framework of Hamiltonian systems which also contains the situation of parallels of hypersurfaces in a Riemannian manifold (cf. [5]).

All maps considered here are differentiable of class C^∞, unless stated otherwise.

1. Optical wave fronts

In this section we shall introduce the notion of optical wave fronts which is a generalization of parallels of hypersurfaces. We need some notions and results on Legendrian singularity theory. For this we refer to [1], [2], [3], [12], [14].

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables. Since we only consider the local situation, the 1-jet bundle $J^1(\mathbb{R}^n, \mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ and the canonical coordinate is given by $(x_1, \cdots, x_n, t, p_1, \cdots, p_n)$. Then the canonical 1-form is given by $\theta = dt - \sum_{i=1}^n p_i\, dx_i$. Let $T^*\mathbb{R}^n$ be the cotangent bundle over $\mathbb{R}^n$ whose canonical coordinate is given by $(x_1, \cdots, x_n, p_1, \cdots, p_n)$. We also have the canonical 1-form $\alpha = \sum_{i=1}^n p_i\, dx_i$ on $T^*\mathbb{R}^n$. There are natural projections, namely,

$$\begin{aligned}
&\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}\ ; \qquad \pi(x, t, p) = (x, t),\\
&\tilde{\pi} : T^*\mathbb{R}^n \to \mathbb{R}^n\ ; \qquad \tilde{\pi}(x, p) = x,\\
&\Pi : J^1(\mathbb{R}^n, \mathbb{R}) \to T^*\mathbb{R}^n\ ; \qquad \Pi(x, t, p) = (x, p),\\
&\pi_2 : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}\ ; \qquad \pi_2(x, t, p) = t.
\end{aligned}$$

Let $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ be a function germ. We call it *a Hamiltonian function germ.* We say that H is *starshaped* if it satisfies $\sum_{i=1}^n p_i \partial H(x, p)/\partial p_i \neq 0$ at any $(x, p) \in H^{-1}(c)$. We remark that if H is starshaped, then $H^{-1}(c)$ is transverse to fibers of $\tilde{\pi}$. It is easy to prove the following proposition.

Proposition 1.1. *If H is starshaped, then $\alpha | H^{-1}(c)$ gives a contact structure on $H^{-1}(c)$, and $\tilde{\pi} | H^{-1}(c) : H^{-1}(c) \to \mathbb{R}^n$ is a Legendrian fibration.*

Let $(l, (x_0, p_0)) \subset (T^*\mathbb{R}^n, (x_0, p_0))$ be a submanifold germ. We say that l is *an optical Legendrian submanifold relative to* H if $\dim l = n - 1$, $\alpha | l = 0$, and $l \subset H^{-1}(c)$. We call the image $\tilde{\pi}(l)$ *an optical wave front relative to* H.

By this proposition, if l is an optical Legendrian submanifold relative to H, then l is a Legendrian submanifold of $H^{-1}(c)$, and $\tilde{\pi}(l)$ is a wave front set in the ordinary sense (see [1], [2], [3], [14]). We now give examples of optical wave front sets.

Example 1.2. Let $g = \sum_{ij} g_{ij}(x)\, dx_i\, dx_j$ be a Riemannian metric on $\mathbb{R}^n$. We define a Hamiltonian function H on $T^*\mathbb{R}^n$ by

$$H(x_1, \cdots, x_n, p_1, \cdots, p_n) = \sum_{ij} g_{ij}(x) p_i p_j.$$

Then we can show that

$$\sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} = 2 \sum_{ik} g_{ik}(x) p_i p_k = 2 \quad \text{on } H^{-1}(1).$$

For a Hamiltonian function germ $H\colon (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$, we define a function germ $\tilde{H} : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, t_0, p_0)) \to (\mathbb{R}, c)$ by $\tilde{H}(x, t, p) = H(x, p)$. We call $\tilde{H}$ *an induced contact Hamiltonian function germ from* H. By the definition, we have the Hamiltonian vector field

$$X_H = -\sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j} \quad \text{on } T^*\mathbb{R}^n.$$

For an isotropic $(n-1)$-dimensional submanifold germ $(l, (x_0, p_0)) \subset (T^*\mathbb{R}^n, (x_0, p_0))$ (i.e., $\alpha|l = 0$), we say that l is *noncharacteristic relative to* H *at* (x_0, p_0) if $X_{H,(x_0,p_0)} \notin T_{(x_0,p_0)} l$. Then we have

Lemma 1.3. *If* $\sum_{i=1}^{n} p_i(\partial H/\partial p_i) \neq 0$ *at* $(x_0, p_0) \in l$, *then* l *is noncharacteristic relative to* H *at* (x_0, p_0).

Proof. If $X_{H,(x_0,p_0)} \in T_{(x_0,p_0)} l$, then

$$0 = \alpha(T_{(x_0,p_0)} l) = \alpha(X_{H,(x_0,p_0)}) = -\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} \quad \text{at } (x_0, p_0),$$

which contradicts to the assumption.

By this lemma, if l is an optical Legendrian submanifold relative to a starshaped Hamiltonian function H, then l is automatically noncharacteristic at any points relative to H. Thus we can apply the characteristic method to solve the Cauchy problem of the first-order partial differential equation $H = c$ with the given initial condition l (see [12]). We consider the embedding $i\colon \mathscr{O} \subset H^{-1}(c)$, where $\mathscr{O} \subset l \times (-\varepsilon, \varepsilon)$ is some open neighborhood of $l \times 0$, $\varepsilon > 0$; here $i(u, t) = T_t(u)$, $u \in l$, $(u, t) \in \mathscr{O}$, and T_t is a one-parameter group of translations along X_H. Hence we have a family of optical wave fronts $\tilde{\pi}(l_t)$.

Example 1.4. We now consider the Hamiltonian function $H = p_1^2 + \cdots + p_n^2$. Let $F: (\mathbb{R}^n, x_0) \to (\mathbb{R}, 0)$ be a submersion germ. We define an $(n-1)$-dimensional submanifold of $T^*\mathbb{R}^n$ by

$$l_F = \left\{ (x, p) | p = \frac{1}{(\sum_{i=1}^n (F_{x_i})^2)^{1/2}} (F_{x_1}, \cdots, F_{x_n}) \text{ and } F(x) = 0 \right\},$$

where $F_{x_i} = \partial F / \partial x_i$.

By the definition, $l \subset H^{-1}(1)$ and $\alpha | l = 0$. Parallels of the hypersurface $F^{-1}(0)$ in $\mathbb{R}^n$ are $\tilde{\pi}(l_t)$ which are constructed by the above argument with the initial submanifold l_F.

Our purpose is to construct a framework to describe the perestroika of singularities of $\tilde{\pi}(l_t)$. Then we consider what the time t is. We can construct a Legendrian submanifold $\mathscr{L} \subset J^1(\mathbb{R}^n, \mathbb{R})$ associated with the Lagrangian submanifold $L \subset T^*\mathbb{R}^n$ in the following way: We fix a point $(x_0, p_0) \in L$. Then for every point $(x, p) \in L$ and every path $\gamma = \gamma(t)$, $\gamma(0) = (x_0, p_0)$, $\gamma(1) = (x, p)$, we define a function

$$S(\gamma, (x, p)) = \int_\gamma \alpha | L.$$

Then

$$\mathscr{L} = \bigcup_{([\gamma], (x, p))} \{(x, S(x, p), p)\},$$

where $[\gamma]$ is the homotopy class represented by the path γ joining (x_0, p_0) and (x, p). Since we consider the local situation, $\mathscr{L}$ does not depend on the choice of γ. For any $(x, p) \in l_t \subset L$, we put $(x', p') = T_{-1}(x, p) \in l$. Then there exists a path γ in l such that $\gamma(0) = (x_0, p_0)$ and $\gamma(1) = (x', p')$. We denote $\phi(t) = (x(t), p(t))$, which is defined by $\phi(\tau) = T_\tau(x', p')$. Since $\alpha | l = 0$,

$$S(x(t), p(t)) = \int_{\phi \cdot \gamma} \alpha | L = \int_0^t \sum_{i=1}^n p_i(\tau) \frac{dx_i}{dt}(\tau)\, d\tau$$
$$= \int_0^t \sum_{i=1}^n p_i(\tau) \frac{\partial H}{\partial p_i}(x(\tau), p(\tau))\, d\tau.$$

Since H is starshaped,

$$\left. \frac{dS(x(t), p(t))}{dt} \right|_{t=0} \neq 0.$$

This means that $\pi_2|\mathscr{L}: \mathscr{L} \to \mathbb{R}$ is a submersion germ. Accordingly we will introduce the notion of graphlike Legendrian unfoldings in the next section.

2. Graphlike Legendrian unfoldings

We introduce the notion of graphlike Legendrian unfoldings and study some fundamental properties. In [10], [11] we defined the notion of Legendrian unfoldings, which is very useful in the theory of first-order partial differential equations. However, it is still too general for our purpose.

Let $(\mathscr{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$ be a Legendrian submanifold germ (i.e., $\dim \mathscr{L} = n$ and $\theta|\mathscr{L} = 0$). We say that $\mathscr{L}$ is *a graphlike Legendrian unfolding* if $\pi_2|\mathscr{L}$ is a submersion germ. We remark that $\mathscr{L}_{t_0} = \mathscr{L} \cap \{t = t_0\}$ is an $(n-1)$-dimensional submanifold, and $\Pi(\mathscr{L}_{t_0}) = l_{t_0}$ is an isotropic submanifold in $(T^*\mathbb{R}^n, (x_0, p_0))$. Here we call $\tilde{\pi}(l_{t_0})$ *a wave front* of l_{t_0}. In order to study perestroikas of wave fronts of graphlike Legendrian unfoldings, we now introduce the following equivalence relation among graphlike Legendrian unfoldings. Let $(\mathscr{L}_i, (x_i, t_i, p_i)) \subset J^1(\mathbb{R}^n, \mathbb{R})$ $(i = 0, 1)$ be graphlike Legendrian unfoldings. We say that $(\mathscr{L}_0, (x_0, t_0, p_0))$ and $(\mathscr{L}_1, (x_1, t_1, p_1))$ are *P-Legendrian equivalent* if there exists a contact diffeomorphism

$$\Phi: (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, t_0, p_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_1, t_1, p_1))$$

of the form

$$\Phi(x, t, p) = (\phi_1(x, t), \phi_2(t), \phi_3(x, t, p))$$

such that $\Phi(\mathscr{L}_0) = \mathscr{L}_1$.

This equivalence relation preserves the diffeomorphic type of perestroikas of wave fronts of graphlike Legendrian unfoldings. We can define the notion of stability with respect to P-Legendrian equivalence in exactly the same way as for the ordinary Legendrian stability (see [1], [2], [3], [14]).

By the Arnol′d-Zakalyukin theory of Legendrian singularities, we can construct generating families of graphlike Legendrian unfoldings. Let $F: (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ be a function germ such that $\partial F/\partial t \neq 0$ and $(d_2F, F)|\{0\} \times \mathbb{R}^n \times \mathbb{R}^k$ is nonsingular, where

$$d_2F(x, q) = \left(\frac{\partial F}{\partial q_1}(x, q), \cdots, \frac{\partial F}{\partial q_k}(x, q)\right).$$

Then (d_2F, F) is a submersion germ, so that $C_*(F) = (d_2F, F)^{-1}(0)$ is an n-dimensional smooth submanifold. We now define a map germ

$$\Phi_F : (C_*(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t, x, q) = \left(x, t, \frac{\partial F/\partial x_1}{\partial F/\partial t}, \cdots, \frac{\partial F/\partial x_n}{\partial F/\partial t}\right).$$

It is easy to show that $\Phi_F^*\theta = 0$ and $\pi_2 \circ \Phi_F$ is a submersion germ, so that Φ_F is a graphlike Legendrian unfolding. By the theory of Arnol'd and Zakalyukin, we can show the following result.

Proposition 2.1. *All graphlike Legendrian unfolding germs are constructed by the above method.*

We call F *a generating family of* Φ_F. We also consider an equivalence relation among generating families of graphlike Legendrian unfoldings. Let

$$F_i : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0) \qquad (i = 0, 1).$$

be generating families of Φ_{F_i}. We say that F_0 and F_1 are *t-P-$\mathscr{K}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0)$$

of the form

$$\Phi(t, x, q) = (\phi_1(t), \phi_2(t, x), \phi_3(t, x, q))$$

such that

$$\langle F_1 \circ \Phi\rangle_{\mathscr{E}_{(t,x,q)}} = \langle F_0\rangle_{\mathscr{E}_{(t,x,q)}},$$

where $\langle F_0\rangle_{\mathscr{E}_{(t,x,q)}}$ denotes the ideal generated by F_0 in the ring of function germs of (t, x, q)-variables at the origin. The definition of *the stable t-P-$\mathscr{K}$-equivalence* is given in the usual way (see [3], [14]).

For a generating family F of Φ_F, we put

$$T_e(\text{P-}\mathscr{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \cdots, \frac{\partial f}{\partial q_k}, f\right\rangle_{\mathscr{E}_{(x,q)}} + \left\langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right\rangle_{\mathscr{E}_x},$$

where $f = F|0 \times \mathbb{R}^n \times \mathbb{R}^k$. We also say that F is *a P-$\mathscr{K}$-versal deformation of f* if

$$\mathscr{E}_{(x.q)} = \left\langle \frac{\partial F}{\partial t}|0 \times \mathbb{R}^n \times \mathbb{R}^k\right\rangle_{\mathbb{R}} + T_e(\text{P-}\mathscr{K})(f).$$

For properties of *P-$\mathscr{K}$*-versal deformations, see [9]. Then we have the following proposition, whose proof is just like that of the ordinary theory of Legendrian singularities ([3], [14]).

Proposition 2.2. (1) *Let* F_i $(i = 0, 1)$ *be generating families of* Φ_{F_i}. *Then* Φ_{F_0} *and* Φ_{F_1} *are P-Legendrian equivalent if and only if* F_0 *and* F_1 *are stably* t*-P-*$\mathscr{K}$*-equivalent.*

(2) *Let* F *be a generating family of* Φ_F. *Then* Φ_F *is stable with respect to the P-Legendrian equivalence if and only if* F *is a P-*$\mathscr{K}$*-versal deformation of* f.

We can classify generic graphlike Legendrian unfoldings by the P-Legendrian equivalence for $n \leq 5$ by the aid of classifications of one-parameter perestroikas of wave fronts. In [15] Zakalyukin has given a generic classification of function germs $F : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ by the stable t-P-$\mathscr{K}$-equivalence. Since the set of function germs $F(t, x, q)$ which satisfy $\partial F/\partial t \neq 0$ is an open subset, such a function germ is stably t-P-$\mathscr{K}$-equivalent to one of the germs in the following list:

$$({}^0A_r) \qquad q_1^{r+1} \pm q_2^2 + \sum_{i=1}^{r} x_i q^{i-1} + t \qquad (1 \leq r \leq n),$$

$$({}^0D_r) \qquad q_1^2 q_2 \pm q_2^{r-1} + \sum_{i=2}^{r} x_i q_2^{i-2} + x_1 q_1 + t \qquad (4 \leq r \leq n),$$

$$({}^1A_r) \quad q_1^{r+1} \pm q_2^2 + q_1^{r-1}(t \pm x_r^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-1} x_i q_1^{i-1} + t \quad (1 \leq r \leq n+1),$$

$$({}^1D_r) \quad q_1^2 q_2 \pm q_2^{r-1} + q_2^{r-2}(t \pm x_r^2 \pm \cdots \pm x_n^2) + \sum_{i=2}^{r-1} x_i q_2^{i-2} + x_1 q_1 + t \qquad (4 \leq r \leq n+1),$$

$$({}^1E_6) \qquad q_1^3 + q_2^4 + (q_1 q_2^2 + 1)t + x_5 q_1 q_2 + x_4 q_2^2 + x_3 q_1 + x_2 q_2 + x_1.$$

Since $(d_2 F, F)|0 \times \mathbb{R}^n \times \mathbb{R}^k$ for germs ${}^1A_1$ and ${}^1A_2$ are not submersive, these germs can be removed from the list of generating families of graphlike Legendrian unfoldings. Thus we obtain

Theorem 2.3. *For* $n \leq 5$, *the generic graphlike Legendrian unfolding is P-Legendrian equivalent to one of the germs of the following type*: ${}^0A_r$ $(1 \leq r \leq n)$, ${}^0D_r$ $(4 \leq r \leq n)$, ${}^1A_r$ $(3 \leq r \leq n)$, ${}^1D_r$ $(4 \leq r \leq n)$, ${}^1E_6$.

We remark that we can choose a generating family of the graphlike Legendrian unfolding Φ_F of the form $F(t, x, q) = f(x, q) - t$ by the implicit function theorem. In this case the function $f(x, q)$ should satisfy the requirement that $(d_2 f, f)$ be nonsingular, so that it gives a generating family of the initial isotropic submanifold $\Phi_F | t = 0$.

3. Optical graphlike Legendrian unfoldings

In this section we return to study perestroikas of optical wave fronts. For our purpose, we now introduce the following: Let $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ be a starshaped Hamiltonian function germ, and $(\mathscr{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$ a graphlike Legendrian unfolding. We say that $\mathscr{L}$ is *optical relative to* H if $\mathscr{L} \subset \tilde{H}^{-1}(c)$, where $\tilde{H}$ is the induced contact Hamiltonian function germ from H. Then we have the following theorem.

Theorem 3.1. (1) *Let* $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ *be a starshaped Hamiltonian function germ, and* $(l, (x_0, p_0)) \subset T^*\mathbb{R}^n$ *be an optical Legendrian submanifold relative to* H. *Then there exists a graphlike Legendrian unfolding* $(\mathscr{L}, (x_0, 0, p_0))$ *which is optical relative to* H *such that* $\tilde{\pi}(\mathscr{L} \cap \{t = 0\}) = l$.

(2) *Let* $(\mathscr{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$ *be a graphlike Legendrian unfolding. Then there exists a starshaped Hamiltonian function germ* $H: (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ *such that* $\mathscr{L}$ *is optical relative to* H.

In order to prove this theorem, we need the following quite useful theorem in symplectic geometry.

Theorem 3.2 (Kostant-Sternberg [8]). *Let* (P, ω) *be a symplectic manifold, L a Lagrangian submanifold, and* β *a smooth* 1*-form on* P *with* $\beta | L = 0$ *and* $d\beta = \omega$. *Then there exist a tubular neighborhood* V *of* L *in* P *and a unique vector bundle isomorphism* $K: V \to (T^*L, \alpha_L)$ *such that* K *is the identity on* L *and* $K^*\alpha_L = \beta$. *Here,* α_L *is the canonical* 1*-form on* T^*L.

Proof. (1) This assertion was proved in §1.

(2) Since $\mathscr{L}$ is a graphlike Legendrian unfolding, we can choose a local parametrization $\phi: (\mathbb{R}^n, (u_0, t_0)) \to (\mathscr{L}, (x_0, t_0, p_0))$ of $\mathscr{L}$ of the form $\phi(u, t) = (x(u, t), t, p(u, t))$, where $u = (u_1, \cdots, u_{n-1})$. Furthermore, we have $\phi^*\theta = 0$, so that it is equivalent to

(a) $$\sum_{i=1}^{n} p_i(u, t) \frac{\partial x_i}{\partial u_j}(u, t) = 0, \qquad j = 1, \cdots, n-1,$$

(b) $$\sum_{i=1}^{n} p_i(u, t) \frac{\partial x_i}{\partial t}(u, t) = 1.$$

We now consider a vector field

$$X = \left(\frac{\partial x}{\partial t}(u, t), 1, \frac{\partial p}{\partial t}(u, t)\right)$$

on $\mathscr{L}$, so that we have

$$\tilde{\pi}_* X = \left(\frac{\partial x}{\partial t}(u, t), \frac{\partial p}{\partial t}(u, t)\right) \in TL,$$

where $L = \tilde{\pi}(\mathscr{L})$. Since L is a Lagrangian submanifold germ in $(T^*\mathbb{R}^n, (x_0, p_0))$, there exist a decomposition $I \cup J = \{1, \cdots, n\}$ and a function germ $S(x_I, p_J)$ such that

$$L = \left\{(x_I, x_J, p_I, p_J) | x_J = -\frac{\partial S}{\partial p_J}, \ p_I = \frac{\partial S}{\partial x_I}\right\}$$

(for details see [3], [14]). If we define a 1-form β on $T^*\mathbb{R}^n$ by $\beta = dS - p_I\, dx_I + x_J\, dp_J$, then we have $\beta | L = 0$. Thus we can use β as the 1-form in Theorem 3.2. It follows that there exist a tubular neighborhood V of L in $T^*\mathbb{R}^n$ and a unique vector bundle isomorphism $K : V \to (T^*L, \alpha_L)$ such that $k|L = \mathrm{id}_L$ and $K^*\alpha_L = \beta$.

Since $\tilde{\pi}_* X$ is a vector field on L, we can lift this vector field $X_{H'}$ on T^*L, where $H' : (T^*L, (x_0, p_0)) \to \mathbb{R}$ is a Hamiltonian function germ. If we adopt the canonical local coordinate $(\bar{x}, \bar{p})$ of T^*L around (x_0, p_0), where $\bar{x}$ is a local coordinate of L, then L is the submanifold defined by $\bar{p}_i = 0$ $(i = 1, \cdots, n)$. We now define a Hamiltonian function germ $H'' : (T^*L, (x_0, p_0)) \to (\mathbb{R}, c)$ by $H''(\bar{x}, \bar{p}) = H'(\bar{x}, \bar{p}) - H'(\bar{x}, 0) + c$. By the definition, we have $H''(L) = c$. The Hamiltonian vector field $X_{H''}$ on T^*L has the following component :

$$\frac{\partial \bar{p}_i}{\partial t} = \frac{\partial H''}{\partial \bar{x}_i}, \quad \frac{\partial \bar{x}_i}{\partial t} = -\frac{\partial H''}{\partial \bar{p}_i}, \qquad i = 1, \cdots, n.$$

On the other hand, we have $\partial H''/\partial \bar{p}_i = \partial H'/\partial \bar{p}_i$ and

$$\frac{\partial H''}{\partial \bar{x}_i}(\bar{x}, 0) = \frac{\partial H'}{\partial \bar{x}_i}(\bar{x}, 0) - \frac{\partial H'}{\partial \bar{x}_i}(\bar{x}, 0) = 0 \quad \text{on } L,$$

so that $X_{H''}|L = X_{H'}|L = \tilde{\pi}_* X$.

Finally, we define a Hamiltonian function germ $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ by $H = H'' \circ K$. Since $K|L = id_L$, the Hamiltonian vector field X_H is an extension of $\tilde{\pi}_* X$ on $T^*\mathbb{R}^n$ and $H(L) = c$. It follows that we have relations

$$\frac{\partial x_i}{\partial t}(u, t) = \frac{\partial H}{\partial p_i}(x(u, t), p(u, t)),$$
$$\frac{\partial p_i}{\partial t}(u, t) = -\frac{\partial H}{\partial x_i}(x(u, t), p(u, t))$$

on L. Since equality (b) implies that

$$1 = \sum_{i=1}^{n} p_i(u, t)\frac{\partial x_i}{\partial t}(u, t) = \sum_{i=1}^{n} p_i(u, t)\frac{\partial H}{\partial p_i}(x(u, t), p(u, t)),$$

H is starshaped near (x_0, p_0). Hence the proof is complete.

This theorem guarantees that the class of graphlike Legendrian unfoldings supplies the correct class which describes perestroikas of optical wave fronts. However, for our first question, we must deal with the case where the Hamiltonian function germ H is fixed.

4. Optical graphlike Legendrian unfoldings relative to a fixed Hamiltonian

In this section we consider the perestroikas of optical wave fronts relative to a fixed Hamiltonian function H. Let $H: (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}^n, 0)$ be a starshaped Hamiltonian function germ. Since $\tilde{\pi} : H^{-1}(c) \to \mathbb{R}^n$ is a Legendrian fibration, there exist a local contact diffeomorphism $K: PT^*\mathbb{R}^n \to H^{-1}(c)$ and a diffeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{\pi} \circ K = h \circ p$, where $p : PT^*\mathbb{R}^n \to \mathbb{R}^n$, is the projective cotangent bundle (see [1], [2], [3], [14]). If l is a Legendrian submanifold germ in $PT^*\mathbb{R}^n$, then $K(l)$ is also a Legendrian submanifold germ in $H^{-1}(c)$. Of course, l and $K(l)$ are Legendrian equivalent, so that any wave front can be realized by an optical one. However, for the perestroikas of wave fronts, the situation is different from the above. Consider the following example.

Example 4.1. Let $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}^n, c)$ be a Hamiltonian function germ defined by $H(x, p) = p_n$, where $(x_0, p_0) = (0, \cdots, 0, 1)$. Let $F(t, x, q) = f(x, q) - t$ be a generating family of the graphlike Legendrian unfolding Φ_F. Suppose that $\Phi_F|t = 0$ is an optical Legendrian submanifold relative to H. Then

$$H\left(x, \frac{\partial f}{\partial x}(x, q)\right) \equiv 1 \mod \left\langle f, \frac{\partial f}{\partial q}\right\rangle_{\mathscr{E}_{(x, q)}}.$$

Because

$$H\left(x, \frac{\partial f}{\partial x}(x, q)\right) = \frac{\partial f}{\partial x_n}(x, q),$$

we have

$$-1 \in \left\langle \frac{\partial f}{\partial x_n} \right\rangle_{\mathscr{E}_x} + \left\langle f, \frac{\partial f}{\partial x} \right\rangle_{\mathscr{E}_{(x,q)}} \subset T_e(p\text{-}\mathscr{K})(f).$$

It follows that $F(t, x, q)$ is not a P-$\mathscr{K}$-versal deformation of $f(x, q)$ if P-$\mathscr{K}$-cod$(f) \geq 1$. Thus, for example, types of ${}^1A_r$ and ${}^1D_r$ in Theorem 2.3 cannot be realized by an optical one relative to H.

Hence, we should assume a kind of nondegeneracy condition on a Hamiltonian function germ. We say that a Hamiltonian function germ $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ is *nondegenerated* if

(1) H is starshaped,

(2) the quadratic form

$$\sum_{1 \leq i, j \leq n} \frac{\partial^2 H}{\partial p_i \partial p_j}(x_0, p_0)\xi_i\xi_j$$

does not vanish on the null space of

$$\left(\frac{\partial H}{\partial p_1}(x_0, p_0), \cdots, \frac{\partial H}{\partial p_n}(x_0, p_0)\right).$$

The second condition of the above definition is given by Duistermaat (Proposition 5.2.1 in [7]) in order to study optical Lagrangian submanifolds. It is easy to check that the Hamiltonian function in Example 1.2 satisfies the above conditions. Thus we obtain the following theorem, which gives an almost complete answer to our first question.

Theorem 4.2. *Let $H : (T^*\mathbb{R}^n, (x_0, p_0)) \to (\mathbb{R}, c)$ be a nondegenerated Hamiltonian function, and $(\mathscr{L}, (x_1, t_1, p_1)) \subset J^1(\mathbb{R}^n, \mathbb{R})$ be a P-Legendrian stable graphlike Legendrian unfolding. Then there exists an optical graphlike Legendrian unfolding $(\mathscr{L}', (x_0, t_0, p_0))$ relative to H such that $\mathscr{L}$ and $\mathscr{L}'$ are P-Legendrian equivalent.*

Proof. Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$. Let

$$G : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a generating family of the graphlike Legendrian unfolding $\mathscr{L}$. Since $(d_2G, G)|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is nonsingular, the set

$$\phi_g = \left\{\left((x_1, \cdots, x_n), \left[\frac{\partial g}{\partial x_1}(x, q); \cdots ; \frac{\partial g}{\partial x_n}(x, q)\right]\right) \middle| d_2g(x, q) = g(x, q) = 0\right\}$$

is a Legendrian submanifold in $PT^*\mathbb{R}^n$, where $g = G|0 \times \mathbb{R}^n \times \mathbb{R}^k$. Thus $l_0 = K(\psi_g)$ is an optical Legendrian submanifold relative to H which is Legendrian equivalent to ψ_g, where K is the contact diffeomorphism as in the previous arguments. It follows from Theorem 3.1 that there exists an optical graphlike Legendrian unfolding $\mathscr{L}'$ relative to H such that $\Pi(\mathscr{L}'|_{t=0}) = l_0$.

We now choose a generating family $F(t, x, q) = f(x, q) - t$ of $\mathscr{L}'$. By definition, $\mathscr{L}'|_{t=0}$ is given by

$$\Phi_F|_{t=0} = \left\{ \left(x, 0, \frac{\partial f}{\partial x_1}(x, q), \dots, \frac{\partial f}{\partial x_n}(x, q) \right) | d_2 f(x, q) = f(x, q) = 0 \right\}$$

so that f and g are stably P-$\mathscr{K}$-equivalent by Arnol′d-Zakalyukin theory [1], [2], [3], [14]). Thus we may assume f and g are P-$\mathscr{K}$-equivalent.

If P-$\mathscr{K}$-cod$(g) = 0$, then P-$\mathscr{K}$-cod$(f) = 0$, so $f(x, q)$ is already a P-$\mathscr{K}$-versal deformation of itself, and $F(t, x, q)$ is also a P-$\mathscr{K}$-versal deformation of $f(x, q)$. From the uniqueness theorem of P-$\mathscr{K}$-versal deformations (see [6], [8]) it follows that $F(t, x, q)$ and $G(t, x, q)$ are t-P-$\mathscr{K}$-equivalent. Hence $\mathscr{L}' = \Phi_F$ and $\mathscr{L} = \Phi_G$ are P-Legendrian equivalent.

Thus, we now assume that P-$\mathscr{K}$-cod$(g) = 1$, so that P-$\mathscr{K}$-cod$(f) = 1$. If $-1 = \partial F/\partial t|_{t=0} \notin T_e(\text{P-}\mathscr{K})(f)$, then we can get the required assertion by the uniqueness of the P-$\mathscr{K}$-versal deformation as in the previous case.

Suppose $1 \in T_e(\text{P-}\mathscr{K})(f)$ for any generating family of $\mathscr{L}'$ of the form $F(t, x, q) = f(x, q) - t$. Since $H(l_0) = c$, we have a relation

$$H\left(x, \frac{\partial f}{\partial x}(x, q)\right) \equiv c \mod \left\langle f, \frac{\partial f}{\partial q} \right\rangle_{\mathscr{E}_{(x,q)}},$$

so that

$$H\left(0, \frac{\partial f}{\partial x}(0, q)\right) \equiv c \mod \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathscr{E}_q},$$

where $f_0 = f|_{x=0}$. We may assume that $f_0 \in \mathfrak{M}_k^3$. Now consider the Taylor polynomial of $H(x, p)$ of degree 2 at (x, p_0) with respect to $p = (p_1, \cdots, p_n)$-variables as follows :

$$H(x, p) = H(x, p_0) + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(x, p_0)(p_i - p_{0,i})$$
$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j}(x, p_0)(p_i - p_{0,i})(p_j - p_{0,j}) + \text{higher terms.}$$

Since $H(0, p_0) = c$, we have

$$\begin{aligned} H\left(0, \frac{\partial f}{\partial x}(0, q)\right) &= c + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(0, \frac{\partial f}{\partial x}(0, p_0)\right)\left(\frac{\partial f}{\partial x_i}(0, q) - p_{0,i}\right) \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, p_0)\left(0, \frac{\partial f}{\partial x}(0, q)\right)\left(\frac{\partial f}{\partial x_i}(0, q) - p_{0,i}\right)\left(\frac{\partial f}{\partial x_j}(0, q) - p_{0,j}\right) \\ &+ \text{higher terms.} \end{aligned}$$

Because

$$H\left(0, \frac{\partial f}{\partial x}(0, q)\right) - c \in \left\langle \frac{\partial f_0}{\partial q}, f_0 \right\rangle_{\mathscr{E}_q} \subset \mathfrak{M}_k^2,$$

the right-hand side of the above equality is a function of $q = (q_1, \cdots, q_k)$-variables whose order is at least 2. Since $F(t, x, q) = f(x, q) - t$ is a generating family of the graphlike Legendrian unfolding $\mathscr{L}'$,

$$\operatorname{rank}\left(\frac{\partial^2 f}{\partial q_i \partial x_j}(0)\right) = k$$

and we may assume that $\partial^2 f(0)/\partial q_i \partial x_j = \delta_{ij}$ for $i, j = 1, \cdots, k$ and $\partial f(0, q)/\partial x_l \in \mathfrak{M}_k^2$ for $l = k+1, \cdots, n$. Thus $\partial f(0, q)/\partial x_i = q_i + \psi(q)$, where $\psi(q) \in \mathfrak{M}_k^2$, so that

$$\sum_{i=1}^{k} \frac{\partial H}{\partial p_i}(0, p_0) q_i = 0.$$

Since H is nondegenerated, we have

$$\sum_{i=1}^{k} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, p_0) q_i q_j \neq 0.$$

On the other hand, the assumption implies that

$$1 \in \left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathscr{E}_q},$$

so that

$$\begin{aligned}\frac{1}{2}\sum_{1\le i,j\le n}&\frac{\partial^2 H}{\partial p_i\partial p_j}(0,p_0)\left(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i}\right)\left(\frac{\partial f}{\partial x_j}(0,q)-p_{0,j}\right)\\ &\equiv H\left(0,\frac{\partial f}{\partial x}(0,q)\right)-c-\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(0,p_0)\left(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i}\right) \mod \mathfrak{M}_k^3\\ &= H\left(0,\frac{\partial f}{\partial x}(0,q)\right)-c-\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}(0,p_0)\frac{\partial f}{\partial x_i}(0,q)\\ &\quad+\sum_{i=1}^{n}p_{0,i}\frac{\partial H}{\partial p_i}(0,p_0)\\ &\in \left\langle\frac{\partial f}{\partial x_1}(0,q),\cdots,\frac{\partial f}{\partial x_n}(0,q)\right\rangle_{\mathbb{R}}+\left\langle f_0,\frac{\partial f_0}{\partial q}\right\rangle_{\mathscr{E}_q} \mod \mathfrak{M}_k^3.\end{aligned}$$

Let $A:\mathbb{R}^n\to\mathbb{R}^n$ be a linear isomorphism. Then

$$\frac{\partial f(Ax,q)}{\partial x_i}=\sum_{j=1}^{n}A_{ij}\frac{\partial f}{\partial x_j}(Ax,q).$$

Since

$$\begin{aligned}\sum_{1\le i,j\le n}&\frac{\partial^2 H}{\partial p_i\partial p_j}(0,p_0)\left(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i}\right)\left(\frac{\partial f}{\partial x_j}(0,q)-p_{0,j}\right)\\ &\equiv\sum_{1\le i,j\le k}\frac{\partial^2 H}{\partial p_i\partial p_j}(0,p_0)q_iq_j \mod \mathfrak{M}_k^3\end{aligned}$$

and the vector space

$$\left\langle\frac{\partial f}{\partial x_1}(0,q),\cdots,\frac{\partial f}{\partial x_n}(0,q)\right\rangle_{\mathbb{R}}+\left\langle f_0,\frac{\partial f_0}{\partial q}\right\rangle_{\mathscr{E}_q} \mod \mathfrak{M}_k^3$$

is invariant under the action of linear isomorphisms $A:\mathbb{R}^n\to\mathbb{R}^n$, all quadratic forms of $q=(q_1,\cdots,q_k)$-variables are contained in the above vector space. If there exists a quadratic form of q-variables such that it is contained in $\langle f_0,\partial f_0/\partial q\rangle_{\mathscr{E}_q} \mod \mathfrak{M}_k^3$, then every quadratic form is contained in it for the same reason as above. In this case, since the vector space $\langle f_0,\partial f_0/\partial q\rangle_{\mathscr{E}_q} \mod \mathfrak{M}_k^3$ has at most dimension k, k should be 1 and f_0 is an A_2-type function germ. It follows from Theorem 2.3 that $F(t,x,q)$ is ${}^0A_2$-type, so that this case is contained in the case P-$\mathscr{K}$-cod$(f)=0$.

We may assume that all quadratic forms of q-variables are contained in

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} \quad \operatorname{mod} \mathfrak{M}_k^3.$$

Since $\mathscr{K}$-cod(f_0) is finite (for the definition of $\mathscr{K}$-finiteness, see [13]), there exists $r \in \mathbb{N}$ such that $\mathfrak{M}_k^r \subset \langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathscr{E}_q}$. By the same arguments as in the previous paragraph, we can assert that every monomial of q-variables of degree 3 is contained in the vector space

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathscr{E}_q} \quad \operatorname{mod} \mathfrak{M}_k^4.$$

If a monomial of degree 3 is contained in $\langle f_0, \partial f_0/\partial q \rangle_{\mathscr{E}_q} \operatorname{mod} \mathfrak{M}_k^4$, then k should be 1, and f_0 is an A_3-type function germ. It follows that

$$\dim_{\mathbb{R}} \left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} \geq 3 \quad \operatorname{mod} \mathfrak{M}_k^4.$$

By Theorem 2.3, $F(t, x, q)$ should be of type ${}^0A_3$. This case is thus contained in the case P-$\mathscr{K}$-cod$(f) = 0$.

For ${}^0A_l$- or ${}^1A_l$-type germs, we can get the same normal forms as Theorem 2.3 without the assumption $n \leq 5$ (cf. Theorem 2.2 in [15]). Hence we can continue this procedure up to degree $r - 1$. Eventually, it is still true that every polynomial of degree $r-1$ is contained in the vector space

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathscr{E}_q} \quad \operatorname{mod} \mathfrak{M}_k^r.$$

Since $\mathfrak{M}_k^r \subset \langle f_0, \partial f_0/\partial q \rangle_{\mathscr{E}_q}$, we have

$$\mathscr{E}_q = \left\langle \frac{\partial f}{\partial x_1}(0, q), \cdots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathscr{E}_q}.$$

It follows that

$$\mathscr{E}_{(x,q)} = T_e(\text{P-}\mathscr{K})(f) + \mathfrak{M}_k \mathscr{E}_{(x,q)},$$

so that

$$\mathscr{E}_{(x,q)} = T_e(\text{P-}\mathscr{K})(f)$$

by the Malgrange preparation theorem. This contradicts the fact that P-$\mathscr{K}$-cod$(f) = 1$. Hence the proof is complete.

Remark. The above proof is motivated by the proof of Proposition 5.2.1 in [7]. In fact, Theorem 4.2 was discovered in the attempt to understand the proof of the proposition.

Acknowledgment

A part of this work was done during the author's stay at the University of Liverpool and was supported mainly by the Anglo-Japanese Scientific Exchange Program (The Royal Society and Japan Society for the Promotion of Science). The author would like to thank Professor C.T.C. Wall and his colleagues in the Department of Pure Mathematics for their kind encouragement and hospitality. The author is also grateful to Professor G. Ishikawa and Dr. T. Ohmoto for useful discussions and advice.

References

[1] V. I. Arnol′d, *Contact geometry and wave propagation*, Monograph. Enseignement Math. **34** (1989).
[2] ——, *Singularities of caustics and wave fronts*, Math. Appl., **62**, Kluwer, Dordrecht, 1990.
[3] V. I. Arnol′d, S. M. Gusein-Zade & A. N. Varchenko, *Singularities of differentiable maps*, Vol. I, Birkhäuser, Basel, 1986.
[4] J. W. Bruce, *Wavefronts and parallels in Euclidean space*, Math. Proc. Cambridge Philos. Soc. **93** (1983) 323–333 .
[5] J. W. Bruce & D. J. Hurley, *Focal sets in certain Riemannian manifolds*, Proc. Edinburgh Math. Soc. **27** (1984) 209–214 .
[6] J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathscr{A}$ and $\mathscr{K}$* , Mem. Amer. Math. Soc. **50-306** (1984).
[7] J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974) 207–281 .
[8] M. Golubitsky & V. Guillemin, *Contact equivalence for Lagrange manifolds*, Advances in Math. **15** (1975) 375–387 .
[9] S. Izumiya, *Generic bifurcations of varieties*, Manuscripta Math. **46** (1984) 137–164 .
[10] ——, *The theory of Legendrian unfoldings and first order differential equations*, Proc. Roy. Soc. Edinburgh, to appear.
[11] ——, *Completely integrable holonomic systems of first order differential equations*, preprint.
[12] V. V. Lychagin, *Local classification of non-linear first order partial differential equations*, Russian Math. Surveys **30** (1975) 105–175 .
[13] C. T. C. Wall, *Finite determinacy of smooth map germs*, Bull. London Math. Soc. **14** (1981) 481–539 .
[14] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Functional Anal. Appl. **10** (1976) 23–31 .
[15] ——, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, J. Soviet Math. **27** (1983) 2713–2735 .

HOKKAIDO UNIVERSITY, SAPPORO