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# ON THE SMOOTH COMPACTIFICATION OF SIEGEL SPACES

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## Introduction

Let $X = \Gamma\backslash\Omega$ be a noncompact locally symmetric Hermitian space, where $\Omega$ is a bounded symmetric domain and $\Gamma$ is an arithmetic subgroup acting on $\Omega$. It is well known that $X$ is quasiprojective[1], and the canonical Bergman metric on $X$ induced from $\Omega$ is a Kähler-Einstein metric of negative curvature if $X$ is smooth (it is the case where $\Gamma$ is neat). Since the smooth compactifications of $X$ were introduced in [1] from the toroidal embeddings, Mumford obtained the following results on $X$ in his proof of noncompact Hirzebruch's proportionality [12]:

1. $X$ is of logarithmic general type.
2. The Bergman metric $g$ on $X$ is a *good* singular Hermitian metric on any smooth toroidal compactification $\overline{X}$ of $X$. In other words, assuming that the boundary $D = \overline{X} - X$ is locally defined as $\prod_{i=1}^{k} z_i = 0$, then the volume form $\Phi$ of $g$ behaves singularly along the boundary $D$ as

$$(|z_1 \cdots z_k|^2\Phi)^{-1} = O(\log^{2N} |z_1 \cdots z_k|)$$

for some integer $N > 0$.

To have broader and deeper applications of the theory on the locally symmetric Hermitian spaces in algebraic and differential geometry (see the references [15], [16] and [9]), people would like to understand more about $X$ and its compactification $\overline{X}$ besides Mumford's work. One would like to completely understand the algebraic structures of the boundary divisor $D$ and the canonical bundle $K_{\overline{X}}$ of $\overline{X}$ and to have a precise singular description of the canonical volume form $\Phi$ along $D$. The goal of this paper is to study these questions for the quotient of Siegel upper half spaces by an intensive investigation of their smooth toroidal compactifications.

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[1] A noncompact variety $V$ is said to be quasiprojective if $V$ is a Zariski open dense subset of a projective variety $\overline{V}$.

## 1. Toroidal compactification of symmetric varieties

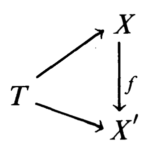
In this section, we briefly recall the construction of smooth compactifications of a locally symmetric variety from the torus embeddings. For the further detailed material on this section, see [1] and [13].

**1.1. Torus embeddings.** Let $T$ be an $n$-dimensional complex torus, i.e,. $T = (\mathbf{C}^*)^n$.

**Definition.** (i) A torus embedding of $T$ is an algebraic variety $X$ such that (1) $X$ contains $T$ as a Zariski open dense subset;

(2) $T$ acts on $X$ extending the natural action on itself defined by translation.

(ii) A morphism between torus embeddings $X$ and $X'$ is a map $f: X \to X'$ such that the diagram



commutes.

We can describe torus embeddings combinatorially.

$T = (\mathbf{C}^*) = \mathrm{Spec}(\mathbf{C}[T_1, T_1^{-1}, T_2, T_2^{-1}, \cdots, T_n, T_n^{-1}])$ as a scheme. Let $M = \mathrm{Hom}(T, \mathbf{C}^*) =$ Character group of $T \simeq \mathbf{Z}^n = \{r = (r_1, r_2, \cdots, r_n) \in \mathbf{Z}^n;\ \chi^r: T \to \mathbf{C}^*\}$ where $\chi^r(t_1, t_2, \cdots, t_n) = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$. $N = \mathrm{Hom}(\mathbf{C}^*, T) =$ group of one-parameter subgroups in $T \simeq \mathbf{Z}^n = \{a = (a_1, a_2, \cdots a_n) \in \mathbf{Z}^n;\ \lambda_a: \mathbf{C}^* \to T\}$ where $\lambda_a(t) = (t^{a_1}, t^{a_2}, \cdots, t^{a_n})$. $M$ and $N$ are dual to each other by the pairing $\langle\ ,\ \rangle: M \times N \to \mathbf{Z}$

$$\langle r, a \rangle = \sum_{i=1}^{n} r_i a_i;$$

then $\chi^r(\lambda_a(t)) = t^{\langle r, a \rangle}$ for $r \in M$, $a \in N$, $t \in \mathbf{C}^*$.

If we identify $\chi^r$ with monomial $\prod_{i=1}^n T_i^{r_i}$, and $S$ is a subsemigroup of $M$ containing 0, then $\mathbf{C}[S] = \mathbf{C}[\chi^r]_{r \in S}$ is a subring of $\mathbf{C}[M] = \mathbf{C}[T_1, T_1^{-1}, \cdots, T_n, T_n^{-1}]$, and $T = \mathrm{Spec}(\mathbf{C}[M]) = N_{\mathbf{C}}/N$ where $N_{\mathbf{C}} = N \otimes \mathbf{C} = \mathbf{C}^n$. Let $\sigma$ be a convex rational polyhedral cone (abbreviated to c.r.p. cone) in $N_{\mathbf{R}} = N \otimes \mathbf{R} = \mathbf{R}^n$ not containing a line. Then

$$\sigma = \{a \in N_{\mathbf{R}};\ \langle r_1, a \rangle \geq 0,\ \ i = 1, \cdots, k,\ \ r_i \in M\},$$

and the dual of $\sigma$ in $M_{\mathbf{R}} = M \otimes \mathbf{R} = \mathbf{R}^n$ is

$$\hat{\sigma} = \{r \in M_{\mathbf{R}};\ \langle r, a \rangle \geq 0,\ \text{ for all } a \in \sigma\}.$$

If $X_\sigma$ is defined to be $\operatorname{Spec}(\mathbf{C}[\hat{\sigma} \cap M])$, then $X_\sigma$ is an affine normal torus embedding of $T$ by $\operatorname{Spec}(\mathbf{C}[M]) \subset \operatorname{Spec}(\mathbf{C}[\hat{\sigma} \cap M])$. Let $\hat{\sigma} \cap M = \mathbf{Z}^+ r_1 + \cdots + \mathbf{Z}^+ r_m$, $r_i \in M$, $i = 1, 2, \cdots, m$ ($m \geq n$, since $\sigma$ not containing a line); hence

$$X_\sigma = \operatorname{Spec}(\mathbf{C}[\chi^{r_1}, \chi^{r_2}, \cdots, \chi^{r_m}]) \subset \mathbf{C}^m .$$

The embedding of $T$ into $X_\sigma$ is given by $i\colon T \to \mathbf{C}^m$, $i(\bar{t}) = (\chi^{r_1}(\bar{t}), \cdots, \chi^{r_m}(\bar{t}))$ where $\bar{t} = (t_1, \cdots t_n) \in T$. $X_\sigma$ is the scheme-theoretic closure of $i(T)$ in $\mathbf{C}^m$. $T$ acts on $X_\sigma$ as

$$\bar{t} \cdot x = (\chi^{r_1}(\bar{t})x_1, \chi^{r_2}(\bar{t})x_2, \cdots, \chi^{r_m} x_m)$$

for $\bar{t} \in T$, $x = (x_1, x_2, \cdots, x_m) \in X_\sigma$. Then $X_\sigma$ = the disjoint union of $T$-orbits in $X_\sigma$, and

$$\{T\text{-orbits in } X_\sigma\} \overset{\text{1-1}}{\longleftrightarrow} \{\text{all faces of } \sigma\} .$$

If $\tau$ is a face of $\sigma$ (we write as $\tau < \sigma$), let $N(\tau)$ be the subset $\{r_i ; \langle r_i , \rangle|_\tau = 0\}$ of $\{r_1, \cdots, r_m\}$, and $O_\tau$ be $T$-orbit in $X_\sigma$ corresponding to $\tau$. Then

$$O_\tau = \{(x_1, x_2, \cdots, x_m) \in X_\sigma ; x_i \neq 0 \text{ if } r_i \in N(\tau) ; x_i = 0 \text{ if } r_i \notin N(\tau)\} ,$$
$$\dim \tau + \dim O_\tau = n = \dim_{\mathbf{C}} T ,$$
$$O_0 = T .$$

If $\Sigma$ is a finite rational partial polyhedral decomposition (abbreviated to a f.r.p.p. decomposition) of $N_{\mathbf{R}}$ (in the future we always assume cone does not contain a line), such that

(i) the face of $\sigma$ is in $\Sigma$ if $\sigma \in \Sigma$;

(ii) for $\sigma_i , \sigma_j \in \Sigma$, $\sigma_i \cap \sigma_j$ is a face of $\sigma_i$ and $\sigma_j$.

Then we can patch $X_{\sigma_i}$ together to form a normal torus embedding of $T$, $X_\Sigma$, by the fact that if $\tau < \sigma$, then $X_\tau \subset X_\sigma$ and $X_\tau \to X_\sigma$ is an open immersion:

$$\begin{array}{ccc} X_\tau & \longrightarrow & X_\sigma \\ \cup & & \cup \\ T & = & T \end{array}$$

and $X_\Sigma = \bigcup\{T\text{-orbits in } X_\Sigma\}$,

$$\{T\text{-orbits in } X_\Sigma\} \overset{\text{1-1}}{\longleftrightarrow} \Sigma .$$

$X_\Sigma$ is smooth if and only if each $\sigma_i$ is regular, i.e., $\sigma_i$ is generated by a part of a basis of $N$.

On the other hand, if $\alpha: N_{\mathbf{C}} \to N_{\mathbf{C}}$ is a linear map which preserves lattice $N$, then it induces an action on $T \simeq N_{\mathbf{C}}/N$ and a map $\alpha^*: (N_{\mathbf{C}})^* \to (N_{\mathbf{C}})^*$ where $(N_{\mathbf{C}})^*$ = dual space of $N_{\mathbf{C}}$ with respect to $\langle\ ,\ \rangle$. If $\alpha^*$ also preserves the dual lattice $M$ of $N$, $\sigma \subset N_{\mathbf{R}}$ is a c.r.p cone, then $\alpha^*$ maps $(\alpha(\sigma))^\frown \cap M$ to $\hat{\sigma} \cap M$, and $\alpha^*: X_\sigma \to X_{\alpha(\sigma))}$ is an extension of $\alpha$ on $T$.

The construction of torus embedding can be used to resolve some type of singularities.

**1.2. Toroidal compactification of the quotient of Siegel spaces.** The theory of toroidal compactification was developed for the locally symmetric varieties in general. We introduce it here for the case of Siegel spaces.

Let $M(n; k) = \{\text{all } n \times n \text{ matrices over } k\}$, $k = \mathbf{C}, \mathbf{r}, \mathbf{Z}, \cdots$,

$$S_n = \{\tau \in M(n; \mathbf{C});\ {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}: \text{ the Siegel space of rank } n,$$

$$G = \operatorname{Sp}(n; \mathbf{R})$$

$$= \left\{ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix};\ A, B, C, D \in M(n; \mathbf{R}),\right.$$

$$\left. {}^tM \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\},$$

the symplectic group of rank $n$.

$G$ acts on $S_n$ by $M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$ for $M \in G$, $\tau \in S_n$; then $S_n = G/K$ as a homogenous space, where $K + \operatorname{Iso}(\sqrt{-1}I_n)$ is a maximal compact subgroup of $G$. We will discuss the compactifications of locally symmetric Hermitian space $\Gamma\backslash S_n = \Gamma\backslash G/K$ for an arithmetic subgroup $\Gamma$ of $G$.

First, we can realize $S_n$ as a symmetric bounded domain $S'_n$ in $M_s(n; \mathbf{C}) = \{\text{all symmetric } n \times n \ \mathbf{C}\text{-matrices}\}$ by $\tau \in S_n \to Z = (\tau - \sqrt{-1}I_n)(\tau + \sqrt{-1}I_n)^{-1} \in S'_n$, $S'_n = \{Z \in M_s(n; \mathbf{C}) I_n - Z\overline{Z} > 0\}$. (This is a Harish-Chandra realization of homogeneous space.) Then $\tau = \sqrt{-1}(Z + I_n)(-Z + I_n)^{-1} \in S_n$ for $Z \in S'_n$.

For simplicity of notation, we will denote both bounded and unbounded realizations of Siegel space as $S_n$ except, if it is needed, we will distinguish their members by $\tau$ and $Z$ as above.

Let $\overline{S_n} = \{Z \in M_s(n; \mathbf{C});\ I - Z\overline{Z} \geq 0\}$ be the topological closure of $S_n$ is $M_s(n; \mathbf{C})$. For $p, q \in \overline{S_n}$, we say "$p \sim q$" if there exist holomorphic maps

$$\alpha_i: S_1 = \{Z \in \mathbf{C};\ |Z| < 1\} \to \overline{S_n} \quad \text{for } i = 1, 2, \cdots, m,$$

such that $\alpha_1(0) = p$, $\alpha_m(0) = q$ and $\alpha_i(S_1) \cap \alpha_{i+1}(S_1) \neq \varnothing$. "$\sim$" defines an equivalence relation on $\overline{S_n}$.

A boundary component of $S_n$ is a maximal subset in $\overline{S_n}$ of mutually equivalent points. Then $\overline{S_n}$ = disjoint union of boundary components. The action of $G$ on $S_n$ extends to $\overline{S_n}$ by

$$M \cdot Z = ((A - \sqrt{-1}C)(Z+1) + (B - \sqrt{-1}D)\sqrt{-1}(Z-1)) \cdot ((A + \sqrt{-1}C)(Z+1) + (B + \sqrt{-1}D)\sqrt{-1}(Z-1))^{-1}$$

for $M = [\begin{smallmatrix} A & B \\ C & B \end{smallmatrix}] \in G$, $Z \in \overline{S_n}$.

**Fact.** (i)

$$F_{n'} = \left\{ \begin{bmatrix} Z' & 0 \\ 0 & I_{n-n'} \end{bmatrix} ; Z' \in S_{n'} \right\}$$

is a boundary component of $\overline{S_n}$ for any $0 \le n' \le n$.

(ii) Any boundary component of $\overline{S_n}$ has the form $g \cdot F_{n'}$ for some $g \in G$, $0 \le n' \le n$.

The boundary component $F = g \cdot F_{n'}$ with $g \in \mathrm{Sp}(n; \mathbf{Q})$ is called rational. Actually, $F$ is rational $\Leftrightarrow \exists g \in \mathrm{Sp}(n; \mathbf{Z})$ such that $F = g \cdot F_{n'}$ for some $0 \le n' \le n$.

**Remark.** Let $S_n^* = \bigcup\{\text{all rational components}\} = \bigcup_{0 \le n' \le n} \mathrm{Sp}(n; \mathbf{Z}) \cdot F_{n'} \subset \overline{S_n}$ ($S_n^*$ is called the rational closure of $S_n$). Then $\Gamma\backslash S_n^*$ with suitable defined topology gives the so-called Stake-Baily-Borel compactification of $\Gamma\backslash S_n$.

Let $F$ be a boundary component of $\overline{S_n}$, and define:

$N(F) = \{g \in G; g \cdot F = F\}$ which will be a parabolic subgroup,

$W(F)$ = the unipotent radical of $N(F)$ (i.e. biggest unipotent normal subgroup of $N(F)$),

$U(F)$ = center of $W(F)$,

$V(F) = W(F)/U(F)$.

If $F' = g \cdot F$ for $g \in G$, then $N(F') = gN(F)g^{-1}$. Therefore, it is enough to know the structures of these groups for $F_{n'}$.

**Fact.** For $F = F_{n'}$, $0 \le n' \le n$, we have the following:

(i)

$$N(F_{n'}) = \left\{ \begin{bmatrix} A' & 0 & B' & * \\ & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \in G; \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}(n'; \mathbf{R}), \quad u \in \mathrm{GL}(n-n'; \mathbf{R}) \right\}.$$

(ii)

$$W(F_{n'}) = \left\{ \begin{bmatrix} I_{n'} & 0 & 0 & E \\ {}^tH & I_{n-n'} & {}^tE & J \\ 0 & 0 & I_{n'} & -H \\ 0 & 0 & 0 & I_{n-n'} \end{bmatrix} ; {}^tEH + J = {}^tHE + {}^tJ \right\}.$$

(iii)

$$U(F_{n'}) = \left\{ \begin{bmatrix} I_{n'} & 0 & 0 & 0 \\ 0 & I_{n-n'} & 0 & J \\ 0 & 0 & I_{n'} & 0 \\ 0 & 0 & 0 & I_{n-n'} \end{bmatrix} = [J]; {}^tJ = J \right\} \simeq M_s(n-n'; \mathbf{R}).$$

Then $v(F_{n'}) = W(F_{n'})/U(F_{n'}) \simeq \{E+\sqrt{-1}H;\ E$ and $H$ are $n\times(n-n')\mathbf{R}$-matrices$\}$ as additive groups.

On the other hand, for the homogeneous space $S_n = G/K$, let $S_n^c = G_{\mathbf{C}}/B$ be the compact dual of $S_n$ where $G_{\mathbf{C}}$ is the complexification of $G$, $B$ is a parabolic subgroup of $G_{\mathbf{C}}$, $S_n \subset S_n^c$.

For a boundary component $F$, we define $S(F) = U(F)_{\mathbf{C}} \cdot S_n \subset S_n^c$. If $F = F_{n'}$, then $U(F_{n'})_{\mathbf{C}} \simeq M_s(n-n'; \mathbf{C}) = \{$symmetric $\mathbf{C}$-matrices of order $n-n'\}$.

$$S(F_{n'}) = U(Fn')_{\mathbf{C}} \cdot S_m = \left\{ \tau = \begin{bmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{bmatrix} \in M(n; \mathbf{C}); \tau_1 \in S_{n'}, \right.$$
$$\left. \tau_3 \in M_s(n-n'; \mathbf{C}) \right\}.$$

$S_n \to S(F_{n'})$ by natural inclusion. $S(F) \simeq F \times V(F) \times U(F)_{\mathbf{C}}$ holomorphically, and $U(F)$ acts on $S(F)$ as the linear translations on the factor $U(F)_{\mathbf{C}}$. In particular,

$$S(F_{n'}) \simeq F_{n'} \times V(F_{n'}) \times U(F_{n'})_{\mathbf{C}},$$
$$\tau = \begin{bmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{bmatrix} \longleftrightarrow (\tau_1, \tau_2, \tau_s), \qquad \tau_1 \in S_{n'},\ \tau_2 \in V(F_{n'}),\ \tau_3 \in U(F_{n'})_{\mathbf{C}},$$

$S_n \to S(F_{n'})$ characterized by

$$J = \operatorname{Im}\tau_3 - {}^t(\operatorname{Im}\tau_2)(\operatorname{Im}\tau_1)^{-1}(\operatorname{Im}\tau_2) > 0$$

as a $(n-n')\times(n-n')$ symmetric $\mathbf{R}$-matrix. (This is called the realization as a Siegel domain of the third kind.)

We define $C(F_{n'}) = \{J \in M_s(n-n'; \mathbf{R}); J > 0\} \subset U(F_{n'})$; then $C(F_{n'})$ is a cone in $U(F_{n'})$, $\Phi: S(F_{n'}) \to U(F_{n'})$, $\Phi(\tau) = J$, $S_n = \Phi^{-1}(C(F_{n'}))$.

We define two subgroups in $N(F_{n'})$:

$$G_h(F_{n'}) = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{n-n'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{n-n'} \end{bmatrix} ; \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}(n'; \mathbf{R}) \right\} \simeq \mathrm{Aut}(F_{n'}),$$

$$G_l(F_{n'}) = \left\{ \begin{bmatrix} I_{n'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{n'} & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} ; u \in \mathrm{GL}(n-n'; \mathbf{R}) \right\}.$$

Let $\mathrm{Aut}(U(F_{n'}), C(F_{n'})) = \{$automorphism of $U(F_{n'})$ which preserves $C(F_{n'})\}$; then $G_l(F_{n'}) = \mathrm{Aut}(U(F_{n'}), C(F_{n'}))$ with the action

$$u(J) = u \cdot J \cdot {}^t u \quad \text{for } u \in G_l(F_{n'}),\ J \in U(F_{n'}),$$

and $N(F_{n'}) = (G_h(F_{n'}) \times G_l(F_{n'})) \cdot W(F_{n'})$ with two projections

$$p_l \colon N(F_{n'}) \to G_l(F_{n'}), \qquad g = \begin{bmatrix} A' & 0 & B' & * \\ & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \to u,$$

$$p_h \colon N(F_{n'}) \to G_h(F_{n'}), \qquad g \to \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}.$$

For the embedding $S_n \to F_{n'} \times V(F_{n'}) \times U(F_{n'})_{\mathbf{C}}$, the action of $N(F_{n'})$ on $S_n$ is of the form:

$$(\tau_1, \tau_2, \tau_3) \to (g(\tau_1), B(\tau_1)\tau_2 + b(\tau_1), A(\tau_3) + a(\tau_1, \tau_2)),$$

where $(\tau_1, \tau_2, \tau_3) \in S_n \subset F_{n'} \times V(F_{n'}) \times U(F_{n'})_{\mathbf{C}}$, $g(\tau_1)$ is the action on $F_{n'}$ induced by $p_h \colon N(F_{n'}) \to G_h(F_{n'})$, and $A(\tau_3)$ is the action on $U(F_{n'})_{\mathbf{C}}$ induced by $p_l \colon N(F_{n'}) \to G_l(F_{n'})$.

Although some group structures and the fact above are introduced for $S_n$ and $F_{n'}$ here only, they can be actually generalized for the bounded symmetric domains by using the theory of Lie groups and Lie algebras (see [1]).

With all set-ups above, we are now able to construct the toroidal compactification of $X = \Gamma\backslash S_n$ for a given arithmetic subgroups $\Gamma$ of $G$.

Let $\Gamma(F) = \Gamma \cap N(F)$ for each rational boundary component $F$,

$$\overline{\Gamma}(F) = p_l(\Gamma(F)) \subset G_l(F),$$

and let $L(F) = \Gamma \cap U(F)$, which is a lattice in vector space $U(F)$. Then given a rational boundary component $F$, we have:

$U(F)$: a real vector space which will correspond to $N_{\mathbf{R}}$ as in §1.1,
$L(F) \subset U(F)$: a lattice which defines a rational structure on $U(F)$ and corresponds to $N$ as in §1.1,
$C(F) \subset U(F)$: a cone,
$\overline{\Gamma}(F)$: arithmetic subgroup of $\mathrm{Aut}(U(F), C(F))$.

The compactification of $X = \Gamma\backslash S_n$ is constructed for a given, so-called, $\Gamma$-admissible family of polyhedral decomposition.

**Definition.** A $\Gamma$-admissible family of polyhedral decomposition is a collection of polyhedral $\Sigma = \{\Sigma_F\}_{F \text{ rational}}$ such that:

1. $\Sigma_F$ is a $\overline{\Gamma}(F)$-admissible polyhedral decomposition of $C(F)$ for every rational boundary component $F$ (that means: $\Sigma_F = \{\sigma_\alpha^F\}_\alpha$). With the rational structure on $U(F)$ by $L(F)$,

 (i) each $\sigma_\alpha^F$ is a convex rational polyhedral cone in $\overline{C}(F)$;
 (ii) $\sigma_\alpha^F \in \Sigma_f$, $\sigma < \sigma_\alpha^F$ (i.e. $\sigma$ is a fact of $\sigma_\alpha^F$) $\Rightarrow \sigma \in \Sigma_F$;
 (iii) $\sigma_\sigma^F, \sigma_\beta^F \in \Sigma_F \Rightarrow \sigma_\alpha^F \cap \sigma_f \beta^F < \sigma_f \alpha^F, \sigma_\beta^F$;
 (iv) $\gamma \in \overline{\Gamma}(F)$, $\sigma_\alpha^F \in \Sigma_F \Rightarrow \gamma \cdot \sigma_\alpha^F \in \Sigma_F$;
 (v) the number of classes of cones module $\overline{\Gamma}(F)$ is finite;
 (vi) $C(F) \subset \bigcup_\alpha \sigma_\alpha^F$ (this is called the rational closure of $C(F)$).

2. If $F_1 = \gamma \cdot F_2$ for $\gamma \in \Gamma$, then $\{\sigma_\alpha^{F_1}\} = \{\gamma \cdot \sigma_\alpha^{F_2}\}$ by the map

$$\gamma \colon U(F_2) \to U(F_1), \qquad g \to \gamma g \gamma^{-1},$$

which is a linear transformation under the realizations of $U(F_1)$ and $U(F_2)$ as vector spaces.

3. If $F_1 < F_2$ (i.e., $F_1 \subset \overline{F_2}$), then $\Sigma_{F_2} = \Sigma_{F_1}|_{U(F_2)}$ by inclusion $U(F_2) \subset U(F_1)$.

The reduction theory of selfadjoint cones gives us the existence of such a $\Gamma$-admissible family since each $C(F)$ is a self-adjoint cone with respect to some inner product on $U(F)$. In fact, it is clear from the definition that a $\Gamma$-admissible family is essentially determined by a $\overline{\Gamma}(F_0)$-admissible polyhedral decomposition of $C(F_0)$ for the smallest standard rational boundary component $F_0$.

Now, supposedly, we are given a $\Gamma$-admissible family; then a toroidal compactification is constructed by the following processes:

**Step 1.** Partial torus compactification for each rational boundary component.

Let $F$ be a rational boundary component, and $\Sigma_F = \{\sigma_\alpha^F\}_\alpha$ be a $\overline{\Gamma}(F)$-admissible polyhedral decomposition of $C(F)$. Then

$$S(F) \simeq F \times V(F) \times U(F)_{\mathbf{C}}, \qquad S_n \subset S(F).$$

Let $S(F)' = S(F)/U(F)_{\mathbf{C}}$; then $S(F)' \simeq F \times V(F)$, and $\prod'_F : S(F) \to S(F)'$ is a principal $U(F)_{\mathbf{C}}$-bundle.

If $T(F) = L(F)\backslash U(F)_{\mathbf{C}}$ is an algebraic torus, then

$$L(F)\backslash S(F) \simeq F \times V(F) \times (L(F)\backslash U(F)_{\mathbf{C}}) \xrightarrow{\overline{\prod}'_F} S(F)' \simeq F \times V(F),$$

and $\overline{\prod}'_F$ is a principal bundle with fibre $T(F)$.

The $\overline{\Gamma}(F)$-admissible polyhedral decomposition $\Sigma_F$ defines a torus embedding:

$$T(F) \subset X_{\Sigma_F}.$$

Then we can construct a fibre bundle

$$(L(F)\backslash S(F))_{\Sigma_F} = (L(F)\backslash S(F)) \times_{T(F)} X_{\Sigma_F}$$

over $S(F)'$ with fibre $X_{\Sigma_F}$. Let

$(L(F)\backslash S_n)_{\Sigma_F}$ = the interior of the closure of $L(F)\backslash S_n$ in $(L(F)\backslash S(F))_{\Sigma_F}$.

**Step 2.** Gluing.

By the definition of $\Gamma$-admissible family and the construction of partial torus compactification, we have the following properties:

(1) if $F_1 < F_2$, $F_1$, $F_2$ are two rational boundary components, then

$$\Sigma_{F_2} = \Sigma_{F_1}|_{U(F_2)}, \qquad U(F_1) \supset U(F_2),$$
$$L(F_2)\backslash S_n \to L(F_1)\backslash S_n, \qquad X_{\Sigma_{F_2}} \to X_{\Sigma_{F_1}},$$

and there is an étale map

$$\prod_{1,2} : (L(F_1)\backslash S_n)_{\Sigma_{F_2}} \to (L(F_1)\backslash S_n)_{\Sigma_{F_1}};$$

(2) if $F_2 = \gamma \cdot F_1$ for $\gamma \in \Gamma$, then $\gamma$ induces an isomorphism

$$\begin{array}{ccc} \gamma\colon U(F_1) & \longrightarrow & U(F_2) \\ \cup & & \cup \\ L(F_1) & \longrightarrow & L(F_2) \\ \gamma|_{C(F_1)}\colon C(F_1) & \longrightarrow & C(F_2), \end{array}$$

and by the definition, $\Sigma_{F_2} = \gamma \cdot \Sigma_{F_1}$. Hence, the action of $\gamma$ on $S_n$ extends to an isomorphism

$$\gamma\colon (L(F_1)\backslash S_n)_{\Sigma_{F_1}} \to (L(F_2)\backslash S_n)_{\Sigma_{F_2}}.$$

Let $(\Gamma\backslash S_n)^{\sim} = \bigcup_{F\,:\,\text{rational}}(L(F)\backslash S_n)_{\Sigma_F}$. An equivalence relation can be defined on $(\Gamma\backslash S_n)^{\sim}$ as follows: $X_1, X_2 \in (\Gamma\backslash S_n)^{\sim}$, assuming that $X_1 \in (L(F_1)\backslash S_n)_{\Sigma_{F_1}}$, $X_2 \in (L(F_2)\backslash S_n)_{\Sigma_{F_2}}$,

$$X_1 \sim X_2 \Leftrightarrow (1)\exists F,\ \text{rational and } \gamma \in \Gamma,\ \text{s.t. } F_1 < F,\ \gamma F_2 < F,$$
$$(2)\exists X \in (L(F)\backslash S_n)_{\Sigma_F}\ \text{s.t.}$$
$$\prod_1 : (L(F)\backslash S_n)_{\Sigma_F} \to (L(F_1)\backslash S_n)_{\Sigma_{F_1}}, \qquad \prod_1(X) = X_1,$$
$$\prod_{\gamma,2} : (L(F)\backslash S_n)_{\Sigma_F} \to (L(\gamma F_2)\backslash S_n)_{\Sigma_{\gamma F_2}}, \qquad \prod_{\gamma,2}(X) = \gamma \cdot X_2.$$

It can be proved that " $\sim$ " is an equivalence relation.

Let $\overline{(\Gamma\backslash S_n)} = (\Gamma\backslash S_n)^{\sim}/\sim$.

**Theorem 1.1** [13]. *$\overline{(\Gamma\backslash S_n)}$ is a Hausdorff analytic variety containing $\Gamma\backslash S_n$ as an open dense subset, and $\overline{(\Gamma\backslash S_n)}$ is a compact algebraic space. $\overline{(\Gamma\backslash S_n)}$ is called a toroidal compactification of $\Gamma\backslash S_n$ and sometimes also a Mumford's compactification.*

It is clear that the compactification constructed above depends on the choice of $\Gamma$-admissible family of polyhedral decompositions and in general only on a compact algebraic variety. For the smoothness and projectivity of these compactifications, we need the following definitions.

**Definition.** 1. A subgroup $\Gamma$ of $G$ is said to be *neat* if the subgroup of $\mathbf{C}^*$ generated by the eigenvalues of all $\gamma \in \Gamma$ is torsion free. (Then, $\Gamma\backslash S_n$ will be smooth.)

2. A $\Gamma$-admissible family of polyhedral decomposition is said to be *projective* if there exists a continuous convex piecewise linear function $f: \Omega \to \mathbf{R}$ where $\Omega = \bigcup_{F \text{ rational}} C(F)$ such that

(1) $f(X) > 0$ for $X \neq 0$,

(2) for each $\sigma_\alpha \in \Sigma_F$, there is a linear function $l_\alpha$ on $U(F)$ such that

(a) $l_\alpha \geq f$ on $C(F)$,

(b) $\sigma_\alpha = \{X \in \overline{C}(F)\,;\, l_\alpha(X) = f(X)\}$,

(3) $f(\Gamma \cap \Omega) \subset \mathbf{Z}$ (a function $f$ with (1), (2) and (3) is called a polar function),

(4) $f$ is $\Gamma$-invariant.

**Theorem 1.2** [13]. 1. *If $\Gamma$ is neat and all cones $\sigma_\alpha^F$ in $\Sigma_F$ are regular with respect to $\Gamma$, i.e., each cone $\sigma_\alpha^F$ is generated by a part of a $\mathbf{Z}$-basis of $L(F) = U(F) \cap \Gamma$, then the compactification $\overline{(\Gamma\backslash S_n)}$ constructed before is smooth.*

2. *If* $\Sigma = \{\Sigma_F\}_{F \text{ rational}}$ *is a projective* $\Gamma$*-admissible family, then* $\overline{(\Gamma\backslash S_n)}$ *is projective.*

**Remarks.** 1. Any arithmetic subgroup $\Gamma$ contains a neat subgroup of finite index. Also, the principal congruence subgroups $\Gamma(k)$ of $\mathrm{Sp}(n;\mathbf{Z})$ are neat for $k \geq 3$:

$$\Gamma(k) \overset{\text{Def}}{=} \{X \in \mathrm{Sp}(n;\mathbf{Z});\, X \equiv I \pmod{k}\}.$$

2. For any $\Gamma$-admissible decomposition $\Sigma$, there exists a refinement $\Sigma'$ such that all cones in $\Sigma'$ are regular, and the toroidal compactification constructed from $\Sigma'$ is a blowing-up of the one from $\Sigma$.

3. A special projective $\Gamma$-admissible family can be obtained from the reduction theory. The resulting family is called the central cone decomposition which we will discuss later in detail.

4. Therefore, for any neat arithmetic subgroup $\Gamma$, there is a nonsingular and projective toroidal compactification of $\Gamma\backslash S_n$.

5. In each partial compactification $(L(F)\backslash S_n)_{\Sigma_F}$, there is an orbit decomposition with respect to the members of $\Sigma_F$. Let

$$O(F) = \bigcup_{\sigma_\alpha \cap C(F) \neq \varnothing} O(\alpha) \subset (L(F)\backslash S_n)_{\Sigma_F},$$

$$\overline{O}(F) = (\Gamma(F)/L(F))\backslash O(F).$$

Recall $\Gamma(F) = \Gamma \cap N(F)$, $L(F) = U(F) \cap \Gamma$. Then, $\overline{(\Gamma\backslash S_n)} = \bigcup_{F \bmod \Gamma} \overline{O}(F)$ as a set where $F$ runs through all rational boundary components, and the map $\prod_F$ from $(L(F)\backslash S_n)_{\Sigma_F}$ to $\overline{(\Gamma\backslash S_n)}$ factors through

$$\prod_F : (L(F)\backslash S_n)_{\Sigma_F} \to (\Gamma(F)/L(F))\backslash (L(F)\backslash S_n)_{\Sigma_F} \xrightarrow{\overline{\prod}_F} \overline{(\Gamma\backslash S_n)},$$

where $\overline{\prod}_F$ is injective near $\overline{O}(F)$, and $\Gamma(F)/L(F)$ acts on $(L(F)\backslash S_n)_{\Sigma_F}$ properly discontinuously.

Notice that $O(F) = S_n$ and $\overline{O}(F) = \Gamma\backslash S_n$ if $F = S_n$.

## 2. The boundary divisors of $\overline{(\Gamma\backslash S_n)}$

Let $X = \Gamma\backslash S_n$ with $\Gamma$ neat, and $\overline{X}$ be a smooth projective toroidal compactification of $X$ constructed from a $\Gamma$-admissible family. Then $\overline{X} - X$ is a divisor of $\overline{X}$ with normal crossing, i.e., $D = \overline{X} - X = \sum_{i=1}^m D_i$ where each $D_i$ is an irreducible smooth divisor of $\overline{X}$, and $D_1, \cdots, D_m$ intersect transversally. In this section, we will discuss the structure of $D$ in the case $\Gamma = \Gamma(k)$ for $k \geq 3$, since those $\Gamma(k)$ are neat.

**2.1. Central cone decomposition.** It is known that, up to $\mathrm{Sp}(n;\mathbf{Z})$, $\{F_{n'}; 0 \le n' \le n\}$ are only rational boundary components of $S_n$. Therefore, $\{(\mathrm{Sp}(n;\mathbf{Z})/\Gamma(k))\cdot F_{n'}; 0 \le n' \le n\}$ will be all inequivalent classes of rational boundary components of $S_n$ under $\Gamma(k)$. Since $F_0 = \{I = n \times n$ identity matrix$\} \le F_{n'}$ and $C(F_{n'}) < C(F_0)$ for any $0 \le n' \le n$, $\gamma\cdot F_0 < \gamma\cdot F_{n'}$ and $\gamma\cdot C(F_{n'}) \subset \gamma\cdot\overline{C}(F_0)$ for any $\gamma \in \mathrm{Sp}(n;\mathbf{Z})/\Gamma(k)$. Then, by the definition, finding a $\Gamma(k)$-admissible family of polyhedral decomposition is reduced to finding a $\overline{\Gamma}(F_0)$-admissible polyhedral decomposition of $C(F_0)$.

Recall the following:

$F_0 = \{I\}$,

$N(F_0) = \left\{\begin{bmatrix} u & * \\ 0 & {}^t u^{-1}\end{bmatrix} \in \mathrm{Sp}(n;\mathbf{R}),\ u \in \mathrm{GL}(n;\mathbf{R})\right\}$,

$U(F_0) = \left\{\begin{bmatrix} I & J \\ 0 & I\end{bmatrix} =: [J];\ J = {}^t J\right\} \simeq M_s(n;\mathbf{R})$,

$C(F_0) = \{J \in U(F_0);\ J > 0\}$ = the set of all positive definite $n \times n$ symmetric $\mathbf{R}$-matrices,

$\Gamma(F_0) = \Gamma \cap N(F_0)$,

$L(F_0) = \Gamma \cap U(F_0) = \{J \in M_s(n;\mathbf{Z}),\ J \equiv 0 \pmod k\}$,

$\overline{\Gamma}(F_0) = p_l(\Gamma(F_0)) = \{u \in \mathrm{GL}(n;\mathbf{Z}),\ u \equiv I \pmod k\} = \mathrm{GL}(n;\mathbf{Z})(k)$,

where $p_l\colon N(F_0) \to G_l(F_0) = \mathrm{Aut}(U(F_0), C(F_0))$.

Since $\overline{\Gamma}(F_0) = \mathrm{GL}(n;\mathbf{Z})(k)$, it will be enough if we can find a $\mathrm{GL}(n;\mathbf{Z})$-admissible decomposition of $C(F_0)$. There are several types of such decompositions. We introduce one here, called "central cone decomposition", for our purpose.

Let $V = M_s(n;\mathbf{R})$ be the vector space of all $n \times n$ symmetric $\mathbf{R}$-matrices, and $C = C(F_0)$ be the cone in $V$. We may consider $V$ as the Lie algebra of $U(F_0)$ and $V$ is isomorphic to $U(F_0)$ by the exponential map. If $B(\ ,\ )$ is the killing form on $V$, then

$$B(X, Y) = \mathrm{Tr}(XY) \overset{\mathrm{Def}}{=} \langle X, Y\rangle.$$

It induces a quadratic form on $U(F_0)$, and $C$ is a self-dual cone in $V$ with respect to $\langle\ ,\ \rangle$.

Let $L = M_s(n;\mathbf{Z})$ be the standard lattice in $V$, $L^*$ be the dual lattice of $L$ w.r.t. $\langle\ ,\ \rangle$; then

$$L_{\mathbf{R}} = L \otimes \mathbf{R} = V,$$
$$L^* = \{Y;\ 2Y \in M_s(n;\mathbf{Z}),\ Y = (y_{ij}),\ y_{ii} \in \mathbf{Z}\}.$$

For $X \in \overline{C}$ and $Y \in C \cap L^*$, define

$$\phi(X) = \min_{Y \in C \cap L^*} \operatorname{Tr}(XY),$$
$$\sigma(Y) = \{X \in \overline{C};\, \phi(X) = \operatorname{Tr}(XY)\}.$$

**Theorem 2.1** [13]. $\Sigma_{\text{cent}} \overset{\text{Def}}{=} \{\sigma = \sigma(Y);\, Y \in C \cap L^*\}$ *is a* $\mathrm{GL}(n; \mathbf{Z})$-*admissible polyhedral decomposition of* $C$ *with rational structure by* $L$, *and* $\Sigma_{\text{cent}}$ *is projective with the polar function* $\phi(X)$.

$\sigma(Y)$, which has same dimension as $C$, is called a central cone, such a $Y$ is called a central element, and $\Sigma_{\text{cent}}$ is called the central decomposition.

Since the rational structure of $C$ induced by $L(F_0) = kL$ is equivalent to one induced by $L$, $\Sigma_{\text{cent}}$ actually also gives us a projective $\overline{\Gamma}(F_0)$-admissible polyhedral decomposition of $C(F_0)$. $\Sigma_{\text{cent}}$ is not regular in general; in fact, it is only regular for $n \leq 3$. We may assume a regular decomposition by refining $\Sigma_{\text{cent}}$ in a suitable way.

There is a regular central cone in $\sum_{\text{cent}}$ for all $n$, which is called the principal cone $\sigma_0$:

$$\sigma_0 = \sigma(Y_0) \quad \text{where } Y_0 = \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \cdot & \cdot & \cdots & \cdot \\ 1/2 & 1/2 & \cdots & 1 \end{bmatrix} \in C \cap L^*.$$

Then

$$\sigma_0 = \left\{ X = (x_{ij});\, X = {}^t X,\ x_{ij} \leq 0 (i \neq j),\, \sum_{j=1}^{n} x_{ij} \geq 0 \right\}$$
$$= \left\{ \sum_{1 \leq i, j \leq n} \lambda_{ij} e_{ij};\ \lambda_{ij} \in \mathbf{R}^+ \right\},$$

where

$$e_{ij} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i \\ \\ \\ \\ j \end{matrix}, \quad 1 \leq i < j \leq n,$$

$$e_{ii} = \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} i, \quad 1 \leq i \leq n.$$

**2.2. The boundary divisor of $\overline{X}$ where $X = \Gamma(k)\backslash S_n$, $k \geq 3$.** Let $F_0 = \{I_n\}$ be the standard rational boundary component of rank 0. Then $U(F_0)$, $C(F_0)$, $L(F_0)$, $\Gamma(F_0)$ and $\overline{\Gamma}(F_0)$ are groups mentioned in §2.2. Moreover,

$\overline{\Gamma}(F_0) \simeq \Gamma(F_0)/L(F_0)$,

$S(F_0) = U(F_0)_{\mathbf{C}} \cdot S_n \simeq U(F_0)_{\mathbf{C}} + M_s(n; \mathbf{C}) \simeq \mathbf{C}^N$, $N = \frac{n(n+1)}{2}$,

$S_n \to S(F_0)$, $\tau = X + \sqrt{-1}Y \to \tau$, $Y > 0$,

$L(F_0)\backslash S_n \to L(F_0)\backslash S(F_0) \simeq (\mathbf{C}^*)^N$,

$S_n \to L(F_0)\backslash S_n$ by $\tau = (\tau_1, \cdots, \tau_N) \to W = (w_1, \cdots, w_N), w_j = e^{i(2\pi/k)\tau_j}$,

$U(F_0)$ acts on $S_n$ as $J \cdot \tau = \tau + J$ for $J \in U(F_0)$, $\tau \in S_n$.

As in §2.2, we view:

$V = U(F_0) = \mathbf{R}^N$ as a vector space,

$C = C(F_0)$ as a cone in $V$,

$\langle\ ,\ \rangle$: the bilinear form on $V$, then $C$ is self-dual,

$\overline{\Gamma}(F_0) = \mathrm{GL}(n; \mathbf{Z})(k) \subset \mathrm{Aut}(V, C)$,

$L = L(F_0) = k \cdot M_s(n; \mathbf{Z}) = (k\mathbf{Z})^N$, a lattice in $V$,

$L^* =$ dual lattice of $L$ w.r.t. $\langle\ ,\ \rangle = \{\frac{1}{k}(x_{ij})_{n\times n};\ x_{ij} \in \frac{1}{2}\mathbf{Z}$ for $i \neq j$, $x_{ii} \in \mathbf{Z}\}$,

$T \overset{\text{Def}}{=} L(F_0)\backslash S(F_0) = L\backslash V_{\mathbf{C}} = (\mathbf{C}^*)^N$, a torus.

If we take $\{e_1, e_2, \cdots, e_N\}$ as a basis for $L$ such that, for any $a = k(a_{ij}) \in L$, $a_{ij} \in \mathbf{Z}$, then $a = \sum_{i=1}^N a_i e_i$ where

$$(a_1, a_2, \cdots, a_N) = (a_{11}, a_{12}, \cdots, a_{1n}, a_{22}, \cdots, a_{2n}, \cdots, a_{nn}).$$

(In fact, $e_l$ is of form $k(b_{st})_{n\times n}$ where $b_{ij} = b_{ji} = 1$ for some pair $(i, j)$, $i \leq j$ and all others $b_{st}$ are 0.) Let $\{e^1, e^2, -\cdots, e^N\}$ be the dual basis for $L^*$. Then, for $r = \frac{1}{k}(r_{ij}) \in L^*$ with $r_{ii} \in \mathbf{Z}$, $r_{ji} \in \frac{1}{2}\mathbf{Z}$ for $i \neq j$, $r = \sum_{i=1}^N r_i e^i$ where

$$(r_1, r_2, \cdots, r_N) = (r_{11}, 2r_{12}, \cdots, 2r_{1n}, r_{22}, 2r_{23}, \cdots, 2r_{2n}, \cdots, r_{nn}).$$

Under these bases, $\langle a, r\rangle = \sum_{i=1}^N a_i r_i$, and we can identify $L$ with $\mathrm{Hom}(\mathbf{C}^*, T)$, denoted as $N$ in §1.1, and $L^*$ with $\mathrm{Hom}(T, \mathbf{C}^*)$, denoted as $M$ in §1.1, by the following maps:

$$a = (a_1, \cdots, a_N) \in L \to \lambda_a: \mathbf{C}^* \to T, \qquad \lambda_a(t) = (t^{a_1}, \cdots t^{a_N}),$$

$$r = (r_1, \cdots, r_N) \in L^* \to \chi_r: T \to \mathbf{C}^*, \qquad \chi_r((t_1, \cdots, t_N)) = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_N}.$$

Therefore, $\chi_r(\lambda_a(t)) = t^{\langle a, r\rangle}$ for $a \in L$, $r \in L^*$, $t \in \mathbf{C}^*$, and

$$T = \mathrm{Spec}(\mathbf{C}_{r\in L^*}[\chi_r]).$$

If $\sigma$ is a rational regular polyhedral cone of maximal dimension in $\overline{C}$,

$$\sigma = \left\{ \sum_{i=1}^{N} \lambda_i v_i ; \lambda_i \geq 0 \right\}$$

with vertices $v_1, v_2, \cdots, v_n$ such that $\{v_1, v_2, \cdots, v_N\}$ is an integral basis for the lattice $L$, let $\{r^1, \cdots, r^N\}$ be dual basis of $\{v_1, \cdots, v_n\}$; then $r^i \in L^*$. Thus

$\hat{\sigma} \cap L^* = \sum_{i=1}^N r^i \mathbf{Z}^+$,

$X_\sigma = \operatorname{Spec}(\mathbf{C}[\hat{\sigma} \cap L^*]) = \operatorname{Spec}(\mathbf{C}[\chi_{r^1}, \cdots, \chi_{r^N}]) \simeq \mathbf{C}^N$,

$T \to X_\sigma$ by $t = (t_1, \cdots, t_N) \to (\chi_{r^1}(t), \cdots, \chi_{r^N}(t))$.

Moreover, $u = (u_1, \cdots, u_N)$ where $u = \chi_{r^i}(t) = t_1^{r_1^i} \cdots t_n^{r_N^i}$ and $r^i = (r_1^i, \cdots, r_n^i)$ will give the complex coordinate system in $X_\sigma$.

**Proposition 2.1.** *The irreducible components of the boundary $D = \overline{X} - X$ are in one-one correspondence with the vertices of all maximal dimensional cones in the $\Gamma$-admissible family.*

*Proof.* Let $v$ be a vertex of a maximal dimensional cone in the $\Gamma$-admissible family. We may assume, without loss of generality, that $v \in \overline{C}(F_0)$.

**Case 1.** If $v \in C(F_0)$, i.e., $v$ lies in the interior of $\overline{C}(F_0)$, it follows that $v$ must be surrounded by finite number of maximal regular cones $\{\sigma_1, \cdots, \sigma_m\}$ up to $\overline{\Gamma}(F_0)$. Let $\tau = \mathbf{R}^+ v$; then $\tau$ is a common face of $\sigma_i$'s, $1 \leq i \leq m$, and $\tau \cap C(F_0) \neq \emptyset$. This implies

$$O(\tau) \subset O(F_0) \to \overline{\Gamma}(F_0) \backslash O(F_0) = \overline{O}(F_0).$$

Since $\dim \tau = 1$, the closure $\overline{O}(\tau)$ is a divisor. In fact, if $u^i = (u_1^i, \cdots, u_N^i)$ is the complex coordinates for each $X_{\sigma_i}$ such that $u_1^i$ is the component associated to vertex $v$, then the divisor $D_v$ associated with $O(\tau)$ is covered by these $m$ coordinate charts, and $D_v$ is defined by $\{u_1^i = 0\}$ in each $X_{\sigma_i}$.

**Case 2.** If $v \in \overline{C}(F_0) - C(F_0)$, then there exists a rational boundary component $F$ of rank $n-1$, i.e., $F = \gamma F_{n-1} 0$, such that $C(F) = \mathbf{R}^+ v$. For any rational boundary component $F'$ such that $F' < F$, we will have the following:

$$U(F') \supset U(F), \qquad \overline{C}(F') \supset C(F),$$

$$\textstyle\prod' : (L(F)\backslash S_n)_{\Sigma_F} \to (L(F')\backslash S_n)_{\Sigma_{F'}},$$

$$O_F(\tau) = O(F) \to O_{F'}(\tau), \qquad \tau = \mathbf{R}^+ v.$$

Thus, the closure of $(\Gamma(F)/L(F))\backslash O(F) = \overline{O}(F)$ in $\overline{X}$ produces an irreducible divisor $D_v$ on the boundary.

The divisors from Cases 1 and 2 represent the different components by Remark 5 at the end of §1.2. On the other hand, from the same remark, the whole boundary $D$ is obtained as the union of all $\overline{O}(F)$ where $F$ runs through all proper rational boundary components of $S_n$. Therefore, each irreducible component of $D$ is obtained from a vertex of either Case 1 or Case 2 above for some rational boundary component $F_0'$ of rank $0$.

Hence the proposition is proved. q.e.d.

Now, we assume that $\overline{X}$ is the compactification of $X = \Gamma(k)\backslash S_n$, $k \geq 3$, from a $\Gamma(k)$-admissible family given by $\Sigma_{\text{cent}}$ or a refinement of $\sum_{\text{cent}}$ if necessary, and $D = \overline{X} - X$. When $n \leq 4$, we have the following.

**Theorem 2.2.** *Let $D = \sum_{i=1}^m D_i$ be the irreducible decomposition of $D$; then the following hold:*

(1) *Each $D_i$ is algebraically isomorphic to $\overline{\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})}$ where $\Gamma' = \mathrm{Sp}(n-1; \mathbf{Z})(k) \times_{\text{semiproduct}} (k\mathbf{Z})^{2(n-1)}$ with group structure such that $\Gamma'$ acts on $S_{n-1} \times \mathbf{C}^{n-1}$ by*

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, (a, b)\right) : (Z_1, Z_2)$$

$$\to ((AZ_1 + B)(CZ_1 + D)^{-1}, (aZ_1 + Z_2 + b)(CB_1 + D)^{-1})$$

*if $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{Sp}(n-1; \mathbf{Z})(k)$, $(a, b) \in (k\mathbf{Z})^{2(n-1)}$ as two row-vectors, and $\overline{\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})}$ is the compactification of $\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})$ induced from same $\Gamma(k)$-admissible family.*

(2) *All $D_i$ intersect along the boundary*

$$\overline{(\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})} - (\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})).$$

*Proof.* It is known that, for $n \leq 3$, $\sum_{\text{cent}}$ is regular and all vertices of maximal dimensional cones in $\sum_{\text{cent}}$ are on the boundary of cone $C(F_0)$ (see [7]). In his thesis [11], McConnell showed that there exists a regular refinement of $\sum_{\text{cent}}$ for $n = 4$ such that all vertices of maximal dimensional cones are also on the boundary of cone $C(F_0)$. Therefore, all irreducible boundary components here are obtained in the manner of Case 2 in Proposition 2.1. Since the vertex in Case 2 of Proposition 2.1 corresponds to a rational boundary component $F$ of rank $n-1$ and all boundary components of same rank are equivalent under the action of $\mathrm{Sp}(n; \mathbf{Z})$, we have that all $D_i$ are isomorphic with each other.

Without loss of generality, we only need to consider the component of $D$ which is produced from the vertex $v$ where

$$v = \begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \dots & 0 \quad 1 \end{bmatrix} \in \overline{C}(F_0).$$

Note that $v$ is a vertex of principal cone $\sigma_0$.

Let $F = F_{n-1} = \{[\begin{smallmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix}];\ Z \in S_{n-1}\}$; then

$$U(F) = \left\{ \begin{bmatrix} I & a' \\ 0 & I \end{bmatrix};\ a' = \begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \dots & 0 \quad a \end{bmatrix},\ a \in \mathbf{R},\ I = I_{n\times n},\ \right\} \subset U(F_0),$$

$$C(F) = \mathbf{R}^+ v \subset \overline{C}(F_0), \quad \text{a 1-dimensional cone},$$
$$L(F) = \Gamma(k) \cap U(F).$$

As described in §1.2, the following hold:

$V(F) = \mathbf{C}^{n-1}$,

$S_n$ is embedded in $S(F) \simeq S_{n-1} \times V(F) \times U(F)_{\mathbf{C}} = S_{n-1} \times \mathbf{C}^{n-1} \times \mathbf{C}$ as a Siegel domain of the third kind,

$(L(F)\backslash S_n)_{\Sigma_F}$ is embedded in $(L(F)\backslash S(F))_{\Sigma_F} \simeq S_{n-1} \times \mathbf{C}^{n-1} \times \mathbf{C}^*$, and $O(F) \simeq S_{n-1} \times \mathbf{C}^{n-1} \times \{0\}$.

Since $\Gamma(F)/L(F) \simeq \Gamma'$ and $\Gamma(F)/L(F)$ acts properly discontinuously on $O(F)$ as defined in the theorem, we have

$$(\Gamma(F)/L(F))\backslash O(F) \simeq \Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1}).$$

$\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})$ has a fiber structure over $\Gamma_{n-1}(k)\backslash S_{n-1}$ where $\Gamma_{n-1}(k) = \mathrm{Sp}(n-1; \mathbf{Z})(k)$. As shown in Proposition 2.1, the irreducible component $D_v$ of $D$ produced from the vertex $v$ is the closure of $(\Gamma(F)/L(F))\backslash O(F)$ in $\overline{X}$. Notice that $\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})$ is itself a locally symmetric space, and a $\Gamma(k)$-admissible family induces a $\Gamma'$-admissible family. Due to the structure of the orbit decomposition in the compactification as stated in Remark 5 of §1.2, the closure of $(\Gamma(F)/L(F))\backslash O(F)$ in the compactification $\overline{X}$ of the whole space $X$ is the same as the induced compactification of $\Gamma'\backslash(S_{n-1} \times \mathbf{C}^{n-1})$. This proves part (1) of the theorem.

Let $D_{v_1}$ and $D_{v_2}$ be two components of $D$ corresponding to two vertices $v_1$ and $v_2$ respectively. Then, by considering the orbit decomposition, $D_{v_1}$ intersects $D_{v_2}$ if and only if $v_1$ and $v_2$ span a 2-dimensional face $\sigma$ of some maximal cone in the $\Gamma(k)$-admissible family. The

2-dimensional face $\sigma$ defines a subvariety $V$ of codimension 2 of $\overline{X}$ in $D_{v_1}$ and $D_{v_2}$ respectively. On the other hand, since $v_1$ is a vertex of $\sigma$, this subvariety $V$ is contained in the boundary of $D_{v_1}$ if $D_{v_1}$ is regarded as a compactified space. Similarly, $V$ is also contained in the boundary of $D_{v_2}$. This completes the proof of Theorem 2.2.

We will discuss this theorem in more detail for $n = 2$ next.

**2.3. The structure of divisor for** $n = 2$. Let $\overline{X}$ be the toroidal compactification of $X = \Gamma(k)\backslash S_2$, $k \geq 3$, by $\sum_{\text{cent}}$. We are going to discuss the boundary divisor of $\overline{X}$ in detail. We will carry out here for $k = 3$, for simplicity of notation. It can be generalized to $\Gamma(k)$ for any $k \geq 3$ without any difficulty.

In the case of $n = 2$, the principal cone $\sigma_0$ is the only maximal cone of $\overline{C}(F_0)$ up to GL(2; **Z**) in $\sum_{\text{cent}}$, and

$$\sigma_0 = \left\{\lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} ; \lambda_i \geq 0,\ 1 \leq i \leq 3\right\}.$$

It can be checked that there are four independent maximal cones up to GL(2; **Z**)(3) $= \overline{\Gamma}(F_0)$. By choosing suitable representatives, they can be pictured as shown in Figure 1.

Then, in the orbits decomposition,

$$(\Gamma(F_0)/L(F_0))\backslash(L(F_0)\backslash S_2)_{\Sigma_{F_0}} \to \bigcup_{i=0}^{3} X_{\sigma_i} = \overline{\Gamma}(F_0)\backslash X_{\Sigma_{F_0}},$$

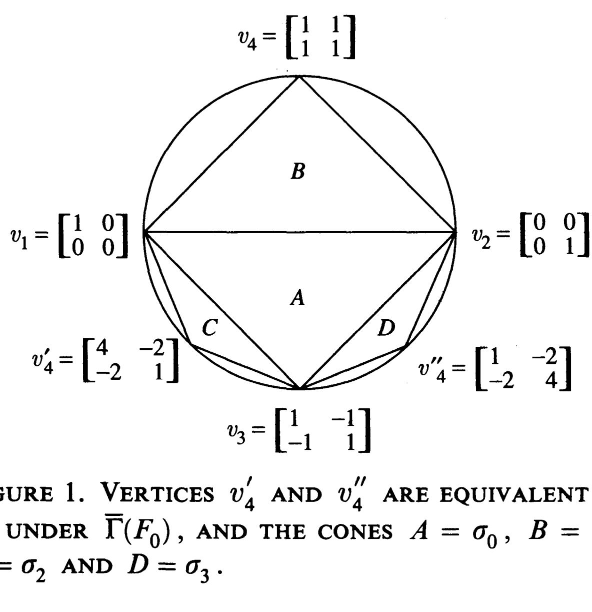



FIGURE 1. VERTICES $v_4'$ AND $v_4''$ ARE EQUIVALENT TO $v_4$ UNDER $\overline{\Gamma}(F_0)$, AND THE CONES $A = \sigma_0$, $B = \sigma_1$, $C = \sigma_2$ AND $D = \sigma_3$.

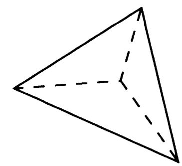
$\bar{\Gamma}(F_0)\backslash X_{\Sigma_{F_0}}$ $\longleftrightarrow$ 

FIGURE 2

(as shown in Figure 2), each face of the tetrahedrons corresponds to a $O_{F_0}(v_i)$ for $1 \le i \le 4$ and each vertex of the tetrahedrons corresponds to a $O_{F_0}(\sigma_i)$ for $0 \le i \le 3$.

First, we have the following.

**Theorem 2.3.** *$\overline{X}$ is the minimal smooth toroidal compactification of $X$, namely, if $\widetilde{X}$ is another smooth toroidal compactification of $X$, then $\widetilde{X}$ is a blowing-up of $\overline{X}$.*

*Proof.* By assuming that $\widetilde{X}$ is a smooth toroidal compactification of $X$ from a $\Gamma$-admissible family $\Sigma$, $\Sigma$ must be regular (see [13]). If $\sigma$ is a maximal cone of $\overline{C}(F_0)$ in $\Sigma_{F_0}$ (such $\sigma$ exists for any $\Sigma$), $\sigma$ is regular and $\sigma \cap \operatorname{Int}\sigma_0 \ne \emptyset$, it can be verified that $\sigma \subset \sigma_0$ because of the position of $\sigma_0$ and the regularity of $\sigma_0$ and $\sigma$. Thus

$\sigma_0 =$ union of some $\sigma_\alpha$ in $\Sigma_{F_0}$

$\Rightarrow$ $\Sigma$ is a refinement of $\Sigma_{\text{cent}}$

$\Rightarrow$ $\widetilde{X}$ is a blowing-up of $\overline{X}$.

Hence the theorem is proved. q.e.d.

If $F_1 = \{[\begin{smallmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix}];\ Z \in \mathbf{C},\ Z\overline{Z} < 1\}$ is the rational boundary component of rank 1, then $F_0 < F_1$ and

$$S(F_1) = U(F_1)_{\mathbf{C}} \cdot S_2 \simeq F_1 \times V_1 \times U(F_1)_{\mathbf{C}} = H \times \mathbf{C} \times \mathbf{C},$$

where $H$ is the upper half-plane, $V_1 \simeq \mathbf{C}$, $U(F_1)_{\mathbf{C}} \simeq \mathbf{C}$. From the structures used in the proof of previous theorem, it follows that

$$C(F_1) = \mathbf{R}^+ v_2 \overset{\text{Def}}{=} C_1, \quad \Gamma_1 = \Gamma \cap N(F_1), \quad L(F_1) = 3 \cdot \mathbf{Z} \overset{\text{Def}}{=} L_1,$$

$\Gamma_1/L_1 = SL(2;\mathbf{Z})(3) \times (3\mathbf{Z})^2 \overset{\text{Def}}{=} \Gamma_1'$ with induced group structure from $\Gamma_1$, so that

$$S_2 \to S(F_1), \qquad \tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \to (\tau_1, \tau_2, \tau_3),$$

$$S_2/L_1 \overset{\exp(i2\pi/3)}{\longrightarrow} S(F_1)/L_1 \simeq H \times \mathbf{C} \times \mathbf{C}^*.$$

Since $C_1$ is a 1-dimensional cone and $\overline{\Gamma} = \overline{\Gamma}(F_1) = \{1\}$, $\Sigma_{F_1} = \{\sigma\}$

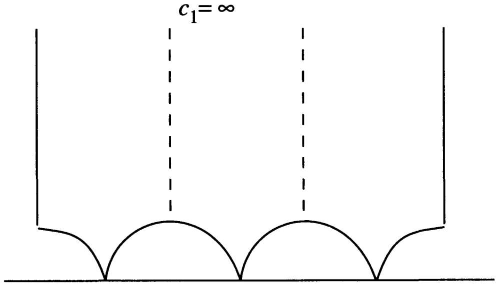



FIGURE 3

where $\sigma = C_1$. Thus

$$(S_2/L_1)_{\Sigma_{F_1}} = (S_2/L_1) \cup H \times \mathbf{C} \times \{0\}$$

and

$$O_{F_1}(\sigma) = H \times \mathbf{C} \times \{0\},$$

$$(\Gamma_1/L_1)\backslash O_{F_1}(\sigma) = \Gamma_1'\backslash O_{F_1}(\sigma) \simeq \Gamma_1'\backslash(H \times \mathbf{C}),$$

where $\Gamma_1'$ acts on $H \times \mathbf{C}$ as defined in Theorem 2.2. Since $\sigma = \mathbf{R}^+ v_2 \subset \overline{C}(F_0)$,

$$O_{F_1}(\sigma) \to O_{F_0}(v_2) \subset \overline{\Gamma}(F_0)\backslash X_{\Sigma_{F_0}}.$$

Let $X_1 = \Gamma_1'\backslash(H \times \mathbf{C})$, $X_2 = SL(2; \mathbf{Z})(3)\backslash H$; then $X_1 \to X_2$ is a fiber space with elliptic fibers. To get the closure of $(\Gamma_1/L_1)\backslash O_{F_1}(\sigma)$ in $\overline{\Gamma}(F_0)\backslash X_{\Sigma_{F_0}}$, we need to understand the compactification of the fiber space $X_1 \to X_2$.

First, actually, $X_2$ is nothing but $\Gamma(3)\backslash S_1$. The fundamental domain of $H$ with respect to $SL(2; \mathbf{Z})(3)$ is shown in Figure 3. Therefore, the compactification of $X_2$ is obtained by adding four cusps, i.e.,

$$\overline{X_2} = X_2 \cup \{c_1, c_2, c_3, c_4\}.$$

There is a natural compactification of $\overline{X_1}$ of $X_1$, such that $\overline{X_1}$ is an elliptic surface over $\overline{X_2}$ with singular fiber over each cusp; each singular fiber is a rational 3-gons (see [1]). On the other hand, each cusp of $\overline{X_2}$ corresponds to a rational boundary component of $S_2$ of rank 0, and they are all $\Gamma(3)$-independent. For example, cusp $c_1 = \infty$ represents the boundary component $F_0$ (in fact, $\overline{X_2}$ is a part of the Baily-Borel boundary

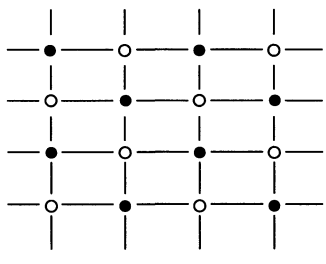


FIGURE 4

of $\Gamma(3)\backslash S_2)$. The rational 3-gon over $c_1$ is nothing but just the boundary of $O_{F_0}(v_2)$ in $\overline{\Gamma}(F_0)\backslash X_{\Sigma_{F_0}}$.

Moreover, if $F_0^2$, $F_0^3$ and $F_0^4$ are the other three boundary components of rank 0 which correspond to $c_2$, $c_3$ and $c_4$ respectively, then $F_1 > F_0^i$, $i = 2, 3, 4$. The cone $C(F_1)$ must be a 1-dimensional member in each $\Sigma_{F_0^i}$. As in the case of $F_0$, each rational 3-gon over $c_i$ for $i = 2, 3, 4$ comes from $O_{F_0^i}(C(F_1))$. It is clear that $\{F_0, F_0^2, F_0^3, F_0^4\}$ are only boundary components of rank 0 which are less than $F_1$. Therefore, the elliptic surface $\overline{X_1}$ is a component of boundary $\overline{(\Gamma(3)\backslash S_2)} - \Gamma(3)\backslash S_2$. Since all boundary components of rank 0 are equivalent under $\mathrm{Sp}(2; \mathbf{Z})$, and the same is true for those of rank 1, all components of $\overline{(\Gamma(3)\backslash S_2)} - \Gamma(3)\backslash S_2$ are the same elliptic surface $\overline{X_1}$.

Furthermore, if we set the following notation:

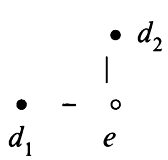
"•" ↔ representing rational boundary component of rank 1,

"∘" ↔ representing rational boundary component of rank 0,

"•–∘" ↔ if "•" > "∘", then under $\Gamma(3)$, the graph of rational boundary components will be as shown in Figure 4.

The certain part of boundary $D$ arising from each of these rational components if illustrated by Figure 5 (next page).

All possible surfaces from Figure 5a gluing together by tetrahedrons from Figure 5b according to the relation in the graph form the boundary $D$. For example, if two surfaces $M_1$ and $M_2$ associated with two "•", say $d_1$, $d_2$, and $d_1$, $d_2$, are connected by a "∘" $e$




then $M_1$ and $M_2$ are glued at one end by the tetrahedron associated to $e$ (see Figure 6).

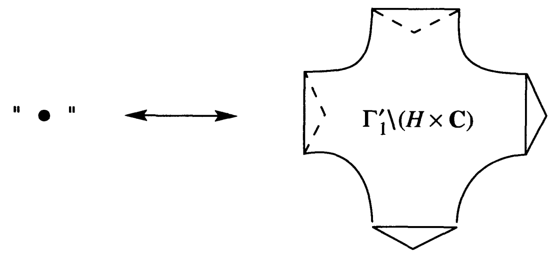



FIGURE 5a

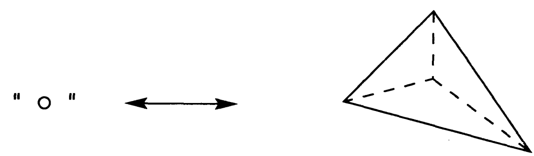



FIGURE 5b

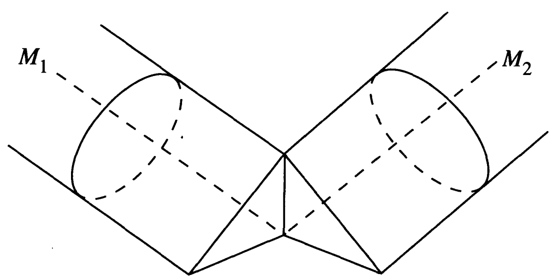



FIGURE 6

On the other hand, if $(\Gamma(3)\backslash S_2)^*$ is the Baily-Borel compactification, then $\overline{(\Gamma(3)\backslash S_2)}$ is the blowing-up of $((\Gamma(3)\backslash S_2)^* - \Gamma(3)\backslash S_2) = \sum_{i=1}^m C_i$ where each $C_i$ is a curve isomorphic to $\overline{X_2}$. If $\alpha\colon \overline{(\Gamma(3)\backslash S_2)} \to (\Gamma(3)\backslash S_2)^*$ is the blowing-up map, then $\alpha\colon D = \sum_{i=1}^m D_i \to \sum_{i=1}^m C_i$ and $\alpha\colon D_i \to C_i$ are given by the fiber map from $\overline{X_1}$ to $\overline{X_2}$.

We summarize the discussion above in the following theorem.

**Theorem 2.4.** *Let $X = \Gamma(k)\backslash S_2$ be the complex quotient manifold of dimension 3, and let $\overline{X}$ be the minimal smooth projective toroidal compactification of $X$, and $D = \overline{X} - X$; then the following hold*:

(1) *$D = \sum_{i=1}^m D_i$ is a divisor with normal crossings only, each $D_i$ is a elliptic surface $S$ with some singular fibers of type ${}_1I_b$ (see* [10] *for the notation), and $\{D_i\}_{i=1}^m$ intersect along the singular fibers.*

(2) *$D = \sum_{i=1}^m D_i$ can be blowing-down to a singular curve $C = \sum_{i=1}^m C_i$ where each $C_i$ is a smooth curver, and the contraction of $D$ into $C$ is given by the fiber map of $S$.*

**Remarks.** 1. The number $m$ is computable by the formula from number theory.

2. The similar structures can be stated for $\Gamma(k)$, $k \geq 3$, if we replace rational 3-gons by rational $k$-gons in general.

3. The structures described in Theorem 2.4 were obtained by Igusa in [7] by studying the desingularization of the Siegel modular forms of genus 2.

## 3. The canonical line bundle of the compactified space for $n = 2$

Let $X = \Gamma(k)\backslash S_2$, $k \geq 3$, $X^*$ be the Satake-Baily-Borel compactification of $X$, and let $\overline{X}$ be the minimal smooth toroidal compactification of $X$. As we have shown in the previous section, $\overline{X}$ is, in fact, the toroidal compactification of $X$ from the central cone decomposition, and $\overline{X}$ is projective. We are going to discuss the canonical line bundle of $\overline{X}$ in this section.

First, we shall recall how $X^*$ can be constructed. Let $A(\Gamma)_l$ be the vector space of all Siegel modular forms of weight $l$ with respect to $\Gamma$ where $\Gamma = \Gamma(k)$ for some $k \geq 3$, i.e.,

$$A(\Gamma)_l = \Big\{ f\colon \text{ holomorphic function on } S_2, \\ f(M\cdot\tau) = \det(C\tau + D)^l f(\tau), \ \forall \tau \in S_2, \ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma \Big\}.$$

Let $A(\Gamma) = \bigoplus_{l\geq 0} A(\Gamma)_l$. Then $A(\Gamma)$ is a positively graded ring and finitely generated over $A(\Gamma)_0 = \mathbf{C}$, and the projective variety associated with $A(\Gamma)$ is the Satake-Baily-Borel compactification of $X$. It has been proved by Igusa in [7] that the minimal toroidal compactification $\overline{X}$ is nothing but the normalization of the blowing-up of $X^*$ with respect to the sheaf of ideals $\mathscr{I}$ defined by all cusp forms in $A(\Gamma)$.

The line bundle $L = \mathscr{O}(1)$ on $X^*$ which corresponds to modular forms of weight one is ample, and

$$\Gamma(X^*, \mathscr{O}(lL)) = A(\Gamma)_l.$$

Let $\alpha\colon \overline{X} \to X^*$ be the blowing-up with respect to $\mathscr{I}$, and $\overline{L} = \alpha^*(L)$. Then

$$\Gamma(\overline{X}, \mathscr{O}(l\overline{L})) \simeq \Gamma(X^*, \mathscr{O}(lL)),$$

since $X^*$ is normal.

On the other hand, Igusa constructed in [6] a cusp form $\varphi = \theta^2$ of weight ten with respect to $\mathrm{Sp}(2;\mathbf{Z})$ where $\theta$ is a function formed by the theta-constants. Therefore

$$\varphi \in \Gamma(X^*, \mathscr{O}(10L)) = \Gamma(\overline{X}, \mathscr{O}(10\overline{L})).$$

If we let $\Delta =$ the diagonal set of $S_2$, i.e., if

$$\Delta = \left\{ \tau = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} ; \operatorname{Im} \tau_i > 0 \right\},$$

Hammond found that the zero set of $\theta$ is the set of $\mathrm{Sp}(2;\mathbf{Z})\cdot\Delta$ (see [5]). Let $\Delta' = \Gamma\backslash(\mathrm{Sp}(2;\mathbf{Z})\cdot\Delta)$, and let $D'$ be the induced compactification of $\Delta'$ in $\overline{X}$, and $D' = \sum D_i'$ be the decomposition of irreducible components. By the similar discussion in §2.4, it can be verified that each $D_i'$ is isomorphic to $W_1 \times W_2$ where $W_i = (SL(2;\mathbf{Z})(k)\backslash H) \cup \{\text{some cusps}\}$ is the standard compactification of $SL(2;\mathbf{Z})(k)\backslash H$. Note that $W_i$ is nonsingular. If $D = \overline{X} - X = \sum D_j$ as in the previous section, then the zero divisor defined by $\varphi = \theta^2$ in $\overline{X}$ is $kD + 2D'$ for $\Gamma = \Gamma(k)$, which implies that

$$10\overline{L} = k[D] + 2[D'].$$

Let $K_{\overline{X}}$ be the canonical line bundle of $\overline{X}$. For

$$\tau \in S_2, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \operatorname{Aut}(S_2) = \mathrm{Sp}(2;\mathbf{R}),$$

it can be derived by direct calculations that

$$dV_{M\tau} = \det(C\tau + D)^{-3}\, dV_\tau,$$

where $dV_{(*)}$ is the standard volume element in the coordinate system $(*)$. (Actually, as we will point out in the next section, the bi-invariant volume form $\Phi$ on $S_2$ is given by $c\, dV_\tau/(\det Y)^3$ if $\tau = X + iY \in S_2$.) Therefore, $\phi^3(\tau)(dV_\tau)^l$ produces a $\Gamma$-invariant $l$-ple 3-form on $S_2$ for a $\phi(\tau) \in A(\Gamma)_l$. If we start with a cusp form $\phi$, then $\phi^3(\tau)(dV_\tau)_l$ defines a $l$-ple 3-form on $\overline{X}$.

On the other hand, if $\tau = [\begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{smallmatrix}] \in S_2$ and we let

$$w_1 = \tau_1, \quad w_2 = \tau_2, \quad w_3 = e^{i(2\pi/k)\tau_3},$$

then, as we have demonstrated in §2.4, the boundary component of $\overline{X}$ arises as $w_3 \to 0$ in the coordinate system $(w_1, w_2, w_3)$, and the whole boundary $D$ of $\overline{X}$ is produced from this manner. Thus

$$dV_\tau = d\tau_1 \wedge d\tau_2 \wedge d\tau_3,$$

$$dV_w = \frac{k}{i2\pi} w_3^{-1} dw_1 \wedge dw_2 \wedge dw_3,$$

and therefore

$$10K_{\overline{X}} = 30\overline{L} - 10[D]$$

by using $\varphi \in A(\Gamma)_{10}$ from above, and hence

$$10K_{\overline{X}} = 30\overline{L} - 10[D] = (3k - 10)[D] + 6[D'].$$

The complete understanding of $D'$ and its intersections with $D$ can be achieved by similar discussions as in the previous section.

Therefore, we have proved the following theorem.

**Theorem 3.1.** *Let $X$, $\overline{X}$ and $D$ be the same as in Theorem 2.4, $K_{\overline{X}}$ be the canonical line bundle of $\overline{X}$, and $D'$ be the closure of the diagonal set of $S_2$ in $\overline{X}$. Then $D'$ is a divisor in $\overline{X}$ and*

$$10K_{\overline{X}} = (3k - 10)[D] + 6[D'].$$

Now we are going to consider an application to the Kodaira dimension of space $X$. We recall that a smooth compact variety $V$ of dimension $n$ is of general type if the transcendence degree of the ring

$$\bigoplus_{m=0}^{\infty} \Gamma(V, mK_V)$$

is $n+1$ (i.e., the Kodaira dimension of $V$ is $n$), where $\Gamma(V, mK_V)$ denotes the space of holomorphic sections of $mK_V$ over $V$. A variety $Y$ is said to be of general type if the smooth compact variety $\overline{Y}$ birational to $Y$ is of general type. $Y$ is said to be of logarithmic general type if there is a smooth compactification $\overline{Y}$ of $Y$ such that $D = \overline{Y} - Y$ is a divisor with normal crossings and the transcendence degree of the ring

$$\bigoplus_{m=0}^{\infty} \Gamma(\overline{Y}, m(K_{\overline{Y}} + D))$$

is $n+1$ (i.e., the logarithmic Kodaira dimension of $Y$ is $n$). Note that saying $Y$ is of logarithmic general type is weaker than saying $Y$ is of general type. By his generalized Hirzebruch's proportionality principle, Mumford proved that a locally symmetric Hermitian variety $\Gamma\backslash\Omega$ with neat arithmetic group $\Gamma$ acting on bounded symmetric domain $\Omega$ is always of logarithmic general type; see [12]. Furthermore, a theorem of Tai in [1] implies that $\Gamma\backslash\Omega$ is of general type if $\Gamma$ is sufficiently small. We prove the following.

**Theorem 3.2.** *$X = \Gamma(k)\backslash S_2$ is of general type for $k \geq 4$.*

*Proof.* We are going to show that the minimal smooth toroidal compactification $\overline{X}$ of $X$ is of general type for $k \geq 4$. It is enough to prove that (see [8]),

$$cm^3 \leq \dim \Gamma(\overline{X}, mm_0K_{\overline{X}})$$

for some positive constant $c$, $m_0 \in \mathbf{Z}^+$ and $m \gg 0$.

Let $\varphi = \theta^2$ be the cusp form of weight ten constructed by Igusa. Then, as we have seen in the proof of Theorem 3.1, $s_0 = \varphi^3(\tau)(dV_\tau)^{10}$ is a section of $10K_{\overline{X}}$ and the divisor defined by $s_0$ is given by

$$\operatorname{div}(s_0) = (3k - 10)D + 6D',$$

where $D$ and $D'$ are divisors of $\overline{X}$ in Theorem 3.1. Therefore, $s_0 \in \Gamma(\overline{X}, 10K_{\overline{X}})$ if $k \geq 4$.

Furthermore, if $f$ is a cusp form of weight $m$ with respect to $\Gamma = \Gamma(k)$, then $s_0^m f^3(\tau)(dV_\tau)^m$ is a section of $11mK_{\overline{X}}$ and is holomorphic when $k \geq 4$, i.e.,

$$s_0^m f^3(\tau)(dv_\tau)^m \in \Gamma(\overline{X}, 11mk_{\overline{X}}).$$

Let $d_m = \dim\{\text{ cusp forms of weight } m \text{ with respect to } \Gamma\}$. A theorem of Mumford in [12] implies that

$$d_m \simeq m^3 \quad \text{for } m \gg 0.$$

Hence,we conclude that

$$cm^3 \leq \dim \Gamma(\overline{X}, 11mK_{\overline{X}})$$

for some positive constant $c$ and $m \gg 0$.

This proves the theorem.

**Remark.** Theorem 3.2 is sharp since G. van der Geer has shown in [4] that $\Gamma(3)\backslash S_2$ is rational.

## 4. The canonical volume form on $X$

In this section, we discuss the canonical volume form of $X$ as a singular volume form on $\overline{X}$ after compactification. In the case of rank 1, it just appears as the Poincaré metric on punctured disc near each cusp. Since the coordinate systems induced from torus embeddings with respect to maximal cones are always related by $\mathrm{Sp}(n; \mathbf{Z})$ (in fact, every $\Gamma(k)$-admissible family is induced from a $\mathrm{Sp}(n; \mathbf{Z})$-admissible family), we can restrict ourself only on $\Gamma(3)$ for the analysis of canonical volume form over those coordinates. Thus

$$S_n = \{\tau = X + \sqrt{-1}Y;\ \tau = {}^t\tau,\ Y > 0\} \subset M(n; \mathbf{C}),$$

$$N = \dim_{\mathbf{C}} S_n = \frac{n(n+1)}{2}.$$

The canonical metric (or Bergman metric) on $S_n$ is given by (see [14])

$$dS_\tau^2 = \mathrm{Tr}(Y^{-1}\,d\tau\,Y^{-1}\,d\overline{\tau}),$$

and induces a metric on $\Gamma(3)\backslash S_n$ which is a complete Kähler-Einstein metric with negative Ricci curvature.

Let $\Phi = (dS_\tau^2)^N = (\mathrm{Tr}(Y^{-1}\,d\tau\,Y^{-1}\,d\overline{\tau}))^N$ be the volume form from $dS_\tau^2$; then a direct calculation shows that

$$\Phi = \frac{c\,dV_\tau}{(\det Y)^{n+1}},$$

where $c$ is a positive constant, and $dV_\tau$ is the standard volume form in term of coordinate system $\tau$.

After compactification, $\Phi$ will be a singular form on $\overline{(\Gamma(3)\backslash S_n}$ with singularity over the boundary $D$ of the compactification. We are interested in the singular behavior of it. By the observations about divisor in earlier sections, we only need to deal with $(\Gamma(F_0)/L(F_0))\backslash(L(F_0)\backslash S_n)_{\Sigma_{F_0}}$, and to understand the singular behavior of $\Phi$ at a point which is the intersection of $N$ components of $D$. In other words, it is enough to see the behavior of $\Phi$ at 0 in $X_\sigma$ for each maximal member of $\Sigma_{F_0}$ up to $\overline{\Gamma}(F_0)$.

As we have seen in previous sections, we first take a projection from

$$S_n \to L(F_0)\backslash S_n \subset T = (\mathbf{C}^*)^N$$

by $w_j = e^{i2\pi\tau_j/3}$, where $\tau = (\tau_1, \tau_2, \cdots, \tau_N)$ and $w = (w_1, w_2, \cdots, w_N)$ are coordinate systems in $S_n$ and $T$ respectively. If we order $\tau$ by

$$\tau = (\tau_{11}, \tau_{12}, \cdots, \tau_{1n}, \tau_{22}, \cdots, \tau_{2n}, \cdots, \tau_{nn})$$

for the symmetric matrix

$$\tau = [\tau_{ij}]_{i,j=1}^n \in S_n.$$

Then, in terms of $w$,

$$\Phi_w = \frac{c_1 dw_1 \wedge \cdots \wedge dw_N \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_N}{\prod_{j=1}^N |w_j|^2 (\det \log|w|)^{n+1}},$$

where $c_1$ is a positive constant, and

$$\log|w| = \begin{bmatrix} \log|w_1| & \log|w_2| & \cdots & \log|w_n| \\ \log|w_2| & \log|w_{n+1}| & \cdots & \log|w_{2n-1}| \\ \cdot & \cdot & \cdots & \cdot \\ \log|w_n| & \log|w_{2n-1}| & \cdots & \log|w_N| \end{bmatrix},$$

which is symmetric.

**Theorem 4.1.** *Let* $\sigma = \{\sum_{i=1}^{N} \mathbf{R}^{+} a_i\}$ *be a regular cone in* $C(F_0)$ *with vertices* $\{a_1, a_2, \cdots, a_N\}$ *and*

$$a_i = [a^i_{jk}]^h_{j,k=1} = [C^i_1, C^i_2, \cdots, C^i_n],$$

*where each* $a_i$ *is symmetric, and* $\{C^i_1, \cdots, C^i_n\}$ *are column vectors of* $a_i$. *Let* $s = (s_1, \cdots, s_N)$ *be the coordinate system in* $X_\sigma$ *such that* $\{s_i = 0\}$ *is the divisor corresponding to vertex* $a_i$. *Then, in terms of* $s$ *system,*

$$\Phi_s = \frac{c\, ds_1 \wedge \cdots \wedge ds_n \wedge d\bar{s}_1 \wedge \cdots \wedge d\bar{s}_N}{\prod_{i=1}^{N} |s_i|^2 \left( \sum_{\substack{(i_1, i_2, \cdots, i_n) \\ 1 \le i_1, \cdots, i_n \le N}} c_{i_1 \cdots i_n} \log|s_{i_1}|^2 \log|s_{i_2}|^2 \cdots \log|s_{i_n}|^2 \right)^{n+1}},$$

*where constant* $c > 0$ *and* $c_{i_1 \cdots i_n} = \det[C^{i_1}_1, C^{i_2}_2, \cdots, C^{i_n}_n]$.

Before proving the theorem, we first would like to consider two applications of the theorem.

**Corollary 1.** *If all vertices* $a_i$ *are in* $\overline{C}(F_0) - C(F_0)$, *and* $a_i = A_i{}^t A_i$ *for an integral vector* $A_i$ *in* $\mathbf{R}^n$ *for each* $i$. *Then*

$$\Phi_s = \frac{c\, ds_1 \wedge \cdots \wedge ds_N \wedge d\bar{s}_1 \wedge \cdots \wedge d\bar{s}_N}{\prod_{i=1}^{N} |s_i|^2 (\sum_{1 \le i_1 < i_2 < \cdots < i_n \le N} c_{i_1 \cdots i_n} \log|s_{i_1}|^2 \log|s_{i_2}|^2 \cdots \log|s_{i_n}|^2)^{n+1}},$$

*where constant* $c > 0$ *and* $c_{i_1 \cdots i_n} = (\det[A_{i_1}, A_{i_2}, \cdots, A_{i_n}])^2$.

*Proof.* Assuming

$$A_i = \begin{bmatrix} a^1_i \\ a^2_i \\ \vdots \\ a^n_i \end{bmatrix},$$

we have $a_i = [a^1_i A_i, a^2_i A_i, \cdots, a^n_i A_i]$ and $C^i_j = a^j_i A_i$, so

$$\det[C^{i_1}_1, C^{i_2}_2, \cdots, C^{i_n}_n] = a^1_{i_1} \cdots a^n_{i_n} \det[A_{i_1}, A_{i_2}, \cdots, A_{i_n}].$$

Applying these identities to the formula in the theorem gives the corollary directly.

**Remark.** From the reduction theory, there is an admissible family for $n \le 4$ such that all cones in the family have the form in Corollary 1 (see the proof of Theorem 2.2).

The geometric interpretations of Corollary 1 can be seen by the next proposition. First, from the discussion about the structures of divisor, we know that if $D_1$ is a component of $D$, which is associated to a vertex $v$ on the boundary of cone, then $D_1 \simeq \overline{(\Gamma' \backslash S_{n-1} \times \mathbf{C}^{n-1})}$. Moreover, $D_1$ is a fiber space over $\overline{(\Gamma_{n-1}(k) \backslash S_{n-1})}$ which is the induced compactification of $\Gamma_{n-1}(k) \backslash S_{n-1}$. On the other hand, as we have pointed out earlier, $\overline{X}$ is a blowing-up of Satake-Baily-Borel compactification $X^* = X \bigcup_{i=0}^{n-1} X_i$ of $X$ along $\bigcup_{i=0}^{n-1} X_i$ where each $X_i$ is the quotient of $\Gamma(k)$-inequivalent classes of $S_i$ of rank $i$. Let $\alpha \colon \overline{X} \to X^*$ be the blowing-up. Then we have

**Proposition 4.1.** *Let $\{a_1, a_2, \cdots, a_n\}$ be the set of vertices from the corollary, and $\{D_{a_1}, D_{a_2}, \cdots, D_{a_n}\}$ be the components of $D$ associated to $\{a_1, a_2, \cdots, a_n\}$ respectively. Then,* $\det[A_1, A_2, \cdots, A_n] \neq 0 \Leftrightarrow D_{a_1} \cap D_{a_2} \cap \cdots \cap D_{a_n}$ *is in the inverse image of $\alpha$ over a copy of $S_0$ only.*

*Proof.* Let $F^i_{n-1}$ be the rational boundary component of rank $n-1$, such that $C(F^i_{n-1}) = \mathbf{R}^+ a_i \subset \overline{C}(F_0)$ (remembering that $F^i_{n-1} = S_{n-1}$). From the construction of compactifications, we may claim that if there exists a rational boundary component $F_{n'}$ of rank $n'$ for some $0 \leq n' \leq n-2$, such that $F_{n'} \subset \overline{F}^i_{n-1}$ $\forall i$, then $D_{a_1} \cap D_{a_2} \cap \cdots \cap D_{a_n}$ corresponds to the fibers over the quotient of some $S_{n'}$, and vice versa. Therefore, proving the proposition is equivalent to proving that $\det[A_1, A_2, \cdots, A_n] = 0 \Leftrightarrow \exists F_{n'}$ with $0 < n' \leq n-2$ such that $F_{n'} \subset \overline{F}^i_{n-1}$ for $1 \leq i \leq n$.

First, let

$$a_i = A_i{}^t A_i, \qquad A_i = \begin{bmatrix} a_i^1 \\ a_i^2 \\ \vdots \\ a_i^n \end{bmatrix} \in \mathbf{Z}^n;$$

then there exists a $B \in \mathrm{GL}(n; \mathbf{Z})$ such that

$$B \cdot [A_1, A_2, \cdots, A_n] = \text{upper triangular}.$$

Let $B \cdot [A_1, A_2, \cdots, A_n] = [A'_1, A'_2, \cdots, A'_n]$, i.e., $BA_i = A'_i$. Then $BA_i{}^t(BA_i) = Ba_i{}^tB = A'_i{}^tA'_i$, and $B$ acts on $a_i$'s just exactly as the action of $\mathrm{GL}(n; \mathbf{Z})$ on the cone $\overline{C}(F_0)$.

If $\det[A_1, A_2, \cdots, A_n] = 0$, we can assume that $A'_n = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \end{bmatrix}$. Since $\sigma$

is chosen up to $\mathrm{GL}(n;\mathbf{Z})$, we may assume that

$$[A_1, A_2, \cdots, A_n] = \begin{bmatrix} \cdot & * & * \\ 0 & \ddots & * \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(upper triangular)}$$

$$\Rightarrow a_i \in \left\{ \begin{bmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ 0 & \ldots & 0 \quad 0 \end{bmatrix} \right\} \cap \overline{C}(F_0).$$

Let

$$F_1 = \left\{ \begin{bmatrix} 1 & & 0 & \\ & \ddots & & \\ 0 & & 1 & \\ & & & Z \end{bmatrix} ; Z \in S_1 \right\}$$

be the rational boundary component of rank 1. Then

$$C(F_1) = \left\{ \begin{bmatrix} & & 0 \\ & b & \vdots \\ 0 & \ldots & 0 \end{bmatrix} ; b \in M_s(n-1;\mathbf{R}), \, b > 0 \right\} \subset \overline{C}(F_0).$$

$$C(F^i_{n-1}) = \mathbf{R}^+ a_i \subset \overline{C}(F_1)$$

$$\Rightarrow F_1 < F^i_{n-1}, \qquad \text{i.e., } S_1 \simeq F_1 \subset \overline{F}^i_{n-1}.$$

Conversely, if there exists a $F_{n'}$ such that $F_{n'} \subset \overline{F}^i_{n-1}$ for some $0 < n' \leq n-2$ and any $i$, then we may assume

$$F_1 = \left\{ \begin{bmatrix} 1 & & 0 & \\ & \ddots & & \\ 0 & & 1 & \\ & & & Z \end{bmatrix} ; Z \in S_1 \right\} < F^i_{n-1} \quad \forall i,$$

$$\Rightarrow C(F^i_{n-1}) = \mathbf{R}^+ a_i \subset \overline{C}(F_1), \qquad a_i \in \overline{C}(F_1) \ \forall i,$$

$$\Rightarrow A_i = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \end{bmatrix} \quad \forall i.$$

Thus $\det[A_1, \cdots, A_n] = 0$, and the proof of Proposition 4.1 is complete.

**Corollary 2.** *Let $\overline{X}$ be the minimal smooth projective compactification of $X = \Gamma(k)\backslash S_2$ from toroidal embeddings, and $D = \overline{X} - X = \sum_{i=1}^m D_i$ be* *a divisor with normal crossings only. Then the canonical volume form* $\Phi$ *of* $X$ *can be represented by*

$$\frac{dV}{\prod_{i=1}^{m}|s_i|^2\left(\sum_{\substack{i\neq j\\ 1\leq i,j\leq m}}\log|s_i|^2\log|s_j|^2\right)^3}$$

*for some Hermitian metric on each* $[D_i]$, *where* $dV$ *is a volume form on* $\overline{X}$, *and* $s_i\in\Gamma(\overline{X}, [D_i])$ *is a section which defines* $D_i$.

*Proof.* By the reduction theory, the principle cone $\sigma_0$ is the only maximal cone in the case of rank 2, and it can be checked that all $c_{i_1\cdots i_n}=1$ in Corollary 1 for all possible index $(i_1, \cdots, i_n)$. Thus Corollary 2 is proved by considering the singularity of the canonical volume form $\Phi$ of $X$ over $\overline{X}$ which is described in Corollary 1.

To prove Theorem 4.1, we need two lemmas.

**Lemma 1.** *If* $\sigma_1$ *and* $\sigma_2$ *are two regular cones with vertices* $\{e_i^1\}_{1\leq i\leq N}$ *and* $\{e_i^2\}_{1\leq i\leq N}$ *respectively, and* $(e_1^2, \cdots, e_N^2)=(e_1^1, \cdots, e_N^1)(a_{ij})$ *where* $(a_{ij})$ *is a* **Z**-*matrix with* $\det(a_{ij})=\pm1$, *and* $\{u_1^i, \cdots, u_N^i\}$ *is the complex coordinate system in* $X_{\sigma_i}$, $i=1, 2$, *such that* $u_j^i$ *is corresponding to vertex* $e_j^i$. *Then*

$$u_i^1=\prod_{j=1}^{N}(u_j^2)^{a_{ij}};$$

*in particular,*

$$\begin{bmatrix}\log|u_1^1|\\ \vdots\\ \log|u_n^1|\end{bmatrix}=(a_{ij})\begin{bmatrix}\log|u_1^2|\\ \vdots\\ \log|u_N^2|\end{bmatrix}.$$

*Proof.* Let $\sigma_1=\{\sum_{i=1}^N \mathbf{R}^+e_i^1\}$, $\sigma_2=\{\sum_{i=1}^N\mathbf{R}^+e_i^2\}$, and let $\{r_1^i, \cdots, r_N^i\}$ be the dual basis of $\{e_1^i, \cdots, e_N^i\}$, $i=1, 2$, i.e., let

$$r_j^1(e_k^1)=\delta_{jk}, \qquad r_j^2(e_k^2)=\delta_{jk}.$$

Since $\{e_1^1, \cdots, e_N^1\}$ and $\{e_1^2, \cdots, e_N^2\}$ are two integral bases, we have

$$\hat{\sigma}_1=\left\{\sum_{i=1}^N\mathbf{R}^+r_i^1\right\}, \qquad \hat{\sigma}_2=\left\{\sum_{i=1}^N\mathbf{R}^+r_i^2\right\}$$

and

$$\hat{\sigma}_j \cap L^* = \left\{ \sum_{i=1}^{N} \mathbf{Z}^+ r_i^j \right\} \quad \text{for } j = 1, 2,$$

$$\begin{bmatrix} r_1^1 \\ \vdots \\ r_N^1 \end{bmatrix} = (a_{ij}) \begin{bmatrix} r_1^2 \\ \vdots \\ r_N^2 \end{bmatrix}.$$

By the construction of $X_{\sigma_i}$,

$$u_j^i = \chi_{r_j^i}, \quad i = 1, 2 \Rightarrow u_i^1 = \prod_{j=1}^{N} (u_j^2)^{a_{ij}}.$$

**Lemma 2.** *Let $\sigma_0$ be the principle cone defined in* §2.1, *i.e.,*

$$\sigma_0 = \left\{ \sum_{i=1}^{N} \mathbf{R}^+ v_i \right\} \quad \text{with } (v_1, v_2, \cdots, v_N)$$
$$= (e_{11}, e_{12}, \cdots, e_{1n}, e_{22}, \cdots, e_{2n}, \cdots, e_{nn}).$$

*Let $u = (u_1, u_2, \cdots, u_N)$ be the coordinate system on $X_{\sigma_0}$ with the induced order from $v_i$'s. Then, in terms of $u$ system on $X_{\sigma_0}$,*

$$\Phi_u = \frac{c\, du_1 \wedge \cdots \wedge du_N \wedge d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_N}{\prod_{j=1}^{N} |u_j|^2 f_u},$$

*where $c > 0$ constant and $f_u = \det \mathscr{M}$, $\mathscr{M}$ denoting the matrix*

$$\begin{bmatrix} \log|u_1 \cdots u_n| & -\log|u_2| & -\log|u_3| & \cdots & -\log|u_n| \\ -\log|u_2| & \log|u_2 u_{n+1} u_{n+2} \cdots u_{2n-1}| & -\log|u_{n+2}| & \cdots & -\log|u_{2n-1}| \\ -\log|u_3| & -\log|u_{n+2}| & \log|u_3 u_{n+2} u_{2n} \cdots u_{3n-3}| & \cdots & -\log|3n-3| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\log|u_n| & -\log|u_{2n-1}| & -\log|u_{3n-3}| & \cdots & \log|u_n u_{2n-1} u_{3n-3} \cdots u_N| \end{bmatrix}.$$

*Proof.* Let $\sigma_0 = \{\sum_{i=1}^{N} \mathbf{R}^+ v_i\}$. If $\{r_1, r_2, \cdots, r_N\}$ is the dual set of $\{v_1, v_2, \cdots, v_N\}$, i.e., if $r_i(v_i) = \delta_{ij}$, then

$$\hat{\sigma}_0 = \left\{ \sum_{i=1}^{N} \mathbf{R}^+ r_i \right\}.$$

With respect to the inner product defined by the trace formula, $\{r_i\}$ can be represented by matrices in the following way

$$\{r_1, r_2, \cdots, r_N\} = \{r_{11}, r_{12}, \cdots, r_{1n}, r_{22}, r_{23}, \cdots, r_{2n}, \cdots, r_{nn}\},$$

where

$$r_{ii} = \begin{bmatrix} & & 1/2 & & \\ 0 & & \vdots & & 0 \\ & & 1/2 & & \\ 1/2 & \dots\ 1/2 & 1 & 1/2\ \dots & 1/2 \\ & & 1/2 & & \\ 0 & & \vdots & & 0 \\ & & 1/2 & & \end{bmatrix}_i, \qquad 1 \le i \le n,$$

$$r_{ij} = \begin{bmatrix} & \vdots & & \vdots & \\ \dots & 0 & \dots & -1/2 & \dots \\ & \vdots & & \vdots & \\ \dots & -1/2 & \dots & 0 & \dots \\ & \vdots & & \vdots & \end{bmatrix}^{i}_{j}, \qquad 1 \le i < j \le n.$$

Then, as we discussed in §2.2,

$$\hat{\sigma}_0 \cap L^* = \left\{ \sum_{i=1}^{N} \mathbf{Z}^+ \left( \frac{1}{3} r_i \right) \right\},$$

$$X_{\sigma_0} = \operatorname{Spec}(\mathbf{C}[\chi_{r_1/3}, \cdots, \chi_{r_N/3}]) \simeq \mathbf{C}^N = \{(u_1, \cdots, u_N)\}, \qquad u_i = \chi_{r_i/3}.$$

Let $w = (w_1, \cdots, w_N)$ be the coordinate system on $L(F_0)\backslash S_n$ as before such that $w = (w_1, \cdots, w_N) = (w_{11}, w_1, \cdots, w_{1n}, w_{22}, w_{23}, \cdots, w_{nn})$ where

$$w_{kj} = e^{i(2\pi/3)\tau_{kj}}.$$

If we set $(u_1, \cdots, u_N)$ to be $(u_{11}, u_{12}, \cdots, u_{1n}, u_{22}, u_{2n}, \cdots, u_{nn})$, then

$$u_{ii} = \chi_{r_{ii}/3}(w_{11} \cdots w_{nn}) = w_{1i} w_{2i} \cdots w_{ii} w_{ii+1} \cdots w_{in}, \qquad 1 \le i \le n,$$

$$u_{ij} = w_{ij}^{-1}, \qquad 1 \le i < j \le n,$$

$$\Rightarrow w_{ij} = u_{ij}^{-1} \quad \text{for } 1 \le i < j \le n,$$

$$w_{ii} = u_{1i} u_{2i} \cdots u_{ii} u_{ii+1} \cdots u_{in} \quad \text{for } 1 \le i \le n.$$

$$\frac{dw_1 \wedge \cdots \wedge dw_n \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_N}{\prod_{i=1}^{N} |w_i|^2} = \frac{du_1 \wedge \cdots \wedge du_N \wedge d\overline{u}_1 \wedge \cdots \wedge d\overline{u}_N}{\prod_{i=1}^{N} |u_i|^2}$$

and

$$\det \begin{bmatrix} \log|w_1| & \log|w_2| & \dots & \log|w_n| \\ \log|w_2| & \log|w_{n+1}| & \dots & \log|w_{2n-1}| \\ \dots & \dots & \dots & \dots \\ \log|w_n| & \log|w_{2n-1}| & \dots & \log|w_N| \end{bmatrix} = \det \mathscr{M}.$$

This proves the lemma by the expression of $\Phi_w$.

Now we are going to prove Theorem 4.1.

*Proof of Theorem* 4.1. Let $\sigma = \{\sum_{i=1}^N \mathbf{R}^+ a_i\}$ be a regular cone. Then

$$a_i = [a^i_{jk}]^n_{j,k=1} = [C^i_1 C^i_2 \cdots C^i_n], \quad n \times n \text{ symmetric } \mathbf{Z} - \text{matrix}.$$

If $\{v_1, \cdots, v_N\} = \{e_{11}, e_{12}, \cdots, e_{1n}, e_{22}, \cdots, e_{nn}\}$ is an integral basis used in Lemma 2, then

$$a_i = (v_1, \cdots, v_N) B_i,$$

where

$$B_i = \begin{bmatrix} a^i_{11} + a^i_{12} + \cdots + a^i_{1n} \\ -a^i_{12} \\ \vdots \\ -a^i_{1n} \\ a^i_{12} + a^i_{22} + \cdots + a^i_{2n} \\ -a^i_{23} \\ \vdots \\ -a^i_{2n} \\ \vdots \\ a^i_{1n} + a^i_{2n} + \cdots + a^i_{nn} \end{bmatrix}.$$

Let

$$B = [B_1, B_2, \cdots, B_n] = (b_{ij})_{N\times N}.$$

Then $(a_1, a_2, \cdots, a_N) = (v_1, v_2, \cdots, v_N) \cdot B$.

If $\{u_1, \cdots, u_N\}$ and $\{s_1, s_N\}$ are the coordinate systems associated with $\{v_1, \cdots, v_N\}$ and $\{a_1, \cdots, a_N\}$ respectively, then by Lemma 1 we obtain

$$u_i = \prod_{j=1}^N s_j^{b_{ij}}.$$

It can be checked directly that

$$\frac{du_1 \wedge \cdots \wedge du_N \wedge d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_N}{\prod_{i=1}^N |u_i|^2} = \frac{ds_1 \wedge \cdots \wedge ds_N \wedge d\bar{s}_1 \wedge \cdots \wedge d\bar{s}_N}{\prod_{i=1}^N |s_i|^2},$$

and

$$\begin{bmatrix} \log|u_1| \\ \log|u_2| \\ \vdots \\ \log|u_N| \end{bmatrix} = B \cdot \begin{bmatrix} \log|s_1| \\ \log|s_2| \\ \vdots \\ \log|s_N| \end{bmatrix}.$$

Thus

$$\begin{aligned} f_u &= \det \mathscr{M} = \det \left[ \sum_{k=1}^{N} a_{ij}^{k} \log|s_k| \right] \\ &= \det \left[ \sum_{i=1}^{N} \log|s_i| C_1^i, \cdots, \sum_{i=1}^{N} \log|s_i| C_n^i \right] \\ &= \sum_{1 \le i_1, \cdots, i_n \le N} \det[C_1^{i_1}, C_2^{i_2}, \cdots, C_n^{i_n}] \log|s_{i_1}| \log|s_{i_2}| \cdots \log|s_{i_n}|. \end{aligned}$$

Hence the theorem is proved by using Lemma 2.

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