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ISOPARAMETRIC SUBMANIFOLDS AND THEIR HOMOGENEOUS STRUCTURES

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0. Introduction

In [4] it was proved that given a submanifold of Euclidean space the representation on the normal space of the holonomy group of the normal connection is an s-representation (i.e., equivalent to the isotropy representation of a semisimple symmetric space). This result has important consequences for the isoparametric problem. For instance, it was used in [2] to give a geomeric characterization of the manifolds which are focal to the isoparametric submanifolds. The purpose of this article is to study in some detail how the homogeneity of isoparametric submanifolds is related to the precise knowledge of the normal holonomy groups of focal manifolds. As a consequence of this we will give a simple and geometric proof, without using Tits buildings, of the remarkable result of Thorbergsson.

Theorem (Thorbergsson [9]). *Every irreducible isoparametric submanifold of a Euclidean space of rank at least* 3 *is an orbit of an* s*-representation.*

Let M be an irreducible isoparametric submanifold of rank ≥ 3 of the Euclidean space. Then, due to the Homogeneous Slice Theorem (see [2]), there are abundantly many submanifolds of M which are homogeneous and isoparametric, namely, those obtained as the fibers of focalization. The problem is how to glue all this information together in order to conclude that M is homogeneous. We use, for this purpose, the techniques developed in [5] together with the results in [4]. Namely, we construct a canonical metric connection in TM such that the second fundamental form is parallel with respect to this connection and the usual connection in the normal bundle. The difference tensor between the Riemannian connection and the new one is also parallel. Then, by the main result in [5], M is an orbit. The method of constructing this connection is by gluing together the canonical connections of the submanifolds of M given by slices.

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It should be remarked that our proof is independent of the homology and tightness of isoparametric submanifolds (see [3]). This information can now be derived from the well-known results for orbits and those for isoparametric hypersurfaces of the sphere.

We hope that the ideas in this paper will be useful to establish the homogeneity of higher rank (rank ≥ 2) isoparametric submanifolds of Hilbert spaces (cf. [7]).

1. Notation and preliminaries

The general references for this section are [6], [7], and [2]. Let M^n be a compact irreducible isoparametric submanifold of $\mathbb{R}^n$ with rank ≥ 3. Let $n_1, \cdots, n_g$ be the principal normals, and $E_1, \cdots, E_g$ be the respective eigendistributions of the shape operator A (i.e., $TM = E_1 \oplus \cdots \oplus E_g$ and $A_\xi X_i = \langle n_i, \xi \rangle X_i$ if X_i lies on E_i). Let, for $1 \leq i, j \leq g$, L_{ij} be the (parallel) vector subbundle of the normal bundle $\nu(M)$ generated by n_i and n_j. Choose, for $1 \leq i, j \leq g$, a parallel $\xi_{ij} \in C^\infty(M, \nu(M))$ such that $\langle \xi_{ij}, n_k \rangle = 1$ if and only if n_k lies on L_{ij}. Let, for $p \in M$, $S_{ij}(p)$ be the leaf through p of the autoparallel distribution $\ker(A_{\xi_{ij}} - \mathrm{Id})$ on M. Then $S_{ij}(p)$ is totally geodesic in M. Moreover, it is a compact full isoparametric submanifold of $\mathbb{V}_{ij}(p) = p + \ker(A_{\xi_{ij}} - \mathrm{Id})(p) \oplus L_{ij}(p)$ with curvature normals $\{n_{k|S_{ij}(p)} : n_k \text{ lies on } L_{ij}\}$. It is homogeneous due to the Homogeneous Slice Theorem of [2] (this theorem has already been used by Thorbergsson in [9]). The rank of $S_{ij}(p)$ is 2 if $i \neq j$, and 1 if $i = j$. In the last case $S_i(p) = S_{ii}(p)$ is a round sphere centered at $p + r_i(p) = p + |n_i|^{-2} \cdot n_i$. In general, there exists a (unique) parallel $r_{ij} \in C^\infty(M, \nu(M))$ lying on L_{ij} such that $S_{ij}(p)$ is contained in the sphere of $\mathbb{V}_{ij}(p)$ with center at $p + r_{ij}(p)$ and radius $|r_{ij}(p)|$. (In the following when two subscripts are equal we shall often omit one of them.) If $\overline{M}^n = M_\xi$ is a parallel manifold to M, where $\xi \in C^\infty(M, \nu(M))$ is parallel, then $\overline{n}_i = (1 - \langle \xi, n_i \rangle)^{-1} n_i$ and $\overline{r}_{ij} = r_{ij} - h_{ij}(\xi)$, where h_{ij} is the orthogonal projection to L_{ij}. From this fact it is easy to see that we may assume, perhaps by considering a parallel manifold, that $\ker(A_{r_{ij}} - \mathrm{Id}) = \ker(A_{\xi_{ij}} - \mathrm{Id})$. In the next sections we shall often assume this.

Let now N be a compact submanifold with constant principal curvatures in $\mathbb{R}^N$. If $p \in N$ and $\xi = (p, \xi_p) \in \nu(N)_p$ then $\mathrm{Hol}_\xi(N)$ is defined to be the subset of $\nu(N)$ obtained by $\nabla^\perp$-parallel displacement of ξ along piecewise differentiable curves starting at p. We have that

$\mathrm{Hol}_\xi(N) \xrightarrow{\pi} N$ is a fiber bundle over N with standard fiber $\Phi_p \cdot \xi$, where $\pi((q, v)) = q$ and Φ_p denotes the normal holonomy group of N at p. There is a natural map i: $\mathrm{Hol}_\xi(N) \to \mathbb{R}^N$ defined by $i((q, v)) = q + v$. The set $N_\xi = i(\mathrm{Hol}_\xi(N))$ is always a compact submanifold with constant principal curvatures. If 1 is not an eigenvalue of the shape operator A_ξ of N, then i is an immersion (actually, it is an imbedding). If, in addition, $\Phi_p \cdot \xi$ is a principal orbit, then N_ξ is isoparametric. When, from the context, there is no possible confusion we shall often identify ξ with ξ_p.

The following result can be easily derived from [2].

Theorem 1.1 (see [2]). *Let $\mathfrak{F}$ be the parallel singular foliation of $\mathbb{R}^N - \{0\}$ induced by a compact irreducible isoparametric submanifold. Let M, $N \in \mathfrak{F}$, $p \in M$, and choose $q \in N$ such that $\xi = q - p$ be normal to M at p. Then $N = M_\xi$ (and hence $q - p$ is normal to N at q and $N_{-\xi} = M$).*

The proof of the following result in a more general context will be included in the Appendix.

Theorem 1.2. *Let N be a submanifold with constant principal curvatures of $\mathbb{R}^N$, let $p \in N$, and let $\xi \in \nu(N)_p$ be such that 1 is not an eigenvalue of the shape operator A_ξ of N. Let Φ and $\tilde{\Phi}$ be the normal holonomy groups of N and N_ξ at p and $p + \xi$ respectively. Then $\tilde{\Phi}$ is the image of the representation of the isotropy subgroup Φ_ξ of Φ at ξ on the normal space, in $\nu(N)$, of $\Phi \cdot \xi$ at ξ.*

2. Homogeneous structures on submanifolds

Let M^n be a submanifold of the Euclidean space. A *homogeneous structure* on M (cf. [10]) is a connection $\tilde{\nabla}$ in the bundle $TM \oplus \nu(M)$ such that:

(i) TM and $\nu(M)$ are parallel subbundles of $TM \oplus \nu(M)$,
(ii) $\tilde{\nabla}$ is a metric connection,
(iii) the second fundamental form of M is $\tilde{\nabla}$-parallel,
(iv) $\tilde{D} = \nabla \oplus \nabla^\perp - \tilde{\nabla}$ is $\tilde{\nabla}$-parallel, where ∇ is the induced Levi-Civita connection on M.

We shall not deal here with such a general situation which turns out to be equivalent to the fact that M be an open part of an extrinsically homogeneous submanifold of the Euclidean space. We shall only consider *normal homogeneous structures*. This means that $\tilde{\nabla} = \nabla^c \oplus \nabla^\perp$ where ∇^c is a metric connection on TM such that $D = \nabla - \nabla^c$ is ∇^c-parallel. With this terminology the main result in [5] can be stated as follows.

Theorem 2 [5]. *A compact full submanifold of the Euclidean space admits a normal homogeneous structure if and only if it is an orbit of an s-representation.*

Let us recall that an s-representation is a representation which is orthogonally equivalent to the isotropy representation of a semisimple symmetric space.

Let (G, K) be a simply connected semisimple symmetric pair of rank 2 and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Denote the Ad-action of K on $\mathfrak{p}$ by a dot. Let $v \in \mathfrak{p}$ and let $M = K.v$ be a principal orbit. Then M is an isoparametric submanifold of $\mathfrak{p}$. Let $n_1, \cdots, n_g$ be the curvature normals, and $E_1, \cdots, E_g$ be the associated eigendistributions of the shape operator A of M. Let $\xi \in C^\infty(M, \nu(M))$ be parallel such that M_ξ is focal and not trivial. Then $M_\xi = K.u$, where $u = v + \xi(v)$ and $K_v \subset K_u$ (isotropy subgroups at v and u).

2.1. We keep the assumptions of §2. As in [5] we have canonical connections ∇^c and $\overline{\nabla}^c$ on M and M_ξ respectively. Observe that, since the second fundamental form of M is ∇^c-parallel, the distributions E_i are ∇^c-parallel. Let X belong to the orthogonal complement, in $\mathfrak{k}$, of the Lie algebra $\mathfrak{k}_v$ of K_v such that $\gamma(t) = \exp(t.X).v$ be a ∇^c-geodesic in $S_j(v)$, where E_j is different from $E_k = \ker(A_\xi - \mathrm{Id})$. Let $w \in E_k(v)$. Then $\exp(t.X)_{*v}(w)$ is ∇^c-parallel along $\gamma(t)$. It is easy to see that $\overline{\gamma}(t) = \exp(t.X).u$ is a $\overline{\nabla}^c$-geodesic in M_ξ and that $\exp(t.X)_{*u}(w)$ is $\nabla^\perp$-parallel along $\overline{\gamma}(t)$ (see [5]). This will be used in §3.

2.2. We keep the assumptions of §2. Let, for $i = 1, \cdots, g$,

$$\widetilde{K}^i = (\{k_{|\mathbf{V}_i(v)} : k \in K \text{ and } k.S_i(v) = S_i(v)\})_0,$$

where $(\ )_0$ denotes the connected component of the identity.

We may assume, as in §1, that $E_i = \ker(A_{r_i} - \mathrm{Id})$, where $r_i = |n_i|^{-2} \cdot n_i$. Let $u_i = v + r_i(v)$; then

$$\widetilde{K}^i = K^i := (\{k_{|\mathbf{V}_i(v)} : k \in K_{u_i}\})_0.$$

It is clear that the left-hand side is contained on the right-hand side. The other inclusion is a consequence of the Homogeneous Slice Theorem and the fact that K^i is the restricted normal holonomy group of M_{u_i} at u_i, if it is full (see [1]). If M_{u_i} is not full, then it is a factor of M and the equality is clear.

The representation of K^i on $\mathbb{V}_i(v)$ is an s-representation (see [1]). Associated to this representation we can construct, as in [5], a canonical connection ∇^i on $S_i(v)$. Then ∇^i coincides with the connection induced by ∇^c in the autoparallel submanifold $S_i(v)$. (To see this notice that the decomposition of [5] coincides with that given in [1, §2] in terms of the eigenspaces of $\mathrm{ad}^2(X)$, where X belongs to a Cartan subalgebra of $\mathfrak{p}$).

Remark. The results in this whole section are also true for rank ≥ 2.

3. The proof of the theorem of Thorbergsson

Let M be as in §1. Then we have the submersions $M \xrightarrow{\pi_{ij}} M_{r_{ij}}$, where $\pi_{ij}(u) = u + r_{ij}(u)$ $(1 \leq i, j \leq g)$. Observe that $(\pi_{ij}^{-1}(\pi_{ij}(u)))_u = S_{ij}(u)$. Suppose now that $i \neq j$. Then we have, for all $q \in M$, that $r_{ij} - r_i$ is constant on $S_i(q)$, because $r_{ij}(q) - r_i(q)$ is orthogonal to $\mathbb{V}_i(q)$. Then it defines, locally, a parallel normal vector field to M_{r_i}. Let us call it η_{ij}, defined in an appropriate neighborhood U of $\pi_i(p) = p + r_i(p)$ where p is a fixed point in M. By Theorem 1.1 we have that $(M_{r_i})_{\eta_{ij}(\pi_i(p))} = M_{r_{ij}}$. There is also a submersion $s \to p_{ij}(s) = s + \eta_{ij}(s)$ defined from U onto an open subset of $M_{r_{ij}}$.

Let now γ be a short horizontal curve in M, with respect to $M \xrightarrow{\pi_{ij}} M_{r_{ij}}$, starting at p. Then γ is also horizontal with respect to $M \xrightarrow{\pi_i} M_{r_i}$. Thus $p_{ij} \circ (\pi_i \circ \gamma) = \pi_{ij} \circ \gamma$, and $\pi_i \circ \gamma$ is horizontal with respect to $U \xrightarrow{p_{ij}} p_{ij}(U)$.

Let, for $q \in M$, ∇_q^{ij} be the canonical connection on $S_{ij}(q)$ induced by the restricted normal holonomy group of $M_{r_{ij}}$ at $\pi_{ij}(q)$. (By [4] this group acts as an s-representation.) Let

$$D_q^{ij} = \nabla - \nabla_q^{ij} : T_qS_{ij}(q) \times T_qS_{ij}(q) \to T_qS_{ij}(q),$$

where ∇ is the Levi-Civita connection on M. (Recall that $S_{ij}(q)$ is totally geodesic in M.) Then $D_q^{ij}(X, \ )$ is skew-symmetric. Let now $X, Y \in T_qM$, $X = \sum_{i=1}^g X_i$, $Y = \sum_{j=1}^g Y_j$, where X_h, $Y_h \in E_h(q)$ $(h = 1, \cdots, g)$. Define the C^∞ tensor D on M by

$$D_q(X, Y) = \sum_{i,j} D_q^{ij}(X_i, Y_j).$$

Notice that $i = j$ is allowed in the above sum.

Lemma 3.1. *If $X, Y \in T_qS_{ij}(q)$, then $D_q(X, Y) = D_q^{ij}(X, Y)$.*

Proof. We may assume that $i = 1$, $j = 2$, and that $T_qS_{12}(q) = E_1(q) \oplus \cdots \oplus E_r(q)$ $(1 < r < g)$. Write $X = X_1 + \cdots + X_r$, $Y = Y_1 + \cdots + Y_r$, where X_k, $Y_k \in E_k(q)$, $1 \le k \le r$. Then

$$D_q(X, Y) = \sum_{k \neq s} D_q^{ks}(X_k, Y_s) + \sum_t D_q^t(X_t, Y_t)$$

where $D_q^t = D_q^{tt}$.

If $k \neq s$, then $S_{ks}(q) = S_{12}(q)$ and $D_q^{ks} = D_q^{12}$. Thus we must only show that $D_q^t(X_t, Y_t) = D_q^{12}(X_t, Y_t)$. We may assume that $t = 1$. Let Φ^{12} be the restricted normal holonomy group of $M_{r_{12}}$ at $\pi_{12}(q)$. Then, by §2.2, the induced canonical connection on $S_1(q)$, as an autoparallel submanifold of $S_{12}(q)$, is determined by the image of the representation, on $\nu(\Phi^{12}z)$, of the isotropy subgroup Φ_z^{12}, where $z = r_1(q) - r_{12}(q)$. But Theorem 1.2 implies that this is the restricted normal holonomy group of M_{r_1} at $\pi_1(q)$. Since $S_1(q)$ is a totally geodesic submanifold of $S_{12}(q)$, we get that $D_q^1(X_1, Y_1) = D_q^{12}(X_1, Y_1)$.

Now it is clear that $D_q(X, \ )$ is a skew-symmetric endomorphism of T_qM. Let us define $\nabla^c = \nabla - D$. Then ∇^c is a metric connection on M.

Lemma 3.2. $\nabla^c\alpha = 0$, *where* α *is the second fundamental form of* M.

Proof. Let $i, j \in \{1, \cdots, g\}$, $i \neq j$. By Lemma 3.1, $S_{ij}(q)$ is a ∇^c-autoparallel submanifold of M. Moreover, E_i is ∇^c-parallel in $S_{ij}(q)$. This is due to the fact that the second fundamental form of $S_{ij}(q)$ is ∇^c-parallel. Thus $\nabla^c_{X_j}X_i$ lies on E_i if X_i lies on E_i and X_j lies on E_j. Since i, j are arbitrary we conclude that the eigendistributions are ∇^c-parallel, so that $\nabla^c\alpha = 0$.

Lemma 3.3. $\nabla^c D = 0$.

Proof. If $X, Y, Z \in T_qS_{ij}(q)$, then $(\nabla^c_X D_q)(Y, Z) = (\nabla^{ij}_X D_q^{ij})(Y, Z) = 0$, where the last equality is due to [5]. Let thus $i \neq j$ and k be such that $E_k(q)$ is not included in $T_qS_{ij}(q)$. Let $\gamma_k\colon [0, 1] \to M$ be a ∇^c-geodesic in $S_k(q)$ with $\gamma_k(0) = q$. Let $X_i \in E_i(q)$ and let $\widetilde{X}_i(t)$ be its ∇^c-parallel transport along $\gamma_k(t)$. This can be carried out in $S_{ki}(q)$. Let us consider $S_{ki}(q) \xrightarrow{\pi_i} (S_{ki})_{r_i}$. Then, by §2.1, $\widetilde{X}_i(t)$ is $\nabla^\perp$-parallel along $\pi_i \circ \gamma_k(t)$ in $(S_{ki}(q))_{r_i}$. It is clear that $\widetilde{X}_i(t)$ is also $\nabla^\perp$-parallel along $\pi_i \circ \gamma_k(t)$ in M_{r_i}. It is not hard to see that $\widetilde{X}_i(t)$ is also $\nabla^\perp$-parallel along $p_{ij} \circ (\pi_i \circ \gamma_k)(t) = \pi_{ij} \circ \gamma_k(t)$ in $M_{r_{ij}}$ (similar to the case of an isoparametric submanifold and a focal manifold in [2, §2]). This argument can be repeated for all $X \in E_h(q)$ such that $E_h(q) \subset T_q(S_{ij}(q))$. Thus for $X \in T_q(S_{ij}(q))$, $\widetilde{X}(t)$ is ∇^c-parallel along $\gamma_k(t)$ in M if and only if it is $\nabla^\perp$-parallel along $\pi_{ij} \circ \gamma_k(t)$ in $M_{r_{ij}}$. Let $p = \gamma_k(1)$ and let $\tau^\perp$ be the $\nabla^\perp$-parallel transport along $\pi_{ij} \circ \gamma_k$. Then $\tau^\perp$ maps $S_{ij}(q)$ into $S_{ij}(p)$ and $\tau_*^\perp(D_q) = D_p$. This is due to the Homogeneous Slice Theorem in [2] and the fact that $\tau_*^\perp$ sends the normal holonomy group of $M_{r_{ij}}$ at q to the corresponding normal holonomy group at p. If $X, Y \in T_q S_{ij}(q)$, then

$$D_p(\tau^c(X), \tau^c(Y)) = D_p(\tau^\perp(X), \tau^\perp(Y)) = \tau^\perp D_q(X, Y) = \tau^c D_q(X, Y),$$

where τ^c denotes the ∇^c-parallel displacement along γ_k. This implies that D is ∇^c-parallel.

Remark. The same methods apply to show the homogeneity of irreducible isoparametric submanifolds of higher rank in spaces of constant (negative) curvature.

Appendix

Let N, M be differentiable manifolds and let $N \xrightarrow{\pi} M$ be a submersion. Let $\mathscr{H}$ be a horizontal distribution with respect to the vertical one, i.e., $TN = \mathscr{H} \oplus \ker(\pi_*)$. Suppose that any piecewise C^1-curve in M has a (unique) horizontal lifting to N with any given basepoint. This is the case where N is compact, or N and M are Riemannian, with N complete and the following condition holds: there exists $c > 0$ such that $\|X\| \leq c.\|\pi_*(X)\|$ for all $X \in x(N)$ which are horizontal.

Given $c\colon [0, 1] \to N$ piecewise differentiable and $s \in [0, 1]$ we denote by $\tilde{c}_s$ the unique horizontal curve in N such that $\pi \circ \tilde{c}_s = \pi \circ c$ and $\tilde{c}_s(s) = c(s)$. Let $f_c\colon [0, 1] \times [0, 1] \to N$ be defined by $f_c(s, t) = \tilde{c}_s(t)$. If $(s_0, t_0) \in [0, 1] \times [0, 1]$, then $f_c(s_0, \ )$ is a horizontal curve and $f_c(\ , t_0)$ is a vertical one. We have also that $f_c(t, t) = c(t)$.

Lemma. *Let N, M be as above and let $E \to N$ be a vector bundle over N with a connection ∇. Suppose that if X, $Y \in x(N)$ are horizontal and vertical (with respect to $\mathscr{H}$) respectively, then the curvature tensor $R^\nabla(X, Y) = 0$. Let $c\colon [0, 1] \to N$ be a piecewise differentiable curve. Then there exists a vertical curve γ (lying in $\pi^{-1}(\pi(c(1)))$ in N such that $\tau_c = \tau_\gamma \circ \tau_{\tilde{c}_0}$, where τ denotes the ∇-parallel transport.*

Proof. Let $f(s, t) = \tilde{c}_s(t)$ be as before. Let $\gamma = f(\ ,1)$. Then γ is vertical. Since

$$\frac{D^\nabla}{\partial t}\frac{D^\nabla}{\partial s} - \frac{D^\nabla}{\partial s}\frac{D^\nabla}{\partial t} = R^\nabla\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = 0,$$

we have $\tau_c = \tau_\gamma \circ \tau_{\tilde{c}_0}$.

Let $f\colon M \to \mathbb{R}^N$ be an immersion. Let

$$\nu(M) = \{(q, w)\colon q \in M, w \in \mathbb{R}^N, \text{ and } w \perp f_*(T_q M)\}$$

be the normal bundle over M induced by f and let $i\colon \nu(M) \to \mathbb{R}^N$ be defined by $i((q, w)) = f(q) + w$. Let $\xi = (p, v) \in \nu(M)$ and suppose that $i|_{\mathrm{Hol}_\xi(M)}$ is an immersion, e.g., that M is compact and v is small. Here $\mathrm{Hol}_\xi(M) \xrightarrow{\pi} M$ denotes the holonomy subbundle (with respect to $\nabla^\perp$) of $\nu(M)$ through ξ, where $\pi((q, w)) = q$ (cf. [2]). Let $\tilde{\xi}$ be the normal vector field to $\mathrm{Hol}_\xi(M)$ defined by $\tilde{\xi}(y) = i(y) - f(\pi(y))$. Then $\tilde{\xi}$ is parallel with respect to the normal connection. The proof of this is similar to that given in [2] when v is a principal vector for the normal holonomy group action. Put on $\mathrm{Hol}_\xi(M)$ the metric induced by i and let $\mathscr{H}$ be the distribution on $\mathrm{Hol}_\xi(M)$ defined by $\mathscr{H} = \ker(\pi_*)^\perp$. Then $\ker(\pi_*) = \ker(\mathrm{Id} - A_{\tilde{\xi}})$, where A is the shape operator of $\mathrm{Hol}_\xi(M)$. Since $\tilde{\xi}$ is parallel, $A_{\tilde{\xi}(y)}$ commutes with all the shape operators at $y \in \mathrm{Hol}_\xi(M)$ because of $R^\perp(w, z)\tilde{\xi} = 0$ and the Ricci identity. Thus all the shape operators at y leave invariant $\ker(\mathrm{Id} - A_{\tilde{\xi}(y)})$ and hence leave invariant $\mathscr{H}_y$. Using again the Ricci identity, we get that $R^\perp(X, Y) = 0$ if $X \in \mathscr{H}_y$, $Y \in \ker(\pi_*)_y$. By the lemma, if $c\colon [0, 1] \to \mathrm{Hol}_\xi(M)$ is piecewise differentiable, then there exist piecewise differentiable curves γ and $\tilde{c}_0$ in $\mathrm{Hol}_\xi(M)$, with γ vertical and $\tilde{c}_0$ horizontal such that $\tau_c^\perp = \tau_\gamma^\perp \circ \tau_{\tilde{c}}^\perp$, where $\tau^\perp$ denotes the $\nabla^\perp$-parallel transport.

Remark. (a) $i_*(T_\xi(\mathrm{Hol}_\xi(M))) = f_*(T_p M) \oplus i_*(T_\xi \Phi.\xi)$ and hence $\mathrm{pr}_2(\nu(\Phi.\xi)_\xi) = \mathrm{pr}_2(\nu(\mathrm{Hol}_\xi(M))_\xi)$, where $\mathrm{pr}_2((q, w)) = w$, and $\nu(\Phi.\xi)_\xi$ has to be seen as the subspace of $\nu(M)_p$ which is orthogonal to $\Phi.\xi$ at ξ.

(b) Let $c(t)$ be a horizontal curve in $\mathrm{Hol}_\xi(M)$ and let $(c(t), X(t))$ be a $\nabla^\perp$-parallel normal vector field to $\mathrm{Hol}_\xi(M)$ along $c(t)$. Let A be the shape operator of $\mathrm{Hol}_\xi(M)$. Then

$$\frac{d}{dt}X(t) = i_*(-A_{X(t)}c'(t)) \in f_*(T_{\pi(c(t))}M).$$

This equality is due to the fact that the shape operators of $\mathrm{Hol}_\xi(M)$ preserve the horizontal foliation. Thus $(\pi \circ \gamma(t), X(t))$ is a $\nabla^\perp$-parallel normal vector field to M along $\pi \circ \gamma(t)$.

(c) Let $\beta(t)$ be a vertical curve in $\mathrm{Hol}_\xi(M)$ starting at ξ and let $(\beta(t), Y(t))$ be a $\nabla^\perp$-parallel normal vector field to $\mathrm{Hol}_\xi(M)$ along $\beta(t)$. Then $(\beta(t), Y(t))$ is a $\nabla^\perp$-parallel normal vector field to $\Phi.\xi$ along $\beta(t)$.

We have the following:

Theorem. *Let $f: M \to \mathbb{R}^N$ be an immersion, let $\xi = (p, \xi_p)$, and let $\mathrm{Hol}_\xi(M) \xrightarrow{\pi} M$ be the holonomy subbundle at ξ of the normal bundle $\nu(M) \xrightarrow{\pi} M$. Suppose that $i: \mathrm{Hol}_\xi(M) \to \mathbb{R}^N$ is an immersion, where $i((q, w)) = f(q) + w$. Let Φ and $\tilde{\Phi}$ be the normal holonomy groups of M and $\mathrm{Hol}_\xi(M)$ at p and ξ respectively. By means of pr_2, identify $\nu(\Phi.\xi)_\xi$ with $\nu(\mathrm{Hol}_\xi(M))_\xi$. Then $\tilde{\Phi}$ is the image of the representation on $\nu(\Phi.\xi)_\xi$ of the isotropy subgroup Φ_ξ of Φ at ξ.*

Proof. Let $c: [0, 1] \to \mathrm{Hol}_\xi(M)$ be piecewise differentiable with $c(0) = \xi = c(1)$. Then from the previous facts it follows that $\tau_c^\perp = \tau_\gamma^\perp \circ \tau_{\tilde{c}_0}^\perp$, where γ is a vertical curve and $\tilde{c}_0$ is a horizontal one. By part (c) of the Remark and by [5] there exists $g \in \Phi$ such that $\tau_\gamma^\perp = g_{|\nu(\Phi.\xi)_{\gamma(0)}}$.

On the other hand, part (b) of the Remark implies

$$\tau_{\tilde{c}_0}^\perp = \tau^\perp_{h|\nu(\mathrm{Hol}_\xi(M))_\xi} = \tilde{g}_{|\nu(\Phi.\xi)_\xi},$$

where $h = \pi \circ \tilde{c}_0 = \pi \circ c$ is a closed curve in M and $\tilde{g} = \tau_h^\perp \in \Phi$. Clearly, $g\tilde{g}.\xi = \xi$ since ξ extends to a parallel normal vector field $\tilde{\xi}$ to $\mathrm{Hol}_\xi(M)$. Thus $\tilde{\Phi} \subset \{g_{|\nu(\Phi.\xi)_\xi} : g \in \Phi_\xi\}$.

Let now $g \in \Phi_\xi$ and let $r: [0, 1] \to M$ be a curve in M such that $\tau_r^\perp = g$. Let $\tilde{r}$ be the horizontal lifting of r to $\mathrm{Hol}_\xi(M)$ with $\tilde{r}(0) = \xi$. Then $\tilde{r}(t) = (r(t), \tilde{\xi}(t))$ is $\nabla^\perp$-parallel along $r(t)$. Since $g.\xi = \xi$, and $\tilde{r}(0) = \tilde{r}(1)$, from part (b) of the Remark it follows that

$$\tau_{\tilde{r}}^\perp = \tau^\perp_{r|\nu(\Phi.\xi)_\xi} = g_{|\nu(\Phi.\xi)_\xi},$$

so that $\tilde{\Phi} \supset \{g_{|\nu(\Phi.\xi)_\xi} : g \in \Phi_\xi\}$.

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