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ON SIMPLY CONNECTED NONCOMPLEX 4-MANIFOLDS

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Abstract

We define a sequence $\{X_n\}_{n\geq 0}$ of homotopy equivalent smooth simply connected 4-manifolds, not diffeomorphic to a connected sum $M_1 \# M_2$ with $b_2^+(M_i) > 0$, $i = 1, 2$, for $n > 0$, and nondiffeomorphic for $n \neq m$. Each X_n has the homotopy type of $7\mathbf{CP}^2 \# 37\overline{\mathbf{CP}}^2$. We deduce that for all but finitely many n the connected sum of X_n with a homotopy sphere is not diffeomorphic to a connected sum of complex surfaces, complex surfaces with reversed orientations and a homotopy sphere.

1. Introduction

A well-known conjecture, asserting that any smooth simply connected 4-manifold is diffeomorphic to a connected sum of complex surfaces and complex surfaces with reversed orientations (S^4 being the trivial connected sum), has recently turned out to be false, by work of Gompf and Mrówka [11]. Gompf and Mrówka produce infinite families of counterexamples via the following construction: in the smooth category it can be assumed that any elliptic fibration on a $K3$ surface has a cusp fiber F, and one can consider the *nucleus* N, a regular neighborhood of the union of F with a smooth section of the fibration. A new surface can be obtained by performing "differentiable logarithmic transforms" on smooth fibers in the interior of N, and it is possible to find three disjoint nuclei in a $K3$ surface corresponding to three different elliptic structures, and do logarithmic transforms inside each of them. Gompf and Mrówka are able to compute, using previous work of Mrówka [13], the values of certain Donaldson invariants of the resulting manifolds. The classification of algebraic surfaces and the values of the invariants show that these manifolds are counterexamples to the conjecture.

The purpose of this paper, which draws part of its inspiration from the above ideas, is to produce new counterexamples to the same conjecture by a slightly different approach: let X be the two-fold branched cover of

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$\mathbf{CP}^2$ along a smooth algebraic curve of degree eight. This can be given the structure of an algebraic surface of general type. We show, using results of Akbulut and Kirby [1], that there is a copy of the nucleus N inside X. Performing one "logarithmic transform" of multiplicity n in the interior of N yields a manifold X_n homeomorphic to X. We use the relative version of the Donaldson invariants (see [2]) and a computation by Donaldson and Kronheimer [4] to show that X_n is not diffeomorphic to X_m if $n \neq m$ (Theorem 4.1). We also show that X_n has a nonzero Donaldson invariant for $n > 0$. It follows that for $n > 0$ X_n is not diffeomorphic to a connected sum $M_1 \# M_2$ with $b_2^+(M_i) > 0$, $i = 1, 2$, and for almost all n is not diffeomorphic to a connected sum of complex surfaces and complex surfaces with reversed orientations (Corollary 4.2). Moreover, these results are stable under connected sums with homotopy spheres (Remark 4.3).

We point out that while our method for producing noncomplex manifolds, that is to do log-transforms on nuclei, is the same as Gompf and Mrówka's, their proof depends on arguments involving L^2-moduli spaces of connections, while ours rests on the algebrogeometric interpretation of the Donaldson invariants.

2. A splitting result

To keep the notation coherent with [10], we shall denote from now on by N_2 the nucleus described in the introduction. By [10] N_2 is orientation-preserving diffeomorphic to the 4-manifold with boundary obtained by attaching two 2-handles to the 4-ball according to the framed link in Figure 1.

Let $\Sigma(2, 3, 11)$ be the Brieskorn homology 3-sphere which is the link of the complex singularity $\{x^2 + y^3 + z^{11}\}$, with its natural orientation. It is well known that ∂N_2 is orientation-preserving diffeomorphic to $-\Sigma(2, 3, 11)$, i.e., $\Sigma(2, 3, 11)$ with reversed orientation.

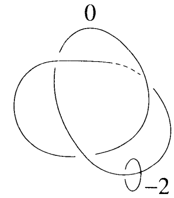

FIGURE 1

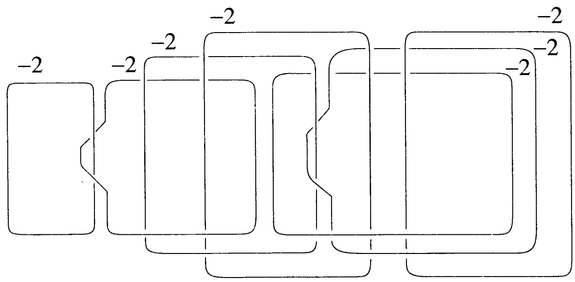

FIGURE 2

Theorem 2.1. *Let X be the two-fold branched cover of* $\mathbf{CP}^2$ *along a smooth algebraic curve of degree 8. Then there is a smooth decomposition* $X = X^+ \bigcup_\Sigma N_2$, *where* $\Sigma = \Sigma(2, 3, 11)$.

Proof. Akbulut and Kirby [1] gave an algorithm to construct a framed link L representing any branched cover of $\mathbf{CP}^2$ along a nonsingular algebraic curve as a 4-ball with 2-handles attached. They explicitly construct L when the curve has degree 6. We shall use their algorithm for a curve of degree 8. Recall that $\mathbf{CP}^2$ can be built from a 0-handle, 2-handle attached to the trivial knot with framing $+1$ and a 4-handle. Given a smooth algebraic curve $F \subset \mathbf{CP}^2$, up to an ambient isotopy of F one can find a decomposition $F = F_0 \cup F_4$, where $F_0 \subseteq \partial$(0-handle) and $F_4 \subseteq \partial$(4-handle), with F_4 consisting of a 2-ball (see [1]). So the p-fold branched cover of $\mathbf{CP}^2$ along F consists of the union of: (i) the p-fold cover of the 0-handle branched over F_0 (with its interior pushed in), (ii) the unbranched p-fold cover of the 2-handle, and (iii) the p-fold cover of the 4-handle branched over an unknotted 2-ball. The 2-handle lifts to p 2-handles and the 4-handle to one 4-handle. It turns out that for a curve of degree d F_0 consists of the Seifert surface for the $(d, d-1)$ torus knot. Akbulut and Kirby show how to construct a framed link presentation L for the p-fold cover of B^4 branched over F_0, with the interior pushed in. The case $p = 2$ is the simplest: it is an exercise to see that for $d = 8$ L contains the framed sublink L' of Figure 2.

By Kirby calculus [12], sliding 2-handles translates into taking band-connected sums, and its can be easily verified that if one orients components of L' and takes orientation-preserving band-connected sums as in Figure 3 (next page) (keeping track of the framings), the results is the framed link of Figure 1.

The existence of the smooth decomposition of X is not immediate.

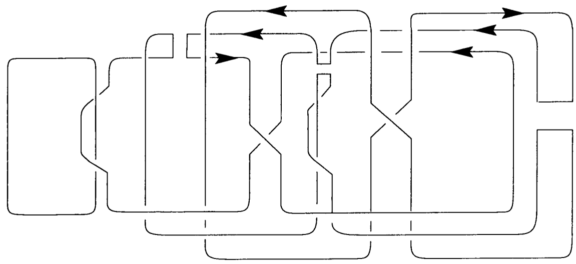

FIGURE 3

3. Computations of Donaldson invariants

Let us recall a few facts about relative Donaldson invariants. Let W be a smooth simply-connected 4-manifold with $b_2^+(W) > 0$. Suppose that ∂W is a $\mathbf{Z}$-homology sphere, and let $I_*(\partial W) = \bigoplus_{i\in \mathbf{Z}_8} I_i(W)$ denote the Floer homology of W [6]. For any integer $r \geq 0$ the *relative Donaldson invariants*

$$\Phi_r(W)\colon \operatorname{Sym}^r H_2(W;\mathbf{Z}) \to I_*(\partial W)$$

are defined [2]. Recall that $I_*(\partial W)$ is endowed with a natural pairing $\cdot\colon I_*(\partial W)\otimes I_*(-\partial W) \to \mathbf{Z}$. Let $X = X^+\bigcup_\Sigma X^-$ a smooth simply connected closed 4-manifold with $b_2^+(X)$ odd which splits smooth along a 3-homology sphere Σ. If $b_2^+(X^+), b_2^+(X^-) > 0$, then the absolute Donaldson invariants $\gamma_c(X)$ [3] can be computed via the Φ_r's and the Poincaré duality pairing on $I_*(\Sigma)$. In fact, let $d = 4c - \frac{3}{2}(1+b_2^+(X))$. Then, using the identification we have

$$\operatorname{Sym}^d H_2(X;\mathbf{Z}) = \bigoplus_{r=0}^{d} \operatorname{Sym}^r H_2(X^+;\mathbf{Z}) \otimes \operatorname{Sym}^{d-r} H_2(X^-;\mathbf{Z}),$$

$$\gamma_c(X) = \sum_{r=0}^{d} \Phi_r(X^+)\cdot\Phi_{d-r}(X^-).$$

Recall that there is a natural isomorphism of algebras between $\operatorname{Sym}^* H^2(X;\mathbf{C})$ and the space $S^*H_2(X,\mathbf{C})$ of symmetric multilinear forms on $H_2(X;\mathbf{C})$ endowed with the usual symmetric product. From now on we shall implicitly use this isomorphism, considering $\gamma_c(X)$, q_X, and any element of $H^2(X)$, as elements of $\operatorname{Sym}^* H^2(X)$.

Let $N_2(n)$ be the 4-manifold with boundary obtained from N_2 by a differentiable logarithmic transform of multiplicity n along a smooth fiber in the interior of $N_2 = N_2(1)$ (see [10] for the definition and a framed link presentation). Recall that, since the log-transform does not alter the boundary, we have $\partial N = \partial N_2(n)$.

Lemma 3.1. $\Phi_0(N_2(n)) = n\Phi_0(N_2)$.

Proof. First suppose $n > 0$. Let S be a smooth elliptic $K3$ surface, and S_n the surface obtained by performing one logarithmic transform of multiplicity $n > 0$ along a smooth fiber. By the results of Friedman and Morgan [9] if c is an integer greater than 3, then the Donaldson invariants $\gamma_c(S_n)$ are polynomials in the intersection form and the canonical class: OB

$$\gamma_c(S_n) = a_l q^l_{S_n} + a_{l-1} q^{l-1}_{S_n} k^2_{S_n} + \cdots ,$$

where $l = d/2$, and $a_l = n(d-1)!!$. In particular, $S_1 = S$, and $\gamma_c(S) = a_l q^l_S$, because $k_S = 0$. By [10] there is a smooth decomposition $S_n = S^+ \bigcup_\Sigma N_2(n)$, and the canonical class of S_n is a rational multiple of the Poincaré dual of the class of a smooth fiber, which is contained in $N_2(n)$. Therefore, for $\alpha \in H_2(S^+)$ with $\alpha \cdot \alpha \neq 0$ and c greater than 3, $\gamma_c(S_n)(\alpha) = n(d-1)!!q^l_{S_n}(\alpha) = n\gamma_c(S)(\alpha)$. Fintushel and Stern [5] have shown that

$$I_*(\Sigma(2, 3, 11)) = \begin{cases} \mathbf{Z} & \text{if } * = 0, 2, 4, 6, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $\alpha \in H_2(S^+)\Phi_c(S^+)(\alpha) = h[\rho]$, $\Phi_0(N_2(n)) = b_n[\sigma]$, where $h, b_n \in \mathbf{Z}$, and $[\rho] \in I_*(\Sigma)$ and $[\sigma] \in I_*(-\Sigma)$ are generators dual to each other. So $\gamma_c(S_n)(\alpha) = \Phi_c(S^+)(\alpha) \cdot \Phi_0(N_2(n)) = hb_n$. Notice that, since $\gamma_c(S_n)(\alpha) \neq 0$, this implies $h \neq 0$. On the other hand $\gamma_c(S)(\alpha) = \Phi_c(S^+)(\alpha) \cdot \Phi_0(N_2) = hb_1$; hence $b_n = nb_1$, and $\Phi_0(N_2(n)) = n\Phi_0(N_2)$.

If $n = 0$ then, $\gamma_2(S_0) = 0$ for all c since by [10] S_0 is diffeomorphic to $3\mathbf{CP}^2 \# 19\overline{\mathbf{CP}}^2$. Hence

$$0 = \gamma_c(S_0)(\alpha) = \Phi_c(S^+)(\alpha) \cdot \Phi_0(N_2(0)) = hb_0,$$

which implies $\Phi_0(N_2(0)) = 0 = 0\Phi_0(N_2)$ since $h \neq 0$. q.e.d.

Let X be the two-fold branched cover of $\mathbf{CP}^2$ along a smooth algebraic curve of degree 8. It is a simply connected algebraic surface of general type. We saw in the previous section that there is a smooth decomposition $X = X^+ \bigcup_\Sigma N_2$. Observe that X^+ has odd intersection form, because X has the homotopy type of $7\mathbf{CP}^2 \# 37\overline{\mathbf{CP}}^2$ while N_2 has even intersection form. For any integer $n \geq 0$ we define $X_n = X^+ \bigcup_\Sigma N_2(n)$. Since $N_2(n)$

has the same rank and signature for all n (see [10]), it follows that the X_n's are all homotopy equivalent and therefore, by the work of Freedman [7], homeomorphic. Now let $q^{(m)} = q^m/m!$.

Proposition 3.2. *Let $k_X \in H_2(X;\mathbf{Z})$ be the Poincaré dual of the canonical class of X, and $\alpha \in H_2(X^+) \subseteq H_2(X;\mathbf{Z})$ orthogonal to k_X. Then $\gamma_7(X_n)(\alpha) = 2nq^{(8)}(\alpha)$.*

Proof. By the computation in [4], §10.4, since α is orthogonal to k_X $\gamma_7(X)(\alpha) = 2q^{(8)}(\alpha)$, so Lemma 3.1 yields

$$\begin{aligned}\gamma_7(X_n)(\alpha) &= \Phi_7(X^+)(\alpha)\cdot\Phi_0(N_2(n)) \\ &= n\Phi_7(X^+)(\alpha)\cdot\Phi_0(N_2) = n\gamma_7(X)(\alpha).\end{aligned}$$

4. Conclusions

We shall now prove the main results. Let X_n, $n \geq 0$, be the manifolds defined in the previous section.

Theorem 4.1. *If X_n is diffeomorphic to X_m then $n = m$.*

Proof. Let $k_X \in H_2(X;\mathbf{Z})$ be the Poincaré dual of the canonical class of X, and $k_X^\perp$ its orthogonal complement with respect to the intersection form. Suppose $f\colon X_n \to X_m$ is a diffeomorphism. For dimensional reasons we can choose $\alpha \in h_2(X^+) \cap k_X^\perp$ with $\alpha\cdot\alpha \neq 0$ and $f_*\alpha \in H_2(X^+) \cap k_X^\perp$. Moreover, since the intersection forms of X_n and X_m are of type $(7, 37)$, f has to be orientation-reserving. Therefore by [3], $\gamma_7(X_n)(\alpha) = \pm\gamma_7(X_m)(f_*\alpha)$, i.e., by Proposition 3.2, $2nq^{(8)}(\alpha) = \pm 2mq^{(8)}(f_*\alpha) = \pm 2mq^{(8)}(\alpha)$, which yields $n = m$.

Corollary 4.2. *X_n is not diffeomorphic to a connected sum $M_1 \# M_2$ with $b_2^+(M_i) > 0$, $i = 1, 2$, for $n > 0$, and for all but finitely many n it is not diffeomorphic to a connected sum of complex surfaces and complex surfaces with reversed orientations.*

Proof. Recall, first of all, that by Kodaira's classification any simply connected complex surface is diffeomorphic to an algebraic one. Hence we may assume that all the surfaces involved are algebraic. By Proposition 3.2 X_n has nonvanishing Donaldson invariants for $n > 0$; hence [3] it is not diffeomorphic to a connected sum $M_1 \# M_2$ with $b_2^+(M_i) > 0$, $i = 1, 2$. This implies that X_n, having the homotopy type of $7\mathbf{CP}^2 \# 37\overline{\mathbf{CP}}^2$, cannot contain a (possibly blown-up) $K3$ summand with either the canonical or the reversed orientation, because such a summand would have $b_2^+ = 3$ or ≥ 19 respectively. Also, X_n cannot contain a (possibly blown-up) elliptic surface Y_n with the canonical orientation, because by the classification of elliptic surfaces $b_2^+(Y_n) = b_2^+(X_n) = 7$ implies $b_2^-(Y_n) \geq 39$. And $-Y_n$ cannot appear as well, because $b_2^+(-Y_n) \geq 9$. Hence the only possible summands are rational and general type surfaces. But the number of non-diffeomorphic 4-manifolds homotopy equivalent to $7\mathbf{CP}^2 \# 37\overline{\mathbf{CP}}^2$ which is possible to construct by taking connected sums of such surfaces is finite, because there are only finitely many diffeomorphism classes of rational or (possibly nonminimal) general type surfaces with $b_2 \leq 44$ (see [8], for instance). Therefore, by Theorem 4.1, for all but finitely many n, X_n is not diffeomorphic to a connected sum of complex surfaces and complex surfaces with reversed orientations.

Remark 4.3. The X_n's stay distinct after taking connected sums with homotopy spheres, because the Donaldson invariants do not change and the proof of Theorem 4.1 goes through. The proof of Corollary 4.2 can be easily adapted to show that, for all but finitely many n, the connected sum of X_n with a homotopy sphere is not diffeomorphic to a connected sum of surfaces (with any choice of the orientations) and a homotopy sphere.

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References

[1] S. Akbulut & R. Kirby, *Branched covers of surfaces in 4-manifolds*, Math. Ann. **252** (1980) 111–131.

[2] M. Atiyah, *New invariants of 3- and 4-dimensional manifolds*, Proc. Sympos. Pure Math., Vol. 48, Amer. Math. Soc., Providence, RI, 1988, 285–299.

[3] S. K. Donaldson, *Polynomial invariants for smooth 4-manifolds*, Topology **29** (1990) 257–315.

[4] S. K. Donaldson & P. B. Kronheimer, *The geometry of four-manifolds*, Clarendon Press, Oxford, 1990.

[5] R. Fintushel & R. Stern, *Instanton homology of Seifert fibered homology three-spheres*, Proc. London Math. Soc. **61** (1990) 109–137.

[6] A. Floer, *An instanton invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988) 215–240.

[7] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982) 357–453.

[8] R. Friedman & J. W. Morgan, *Algebraic surfaces and 4-manifolds: some conjectures and speculations*, Bull. Amer. Math. Soc. (N.S.) **18** (1988) 1–19.

[9] ——, *Smooth four-manifolds and complex surfaces*, to appear.

[10] R. E. Gompf, *Nuclei of elliptic surfaces*, Topology **30** (1991) 479–511.

[11] R. E. Gompf & T. Mrówka, *Irreducible four-manifolds need not be complex*, preprint.

[12] R. Kirby, *A calculus for framed links in* S^3, Invent. Math. **45** (1978) 36–56.

[13] T. Mrówka, *A Mayer-Vietoris principle for Yang-Mills moduli spaces*, Ph.D. thesis, Berkeley, 1989.

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